Upper and lower bounds for stochastic Bellman functions by nodal decomposition

Application to the decentralized optimization of urban micro-grids

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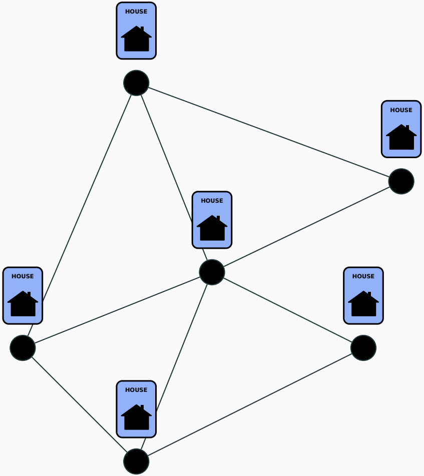
Motivation

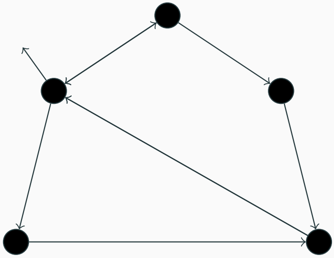
We consider a *peer-to-peer* community, where different buildings exchange energy



- Each node is a decision center
- Power flows through edges
- Multistage decisions
- Large-scale problem

Problem statement

Modeling exchanges between nodes



The grid is represented by a graph

Let $T \in \mathbb{N}^\star$ be a horizon

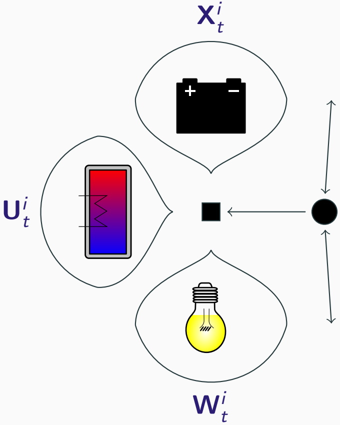
At each time $t \in [\![0, T-1]\!]$ we consider a coupling between the nodal subproblems

$$\sum_{i=1}^{N} \Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i) = 0$$

with $\Theta_t^i : \mathbb{X}_t^i \times \mathbb{U}_t^i \to \mathbb{R}^p$ inducing p *coupling constraints*

Production at each node of the grid

At each node i of the grid, at each time t, we have



- $\mathbf{X}_t^i \in \mathbb{X}_t^i$: state variable (battery, hot water tank)
- $\mathbf{U}_t^i \in \mathbb{U}_t^i$: control variable (energy production)
- $\mathbf{W}_t^i$: noise (consumption, renewable)

A stochastic optimization problem decoupled in space

At time t, we consider at node i

- An *instantaneous cost*

$$L_t(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}^i)$$

- A *dynamic* constraint

$$\mathbf{X}_{t+1}^i = f_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}^i)$$

- A *non-anticipativity* constraint

$$\sigma(\mathbf{U}_t^i) \subset \sigma(\mathbf{W}_0^i, \cdots, \mathbf{W}_t^i) = \mathcal{F}_t^i$$

Writing down the global optimization problem

We aim at minimizing the operational costs over the nodes $i \in [\![1, N]\!]$

$$\min_{\mathbf{X},\mathbf{U}} \mathbb{E}\Big[\sum_{i=1}^{N} \sum_{t=0}^{T-1} L_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}^i) + K^i(\mathbf{X}_T^i)\Big]$$

subject to, for all $t \in [\![0, T-1]\!]$

i) The nodal dynamics constraints

$$\mathbf{X}_{t+1}^i = f_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i, \mathbf{W}_{t+1}^i) \quad \forall i \in [\![1, N]\!]$$

ii) The non-anticipativity constraints

$$\sigma(\mathbf{U}_t^i) \subset \sigma(\mathbf{W}_0^i, \cdots, \mathbf{W}_t^i) \quad \forall i \in [\![1, N]\!]$$

iii) The coupling constraint

$$\sum_{i=1}^{N} \Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i) = 0$$

What do we plan to do

- We have formulated a stochastic optimization problem
- Without coupling, the problem could be decomposed
- We will handle the coupling constraint by two methods:
  price and quantities decompositions
- We will show that decomposition leads to *lower* and *upper* bounds for the original problem
  - Price decomposition yields a lower bound
  - Quantities decomposition yields an upper bound
- Those bounds can be obtained by Dynamic Programming
  with nodal Bellman functions in low dimension

Lecture outline

Decomposition by prices and quantities

We consider an abstract problem

Let, for $i \in [\![1, N]\!]$

- $u^i \in \mathbb{R}^{m_i}$ be a decision variable
- $J^i : \mathbb{R}^{m_i} \to \mathbb{R},\ i \in [\![1, N]\!]$ be a proper function
- $\Theta^i : \mathbb{R}^{m_i} \to \mathbb{R}^p$ be a coupling constraint

We consider the following problem

$$V^\sharp = \inf_{u^1, \cdots, u^N} \sum_{i=1}^{N} J^i(u^i)$$

$$\text{s.t.} \ \sum_{i=1}^{N} \Theta^i(u^i) = 0$$

Price decomposition

$$
\begin{aligned}
V^\sharp &= \inf_{u^1,\cdots,u^N} \sup_{\lambda} \sum_{i=1}^{N} J^i(u^i) + \left\langle \lambda \,, \Theta^i(u^i) \right\rangle \\
&\geq \sup_{\lambda} \inf_{u^1,\cdots,u^N} \sum_{i=1}^{N} \Big(J^i(u^i) + \left\langle \lambda \,, \Theta^i(u^i) \right\rangle \Big) \\
&= \sup_{\lambda} \sum_{i=1}^{N} \underbrace{\inf_{u^i} \Big(J^i(u^i) + \left\langle \lambda \,, \Theta^i(u^i) \right\rangle \Big)}_{\underline{V}^i(\lambda)}
\end{aligned}
$$

Quantities decomposition

$$
\begin{aligned}
V^\sharp &= \inf_{u^1,\cdots,u^N} \sum_{i=1}^{N} J_i(u_i) \\
&= \inf_{\substack{q^1,\cdots,q^N \\ q^1+\cdots+q^N=0}} \inf_{\substack{u^1,\cdots,u^N \\ \Theta^i(u_i)=q^i}} \sum_{i=1}^{N} J_i(u_i) \\
&= \inf_{\substack{q^1,\cdots,q^N \\ q^1+\cdots+q^N=0}} \sum_{i=1}^{N} \underbrace{\inf_{\substack{u_i \\ \Theta^i(u_i)=q^i}} J_i(u_i)}_{\overline{V}^i(q^i)} \\
&\leq \sum_{i=1}^{N} \overline{V}^i(q^i) \quad \text{s.t. } q^1+\cdots+q^N=0
\end{aligned}
$$

Bounds on decomposed functions

Theorem

For any

- *multiplier* $\lambda \in \mathbb{R}^p$
- *allocation* $q = (q^1, \cdots, q^N)$ *such that* $q^1 + \cdots + q^N = 0$

we have

$$\sum_{i=1}^{N} \underline{V}^i(\lambda) \leq V^\sharp \leq \sum_{i=1}^{N} \overline{V}^i(q^i)$$

Decomposition of Bellman functions

Global value functions

Let $x_t = (x_t^1, \cdots, x_t^N)$ be the global state, lying in $\mathbb{X}_t = \mathbb{X}_t^1 \times \cdots \times \mathbb{X}_t^N$

The global value function $V_t : \mathbb{X}_t \to \mathbb{R}$ writes

$$
\begin{aligned}
V_t(x_t) = \min_{\mathbf{X},\mathbf{U}} \ & \mathbb{E}\Big[\sum_{i=1}^{N}\Big(\sum_{s=t}^{T-1} L_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) + K^i(\mathbf{X}_T^i)\Big)\Big] \\
\text{s.t.} \quad & \mathbf{X}_{s+1}^i = f_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) \ , \ \ \mathbf{X}_t^i = x_t^i \quad \forall i \in [\![1, N]\!] \\
& \sigma(\mathbf{U}_s^i) \subset \mathcal{F}_s^i \quad \forall i \in [\![1, N]\!] \\
& \sum_{i=1}^{N} \Theta_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i) = 0
\end{aligned}
$$

Two decomposition schemes to decouple the problem

Nodal subproblems are coupled via the constraints

$$\sum_{i=1}^{N} \Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i) = 0$$

We decompose global optimization problem by

1. *Price decomposition*: we dualize the coupling constraint via a multiplier $\boldsymbol{\lambda}$

$$\sum_{i=1}^{N} \Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i) = 0 \quad \rightsquigarrow \boldsymbol{\lambda}_t$$

2. *Quantities decomposition*: for any allocation $\mathbf{Q}_t = (\mathbf{Q}_t^1, \cdots, \mathbf{Q}_t^N)$ such that $\sum_{i=1}^{N} \mathbf{Q}_t^i = 0$ we put

$$\Theta_t^i(\mathbf{X}_t^i, \mathbf{U}_t^i) = \mathbf{Q}_t^i$$

Decomposition of Bellman functions

Price nodal decomposition

Price nodal value functions

Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_{T-1})$ be a stochastic process

We define the price nodal value function at time $t = 0$

$$\underline{V}_0^i[\boldsymbol{\lambda}](x_0^i) = \min_{\mathbf{X}^i, \mathbf{U}^i} \mathbb{E}\Big[\sum_{s=0}^{T-1} L_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) + \underbrace{\langle \boldsymbol{\lambda}_s\,, \Theta_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i)\rangle}_{coupling} + K^i(\mathbf{X}_T^i)\Big]$$

$$\begin{aligned} \text{w.r.t.} \quad & \mathbf{X}_{s+1}^i = f_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) \\ & \mathbf{X}_0^i = x_0^i \\ & \sigma(\mathbf{U}_s^i) \subset \mathcal{F}_s^i \end{aligned}$$

Price nodal value functions are lower bounds

Theorem

For all multipliers $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_{T-1})$

$$\sum_{i=1}^{N} \underline{V}_0^i[\boldsymbol{\lambda}](x_0^i) \leq V_0(x_0) \;, \qquad \forall\, x_0 = (x_0^1, \cdots, x_0^N)$$

Proof

Adaptation of the previous proof, considering

$$\underline{V}_0[\boldsymbol{\lambda}](x_0) = \sum_{i=1}^{N} \underline{V}_0^i[\boldsymbol{\lambda}](x_0^i)$$

Solving price nodal value functions by Dynamic Programming

We are able to solve $\underline{V}_0^i$ node by node by Dynamic Programming if

- noises $\mathbf{W}_0^i, \cdots, \mathbf{W}_T^i$ are independent
- the random process $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_{T-1})$ is
  - either a constant random process
  - or such that $\boldsymbol{\lambda}_t = \phi_t(\mathbf{W}_{t+1})$
    (supposing that $\mathbf{W}_{t+1}^i = \mathbf{W}_{t+1}$ for all i)

Then for all $t = T-1, \cdots, 0$ we define recursively
the price nodal value functions at time t

$$\underline{V}_t^i[\boldsymbol{\lambda}](x_t^i) = \min_{u_t^i} \mathbb{E}\Big[L_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}^i) + \langle \boldsymbol{\lambda}_t \,, \Theta_t^i(x_t^i, u_t^i)\rangle + \underline{V}_{t+1}^i[\boldsymbol{\lambda}]\big(f_t(x_t^i, u_t^i, \mathbf{W}_{t+1}^i)\big)\Big]$$

Price nodal value functions are lower-bounds

Theorem

Let $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_{T-1})$ be a multiplier among one of the two previous classes
For all $t \in [\![0, T-1]\!]$, we have

$$\sum_{i=1}^{N} \underline{V}_t^i[\boldsymbol{\lambda}](x_t^i) \leq V_t(x_t) \; , \qquad \forall \, x_t = (x_t^1, \cdots, x_t^N)$$

Proof

By induction

Decomposition of Bellman functions

Quantities nodal decomposition

Quantities nodal value functions

Let $\mathbf{Q} = (\mathbf{Q}_0, \cdots, \mathbf{Q}_{T-1})$ be an allocation process such that

$$\mathbf{Q}_t^1 + \cdots + \mathbf{Q}_t^N = 0$$

We define the quantities nodal value function at time $t = 0$

$$\begin{aligned} \overline{V}_0^i[\mathbf{Q}](x_0^i) = \min_{\mathbf{X}^i, \mathbf{U}^i} \; & \mathbb{E}\Big[\sum_{s=0}^{T-1} L_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) + K^i(\mathbf{X}_T^i)\Big] \\ \text{w.r.t.} \quad & \mathbf{X}_{s+1}^i = f_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i, \mathbf{W}_{s+1}^i) \\ & \mathbf{X}_0^i = x_0^i \\ & \sigma(\mathbf{U}_s^i) \subset \mathcal{F}_s^i \\ & \underbrace{\Theta_s^i(\mathbf{X}_s^i, \mathbf{U}_s^i) = \mathbf{Q}_s^i}_{\textit{coupling}} \end{aligned}$$

Quantities nodal value functions are upper-bounds

Theorem

For all stochastic process $\mathbf{Q} = (\mathbf{Q}_0, \cdots, \mathbf{Q}_{T-1})$
such that $\mathbf{Q}_t^1 + \cdots + \mathbf{Q}_t^N = 0$

$$V_0(x_0) \leq \sum_{i=1}^{N} \overline{V}_0^i[\mathbf{Q}](x_0^i) \;, \qquad \forall\, x_0 = (x_0^1, \cdots, x_0^N)$$

Proof

Adaptation of the previous proof, considering

$$\overline{V}_0[\mathbf{Q}](x_0) = \sum_{i=1}^{N} \overline{V}_0^i[\mathbf{Q}](x_0^i)$$

Solving quantities nodal value functions by DP

With the same assumptions as in price nodal value functions ($\mathbf{Q}$ is constant or such that $\mathbf{Q}_t = \psi_t(\mathbf{W}_{t+1})$) we are able to solve $\overline{V}_0^i$ node by node by Dynamic Programming

We define recursively the quantities nodal value function at time t

$$\overline{V}_t^i[\mathbf{Q}](x_t^i) = \min_{u_t^i} \mathbb{E}\Big[L_t^i(x_t^i, u_t^i, \mathbf{W}_{t+1}^i) + \overline{V}_{t+1}^i[\mathbf{Q}]\big(f_t(x_t^i, u_t^i, \mathbf{W}_{t+1}^i)\big)\Big]$$
$$\text{s.t. } \Theta_t^i(x_t^i, u_t^i) = \mathbf{Q}_t^i$$

Theorem

For all allocation $\mathbf{Q} = (\mathbf{Q}_0, \cdots, \mathbf{Q}_{T-1})$ such that $\mathbf{Q}_t^1 + \cdots + \mathbf{Q}_t^N = 0$ among the two previous classes. For all $t \in [\![0, T-1]\!]$, we have

$$V_t(x_t) \leq \sum_{i=1}^{N} \overline{V}_t^i[\mathbf{Q}](x_t^i) , \qquad \forall\, x_t = (x_t^1, \cdots, x_t^N)$$

We obtain upper and lower bounds for the original problem :)

Theorem

Let $t \in [\![0, T]\!]$

- *For all multiplier* $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_0, \cdots, \boldsymbol{\lambda}_{T-1})$ *such that* $\boldsymbol{\lambda}_t$ *is constant or* $\boldsymbol{\lambda}_t = \phi_t(\mathbf{W}_{t+1})$
- *For all allocation* $q = (\mathbf{Q}_0, \cdots, \mathbf{Q}_{T-1})$ *such that* $\mathbf{Q}_t$ *is constant or* $\mathbf{Q}_t = \psi_t(\mathbf{W}_{t+1})$, *satisfying* $\mathbf{Q}_t^1 + \cdots + \mathbf{Q}_t^N = 0$

we have

$$\sum_{i=1}^{N} \underline{V}_t^i[\boldsymbol{\lambda}](x_t^i) \ \leq \ V_t(x_t) \ \leq \ \sum_{i=1}^{N} \overline{V}_t^i[\mathbf{Q}](x_t^i) \ , \qquad \forall \ x_t = (x_t^1, \cdots, x_t^N)$$

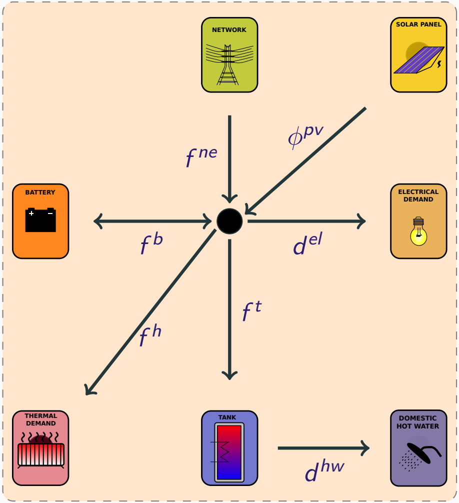
Where are we heading to?

- We have established upper and lower bounds
  for the global optimization problem

- Now we illustrate these results with numerical examples
  - We apply nodal decomposition
    to the management of urban microgrid
  - We obtain surprisingly tight bounds!

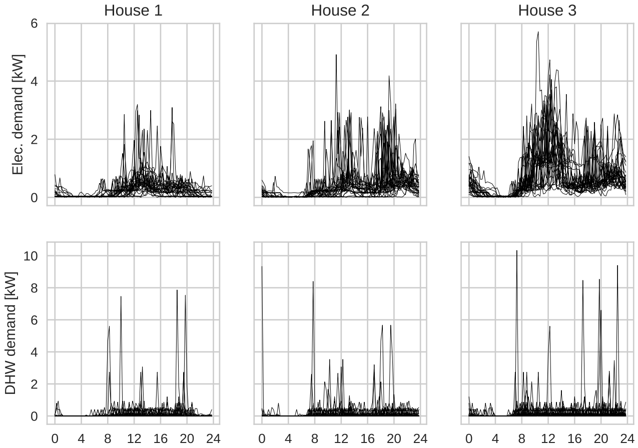
Application to the management of microgrids

Each house owns different devices



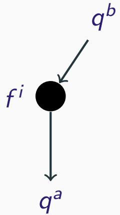
- **Stock variables** $\mathbf{X}_t = (\mathbf{B}_t, \mathbf{H}_t, \boldsymbol{\theta}^i_t, \boldsymbol{\theta}^w_t)$
  - $\mathbf{B}_t$, battery level (kWh)
  - $\mathbf{H}_t$, hot water storage (kWh)
  - $\boldsymbol{\theta}^i_t$, inner temperature ($^\circ C$)
  - $\boldsymbol{\theta}^w_t$, wall's temperature ($^\circ C$)
- **Control variables** $\mathbf{U}_t = (\mathbf{F}_{\mathbf{B},t}, \mathbf{F}_{T,t}, \mathbf{F}_{\mathbf{H},t})$
  - $\mathbf{F}_{\mathbf{B},t}$, energy exchange with the battery (kW)
  - $\mathbf{F}_{T,t}$, energy used to heat the hot water tank (kW)
  - $\mathbf{F}_{\mathbf{H},t}$, thermal heating (kW)
- **Uncertainties** $\mathbf{W}_t = (\mathbf{D}^{el}_t, \mathbf{D}^{hw}_t)$
  - $\mathbf{D}^{el}_t$, electrical demand (kW)
  - $\mathbf{D}^{hw}_t$, domestic hot water demand (kW)

Electrical and thermal demands are uncertain



These scenarios are generated with `StRoBE`, a generator open-sourced by KU-Leuven

Connecting house to the remaining graph



At each node, we consider injection flow f

$$f^i = \sum_{\ell \in \epsilon(i)} q^\ell$$

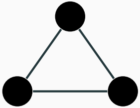
with $\epsilon(i)$ set of edges connected to i
and q^ℓ flow through arcs ℓ

The load balance equation at node i writes

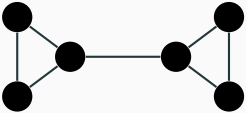
$$\underbrace{\mathbf{F}^i_{NE,t+1}}_{Network} = \underbrace{{\mathbf{D}^{el}_{t+1}}^i}_{Demand} + \underbrace{\mathbf{F}^i_{\mathbf{B},t}}_{Battery} + \underbrace{\mathbf{F}^i_{\mathbf{H},t}}_{Heating} + \underbrace{\mathbf{F}^i_{T,t}}_{Tank} - \underbrace{\phi^{pv,i}_t}_{Solar\ \ panel} + \underbrace{\mathbf{F}^i_t}_{Injection}$$

We consider four different networks

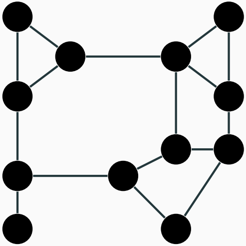
A



B



C



D

Configurations

We consider three kinds of houses, with different devices

H1: Solar panels + battery + hot water tank

H2: Solar panels + hot water tank

H3: Hot water tank

Batteries are mutualized, thus favoring the exchanges between houses H1 and houses H2 and H3

The different graphs have growing state dimensions

Graph	N (nodes)	L (arcs)	$dim(\mathbb{X})$	$dim(\mathbb{W})$	$card(\mathbb{W})$
A	2	1	7	4	10^2
B	3	3	10	6	10^3
C	6	7	20	12	10^6
D	12	15	40	24	10^{12}

Results on the two nodes graph

A

- $x_0 = (x_0^1, x_0^2) \in \mathbb{R}^7$ the initial position
- $V_0(x_0)$ the exact solution of the problem (unknown)

We get the following results

ALGO	Lower bound		Upper bound	Gap
NODAL	1.16	$\leq V_0(x_0) \leq$	1.18	1.7 %
SDDP	1.17	$\leq V_0(x_0) \leq$	?	?

Displaying all results for nodal decomposition

Graph	$dim(\mathbb{X}_t)$	Lower bound	Upper bound	Gap
2 nodes	7	1.16	1.18	1.7 %
3 nodes	10	3.09	3.14	1.6 %
6 nodes	20	6.18	6.28	1.6 %
12 nodes	40	12.37	12.58	1.7 %

Computation time

We denote by $\widehat{V}_t$ the value functions computed by SDDP (when possible)

Algo	Graph A	Graph B	Graph C	Graph D
$\sum_i \underline{V}_0^i(x_0)$	1.16 6'	3.09 11'	6.18 26'	12.37 42'
$\widehat{V}_0(x_0)$	1.17 6'	3.11 37'	? ?	? ?
$\sum_i \overline{V}_0^i(x_0)$	1.18 7'	3.14 10'	6.28 28'	12.58 79'

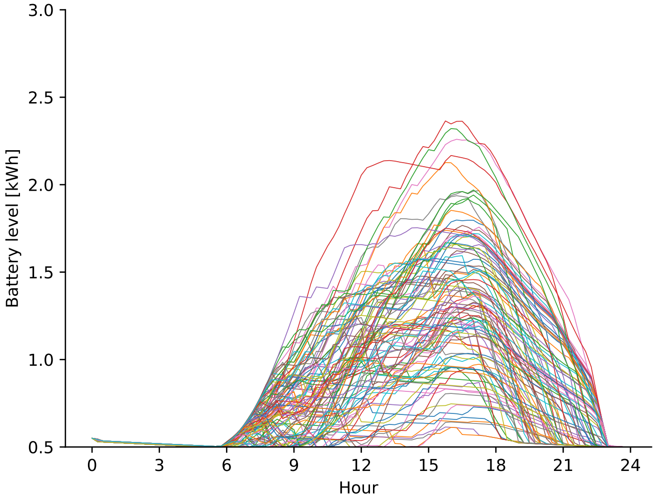
Conclusion

Conclusion

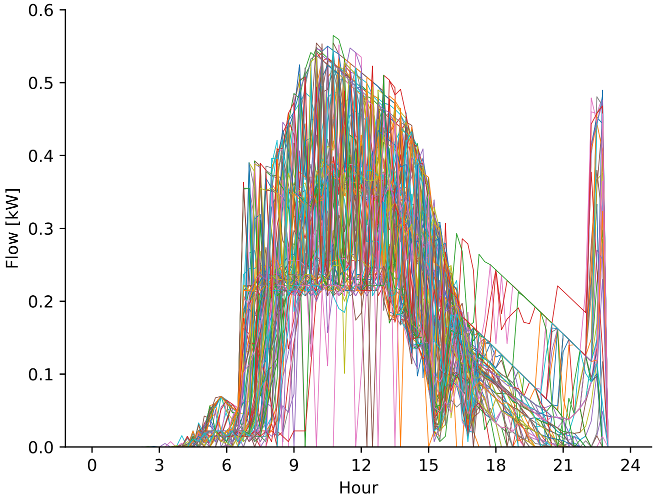
- With problems with state dimension up to 40,
  we obtain tight bounds (less than 2%)
  for a running time up to 1h

- Can we obtain tighter bounds?
  If we select properly the stochastic processes $\mathbf{Q}$ and $\boldsymbol{\lambda}$,
  we can obtain nodal value functions but with an extended local state

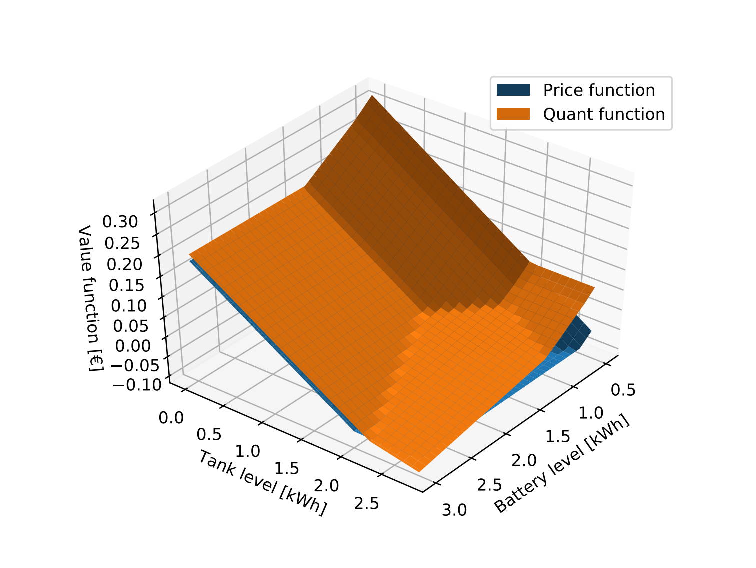
Battery level



Flow through arc



Comparing prices with quantities nodal value functions

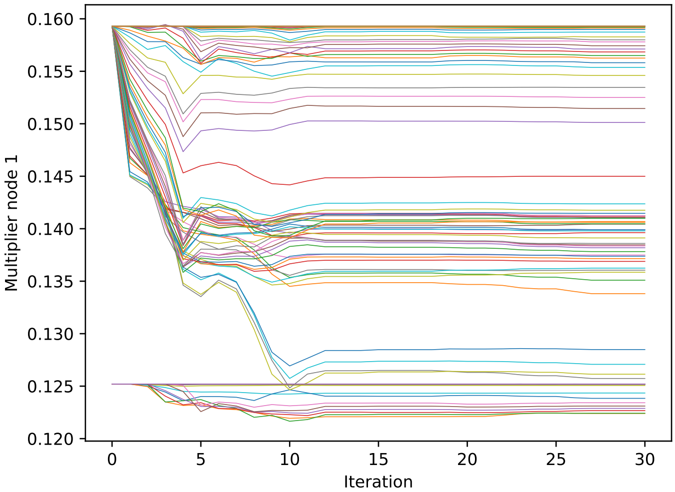


Numerical results

Implementation

- Gradient descent is performed with IPOPT (L-BFGS-B)
- Dynamic Programming is solved by SDDP
- QP subproblems are solved with Gurobi 7.02
- The glue code is implemented with Julia 0.6

Convergence of multipliers



Displaying optimal multipliers