# Homeworks

*SMDS-2-2020*
*https://elearning2.uniroma1.it/course/view.php?id=8609*

***deadline 22/06/2020 (23:55)*** *2019-2020*
*(write your answers and provide your R code for the numerical solutions)*

**Your Last+First Name ___LLLLLLLLL___FFFFFF___ Your Matricola 99999**

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1) Acceptance-Rejection algorithm

a. Use the acceptance-rejection algorithm to provide 10000 i.i.d. simulations from a target standard normal distribution $\pi(\cdot)$ using a Cauchy distribution as auxiliary density $q(\cdot)$.
b. How can you numerically evaluate the efficiency of this simulation strategy? Provide your numerical evaluation and explain how you can compute it exactly or approximately.

2) Infinitely exchangeable sequence of binary random variables $X_1, ..., X_n, ...$

a. Provide the definition of an infinitely echangeable binary sequence of random variables $X_1, ..., X_n, ....$. Consider the De Finetti representation theorem relying a suitable distribution $\pi(\theta)$ on $[0, 1]$ and show that
   - $E[X_i] = E_\pi[\theta]$
   - $E[X_i X_j] = E_\pi[\theta^2]$
   - $Cov[X_i X_j] = Var_\pi[\theta]$

b. Prove that any couple of random variabes in that sequence must be non-negatively correlated.

c. Find what are the conditions on the distribution $\pi(\cdot)$ so that $Cor[X_i X_j] = 1$.

d. What do these conditions imply on the type and shape of $\pi(\cdot)$? (make an example)

3a) Illustrate the characteristics of the statistical model for dealing with the *Dugong*'s data. Lengths ($Y_i$) and ages ($x_i$) of 27 dugongs ([sea cows](https://en.wikipedia.org/wiki/Dugong)) captured off the coast of Queensland have been recorded and the following (non linear) regression model is considered in Carlin and Gelfand (1991)

$$\begin{aligned} Y_i &\sim N(\mu_i, \tau^2) \\ \mu_i = f(x_i) &= \alpha - \beta\gamma^{x_i} \end{aligned}$$

Model parameters are $\alpha \in \Re$, $\beta \in \Re$, $\gamma \in (0, 1)$, $\tau^2 \in (0\infty)$. Let us consider the following prior distributions:

$$\alpha \sim N(0, \sigma^2_\alpha) \ ; \ \ \beta \sim N(0, \sigma^2_\beta) \ ; \ \ \gamma \sim Unif(0, 1) \ ; \ \ \tau^2 \sim IG(a, b))(InverseGamma)$$

3b) Derive the corresponding likelihood function

3c) Compute numerically the maximum likelihood estimate $\hat{\theta}_{ML}$ for the vector of parameters of interest $(\alpha, \beta, \gamma, \tau^2)$ and compare it with the Maximum-a-Posterori estimate $\hat{\theta}_{MaP}$. (Suggestion: use the 'optim(...)' function or 'fitdistr(...)' in R )

3d) Write down the expression of the joint prior distribution of the parameters at stake and illustrate your suitable choice for the hyperparameters.

3e) Write down the expression of the poterior density up to a proportionality constant and derive all the expressions of the 4 full conditional distributions up to proportionality constants. For which components can you recognize some known parametric distributions?

3f) Implement an MCMC algorithm based on a Gibbs cycle in which

$$\begin{aligned} &\alpha_{t+1} && \sim \pi(\alpha|\beta_t, \gamma_t, \tau^2_t) \\ &\beta_{t+1} && \sim \pi(\beta_{t+1}|\alpha_{t+1}, \gamma_t, \tau^2_t) \\ &\gamma_{t+1} && \sim \pi(\gamma_{t+1}|\alpha_{t+1}, \beta_{t+1}, \tau^2_t) \\ &\tau^2_{t+1} && \sim \pi(\tau_{t+1}|\alpha_{t+1}, \beta_{t+1}, \gamma_{t+1}) \end{aligned}$$

replacing, when the full conditional distribution is not recognised as a known type of distribution with the simulation based on a Metropolis Hastings algoritm having as a univariate target density the full conditional density and as a proposal a fixed (independendent) proposal in the appropriate univariate support (e.g Uniform if the component takes value in $(0, 1)$ or gaussian with fixed parameters if the component takes value in $\Re$).

3g) Show the 4 univariate trace-plots of the simulations of each parameter

3h) Evaluate graphically the behaviour of the empirical averages $\hat{I}_t$ with growing $t = 1, ..., T$

3i) Provide estimates for each parameter together with the approximation error and explain how you have evaluated such error

3j) Which parameter has the largest posterior uncertainty? How did you measure it?

3k) Which couple of components of the parametr vector $(\alpha, \beta, \gamma, \tau^2)$ has the largest correlation (in absolute value)?

3l) Use the Markov chain to approximate the posterior predictive distribution of the length of a dugong with age of 20 years.

3m) Provide the prediction of another dugong with age 30

3n) Which prediction is less precise?

4) Suppose `mu` is a column vector with the following entries

```
mu
```

```
## [1] 0.19 0.53 0.28
```

representing the probability mass distribution over a suitable finite discrete state space $\mathcal{S} = \{1, 2, ..., d\}$ of the random starting state of a Markov chain $X_0$ with transition probability matrix `P`

```
P
```

```
##      [,1] [,2] [,3]
## [1,] 0.47 0.28 0.25
## [2,] 0.34 0.32 0.34
## [3,] 0.04 0.53 0.43
```

Write the most compact R code to obtain the following ditributions as R ojects:

a. `m_0` – the marginal distribution of $X_0$ as a column vector
b. `m_1` – the marginal distribution of $X_1$ as a column vector
c. `J` – the joint distribution of $(X_0, X_1)$ as a matrix with generic entry at row $r$ and column $c$ equal to $Pr\{X_1 = r, X_0 = c\}$
d. `C_1_0` – all conditional distributions of $X_1|X_0$ as a matrix with generic entry at row $r$ and column $c$ equal to $Pr\{X_1 = c|X_0 = r\}$
e. `C_0_1` – all conditional distributions of $X_0|X_1$ as a matrix with generic entry at row $r$ and column $c$ equal to $Pr\{X_0 = c|X_1 = r\}$

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## This homework will be graded and it will be part of your final evaluation
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## Last update by LT: Fri Jun 12 19:22:46 2020
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