MARRIAGE AND CLASS*

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Here we consider a matching model where agents are heterogeneous and utilities nontransferable. We utilize this framework to study how equilibrium sorting takes place in marriage markets. We impose conditions that guarantee the existence of a steady state equilibrium and then characterize it. Several examples are developed to illustrate the richness of equilibria. The model reveals an interesting sorting externality that can support multiple steady state equilibria, even with constant returns to matching.

The matching framework, developed by Mortensen [1982, 1985], Diamond [1982], and Pissarides [1990], has proved to be a useful tool in labor economics, macroeconomics, and monetary theory. Here we consider a matching model where agents are heterogeneous and utilities are nontransferable. We utilize this framework to study how equilibrium sorting takes place in marriage markets. We impose conditions that guarantee the existence of a steady state equilibrium and then characterize it.

The matching framework appears ideally suited for the analysis of marriage markets. First, singles clearly face difficulty contacting each other: a defining characteristic of matching models. Second, each single's objective, like employers and workers in a labor market, is to form a long-term relationship: the typical case in matching models. Third, singles compete, in the sense that they all do the best they can given their expectations and the constraints they face. It should be noted that the focus is on the macroeconomics of marriage markets rather than on the decision problems faced by the participants. Indeed, the individual decision problem will be kept as simple as possible.

Although the focus of attention of most studies on the economics of marriage has been on the individual decision problem, it has been noted at some length that positive assortative mating is a predicted characteristic of equilibrium. Positive assortative mating indicates a positive association between the traits of partners. Becker [1973, 1974] shows that there is strong empirical evidence of a positive correlation between intelligence, education, height, attractiveness, nonhuman wealth, etc. He suggests how this can come about within the context of a full-information mar-

*We would like to thank an anonymous referee, Patricia Dillon, Lones Smith, Randall Wright, and Dale Mortensen for more than helpful comments. Of course, we are responsible for remaining errors.

The Quarterly Journal of Economics, February 1997.

ket model. Here we construct a decentralized marriage market model that predicts a very special kind of positive assortative mating as an equilibrium outcome.

Looking for a marriage partner is clearly a complex task. Singles of the opposite sex possess many different characteristics that make them more or less desirable. Evaluating these characteristics to determine whether an individual will make a good partner is complicated, especially as both people evaluate each other at the same time. The particular characteristics individuals use in evaluating others is not discussed here. Instead, it is assumed throughout that each individual, man or woman, can be characterized by a real number; we term this the individual's pizazz.[1] If a man and woman marry, the utility flow obtained by the man equals the woman's pizazz, whereas the woman's utility flow equals the man's pizazz.[2] Another major simplification used throughout is that an individual's pizazz is instantly observed on contact.

In the market studied, singles only contact singles of the opposite sex every now and then. When this happens, each observes the other's pizazz. Then, both decide whether to propose or not. If both propose, they marry and leave the market. If at least one does not propose, they separate and continue to look for another partner. To whom should a person propose if contacted? The utility-maximizing solution depends on how quickly singles are contacted and who is expected to propose to this single if they make contact. In equilibrium, all singles must have correct expectations.

It is assumed throughout that the flow of new singles into the market is constant per unit of time. Further, the distribution of pizazz among new women and men is assumed constant through time (although the two distributions need not be the same). Of course, the strategies of all participating singles determine the rate at which they leave the market. In a steady state equilibrium the flow-in of new singles equals the number who leave the market, and the distribution of pizazz among both women and men in the market remains constant.

Given that a steady state exists, it will be shown that the

1. *Webster's Ninth New Collegiate Dictionary* [1985] defines pizazz on page 897 as the quality of being exciting or attractive.

2. Burdett and Wright [1994] consider a model similar to that described here. However, they assume that there is no agreement about the ranking of individuals by pizazz. Tastes are idiosyncratic.

best strategies of singles are such that they partition themselves into classes according to their pizazz levels. To illustrate, let $[\underline{p}_j, \bar{p}_j]$ denote the smallest closed set that contains the pizazz of all j, $j = m$ (men) and w (women). An n partition can be represented by $[p_{jn}, p_{jn-1}, \ldots, p_{j2}, p_{j1}, p_{j0}]$, $\underline{p}_j = p_{jn}$, and $\bar{p}_j = p_{j0}$. A j single with pizazz $p \in [p_{jk}, p_{jk+1})$ is a member of class k. In a steady state equilibrium women in class k only marry men in class k and vice versa. In such an equilibrium it is possible that one sex has one more class than the other. In this case, those in the $n + 1$ class, where the other sex has n classes, never marry.

Similar equilibrium characteristics have been obtained by others within the context of different but related models. In this literature the matching problem is simplified by assuming that any pair who leaves the market are immediately replaced by clones (see, for example, MacNamara and Collins [1990], Bloch and Ryder [1994], and Morgan [1994]). By ensuring that the steady state pizazz distributions are independent of the matching decisions of agents, this cloning assumption guarantees a simple existence and uniqueness result. In this study clones do not automatically replace those who exit; there is an exogenous flow-in of new singles into the market. Of course, in a steady state, equilibrium requires that the (endogenous) distribution of pizazz of those who exit equal the flow-in distribution.[3] This extension is important as it not only best captures the marriage market, but it allows the marriage decisions to influence the steady state distributions of pizazz of singles in the market. The model reveals an interesting sorting externality that can support multiple steady state equilibria, even with constant returns to matching. When there are multiple equilibria, individuals with high pizazz are more selective in some equilibria than in others. By becoming more selective, the matching rate of these types is lower, and consequently in a steady state their number is greater. This sorting effect can support their decision to be more selective, and multiple equilibria are possible with interesting welfare implications.

The results obtained have clear relevance to the analysis of labor markets (for example, see Sattinger [1995] and Davis [1995]). In the labor market context both workers and employers

3. The closest paper in the literature to this case is Smith [1995]. He assumes no entry of new singles into the market and considers the resulting nonsteady state matching dynamics through time. He also shows that, when singles have more general preferences than those used here, the class partition result may not exist.

seek long-term relationships. Further, some workers may well be more productive than others, and some firms have more productive jobs. The major difference between the marriage market and a labor market is that workers and employers can negotiate over wages; i.e., utility is transferable. Although this is a nontrivial extension, intuition suggests that similar sorting externalities will arise. Lu and McAfee [1995] have used a similar approach to that proposed here to analyze a housing market.

The paper is organized as follows. In Section I the basic framework is outlined. In Section II we assume that singles have partially rational expectations. In particular, we assume that singles are perfectly rational except all expect the two current distributions of pizazz among singles in the market to remain constant through time. Given these near rational expectations, we characterize the unique equilibrium for any given pair of flow-in distributions. At such an equilibrium it is possible to calculate the distribution of pizazz of each sex among those who leave the market through marriage (or death) each period dt, as well as the number of each sex who leave. If these flow-out distributions and the number who leave equal the exogenous flow-in distributions and number who flow in, the near rational expectations equilibrium is fully rational as the distributions of pizazz among singles in the market will indeed remain constant through time. Conditions that guarantee the existence of such a fully rational steady state equilibrium are established in Section III. In Section IV we discuss some of the many implications and special cases that follow. These examples reveal a rich framework that leads to many testable predictions. In the final section other extensions are briefly discussed.

I. The Framework

Suppose that a large and equal number of single men and single women participate in a marriage market. Let $N(t)$ denote the number (measure) of women in the market at time t (which also equals the number of men). Assume that each individual can be characterized by a real number, the individual's pizazz. The importance of an individual's pizazz can be explained as follows. If a man and a woman decide to marry, the woman's utility from the marriage equals the man's pizazz, whereas the man's utility from the marriage equals the woman's pizazz. It will be shown that the advantage of having high pizazz is that it enables one to attract someone of the opposite sex who also has high pizazz. This assumption, however, rules out narcissism—looking in the mirror to admire one's own pizazz does not increase utility.

Unfortunately, singles in this market face difficulty contacting singles of the opposite sex. Let α be the arrival rate of singles of the opposite sex faced by a single of either sex, where α is the parameter of a Poisson process. As α is assumed to be independent of the number of participating singles, we have what is termed constant returns in the matching function. When two singles of the opposite sex meet, both observe the other's pizazz. If both propose marriage, they form a match and leave the market. There is no search while married. On the other hand, if at least one does not propose, they separate, and both continue to look for a partner.

The life of any individual is described by an exponential random variable with parameter $\delta > 0$. Hence, δdt denotes the probability that any individual dies in small time interval dt. To simplify the turnover dynamics, assume that an agent never returns to this market once a match has been formed (including the case when the partner has died). All agents discount at rate $r > 0$, and assume that both men and women obtain zero utility flow when single.

Let βdt denote the number of new single men and new single women who enter the market in any time interval dt. Among the set of new entrants at any time, let $F_w(z)$ denote the probability any new woman has pizazz no greater than z, whereas $F_m(z)$, denotes the probability any new man has pizazz no greater than z. Keeping things as simple as possible, assume that F_j is twice differentiable and strictly increasing over the interval $[\underline{x}_j, \bar{x}_j]$, where $\underline{x}_j$ and $\underline{w}_j$ indicate the infimum and supremum of its support, and $\underline{x}_j > 0, j = s, m$.

Of course, the distribution of pizazz among all single women in the market may not equal F_w. Let $G_w(.,t)$ denote the distribution of pizazz among single women in the market at time t. Similarly, $G_m(.,t)$ denotes the distribution of pizazz among single men at t. If a single woman meets a single man at time t, then $G_m(\bar{z},t)$ denotes the probability that the single man's pizazz is no greater than $\bar{z}$, whereas if a single man contacts a woman at t, $G_w(\bar{z},t)$ is the probability the woman's pizazz is no greater than $\bar{z}$. In this sense search is random.

II. The Stationary Environment

Throughout this section it is assumed that all singles believe the market can be characterized by (G_w, G_m), where

(R1) $$G_w(z,t) = G_w(z) \text{ for all } z \text{ and all } t;$$

(R2) $$G_m(z,t) = G_m(z) \text{ for all } z \text{ and all } t.$$

Assume that G_j is continuous and has support $\underline{x}_j, \overline{x}_j]$, $j = m, w$.[4]

(R1)–(R2) define a stationary market environment. A partial rational expectations equilibrium (PREE) is defined as follows. Conditional on the belief (R1)–(R2), a PREE requires that all agents use utility-maximizing strategies, given the behavior of other agents. We call this "partial" as it may not necessarily be the case that the flow-out of the market, which is generated by the resulting matching behavior, equals the flow-in (described by (F_w, F_m, β)). By characterizing a PREE for any (G_w, G_m), Section IV will then identify those (G_w, G_m) which imply that the flow-out distribution equals the flow-in distribution that therefore identifies all possible steady state equilibria.[5]

For simplicity, assume that all agents use stationary strategies, where a strategy is a list of people to whom a particular single will propose to on contact.[6] Hence, the set of individuals who will propose to a particular single if they make contact is well defined. Let $G_w(\cdot \mid x)$ denote the distribution of pizazz among single women who will propose to a man with pizazz x if they meet. In a similar fashion, define $G_m(\cdot \mid x)$ for all x. Further, the arrival rate of proposals faced by a single j with pizazz x at any time t, $\alpha_j(x)$, $j = w, m$, is also well defined.

Let $U_w(x)$ denote this woman's expected discounted lifetime utility when single. Given that utility x' is obtained if she marries someone with pizazz x', standard dynamic programming arguments imply that

4. A more general approach would not restrict G_j in this way. However, there is no loss of generality as the steady state analysis in the next section automatically implies that G_j and F_j have the same support, and no mass points in F_j implies no mass points in G_j, $j = w, m$.

5. Much of the literature has avoided this issue by assuming "clones," where those that exit the market are immediately replaced by identical agents. Proposition 1 below describes the equilibrium for this simpler case.

6. The following equilibrium with stationary strategies is in fact the unique PREE.

$$(1 + rdt)U_w(x) = (1 - \delta dt)\left[\alpha_w(x)\,dtE\max\{\tilde{x}, U_w(x) \mid x \text{ and offer}\}\right] + (1 - \delta dt)\left[(1 - \alpha_w(x)\,dt)U_w(x)\right],$$

where, given that an offer is made, $\tilde{x} \sim G_m(\cdot \mid x)$. Manipulating and letting $dt \to 0$ yields

$$(1)\quad (r + \delta)U_w(x) = \alpha_w(x)\left[E\max\{\tilde{x}, U_w(x) \mid x \text{ and offer}\} - U_w(x)\right].$$

Note that (1) implies a reservation match strategy is optimal, where $R_w(x) = U_w(x)$, as

$$(2)\qquad x' \gtrless U_w(x) \text{ as } x' \gtrless R_w(x).$$

Hence, this woman will propose to a man with pizazz x' if and only $x' \geq R_w(x)$. Of course, women with different pizazz may have different minimum standards.

Substituting (2) into (1) and manipulating yields an equation that defines the reservation match quality

$$(3)\qquad R_w(x) = \frac{\alpha_w(x)}{r + \delta}\int_{R_w(x)}^{\bar{x}_m}\left[1 - G_m(\tilde{x} \mid x)\right]d\tilde{x}.$$

As the situation is entirely symmetric, the reservation match quality of a man with pizazz x, $R_m(x)$, satisfies

$$(4)\qquad R_m(x) = \frac{\alpha_m(x)}{r + \delta}\int_{R_m(x)}^{\bar{x}_w}\left[1 - G_w(\tilde{x} \mid x)\right]d\tilde{x}.$$

Note that (3) describes the reservation match strategy of a woman with pizazz x, given the expected rate of proposals by men. In a PREE the arrival rate of proposals and the conditional offer distribution $G_m(\cdot \mid x)$ must be consistent with the reservation match strategy of men, described by (4). Similarly for men.

Equilibrium immediately implies that the reservation match strategies $R_j(\cdot)$ are nondecreasing, $j = m, w$. Any man willing to propose to a woman with pizazz x' is also willing to propose to a woman with pizazz x, where $x > x'$. By receiving at least the same offers, it follows that $U_w(x) \geq U_w(x')$. Hence $R_w(x) \geq R_w(x')$. Symmetry implies that $R_m(\cdot)$ is also nondecreasing.

The first proposition shows that in a PREE the participants partition themselves into n distinct classes. Proposition 1 shows that a woman in class n will only propose to men who are in the same class or higher, and will always reject a man from a lower class. Men do the same. In equilibrium only men and women from the same class marry.

PROPOSITION 1. Given (G_w, G_m), a PREE implies existence of a unique doubleton $(\{y_w(n)\}_{n=0}^{J_w}, \{y_m(n)\}_{n=0}^{J_m})$ such that

$$y_m(n) = \frac{\alpha}{r+\delta}\int_{y_m(n)}^{y_m(n-1)}\left[G_m\left(y_m(n-1)\right) - G_m(x)\right]dx \tag{5}$$

$$y_w(n) = \frac{\alpha}{r+\delta}\int_{y_w(n)}^{y_w(n-1)}\left[G_w\left(y_w(n-1)\right) - G_w(x)\right]dx, \tag{6}$$

where (a) $y_m(0) = \bar{x}_m$, (b) $y_w(0) = \bar{x}_w$, (c) $y_m(J_m) \leq \underline{x}_m$, and (d) $y_w(J_w) \leq \underline{x}_w$. Further, for any n such that $0 < n \leq \min\{J_w, J_m\}$: (i) a single woman with pizazz $x \in [y_w(n), y_w(n-1))$ proposes to a single man she contacts if and only if his pizazz $x \geq y_m(n)$; (ii) a single man with pizazz $x \in [y_m(n), y_m(n-1))$ proposes to a single woman he contacts if and only if her pizazz $x \geq y_w(n)$; (iii) if a couple make contact, they marry if and only if there is an n such that the woman's $x \in [y_w(n), y_w(n-1))$ and the man's $x \in [y_m(n), y_m(n-1))$; (iv) any single with pizazz $x \in [y_j(n'), y_j(n'-1))$, for $n' > \min\{J_w, J_m\}$, never marries.

Proof of Proposition 1. We derive the desired result by establishing the following lemmas. A single j with pizazz $\bar{x}_j$ is termed the most desirable j.

LEMMA 1.

$$R_w(\bar{x}_w) = \frac{\alpha}{r+\delta}\int_{R_w(\bar{x}_w)}^{\bar{x}_m}\left[1 - G_m(\tilde{x})\right]d\tilde{x} < \bar{x}_m \tag{7}$$

$$R_m(\bar{x}_m) = \frac{\alpha}{r+\delta}\int_{R_m(\bar{x}_m)}^{\bar{x}_w}\left[1 - G_w(\tilde{x})\right]d\tilde{x} < \bar{x}_w. \tag{8}$$

Proof of Lemma 1. It is immediate that all men (women) propose if they contact a woman (man) with pizazz $\bar{x}_w(\bar{x}_m)$. This implies that $\alpha_w(\bar{x}_w) = \alpha$ and $G_m(\cdot \mid \bar{x}_w) = G_m(\cdot)$. Substitution into (3) establishes (7). It is now straightforward to show $R_w(\bar{x}_w) < \bar{x}_m$. The same argument holds for the most desirable man $\bar{x}_m$. This completes the proof.

Note that y_m (1) defined in Proposition 1 equals $R_w(\bar{x}_w)$ defined by (7). Lemma 1 has established that the most desirable woman will propose to any man with pizazz $x \geq y_m$ (1). As $R_w(\cdot)$ is nondecreasing, this implies that all women will propose to such men.

Lemma 2 shows that, if the most desirable woman or man will propose to a single, then that individual will have the same reservation utility level as the most desirable of his or her sex.

LEMMA 2. Men with $x \in [y_m(1), \bar{x}_m)$ have $R_m(x) = R_m(\bar{x}_m) = y_w(1)$. Women with $x \in [y_w(1), \bar{x}_w)$ have $R_w(x) = R_w(\bar{x}_w) = y_m(1)$.

Proof of Lemma 2. Consider a single man with pizazz $x \in [y_m(1), \bar{x}_m]$. Since any woman will propose to such a man (given contact), then $\alpha_m(x) = \alpha$, $G_w(\cdot \mid x) = G_w(\cdot)$. Hence $R_m(x) = R_m(\bar{x}_m)$. Symmetry implies the same for all single women with pizazz $x \in [y_w(1), \bar{x}_w)$. This completes the proof.

Men with $x \in [y_m(1), \bar{x}_m)$ are termed class 1 men, whereas women with $x \in [y_w(1), \bar{x}_w)$ are called class 1 women. The next lemma follows directly from Lemma 2.

LEMMA 3. Men with $x \in [y_m(1), \bar{x}_m)$ and women with $x \in [y_w(1), \bar{x}_w)$ form a class (class 1) in that they propose to each other (and marry), rejecting all others.

It is now straightforward to construct an induction proof of Proposition 1. Consider a woman not in class 1; i.e., $x < y_w(1)$. The crucial insight is that she is automatically rejected by class 1 men, and so the only contacts of interest to her are with those men who are not class 1. This contact rate is given by $\alpha G_m(y_m(1))$. Given a random contact, the conditional pizazz distribution of men who are not class 1 is $G_m(\cdot)/G_m(y_m(1))$.

LEMMA 4. Given $y_w(2)$, $y_m(2)$ defined in Proposition 1:

(a) any single woman with pizazz $x \in [y_w(2), y_w(1))$ has $R_w(x) = y_m(2)$;

(b) any single man with pizazz $x \in [y_m(2), y_m(1))$ has $R_m(x) = y_w(2)$.

Proof of Lemma 4. Consider a desirable woman not in class 1, i.e., $x = y_w(1) - \varepsilon$, where $\varepsilon > 0$ but is arbitrarily small. All men not in class 1 will propose to such women (given contact). Hence for such women, $\alpha_w(x) = \alpha G_m(y_m(1))$, while $G_m(z \mid x) = G_m(z)/G_m(y_m(1))$ for $z \leq y_m(1)$. Substituting this into (3) implies that

$$R_w(x) = \frac{\alpha}{r + \delta} \int_{R_w(x)}^{y_m(1)} \left[G_m\left(y_m(1)\right) - G_m(x)\right] dx.$$

By definition, $y_m(2) = R_w(x)$, which denotes the lowest man pizazz that is acceptable to a highly desirable woman not in class 1. $R_w(\cdot)$ nondecreasing implies that all women not in class 1 will propose to a man whose pizazz $x \geq y_m(2)$. By symmetry, all men not in class 1 will propose to women whose pizazz $x \geq y_w(2)$. Types satisfying $x \in [y_j(2), y_j(1))$ form class 2—they always propose to each other, reject others with a lower pizazz, and are rejected by others from class 1. This completes the proof.

The nature of the general solution is now clear. The set of single men and the set of women can be partitioned into J_w and J_m classes. A woman is of class n if and only if her pizazz $x \in [y_w(n), y_w(n-1))$. Similarly, a man is of class n if and only if his pizazz $x \in [y_m(n), y_m(n-1))$. For any woman with $x \in [y_w(n), y_w(n-1))$, $R_w(x) = y_m(n)$, while for any man with $x \in [y_m(n), y_m(n-1))$, $R_m(x) = y_w(n)$. Such agents form a class (class n) as each will only marry a member from the opposite sex who is in the same class. The claims made in Proposition 1 follow by a straightforward induction proof. Uniqueness of the partition follows by direct inspection. A simple contradiction argument (using $\underline{x}_j > 0$) shows that J_j must be finite. This completes the proof.

Having characterized a PREE for any (G_w,G_m), the next section considers steady state equilibria.

III. Existence of Steady State Equilibria

Given (G_w,G_m), from Proposition 1, it follows that a PREE generates a unique partition $(\{y_w(n)\}_{n=0}^{J_w}, \{y_m(n)\}_{n=0}^{J_m})$. This partition implies a unique distribution of pizazz among those who flow out: $H_j(\cdot)$, $j = w, m$. This partition and N, the number of singles in the market, also implies a number of each sex who flow out of the market per period dt (through either marriage or death) denoted by $_{\beta}dt$.

DEFINITION. Given (F_w,F_m,β), a steady state equilibrium (SSE) is a triple (G_w,G_m,N) where
(i) the agent strategies are consistent with a PREE, and
(ii) there is balanced flow, where for any $[z_1,z_2) \subseteq [\underline{x}_j,\bar{x}_j]$,

$$\hat{\beta}\left[H_j(z_2) - H_j(z_1)\right] = \beta\left[F_j(z_2) - F_j(z_1)\right].$$

Given any continuous distributions (G_w,G_m), let λ_{mn} denote the proportion of men who are in class n, which in a PREE is defined by

$$\lambda_{mn} = G_m\big(y_m(n-1)\big) - G_m\big(y_m(n)\big). \tag{9}$$

Similarly for λ_{wn}. Now consider a single women in class n. The probability in small time interval dt that this woman meets a class n man and therefore marries is $\alpha\lambda_{mn}dt$. It follows that the number of women with pizazz $x \in [z_1, z_2)$ who flow out of the market is $[\alpha\lambda_{mn} + \delta][G_w(z_2) - G_w(z_1)]N$. Given (F_w, F_m, β), the flow-in of new single women with pizazz $x \in [z_1, z_2)$ is given by $\beta[F_w(z_2) - F_w(z_1)]$. Hence balanced flow is satisfied for $j = w$ if and only if

$$(10) \qquad (\alpha\lambda_{mn} + \delta)\big[G_w(z_2) - G_w(z_1)\big]N = \beta\big[F_w(z_2) - F_w(z_1)\big]$$

for all $[z_1, z_2) \subset [y_w(n), y_w(n-1))$, and for all n. As F_w is differentiable, (10) implies that

$$(11) \qquad G_w'(z_2) = \frac{\beta}{(\alpha\lambda_{mn} + \delta)N} F_w'(z_2);$$

i.e., within a class the steady state density function is simply the entry density function appropriately rescaled. In particular, the higher the exit rate of women in class n (which depends on λ_{mn}), the lower their density. Similarly for men. The fact that class size λ_{jn} varies with n implies that the steady state density function $G_j'(\cdot)$ is discontinuous at the class boundaries.

To identify a SSE, set $[z_1, z_2) = [y_j(n), y_j(n-1))$ and note that (10) implies

$$(12) \qquad [\alpha\lambda_{mn} + \delta]\lambda_{wn}N = \beta\big[F_w(y_w(n-1)) - F_w(y_w(n))\big]$$

$$(13) \qquad [\alpha\lambda_{wn} + \delta]\lambda_{mn}N = \beta\big[F_m(y_m(n-1)) - F_m(y_m(n))\big],$$

which describe the steady state class sizes.

Now consider a candidate SSE (G_w, G_m, N). A PREE implies a unique partition $(\{y_w(n)\}_{n=0}^{J_w}, \{y_m(n)\}_{n=0}^{J_m})$, while balanced flow implies that $G_j(.)$ must satisfy (10) within a class. Using (10) and (12) to substitute out G_j in Proposition 1, it follows that the partition $(\{y_w(n)\}_{n=0}^{J_w}, \{y_m(n)\}_{n=0}^{J_m})$ is part of a SSE if and only if

$$(14) \qquad y_m(n) = \frac{\alpha\lambda_{mn}}{r + \delta}\int_{y_m(n)}^{y_m(n-1)} \frac{F_m(y_m(n-1)) - F_m(x)}{F_m(y_m(n-1)) - F_m(y_m(n))}\,dx$$

$$(15) \qquad y_w(n) = \frac{\alpha\lambda_{wn}}{\delta + r}\int_{y_w(n)}^{y_w(n-1)} \frac{F_w(y_w(n-1)) - F_w(x)}{F_w(y_w(n-1)) - F_w(y_w(n))}\,dx,$$

where λ_{jn} are defined by (12) and (13), and the boundary conditions are (a) $y_w(0) = \bar{x}_w$, (b) $y_m(0) = \bar{x}_m$, (c) $y_w(J_w) \leq \underline{x}_w$, and (d) $y_m(J_m) \leq \underline{x}_m$.

Note that $y_w(n)$ in (15) is the reservation match of all class n men. This reservation match depends on $\alpha\lambda_{wn}$, the rate at which class n singles meet, and

$$\frac{F_w(x) - F_w\big(y_w(n)\big)}{F_w\big(y_w(n-1)\big) - F_w\big(y_w(n)\big)},$$

which is the conditional distribution of women's pizazz in this class. Therefore, (15) is simply the steady state analog of (6).

Proposition 2 now gives conditions that fully characterize a SSE.

PROPOSITION 2. (Characterization of SSE). Given (F_w,F_m,β), then (G_w,G_m,N) defines a SSE if and only if (G_w,G_m) satisfy (11), where $\lambda_{jn} \geq 0$ satisfies the partition defined by (12)–(15) and the boundary conditions (a)–(d), and

$$\sum_{n=1}^{J_w}\lambda_{wn} = \sum_{n=1}^{J_m}\lambda_{mn} = 1. \tag{16}$$

Proof of Proposition 2. By definition of λ_{jn}, (16) guarantees that $G_j(\bar{x}_j) = 1$ and $j = w, m$. Clearly, by construction, any SSE must satisfy the conditions described by Proposition 2. Further, any solution to Proposition 2 implies that (G_w,G_m) have the required properties of steady state distribution functions, in that G_j is increasing over interval $[\underline{x}_j,\bar{x}_j]$ with $G_j(\underline{x}_j) = 0$ and $G_j(\bar{x}_j) = 1$, $j = m, w$. Hence, such a solution identifies a SSE as it satisfies PREE and balanced flow. This completes the proof.

We now provide a formal existence proof of SSE. Unfortunately, there are many difficulties associated with constructing a general fixed point argument (see Smith [1995] for a discussion of those issues). Our proof establishes conditions under which a triple (G_w,G_m,N) can be found that satisfies Proposition 2. To do this, we need to assume that $(1 - F_j)$, $j = w, m$ are log-concave.[7] Proposition 3 shows why.

PROPOSITION 3. Given (F_w,F_m), where $(1 - F_j)$ is log concave $j = w, m$, then for any $N > 0$, a unique partition $(\{y_w(n), \lambda_{wn}\}_{n=0}^{J_w}, \{y_m(n), \lambda_{mn}\}_{n=0}^{J_m})$ exists satisfying (12)–(15) and the boundary

7. That is, $\log(1 - F_j(\cdot))$ is concave.

conditions (a)–(d). Further, for $n \leq J_j$, this recursion defines a positive and strictly decreasing sequence $y_j(n)$ and a positive sequence λ_{jn}, $j = w, m$.

Proof of Proposition 3. See the Appendix.

Log-concavity of the survivor function implies that given any value of N, there is a unique solution to the partition (12)–(15), (a)–(d), and hence by (11) unique candidate functions G_j. Of course, for an arbitrary choice of N, the resulting partition may not satisfy (16) (and $G_j(\bar{x}_j) \neq 1$). The proof of Theorem 1 below effectively shows that, as N changes, the class sizes (defined by (12)–(15) and (a)–(d)) are continuous functions of N. This continuity allows us to show that a solution to (16) must exist. Without log-concavity of the survivor function, there may be multiple solutions to (12)–(15), and it does not follow that a continuous solution for λ_{jn} exists (for example, a solution may disappear for some N).

THEOREM 1. If $(1 - F_j)$ is log concave, $j = w, m$, a steady state equilibrium exists.

Proof of Theorem 1. See the Appendix.

Establishing the stability properties of SSE is beyond the scope of this paper. The dynamics are particularly complex. Outside of SSE, the distributions are time varying, which implies that the agents' reservation match strategies are also time varying. Unfortunately, this implies that the offer distributions $G_j(z,t \mid x)$ are not continuous functions of time.[8] Such work is left for future research. In the remainder of the study we explore some of the many implications that follow given the equations of Proposition 2.

IV. EXAMPLES

Example 1: The Uniform Distribution

To understand better the mechanics of Proposition 3, we first examine a particularly simple case, where x_j is uniformly distributed over $[\underline{x},\bar{x}]$, $j = m, w$. Given N (which exists by Theorem 1),

8. Smith [1995] analyses nonsteady state dynamics when there is no entry. Burdett and Coles [1995] analyze the dynamic equilibria of the two-types example described later in the study. Both SSE described in Theorem 2 are in fact stable nodes.

there exists a unique solution to (12)–(15), where $\lambda_{jn} = \lambda_n$, $y_j(n) = y(n)$ are defined by

$$y(n) = \frac{\alpha\lambda_n}{2(r+\delta)}\left[y(n-1) - y(n)\right]$$

$$\lambda_n[\delta + \alpha\lambda_n] = \frac{\beta}{N[\bar{x} - \underline{x}]}\left[y(n-1) - y(n)\right]$$

(assuming that $y(n) > \underline{x}$) and $y(0) = \bar{x}$). The recursion ends as soon as the solution above implies that $y(n) \leq \underline{x}$ which defines J the number of classes and λ_J satisfies $\lambda_J[\delta + \alpha\lambda_J] = \beta y(J-1)/[N(\bar{x} - \underline{x})]$. It is straightforward to show that this pair of equations describes positive and strictly decreasing sequences for the class boundaries $y(n)$, for the class widths $[y(n-1) - y(n)]$ and for the class sizes λ_n.[9] More interestingly, (11) implies that the steady state density function is constant within each class, while λ_n strictly decreasing with n implies that it is discontinuous at the class boundaries where $G'(y(n)^-) > G'(y(n)^+)$. As this density function is monotonically decreasing, this also implies that the (uniform) distribution of pizazz of entrants first order stochastically dominates the distribution of pizazz of unmatched singles. Hence this market has two further equilibrium characteristics: high pizazz singles tend to marry and leave the singles market relatively quickly; while unmatched singles notice that there is a greater preponderance of low pizazz members of the opposite sex (relative to their entry distribution).

Example 2: Old Maids and Child Brides

It is not necessarily true that everybody will marry in a SSE if $F_m \neq F_w$. To see this, let μ_j denote the mean of the distribution $F_j(.)$, and suppose that $\underline{x}_w/\mu_w < \underline{x}_m/\mu_m$. A simple interpretation of this condition is that relative to their own cohorts, the lowest pizazz woman is relatively less desirable than the lowest pizazz man. Now consider α, where

$$\frac{\underline{x}_w}{\mu_w - \underline{x}_w} < \frac{\alpha}{r+\delta} < \frac{\underline{x}_m}{\mu_m - \underline{x}_m}$$

(which is a nonempty interval given that $\underline{x}_w/\mu_w < \underline{x}_m/\mu_m$). Noting that μ_j can be written as $\mu_j = \underline{x}_j + \int_{\underline{x}_j}^{\bar{x}_j} [1 - F_j(x)]dx$, this restriction

9. Bloch and Ryder [1994] obtain the same result when clones replace exiting agents.

and the conditions in Proposition 3 imply that $\lambda_{m1} = 1$. There is a single class of men where each woman proposes to the first man she meets. However, there are two classes of women. If y denotes the boundary of the first class $y = y_w(1)$, then

$$y = \left[\frac{\alpha}{r + \delta}\right]\left[\frac{\delta}{\delta + \alpha F_w(y)}\right]\int_y^{\bar{x}_w}\left[1 - F_w(x)\right]dx.$$

Further, $\lambda_{w1} = \delta[1 - F(y)]/[\delta + \alpha F(y)]$. It is straightforward to show that there is a unique solution for $y \in (\underline{x}_w, \bar{x}_w)$ and that $\lambda_{w1} \in (0, 1)$ and $\lambda_{w2} = 1 - \lambda_{w1}$. In this case, men refuse to marry any woman with pizazz $x \in [\underline{x}_w, y)$.[10] Further, women in class 1 marry more quickly than men; the marriage rate of such women being α while that of men is $\alpha\lambda_{w1}$. As a consequence, women who marry will tend to be younger than their partners, while there are other women who face a zero probability of marriage.

Example 3: Two Types

To establish Proposition 2, it was assumed that F_w and F_m were continuous. It is possible to modify the analysis to deal with the situation where there are only a finite number of pizazz types among men and women. We here utilize such an example to illustrate how multiple equilibria can occur.

Keeping things as simple as possible, suppose that there are only two types: those with high pizazz x_H, and those with low pizazz x_L, where $x_H > x_L > 0$. Further, assume that the distribution of pizazz among women who flow into the market in any time interval is the same as that among men; i.e., $F_w(.) = F_m(.) = F(.)$. In particular, assume that

$$F(x) = \begin{cases} 0, & \text{if } x < x_L \\ 1 - \pi, & \text{if } x_L \le x < x_H \\ 1, & \text{if } x \ge x_H. \end{cases}$$

Hence, a proportion π of all new entrants have high pizazz.

We focus on symmetric equilibria, where $y_j(n) = y(n)$ and $\lambda_{jn} = \lambda_n, j = w, m$. There are two possible types of such equilibria, characterized by whether high pizazz types are willing to marry low pizazz types or not.

10. An interesting side-issue is that SSE do not exist for these parameter values when $\delta = 0$. As some women never marry, a positive death rate is necessary for existence. Generic existence with $\delta = 0$ requires symmetry $F_j(\cdot) = F(\cdot), j = m, w$.

In any equilibrium high pizazz types will always marry each other on contact. Now suppose that all types expect high pizazz types also to marry low pizazz types on contact. If this is an equilibrium belief, then low pizazz types must also be willing to marry each other.[11] Let $\eta(t)$ indicate the proportion of singles (of both sexes) who have pizazz x_H at time t, while $N_H(t) = \eta(t)N(t)$ denotes the number of high pizazz types (of each sex) at time t. Given the beliefs described above, $N(t)$ and $N_H(t)$ are expected to evolve according to $N'(t) = \beta - (\alpha + \delta)N(t)$ and $N_H'(t) = \beta\pi - (\alpha + \delta)N_H(t)$. Together these imply that $\eta'(t) = (\beta/N(t))(\pi - \eta(t))$. Hence, a stationary state occurs if and only if $N(t) = \beta/(\alpha + \delta)$ and $\eta(t) = \pi$ for all t. Of course, this stationary state describes a steady state equilibrium if and only if high pizazz agents wish to marry low pizazz agents, which requires that $y(1) \leq x_L$. In that case $\lambda_1 = 1$ (there is a single class), and (14) implies that $y(1) \leq x_L$ if and only if

$$y(1) = \left[\alpha/(r + \delta + \alpha)\right]\left[\pi x_H + (1 - \pi)x_L\right] \leq x_L,$$

which requires that

$$\alpha \leq \left(\frac{r + \delta}{\pi}\right)\left[\frac{x_L}{x_H - x_L}\right]. \tag{17}$$

Hence if the contact rate is low enough, those with high pizazz will marry low pizazz types in a steady state equilibrium. We will refer to this equilibrium as a Single Class Equilibrium (SCE).[12]

Now we change the initial beliefs of these agents. Suppose that all expect high pizazz types will not marry low pizazz types on contact. Again, if this is an equilibrium belief, low pizazz types must be willing to marry each other (remember being single yields zero utility). Given these beliefs, $N(t)$ and $N_H(t)$ are now expected to evolve according to $N'(t) = \beta - [\delta + \alpha[\eta(t)^2 + (1 - \eta(t))^2]]N(t)$ and $N_H'(t) = \beta\pi - [\alpha\eta(t) + \delta]N_H(t)$. Together these imply that $\eta'(t) = (\beta/N(t))(\pi - \eta(t)) + 2\alpha\eta(t)(1 - \eta(t))(0.5 - \eta(t))$. In this case the stationary state is $\eta(t) = \overline{\eta}(\pi)$ and $N(t) = \overline{N}$ for all t, where

11. Section III showed that in any PREE, the reservation match of any type increases with pizazz.

12. More formally, assuming (17), $G_j = F$ and $N = \beta/(\alpha + \delta)$ satisfy Proposition 2 which imply that $\lambda_1 = 1$, $y(1) \leq x_L$, and $J = 1$.

$$\pi = \frac{\bar{\eta}(\alpha\bar{\eta} + \delta)}{\delta + \alpha\bar{\eta}^2 + \alpha(1 - \bar{\eta})^2} \quad \text{and} \quad \frac{\bar{N} = \beta}{\delta + \alpha\bar{\eta}^2 + \alpha(1 - \bar{\eta})^2}. \tag{18}$$

Straightforward algebra shows that a solution $\bar{\eta}(\pi) \in [0,1]$ always exists and is unique. Furthermore, if $\pi < 0.5$, then $\bar{\eta}(\pi) \in (\pi, 0.5)$; while if $\pi > 0.5$, then $\bar{\eta}(\pi) \in (0.5, \pi)$. Such a stationary state is a steady state equilibrium if and only if high pizazz types prefer to reject low pizazz types; i.e., that $y(1) > x_L$. In that case, there are two classes where $\lambda_1 = \bar{\eta}(\pi)$ and $\lambda_2 = 1 - \bar{\eta}(\pi)$ and (14) implies that $y(1) > x_L$ if and only if

$$y(1) = \left[\frac{\alpha\bar{\eta}(\pi)x_H}{(r + \delta + \alpha\bar{\eta}(\pi))}\right] > x_L, \tag{19}$$

which requires that

$$\alpha > \left(\frac{r + \delta}{\bar{\eta}(\pi)}\right)\left[\frac{x_L}{x_H - x_L}\right]. \tag{20}$$

This time, selective behavior by high pizazz types can occur in a steady state equilibrium if the contact rate is sufficiently high. We term such a steady state equilibrium an Elitist Equilibrium (EE).[13]

We now show that there exist parameter values where both steady-state equilibria exist.

THEOREM 2. Both a SCE and an EE exist if and only if $0 < \pi < 0.5$, and

$$\alpha \in \left(\left(\frac{r + \delta}{\bar{\eta}(\pi)}\right)\left[\frac{x_L}{x_H - x_L}\right], \left(\frac{r + \delta}{\pi}\right)\left[\frac{x_L}{x_H - x_L}\right]\right),$$

where $\bar{\eta}(\pi) > \pi$.

Proof of Theorem 2. The above has shown that both steady state equilibria exist if and only if α lies in the interval stated. The definition of $\bar{\eta}(\pi)$ implies that this interval is nonempty if and only if $\pi < 0.5$. This completes the proof.

The intuition is as follows. In a SCE the proportion of singles

13. More formally, assuming that (20) holds, $G_j(x) = 1 - \bar{\eta}(\pi)$ for $x_L \leq x < x_H$ (where $G_j(x_L) = 0$, $G_j(x_H) = 1$) and $N = \bar{N}$ satisfy Proposition 2 which imply that $\lambda_1 = \bar{\eta}(\pi)$ and $y(1) \in (x_L, x_H)$, and that $\lambda_2 = 1 - \bar{\eta}(\pi)$, $y(2) < x_L$ and $J = 2$.

with high pizazz is π. In an EE, however, if $\pi < 0.5$, the proportion of high pizazz singles is greater than π (but less than 0.5). At the relevant parameters this higher proportion of high pizazz singles justifies them being more selective. On the other hand, at an EE when $\pi > 0.5$, the proportion of high pizazz singles is lower than π, and therefore cannot at the same time support a SCE.

It can be shown that N is larger (there are more singles) in the EE than in the SCE. The greater selectivity of high pizazz types in the EE reduces their own matching rates as well as the matching rates of the low pizazz types. The number of singles has to be greater to ensure that the aggregate matching rate continues to equal the flow in.

The welfare implications of these two steady state equilibria are interesting. These steady states are not Pareto rankable: the high pizazz types prefer the EE; the low pizazz types prefer the SCE. Underpinning these results is a sorting externality. In the SCE a high pizazz type makes other high pizazz agents worse off by being willing to marry low pizazz types. The converse is the case in the EE. Elitist behavior makes the elite better off at the cost of the lower pizazz types. From a utilitarian perspective this is inefficient. The utilitarian Social Planner does not care about who marries whom as marriage rejections simply reduce the total discounted flow of utility. Of course, the high pizazz types are not concerned about that.

Example 4: Uniqueness of Class Equilibria

Given the multiplicity result in the two-types case, we now turn to finding sufficient restrictions that imply uniqueness of a steady state equilibrium. For simplicity, we restrict attention to the symmetric case, where $F_j(.) = F(.), j = w, m$, and this time set $\delta = 0$.

PROPOSITION 4. If $xF'(x)$ is increasing in x, then there is a unique SSE.

Proof of Proposition 4. See the Appendix.

This is a strong restriction on F.[14] However, we conjecture that a weaker assumption does not exist which guarantees

14. Suppose that $F'(x) = g(x)/x$, where g is a strictly positive, increasing function. Integration implies that F must increase faster than $\log x$. The restriction $xF'(x)$ increasing in x rules out all distributions with unbounded support. Conversely, the uniform distribution satisfies this criterion.

uniqueness for the general (symmetric) case. To see why, consider the following proposition.

PROPOSITION 5. $xF'(x)$ increasing in x guarantees that class size is decreasing in a SSE; i.e., that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_J > 0$.

Proof of Proposition 5. See Lemma 11 in the Appendix.

Compare this result with the two-types example above. There it was shown that multiple equilibria can occur only if $\pi < 0.5$, as this supports the appropriate feedback effect—that if high pizazz types are choosy, they increase in proportion to the rest of the market. Further, in the EE the restriction $\pi < 0.5$ implies that $\lambda_1 = \overline{\eta}(\pi) < 0.5 < 1 - \overline{\eta}(\pi) = \lambda_2$; i.e., there is increasing class size. Conversely, $\pi > 0.5$ guarantees uniqueness and also implies decreasing class size in the EE. This suggests that the concept of decreasing class size is closely related to that of uniqueness of SSE. Although not formally established, the proof of Lemma 11 given in the Appendix can be extended to show that $xF'(x)$ increasing in x is also necessary to ensure that class size is decreasing in a SSE for all possible parameter values.[15]

To understand why decreasing class size is so important for uniqueness, consider the thought experiment that women in class $n < J$ deviate from the steady state and become slightly more choosy and reject men with pizazz $x \in [y(n), y(n) + \epsilon)$, where $\epsilon > 0$ but arbitrarily small. Consider the first effect of this deviation. The exit rate of the remaining class n men is unchanged. However, the newly rejected men can now only match with class $n + 1$ women, and crucially their exit rate changes a discrete amount from $\alpha\lambda_n$ to $\alpha\lambda_{n+1}$. If $\lambda_{n+1} > \lambda_n$ (the case of increasing class size), the first effect of this deviation is that these types exit more quickly and so their numbers decline. Hence, the conditional steady state distribution $G_m(x \mid x \in [y(n), y(n-1)))$ with this deviation first-order stochastically dominates the original one. This implies that the reservation match quality of class n women does indeed increase which potentially supports multiple equilibria. Conversely, if $\lambda_{n+1} < \lambda_n$, the newly rejected types exit even more slowly and hence build up relatively more in number. The shift in the distribution of pizazz of men in class n causes class n women's

15. If α is arbitrarily large, it can be shown that the class widths $[y(n), y(n - 1))$ become arbitrarily small. If $xF'(x)$ is strictly decreasing in x over some nonempty interval, it follows by the proof of Lemma 11 that the classes in this interval must be strictly increasing in size.

reservation match to decrease, which does not support the original supposed increase. Guaranteeing uniqueness requires a relatively severe restriction on F which ensures decreasing class size.

V. Conclusion

This paper has provided a simple proof for why class partitions may arise in a marriage market with two-sided heterogeneity. With exogenous inflows of new agents, we have shown that there are sorting externalities which can support multiple class equilibria, even when there are constant returns to matching.

There are several directions for future research. For simplicity, this paper has focused on characterizing steady state equilibria and has shown that guaranteeing uniqueness requires some strong restrictions. Characterizing the nonsteady state dynamics about such steady state equilibria is an important, though difficult extension. Smith [1995] has examined equilibrium dynamics when there is no inflow of new agents, while Burdett and Coles [1995] have done this for the two-types case described above. They find that when two steady state equilibria exist, both of them are stable. Indeed, for some initial values, they show that there are multiple dynamic equilibria.

Another important extension is to allow singles to bargain over the surplus of the match. This, of course, describes a simple labor market situation. Initial work indicates that the sorting externality identified here can support multiple equilibria. This potentially provides new insights into how the labor market might segment when there are heterogeneous firms and workers. A further direction is to endogenize the flow-in distributions of pizazz in different ways. Two variations appear interesting at first blush. One is to assume that agents are born with some pizazz drawn from some exogenous distribution, but at some cost can obtain greater pizazz. It is the postpurchase distributions of pizazz that flows into the marriage market. Given the class partition result, if a single purchases pizazz, he or she only buys that which puts them at the lower bound of a higher class—there is no benefit to purchasing any more within a class. This suggests that clustering will occur at class boundaries. Another variation is to assume that when a couple marries they have two children whose pizazz is a weighted average of their parents' pizazz. In this case the pizazz distributions of singles will evolve through time depending on previous marriage decisions.

Appendix

Proof of Proposition 3 and Theorem 1

We establish Proposition 3 and Theorem 1 by a series of lemmas. The first of these establishes a technical result that plays a role later. For $x_2 \leq x_1$, define

$$\Gamma(x_2, x_1) = \left[F(x_1) - F(x_2)\right]^2 - F'(x_2)\int_{x_2}^{x_1}\left[F(x_1) - F(x)\right]dx.$$

LEMMA 5. $1 - F(x)$ log-concave implies that $\Gamma(\cdot) \geq 0$ for all $x_2 \leq x_1$.

Proof of Lemma 5. First, note that $\Gamma(x_2, x_1) = 0$ if $x_2 = x_1$. Hence, the claim made is established if we can show $\partial\Gamma/\partial x_2 \leq 0$ for all $x_2 < x_1$. Differentiating $\Gamma(\cdot)$ with respect to x_2 implies that

$$\frac{\partial \Gamma}{\partial x_2} = -\left[F(x_1) - F(x_2)\right]F'(x_2) - F''(x_2)\int_{x_2}^{x_1}\left[F(x_1) - F(x)\right]dx.$$

However, if $(1 - F(x))$ is log-concave, then $(1 - F(x))F''(x) + F'(x)^2 \geq 0$, and therefore

$$-F''(x_2) \leq \frac{F'(x_2)^2}{1 - F(x_2)}.$$

It follows that

$$\frac{\partial \Gamma}{\partial x_2} \leq -\left[F(x_1) - F(x_2)\right]F'(x_2) + \frac{F'(x_2)^2}{1 - F(x_2)}\int_{x_2}^{x_1}\left[F(x_1) - F(x)\right]dx.$$

Further, if $1 - F(x)$ is log-concave, $F'(x)/(1 - F(x))$ is increasing in x, and therefore

$$-F(x) \leq -1 + \frac{1 - F(x_2)}{F'(x_2)}F'(x) \text{ for } x > x_2.$$

Hence,

$$\frac{\partial \Gamma}{\partial x_2} \leq -\left[F(x_1) - F(x_2)\right]F'(x_2) + \frac{F'(x_2)^2}{1 - F(x_2)}\int_{x_2}^{x_1}\left[F(x_1) - 1 + \frac{1 - F(x_2)}{F'(x_2)}F'(x)\right]dx.$$

Integrating this last expression yields

$$\frac{\partial \Gamma}{\partial x_2} \le -[x_1 - x_2]F'(x_2)^2 \frac{1 - F(x_1)}{1 - F(x_2)} \le 0,$$

which is the desired result. This completes the proof of Lemma 5.

Consider now the following function:

$$\phi_j(y_j(n), y_j(n-1)) = \int_{y_j(n)}^{y_j(n-1)} \frac{F_j(y_j(n-1)) - F_j(x)}{F_j(y_j(n-1)) - F_j(y_j(n))}\, dx.$$

Defined for $y_j(n-1) > \underline{x}_j \cdot F_j(\cdot)$ strictly increasing and twice differentiable over its support implies that $\phi_j(\cdot)$ is well defined, continuous, and twice differentiable almost everywhere (but only consider right differentiation at $y_j(n) = \underline{x}_j$). It can be shown that $\phi_j(\cdot) \to 0$ as $y_j(n) \to y_j(n-1)$ and $\phi_j(\cdot) \to \underline{x}_j - y_j(n)$ as $y_j(n-1) \to \underline{x}_j$. Furthermore, given that $1 - F_j(x)$ is log-concave, Lemma 5 implies that $\phi_j(\cdot)$ is decreasing in $y_j(n)$. Direct inspection establishes that $\phi_j(\cdot)$ is strictly increasing in $y_j(n-1)$.

We establish Proposition 3 and existence as follows. First, fix any $N > 0$, and let $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ and $\hat{J}_j(N)$ denote the solution to (12)–(15) and the boundary conditions (a)–(d) described in Proposition 2; i.e., they satisfy the following recursive equations:

(i) $\hat{y}_{j0}(N) = \bar{x}_j$

(ii) if $\hat{y}_{y\mathrm{N}-1}(n) > \underline{x}_j$ (both $j = m$ and w), then $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ satisfy

$$\hat{y}_{mn}(N) = \left[\frac{\alpha \hat{\lambda}_{mn}(N)}{r + \delta}\right] \phi_m(\hat{y}_{mn}(N), \hat{y}_{m,n-1}(N)) \tag{21}$$

$$\hat{y}_{wn}(N) = \left[\frac{\alpha \hat{\lambda}_{wn}(N)}{r + \delta}\right] \phi_w(\hat{y}_{wn}(N), \hat{y}_{w,n-1}(N)) \tag{22}$$

$$N[\alpha \hat{\lambda}_{mn}(N) + \delta]\hat{\lambda}_{wn}(N) = \beta[F_w(\hat{y}_{w,n-1}(N)) - F_w(\hat{y}_{wn}(N))] \tag{23}$$

$$N[\alpha \lambda_{wn}(N) + \delta]\lambda_{mn}(N) = \beta[F_m(\hat{y}_{m,n-1}(N)) - F_m(\hat{y}_{mn}(N))] \tag{24}$$

(iii) if $\hat{y}_{j,n-1}(N) \le \underline{x}_j$, then $\hat{y}_{jn}(N) = \hat{\lambda}_{jn}(N) = 0$,

(iv) if $\hat{y}_{j,n-1}(N) > \underline{x}_j$ and $\hat{y}_{k,n-1}(N) \le \underline{x}_k$ (for $k \ne j$), then $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ satisfy

$$\hat{y}_{jn}(N) = \left[\frac{\alpha \hat{\lambda}_{jn}(N)}{r + \delta}\right] \phi_j(\hat{y}_{jn}(N), \hat{y}_{j,n-1}(N)) \tag{25}$$

$$\hat{\lambda}_{jn}(N) = \frac{\beta}{\delta N}\Big[F_j\big(\hat{y}_{j,n-1}(N)\big) - F_j\big(\hat{y}_{jn}(N)\big)\Big] \tag{26}$$

(v) $\hat{J}_j(N) = \max\{n: \hat{y}_{jn}(N) > 0\}$.

Lemmas 6, 7, and 8 below prove Proposition 3 by showing that for any $N > 0$, there exists a unique solution for $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ and that such a solution satisfies $\hat{y}_{jn}(N) < \hat{y}_{j,n-1}(N)$ and $\hat{\lambda}_{jn}(N) > 0$ for $n \le \hat{J}_j(N)$. These lemmas also establish that $\hat{y}_{jn}$, $\hat{\lambda}_{jn}$ are continuous functions of N. Lemma 9 then establishes that there exists some $N^c > 0$ such that $\sum_n \hat{\lambda}_{jn}(N^c) = 1$, for $j = m, w$. This proves that Theorem 1 as a solution to Proposition 2 then exists for $N = N^c$, where $y_j(n) = \hat{y}_{jn}(N^c)$, $\lambda_{jn} = \hat{\lambda}_{jn}(N^c)$, $J_j = \hat{J}_j(N^c)$, and G_j are defined by (11), $j = m, w$.

LEMMA 6. If $\hat{y}_{j,n-1}(N) > \underline{x}_j$ and continuous at N for some $N > 0$ for both $j = w, m$, then there exists a unique solution for $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$, where
(i) $0 < \hat{y}_{jn}(N) < \hat{y}_{j,n-1}(N)$, $\hat{\lambda}_{jn}(N) > 0$ and are continuous at N;
(ii) $\hat{y}_{jn}(N) \to 0$, $\hat{\lambda}_{jn}(N) \to 0$ as $\hat{y}_{j,n-1}(N) \to \underline{x}_j$.

Proof of Lemma 6. Substituting out $\hat{\lambda}_{jn}(N)$, (21)–(24) imply that $\hat{y}_{jn}(N)$ satisfy

$$(r+\delta)\frac{\hat{y}_{mn}}{\phi_m(.)} + \delta = \frac{\alpha\beta}{(r+\delta)N}\frac{\phi_w(.)}{\hat{y}_{wn}}[F_w(\hat{y}_{w,n-1}) - F_w(\hat{y}_{wn})] \tag{27}$$

$$(r+\delta)\frac{\hat{y}_{mn}}{\phi_m(.)} + \delta = \frac{\alpha\beta}{(r+\delta)N}\frac{\phi_w(.)}{\hat{y}_{wn}}[F_w(\hat{y}_{w,n-1}) - F_w(\hat{y}_{wn})]$$

$$\left(r+\delta+\delta\frac{\phi_m(.)}{\hat{y}_{mn}}\right)(F_m(\hat{y}_{m,n-1}) - F_m(\hat{y}_{mn})) \tag{28}$$

$$= \left(r+\delta+\delta\frac{\phi_w(.)}{\hat{y}_{wn}}\right)(F_w(\hat{y}_{w,n-1}) - F_w(\hat{y}_{wn})).$$

which are two equations in two unknowns $\hat{y}_{jn}(N)$.

Consider first (27). Log-concavity implies that the left-hand side is a strictly increasing, continuous function of $\hat{y}_{mn}$ for $\hat{y}_{mn} < \hat{y}_{m,n-1}(N)$, while log-concavity and continuity of $F_w(.)$ implies that the right-hand side is a continuous strictly decreasing function of $\hat{y}_{wn}$ for $\hat{y}_{wn} > 0$. Hence (27) defines $\hat{y}_{mn} = \sigma_1(\hat{y}_{wn}; N, \hat{y}_{y,n-1}(N))$, where $\sigma_1(.)$ is continuous and is strictly decreasing in $\hat{y}_{wn}$. Furthermore, $\hat{y}_{mn} \to \hat{y}_{\mathrm{ym},n-1}(N)$ as $\hat{y}_{wn} \to 0$, and $\hat{y}_{mn} < 0$ (exists) if $\hat{y}_{wn} = \hat{y}_{w,n-1}(N)$.

In (28), the left-hand side is a continuous strictly decreasing function of $\hat{y}_{mn}$ for $\hat{y}_{mn} > 0$, while the right-hand side is a continuous strictly decreasing function of $\hat{y}_{wn}$ for $\hat{y}_{wn} > 0$. Hence this equation defines $\hat{y}_{mn} = \sigma_2(\hat{y}_{wn}; N, \hat{y}_{y,n-1}(N))$, where $\sigma_2(.)$ is continuous and strictly increasing with $\hat{y}_{wn}$. Furthermore, $\hat{y}_{mn} \to 0$ as $\hat{y}_{wn} \to 0$, and $\hat{y}_{mn} = \hat{y}_{m,n-1}(N)$ if $\hat{y}_{wn} = \hat{y}_{w,n-1}(N)$. Existence and uniqueness of $\hat{y}_{jn}(N)$ (given N, $\hat{y}_{j,n-1}(N) > \underline{x}_j$) follows straightforwardly from these facts, and $0 < \hat{y}_{jn}(N) < \hat{y}_{j,n-1}(N)$. Further, as $\phi_j(.)$ are continuous functions of $\hat{y}_{j,n-1}(N)$ which by assumption are continuous at N, then $\sigma_i(.)$ must be continuous functions at N, and hence $\hat{y}_{jn}(N)$ must also be continuous at N. Given that, it is straightforward to show that there exists a unique (positive) solution for $\hat{\lambda}_{jn}(N)$ which is also continuous at N. $\hat{y}_{m,n-1}(N) \to \underline{x}_m$ implies that $\phi_m(\hat{y}_{mn}, \hat{y}_{m,n-1}) \to [\underline{x}_m - \hat{y}_{mn}]$. Substituting this in (21)–(24) and using the fact that $\hat{\lambda}_{wn}(N) \geq 0$ implies the lemma. This completes the proof.

LEMMA 7. Suppose that only one sex satisfies $\hat{y}_{j,n-1}(N) > \underline{x}_j$ for some $n - 1$ and $N > 0$. Further, assume that $\hat{y}_{j,n-1}(N)$ is continuous at N. Then there exists a unique solution for $\hat{y}_{j,n}(N)$ and $\hat{\lambda}_{jn}(N)$, where
(i) $0 < \hat{y}_{jn}(N) < \hat{y}_{j,n-1}(N)$, $\hat{\lambda}_{jn}(N) > 0$ and are continuous at N;
(ii) $\hat{y}_{jn}(N) \to 0$, $\hat{\lambda}_{jn}(N) \to 0$ as $\hat{y}_{j,n-1}(N) \to \underline{x}_j$.

Proof of Lemma 7. Substituting out $\hat{\lambda}_{jn}(N)$ in (25) and (26), the properties of $\phi_j(.)$ are sufficient to show that $\hat{y}_{jn}(N)$ must have the properties stated in the lemma. Equation (26) then implies that $\hat{\lambda}_{jn}(N)$ has the same properties. This completes the proof.

LEMMA 8. For all $N > 0$, there exists a unique solution for $\hat{y}_{j,n}(N)$, $\hat{\lambda}_{jn}(N)$, where
(i) $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ are continuous functions ($j = w$, m and $n \geq 0$),
(ii) $\sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{wn}(N) = \sum_{n=1}^{\hat{J}_m(N)} \hat{\lambda}_{mn}(N)$.
(iii) $\hat{y}_{jn}(N) < \hat{y}_{j,n-1}(N)$ and $\hat{\lambda}_{jn}(N) > 0$ if $\hat{y}_{j\mathrm{n}-1}(N) > \underline{x}_j$.

Proof of Lemma 8. (i) Follows by induction. By definition $\hat{y}_{j0}(N) = \bar{x}_j$ is a continuous function of N for all N. Lemmas 6 and 7 imply the appropriate induction step.

(ii) If $\hat{J}_j(N)$ is finite for all $N > 0$, the lemma follows by summing (23) and (24) over n and noting that $\sum_{n=1}^{\hat{J}_j(N)} [F_j(\hat{y}_{j,n-1}(N)) - F_j(\hat{y}_{jn}(N))] = 1$. A simple contradiction proof establishes that $\hat{J}_j(N)$ is finite for $N > 0$. (Given that $N > 0$, the assumption $\underline{x}_j > 0$ ensures that while $\hat{y}_{j,n}(N) > \underline{x}_j$, the difference $\hat{y}_{j,n-1}(N) - \hat{y}_{jn}(N)$ defined by (21)–(24) or (25)–(26) cannot be arbitrarily small).

(iii) Follows directly from Lemmas 6 and 7. This completes the proof of Proposition 3. We now establish Theorem 1.

LEMMA 9. There exists $N > 0$ such that

$$\sum_{n=1}^{\hat{J}_{\hat{w}}(N)} \hat{\lambda}_{wn}(N) = \sum_{n=1}^{\hat{J}_m(N)} \hat{\lambda}_{mn}(N) = 1.$$

Proof of Lemma 9. First notice that Lemma 8 and the definition of $\hat{J}_w(N)$ imply that $\sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{wn}(N) = \sum_{n=1}^{\hat{J}_w(N)+1} \hat{\lambda}_{wn}(N)$, which must be a continuous function (if $\hat{J}_w(N)$ increases by one, $\hat{\lambda}_{w,\hat{J}w+1}(N) = 0$).

Now consider N arbitrarily large. The solution to (21)–(26) implies that $\hat{J}_w(N) = \hat{J}_m(N) = 1$, where $\hat{y}_{j1}(N)$, $\hat{\lambda}_{j1}(N)$ are both arbitrarily close to zero. This implies that $\sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{wn}(N) = \sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{mn}(N) < 1$ for N large enough.

Suppose instead that N is arbitrarily small (but positive). The solution to (21)–(24) implies that $\hat{y}_{j1}(N)$ is arbitrarily close to $\bar{x}_j$. Taylor expanding appropriately, it can be shown that

$$\hat{\lambda}_{w1} \to \left[\frac{2\beta(r + \delta)[F_m'(\bar{x}_m)\bar{x}_m]^2}{\alpha^2 N[F_w'(\bar{x}_w)\bar{x}_w]}\right]^{1/3} \to \infty \quad \text{as } N \to 0.$$

Similarly for $\hat{\lambda}_{wn}(N)$, $n > 1$. Hence $\sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{wn}(N) = \sum_{n=1}^{\hat{J}_w(N)} \hat{\lambda}_{mn}(N) > 1$ for N small enough. Hence continuity establishes the lemma.

Lemma 9 and the construction of $\hat{y}_{jn}(N)$, $\hat{\lambda}_{jn}(N)$ and $\hat{J}_j(N)$ establish Theorem 1.

Proof of Proposition 4. For the symmetric case, with $\delta = 0$, (12)–(16) can be reduced to

$$(29) \quad y(n) = \left(\frac{\alpha\beta}{r^2 N}\right)^{1/2} [F(y(n-1)) - F(y(n))]^{1/2} \, \phi(y(n), y(n-1))$$

$$(30) \quad \lambda_n = \left[\frac{\beta}{\alpha N}\right]^{1/2} [F(y(n-1)) - F(y(n))]^{1/2}$$

$$(31) \quad \sum_{n=1}^{J} \lambda_n = 1,$$

where the boundary conditions become (a) $y(0) = \bar{x}$ and (b) $y(J) \le \underline{x}$. To prove uniqueness, we use the same approach taken to prove existence. Fix $N > 0$, and define the functions $\hat{y}_n(N)$, $\hat{\lambda}_n(N)$,

and $\hat{J}(N)$ as follows:
(i) $\hat{y}_0(N) = \bar{x}$;
(ii) if $\hat{y}_{n-1}(N) > \underline{x}$, then $\hat{y}_n(N)$, $\hat{\lambda}_n(N)$ satisfy

$$\hat{\lambda}_n(N) = \left[\frac{\alpha\beta}{r^2 N}\right]^{1/2} [F(\hat{y}_{n-1}(N)) - F(\hat{y}_n(N))]^{1/2} \phi(\hat{y}_n(N), \hat{y}_{n-1}(N)) \tag{32}$$

$$\hat{\lambda}_n(N) = \left[\frac{\beta}{\alpha N}\right]^{1/2} [F(\hat{y}_{n-1}(N)) - F(\hat{y}_n(N))]^{1/2} \tag{33}$$

(iii) if $\hat{y}_{n-1}(N) \le x$, then $\hat{y}_n(N) = \hat{\lambda}_n(N) = 0$;
(iv) $\hat{J}(N) = \max\{n\colon \hat{y}_n(N) > 0\}$.

A simple induction argument on (32) and (33) establishes that these functions are continuous for $N > 0$. Using the same arguments that proved Theorem 1, it is straightforward to establish existence of a solution $N^c > 0$, where $\sum_{n=1}^{J(N)} \hat{\lambda}_n(N^c) = 1$. The following establishes that $\sum_{n=1}^{J(N)} \hat{\lambda}_n(N)$ is strictly decreasing in N, which implies uniqueness.

LEMMA 10. $\hat{y}_n(\cdot)$ is decreasing and differentiable (left differentiable when $\hat{y}_n(\cdot) = \underline{x}$), and is strictly decreasing for $0 < n \le \hat{J}(N)$.

Proof of Lemma 10. Differentiability is established with a simple induction argument on (32). $1 - F$ log-concave, Lemma 5, and induction on (32) also ensures that $\hat{y}_n(N)$ is strictly decreasing in N for $0 < n \le \hat{J}(N)$.

LEMMA 11. $\hat{\lambda}_{n-1}(N) \ge \hat{\lambda}_n(N)$ for any $N > 0$.

Proof of Lemma 11. Fix $N > 0$. The lemma is trivial for $n - 1 \ge \hat{J}(N)$, and so consider $n - 1 < \hat{J}(N)$. Any class with boundaries $[y_L, y_H)$ must satisfy (32) which can be rewritten as $y_L = k\theta(y_L, y_H)$, where $k = [\alpha\beta/r^2N]^{1/2}$ and

$$\theta(\cdot) = [F(y_H) - F(y_L)]^{-1/2} \int_{y_L}^{y^H} F(y_H) - F(x)dx.$$

The number of people in such a class is $rkN[F(y_H) - F(y_L)]^{1/2}$. The lemma is established by proving for any $N > 0$ (fixed), $[F(y_H) - F(y_L)]$ is increasing in y_H where $y_L = k\theta(y_L, y_H)$.

Differentiation implies that

$$[1 - k\theta_1]dy_L = k\theta_2 dy_H \tag{34}$$

and so

$$\frac{d}{dy_H}[F(y_H) - F(y_L)] = F'(y_H) - F'(y_L)\frac{k\theta_2}{1 - k\theta_1}. \tag{35}$$

Using $k = y_L/\theta$, calculating θ_1, θ_2, and substituting in (35), it can be shown that

$$\frac{d}{dy_H}[F(y_H) - F(y_L)] = A(y_L, y_H)\int_{y_L}^{y_H} xF'(x) - y_L F'(y_L)dx,$$

where $A(\cdot)$ is a positive function. $xF'(x)$ increasing in x establishes the lemma.

LEMMA 12. $\sum_{n=1}^{J(N)} \hat{\lambda}_n(N)$ is strictly decreasing in N.

Proof of Lemma 12. For $n < \hat{J}(N)$, total differentiation of (33) implies that

$$\frac{d\lambda_n}{dN} = \frac{\lambda_n}{2N} + \left[\frac{\beta}{2\alpha N\hat{\lambda}_n}\right]\left[F'(\hat{y}_{n-1})\frac{d\hat{y}_{n-1}}{dN} - F'(\hat{y}_n)\frac{d\hat{y}_n}{dN}\right];$$

while for $n = \hat{J}(N)$;

$$\frac{d\hat{\lambda}_n}{dN} = \frac{\hat{\lambda}_n}{2N} + \left[\frac{\beta}{2\alpha N\hat{\lambda}_n}\right]F'(\hat{y}_{n-1})\frac{d\hat{y}_{n-1}}{dN}.$$

Summing over n implies that

$$\frac{d}{dN}\sum_{n=1}^{\hat{J}(N)}\hat{\lambda}_n(N)$$

$$= \sum_{n=1}^{\hat{J}(N)-1}\left[-\frac{\hat{\lambda}_n}{2N} + \left[\frac{\beta}{2\alpha N}\right]\left[\frac{1}{\hat{\lambda}_{n+1}} - \frac{1}{\hat{\lambda}_n}\right]F'(\hat{y}_n)\frac{d\hat{y}_n}{dN}\right] - \frac{\hat{\lambda}_J}{2N}$$

(where if $\hat{J}(N)$ increases by one to a small change in N, $\hat{\lambda}_{j\hat{J}}(N) = 0$). Now $\hat{\lambda}_n > 0$, $\hat{\lambda}_n \geq \hat{\lambda}_{n+1}$ (by Lemma 11) and $\hat{y}_n$ decreasing in N (by Lemma 10) for $n \leq \hat{J}(N)$, establishes the lemma.

Lemma 12 establishes Proposition 4.

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