Value at Risk and Expected Shortfall

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Risk

What is Risk?

risk, *noun*, situation involving exposure to danger.

Origin: Mid 17th century: from French risque (noun), risquer (verb), from Italian risco 'danger' and rischiare 'run into danger'.

Oxford English Dictionary

risk management is the identification, assessment, and prioritization of risks followed by coordinated and economical application of resources to minimize, monitor, and control the probability and/or impact of unfortunate events or to maximize the realization of opportunities.

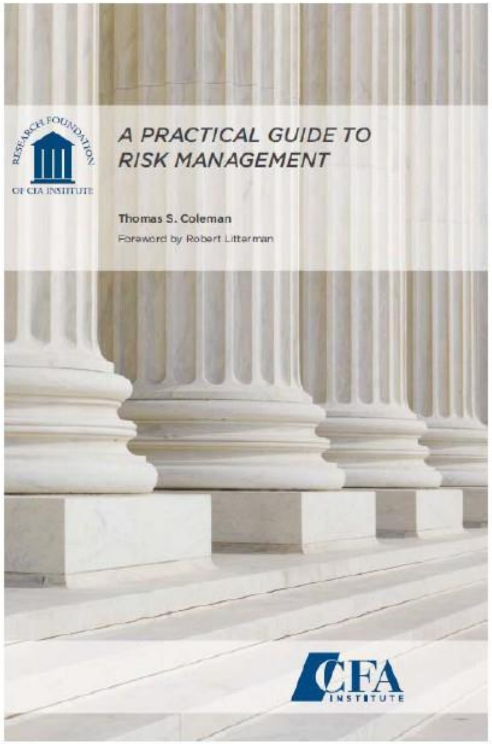
Hubbard, Douglas (2009). *The Failure of Risk Management: Why It's Broken and How to Fix It*. John Wiley & Sons.

Risk

The 1960 Valdivia earthquake or Great Chilean earthquake (Gran terremoto de Chile) of Sunday, 22 May 1960 was **the most powerful earthquake ever recorded**, rating a 9.5 on the moment magnitude scale.

It occurred in the afternoon (19:11 GMT, 15:11 local time), and lasted approximately 10 minutes. **The resulting tsunami** affected southern Chile, Hawaii, Japan, the Philippines, eastern New Zealand, southeast Australia, and the Aleutian Islands.

Financial Risk



Coleman, Tom, A Practical Guide to Risk Management (July 27, 2011). CFA Institute Research Foundation M2011-2.
Available at SSRN:
http://ssrn.com/abstract=2586032

Risk measurement has three goals:

- **Uncovering "known" risks** faced by the portfolio or the firm. By "known" risks, I mean risks that can be identified and understood with study and analysis because these or similar risks have been experienced in the past by this particular firm or others. Such risks often are not obvious or immedi-ately apparent, possibly because of the size or diversity of a portfolio, but these risks can be uncovered with diligence.
- **Making the known risks easy to see**, understand, and compare—in other words, the effective, simple, and transparent display and reporting of risk. Value at risk, or VaR, is a popular tool in this arena, but there are other, complementary, techniques and tools.
- **Trying to understand and uncover the "unknown"** or unanticipated risks—those that may not be easy to understand or anticipate, for example, because the organization or industry has not experienced them before.

The Axioms of Risk

A Risk measure, $\mathbf{R}$, is *coherent* if it satisfies the following (Artzner et al. 1999)

- **Monotonicity:**

  If portfolio $Y \geq X$ always, then $\mathbf{R}(Y) \leq \mathbf{R}(X)$

- **Subadditivity:**

  $\mathbf{R}(X + Y) \leq \mathbf{R}(X) + \mathbf{R}(Y)$

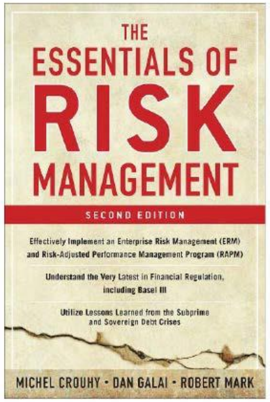
- **Positive Homogeneity:**

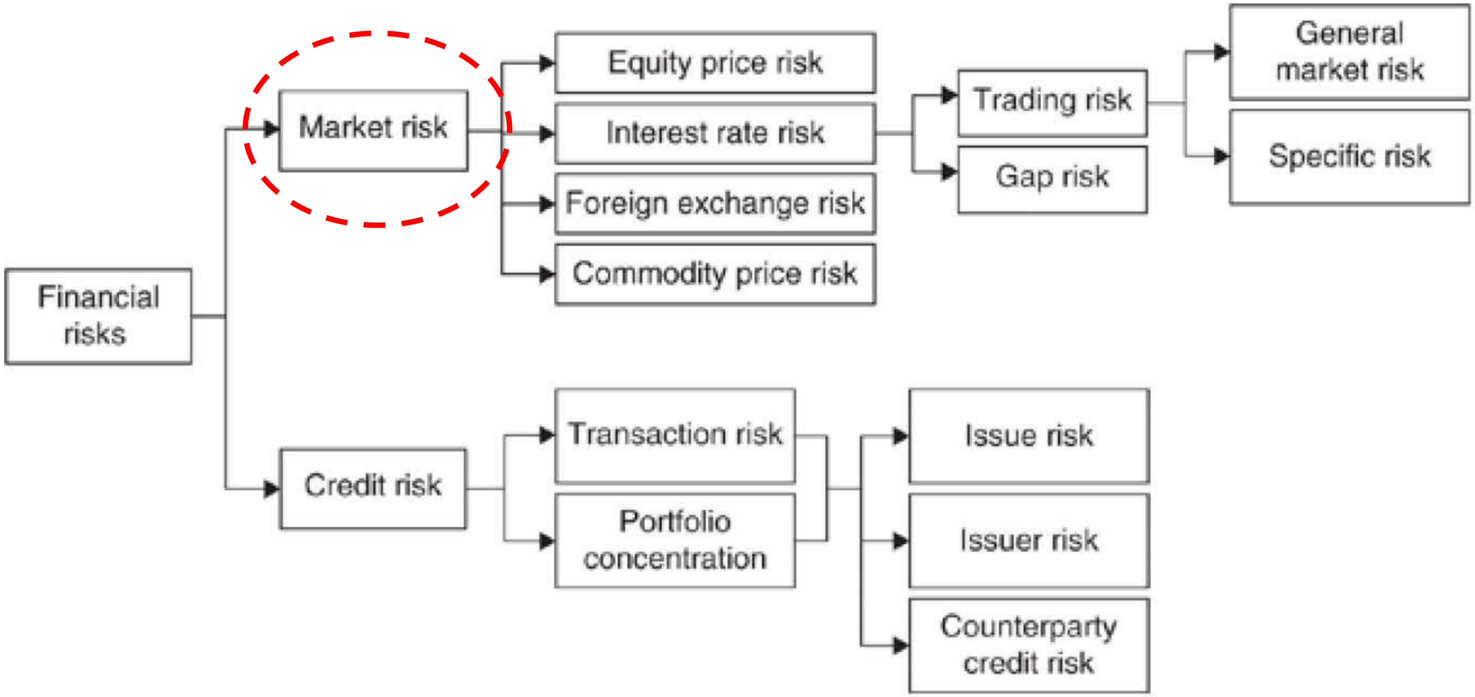
  For a constant $h \geq 0$, $\mathbf{R}(hX) = h\,\mathbf{R}(X)$

- **Translational Invariance:**

  For a risk-free amount n, $\mathbf{R}(X + n) = \mathbf{R}(X) - n$

Financial Risks





What is Market Risk?

*"Market risk encompasses the risk of **financial loss** resulting from **movements in market prices**."*

Market risk is rated on assessment of the following evaluation factors:

The sensitivity of the financial institution's earnings (or the economic value of its capital) to adverse changes in interest rates, FX rates, commodity prices, or equity prices.

https://www.federalreserve.gov/supervisionreg/topics/market_risk_mgmt.htm

The ability of management to identify, measure, monitor, and control exposure to market risk given the institution's size, complexity, and risk profile.

The nature and complexity of interest rate risk exposure arising from nontrading positions.

The nature and complexity of market risk exposure arising from trading and foreign operations.

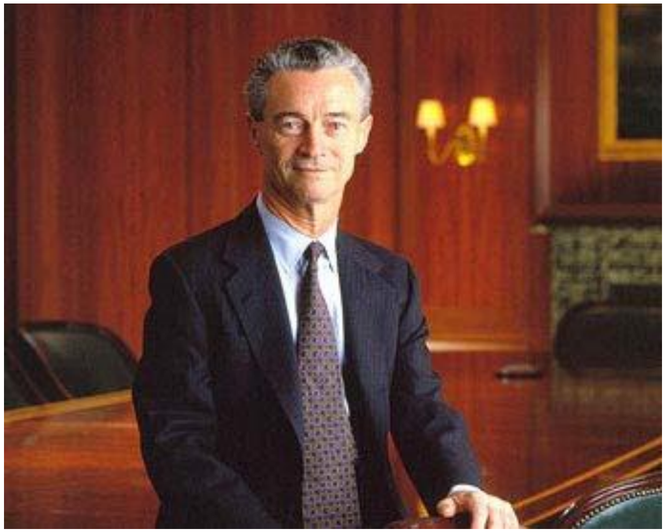
Market Risk Management Supervisory Policy and Guidance

https://www.federalreserve.gov/supervisionreg/topics/market_risk_mgmt.htm

Value at Risk

FitchLearning

“Invention” of Value at Risk



JPMorgan is credited wìth helping to make VaR a widely used measure. The Chairman, Sir Dennis Weatherstone was dissatisfied with the long-risk reports he received every day. These contained a huge amount of detail on the Greek letters for different exposures, but very little that was really useful to top management. He asked for something simpler that focused on the bank's total exposure over the next 24 hours measured across the bank's entìre trading portfolio.

At first his subordinates said this was impossible, but eventually they adapted the Markowìtz portfolio theory to develop a VaR report. This became known as the 4:15 report because it was placed on the chairman's desk at 4:15 pm every day after the close of trading.

John C. Hull, Risk Management and Financial Institutions, Wiley, 2012

Value at Risk (VaR)

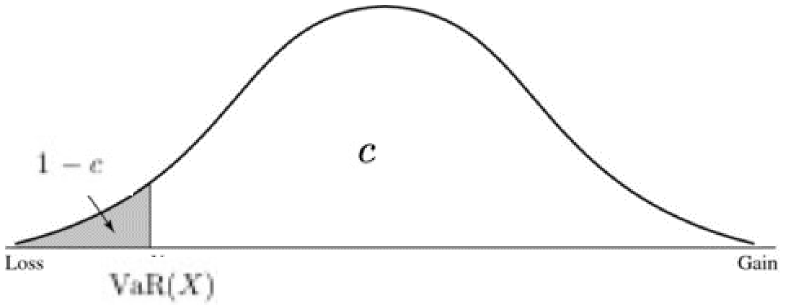
Definition: Value at Risk (VaR)

Value-at-risk (VaR) defined as the worst loss that might be expected from holding a security or portfolio over a given period of time (usually 10 days for the purpose of regulatory capital reporting), given a specified level of confidence.

Example:

A position has **a daily** VaR of $10m at the 99% confidence. We mean that the realised daily losses will be higher than $10m on only one day within 100 trading day period (ie, 2-3 days each year).

$$\Pr[x \leq \mathrm{VaR}(X)] = 1 - c$$



Interpretation of VaR

VaR is **not the answer** to the question: “How much can I lose on my portfolio over a given period of time?”

$$\Pr(\phi \leq \frac{\mathrm{VaR}(X) - \mu}{\sigma}) \quad = \quad 1 - c$$

Instead, VaR offers a **probability statement** about the potential change in the value of a portfolio resulting from a change in market factors over a specified period of time. Factor values are usually standardised. 99% Standard VaR (loss) is always equal to -2.32635.

VaR does not state by how much actual losses are likely to exceed the VaR figure (that is Expected Shortfall’s business). The statement is about how likely (or unlikely) the threshold, eg 99th percentile, will be exceeded.

VaR and time horizons

VaR is often used to manage market risk over a 1-day time horizon. For this purpose, it's necessary to derive VaR from the daily distribution of the portfolio returns.

However, regulators have set a time horizon of 10 days for the purpose of VaR calculations that are used to report regulatory capital requirements. Ideally, this ‘10-day VaR’ would be derived from a corresponding distribution of results over a 10-day horizon. **That is problematic**, however, as it implies the time series of data used for the analysis must be much longer (plus disttribution propagation).

The simple workaround is multiplying the **daily VaR** by the square root of time.

Why?

Explorations in Asset Returns

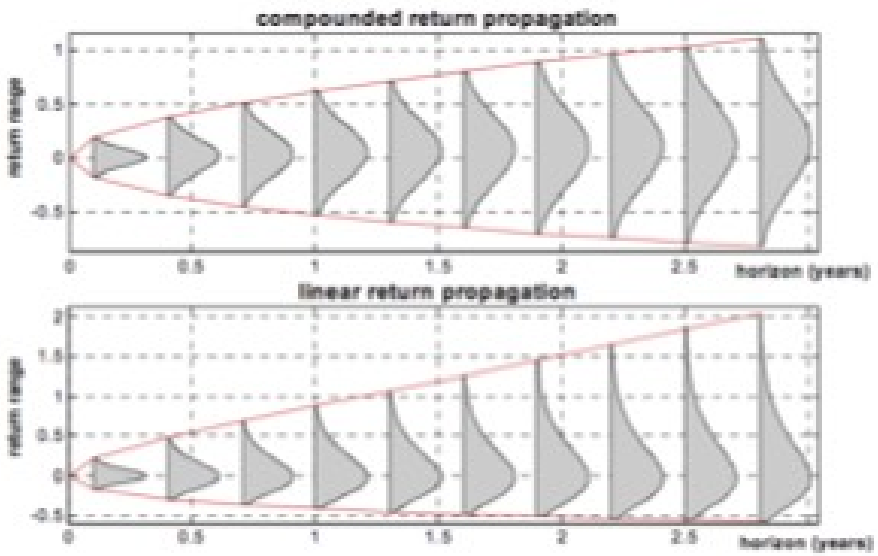
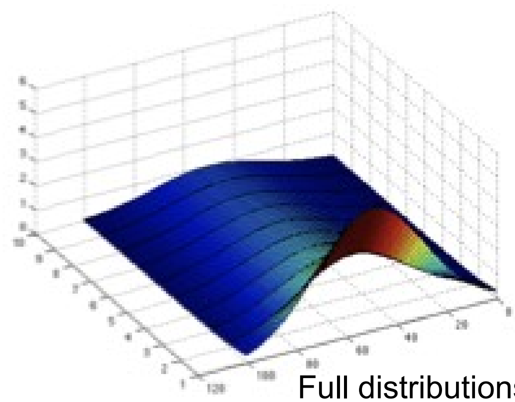
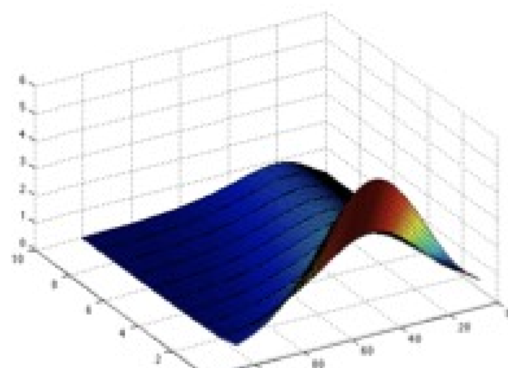


Figure 1: From: Linear versus Compounded Returns by Attillio Meucci (2010).





Full distributions from: *Explorations in Assest Returns* by Richard Diamond (2015)

Expected Shortfall (ES)

Expected Shortfall (ES)

Expected Shortfall also sometimes referred to as conditional value at risk, conditional tail expectation, or expected tail loss.

Whereas VaR asks the question: "How bad can things get?" expected shortfall asks: **"If things do get bad, what is the expected loss?**

Expected shortfall, like VaR, is a function of two parameters: T (the time horizon) and X (the confidence level).

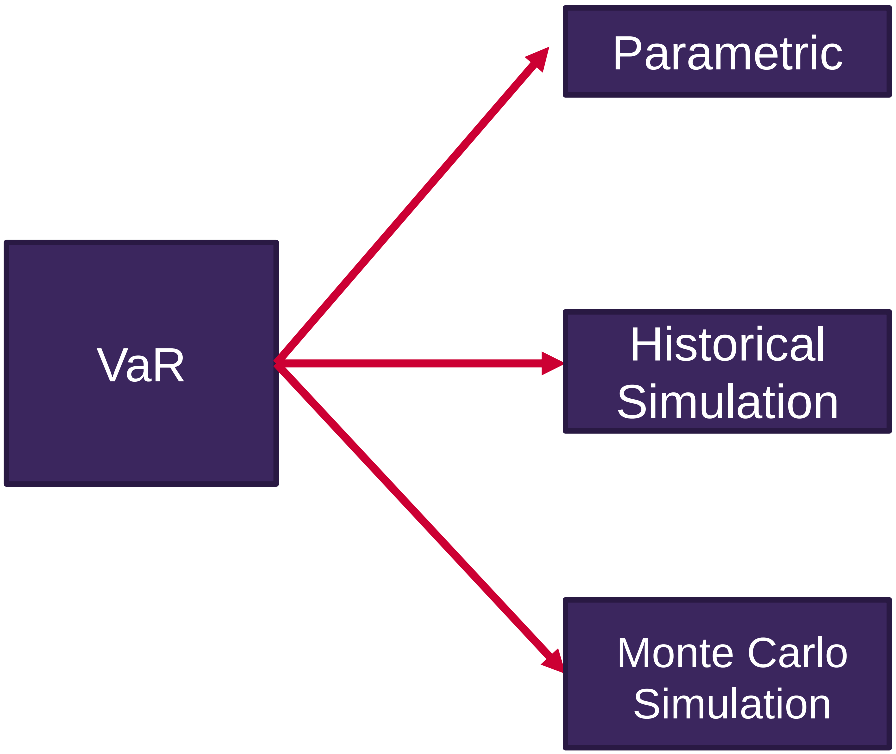
It is the expected loss during time T conditional on the loss being greater than the Xth percentile of the loss distribution.

$$\mathrm{ES}_c(X) \quad = \quad \mathbb{E}[\, X \,|\, X \leq \mathrm{VaR}_c(X)]$$

Expected Shortfall (ES)

- Expected value of all changes in the portfolio value **in the tail** of the distribution conditional on these changes **exceeding the VaR**
  - Requires a simulation (e.g. historical, Monte Carlo) to calculate
  - Also known as conditional VaR or Expected Tail Loss
- Takes into account the severity of a loss rather than just its probability
- ES is a ‘coherent’ risk measure, VaR is not
  - Reflects the risk reduction from diversification effects (‘subadditivity’)
- ES just gives more detail about a particular distribution rather than all potential sources of risk
- ES is much more difficult to backtest than VaR

Methods for Estimating VaR



Methods for Estimating VaR

First Method: Analytic variance-covariance approach (aka parametric VaR)

Under the analytic variance-covariance approach or ‘delta normal’ approach, we assume the risk factors (portfolio values) are log-normally distributed.

That makes the calculation much simpler, since the normal distribution is completely characterised by its first two moments. We need the mean and the variance of the portfolio return distribution from:

a. The multivariate distribution of the risk factors
b. The composition of the portfolio

$$\mathrm{VaR}(X) \quad = \quad \mu + \Phi^{-1}(1-c) \times \sigma$$

Methods for Estimating VaR

Second Method: historical simulation (aka non-parametric VaR)

The historical simulation approach to VaR calculation is conceptually simple and does not oblige the user to make assumptions about the distribution.

However, at least 2-3 years of historical data are necessary to produce meaningful results.

1. Select a sample of actual daily risk factor changes over a given period of time, say 500 days (ie, two years' worth of trading days).

2. Apply those daily changes to the current value of the risk factors, revaluing the current portfolio as many times as the number of days in the historical sample. Sum these changes across all positions, keeping the days synchronized.

3. Construct the histogram of portfolio values and identify the VaR that isolates the first percentile of the distribution in the left-hand tail (assuming VaR is derived at the 99% confidence level).

Advantages

- The major attraction of historical simulation is that the method is completely nonparametric (i.e., we don't need to worry about setting parameters) and does not depend on any assumptions about the distribution of the risk factors.
- The nonparametric nature of historical simulation also obviates the need to estimate volatilities and correlations. Historical volatilities and correlations are already reflected in the data set, so all we need to calculate are the synchronous risk-factor returns over a given historical period.
- Historical simulation has also no problem accommodating fat tails in distributions, since the historical returns already reflect actual synchronous moves in the market across all risk factors.

Disadvantages

- The main drawback of historical simulation is its complete dependence on a particular set of historical data.
- The underlying assumption is that the past, as captured in this historical data set, is a reliable representation of the future.

Methods for Estimating VaR

Third Method: Monte Carlo Simulation

Consists of repeatedly simulating the random processes that govern market prices and rates. Each simulation (scenario) generates a possible value for the portfolio at the target horizon (eg, 10 days).

a. Specify all the relevant riskfactors, their stochastic processes and parameter estimates

b. Construct price paths. Price paths are constructed using random numbers produced and advancing one step at a time (daily) the numerical solution to the stochastic processes

c. Value the portfolio for each path (scenario).

The process is repeated a large number of times, say 10,000 times, to generate the distribution, at the risk horizon, of the portfolio return.

VaR at the 99% confidence level is then simply derived as the distance to the mean of the first percentile of the distribution.

VaR Examples

Example 1: VaR for One Asset (parametric method)

Position:	USD100,000 in QQQ (Nasdaq 100)
Standard deviation of daily returns:	2.21%
99% confidence level:	2.33 standard deviations

Assignment:

1. Calculate 1 day VaR for this asset at 99% confidence
2. How would you then scale this up to 10 days?
3. List three limitations to the use of VaR as a measurement of market risk

VaR formula for a single asset = Vol x Square Root [T] x CL x N

Vol:	Daily volatility of the underlying asset
Square Root [T]:	Square root of time (days)
CL:	Confidence level (in standard deviations)
N:	Notional or underlying amount

Example 1: VaR for One Asset (parametric method)

1 day VaR = 2.21% * Square Root [1] * 2.33 * $100,000 = $5,149

10 day VaR = 2.21% * Square Root [10] * 2.33 * $100,000 = $16,284

- Note: Complexities arise at portfolio level as we have to take into account correlations between different positions

Limitations:

- Past not always good indicator of the future
- May not cover unusual market conditions (e.g. LTCM, Lehman default)
- Some market risks are not easily captured by VaR e.g.
  - Non-linear pay-offs such as options
  - Managed rates
  - Relative value risks e.g. basis, correlation

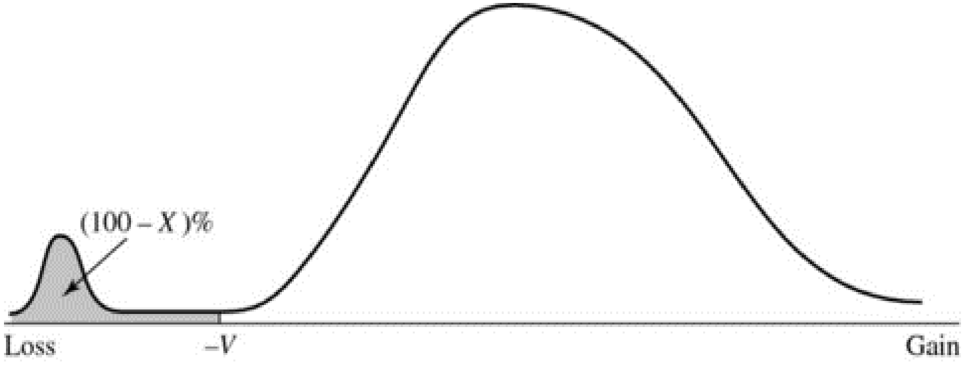
Assumptions about liquidity – time horizon may be inappropriate

Assumes continuous, small moves in value (no discontinuity)

Example 2: Expected Shortfall (parametric method)

Suppose that X=99, T is 10 days, and the VaR is $64 million. The expected shortfall is the average amount lost over a 10-day period assuming that the loss is greater than $64 million. The parametric formula for ES for one asset is:

$$ES = \mu + \sigma \frac{e^{-Y^2/2}}{\sqrt{2\pi}(1-X)}$$



Risk Management in Financial Institutions. John Hull, 4 ED

Example 3: VaR for One Asset (historical simulation)

Lab_VaR_ES_Example_ FTSE

K11 =AVERAGE(E2:E26)

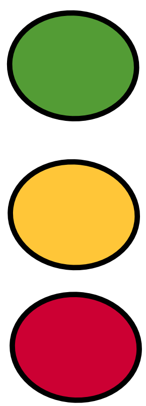
	A	B	C	D	E	G	H	J	K
1	30-Jun-10	4916.87	RETURNS	Rank	SORTED	Current Price	6169.74		
2	01-Jul-10	4805.75	-2.26%	1	-10.87%	N=	2524		
3	02-Jul-10	4838.09	0.67%	2	-7.69%	total sample size			
4	05-Jul-10	4823.53	-0.30%	3	-5.25%				
5	06-Jul-10	4965	2.93%	4	-4.67%	n=	25		
6	07-Jul-10	5014.82	1.00%	5	-4.67%				
7	08-Jul-10	5105.45	1.81%	6	-4.49%				
8	09-Jul-10	5132.94	0.54%	7	-4.05%	**with a probability of**			
9	12-Jul-10	5167.02	0.66%	8	-4.01%		**1.00%**		
10	13-Jul-10	5271.02	2.01%	9	-3.99%				
11	14-Jul-10	5253.52	-0.33%	10	-3.83%	**VaR =**	**-3.11%**	**ES =**	**-4.17%**
12	15-Jul-10	5211.29	-0.80%	11	-3.79%	Var(1d, 99%)=	£ 3.11	ES(1d,99%)=	£ 4.17
13	16-Jul-10	5158.85	-1.01%	12	-3.62%	for £100 invested		for £100 invested	
14	19-Jul-10	5148.28	-0.20%	13	-3.58%	*there is a 99% probability*			
15	20-Jul-10	5139.46	-0.17%	14	-3.50%	*that losses will not go beyond £3.11*			
16	21-Jul-10	5214.64	1.46%	15	-3.49%	*in one day*			
17	22-Jul-10	5313.81	1.90%	16	-3.46%				
18	23-Jul-10	5312.62	-0.02%	17	-3.43%	**VaR =**	**£ 191.88**		
19	26-Jul-10	5351.12	0.72%	18	-3.39%	Var(1d, 99%)=			
20	27-Jul-10	5365.67	0.27%	19	-3.34%	for £6169.74 invested			
21	28-Jul-10	5319.68	-0.86%	20	-3.34%	current stock value			
22	29-Jul-10	5313.95	-0.11%	21	-3.23%	*there is a 99% probability*			
23	30-Jul-10	5258.02	-1.05%	22	-3.18%	*that losses will not go beyond £191.88*			
24	02-Aug-10	5397.11	2.65%	23	-3.15%	*in one day*			
25	03-Aug-10	5396.48	-0.01%	24	-3.15%				
26	04-Aug-10	5386.16	-0.19%	25	-3.11%	**VaR =**	**£ 606.78**		
27	05-Aug-10	5365.78	-0.38%	26	-3.05%	Var(10d,99%)=			
28	06-Aug-10	5332.39	-0.62%	27	-3.03%	*there is a 99% probability*			
29	09-Aug-10	5410.52	1.47%	28	-2.98%	*that losses will not go beyond £606.78*			
30	10-Aug-10	5376.41	-0.63%	29	-2.96%	*in ten days*			
31	11-Aug-10	5245.21	-2.44%	30	-2.83%				
32	12-Aug-10	5266.06	0.40%	31	-2.83%				
33	13-Aug-10	5275.44	0.18%	32	-2.83%				

DATA | ANALYSIS

Backtesting Value at Risk

- Backtesting tests validity of VaR model
  - Regulatory requirement if using model to calculate capital
- Compare VaR figures to actual outcomes
- More rigorous approach is to compare VaR to ‘hypothetical’ P&L
  - Based on holding end of day position throughout next day

‘Traffic lights’ – Basel I Approach



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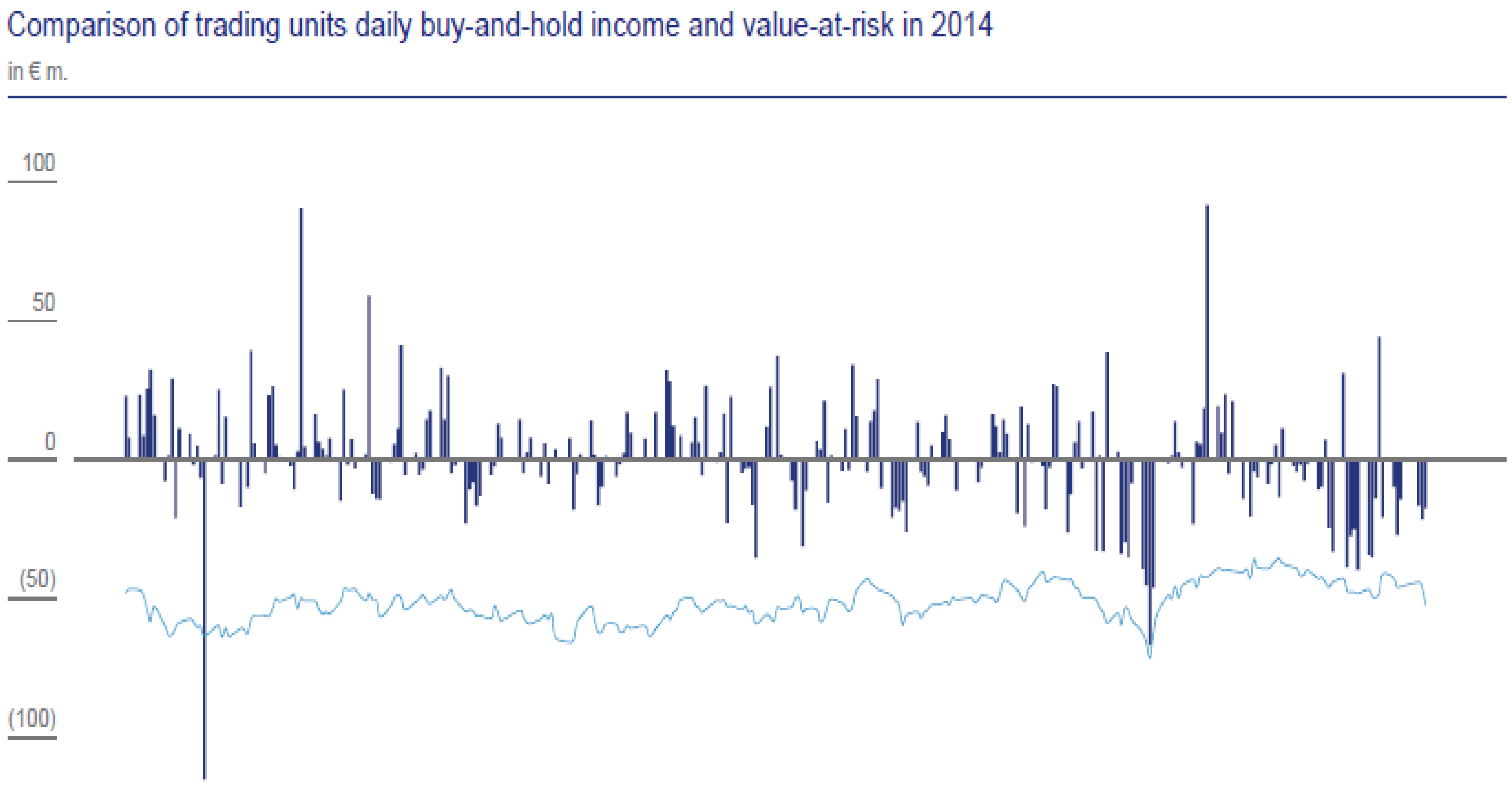
Backtesting Value at Risk: Deutsche Bank

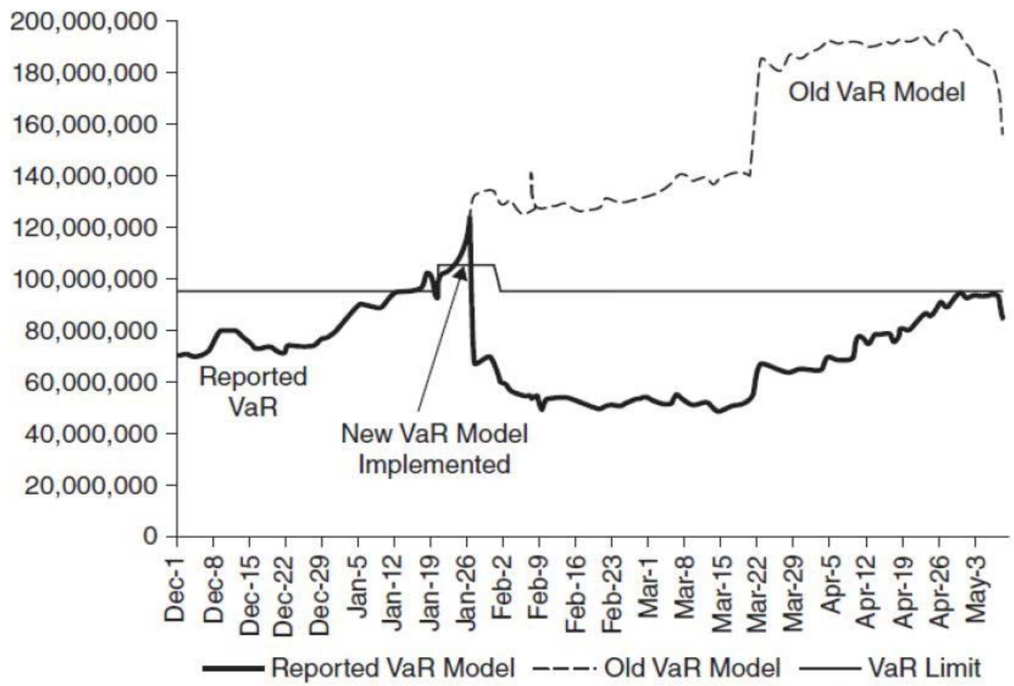


Source: Deutsche Bank Annual Report 2014, page 194

Case Study: The London Whale

'London Whale' Rattles Debt Market
by Wall Street Journal (Published on 6 Apr 2012)
https://www.youtube.com/watch?v=pen_3uYu0s4

Case Study: The London Whale



United States Senate Permanent Subcommittee on Investigations, *JP Morgan Chase Whale Trades: A Case History of Derivatives Risks and Abuses*, Hearing, March 15, 2013, Exhibits.

During the first half of 2012, JPMorgan Chase lost billions of dollars from exposure to a massive credit derivative portfolio.

The losses were the result of the so-called "London Whale" trades executed by traders in its London office. Initially dismissed by the bank's chief executive as a "tempest in a teapot," the trading losses quickly doubled and then tripled.

In contrast to JPMorgan Chase's reputation for best-in-class risk management, the whale trades exposed a bank culture in which risk limit breaches were routinely disregarded, risk metrics were frequently criticized or downplayed, and risk evaluation models were targeted by bank personnel seeking to produce artificially lower capital requirements."

Portfolio VaR and ES

Harry Markowitz

Harry Max Markowitz (born August 24, 1927) is an American economist, and a recipient of the 1989 John von Neumann Theory Prize and the 1990 Nobel Memorial Prize in Economic Sciences.

https://www.thefamouspeople.com/profiles/images/harry-markowitz-1.jpg

PORTFOLIO SELECTION*

HARRY MARKOWITZ
The Rand Corporation

THE PROCESS OF SELECTING a portfolio may be divided into two stages. The first stage starts with observation and experience and ends with beliefs about the future performances of available securities. The second stage starts with the relevant beliefs about future performances and ends with the choice of portfolio. This paper is concerned with the second stage. We first consider the rule that the investor does (or should) maximize discounted expected, or anticipated, returns. This rule is rejected both as a hypothesis to explain, and as a maximum to guide investment behavior. We next consider the rule that the investor does (or should) consider expected return a desirable thing *and* variance of return an undesirable thing. This rule has many sound points, both as a maxim for, and hypothesis about, investment behavior. We illustrate geometrically relations between beliefs and choice of portfolio according to the "expected returns—variance of returns" rule.

One type of rule concerning choice of portfolio is that the investor does (or should) maximize the discounted (or capitalized) value of future returns.[1] Since the future is not known with certainty, it must be "expected" or "anticipated" returns which we discount. Variations of this type of rule can be suggested. Following Hicks, we could let "anticipated" returns include an allowance for risk.[2] Or, we could let the rate at which we capitalize the returns from particular securities vary with risk.

The hypothesis (or maxim) that the investor does (or should) maximize discounted return must be rejected. If we ignore market imperfections the foregoing rule never implies that there is a diversified portfolio which is preferable to all non-diversified portfolios. Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim.

Modern Portfolio Theory

We consider a universe of n assets. Let $x = (x_1, \ldots, x_n)$ be the vector of weights in the portfolio. We assume that the portfolio is fully invested meaning that $\sum_{i=1}^{n} x_i = \mathbf{1}^\top x = 1$. We denote $R = (R_1, \ldots, R_n)$ the vector of asset returns where R_i is the return of asset i. The return of the portfolio is then equal to $R(x) = \sum_{i=1}^{n} x_i R_i$. In a matrix form, we also obtain $R(x) = x^\top R$. Let $\mu = \mathbb{E}[R]$ and $\Sigma = \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right]$ be the vector of expected returns and the covariance matrix of asset returns. The expected return of the portfolio is:

$$\mu(x) = \mathbb{E}[R(x)] = \mathbb{E}\left[x^\top R\right] = x^\top \mathbb{E}[R] = x^\top \mu$$

**Extract from Thierry Roncalli, Introduction to Risk Parity and Budgeting, CRC Press, 2014*

Modern Portfolio Theory

whereas its variance is equal to:

$$\begin{aligned}
\sigma^2(x) &= \mathbb{E}\left[(R(x)-\mu(x))(R(x)-\mu(x))^\top\right] \\
&= \mathbb{E}\left[\left(x^\top R - x^\top \mu\right)\left(x^\top R - x^\top \mu\right)^\top\right] \\
&= \mathbb{E}\left[x^\top (R-\mu)(R-\mu)^\top x\right] \\
&= x^\top \mathbb{E}\left[(R-\mu)(R-\mu)^\top\right] x \\
&= x^\top \Sigma x
\end{aligned}$$

Modern Portfolio Theory

We can then formulate the investor's financial problem as follows:

1. Maximizing the expected return of the portfolio under a volatility constraint (σ-problem):

$$\max \mu(x) \quad \text{u.c.} \quad \sigma(x) \leq \sigma^{\star} \tag{1.1}$$

2. Or minimizing the volatility of the portfolio under a return constraint (μ-problem):

$$\min \sigma(x) \quad \text{u.c.} \quad \mu(x) \geq \mu^{\star} \tag{1.2}$$

Portfolio Risk Measures

By definition, the loss of the portfolio is $L(x) = -R(x)$ where $R(x)$ is the return of the portfolio. We consider then different risk measures:

- Volatility of the loss

$$\mathcal{R}(x) = \sigma(L(x)) = \sigma(x)$$

The volatility of the loss is the portfolio's volatility.

- Standard deviation-based risk measure

$$\mathcal{R}(x) = \mathrm{SD}_c(x) = \mathbb{E}[L(x)] + c \cdot \sigma(L(x)) = -\mu(x) + c \cdot \sigma(x)$$

To obtain this measure, we scale the volatility by factor $c > 0$ and subtract the expected return of the portfolio.

**Extract from Thierry Roncalli, Introduction to Risk Parity and Budgeting, CRC Press, 2014, pp. 71*

Portfolio Risk Measures

- Value-at-risk

$$\mathcal{R}(x) = \mathrm{VaR}_\alpha(x) = \inf\{\ell : \Pr\{L(x) \le \ell\} \ge \alpha\}$$

The value-at-risk is the α-quantile of the loss distribution $\mathbf{F}$ and we note it $\mathbf{F}^{-1}(\alpha)$.

- Expected shortfall

$$\mathcal{R}(x) = \mathrm{ES}_\alpha(x) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(x)\,\mathrm{d}u$$

The expected shortfall is the average of the VaRs at level α and higher (Acerbi and Tasche, 2002). We note that it is also equal to the expected loss given that the loss is beyond the value-at-risk:

$$\mathrm{ES}_\alpha(x) = \mathbb{E}\left[L(x) \mid L(x) \ge \mathrm{VaR}_\alpha(x)\right]$$

Portfolio Risk Measures

Let us assume that the asset returns are normally distributed: $R \sim \mathcal{N}(\mu, \Sigma)$. We have $\mu(x) = x^\top \mu$ and $\sigma(x) = \sqrt{x^\top \Sigma x}$. It follows that the standard deviation-based risk measure is:

$$\mathrm{SD}_c(x) = -x^\top \mu + c \cdot \sqrt{x^\top \Sigma x} \tag{2.1}$$

For the value-at-risk, we have $\Pr\{L(x) \leq \mathrm{VaR}_\alpha(x)\} = \alpha$. Because $L(x) = -R(x)$ and $\Pr\{R(x) \geq -\mathrm{VaR}_\alpha(x)\} = \alpha$, we deduce that:

$$\Pr\left\{\frac{R(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}} \leq \frac{-\mathrm{VaR}_\alpha(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}}\right\} = 1 - \alpha$$

It follows that:

$$\frac{-\mathrm{VaR}_\alpha(x) - x^\top \mu}{\sqrt{x^\top \Sigma x}} = \Phi^{-1}(1 - \alpha)$$

We finally obtain:

$$\boxed{\mathrm{VaR}_\alpha(x) = -x^\top \mu + \Phi^{-1}(\alpha)\sqrt{x^\top \Sigma x}} \tag{2.2}$$

Portfolio Risk Measures

This is a special case of the standard deviation-based risk measure with $c = \Phi^{-1}(\alpha)$. It implies that the value-at-risk is a coherent and convex risk measure if the asset returns are normally distributed. The expression of the expected shortfall is:

$$\mathrm{ES}_\alpha(x) = \frac{1}{1-\alpha}\int_{-\mu(x)+\sigma(x)\Phi^{-1}(\alpha)}^{\infty} \frac{u}{\sigma(x)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{u+\mu(x)}{\sigma(x)}\right)^2\right) \mathrm{d}u$$

With the variable change $t = \sigma(x)^{-1}(u+\mu(x))$, we obtain:

$$\begin{aligned}
\mathrm{ES}_\alpha(x) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} (-\mu(x)+\sigma(x)t)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= -\frac{\mu(x)}{1-\alpha}[\Phi(t)]_{\Phi^{-1}(\alpha)}^{\infty} + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} t\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= -\mu(x) + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^2\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= -\mu(x) + \frac{\sigma(x)}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^2\right)
\end{aligned}$$

Portfolio Risk Measures

The expected shortfall of portfolio x is then:

$$\boxed{\mathrm{ES}_\alpha(x) = -x^\top \mu + \frac{\sqrt{x^\top \Sigma x}}{(1-\alpha)} \phi\left(\Phi^{-1}(\alpha)\right)} \tag{2.3}$$

Like the value-at-risk, it is a standard deviation-based risk measure with $c = \phi\left(\Phi^{-1}(\alpha)\right) / (1-\alpha)$.

Example: Portfolio VaR and ES

Example *We consider three stocks A, B and C. Their current prices are respectively* 244, 135 *and* 315 *dollars. We assume that their expected returns are equal to* 50 *bps,* 30 *bps and* 20 *bps on a daily basis, whereas their daily volatilities are* 2%, 3% *and* 1%. *The correlation of asset returns is given by the following matrix:*

$$\rho = \begin{pmatrix} 1.00 & & \\ 0.50 & 1.00 & \\ 0.25 & 0.60 & 1.00 \end{pmatrix}$$

**Extract from Thierry Roncalli, Introduction to Risk Parity and Budgeting, CRC Press, 2014, pp. 75*

Example: Portfolio VaR and ES

We consider Portfolio #1 composed of two stocks A, one stock B and one stock C. The value of the portfolio is then equal to 938 dollars. We deduce that the weights x are $(52.03\%, 14.39\%, 33.58\%)$. Computation gives $\mu(x) = 37.0$ bps and $\sigma(x) = 1.476\%$. In Table 2.1, we report the values taken by the value-at-risk and the expected shortfall for different confidence levels α. Because these risk measures are computed using Formulas (2.2) and (2.3) with the portfolio weights, the values are expressed as a percentage. For example, the value-at-risk of the portfolio for $\alpha = 99\%$ is equal to[2]:

$$\mathrm{VaR}_{99\%}(x) = -0.370\% + 2.326 \times 1.476\% = 3.06\%$$

For the expected shortfall, we obtain[3]:

$$\mathrm{ES}_{99\%}(x) = -0.370\% + 2.667 \times 1.476\% = 3.56\%$$

Example: Portfolio VaR and ES

TABLE 2.1: Computation of risk measures $\mathrm{VaR}_\alpha(x)$ and $\mathrm{ES}_\alpha(x)$

Portfolio	$\mathcal{R}(x)$		α 90%	95%	99%	99.5%
#1	VaR	(in %)	1.52	2.06	3.06	3.43
		(in \$)	14.27	19.30	28.74	32.20
	ES	(in %)	2.22	2.67	3.56	3.90
		(in \$)	20.83	25.09	33.44	36.58

```
# LAB: VAR and ES portfolio
# Thierry Roncalli, Introduction to Risk Parity and Budgeting, CRC Press, 2014
# Example page 75
#
# Developed Alonso Pena, Fitch Learning UK, Jan 2018

import math
import numpy as np
from scipy.stats import norm

# STEP 1 input data three stocks A,B,C
x = np.matrix([[0.5203], [0.1439], [0.3358]])
mu = np.matrix([[50/10000],[30/10000], [20/10000]])
vol = np.matrix([[2/100], [3/100], [1/100]])

rho = np.matrix([[1, 0.5, 0.25],
[0.5, 1, 0.6],
[0.25, 0.6, 1]])

Sigma = np.matrix([[4.0000e-004, 3.0000e-004, 5.0000e-005],
[3.0000e-004,9.0000e-004, 1.8000e-004],
[5.0000e-005,1.8000e-004,1.0000e-004]])

alpha=0.99

# STEP 2 calculations
# compute VAR
VAR_x = -x.T*mu + norm.ppf(alpha)*math.sqrt(x.T*Sigma*x)

# compute ES
ES_x = -x.T*mu + (math.sqrt(x.T*Sigma*x))/(1-alpha)*norm.pdf(norm.ppf(alpha))

# STEP 3 output results
print('VAR_x  = ', VAR_x )
print('ES_x  = ', ES_x )
```

Appendix:
Risk Allocation

Risk Allocation

Let $w = (w_1, w_2, \ldots, w_N)'$ denote the vector of portfolio weights on N assets, instruments, asset classes, or managers, and let $\sigma(w)$ and $\mathrm{VaR}(w)$ denote the portfolio standard deviation and value-at-risk, which depend on the positions or weights w_i. Imagine multiplying all of the portfolio weights by the same constant, k, that is, consider the vector of portfolio weights $kw = (kw_1, kw_2, \ldots, kw_N)'$ and the associated portfolio standard deviation $\sigma(kw)$ and value-at-risk $\mathrm{VaR}(kw)$. A key property of the portfolio standard deviation is that scaling

Extract from Neil D. Pearson, Risk budgeting: Portfolio Problem Solving with Value-at-Risk, Wiley 2002, pages 153-163

Risk Allocation

all positions by the common factor k scales the standard deviation by the same factor, implying $\sigma(kw) = k\sigma(w)$. This is also true of the value-at-risk, because scaling every position by k clearly scales every profit or loss by k, and thus scales the value-at-risk by k.

In mathematical terminology, the result that $\mathrm{VaR}(kw) = k\mathrm{VaR}(w)$ for $k > 0$ means that function giving the value-at-risk is *homogenous of degree 1*, or *linear homogenous*. From a financial perspective, this property of value-at-risk is almost obvious: if one makes the same proportional change in all positions, the value-at-risk also changes proportionally.

Though very nearly obvious, this property has an important implication. If value-at-risk is linear homogenous, then Euler's law (see the notes to this chapter) implies that both the portfolio standard deviation and VaR can be decomposed as

$$\sigma(w) = \frac{\partial\sigma(w)}{\partial w_1}w_1 + \frac{\partial\sigma(w)}{\partial w_2}w_2 + \cdots + \frac{\partial\sigma(w)}{\partial w_N}w_N \tag{10.1}$$

and

$$\boxed{\mathrm{VaR}(w) = \frac{\partial\mathrm{VaR}(w)}{\partial w_1}w_1 + \frac{\partial\mathrm{VaR}(w)}{\partial w_2}w_2 + \cdots + \frac{\partial\mathrm{VaR}(w)}{\partial w_N}w_N,} \tag{10.2}$$

Risk Allocation

respectively. The ith partial derivative, $\partial\sigma(w)/\partial w_i$ or $\partial \mathrm{VaR}(w)/\partial w_i$, is interpreted as the effect on risk of increasing w_i by one unit; in particular, changing the ith weight by a small amount, from w_i to w_i^*, changes the risk by approximately $(\partial\sigma(w)/\partial w_i)(w_i^* - w_i)$, or $(\partial \mathrm{VaR}(w)/\partial w_i)(w_i^* - w_i)$. The ith term, $(\partial\sigma(w)/\partial w_i)w_i$ or $(\partial \mathrm{VaR}(w)/\partial w_i)w_i$, is called the *risk contribution* of the ith position and can be interpreted as measuring the effect of percentage changes in the portfolio weight w_i. For example, the change from w_i to w_i^* is a percentage change of $(w_i^* - w_i)/w_i$, and the change in portfolio standard deviation resulting from this change in the portfolio weight is

$$\frac{\partial\sigma(w)}{\partial w_i}(w_i^* - w_i) = \frac{\partial\sigma(w)}{\partial w_i}w_i \times \frac{(w_i^* - w_i)}{w_i},$$

the product of the risk contribution and the percentage change in the weight.

Risk Allocation

A key feature of the risk contributions is that they sum to the portfolio risk, permitting the portfolio risk to be decomposed into the risk contributions of the N positions w_i. Similarly, we can define $((\partial\sigma(w)/\partial w_i)w_i)/\sigma(w)$ or $((\partial \text{VaR}(w)/\partial w_i)w_i)/\text{VaR}(w)$ to be the percentage contribution to portfolio risk of the ith position. It is straightforward to compute these risk contributions when risk is measured by standard deviation.