**EPFL**
School of Computer and Communication Sciences
**Martin Jaggi & Nicolas Flammarion**
github.com/epfml/OptML_course

# Problem Set 10, May 14, 2022 (Convex conjugate)

For a function $f : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ (which is not necessarily convex !), we consider its **convex conjugate** which for $y \in \mathbb{R}^d$ is defined as

$$f^*(y) = \sup_{x \in \mathbb{R}^d} (\langle x, y \rangle - f(x)) \in \mathbb{R} \cup \{+\infty\}$$

Prove the following properties.

1. Show that $f^*$ is convex.
2. Show that for $x, y \in \mathbb{R}^d$, $f(x) + f^*(y) \geq \langle x, y \rangle$. This is known as Fenchel's inequality.
3. Show that the biconjugate $f^{**}$ (the conjugate of the conjugate) is such that $f^{**} \leq f$.

The Fenchel-Moreau theorem (which we will not prove here) states that $f = f^{**}$ if and only if $f$ is convex and closed. It will turn out to be useful to show the following property.

4. Assume that $f$ is closed and convex. Then show that for any $x, y \in \mathbb{R}^d$,

$$\begin{aligned} y \in \partial f(x) &\Leftrightarrow x \in \partial f^*(y) \\ &\Leftrightarrow f(x) + f^*(y) = \langle x, y \rangle \end{aligned}$$