Big Data Computing Homework 3

Fabio Fraschetti 1834942

November 2023

[Fabio Fraschetti]

1 Exercise

I wrote a colab called BDHW3EX1Fraschetti that is in the folder.

2 Exercise

2.1

We know by the properties of the symmetric matrix $V^T AV = \Lambda$ and so we can write this:

$$A = V\Lambda V^T = \sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^T$$

Now we try to compute A^2

$$A^2 = A \cdot A = V\Lambda V^T V\Lambda V^T = V\Lambda^2 V^T$$

Where $V^T V = I$ because V is orthogonal. Thus we can generalize it by computing A^k

$$A^k = A \cdot A \ldots A = V\Lambda V^T V\Lambda V^T \ldots V\Lambda V^T = V\Lambda^k V^T$$

Where we have multiplied A k-times. Now we can transform it in terms of $\boldsymbol{u}_i$ and λ_i

$$V\Lambda^k V^T = \sum_{i=1}^{n} \lambda_i^k \boldsymbol{u}_i \boldsymbol{u}_i^T$$

2.2

I guess that

$$A^{-1} = \sum_{j=1}^{n} \frac{1}{\lambda_j} \boldsymbol{u}_j \boldsymbol{u}_j^T$$

$Proof$: The matrix A^{-1} is the inverse of A if $A \cdot A^{-1} = I$. So we write that:

$$A \cdot A^{-1} = \sum_{i=1}^{n} \lambda_i \boldsymbol{u}_i \boldsymbol{u}_i^T \sum_{j=1}^{n} \frac{1}{\lambda_j} \boldsymbol{u}_j \boldsymbol{u}_j^T = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\lambda_i}{\lambda_j} \boldsymbol{u}_i \boldsymbol{u}_i^T \boldsymbol{u}_j \boldsymbol{u}_j^T$$

Where $\boldsymbol{u}_i^T \boldsymbol{u}_j = \delta_{ij}$ because it's an orthonormal basis. I applied only the definitions. Now we can substitute δ_{ij} and simplify the summation of j:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{u}_i \delta_{ij} \boldsymbol{u}_j^T = \sum_{i=1}^{n} \boldsymbol{u}_i \boldsymbol{u}_i^T = \boldsymbol{U}\boldsymbol{U}^T = I$$

So we proved that $A \cdot A^{-1} = I$ and so $A^{-1} = \sum_{i=1}^{n} \frac{1}{\lambda_i} \boldsymbol{u}_i \boldsymbol{u}_i^T$

3 Exercise

3.1

Given $A = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$ we first compute $A \cdot A^T$ as:

$$A \cdot A^T = \sum_{i=1}^{r} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \sum_{j=1}^{r} \sigma_j \boldsymbol{v}_j \boldsymbol{u}_j^T = \sum_{i,j=1}^{r} \sigma_i \sigma_j \boldsymbol{u}_i \boldsymbol{v}_i^T \boldsymbol{v}_j \boldsymbol{u}_j^T$$

Where $\boldsymbol{v}_i^T \boldsymbol{v}_j$ is the scalar product of 2 vectors that is always 0 unless $i = j$. When $i = j$ the result of the scalar product is 1, because this are 2 orthonormal vectors. So we can continue the equality with:

$$\sum_{i,j=1}^{r} \sigma_i \sigma_j \boldsymbol{u}_i \boldsymbol{v}_i^T \boldsymbol{v}_j \boldsymbol{u}_j^T = \sum_{i=1}^{r} \sigma_i^2 \boldsymbol{u}_i \boldsymbol{u}_i^T$$

Now that we have calculated AA^T we can compute $(AA^T)^k$ as:

$$(AA^T)^k = \left(\sum_{i=1}^{r} \sigma_i^2 \boldsymbol{u}_i \boldsymbol{u}_i^T \right)^k = \sum_{i=1}^{r} \sigma_i^{2k} \boldsymbol{u}_i \boldsymbol{u}_i^T$$

for the same reason of the exercise 2.1 above.

3.2

A matrix Q that has orthonormal columns $q_1, q_2 \dots q_n$ such that so $q_i^T q_j = 1$ for all $i = j$ and $q_i^T q_j = 0$ for all $i \neq j$, because the columns are orthonormal basis. As result we compute $Q^T Q$ that is a diagonal matrix, in particular $Q^T Q = I$. This implies that the rows of Q^T (which are the columns of Q) are orthonormal. By considering $QQ^T = I$ in the same way, we see the columns of Q^T , and thus the rows of $(Q^T)^T = Q$ also form an orthonormal basis. In other words from the uniqueness of the inverted matrix we know that the right inverse matrix is equal to the left inverse matrix so $Q^T Q = QQ^T = I$. Therefore, if a square matrix Q has columns forming an orthonormal basis, its rows are also orthonormal.

3.3

To prove that B is the inverse of A we need to show that $A \cdot B = B \cdot A = I$ so we write this:

$$A \cdot B = \sum_{i=1}^{n} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \sum_{j=1}^{n} \frac{1}{\sigma_j} \boldsymbol{v}_j \boldsymbol{u}_j^T = \sum_{i,j=1}^{n} \frac{\sigma_i}{\sigma_j} \boldsymbol{u}_i \boldsymbol{v}_i^T \boldsymbol{v}_j \boldsymbol{u}_j^T = \sum_{i=1}^{n} \boldsymbol{u}_i \boldsymbol{u}_i^T = I$$

Where $\boldsymbol{v}_i^T \boldsymbol{v}_j = \delta_{ij}$ because they form an orthonormal basis, $\frac{\sigma_i}{\sigma_j} = 1$ when $i = j$ that are the only possible values because of δ_{ij} and the last equality is true because the sum of orthonormal vectors is equal to identity matrix. In a similar way we can show that $B \cdot A = I$.

$$B \cdot A = \sum_{i=1}^{n} \frac{1}{\sigma_i} \boldsymbol{v}_i \boldsymbol{u}_i^T \sum_{j=1}^{n} \sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T = \sum_{i,j=1}^{n} \frac{\sigma_j}{\sigma_i} \boldsymbol{v}_i \boldsymbol{u}_i^T \boldsymbol{u}_j \boldsymbol{v}_j^T = \sum_{i=1}^{n} \boldsymbol{v}_i \boldsymbol{v}_i^T = I$$

Where now we consider $\boldsymbol{u}_i^T \boldsymbol{u}_j = \delta_{ij}$. Therefore we can say that B is the inverse of A.

4 Exercise

4.1

Since $x^T Bx \geq 0$ we need to prove this $x^T AA^T x \geq 0$ to be PSA.

$$x^T AA^T x = (A^T x)^T (A^T x) = ||A^T x||_2^2 \geq 0$$

For all $\boldsymbol{x} \in R^m$
Where $||A^T x||_2^2$ is the squared euclidean norm. The last inequality is true because it's the dot product with itself and it's always non negative. Therefore we conclude that AA^T is PSD.

4.2 Bonus

We know that if λ is an eigenvalue of A we can write $Ax = \lambda x$ where x is the eigenvector. So we can rewrite it as:

$$x^T Ax = \lambda x^T x = \lambda ||x||^2 \geq 0$$

Where $||x||^2 \geq 0$ because it's the squared norm of x. Therefore λ must be greater then 0 otherwise the definition of PSA isn't satisfied. We get

$$\lambda = \frac{x^T Ax}{||x||^2} \geq 0$$