ETH zürich

Exercise 6: Error Correction and Error-Rate Bounds

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- **Exercise: Thursday, 14 November 2024**
- This exercise will be assisted in CAB G11 from 8:15 h to 10:00 h
- Problems 3 and 4 require a computer with MATLAB (or Octave) installed
- **Due date:** Monday, 25 November 2024, 08:00 h
- **Submission instructions:** Upload your solutions (answers and MATLAB code) to Moodle as a *single* zip-file with a file name `⟨last-name⟩-⟨first-name⟩-e6.zip`. Your homework should consist of a single pdf file for the written answers and one MATLAB script per problem part. The MATLAB file must be named according to the syntax `E6_P3_4.m` (for "Exercise 6 Problem 3 Part 4").
- **Rules:** You may collaborate with other students on this exercise, but each student must turn in their own assignment. You must specify the names of your collaborators! Copying answers from previous years' exercises is strictly forbidden.

Problem	Maximum Points	Points Received
1	10	
2	12	
3	28	
4	50	
Total	**100**	

Problem 1: A Simple Linear Block Code (10pts)

Consider a $(n, k, d)_2$ linear block code in GF(2) with the following generator matrix:

$$\boldsymbol{G} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{1}$$

Part 1 (2pts). Specify n, k, and the rate R of this code.

Part 2 (2pts). Compute the minimum distance $d_{\min}(\mathcal{C}) = d$ of this code.

Part 3 (2pts). How many errors can this code detect? How many errors can this code correct?

Part 4 (4pts). Specify the parity check matrix $\boldsymbol{H}$ for this code. What is the parity-check matrix doing? Explain its functionality with words.

Problem 2: Log-Likelihood Ratio (LLR) Values (12pts)

Consider an AWGN channel modeled as $\mathsf{y} = \mathsf{s} + \mathsf{n}$, where $\mathsf{s} \in \{-1, +1\}$ are BPSK transmit symbols that are equally likely and $\mathsf{n} \sim \mathcal{N}(0, \sigma^2)$ is AWGN. Assume that the input bit $\mathsf{b} = 0$ is mapped to the BPSK symbol $+1$ and the bit $\mathsf{b} = 1$ to the BPSK symbol -1. Practical systems often use data detectors that compute *soft-output information* on the transmitted bits instead of hard-output estimates in the set $\{0, 1\}$. The most common soft-output data detector computes *log-likelihood ratio* (LLR) values (or logits), which are defined as follows:

$$LLR(y) \triangleq \ln\left(\frac{\mathbb{P}[\mathsf{b} = 1 \mid \mathsf{y} = y]}{\mathbb{P}[\mathsf{b} = 0 \mid \mathsf{y} = y]}\right). \tag{2}$$

LLR values represent the likelihood of the transmitted bit b being either 1 or 0 given the observed channel output $\mathsf{y} = y$. Large positive LLR values indicate strong confidence that the transmitted bit b was 1; large negative LLR values indicate strong confidence that the transmitted bit b was 0; an LLR of zero indicates that one is unsure whether a bit 1 or 0 was transmitted.

Part 1 (6pts). Compute the LLR value for the given AWGN channel and simplify the expression as much as you can. How does the result depend on the SNR?

Part 2 (6pts). Assume that the LLR value $LLR(y)$ is given, but you are interested in the probability of a 1 being transmitted, i.e., you wish to express the quantity $\mathbb{P}[\mathsf{b} = 1 \mid \mathsf{y} = y]$ using the LLR value. Compute $\mathbb{P}[\mathsf{b} = 1 \mid \mathsf{y} = y]$ and $\mathbb{P}[\mathsf{b} = 0 \mid \mathsf{y} = y]$ given $LLR(y)$ and simplify your expressions.

Problem 3: Repetition Coding in AWGN (28pts)

In class, you have seen how symbol-level repetition coding works. You are now supposed to compute the error probability of a simple AWGN system that performs K repetitions of the same BPSK symbol.

Consider a simple AWGN channel modeled as $\mathsf{y}_k = \mathsf{s} + \mathsf{n}_k$ with $k = 1, 2, \ldots, K$, where $\mathsf{s} \in \{-1, +1\}$ are equally likely BPSK symbols and $\mathsf{n}_k \sim \mathcal{N}(0, \sigma^2)$ is AWGN.

Part 1 (6pts). Derive the optimal detection rule for repetition coding.

Part 2 (2pts). Compute the SNR for the AWGN repetition coding system.

Part 3 (8pts). Compute the error probability for the optimal detection rule. Provide a simplified expression that depends on the SNR. Explain what happens with increasing K.

Part 4 (12pts). Finally, simulate the symbol error rate of this repetition coding scheme. Write a MATLAB script that performs a Monte–Carlo simulation for the symbol error rate (SER) performance with repetition coding for $K = 1, 2, 4, 8, 16$ repetitions. Perform $10,000$ Monte–Carlo trials for each SNR and overlay all 5 SER curves in a single plot. Also include the analytical results from Part 3 in the same doubly-logarithmic plot. Show an SNR range from -20 dB to $+20$ dB and an SER range from 10^{-4} to 10^{0}. Briefly explain your observations.

Problem 4: Bit- and Symbol Error Rate of 16-QAM Constellation (50pts)

In this problem, we are analyzing the bit error rate (BER) and symbol error rate (SER) of a 16-QAM constellation, as shown in Figure 1. We again consider the AWGN channel

$$\mathsf{y} = \mathsf{s} + \mathsf{n} \tag{3}$$

with equally likely transmit symbols $\mathsf{s} \in \mathcal{C}$ and circularly symmetric complex white Gaussian noise $\mathsf{n} \sim \mathcal{CN}(0, N_0)$.

Part 1 (2pts). What is the average signal energy $\mathbb{E}\left[|\mathsf{s}|^2\right]$? Normalize the constellation to unit-energy and proceed with the normalized constellation in the following. What is the horizontal and diagonal distance to neighboring constellation points?

Part 2 (8pts). Derive the union bound approximation for the error probability of symbol s_1 in the normalized constellation.

Hint: Proceed analogous to the derivation shown in the lecture notes in Section 5.5.

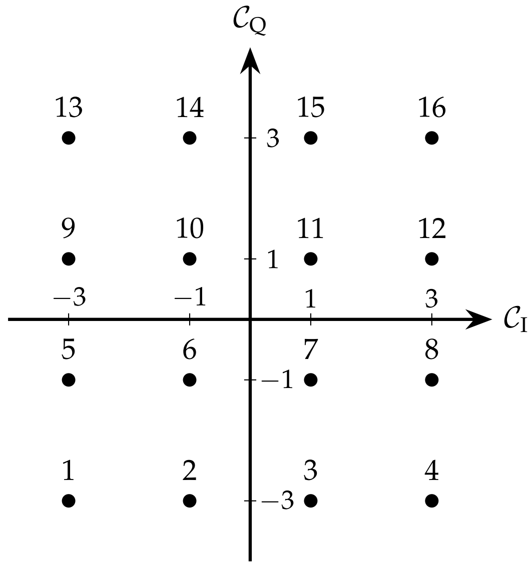

Figure 1: Unscaled 16-QAM constellation.

Part 3 (4pts). Derive the improved union bound approximation for constellation symbol $\mathsf{m} = 1$, and also for all constellation symbols assuming equally likely symbols.

Part 4 (8pts). Write a MATLAB script that simulates the symbol error rate assuming the maximum likelihood detection rule. Compare the simulated SER with the union bound and the improved union bound approximation. To this end, plot the bounds and SER vs. the SNR in dB (i.e., the x-axis is $10\log_{10}(1/N_0)$ [dB]).

Part 5 (8pts). Next, we associate the normalized 16-QAM constellation with the bit labeling depicted in Figure 2, which originates from a Gray code. The goal of the subsequent sub-problems is to analyze the bit error probability, analogously to the preceding SER analysis.

Derive the union bound approximation for the bit error probability of bit b_1.

Hint: You can simplify the analysis significantly when taking into account the special structure of the Gray code. Where is $\mathsf{b}_1 = 1$*?*

Part 6 (4pts). Derive the improved union bound approximation for B_1.

Part 7 (8pts). Write a MATLAB script that simulates the bit error rate assuming the maximum likelihood detection rule. Compare the simulated BER of b_1 with the union bound and the improved union bound approximation. Also compare the average BER over all bits b_i with $i \in \{1, \ldots, 4\}$ in another plot. Plot all bounds and BER vs. the SNR in dB (i.e., the x-axis is $10\log_{10}(1/N_0)$ [dB]).

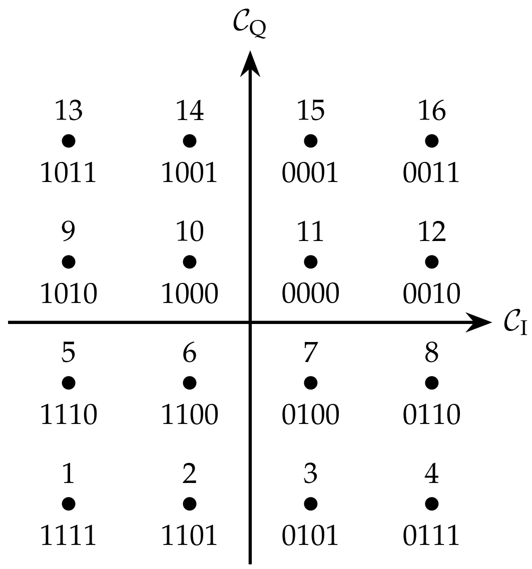

Figure 2: 16-QAM constellation with Gray labeling for the bits $\{b_1, b_2, b_3, b_4\}$.

Part 8 (8pts). Finally, use any other (e.g., random) bit mapping and compare the simulated BER performance with that of the Gray mapping. Write a MATLAB script that plots a comparison of the results.