ETH zürich

Exercise 8: Estimators, Channel Estimation, and Inter-Symbol-Interference Communication

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- **Exercise: Thursday, 28 November 2024**
- This exercise will be assisted in CAB G11 from 8:15 h to 10:00 h
- Problems 1 and 4 require a computer with MATLAB (or Octave) installed
- **Due date:** Monday, 9 December 2024, 08:00 h
- **Submission instructions:** Upload your solutions (answers and MATLAB code) to Moodle as a *single* zip-file with a file name `⟨last-name⟩-⟨first-name⟩-e8.zip`. Your homework should consist of a single pdf file for the written answers and one MATLAB script per problem part. The MATLAB file must be named according to the syntax `E8_P1_4.m` (for "Exercise 8 Problem 1 Part 4").
- **Rules:** You may collaborate with other students on this exercise, but each student must turn in their own assignment. You must specify the names of your collaborators! Copying answers from previous years' exercises is strictly forbidden.

Problem	**Maximum Points**	**Points Received**
1	31	
2	25	
3	20	
4	24	
Total	**100**	

Problem 1: Linear and Non-Linear Minimum Mean-Squared Error (MMSE) Estimators (31pts)

The goal of this problem is to compare various linear and nonlinear estimators that minimize the mean-squared error (MSE), which is defined as

$$MSE = \mathbb{E}\left[|\mathsf{x} - \check{\mathsf{x}}|^2\right]. \tag{1}$$

Consider a binary-valued source with prior PDF

$$f_{\mathsf{x}}(x) = \frac{1}{2}\,\delta(x-1) + \frac{1}{2}\,\delta(x+1); \tag{2}$$

this prior probability models equally likely BPSK signals taken from the set $\{-1, +1\}$. Consider a simple real-valued AWGN channel $\mathsf{y} = \mathsf{x} + \mathsf{n}$, where $\mathsf{n} \sim \mathcal{N}(0, \sigma^2)$.

Part 1 (8pts). Let us start by deriving the optimal nonlinear minimum mean-squared error (MMSE) estimator for this system, which is given by the posterior mean (PM) estimator

$$\check{x}^{\text{PM}}(y) = \mathbb{E}[\mathsf{x}|\mathsf{y} = y] = \int_{x\in\mathbb{R}} x\, f_{\mathsf{x}|\mathsf{y}}(x|y)\,\mathrm{d}x. \tag{3}$$

Derive a closed-form expression for the PM estimator.

Hint: Rewrite $f_{\mathsf{x}|\mathsf{y}}(x|y)$ *using Bayes' rule and explicitly evaluate the resulting integral(s). Also remember that* $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

Part 2 (8pts). Derive a closed-form expression for the best possible *linear* minimum MSE estimator of the form

$$\check{\mathsf{x}}^{\text{LMMSE}} = a\mathsf{y} + b \tag{4}$$

with some real-valued constant a and b. What assumptions are required so that this linear estimator would be the optimal (posterior mean) estimator?

Hint: Substitute (4) *for* $\check{\mathsf{x}}$ *in* (1) *and minimize the MSE with respect to a and b by setting the partial derivatives to zero.*

Part 3 (5pts). Derive the MAP data detection rule.

Hint: You have done this before.

Part 4 (10pts). Use MATLAB to evaluate the MSE of the PM and optimal linear estimators, as well as the MAP data detection rule, via Monte–Carlo simulations.

The MSE is defined as $MSE = \mathbb{E}_{\mathsf{x},\mathsf{n}}\left[(\check{\mathsf{x}} - \mathsf{x})^2\right]$, where $\check{\mathsf{x}}$ is either the PM, or the linear MSE estimate, or the output of the MAP data detector.

Generate a plot in which you sweep the SNR defined as $SNR = 1/\sigma^2$ in the range $[-10, +20]$ decibels. Plot three MSE curves (the one from the PM estimator derived in Part 1, that of the LMMSE estimator derived in Part 2, and that of the MAP detection rule derived in Part 3) on the y-axis in logarithmic scale from 10^{-4} to 10^{2}. Explain what you observe.

Remark: As you will see, exploiting the signal prior often enables significantly better estimators than linear MSE estimators (e.g., the LMMSE estimator). Unfortunately, in most cases the PM estimator does not admit a closed-form expression. Hence, one often resorts to LMMSE estimators in practical systems. Furthermore, you should observe that the MAP detection rule is *not* optimal in terms of the MSE (but it would be if we were to evaluate the symbol error probability instead of the MSE).

Problem 2: Channel Estimation (25pts)

Consider the following discrete-time, real, and noisy LTI system:

$$\mathsf{r}[k] = \sum_{\ell=0}^{L-1} h[\ell]\, \mathsf{s}[k-\ell] + \mathsf{z}[k], \quad k = 0, 1, \ldots, T-1, \tag{5}$$

where the noise is IID Gaussian with $\mathsf{z}[k] \sim \mathcal{N}(0, N_0)$ for $k = 0, 1, \ldots, T-1$, and $T \geq L$. In this problem, the impulse response $h[\ell]$, $\ell = 0, 1, \ldots, L-1$, is not known to the receiver and must be estimated. We will study various approaches to do so.

Part 1 (5pts). Assume that the transmitter sends the pilot signal $s[k] = \sqrt{E_\mathrm{s}}\, \delta[k]$, $k = 0, 1, \ldots, T-1$, which is known at the receiver. Here E_s is the transmit energy of the pilot signal and $\delta[k]$ is the Kronecker delta function. Furthermore, assume that the statistics of the impulse response is unknown and, hence, the receiver is using least-squares (LS) channel estimation. Derive a closed-form expression for the LS estimate $\check{\mathsf{h}}[\ell]$, $\ell = 0, 1, \ldots, L-1$, of the channel's impulse response.

Part 2 (5pts). Consider the LS estimate from Part 1. Compute the associated mean-squared error (MSE), which we define as follows:

$$MSE = \mathbb{E}_{\mathsf{z}}\left[\frac{1}{T}\sum_{k=0}^{T-1} \left|h[k] - \check{\mathsf{h}}[k]\right|^2\right]. \tag{6}$$

Hint: Always make sure that you perform sanity checks on your results. For example, check what happens if the noise variance N_0 approaches zero or if the signal energy E_s approaches infinity.

Part 3 (5pts). Assume that you are allowed to repeat the pilot sequence M times. Derive an improved LS estimator that takes into account all M transmissions of the pilot sequence $s[k]$, $k = 0, 1, \ldots, T-1$.

Part 4 (5pts). Consider the estimator from Part 3. Derive the associated MSE.

Part 5 (5pts). Assume that you are allowed to send one training sequence only (i.e., $M = 1$), but you are allowed to double the transmit energy $2E_s$. Is this channel estimation strategy better in terms of the MSE than sending two repetitions $M = 2$ with total power $2E_s$ over both transmitted training sequences (i.e., with energy E_s per sequence)? Note that the total transmitted energy for both approaches is the same.

Problem 3: Single-Carrier Transmission with ISI (20pts)

Consider a real-valued inter-symbol interference (ISI) communication system modeled as follows:

$$\mathbf{y} = \boldsymbol{H}\mathbf{s} + \mathbf{z}. \tag{7}$$

Here, $\mathbf{y} \in \mathbb{R}^2$ is the received vector and $\boldsymbol{H} \in \mathbb{R}^{2\times 2}$ is the Toeplitz channel matrix defined as

$$\boldsymbol{H} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \tag{8}$$

where $\alpha > 0$ is a real-valued parameter. We assume that $\boldsymbol{H}$ is known perfectly at the receiver.

The transmit vector is $\mathbf{s} \in \mathcal{X}^2$, and the entries are taken with equal probability from a BPSK constellation $\mathcal{X} = \{-1, +1\}$. The vector $\mathbf{z} \sim \mathcal{N}(\mathbf{0}, \sigma^2\boldsymbol{I}_T)$ models IID zero-mean symmetric Gaussian noise with variance σ^2 per entry.

Part 1 (5pts). Assume that you are using successive interference cancellation (SIC). In the first step, SIC performs detection of the first transmit symbol as follows:

$$\check{\mathsf{s}}_1 = \arg\min_{s\in\mathcal{X}} |\mathsf{y}_1 - 1\cdot s|^2. \tag{9}$$

For this detector, compute the error probability of the first symbol, i.e.,

$$P_{1,\text{error}} = \mathbb{P}[\mathsf{s}_1 \neq \check{\mathsf{s}}_1]. \tag{10}$$

Part 2 (5pts). Assume that the decoded symbol in the first step of SIC was correct, i.e., $\check{\mathsf{s}}_1 = \mathsf{s}_1$. For this situation, compute the error probability $P_{2,\text{error}}$ of the detection step of SIC in the second time step. Remember that the detection rule for SIC is

$$\check{\mathsf{s}}_2 = \arg\min_{s\in\mathcal{X}} |\mathsf{y}_2 - \alpha\check{\mathsf{s}}_1 - 1\cdot s|^2. \tag{11}$$

Part 3 (10pts). Now assume that the decoded symbol in the first step of SIC was *incorrect*, i.e., $\check{\mathsf{s}}_1 \neq \mathsf{s}_1$. For this situation, compute the error probability $P_{2,\text{SICerror}}$ of the detection step of SIC in the second time step.

Hint: It may helpful to carry out the derivations using the binary decision rule.

Problem 4: Linear MMSE Equalization and ML Detection over a Noisy ISI Channel (24pts)

The goal of this problem is to implement a Monte–Carlo simulation that computes the symbol error rate (SER) of the linear MMSE estimator and the ML detector for a real-valued system with ISI and AWGN.

We consider BPSK transmission over $T = 10$ time slots and an impulse response of length $L = 2$ samples with the following nonzero taps $h[0] = 1$ and $h[1] = 0.75$. The real-valued channel model is

$$\mathsf{r}[k] = \sum_{\ell=0}^{1} h[\ell]\mathsf{s}[k-\ell] + \mathsf{z}[k], \quad k = 0, 1, \ldots, 9, \tag{12}$$

with IID real-valued Gaussian noise $\mathsf{z}[k] \sim \mathcal{N}(0, \sigma^2)$, $k = 0, 1, \ldots, 9$.

In the following, you will write a MATLAB script named "`E8_P4.m`" that simulates the SER of both detectors. Perform 10,000 Monte–Carlo trials for each SNR value. Plot the resulting SER in MATLAB as a double-logarithmic plot from -10 dB to $+20$ dB and with an SER range from 10^{-4} to 10^{0}. Which of the two methods performs better? Which one is the slowest? Which one would you use in practice?

Part 1 (12pts). The linear minimum mean-squared error (LMMSE) detector finds an optimal trade-off between noise enhancement and ISI removal. Instead of directly inverting the Toeplitz matrix $\boldsymbol{H}$ that models the ISI channel, as it is the case for ZF equalization, one computes the following regularized matrix inversion

$$\boldsymbol{W} = \left(\boldsymbol{H}^{\mathsf{H}}\boldsymbol{H} + \boldsymbol{I}\frac{\sigma^2}{E_{\mathrm{s}}}\right)^{-1}\boldsymbol{H}^{\mathsf{H}}, \tag{13}$$

where $E_{\mathrm{s}} = \mathbb{E}\left[|\mathsf{s}[k]|^2\right]$ and $\sigma^2 = \mathbb{E}\left[|\mathsf{z}[k]|^2\right]$. Instead of computing $\check{\mathbf{s}} = \boldsymbol{H}^{-1}\mathbf{y}$ as for ZF equalization, one applies this LMMSE matrix to the received vector as $\check{\mathbf{s}} = \boldsymbol{W}\mathbf{y}$ followed by entry-wise data detection. Implement this detector and simulate its SER.

Part 2 (12pts). Implement the ML detector and simulate its SER.