Exercise 4: Amplitude Quantization

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- **Exercise: Thursday, 31 October 2024**
- This exercise will be assisted in CAB G11 from 8:15 h to 10:00 h
- Problems 2 and 3 require a computer with MATLAB (or Octave) installed
- **Due date:** Monday, 11 November 2024, 08:00 h
- **Submission instructions:** Upload your solutions (answers and MATLAB code) to Moodle as a *single* zip-file with a file name `⟨last-name⟩-⟨first-name⟩-e4.zip`. Your homework should consist of a single pdf file for the written answers and one MATLAB script per problem part. The MATLAB file must be named according to the syntax `E4_P2_1.m` (for "Exercise 4 Problem 2 Part 1").
- **Rules:** You may collaborate with other students on this exercise, but each student must turn in their own assignment. You must specify the names of your collaborators! Copying answers from previous years' exercises is strictly forbidden.

Problem	**Maximum Points**	**Points Received**
1	20	
2	40	
3	25	
4	15	
Total	**100**	

Problem 1: Analog-to-Digital Converter (A/D) Datasheet (20pts)

On Moodle, you can find the datasheet of the Texas Instruments (TI) single-ended analog-to-digital (A/D) converter PCM1808-Q1 suitable for low-end audio applications. The goal of this problem is to learn where to find the relevant information for a given application.

Part 1 (10pts). What is the maximum sampling rate f_s in Hz and the maximum number of quantization bits B of this A/D converter? Assuming the quantizer delivers exactly these specifications, what would the highest frequency $f_{\max}$ be that you could sample without losing information? And, what would the theoretical SQR value in decibels be?

Part 2 (10pts). Real-world data converters do not exactly achieve the theoretical specifications. Assume a sampling rate of $f_s = 44.1\,\text{kHz}$. What is the attenuation of the A/D converter at $f_s/2$? What is the signal-to-quantization-noise ratio (SQR) of the A/D converter at this sampling rate?

Hint: Many datasheets call the actually achieved SQR the signal-to-noise ratio.

Problem 2: Comparison of 1-bit Quantization Models (40pts)

The literature describes a number of amplitude quantization models, which all have advantages and disadvantages. In this problem, you will use Monte–Carlo simulations in order to compare some of the most prominent models for 1-bit quantizers. A 1-bit quantizer is simply the signum (or sign) function, i.e., $\mathsf{y} = \text{sign}(\mathsf{x})$, where $y = +1$ if the input signal $x \geq 0$ and $y = -1$ if $x < 0$.

The most common quantization models are as follows:

1. The *naïve quantization model*, which is also known as *additive quantization noise (AQN) model* or *pseudo quantization noise (PQN) model*. For this model, the quantization "noise" q is given by
$$\mathsf{y} = \text{sign}(\mathsf{x}) = \mathsf{x} + \mathsf{q}. \tag{1}$$
2. The *Bussgang-based quantization model*. For this model, the residual distortion d is given by
$$\mathsf{y} = \text{sign}(\mathsf{x}) = \alpha\mathsf{x} + \mathsf{d}, \tag{2}$$
where α is the so-called *Bussgang gain* that depends on the input signal and the quantizer function. The Bussgang gain α is chosen so that the cross-correlation $\rho_{\mathsf{x},\mathsf{d}} = \mathbb{E}[\mathsf{x}\mathsf{d}]$ between the input signal x and the distortion d is zero (uncorrelated).

3. The *SQR-optimal quantization model* (SQR stands for *signal-to-quantization-noise ratio*) utilizes the same model as in (2), but α is chosen so that the SQR
$$SQR = \frac{\mathbb{E}\left[(\alpha \mathsf{x})^2\right]}{\mathbb{E}\left[\mathsf{d}^2\right]} \tag{3}$$
is maximal.

For simplicity, we model the input signal x as a standard normal random variable (i.e., a zero-mean Gaussian random variable with unit variance $\sigma_{\mathsf{x}}^2 = 1$).

Part 1 (10pts). For the naïve quantizer in (1), use Monte–Carlo simulations to numerically calculate the variance of the quantization "noise" $\sigma_{\mathsf{q}}^2 = \mathbb{E}\left[\mathsf{q}^2\right]$ and the correlation $\rho_{\mathsf{x},\mathsf{q}} = \mathbb{E}\left[\mathsf{xq}\right]$ between the Gaussian input signal x and the quantization "noise." Write a MATLAB script named "`E4_P2_1.m`" that calculates σ_{q}^2, the SQR (in dB), and $\rho_{\mathsf{x},\mathsf{q}}$. To this end, perform $T = 100,000$ Monte–Carlo trials to empirically calculate all of the necessary expectation expressions.

Hint: Standard normal random variables can be generated using MATLAB's `randn` *command.*

Part 2 (10pts). For the Bussgang-based quantizer in (2), start by computing the optimal value for α analytically. To do so, first minimize the mean squared error (MSE)
$$\sigma_{\mathrm{MSE}}^2 = \mathbb{E}\left[\mathsf{d}^2\right] = \mathbb{E}\left[(\mathsf{y} - \alpha \mathsf{x})^2\right], \tag{4}$$
and second prove that for the optimal choice of α, the correlation $\rho_{\mathsf{x},\mathsf{d}} = \mathbb{E}\left[\mathsf{xd}\right]$ between the Gaussian input x and the distortion d is zero.

Hint: Do not evaluate the expectations explicitly.

Next write a MATLAB script named "`E4_P2_2.m`" that uses $T = 100,000$ Monte–Carlo trials to numerically calculate the optimal α, the variance of the distortion $\sigma_{\mathsf{d}}^2 = \mathbb{E}\left[\mathsf{d}^2\right]$, the SQR (in dB), and the correlation $\rho_{\mathsf{x},\mathsf{d}} = \mathbb{E}\left[\mathsf{xd}\right]$.

Hint: Your numerically obtained value for the optimal scaling factor α should be close to $\alpha_{\mathrm{Bussgang}} = \sqrt{2/\pi} \approx 0.7979$.

Part 3 (10pts). For the SQR-optimal quantization model, first compute the optimal value for α analytically and then write a MATLAB script named "`E4_P2_3.m`" that simulates the required quantity to determine the optimal value of α. Finally, evaluate σ_{d}^2, the SQR, and $\rho_{\mathsf{x},\mathsf{d}}$ using $T = 100,000$ Monte–Carlo trials.

Hint: It is much easier to minimize the reciprocal SQR. Moreover, your numerically obtained value for the optimal α should be close to $\alpha_{\mathrm{SQR}} = \sqrt{\pi/2} \approx 1.2533$.

Part 4 (10pts). Finally, implement a MATLAB script named "`E4_P2_4.m`" that uses $T = 100,000$ Monte–Carlo trials to plot the SQR, the mean squared error (MSE), and the cross-correlation $\rho_{\mathsf{x},\mathsf{d}}$ for varying values of $\alpha \in [0, 2]$. Compare the naïve, the Bussgang-based, and the SQR-optimal quantizer. Discuss your results: How do the different quantization models differ from each other? Which one would you consider to be more practical?

Problem 3: Probability and More Quantization (25pts)

In Problem 2, we analyzed three different quantization models for 1-bit quantizers. Let us now numerically analyze the performance of quantizers with more than 1-bit. We will consider the following quantization functions

$$y = Q_B(x) \tag{5}$$

$$= \begin{cases} \mathrm{sign}(x) & B = 1, \\ \frac{1}{2^{B-1}-\frac{1}{2}} \cdot \left(\mathrm{round}\left(\min\{\max\{x,-1\},+1\}(2^{B-1}-1)-\frac{1}{2}\right)+\frac{1}{2}\right) & B \in \{2,3,\ldots\}. \end{cases} \tag{6}$$

Here, $\mathrm{round}(r)$ rounds the real number r to the nearest integer and $B \in \mathbb{N}$ denotes the number of quantization bits. For $B = 3$ bits, the above quantization function looks as shown in Figure 1. Note that these quantizers only quantize values that are in the

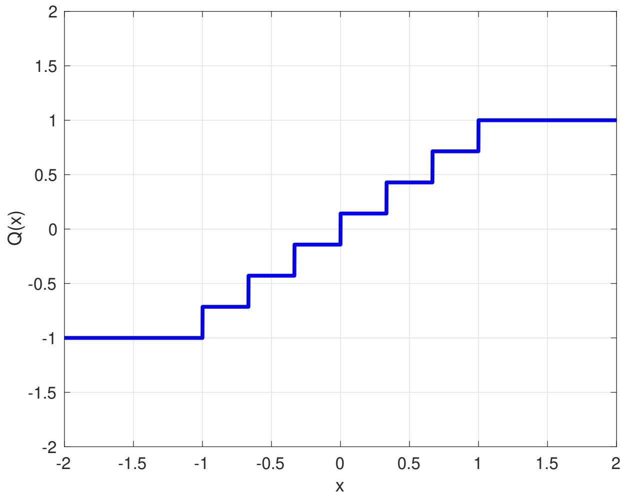

Figure 1: Quantization function (6) for $B = 3$.

range $x \in [-1, 1]$; all values exceeding this range are clipped to either $+1$ or -1.

Write a MATLAB script named "`E4_P3.m`" that performs the following steps. For $B = 1, 2, \ldots, 24$ bits, use 1,000,000 Monte-Carlo trials to simulate the following quantities:

- $\sigma_\mathsf{q}^2 = \mathbb{E}\left[\mathsf{q}^2\right]$,
- $\sigma_\mathsf{x}^2 = \mathbb{E}\left[\mathsf{x}^2\right]$,
- $\rho_{\mathsf{x},\mathsf{q}} = \mathbb{E}[\mathsf{xq}]$, and
- $\mathrm{SQR} = \frac{\sigma_\mathsf{x}^2}{\sigma_\mathsf{q}^2}$.

Assume the naïve quantization-noise model $\mathsf{y} = \mathsf{x} + \mathsf{q}$ when numerically computing these quantities.

In contrast to Problem 2, we now assume that the input signal x is a uniform random variable in the range $[-1, +1]$ (we do this to make sure that we do not exceed the input range of our quantizer).

Generate two plots that contain the following. The first plot "`E4_P3_1a.pdf`" should contain the simulated SQR values in decibels (dB) on the y-axis and the number of bits B on the x-axis. As a reference, include the analytical SQR-expression for a full-scale sinusoidal input from class:

$$\text{SQR} \approx 6.02\,B + 1.76 \quad [\text{dB}]. \tag{7}$$

Note that the simulated and analytical values should be quite close if you implemented everything correctly.

The second plot "`E4_P3_1b.pdf`" should contain the simulated correlation $\rho_{\mathsf{x},\mathsf{q}}$ on the y-axis and the number of bits B on the x-axis. You should see that the correlation between the signal x and the quantization noise q decays very quickly—only a few bits are sufficient to have uncorrelated quantization "noise," even with the naïve quantization model. Thus, the alternative two models from Problem 2 are useful only for coarsely quantized systems.

Problem 4: Total Harmonic Distortion Plus Noise (THD+N) (15pts)

A typical performance measure of A/D converters is the so-called *total harmonic distortion plus noise* (THD+N). The THD+N is measured by applying a pure sinusoidal signal $y(t) = V\sin(2\pi f_1 t)$ with peak voltage V at a test frequency f_1 (in Hz) to the A/D converter. Since real-world data converters distort the amplitude of the input signal, the sampled signal will have harmonics that were not present in the original signal. Harmonics are sinusoids at integer multiples of the input frequency. In addition to harmonic distortions, the output will also contain quantization noise. The THD+N measures the following quantity:

$$\text{THD+N} = \frac{\sigma_q + \sqrt{V_2^2 + V_3^2 + \cdots}}{V_1}, \tag{8}$$

where V_1, V_2, V_3, etc. are the voltage levels associated with the 1st, 2nd, 3rd, etc. harmonics at the output, respectively. The quantity σ_{q} is the quantization noise standard deviation (the square root of the variance). In the datasheet from Problem 1 on Page 9 in Figure 11, one can find the THD+N for this particular A/D converter for different sampling rates. For example, for $f_s = 44.1\,\text{kHz}$, we have THD+N $\approx -93.2\,\text{dB}$.

Instead of skimming through a datasheet to find more quantities as in Problem 1, let us numerically calculate the THD+N for a simple nonlinear function $f(x) = x^3$ assuming that the input signal was the sinusoid $y(t) = V\sin(2\pi f_1 t)$ and for $\sigma_{\mathsf{q}} = 0$. Calculate V_1, V_2, and V_3 at the output of the nonlinear function $f(x) = x^3$ and derive an analytical expression for the THD+N. Does the THD+N depend on f_1 and V?