Keen model with exogenous shocks

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1 Introduction

In his paper [4], Keen proposed what has come to be known as the Keen model: a system of ordinary differential equations (ODEs) modelling wages, employment, and debt in a closed economy. The Keen model is an important foundation from which to begin as it seeks to mathematically model Minsky's instability hypothesis in a simplified way. Grasselli and Costa Lima provided a thorough analysis of the model in their paper [2], and this report follows their work in particular. Section 2 describes the set-up of our macroeconomic model including the balance sheet and transactions for the closed three sector economy. Section 3 introduces an extension to the model, whereby we allow for the possibility of defaulting in the banking sector, and describe how the resulting shocks affect the model going forward. This investigation into the shocks in the system is the principal motivation for this report. In section 4 we discuss a numerical example and provide some preliminary results. Finally, section 5 describes how a banking network might be implemented into the model in order to endogenize the shocks to the system, and includes a mechanism for clearing payments based on the work by Eisenberg and Noe [1] to be used to settle interbank debts, in particular if defaults occur in the banking sector.

2 The Keen Model

We begin with a closed economy with three sectors: firms, banks, and households. The firm sector produces one homogenous good, used for both consumption (household spending) and investment by the firms, and we denote the output from the firm sector by Y. Let K denote the total stock of capital in the economy and we assume it determines the output Y according to

$$Y = \frac{K}{\nu}, \tag{1}$$

where ν is a constant capital-to-output ratio. Assume capital changes with respect to time t according to

$$\dot{K} = I - \delta K, \tag{2}$$

where I denotes capital investment in real terms and δ the depreciation rate. Total sales demand Y_d is entirely determined by capital investment I and the total household consumption C

$$Y_d = I + C, \tag{3}$$

and we assume the output always meets the total sales demand, that is $Y = Y_d$.

Denote further, the nominal wage bill by W, the total working age population by P, and the number of employed workers by E ($E \leq P$). Define the average productivity per worker a, the employment rate λ, and the nominal wage rate w as

$$a = \frac{Y}{E}, \qquad \lambda = \frac{E}{P} = \frac{Y}{aP}, \qquad w = \frac{W}{E} \tag{4}$$

and define the unit cost of production - the wage bill divided by the output produced - as

$$c = \frac{W}{Y} = \frac{w}{a}. \tag{5}$$

Worker productivity and the working age population are assumed to grow outside the scope of the model according to

$$\frac{\dot{a}}{a} = \alpha, \qquad \frac{\dot{P}}{P} = \beta. \tag{6}$$

Nominal output, after denoting the unit price level for the homogenous good by p produced by the firm sector, is given by

$$pY = p(I + C) = pI + pC \tag{7}$$

Finally, for the banking sector, denote the household deposits and the loans to firms by Δ and Λ respectively. We now arrive at the balance sheet, transactions, and flow of funds for this economy as described in Table 1. Note in the balance sheet that we are assuming households do not borrow from banks (no household loans) and firms do not keep deposits in banks, opting to use any remaining balance to repay their loans instead. We also account for inflation and normalize the nominal variables in the system. After dividing the wage bill W, and the loans Λ by the nominal output pY, we obtain the wage share ω and the debt ratio for firms ℓ

$$\omega = \frac{W}{pY} = \frac{c}{p} = \frac{w}{p\,a}, \qquad \ell = \frac{\Lambda}{pY}. \tag{8}$$

	Households	Firms		Banks	Sum
Balance Sheet					
Capital Stock		$+pK$			$+pK$
Deposits	$+\Delta$			$-\Delta$	0
Loans		$-\Lambda$		$+\Lambda$	0
Sum (net worth)	X_h	X_f		X_b	pK
Transactions		current	capital		
Consumption	$-pC$	$+pC$			0
Capital Investment		$+pI$	$-pI$		0
Wages	$+W$	$-W$			0
Depreciation		$-p\delta K$	$+p\delta K$		0
Interest on deposits	$+r_d\Delta$			$-r_d\Delta$	0
Interest on loans		$-r\Lambda$		$+r\Lambda$	0
Dividends	$+\Pi_b$			$-\Pi_b$	0
Financial Balances	S_h	S_f	$-p(I-\delta K)$	S_b	0
Flow of Funds					
Change in Capital Stock		$+p(I-\delta K)$			$+p(I-\delta K)$
Change in Deposits	$+\dot{\Delta}$			$-\dot{\Delta}$	0
Change in Loans		$-\dot{\Lambda}$		$+\dot{\Lambda}$	0
Column sum	S_h	S_f		S_b	$p(I-\delta K)$

Table 1: Balance sheet and transaction flows for the three sector economy

From Table 1 we see that the net profits (or savings) for firms, S_f, is given by

$$\begin{aligned} S_f &= pC + pI - W - p\delta K - r\Lambda \\ &= p(C+I) - W - r\Lambda - p\delta K \\ &= pY - W - r\Lambda - p\delta K \end{aligned} \tag{9}$$

the nominal output minus wages, interest on debt (r is the fixed interest rate on loans), and depreciation of capital, and these savings are the funds available to the firms for investment. The difference then, between the nominal investment (corresponding to the change in capital stock) and the savings S_f is exactly the amount of external financing (by way of loans from the banking sector) needed by the firms. That is

$$\begin{aligned} \Lambda^d &= p\dot{K} - S_f \\ &= p(I^d - \delta K) - S_f \quad \text{from (2)} \\ &= pI^d - \Pi_p, \end{aligned} \tag{10}$$

where pI^d is the nominal investment demand and $\Pi_p = S_f + p\delta K = pY - W - r\Lambda$, the (nominal) pre-depreciation profit of the firms. However, in our model we assume that banks will attempt to maintain a minimum regulatory capital (equity) at all times. That is

$$\begin{aligned} X_b &\coloneqq \Lambda - \Delta \\ &= k_r\Lambda \end{aligned} \tag{11}$$

for some constant target capital adequacy ratio $0 < k_r < 1$. If, as we'll see later, the banks find themselves in a situation where $X_b < k_r\Lambda$, we assume they will only be able to meet a portion of the demand for credit. Let $k = X_b/\Lambda$ be the equity ratio and say the actual change in credit (loans) is

$$\dot{\Lambda} = R(k)\Lambda^d \tag{12}$$

where $R(\cdot)$ is a function of k given by

$$R(k) \coloneqq \begin{cases} \frac{k}{k_r} & 0 \le k \le k_r \\ 1 & k_r \le k \end{cases} \tag{13}$$

The actual nominal investment is then

$$\begin{aligned} pI &= \dot{\Lambda} + \Pi_p \\ &= R(k)\Lambda^d + \Pi_p \\ &= R(k)(pI^d - \Pi_p) + \Pi_p \\ &= R(k)pI^d + \Pi_p(1 - R(k)), \end{aligned} \tag{14}$$

a linear combination of the (nominal) investment demand and the pre-depreciation profit. We further assume the investment demand is given by

$$I^d = \kappa(\pi)Y, \tag{15}$$

where $\kappa(\cdot)$ - the rate of new investment - is a function of the profit share π, itself obtained after normalizing the pre-depreciation profits Π_p

$$\pi = \frac{\Pi_p}{pY} = 1 - \omega - r\ell. \tag{16}$$

Thus, from (13),(14),(15), we have an expression for investment I given by

$$I = I_k = \begin{cases} Y\Big(\frac{k}{k_r}\big(\kappa(\pi) - \pi\big) + \pi\Big) & 0 < k < k_r \\ Y\kappa(\pi) & k \geq k_r. \end{cases} \tag{17}$$

Then, in (2) we have

$$\begin{aligned} \frac{\dot{K}}{K} &= \frac{I_k}{K} - \delta \\ &= \begin{cases} \frac{k\big(\kappa(\pi)-\pi\big)+\pi}{k_r\nu} - \delta & k < k_r \\ \frac{\kappa(\pi)}{\nu} - \delta & k \geq k_r. \end{cases} \\ &:= g(\omega, \ell; k) \end{aligned} \tag{18}$$

Notice, from (1) that $\dot{K}/K = \dot{Y}/Y$, so (18) also describes the dynamics of Y.

Now we address the price dynamics, $\dot{p}$, and introduce two parameters. We assume the long-run equilibrium price is given by a constant markup $m \geq 1$ multiplied by the (instantaneous) unit labor cost c from (5): $\bar{p} = mc$, and the observed prices converge to $\bar{p}$ through a lagged adjustment with speed $\eta > 0$. Thus we have

$$\begin{aligned}\dot{p} &= \eta(\bar{p} - p)\\ \frac{\dot{p}}{p} &= \eta(\frac{mc}{p} - 1)\\ \frac{\dot{p}}{p} &= \eta(m\omega - 1) := i(\omega). \qquad \text{from (8)}\end{aligned} \tag{19}$$

Regarding the wage rate w we assume the following dynamics

$$\frac{\dot{w}}{w} = \Phi(\lambda) + \gamma\frac{\dot{p}}{p}, \tag{20}$$

for constant $0 \leq \gamma \leq 1$. $\Phi(\cdot)$ is the Philips curve which models an inverse relationship between employment rates and inflation, and the assumption here is that workers bargain for wages according to the current employment rate λ as well as the observed inflation rates described in (19). The parameter γ represents the degree of money illusion, with $\gamma = 1$ corresponding to the case where workers think of currency entirely in real terms and fully account for inflation in their bargaining as a result.

This leads us into the dynamics of our state variables starting with the wage share $\omega = w/(pa)$. From (6), (8), (19), (20), we have

$$\begin{aligned}\frac{\dot{\omega}}{\omega} &= \frac{\dot{w}}{w} - \frac{\dot{p}}{p} - \frac{\dot{a}}{a}\\ &= \Phi(\lambda) + \gamma i(\omega) - i(\omega) - \alpha\\ &= \Phi(\lambda) - \alpha - i(\omega)(1-\gamma).\end{aligned} \tag{21}$$

Next is the employment rate $\lambda = Y/(aP)$. From (4), (6), (18), we have

$$\begin{aligned}\frac{\dot{\lambda}}{\lambda} &= \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{P}}{P}\\ &= g(\omega, \ell; k) - \alpha - \beta.\end{aligned} \tag{22}$$

Lastly, the debt ratio is given by $\ell = \Lambda/pY$. From (8), (12), (17), (18), (19), we have

$$\begin{aligned}\frac{\dot{\ell}}{\ell} &= \frac{\dot{\Lambda}}{\Lambda} - \frac{\dot{p}}{p} - \frac{\dot{Y}}{Y}\\ &= \frac{pI - \Pi_p}{\Lambda} - i(\omega) - g(\omega, \ell; k)\\ &= \frac{\frac{I_k}{Y} + \pi}{\ell} - i(\omega) - g(\omega, \ell; k) \qquad \text{from (16) after factoring } pY\\ &= R(k)\frac{\kappa(\pi) - \pi}{\ell} - i(\omega) - g(\omega, \ell; k).\end{aligned} \tag{23}$$

Combining (21), (22), (23) we arrive at our system of ordinary differential equations known as the Keen Model.

$$\begin{cases} \dot{\omega} = \omega \left(\Phi(\lambda) - \alpha - i(\omega)(1-\gamma)\right) \\ \dot{\lambda} = \lambda \left(g(\omega, \ell; k) - \alpha - \beta\right) \\ \dot{\ell} = -\ell \left(i(\omega) + g(\omega, \ell; k)\right) + R(k)\big(\kappa(\pi) - \pi\big) \end{cases} \tag{24}$$

It is standard to assume that the banking sector aggregately distributes all of their profits to the housing sector in the form of dividends, in order to maintain a constant net worth or equity $\dot{X}_b = 0$. In the flow of funds section in Table 1 this means that the banks financial balance doesn't change:

$$\begin{aligned} S_b = \dot{X}_b = \dot{\Lambda} - \dot{\Delta} = 0 \iff \dot{\Lambda} = \dot{\Delta} \\ \implies \Pi_b = r\Lambda - r_d\Delta. \end{aligned} \tag{25}$$

where r_d is the interest rate on the deposits. However, this assumption is not suitable for this report, where we are interested in the strategic behavior of banks under varying circumstances. We've already adopted the assumption in (11) that banks will distribute enough profits as dividends in order to maintain a constant equity-to-loans ratio. From (11) it follows that

$$\begin{aligned} S_b = \dot{X}_b &= k_r \dot{\Lambda} \\ &= k_r\, R(k)\big(\kappa(\pi) - \pi\big)pY \end{aligned} \tag{26}$$

and

$$\dot{\Delta} = \dot{\Lambda} - \dot{X}_b = \big(1 - k_r\big)\dot{\Lambda}. \tag{27}$$

Now we can determine the dynamics for the deposit ratio $d = \Delta/pY$ from (26), (27)

$$\begin{aligned}
\frac{\dot{d}}{d} &= \frac{\dot{\Delta}}{\Delta} - \frac{\dot{p}}{p} - \frac{\dot{Y}}{Y} \\
\dot{d} &= \frac{(1-k_r)\dot{\Lambda}}{pY} - d\big(i(\omega) + g(\omega, \ell; k)\big) \\
&= (1-k_r)\Big(\dot{\ell} + \ell\big(i(\omega) + g(\omega, \ell; k)\big)\Big) \\
&\qquad - d\Big(i(\omega) + g(\omega, \ell; k)\Big) \\
&= (1-k_r)\dot{\ell} + \big((1-k_r)\ell - d\big)\big(i(\omega) + g(\omega, \ell; k)\big) \\
&= (1-k_r)\Big(-\ell\,\big(i(\omega) + g(\omega, \ell; k)\big) + R(k)\big(\kappa(\pi) - \pi\big)\Big) \\
&\qquad + \big((1-k_r)\ell - d\big)\big(i(\omega) + g(\omega, \ell; k)\big) \\
&= \big(1-k_r\big)R(k)\big(\kappa(\pi) - \pi\big) - d\big(i(\omega) + g(\omega, \ell; k)\big),
\end{aligned} \tag{28}$$

so S_b, Λ, and Δ can be calculated in terms of the state variables (ω, λ, ℓ) from (24) and (p, d) from (31) together. This now completes the model, and the expression for the dividends Π_b paid by the banking sector to the households is given by (in contrast to (25))

$$\begin{aligned}
\Pi_b &= r\Lambda - r_d\Delta - S_b \\
&= r\Lambda - r_d\Delta - k_r\dot{\Lambda}
\end{aligned} \tag{29}$$

We see that even if the equity ratio $k = X_b/\Lambda$ is less than k_r; $k < k_r$, the banks will respond by not lending as much as they would like, and this corresponds to a decrease in the amount of dividends paid to households. This can be seen in the dynamics for k

$$\begin{aligned}
\frac{\dot{k}}{k} &= \frac{\dot{X}_b}{X_b} - \frac{\dot{\Lambda}}{\Lambda} \\
&= \frac{k_r\dot{\Lambda}}{X_b} - \frac{\dot{\Lambda}}{\Lambda} \\
&= k_r\dot{\Lambda}\Big(\frac{1}{X_b} - \frac{1}{k_r\Lambda}\Big),
\end{aligned} \tag{30}$$

ie: if $X_b < k_r\Lambda$ (that is $k < k_r$) then we see that long-term $\dot{k}/k > 0$ (since $k_r\dot{\Lambda} > 0$ in the long term) and (30) will converge to k_r. Likewise if $X_b > k_r\Lambda$ corresponding to the case where the equity is larger than the banks desire, they will cap their lending and increase the dividends they pay out to households. Thus, k will again converge to k_r in the long run.

We obtain a trajectory for $\big(p(t), d(t)\big)$ by solving the ancilliary system (from (19), (28))

$$\begin{cases} \dot{p} = p\, i(\omega) \\ \dot{d} = \big(1 - k_r\big) R(k) \big(\kappa(\pi) - \pi\big) - d\big(i(\omega) + g(\omega, \ell; k)\big) \end{cases} \tag{31}$$

and compute the total output $Y(t)$ from a given trajectory of the original system $\big(\omega(t), \lambda(t), \ell(t)\big)$ as

$$Y(t) = \lambda(t) a_0 P_0 e^{(\alpha+\beta)t} \tag{32}$$

for some constants a_0, the initial worker productivity, and P_0 the initial initial working age population. From this we can now compute the levels of loans and deposits in the banking sector as $\Lambda = \ell pY$, $\Delta = dpY$ respectively.

3 An extension to the model

In the previous section we've assumed that the loans $\Lambda(t)$ and the deposits $\Delta(t)$ solved from the ODE systems (24) and (31) will be continuous functions of time. However, in the event of a default from within the banking sector, there could be a sudden instantaneous loss of capital in the system in the form of *shocks* leading to jump discontinuities. These shocks could then possibly alter trajectories of our state variables $\big(\omega, \lambda, \ell, p, d\big)$ in unexpected ways. For now we consider exogeneous shocks to the aggregated loans Λ and aggregated deposits Δ from all the banks together.

We begin with the assumption that banks can only default at set times $T_1, T_2, ...$, interpreted as final settlement times. In the interval $[T_{i-1}, T_i)$ we assume that there is no potential for defaulting, so (24) and (31) hold for $\forall t \quad T_{i-1} \leq t < T_i$. At settlement time T_i, the pre-settlement initial conditions are given as

$$\big(\omega(T_i^-), \lambda(T_i^-), \ell(T_i^-), p(T_i^-), d(T_i^-)\big). \tag{33}$$

Here we add the possibility of defaulting in the banking sector, and thus shocks to system. This is done in the following way: at settlement time T_i, the pre-settlement total loans and deposits are given as

$$\Lambda(T_i^-) = \ell(T_i^-) p(T_i^-) Y(T_i^-) \qquad \Delta(T_i^-) = d(T_i^-) p(T_i^-) Y(T_i^-), \tag{34}$$

or, simply the values obtained from the ODEs for $t \in [T_{i-1}, T_i)$. Post-settlement, a possible default in the banking sector may lead to a sudden loss in the total loans and deposits in the system. That is, post-settlement at time T_i we have

$$\Lambda(T_i) = f_{\ell,t}\,\ell(T_i^-)p(T_i^-)Y(T_i^-) \qquad \Delta(T_i) = f_{d,t}\,d(T_i^-)p(T_i^-)Y(T_i^-), \tag{35}$$

where $f_{\ell,t}$ and $f_{d,t}$ have the following properties:

$$\begin{gathered} 0 \leq f_{\ell,t}, f_{d,t} \leq 1 \\ \forall t \in \mathbb{R}^+ \, \forall i \in \mathbb{N} \quad t \neq T_i \implies f_{\ell,t} = f_{d,t} = 1. \end{gathered} \tag{36}$$

Since post-settlement there can only be a sudden loss in the total loans and deposits in the system (as oppose to a sudden gain), we require that $f_{\ell,t}$,$f_{d,t} \leq 1$ in order for $\Lambda(T_i) \leq \Lambda(T_i^-)$ and $\Delta(T_i) \leq \Delta(T_i^-)$. The second condition in (36) is simply ensuring that there is no possibility for a default except at the specified settlement times T_i. A value of $f_{\ell,t} = 1$ for example simply represents the situation that post settlement there was no defaulting that occurred and the system proceeds normally so that $\Lambda(T_i) = \Lambda(T_i^-)$. In addtion, we have the following restriction on $f_{\ell,t}$, $f_{d,t}$:

$$f_{d,T_i}\,d(T_i^-) < f_{\ell,T_i}\,\ell(T_i^-), \tag{37}$$

since we require that $\Delta(t) < \Lambda(t)$ at all times. A situation where deposits are greater than loans would represent bankruptcy, so we impose (37) to avoid this. Therefore post-settlement at time T_i, the initial conditions for (24), (31) are

$$\big(\omega(T_i^-), \lambda(T_i^-), f_{\ell,T_i}\,\ell(T_i^-), p(T_i^-), f_{d,T_i}\,d(T_i^-)\big), \tag{38}$$

and once again this is valid $\forall t \in [T_i, T_{i+1})$ or until the next settlement date.

4 Numerical Example and Early Results

To begin, we'll base the choice of parameters on [2] and [3] for the sake of later comparison and take the fundamental economic constants to be

$$(\alpha, \beta, \delta, \nu, r) = (0.025, 0.02, 0.01, 3, 0.03). \tag{39}$$

Likewise, as in [4] we take the Phillip's curve Φ to be

$$\Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0 \tag{40}$$

where

$$(\phi_0, \phi_1) = \Big(\frac{0.04}{1-0.04^2}, \frac{0.04^3}{1-0.04^2}\Big). \tag{41}$$

$\kappa(\cdot)$, the function describing the rate of investment, is taken to be

$$\kappa(\pi) = \kappa_0 + \exp(\kappa_1 + \kappa_2\pi), \tag{42}$$

with

$$(\kappa_0, \kappa_1, \kappa_2) = (-0.0065, -5, 20) \tag{43}$$

such that they satisfy the conditions described in [2]. Additionally, we must select the parameters related to inflation in $i(w)$ and the degree of money illusion in the dynamics for the wage rate (20), the choices for these parameters will be based on [3]. Grasselli and Huu show that the parameter for the markup m determines whether the preferable (or *good*) equilibrium is inflationary or deflationary. Clearly from (19) we see that if $m\bar{\omega}_1 > 1$ then $\dot{p}/p > 0$ corresponding to the situation with positive inflation, and likewise if $m\bar{\omega}_1 < 1$ then $\dot{p}/p < 0$ which corresponds to the deflationary situation. In order to obtain an economically desirable good equilibrium to be inflationary we select m such that

$$m > \frac{1}{\bar{\omega}_1}, \tag{44}$$

Following their lead, they choose $m = 1.2$ as their markup. The parameter η corresponds to the speed at which the price p adjusts to its long term target and Grasselli and Huu take $\eta = 4$ corresponding to an average adjustment period of three months. Their selections for the additional parameters from [3] are then

$$(\eta, p, \gamma) = (4, 1.2, 0.8). \tag{45}$$

Concerning the shock paramters f_ℓ and f_d we draw them from a Rayleigh distribution with scaling parameter $\sigma > 0$. We include additional logic to make sure the shocks stay within our specified bounds, in particular if f_ℓ,$f_d \geq 1$ then f_ℓ,$f_d := 1$. We observe that for small σ close to zero there is not only a higher probability of non-trivial shocks occurring (f_ℓ,$f_d < 1$) but also more likely to observe severe shocks than for large σ. At this stage the actual distribution and associated parameters are chosen based on the severity of their impact rather than their ability to emulate realistic defaulting scenarios. Finally we take $k_r = 0.08$ and say that banks will always try to maintain a constant equity to loans ratio of 8%.

In Figure 1a we observe a sample trajectory of the model that converges to the bad equilirbium described in [3] with the same choice of parameters

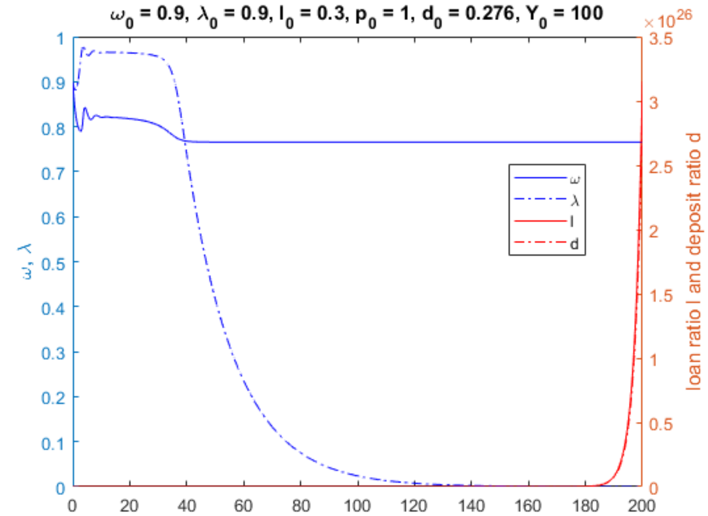
$$(\bar{\omega}_3, \bar{\lambda}_3, \bar{\ell}_3, \bar{d}_3) = (0.7656, 0, +\infty, +\infty) \tag{46}$$

corresponding to an deflationary situation with infinite debt and zero employment in the economy. After introducing a frequent series of shocks, where we take the scaling parameter $\sigma = 0.75$ in the Rayleigh distribution to define f_ℓ and f_d, the shocks have the effect of shifting the trajectory from the bad stable equilibrium to the good equlibrium described by Grasselli and Huu, namely

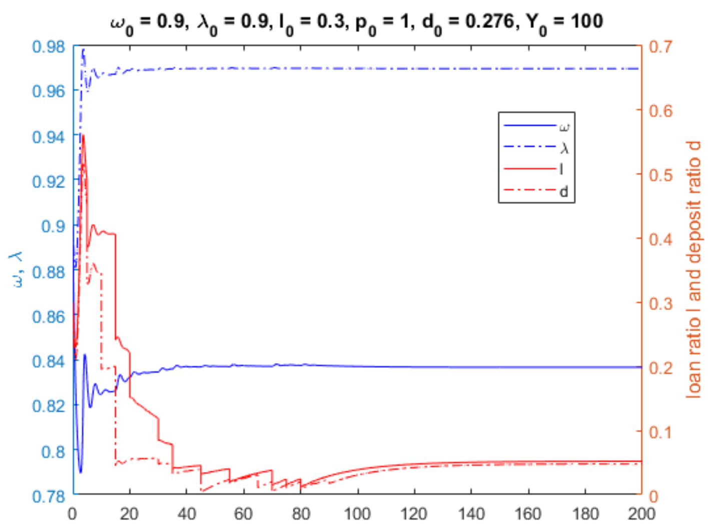
$$(\bar{\omega}_1^+, \bar{\lambda}_1^+, \bar{\ell}_1^+, \bar{d}_1^+) = (0.8366, 0.9693, 0.0521, 0.0478). \tag{47}$$

The notation is borrowed from that paper. This equilibrium corresponds to a situation with finite debt levels and a positive inflation rate of $\bar{i}_1^+ = 1.6\%$ (Figure 1b). Notice also that $\bar{d}_1^+ = (1 - k_r)\bar{\ell}_1^+$, signifying that the equity converges to the pre-specified ratio $X_b = k_r\Lambda$. The evolution of the equity ratio k can be seen in Figure 2 and after each jump (the steep vertical lines are the result of the shocks), the equity ratio tries to return to k_r. We have taken a bad situation in Figure 1a and after adding shocks, moved into a good situation in Figure 1b. This is a counterintuitive result, but one intepretation is that by shocking the system early and often, the debt levels never grow to be infeasible, and the shocks then prevent an overleveraging of the economy where the debt levels are too high relative to the output.

Of particular interest is the situation where $X_b < k_r\Lambda$ or $k < k_r$. A series of shocks that can frequently keep the equity in the banking sector below its target value of $k_r\Lambda$ has the potential to alter trajectories for the worse. If we are to have $k < k_r$, then post-settlement



(a) Wage rate ω, employment rate λ, loan and deposit ratios ℓ, d converging to an equilibrium with infinite debt in the Keen model. No shocks present.



(b) Keen model with same initial conditions. After shocks introduced wages, employment, and debt now converge to a desirable equilibrium with finite debt.

Figure 1: Example trajectories in the Keen model before and after shocks are introduced

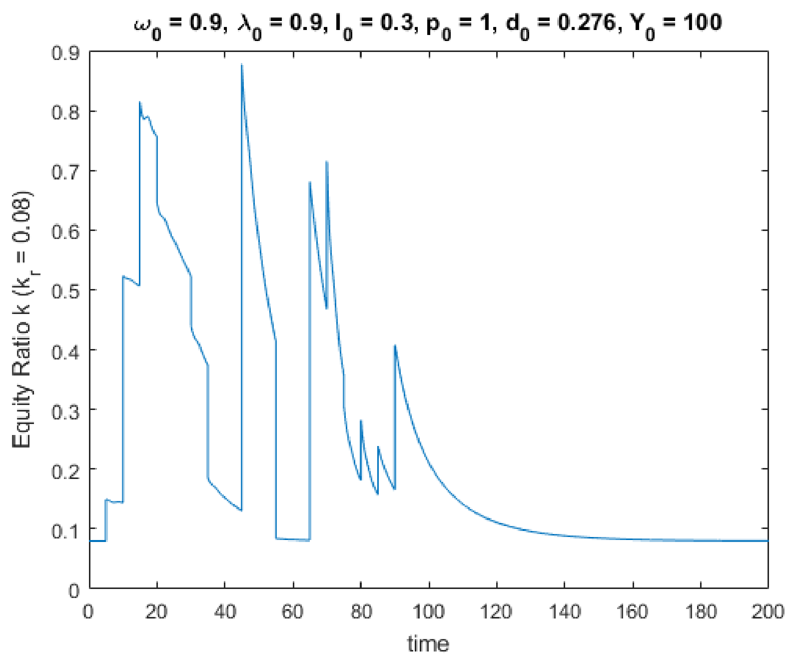


Figure 2: Companion graph to be compared with Figure 1(b). The equity ratio k after frequent shocks are applied to the system will still converge to k_r in the long term.

we need

$$\begin{aligned} k &= 1 - \frac{f_d\, d}{f_\ell\, \ell} < k_r \\ &= 1 - k_r < \frac{f_d\, d}{f_\ell\, \ell} \\ &= (1 - k_r) f_\ell\, \ell < f_d\, d \\ &= (1 - k_r) f_\ell \frac{\ell}{d} < f_d \\ &= \frac{1 - k_r}{1 - k_{pre}} f_\ell < f_d, \end{aligned} \tag{48}$$

where k_{pre} refers to the equity ratio pre-settlement. In this situation, with severe and frequent shocks to the system, it may be possible to alter a trajectory headed towards the good equilibrium with finite debt to a bad one with infinite debt. We have not seen an example illustrating this situation yet, and with the current construction and hypotheses of the model it may not be possible. Further investigation is required in order to engineer an illustrative example.

5 Future Work

In section 3 we considered exogeneous shocks to the cumulative loans and deposits in the system, the source of which was outside the scope of the model. The next step is to consider each individual bank as part of larger network, then to consider them aggregately. Suppose there is a network of N banks that comprise the banking sector. Each bank has external assets A_i in the form of loans given to the firms such that cumulatively this is the total amount of loans in the system,

$$\Lambda(t) = \sum_i^N A_i(t). \tag{49}$$

Likewise, each bank has external liabilities L_i in the form of deposits from the households such that cumulative this is the total amount of deposits in the system,

$$\Delta(t) = \sum_i^N L_i(t). \tag{50}$$

The banks can also have internal assets and liabilities too, whereby bank i has an interbank deposit L_{ji} in bank j - an asset for bank i and a liability for bank j. The total interbank assets of bank i are then

$$\hat{A}_i(t) = \sum_j^N = L_{ji}(t) \tag{51}$$

and the total interbank liabilities of bank i are

$$\hat{L}_i(t) = \sum_j^N = L_{ij}(t). \tag{52}$$

Therefore, the total capital of bank i is then

$$X_i = A_i(t) + \hat{A}_i(t) - L_i(t) - \hat{L}_i(t) \tag{53}$$

and observe that the total capital (or equity) in the banking sector then is the sum of each banks individual capital

$$X_b = \sum_i^N X_i = \Lambda(t) - \Delta(t), \tag{54}$$

and therefore the sum of the interbank liabilities must equal the sum of the interbank assets, or

$$\sum_i^N \hat{A}_i(t) = \sum_i^N \sum_j^N = L_{ji}(t) = \sum_i^N \sum_j^N = L_{ij}(t) = \sum_i^N \hat{L}_i(t). \tag{55}$$

The banking network can then be represented as a pair of $N \times N$ Asset and Liability matrices:

$$\begin{aligned} \mathbb{A} &= (A_{ij}), \quad A_{ii} = A_i, \quad A_{ij} = L_{ji}, \\ \mathbb{L} &= (L_{ij}), \quad L_{ii} = L_i \end{aligned} \tag{56}$$

for $i = 1, ..., N$.

Returning to the set-up described in section 3, we once again assume that banks can only default at the known settlement times $T_1, T_2, ...$ and say bank i is in default at settlement time T_k if its capital is negative: the external and interbank liabilities exceed thier external and internal assets:

$$A_i(T_k) + \hat{A}_i(T_k) \leq L_i(T_k) + \hat{L}_i(T_k). \tag{57}$$

Here, bank i will only be able to pay a proportion g_i of their total liabilities and consequently this will affect the internal assets of other banks in the system, possibly leading to further defaults. We will use the clearing payment mechanism described by Eisenberg and Noe in their paper [1] for the settlement process. A clearing payment vector characterizes the payments of each bank to the other banks in the system and is given by $\mathbf{g} = (g_1, ..., g_N)$ such that (recall the definition of $\hat{A}_i(t)$ in (51))

$$G_i(\mathbf{g}) := min\Big(\frac{A_i(T_k) + \sum_j^N L_{ji}(T_k)\, g_j}{L_i(T_k) + \hat{L}_i(T_k)}, 1\Big) = g_i. \tag{58}$$

That is, $\mathbf{G}(\mathbf{g}) = \big(G_1(\mathbf{g}), ..., G_N(\mathbf{g})\big) = \mathbf{g}$, so $\mathbf{g}$ is a fixed point of the map $\mathbf{G}$, which can be interpreted as the proportion of total funds that will be put towards satisfying debt obligations. Given certain conditions described in [1], the existence and uniqueness of this clearing vector can be shown. Included in the total funds available to bank i are the payments it would receive in this scenario from other nodes in the system as well as their external assets. Given a clearing payment vector $\mathbf{g}$ we say that bank i is in default if $g_i < 1$. Post settlement at time T_k, then, we obtain the total loans and deposits as

$$\begin{aligned} \Lambda(T_k) &= \sum_i^N A_i(T_k) \\ \Delta(T_k) &= \sum_i^N L_i(T_k) \end{aligned} \tag{59}$$

which may exhibit discontinuities as a result of the defaulting banks. Consequently the loan and deposit ratios ℓ and d

$$\ell(T_k) = \frac{\Lambda(T_k)}{p(T_k)Y(T_k)} \quad d(T_k) = \frac{\Delta(T_k)}{p(T_k)Y(T_k)}, \tag{60}$$

may exhibit discontinuities as well, and these discontinuities will propagate throughout the other variables in the system indirectly. Just as before, we say the initial conditions for the ODE system (24) and (31) post settlement are given by $\big(\omega(T_k), \lambda(T_k), \ell(T_k), p(T_k), d(T_k)\big)$ and this is valid for $t \in [T_k, T_{k+1})$. With a banking network in place, we can endogenize the shocks as well as assess the systemic risk in the network. What network structures are particularly fragile or robust to chains of defaults? A chain of defaults would naturally lead to a larger shock to the system, so we can begin to look into what network structures pose the most risk to the ODE side of the model, ie: for a given network, can one default trigger a large enough shock to move a trajectory from a good equilibrium to a bad one.

References

[1] Larry Eisenberg and Thomas H. Noe. Systemic risk in financial systems. *Management Science*, 47(2):236–249, 2001.

[2] M. R. Grasselli and B. Costa Lima. An analysis of the keen model for credit expansion, asset price bubbles and financial fragility. *Mathematics and Financial Economics*, 6(3):191–210, Jun 2012.

[3] Matheus R. Grasselli and Adrien Nguyen-Huu. Inflation and speculation in a dynamic macroeconomic model. *Journal of Risk and Financial Management*, 8(3):1–26, 2015.

[4] Steve Keen. Finance and economic breakdown: Modeling minskys financial instability hypothesis. *Journal of Post Keynesian Economics*, 17(4):607–635, 1995.