Theory of Computing Notes

Chapter 1: Introduction

Fraser Watt

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Chapter Notes

Sets and Notation

With set notation being represented with curly braces, an empty set would be $\{\}$, whilst a collection which only holds that empty set would be $\{\{\}\}$. Other important set notation:

- In / not in: $\in$, $\notin$.
- x as a subset of y: $x \subset y$.
- Natural numbers (positive integers): N.

Set logic might read something like this:

$$\{2n | n \in N\}$$

The above would be read as *"2n, given n is an element within the set of all natural numbers."*

Fraser's Q: Why does $N - \{0\}$ denote the set of all positive numbers, and not something like $N \neq 0$?

- They're equivalent. Whilst $N - \{0\}$ means the set of all natural numbers *except* a set only containing 0, $N \neq 0$ is just all natural numbers which do not equal 0.
- You might prefer the first format as it's more flexible - you can just add more numbers to exclude. E.g. $N - \{0, 1, 2\}$

Joining Sets

- Intersection: Take elements common to both sets, written as $A \cap B$. This is like a SQL `JOIN`
- Union: New set includes all unique / distinct values of component sets, written as $A \cup B$. This is like a SQL `FULL JOIN`
- Cartesian Product: Takes elements from both sets and returns every possible combination from them, written as $A \times B$. This is like a SQL `CROSS JOIN`

Relations

Throughout the book, there's a focus on using strings. Strings (words) are a collection of symbols where the symbols are all taken from a fixed alphabet. The alphabet can be thought of as a set!

- Length of word w would be represented as $|w|$.
- Reverse of word w would be represented as w^R (palendrome).

The complement of a set is *everything that isn't in a given set*. So if A was a set of all natural numbers, $A' = \{n|n \notin N\}$. This is sort of like writing `!(yourLogicalStatement)` in Javascript, but ! in mathematical notation is factoral multiplication so wouldn't make sense here.

Transitive, Reflexive, and Symmetric relations.

- **Transitive:** Modular arithmetic (`%` in Python) useful to deconstruct this: if 4 mod 3 = 10 mod 3 and 10 mod 3 = 13 mod 3, then there is a transitive relation which says that 4 mod 3 = 13 mod 3.
- **Symmetric:** If you have equivalence from $a \to b$ and another arrow from $b \to a$, then this is symmetric. Again, all or nothing — it is symmetric if for every arrow drawing equivalence, there is an equal and opposing relation backwards.
- **Reflexive:** If we are also going to use modular arithmetic here too, then we would be looking more like \$10 mod 3 = 10 mod 3 \$.
  - If you add more arrows going to other items, it doesn't matter — still reflexive! On the other hand, if you have two sets $\{a, b, c, d\}$, and you have arrows going from both a's, b's, & c's — but no equivalence between d_1 and d_2 — this is not reflexive. It's all or nothing!
- **Equivalence:** If it has a Transitive, Symmetric and Reflexive relation, then we can say it has an equivalence relation. I.e. $A = B$

Exercises

Exercise 1.1: Give the notation for a set that contains only the page number of this exercise (page 9).

$$R = \{9\}$$

Exercise 1.2: True or False

a) $\{2, 4, 5\} \subset \{2n \mid n \in N\}$ False

b) $\{e, i, t, c\} \subset \{a, ..., z\}$ True

c) $\{2n \mid \bar{n} \in N\} = \{2n + 1 \mid n \in N\}$ True

d) $\{1, 2, 3, 4, 5\} = \{1, ..., 5\}$ True

Exercise 1.3: Give a representation for the odd natural numbers.

$$\{n|\frac{n}{2} \notin N\}$$

Exercise 1.4: For each of the strings w below, exhibit both $|w|$ and w^R:

a) dooGyreV: $|a| = 8,\ a^R =$ VeryGood
b) kaerBAekaT: $|b| = 10,\ b^R =$ TakeABreak
c) kcanSAevaH: $|c| = 10,\ c^R =$ HaveASnack
d) ysaEooTsIsihT: $|d| = 13,\ d^R =$ ThisIsTooEasy

Exercise 1.5: Let $A = \{1, 2, ..., 100\}$ and $B = \{a, b, c, d\}$. How large is $A \times B$?

$N = 400$

Exercise 1.6: Let $R = \{a, b, c, d\}$ and $S = \{0, 1\}$. What is $R \times S$?

$$\{(a, 0), (b, 0), (c, 0), (d, 0), (a, 1), (b, 1), (c, 1), (d, 1)\}$$

Exercise 1.7: Define a relation R on $N \times N$ by

$$R = \{(x, y) \mid x \in N, y \in N \text{ and } y \text{ is even}\}$$

Prove or disprove: R is an equivalence relation:

- R can be understood to mean "given (x, y), R is the set of all number pairs where x and y are both natural numbers (N), and the sum of $x + y$ is even."
- $R_{(1,3)}$: x and y are both natural numbers. $1 + 3 = 4$ and $\frac{4}{2} = 2$, therefore $(1, 3)$ is in R.
- $R_{(4,6)}$: x and y are both natural numbers. $4 + 6 = 10$ and $\frac{10}{2} = 5$, therefore $(4, 6)$ is also in R.
- $2 \neq 5$, therefore $R_{(4,6)} \neq R_{(1,3)}$. If these two items are not reflexive, then this set does not have a reflexive property.
- If the set does not have a reflexive relation, it cannot have an equivalence relation.