Theory of Computing Notes

Chapter 1: Introduction

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Chapter Notes

Sets and Notation

With set notation being represented with curly braces, an empty set would be $\{\}$, whilst a collection which only holds that empty set would be $\{\{\}\}$. Other important set notation:

- In / not in: $\in$, $\notin$.
- x as a subset of y: $x \subset y$.
- Natural numbers (positive integers): N.

Set logic might read something like this:

$$\{2n|n \in N\}$$

The above would be read as *"$2n$, given n is an element within the set of all natural numbers."*

Fraser's Q: Why does $N - \{0\}$ denote the set of all positive numbers, and not something like $N \neq 0$?

- This is because N is a set, whilst 0 is a number. So although $\{0\}$ might only be a set with a length of one, $\{0\} \neq 0$.
- $N \neq 0$ is true! But because a set is not a number. This is why we would say $N - \{0\}$ when we are talking about the set of all positive numbers.

Joining Sets

- Intersection: Take elements common to both sets, written as $A \cap B$. This is like a SQL `JOIN`
- Union: New set includes all unique / distinct values of component sets, written as $A \cup B$. This is like a SQL `FULL JOIN`
- Cartesian Product: Takes elements from both sets and returns every possible combination from them, written as $A \times B$. This is like a SQL `CROSS JOIN`

Relations

Throughout the book, there's a focus on using strings. Strings (words) are a collection of symbols where the symbols are all taken from a fixed alphabet. The alphabet can be thought of as a set!

- Length of word w would be represented as $|w|$.
- Reverse of word w would be represented as w^R (palendrome).

The complement of a set is *everything that isn't in a given set*. So if A was a set of all natural numbers, $A' = \{n | n \notin N\}$. This is sort of like writing `!(yourLogicalStatement)` in Javascript, but ! in mathematical notation is factoral multiplication so wouldn't make sense here.

Transitive, Reflexive, and Symmetric relations.

- **Transitive:** Modular arithmetic (`%` in Python) useful to deconstruct this: if 4 mod 3 = 10 mod 3 and 10 mod 3 = 13 mod 3, then there is a transitive relation which says that 4 mod 3 = 13 mod 3.
- **Symmetric:** If you have equivalence from $a \to b$ and another arrow from $b \to a$, then this is symmetric. Again, all or nothing — it is symmetric if for every arrow drawing equivalence, there is an equal and opposing relation backwards.
- **Reflexive:** If we are also going to use modular arithmetic here too, then we would be looking more like 10 mod 3 = 10 mod 3.
  - If you add more arrows going to other items, it doesn't matter — still reflexive! On the other hand, if you have two sets $\{a, b, c, d\}$, and you have arrows going from both a's, b's, & c's — but no equivalence between d_1 and d_2 — this is not reflexive. It's all or nothing!
- **Equivalence:** If it has a Transitive, Symmetric and Reflexive relation, then we can say it has an equivalence relation. I.e. $A = B$

Exercises

Exercise 1.1: Give the notation for a set that contains only the page number of this exercise (page 9).

$$R = \{9\}$$

Exercise 1.2: True or False

a) $\{2, 4, 5\} \subset \{2n \mid n \in N\}$ False

b) $\{e, i, t, c\} \subset \{a, ..., z\}$ True

c) $\{2n \mid \bar{n} \in N\} = \{2n + 1 \mid n \in N\}$ False

Originally had this as True, but did not consider that the complement of $\{2n \mid n \in N\}$ would also include non-natural numbers, whereas $\{2n \mid n \in N\}$ specifically states all members of the set would be natural.

d) $\{1, 2, 3, 4, 5\} = \{1, ..., 5\}$ True

Exercise 1.3: Give a representation for the odd natural numbers.

My original answer to this was:

$$\{n|\frac{n}{2} \notin N\}$$

This is incorrect as (e.g.) 4.5 would work; $\frac{4.5}{2} = 2.25$ is not a neutral number. Instead we want to include a statement that n is also a natural number, like so:

$$\{n|n \in N, \frac{n}{2} \notin N\}$$

However, a more conventional way of writing it is:

$$\{2n - 1 \mid n \in N\}$$

Not only is this slightly more concise, all I've done in the first definition is define what my set is *not*, as opposed to defining what it *is*.

Exercise 1.4: For each of the strings w below, exhibit both $|w|$ and w^R:

a) dooGyreV: $|a| = 8,\ a^R =$ VeryGood
b) kaerBAekaT: $|b| = 10,\ b^R =$ TakeABreak
c) kcanSAevaH: $|c| = 10,\ c^R =$ HaveASnack
d) ysaEooTsIsihT: $|d| = 13,\ d^R =$ ThisIsTooEasy

Exercise 1.5: Let $A = \{1, 2, ..., 100\}$ and $B = \{a, b, c, d\}$. How large is $A \times B$?

$N = 400$

Exercise 1.6: Let $R = \{a, b, c, d\}$ and $S = \{0, 1\}$. What is $R \times S$?

$$\{(a, 0), (b, 0), (c, 0), (d, 0), (a, 1), (b, 1), (c, 1), (d, 1)\}$$

Exercise 1.7: Define a relation R on $N \times N$ by

$$R = \{(x, y) \mid x \in N, y \in N, x + y \text{ is even}\}$$

Prove or disprove: R is an equivalence relation:

- R can be understood to mean "given (x, y), R is the set of all number pairs where x and y are both natural numbers (N), and the sum of $x + y$ is even."
- In order to prove R is an equivalence relation, we need to prove that it is symmetric, reflexive and transitive.
  - It is symmetric as if $x + y$ is even, then so is $y + x$.
  - It is reflexive if for all $x \in N$, $(x, x) \in R$. As any x would need to be an even natural number, this would satisfy the constraints that $y \in N, y$ is even.
  - It is transitive if $(x, y) \in R, (y, z) \in R$, and $(x, z) \in R$. The addition of any two even numbers is always another even number, therefore any (x, y) that satisfies the constraints of this set would also satisfy them in the case of (y, z) and (x, z).
- As this set satisfies symmetric, reflexive and transitive relations, we can prove R is an equivalence relation.