Signal Processing and Optimization for Big Data

15/10/2025

1) Let us consider the LASSO problem

$$\min_{\boldsymbol{x}} \|\mathbf{y} - \boldsymbol{A}\boldsymbol{x}\|^2 + \alpha\|\boldsymbol{x}\|_1, \tag{1}$$

where $\boldsymbol{A} \in \mathbb{R}^{N\times M}$, $\boldsymbol{x} \in \mathbb{R}^{M\times 1}$ and $\alpha > 0$. Write two functions, using a programming language of your choice (e.g., Matlab/Python) to solve the problem through **ISTA** and **ADMM**.

2) Test the functions considering a Gaussian random matrix $\boldsymbol{A}$ and generating $\boldsymbol{y}$ from a 3-sparse vector $\boldsymbol{x_0}$. Let K be the number of **iterations**

- Plot the temporal behavior of the reconstruction MSE $E[k] = \|\boldsymbol{x}[k] - \boldsymbol{x_0}\|^2$ and the objective function $J[k]$.
- Plot the sparsity of the obtained solution and $\boldsymbol{x}[K]$ and the final reconstruction error $E[K]$ for different values of α.
- Plot the temporal behavior of the two terms of the cost function for different values of α.

Hint: Use, for example, N=10, M=20, K=200.

3) Let us consider the following linear observation model

$$\mathbf{y} = \mathbf{A}\mathbf{x}_0 + \mathbf{n}, \tag{2}$$

where $\mathbf{y} \in \mathbb{R}^N$, $\mathbf{A} \in \mathbb{R}^{N\times M}$ and $\mathbf{n} \sim \mathcal{N}(0, \sigma_n^2\mathbf{I})$. Let us also consider the Ridge regression problem

$$\min_{\boldsymbol{x}} \|\mathbf{y} - \boldsymbol{A}\boldsymbol{x}\|^2 + \alpha\|\boldsymbol{x}\|_2, \tag{3}$$

- Write a function to solve the Ridge regression problem trough an iterative algorithm.
- Fix a noise variance σ_n of your choice, and generate some synthetic data according to the linear model (2). Choose $\mathbf{A}$ as a random matrix, and assume $\mathbf{x}_0 \sim \mathcal{N}(0, \sigma_0\mathbf{I})$.
- Plot the temporal behaviour of the cost function $J[k]$ for different values of the regularization parameter α and compare the convergence speed of Ridge regression solved through GD and LASSO solved through the ISTA method.
- Considering the same observation model, solve the LASSO problem for different values of the regularization parameter α and plot: i) the temporal behaviour of the cost function and the reconstruction MSE for different values of α; ii) the sparsity of the solution as a function of the parameter α; .

4) Let us consider the *mpg-dataset*. This data-set contains **398** instances of car models for which we want to predict the fuel consumption $\boldsymbol{y}$, expressed in miles per gallon (MPG). We have 7 features, used to build the design matrix $\boldsymbol{A}$. Choose one method between ISTA and ADMM and apply it to solve this regression problem.

- Select the best trade-off parameter α through cross-validation, considering N_{tr} samples for training and N_{val} samples for validation.
- For each value of α, evaluate the sparsity of the obtained solution and the generalization capabilities of the model.
- Plot the temporal behavior of the terms of the cost function for different values of α.

Hint 1: A possible way to evaluate a regression model is given by the R2 score, which is defined by

$$R2 = 1 - \frac{RSS}{TSS}, \tag{4}$$

where $RSS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ is the Residual Sum of Squares, while $TSS = \sum_{i=1}^{n}(y_i - \overline{y})^2$ is the Total Sum of Squares. Specifically, $\boldsymbol{y}, \hat{\boldsymbol{y}} \in \mathbb{R}^{N_{val}}$ are the observations and the predictions vectors respectively, while $\overline{y}$ is the mean of the observations.

Hint 2: it could be useful to normalize the data to help the convergence of the algorithms.

5) Given the following power allocation problem

$$\max_{\{p_i\}_{i=1}^{N}} \sum_{i=1}^{N} \mathrm{B_c} \log_2(1 + \gamma_i p_i), \tag{5}$$

$$\text{s.t.} \sum_{i=1}^{N} p_i \le P_{\max}$$

$$p_i \ge 0 \qquad \forall i = 1, \ldots, N \tag{6}$$

where N is the number of sub-carriers, B_c is the bandwidth, $\{p_i\}_{i=1}^{N}$ are the tx powers, $P_{\max}$ is the power budget and $\{\gamma_i\}_{i=1}^{N}$ model the status of the channels.

- Write a function to obtain the optimal solution
- Draw a waterfill plot to show the obtained power allocation

Hint: in Matlab you can use the function waterfill for sanity-check. Otherwise, you can compare the results with the ones obtained through CVX.

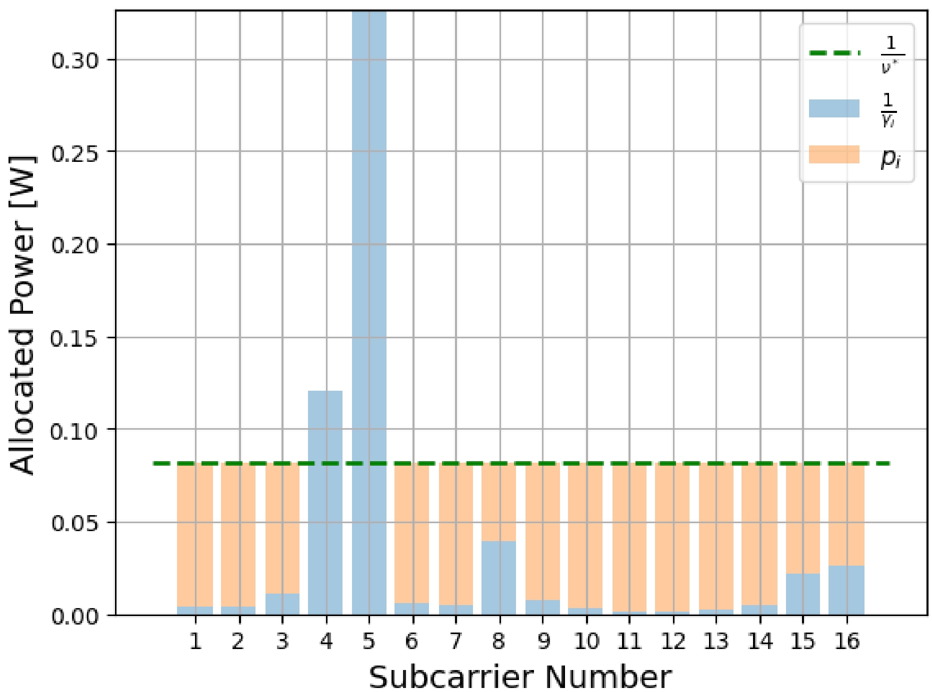

Figure 1: Example of Waterfill plot