Chapter 21 Solusion

github.com/yirong-c/CLRS

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21.1

21.1-1

Edge processed	Collection of disjoint sets										
initial sets	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$	$\{i\}$	$\{j\}$	$\{k\}$
(d,i)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d,i\}$	$\{e\}$	$\{f\}$	$\{g\}$	$\{h\}$		$\{j\}$	$\{k\}$
(f,k)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d,i\}$	$\{e\}$	$\{f,k\}$	$\{g\}$	$\{h\}$		$\{j\}$	
(g,i)	$\{a\}$	$\{b\}$	$\{c\}$	$\{d,g,i\}$	$\{e\}$	$\{f,k\}$		$\{h\}$		$\{j\}$	
(b,g)	$\{a\}$	$\{b,d,g,i\}$	$\{c\}$		$\{e\}$	$\{f,k\}$		$\{h\}$		$\{j\}$	
(a,h)	$\{a,h\}$	$\{b,d,g,i\}$	$\{c\}$		$\{e\}$	$\{f,k\}$				$\{j\}$	
(i,j)	$\{a,h\}$	$\{b,d,g,i,j\}$	$\{c\}$		$\{e\}$	$\{f,k\}$					
(d,k)	$\{a,h\}$	$\{b,d,f,g,i,j,k\}$	$\{c\}$		$\{e\}$						
(b,j)	$\{a,h\}$	$\{b,d,f,g,i,j,k\}$	$\{c\}$		$\{e\}$						
(d,f)	$\{a,h\}$	$\{b,d,f,g,i,j,k\}$	$\{c\}$		$\{e\}$						
(g,j)	$\{a,h\}$	$\{b,d,f,g,i,j,k\}$	$\{c\}$		$\{e\}$						
(a,e)	$\{a,e,h\}$	$\{b,d,f,g,i,j,k\}$	$\{c\}$								

21.1-2

Proof. By contents in B.4, we know that the connected components of a graph are the equivalence classes of vertices under the "is reachable from" relation. The collection of the disjoint sets is exactly the quotient set of $G.V$ by the "is reachable from" relation. It is not hard to find out that CONNECTED-COMPONENTS construct such the quotient set since the procedure unions vertices based on all edges, and edges connect two reachable vertices with the smallest length of the path (recall that a equivalence relation must be transitive). Two vertices are in the same connected component if and only if they are reachable from each other. $\square$

21.1-3

FIND-SET: $2 \cdot |E|$
UNION: $|V| - k$

21.2

21.2-1

```
struct Set
{
    Node *head;
    Node *tail;
    int size;
};

struct Node
{
    int key;
    Set *set;
    Node *next;
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->set = new Set;// need to be freed
    x->set->head = x;
    x->set->tail = x;
    x->set->size = 1;
}

Node* FindSet(Node *x)
{
    return x->set->head;
}

void Union(Node *x, Node *y)
{
    Node *node;
    if (x->set->size < y->set->size)
    {
        Union(y, x);
    }
    else
```

```
    {
        node = y->set->head;
        x->set->size += y->set->size;
        x->set->tail->next = node;
        x->set->tail = y->set->tail;
        delete y->set;
        while (node)
        {
            node->set = x->set;
            node = node->next;
        }
    }
}
```

21.2-2

collection before line 3:

$$\{\{x_1\},\{x_2\},\{x_3\},\{x_4\},\{x_5\},\{x_6\},\{x_7\},\{x_8\},\{x_9\},\{x_{10}\},\{x_{11}\},\{x_{12}\},\{x_{13}\},\{x_{14}\},\{x_{15}\},\{x_{16}\}\}$$

collection before line 5:

$$\{\{x_1,x_2\},\{x_3,x_4\},\{x_5,x_6\},\{x_7,x_8\},\{x_9,x_{10}\},\{x_{11},x_{12}\},\{x_{13},x_{14}\},\{x_{15},x_{16}\}\}$$

collection before line 7:

$$\{\{x_1,x_2,x_3,x_4\},\{x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12}\},\{x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 8:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12}\},\{x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 9:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\},\{x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16}\}\}$$

collection before line 10:

$$\{\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10},x_{11},x_{12},x_{13},x_{14},x_{15},x_{16}\}\}$$

Hence FIND-SET(x_2) and FIND-SET$(x_1 1)$ return a pointer points to x_1.

21.2-3

Lemma 1. Using the linked-list representation of disjoint sets and the weighted-union heuristic, a sequence of h UNION operations on a disjoint set that has never been operated UNION takes $O(h \lg h)$ time.

Proof. We claim that, after h UNION operations, the largest set has at most $h+1$ members. Notice that the number of sets decreases by one each time UNION is called. Suppose that we have n sets in which each set contains one member in the beginning. After the h UNION operations, we have $(n-h)$ sets. Note that each set must contain one member. In order to maximize the number of members in the the largest set, we let $(n-h-1)$ sets contains one member, and let the remaining set contains all remaining members. Then the remaining set contains

$$n-(n-h-1)=h+1$$

members.

We claim that each object's pointer back to its set object is updated at most $\lceil \lg h \rceil$ times over all the UNION operations. Let x be an arbitrary object. By the similar approach in the proof of Theorem 21.1, we know that for any $k \leq h+1$, after x's pointers has been updated $\lceil \lg k \rceil$ times, the resulting set must have at least k members. Since the largest set has at most $h+1$ members, each object's pointer is updated at most $\lg(h+1)$ times over all the UNION operations.

We claim that there are h elements have been updated their pointers back to their set objects at least once. Consider a set contains k members. Then within this set, there are $(k-1)$ members have been updated their pointers back to their set objects at least once since there must exists exactly $(k-1)$ members updated their pointers from the initial pointer to the current one. Let $\mathcal{S}$ be our collection of sets. Then after the h UNION operations, the number of elements have been updated their pointers is

$$\sum_{A \in \mathcal{S}} (|A|-1) = \sum_{A \in \mathcal{S}} |A| - |\mathcal{S}| = n-(n-h) = h$$

Since each object's pointer is updated at most $\lceil \lg h \rceil$ times and there are h elements have been updated their pointers, we conclude h UNION operations on a disjoint set that has never been operated UNION takes $O(h \lg h)$ time. □

Claim 2. The amortized time of MAKE-SET and FIND-SET is $O(1)$, and the amortized time of UNION is $O(\lg n)$.

Proof. Suppose that we performed h UNION operations. Since n MAKE-SET operations are performed, we know $(m-n-h)$ FIND-SET operations are performed By the lemma, we know that the total actual cost of UNION is $O(h \lg h)$. Hence the total actual cost of the sequence is

$$O(\underbrace{n}_{\text{MAKE-SET}} + \underbrace{(m-n-h)}_{\text{FIND-SET}} + \underbrace{h \lg h}_{\text{UNION}}) = O(m-h+h \lg h)$$

The total amortized cost of the sequence is

$$O(\underbrace{n}_{\text{MAKE-SET}} + \underbrace{(m-n-h)}_{\text{FIND-SET}} + \underbrace{h \lg n}_{\text{UNION}}) = O(m-h+h \lg n)$$

Since $h < n$, we have showed the claim successfully. □

21.2-4

In the ith UNION operation, we call UNION(x_{i+1}, x_i). At this time, the size of set contains x_i contains i members, and the size of set contains x_{i+1} contains 1 members. Then we notice, for all $i \geq 2$, we append the list contains x_{i+1} onto the list contains x_i with the weighted-union heuristic, and this only takes $\Theta(1)$ time for each operation. We operate n times MAKE-SET and $(n-1)$ times UNION, so the sequence takes $\Theta(n + (n-1)) = \Theta(n)$ time.

21.2-5

```
struct Node
{
    int key;
    Node *next;
    // let the tail element be the set's representative
    union
    {
        Node *tail;// for non-tail elements
        Node *head;// for the tail element
    } representative;
    int size;// only for the tail element
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->representative.head = x;
    x->size = 1;
}

Node* FindSet(Node *x)
{
    return x->next ? x->representative.tail : x;
}

void Union(Node *x, Node *y)
{
    Node **node, *x_head, *y_head, *x_representative, *y_representative;
    if (x->representative.tail->size < y->representative.tail->size)
    {
        Union(y, x);
```

```
    }
    else
    {
        x_representative = FindSet(x);
        y_representative = FindSet(y);
        x_head = x_representative->representative.head;
        y_head = y_representative->representative.head;
        x_representative->size += y_representative->size;
        node = &y_head;
        while (*node)
        {
            (*node)->representative.tail = x_representative;
            node = &((*node)->next);
        }
        *node = x_head;
        x_representative->representative.head = y_head;
    }
}
```

21.2-6

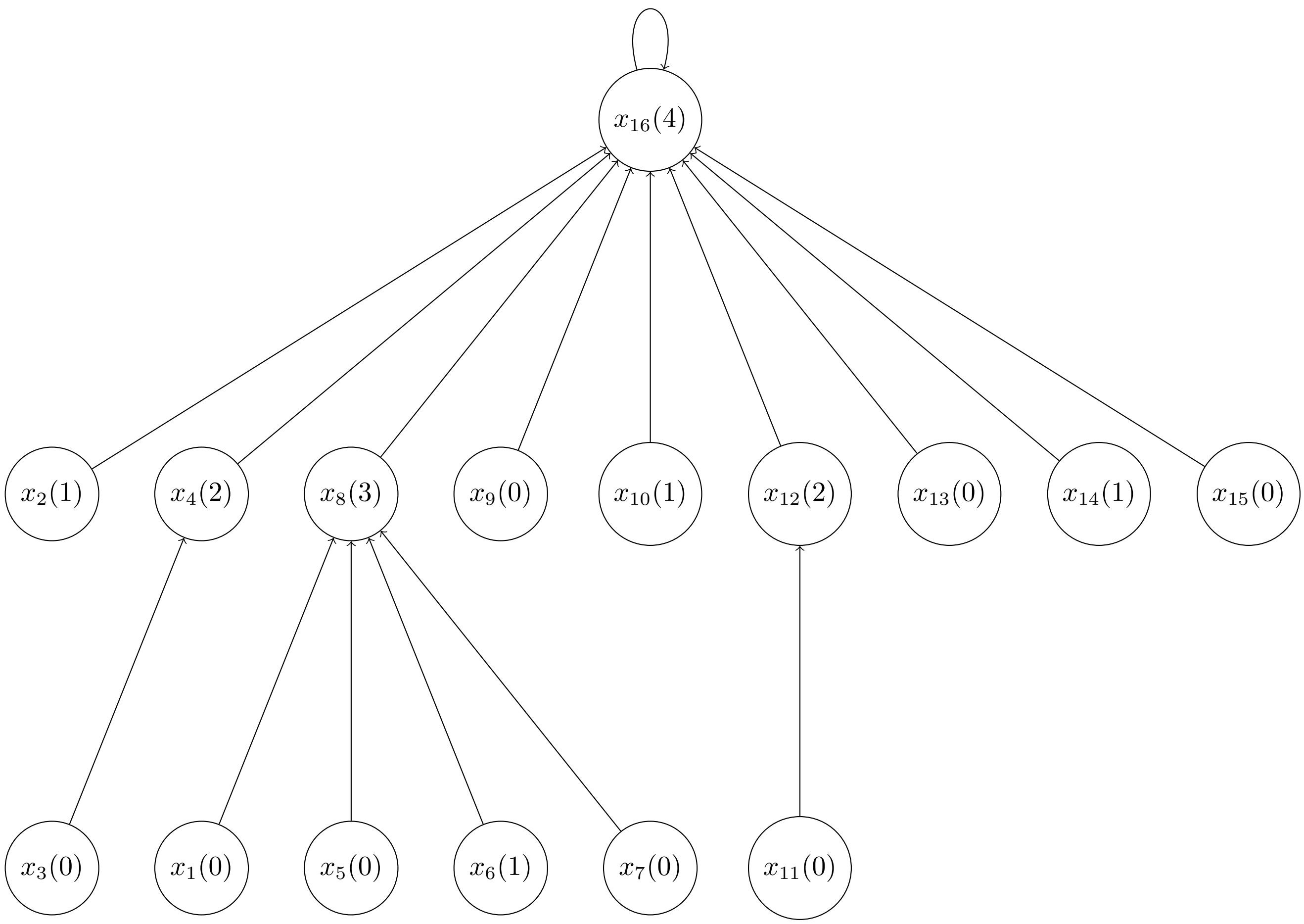
```
struct Set
{
    Node *head;
    int size;
};

struct Node
{
    int key;
    Set *set;
    Node *next;
};

void MakeSet(Node *x)
{
    x->next = nullptr;
    x->set = new Set;// need to be freed
    x->set->head = x;
    x->set->size = 1;
```

```
}

Node* FindSet(Node *x)
{
    return x->set->head;
}

void Union(Node *x, Node *y)
{
    Node **node, *x_second;
    if (x->set->size < y->set->size)
    {
        Union(y, x);
    }
    else
    {
        x->set->size += y->set->size;
        x_second = x->next;
        x->next = y->set->head;
        node = &(x->next);
        delete y->set;
        while (*node)
        {
            (*node)->set = x->set;
            node = &((*node)->next);
        }
        *node = x_second;
    }
}
```

21.3

21.3-1

The data structure in the end (rank of each node is in the parentheses):



Hence FIND-SET(x_2) and FIND-SET($x_1 1$) return a pointer points to $x_1 6$.

21.3-2

```
Node* FindSet(Node *x)
{
    Node *representative, *tmp;
    representative = x;
    while (representative != representative->p)
    {
        representative = representative->p;
    }
    while (x->p != representative)
    {
        tmp = x->p;
        x->p = representative;
        x = tmp;
    }
```

```
    return representative;
}
```

21.3-3

Note that we are proving that the upper bound $O(m \lg n)$ is tight (least upper bound), instead of prove it is a tight bound $\Theta(m \lg n)$. In order to prove it, we just need to find an example that takes $\Omega(m \lg n)$ time, which is what the question is asking for. We want our sequence to take as much as possible time. WLOG, assume that $n = 2^k$ for some $k \in \mathbb{N}$. Consider the following sequence:

$$\begin{aligned}
\langle &\textsc{Make-Set}(x_1), \textsc{Make-Set}(x_2), \cdots, \textsc{Make-Set}(x_n), \\
&\textsc{Union}(x_1, x_2), \textsc{Union}(x_3, x_4), \cdots, \textsc{Union}(x_{n-1}, x_n), \\
&\textsc{Union}(x_1, x_3), \textsc{Union}(x_5, x_7), \cdots, \textsc{Union}(x_{n-3}, x_{n-1}), \\
&\textsc{Union}(x_1, x_5), \textsc{Union}(x_9, x_13), \cdots, \textsc{Union}(x_{n-7}, x_{n-3}), \\
&\qquad\qquad \vdots \\
&\textsc{Union}(x_1, x_{n/2+1}), \\
&\textsc{Find-Set}(x_1) \cdots \qquad\qquad \text{(until all } m \text{ operations are performed)} \rangle
\end{aligned}$$

We performed n Make-Set, $(n-1)$ Union, and $(m-2n+1)$ Find-Set. We observed we performed $n/2$ $\textsc{Union}(x_i, x_{i+1})$, $n/4$ $\textsc{Union}(x_i, x_{i+2})$, $n/8$ $\textsc{Union}(x_i, x_{i+4})$, $\cdots$. We conclude we performed $n/2^j$ $\textsc{Union}(x_i, x_{i+2^{j-1}})$ for all $j = \{1, 2, \cdots, \lg n\}$. For each j, the height of each tree increases by 1. Hence after all $(n-1)$ Union operations, the height of the tree (for the only set) is $\lg n$. Note that x_1 is the deepest element in the tree. Then, each of $\textsc{Find-Set}(x_1)$ operation takes $\Theta(\lg n)$ time, and we perform $(m - 2n + 1)$ times $\textsc{Find-Set}(x_1)$ operation. Hence all of $\textsc{Find-Set}(x_1)$ take $(m - 2n + 1)\Theta(\lg n) = \Theta(m \lg n)$ time. We successfully find a sequence that takes $\Omega(m \lg n)$ time.

21.3-4

We just need to modify Link procedure to maintain the data structure.

```
void Link(Node *x, Node *y)
{
    Node *y_next;
    if (x->rank > y->rank)
    {
        y->p = x;
    }
    else
    {
        x->p = y;
        if (x->rank == y->rank)
```

```
            ++y->rank;
    }
    // maintain the circular list
    y_next = y->next;
    y->next = x->next;
    x->next = y_next;
}

std::list<Node*> PrintSet(Node *x)
{
    Node *node;
    std::list<Node*> result;
    result.push_back(x);
    for (node = x->next; node != x; node = node->next)
    {
        result.push_back(node);
    }
    return result;
}
```

21.3-5

Let a_i be the number of nodes with depth greater than 0 (i.e. non-root) in the forest after the ith operation. Let b_i be the number of nodes with depth greater than 1 (i.e. non-root and non-child-of-root) in the forest after the ith operation. Suppose that we start to perform FIND-SET operations in the kth operation. Let the potential function be

$$\Phi(D_i) \begin{cases} a_i & \text{if } i < k \\ b_i & \text{if } i \geq k \end{cases},$$

where D_i is the disjoint forest after the ith operation. Let $\Phi(D_0) = 0$. Observed each of MAKE-SET and LINK takes $O(1)$ time. Denote $depth_i(x)$ as the depth of the node x in the tree after the ith operation. Then FIND-SET(x) moves $\max(0, depth_{i-1}(x) - 1)$ nodes to be the children of the root node. Denote c_i as the cost of the ith operation. Then we assume

$$c_i = \begin{cases} 1 & \text{if MAKE-SET is performed in the } i\text{th operation,} \\ 1 & \text{if LINK is performed in the } i\text{th operation,} \\ \max(1, depth_{i-1}(x)) & \text{if FIND-SET}(x) \text{ is performed in the } i\text{th operation} \end{cases}$$

Case 1. MAKE-SET is performed in the ith operation.

Then $a_i = a_{i-1}$, so

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 0 = 1$$

Case 2. LINK(x, y) is performed in the ith operation.

Note that x and y must be root nodes of different tree. This operation make either x or y be a non-root node. Then $a_i = a_{i-1} + 1$, so

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) = 1 + 1 = 2$$

Case 3. FIND-SET(x) is performed in the ith operation.

Note that $b_i \leq a_i$ for all i. If $i = k$, then

$$\Phi(D_i) - \Phi(D_{i-1}) = b_i - a_{i-1} \leq b_i - b_{i-1}$$

If $i > k$, then

$$\Phi(D_i) - \Phi(D_{i-1}) = b_i - b_{i-1}$$

Note that

$$b_i - b_{i-1} = -(\max(1, depth_{i-1}(x)) - 1) = 1 - \max(1, depth_{i-1}(x))$$

Hence

$$\hat{c_i} = c_i + \Phi(D_i) - \Phi(D_{i-1}) \leq c_i + b_i - b_{i-1} = \max(1, depth_{i-1}(x)) + (1 - \max(1, depth_{i-1}(x))) = 1$$

We have shown that amortized of each operation is $O(1)$ with the path-compression heuristic, no matter whether we use union by rank or not. Hence the sequence takes $O(m)$ time with the path-compression heuristic, no matter whether we use union by rank or not.

21.4

21.4-1

Proof. We prove by induction on the number of operations.

(Base) Initially, there is no element in the disjoint sets, so it is trivial.

(Induction) Denote D_i be the data structure after ith operation. Denote $p_i(x)$ be $x.p$ after ith operation. Denote $rank_i(x)$ be $x.rank$ after ith operation. Suppose that, for all $x \in D_{i-1}$, ① $rank_{i-1}(x) < rank_{i-1}(p_{i-1}(x))$ if $x \neq p_{i-1}(x)$, ② $rank_{i-1}(x) = rank_{i-2}(x)$ if $x \neq p_{i-1}(x)$, and ③ $rank_{i-1}(p_{i-1}(x)) \geq rank_{i-2}(p_{i-2}(x))$.

Case 1. MAKE-SET(y) is performed in the ith operation.

Then all structures of trees in D_{i-1} remain same in D_i, and ①②③ are vacuously true for y.

Case 2. FIND-SET(y) is performed in the ith operation.

Since no rank changes in FIND-SET, ② holds. Let z be the root of the tree contains y. Let A be the set contains all the nodes on the simple path from y to z except z in D_{i-1}. Let $w \in A$ be arbitrary choice. We have $w \neq p_{i-1}(w)$. By ①, since $p_i(w) = z$, we have

$$rank_{i-1}(w) < rank_{i-1}(p_{i-1}(w)) \leq rank_{i-1}(z) = rank_{i-1}(p_i(w))$$

Since no rank changes in FIND-SET, we have $rank_i(w) < rank_i(p_i(w))$ and $rank_i(p_i(w)) \geq rank_{i-1}(p_{i-1}(w))$. For all elements not in A, their p are not changed in the ith operation. Thus, we conclude ①③ holds.

Case 3. UNION(y, z) is performed in the ith operation. Let f be the root of the tree contains y, and let g be the root of the tree contains z. UNION(y, z) is acutally perfroming the following sequence

$$\langle \text{FIND-SET}(y), \text{FIND-SET}(z), \text{LINK}(f, g)$$

By case 2, we know IH holds after FIND-SET. Then we assume ①②③ holds for $(i-1)$th operation and we are performing LINK(f, g) in the ith operation. There are three subcases: $rank_{i-1}(f) > rank_{i-1}(g)$, $rank_{i-1}(f) < rank_{i-1}(g)$, and $rank_{i-1}(f) = rank_{i-1}(g)$; we notice the first two are similar, so we ignore the first subcase. First, we suppose that $rank_{i-1}(f) < rank_{i-1}(g)$. Notice that f is the only node that p attribute is changed in the ith operation. We have $p_{i-1}(f) = f$ and $p_i(f) = g$. Then

$$rank_{i-1}(p_{i-1}(f)) = rank_{i-1}(f) < rank_{i-1}(g) = rank_{i-1}(p_i(f))$$

Since no rank changes in this subcase, we have $rank_i(f) < rank_i(p_i(f))$ and $rank_i(p_i(f)) \geq rank_{i-1}(p_{i-1}(f))$. Thus, we conclude ①②③ holds. Now, we suppose that $rank_{i-1}(f) = rank_{i-1}(g)$. By the similar approach to the last subcase, we have

$$rank_{i-1}(p_{i-1}(f)) = rank_{i-1}(f) = rank_{i-1}(g) = rank_{i-1}(p_i(f))$$

By line 5 of the procedure, we have $rank_i(g) = rank_{i-1}(g) + 1$, and ranks of all nodes except g remain same in the ith operation. Then

$$rank_{i-1}(p_{i-1}(f)) = rank_i(f) = rank_{i-1}(f) = rank_{i-1}(g) = rank_i(g) - 1 < rank_i(g) = rank_i(p_i(f))$$

Hence ①②③ holds for f. Since $p_{i-1}(g) = p_i(g) = g$, we have

$$rank_{i-1}(p_{i-1}(g)) = rank_{i-1}(g) = rank_i(g) - 1 < rank_i(g) = rank_i(p_i(g))$$

Hence ①②③ holds for g (①② are vacuously true). For all nodes other than f and g, rank and p attributes remain same in ith operation. Thus, we conclude ①②③ holds. □

21.4-2

Lemma 3. If there exist a node has rank of k, then there exists at least 2^k nodes in the forest.

Proof. *(Base)* If there exist a node has rank of 0, there is at least one node in the forest.

(Induction) Suppose that the lemma is true for $k = h$ for some $h \geq 0$. Consider $k = h + 1$. If there exist a node has rank of $h + 1$, then, by line 4 of LINK, there must were at least two node have rank of h before. By the inductive hypothesis, there exists at least $2 \cdot 2^k = 2^{k+1}$ nodes in the forest. □

Claim 4. In an n disjoint sets using union by rank and path compression heuristic, every node has rank at most $\lfloor \lg n \rfloor$.

Proof. Suppose that their exist a node has rank of $\lfloor \lg n \rfloor + 1$, for the purpose of contradiction. By the lemma, there exists at least $2^{\lfloor \lg n \rfloor + 1}$ nodes in the forest.

$$2^{\lfloor \lg n \rfloor + 1} > 2^{\lg n} = n$$

Contradiction. □

21.4-3

We need $\lceil \lg(k+1) \rceil$ bits to store value k. Hence $\lceil \lg(\lfloor \lg n \rfloor + 1) \rceil$ bits are necesary to store $x.rank$.

21.4-4

Proof. Taking advantage of Lemma 21.7, we convert the sequence into a sequence of m MAKE-SET, LINK, and FIND-SET. Without path compression, rank of each node is exactly the height of the node. Observed MAKE-SET and LINK take $O(1)$ time, and FIND-SET(x) takes $O(x.rank)$ time. Hence the sequence run in $O(m \lg n)$ time. □

21.4-5

No. Consider $x.rank = 1$, $x.p.rank = 7$, and $x.p.p.rank = 8$. Clearly, $x.rank > 0$ and $x.p$ is not a root. Since

$$A_2(x.rank) = A_2(1) = 7$$

and

$$A_3(x.rank) = A_3(1) = 2047 \quad ,$$

we have

$$A_2(x.rank) \le x.p.rank < A_3(x.rank) \quad ,$$

so

$$\text{level}(x) = 2 \quad .$$

Since

$$A_0(x.p.rank) = A_0(7) = 8$$

and

$$A_1(x.p.rank) = A_1(7) = 15 \quad ,$$

we have

$$A_0(x.p.rank) \le x.p.p.rank < A_1(x.p.rank) \quad ,$$

so

$$\text{level}(x.p) = 0 \quad .$$

Therefore,

$$\text{level}(x) > \text{level}(x.p) \quad .$$

21.4-6

Since $A_3(1) = 2047$, $\alpha'(n) \leq 3$ implies

$$A_3 = 2047 \geq \lg(n+1) \quad .$$

Then

$$n \leq 2^{2047} - 1 = \frac{2^{2048}}{2} - 1 = \frac{(2^4)^{512}}{2} - 1 = 16^{511} - 1 \gg 10^{80} \quad .$$

Replace all $\alpha(n)$ with $\alpha'(n)$ in the argument. The only modification we need to make is at the bound (21.1). We claim that

$$0 \leq \text{level}(x) < \alpha'(n) \quad .$$

We need to modify the argument to prove $\text{level}(x) < \alpha'(n)$.

$$\begin{aligned}
A_{\alpha'(n)}(x.rank) &\geq A_{\alpha'(n)}(1) && \text{(bacause } A_k(j) \text{ is strictly increasing)} \\
&\geq \lg(n+1) && \text{(by the definition of } \alpha'(n)\text{)} \\
&> \lfloor \lg(n) \rfloor \\
&\geq x.p.rank && \text{(by exercise 21.4-2)}
\end{aligned}$$

Chapter 21 Problems

21-1

(a)

$[4, 3, 2, 6, 8, 1]$

(b)

Proof. *(Base)* Let $j \in \{1, 2, \cdots, m\}$, and let $t \in K_j$. Suppose that $extracted[k] = t$ for some $k \in \{1, 2, \cdots, j-1\}$, for the purpose of contradiction. Then $extracted[k]$ is extracting the value from K_j where $j > k$. Contradiction.

(Induction) Assume the value is removed from the set after we extract it. Suppose that ①

$$\forall j \in \{1, 2, \cdots, m\}, \forall t \in K_j, extracted[k] = t \text{ for some } k \in \{j, j+1, \cdots, m\} \text{ or } t \text{ will not be extracted} \quad ,$$

and suppose that ② any values in $\{1, 2, \cdots, i-1\}$ have been extracted correctly (hence these values are removed from $\bigcup_{j \in \{1,2,\cdots,m\}} K_j$). Then at this time, i is the smallest value that is not in the *extracted* array. Now, we determine j such that $i \in K_j$. By the hypothesis ①, we know that $extracted[k] = i$ for some $k \in \{j, j+1, \cdots, m\}$ or i will not be extracted. Since $extracted[j]$ extracted value early than $extracted[j+1], extracted[j+2], \cdots, extracted[m]$, we know $extracted[j]$ extracted the smallest possible value i, and this value is what OFF-LINE-MINIMUM will choose, so we have shown that i is extracted correctly. Hence ② holds. Let l be the smallest value greater than j for which set K_l exists (i.e. *extracted*$[l]$ was empty). Let $A = K_j$ for facilitating our analysis to K_j after destroying K_j. Now, we performed $K_l = K_k \cup K_l$ and destroyed K_j, we claim the hypothesis ① holds. By the way we choosed l, we knew $extract[j], extracted[j+1], \cdots, extracted[l-1]$ were nonempty, so for all $t \in A$, $extracted[k] = t$ only if $k \neq j, j+1, \cdots, l-1$. Then for all $t \in A$, $extracted[k] = t$ for some $k \in \{j, j+1, \cdots, m\}$ or t will not be extracted. Hence ① holds. □

(c)

```cpp
struct Node
{
    int p;
    int rank;
    // set info:
    int subsequence_index_lower;// only root
    int subsequence_index_upper;// only root
    int prev;// only root
    int next;// only root
};

int FindSet(std::vector<Node>& forest, int x)
{
    if (forest[x].p != x)
        forest[x].p = FindSet(forest, forest[x].p);
    return forest[x].p;
}

// keep y's set info
void Link(std::vector<Node>& forest, int x, int y)
{
    forest[forest[x].prev].next = forest[x].next;
    forest[forest[x].next].prev = forest[x].prev;
    if (forest[x].rank > forest[y].rank)
    {
        forest[y].p = x;
        forest[x].subsequence_index_lower = forest[y].subsequence_index_lower;
        forest[x].subsequence_index_upper = forest[y].subsequence_index_upper;
        forest[x].prev = forest[y].prev;
        forest[x].next = forest[y].next;
```

```
        forest[forest[x].prev].next = x;
        forest[forest[x].next].prev = x;
    }
    else
    {
        forest[x].p = y;
        if (forest[x].rank == forest[y].rank)
            ++forest[y].rank;
    }
}

void Union(std::vector<Node>& forest, int x, int y)
{
    return Link(forest, FindSet(forest, x), FindSet(forest, y));
}

// operations: E represents by -1
// n: domain size
std::vector<int> OffLineMinimum(const std::vector<int>& operations, int n)
{
    int i, j, m, root, last_root;
    std::vector<Node> forest(n + 1);// index start by 1
    // init disjoin-set forest
    for (i = 1; i <= n; ++i)
    {
        forest[i].p = i;
        forest[i].rank = 0;
    }
    // init subsequence
    m = 1;
    root = 0;
    last_root = 0;
    for (i = 0; i < operations.size(); ++i)
    {
        if (operations[i] < 0)
        {
            // extract
            forest[root].subsequence_index_upper = m;
            ++m;
            last_root = root;
```

```
        }
        else if (last_root == root)
        {
            // first insert in the subsequence
            root = operations[i];
            forest[root].subsequence_index_lower = m;
            forest[root].prev = last_root;
            forest[last_root].next = root;
        }
        else
        {
            // non-first insert in the subsequence
            forest[operations[i]].p = root;
            forest[root].rank = 1;
        }
    }
    m = forest[last_root].subsequence_index_upper;
    forest[last_root].next = 0;// for situation of that the last operation is extract
    forest[0].subsequence_index_lower = m + 1;
    // compute extracted array
    std::vector<int> extracted(m);
    for (i = 1; i <= n; ++i)
    {
        root = FindSet(forest, i);
        j = forest[root].subsequence_index_lower;
        if (j <= m)
        {
            extracted[j - 1] = i;
            if (forest[root].subsequence_index_lower < forest[root].subsequence_index_upper)
                ++forest[root].subsequence_index_lower;
            else
                Union(forest, root, forest[root].next);
        }
    }
    return extracted;
}
```

In the worst-case, the running time is $O(n\alpha(n))$.

21-2

(a)

Proof. Suppose that we performed n times MAKE-TREE. To maximize the depth we performed $n-1$ times GRAFT to form a tree where the depth of leaf is $n-1$ (the tree become a linked list contains all n nodes). Then we performed $m-2n+1$ FIND-DEPTH on the leaf. Each of MAKE-TREE and GRAFT takes $\Theta(1)$ time, and each of FIND-DEPTH takes $\Theta(n)$ time. Hence the sequence takes

$$\underbrace{n \cdot \Theta(1)}_{\text{MAKE-TREE}} + \underbrace{(n-1) \cdot \Theta(1)}_{\text{GRAFT}} + \underbrace{(m-2n+1) \cdot \Theta(n)}_{\text{FIND-DEPTH}} = O(mn)$$

time in total. Consider $n = \frac{m}{3}$. In this case, the sequence takes $\Theta(m^2)$ time. □

(b)

```
void MakeTree(Node *x)
{
    x->p = x;
    x->rank = 0;
    x->d = 0;
}
```

(c)

```
// after the function return, v->p is the root of the tree in the disjoint sets
void Compression(Node *v)
{
    if (v->p != v->p->p)
    {
        Compression(v->p);
        v->d += v->p->d;
        v->p = v->p->p;
    }
}

// after the function return, v->p is the root of the tree in the disjoint sets
int FindDepth(Node *v)
{
    Compression(v);
    return (v == v->p) ? v->d : (v->d + v->p->d);
}
```

(d)

```
void Graft(Node *r, Node *v)
{
    Node *r_set, *v_set;
    Compression(r);
    // r_set is the root node of the disjoint set contains r
    r_set = r->p;
    // add depths of all nodes in the tree contains node r by the depth of node v
    // correctness: disjoint set contains r is
    //              exactly the set contains all elements in the tree contains r
    r_set->d += (FindDepth(v) + 1);
    // v_set is the root node of the disjoint set contains v
    v_set = v->p;
    if (r_set->rank > v_set->rank)
    {
        v_set->p = r_set;
        // since r_set becomes parent of v_set in the disjoint set,
        // we need to subtract pseudodistance of v_set by that of r_set
        v_set->d -= r_set->d;
    }
    else
    {
        r_set->p = v_set;
        // since v_set becomes parent of r_set in the disjoint set,
        // we need to subtract pseudodistance of r_set by that of v_set
        r_set->d -= v_set->d;
        if (r_set->rank == v_set->rank)
            ++v_set->rank;
    }
}
```

(e)

$\Theta(m\alpha(n))$ where n is the number of nodes in the forest.

21-3

(a)

Proof. LCA is actually doing a postorder tree walk (the walk executes lines 8 - 10 for the root after doing so for the subtrees). Then each pair in P will be processed twice at line 8, 9. Note that the procedure blanken the node after the node is visted. Let $\{u, v\}$ be an arbitrary choice. WLOG, assume u is visited before v. At the time of u is being visiting, v has not been visited, so $v.color \neq BLACK$, and line 10 does not execute for the pair $\{u, v\}$, therefore. At the time of v is being visiting, v has already been visited, so $v.color == BLACK$, and line 10 executes for the pair $\{u, v\}$, therefore. Thus, line 10 executes exactly once for each pair $\{u, v\} \in P$. □

(b)

Proof. Denote $height(u)$ as the height of the node u in T, and denote $depth(u)$ as the depth of the node u in T.

(Base) Let $u \in T$ where $height(u) = 0$. Then u do not have any descendant. Suppose that at the time of the call LCA(u), the number of sets is $depth(u)$. Then at the time of LCA(u) return, the number of sets is $depth(u) + 1$, where the new set is from Make-Set(u) at line 1. Thus the inductive hypothesis (see below) holds for all $u \in T$ where $height(u) = 0$.

(Induction) Inductive hypothesis: for all $u \in T$ where $height(u) \leq k$, if at the time of the call LCA(u), the number of sets is $depth(u)$, then at the time of the call LCA(v) where v is a descendant of u, the number of sets is $depth(v)$, and at the time of LCA(u) return, the number of sets is $depth(u) + 1$.

Let $u \in T$ where $height(u) = k + 1$. Suppose that at the time of the call LCA(u), the number of sets is $depth(u)$. Then at the time after line 2 of LCA(u) is executed, the number of sets is $depth(u) + 1$. We claim that after the **for** loop (line 3 - 6), the number of sets is $depth(u) + 1$. At line 4, we call LCA(v) where v is a child of u. Then $height(v) < height(u)$, so $height(v) \leq k$. And the number of sets is $depth(u) + 1 = depth(v)$. By the inductive hypothesis, at the time of LCA(v) return, the number of sets is $depth(v) + 1 = (depth(u) + 1) + 1 = depth(u) + 2$. And we have at the time of the call LCA(w) where w is a descendant of v, the number of sets is $depth(w)$. At line 5, we perform UNION(x, y), so at this time, the number of sets is $depth(u) + 2 - 1 = depth(u) + 1$. Thus, after the **for** loop, the number of sets is $depth(u) + 1$, and the number of sets is same at the time of LCA(u) return. Hence the inductive hypothesis holds for all $u \in T$ where $height(u) \leq k + 1$.

(Conclusion) By mathmatical induction, since at the time of the call LCA($T.root$), the number of sets is $depth(T.root) = 0$, and all nodes in T is a descendant of $T.root$, we conclude at the time of the call LCA(u), the number of sets is $depth(u)$. □

(c)

Denote T_u as the subtree rooted at u.

Lemma 5. Suppose that the program is currently running in LCA(a) (i.e. the top of the call stack is LCA(a)) at the end of the line 6. Let B be the set contains all black color children of a. Let $A = a \cup \bigcup_{b \in B} T_b$. Then the disjoint set of node a is exactly set A and FIND-SET(a).$ancestor = a$.

Proof. *(Base)* Let $a \in T$ where $height(a) = 0$. Then a does not have any children. Thus, at the end of line 6 (consider the start of line 7 here) of LCA(a), the disjoint set of node a only contains

a, and FIND-SET$(a).ancestor = a$ by line 2.

(Induction) Suppose that the lemma is true for all a where $height(a) \leq k$. Let $a \in T$ where $height(a) = k + 1$. Let B be the set contains all black color children of a. Then for all $b \in B$, $height(b) \leq k$. Since $b.color$ is black, the colors of all children of b are black. By the hypothesis, all elements in T_b are in a same disjoint set. At line 5 - 6, we union all elements in T_b with a. Thus, the disjoint set of node a is $a \cup \bigcup_{b \in B} T_b$. We have FIND-SET$(a).ancestor = a$ by line 6 immediately. □

Claim 6. Suppose the program is currently running in LCA(u) (i.e. the top of the call stack is LCA(u)). Then for all $v \in T$ where $u \neq v$, if $v.color$ is BLACK, then FIND-SET$(v).ancestor$ is the least common ancestor of u and v.

Note that it is impossible to have v to be a proper ancestor of u.

Proof. Let $u, v \in T$ where $u \neq v$, and let a be the least common ancestor of u and v.

Suppose that the program is currently running in LCA(a), u has not been visited, and v has been visited. Then $u.color$ is WHITE, and $v.color$ is BLACK. Let $b \in T$ such that b is a child of a and v is a descendant of b (i.e. $v \in T_b$). Since the program is currently running in LCA(a), $b.color$ must be BLACK also. By the lemma, we have FIND-SET(v) = FIND-SET(a) and FIND-SET$(v).ancestor$ = FIND-SET$(a).ancestor = a$.

Suppose that the program keeps running and is currently running in LCA(u). When the top of the call stack is not LCA(u) and the call stack still contains LCA(u), the disjoint set contains a must remains same. Thus, FIND-SET$(v).ancestor$ and FIND-SET$(a).ancestor$ still equal to a. □

(d)

operation	location	times
MAKE-SET	line 1	$\lvert T\rvert$
FIND-SET	line 2	$\lvert T\rvert$
UNION	line 5	$\lvert T\rvert - 1$
FIND-SET	line 6	$\lvert T\rvert - 1$
FIND-SET	line 10	$\lvert P\rvert$ (by part (a))

Hence LCA takes $O((|T| + |P|)\alpha(|T|))$. Note that we can bound $|P|$ from above:

$$|P| \leq \binom{|T|}{2} = \Theta(|T|^2)$$

Then we can say that LCA takes $O(|T|^2\alpha(|T|))$ also.