# Wavelets

## 4C8: Digital Media Processing

Ussher Assistant Professor François Pitié

2021/2022

Department of Electronic & Electrical Engineering , Trinity College Dublin

*adapted from original material written by Prof. Anil Kokaram.*

# Overview

This lecture develops the idea of the Haar Transform into an introduction to wavelet theory. The goals here are to

- introduce Filter Banks
- introduce the 1D and 2D discrete wavelet transform
- observe the use of wavelets for compression
- introduce the use of wavelet analysis for de-noising
- understand that the wavelet decomposition is not necessarily shift-invariant

# Filter-Banks

# Haar: 2-band filter bank

Recall the 1-D Haar transform from earlier lectures.

$$\begin{bmatrix} y(1) \\ y(2) \end{bmatrix} = T \begin{bmatrix} x(1) \\ x(2) \end{bmatrix} \qquad \text{where } T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

now, if the sequence is long enough, the results can be placed in two separate sequences $y0$ and $y1$:

$$y_0(n) = \frac{1}{\sqrt{2}} \left( x(n-1) + x(n) \right)$$

$$y_1(n) = \frac{1}{\sqrt{2}} \left( x(n-1) - x(n) \right)$$

These can be expressed as 2 FIR filters:

$$Y_0(z) = H_0(z)X(z) \qquad \text{where} \quad H_0(z) = \frac{1}{\sqrt{2}}(z^{-1} + 1)$$

$$Y_1(z) = H_1(z)X(z) \qquad \text{where} \quad H_1(z) = \frac{1}{\sqrt{2}}(z^{-1} - 1)$$

# Haar: Analysis

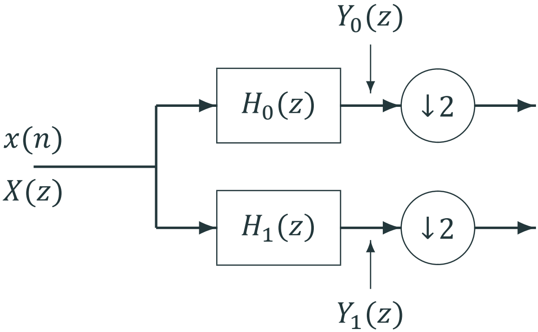
In Haar, every pair of input samples $(x_n,\ x_{n-1})$ is transformed into an output pair $(y_n,\ y_{n-1})$.

All pairs $(x_n, x_{n-1}),\ (x_{n-2}, x_{n-3})$ are treated separately. Thus the total number of output samples is the same as the total number of input samples. Thus only the output $y_0(n), y_0(n-2), y_0(n-4), \cdots$ and $y_1(n), y_1(n-2), y_1(n-4), \cdots$ are in fact ever calculated.

$$
\begin{aligned}
y_n &= y_0(n) \\
y_{n+1} &= y_1(n) \\
y_{n+2} &= y_0(n+2) \\
y_{n+3} &= y_1(n+2) \\
&\vdots
\end{aligned}
$$

We may thus represent the Haar transform operation by a pair of filters followed by downsampling (see next slide).

# Haar: Analysis



# Haar: Reconstruction

Similarly, to reconstruct $\hat{x}(n)$ from $\hat{y}(n)$, we can write for long sequences:

$$\hat{x}(2n-1) = \frac{1}{\sqrt{2}}\left(\hat{y}_0(2n) + \hat{y}_1(2n)\right)$$
$$\hat{x}(2n) = \frac{1}{\sqrt{2}}\left(\hat{y}_0(2n) - \hat{y}_1(2n)\right)$$

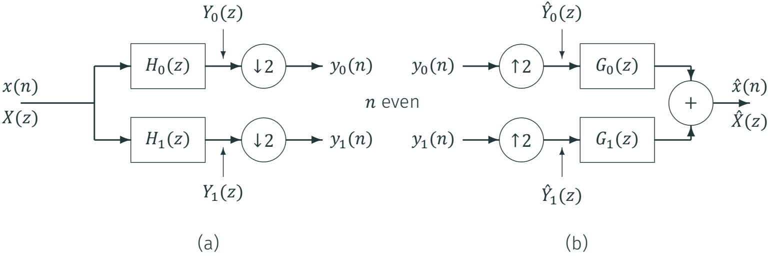
$\hat{y}_0(n)$ and $\hat{y}_1(n)$ are only calculated at even values of $n$, thus we may assume $\hat{y}(2n+1) = 0$ and we can now write:

$$\hat{x}(n) = \frac{1}{\sqrt{2}}[\hat{y}_0(n+1) + \hat{y}_0(n)] + \frac{1}{\sqrt{2}}[\hat{y}_1(n+1) - \hat{y}_1(n)]$$

or as z-transforms:

$$\hat{X}(z) = G_0(z)\,\hat{Y}_0(z) + G_1(z)\,\hat{Y}_1(z) \quad \text{with} \quad \begin{cases} G_0(z) = \frac{1}{\sqrt{2}}(z+1) \\ G_1(z) = \frac{1}{\sqrt{2}}(z-1) \end{cases}$$

# Haar: 2-Band Filter Bank




Two-band filter banks for analysis (a) and reconstruction (b).

# Haar: upsampling/downsampling

Let's look at the upsampling/downsampling part:

$$Y_0(z) \longrightarrow \downarrow 2 \longrightarrow \uparrow 2 \longrightarrow \hat{Y}_0(z)$$

We can show (next slide) that $Y_0(z)$ is simply related to $\hat{Y}_0(z)$ as follows:

$$\hat{Y}_0(z) = \frac{1}{2}[Y_0(z) + Y_0(-z)]$$

Let's convince ourselves that this is the case. Let's first expand the $z$-transforms $Y_0(z)$ and $Y_0(-z)$:

$$\frac{1}{2}[Y_0(z) + Y_0(-z)] = \frac{1}{2}\sum_n y_0(n)\ z^{-n} + y_0(n)\ (-z)^{-n}$$

Then split the even and odd terms:

$$= \frac{1}{2}\sum_n y_0(2n)\ z^{-2n} + y_0(2n+1)\ z^{-2n-1} +$$

$$y_0(2n)\ (-z)^{-2n} + y_0(2n+1)\ (-z)^{-2n-1}$$

$$= \sum_n y_0(2n)\ z^{-2n}$$

On the other hand we have:

$$\hat{Y}_0(z) = \sum_n \hat{y}_0(n) z^{-n} = \sum_n \hat{y}_0(2n) z^{-2n} + \hat{y}_0(2n+1) z^{-2n-1}$$

since $\hat{y}_0(2n) = y_0(2n)$ and $\hat{y}_0(2n+1) = 0$.

$$\hat{Y}_0(z) = \sum_n \hat{y}_0(2n) z^{-2n}$$

Thus have indeed $\frac{1}{2}[Y_0(z) + Y_0(-z)] = \hat{Y}_0(z)$

A note here: we've already seen that with resampling.

Recall that the Fourier Transform for frequency $\omega$ radians/pixel can be computed by taking the $z$-Transform at $z = e^{j\omega}$.

Hence

$$X(-z) = X(-e^{j\omega}) = X\left(e^{j(\omega+\pi)}\right)$$

The term $X(-z)$ then corresponds to a copy of the original spectrum shifted by $\pi$. Now $\pi$ corresponds to half the sampling frequency. Thus,

$X(-z)$ is simply that replication of the spectrum at half the sampling frequency that occurs when downsampling by 2.

# Perfect Reconstruction

We are not restricted to using the Haar filters, and we could get better compression with more elaborate filters.

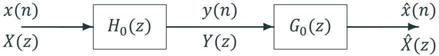
We need to choose 2 low pass filters ($H_0(z)$ and $G_0(z)$) and 2 high pass filters ($H_1(z)$ and $G_1(z)$).

But the reconstruction filters must give exactly the same signal as the input to the analysis filter bank.

This is known as perfect reconstruction (PR).

# Perfect Reconstruction (PR)

Let's stop for a second and realise that PR is non-trivial by considering the 1-band case:




For the perfect reconstruction to hold, we need to have

$$G_0(z)H_0(z) \equiv 1 \quad \text{and thus} \quad G_0(z) = \frac{1}{H_0(z)}$$

which is simply not possible to achieve with non-trivial FIR filters: if $H_0$ is FIR then $G_0$ is IIR. Also if $H_0$ cuts off frequencies, then there is no way for $G_0$ to recover these lost frequencies.

Thus PR is non-trivial and it is only by splitting the signal into multiple bands that we can achieve PR with FIR filters.

# Perfect Reconstruction

Let's go back to the two-band problem. We are now able to generalise our analysis for arbitrary filters $H_0$, $H_1$, $G_0$ and $G_1$.

$$
\begin{aligned}
\hat{X}(z) = & \ \frac{1}{2}G_0(z)\hat{Y}_0(z) + \frac{1}{2}G_1(z)\hat{Y}_1(z) \\
= & \ \frac{1}{2}G_0(z)[Y_0(z) + Y_0(-z)] + \frac{1}{2}G_1(z)[Y_1(z) + Y_1(-z)] \\
= & \ \frac{1}{2}G_0(z)H_0(z)X(z) + \frac{1}{2}G_0(z)H_0(-z)X(-z) \\
& + \frac{1}{2}G_1(z)H_1(z)X(z) + \frac{1}{2}G_1(z)H_1(-z)X(-z) \\
= & \ \frac{1}{2}X(z)[G_0(z)H_0(z) + G_1(z)H_1(z)] \\
& + \frac{1}{2}X(-z)[G_0(z)H_0(-z) + G_1(z)H_1(-z)]
\end{aligned}
$$

# Perfect Reconstruction

The Perfect Reconstruction (PR) condition requires $\hat{X}(z) \equiv X(z)$, which is given by:

$$G_0(z)H_0(z) + G_1(z)H_1(z) \equiv 2 \tag{1}$$

and

$$G_0(z)H_0(-z) + G_1(z)H_1(-z) \equiv 0 \tag{2}$$

The Identity (1) is known as the no-distortion condition because the term in $X(z)$ corresponds to distortion caused by the filters.

The Identity (2) is known as the anti-aliasing condition because the term in $X(-z)$ is the unwanted aliasing term caused by down-sampling $\mathbf{y}_0$ and $\mathbf{y}_1$ by 2.

# PR for the Haar Wavelet

Let's see how the Perfect Reconstruction works for Haar.

$$H_0(z) = \frac{1}{\sqrt{2}}(z^{-1} + 1) \qquad G_0(z) = \frac{1}{\sqrt{2}}(z + 1)$$

$$H_1(z) = \frac{1}{\sqrt{2}}(z^{-1} - 1) \qquad G_1(z) = \frac{1}{\sqrt{2}}(z - 1)$$

no-distortion:

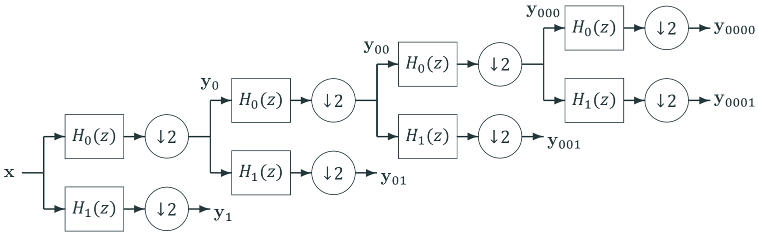
$$\begin{aligned} G_0(z)H_0(z) + G_1(z)H_1(z) &= \frac{1}{2}(z + 1)(z^{-1} + 1) + \frac{1}{2}(z - 1)(z^{-1} - 1) \\ &= \frac{1}{2}(z^{-1} + z + 2) + \frac{1}{2}(-z^{-1} - z + 2) = 2 \end{aligned}$$

no-aliasing:

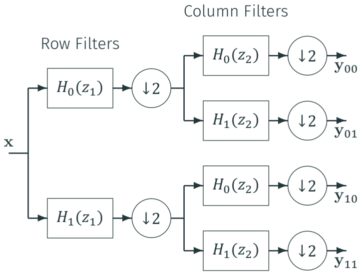
$$\begin{aligned} G_0(z)H_0(-z) + G_1(z)H_1(-z) &= \frac{1}{2}(z + 1)(-z^{-1} + 1) + \frac{1}{2}(z - 1)(-z^{-1} - 1) \\ &= \frac{1}{2}(-z^{-1} + z) + \frac{1}{2}(+z^{-1} - z) = 0 \end{aligned}$$

# Discrete Wavelet Transform

We may achieve greater compression if the low band is further split into two. This may be repeated a number of times to give the binary filter tree.

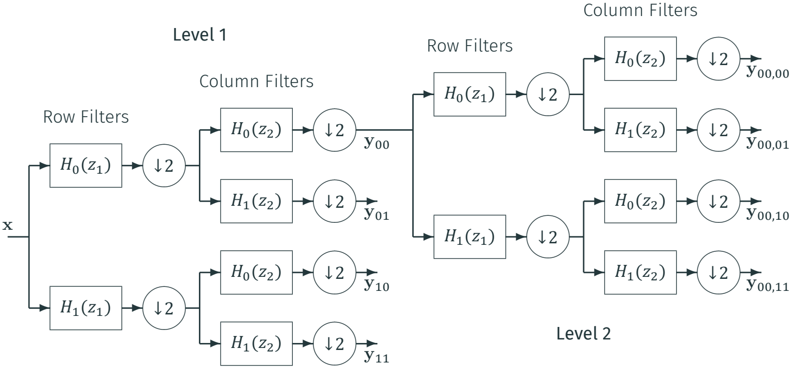


# 2D Wavelet Transform




$y_{00}$ is mapped to the LoLo, $y_{01}$ to the HiLo, $y_{10}$ to the LoHi and $y_{11}$ to the HiHi.

# The Multilevel 2D Discrete Wavelet Xform




Two levels of a 2-D filter tree, formed from 1-D lowpass ($H_0$) and highpass ($H_1$) filters.

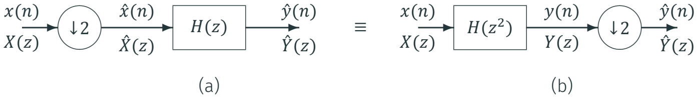
# What does it do to a signal?

We need to work out the impulse response of each filter output $\mathbf{y}_1$, $\mathbf{y}_{01}$, $\mathbf{y}_{001}$, $\mathbf{y}_{000}$, etc.

But for this, we need to understand the combined effect of the down-sample operations.

# Multi-Rate Filtering Theorem

The downsample–filter operation of (a) is equivalent to the filter–downsample of (b) if the filter is changed from $H(z)$ to $H(z^2)$.



# Multi-Rate Filtering Theorem: Proof

From (a), we derive

$$\hat{y}_{(a)}[n] = \sum_i \hat{x}[n-i]\ h[i] = \sum_i x[2n-2i]\ h[i]$$

From (b), we derive

$$\hat{y}_{(b)}[n] = y[2n] = \sum_i x[2n-i]\ h'[i]$$

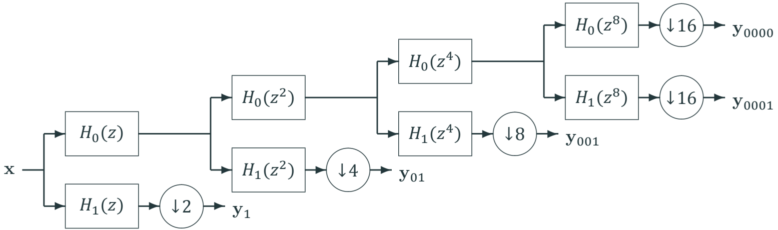
where $h'$ is the impulse response of $H(z^2)$.

$$H(z^2) = \sum_n h'[n] z^{-n} = \sum_n h[n] z^{-2n}$$

note that $h'[2n] = h[n]$ and $h'[2n+1] = 0$, thus

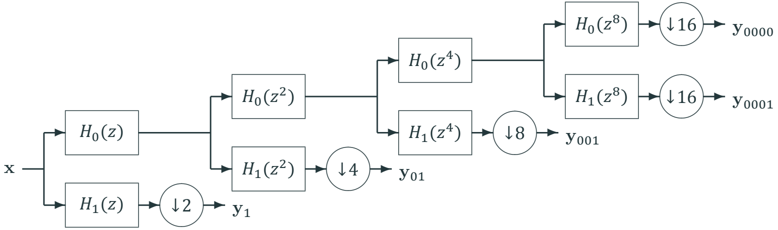
$$\begin{aligned}\hat{y}_{(b)}[n] &= \sum_i x[2n-i]\ h'[i] = \sum_i x[2n-2i]\ h'[2i] + x[2n-2i]\ h'[2i+1] \\ &= \sum_i x[2n-2i]\ h[i] = \hat{y}_{(a)}[n]\end{aligned}$$

# What does it do to a signal?




Binary filter tree, transformed so that all downsampling operations occur at the outputs.

# What does it do to a signal? (Haar)



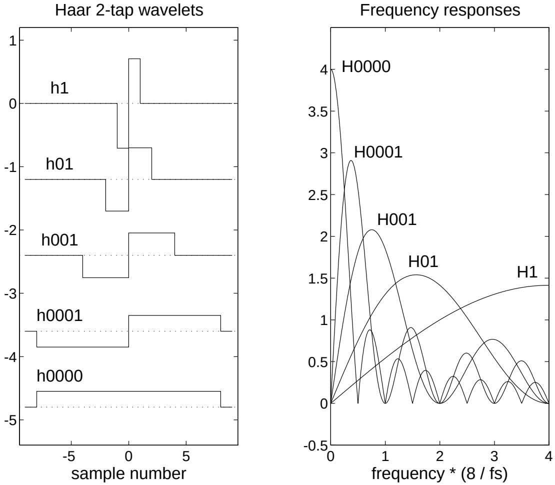

$$
\begin{aligned}
H_{01}(z) &= H_0(z)\, H_1(z^2) \\
H_{001}(z) &= H_0(z)\, H_0(z^2)\, H_1(z^4) \\
H_{0001}(z) &= H_0(z)\, H_0(z^2)\, H_0(z^4)\, H_1(z^8) \\
H_{0000}(z) &= H_0(z)\, H_0(z^2)\, H_0(z^4)\, H_0(z^8)
\end{aligned}
$$

# What does it do to a signal? (Haar)

Transfer functions to the outputs of the 4-level Haar:

$$
\begin{aligned}
H_{01}(z) &= \frac{1}{2}\,[\,(z^{-3}+z^{-2})-(z^{-1}+1)\,]\\
H_{001}(z) &= \frac{1}{2\sqrt{2}}\,[\,(z^{-7}+z^{-6}+z^{-5}+z^{-4})-(z^{-3}+z^{-2}+z^{-1}+1)\,]\\
H_{0001}(z) &= \frac{1}{4}\,[\,(z^{-15}+z^{-14}+z^{-13}+z^{-12}+z^{-11}+z^{-10}+z^{-9}+z^{-8})\\
&\qquad -\ (z^{-7}+z^{-6}+z^{-5}+z^{-4}+z^{-3}+z^{-2}+z^{-1}+1)\,]\\
H_{0000}(z) &= \frac{1}{4}\,[\,z^{-15}+z^{-14}+z^{-13}+z^{-12}+z^{-11}+z^{-10}+z^{-9}+z^{-8}\\
&\qquad +\ z^{-7}+z^{-6}+z^{-5}+z^{-4}+z^{-3}+z^{-2}+z^{-1}+1\,]
\end{aligned}
$$

# Impulse Responses (Haar)




Impulse responses and frequency responses of the 4-level Haar filters.

# Wavelet Transform

The process of creating $y_1$, $y_{01}$, etc. is the Wavelet Transform.

The "Wavelet" refers to the impulse response of the cascade of filters.

The shape of impulse response is similar at each level. It is derived from something called a "Mother wavelet".

The low pass Impulse response to level k is called the "scaling function at level k".

# Good Wavelets for Compression

There are better filters than the Haar filters.

Wavelet filter design is art and science, and we won't go into this at all in this course. You will just be exposed to a couple of wavelets that are used in the literature. But there are many wavelets! Only some are good for compression and others for analysis.

# Wavelet Design

Recall the PR conditions:

$$G_0(z)H_0(z) + G_1(z)H_1(z) \equiv 2$$
$$G_0(z)H_0(-z) + G_1(z)H_1(-z) \equiv 0$$

The usual way of satisfying the anti-aliasing condition is to set:

$$H_1(z) = z^{-k}G_0(-z) \qquad \text{and} \qquad G_1(z) = z^k H_0(-z)$$

where $k$ must be odd. Now we define the lowpass product

$$P(z) = H_0(z)G_0(z)$$

and the no-distortion condition becomes:

$$P(z) + P(-z) = 2$$

which means that $P(z)$ is of the form $P(z) = 1 + \sum_n p_{2n+1} z^{-2n-1}$. And since $P(z)$ should also be zero phase, $P(z)$ should be symmetric:

$$P(z) = \cdots + p_3 z^{-3} + p_1 z^{-1} + 1 + p_1 z^{-1} + p_3 z^{-3} + \cdots$$

# Wavelet Design

The design of a set of $H_0(z)$, $H_1(z)$, $G_0(z)$, $G_1(z)$ filters can now be summarized as follows:

1. choose $p_1$, $p_3$, etc. to design a zero-phase lowpass filter $P(z)$ with some desired qualities.
2. factorize $P(z)$ into $H_0(z)$ and $G_0(z)$ to have similar frequency response
3. derive $H_1(z)$ and $G_1(z)$ from previous slide equation

# Wavelet Design

Wavelet filter design is clearly non-trivial for filters of any use. A Belgian mathematician, Ingrid Daubechies, did much pioneering work on wavelets in the 1980s. She discovered that to achieve smooth wavelets after many levels of the binary tree, the lowpass filters $H_0(z)$ and $G_0(z)$ must both have a number of zeros at $z = -1$.

The simplest case is

$$P(z) = \frac{1}{2}(z + 2 + z^{-1}) = \frac{1}{2}(z+1)(1+z^{-1}) = G_0(z)\, H_0(z)$$

which gives the familiar Haar filters.

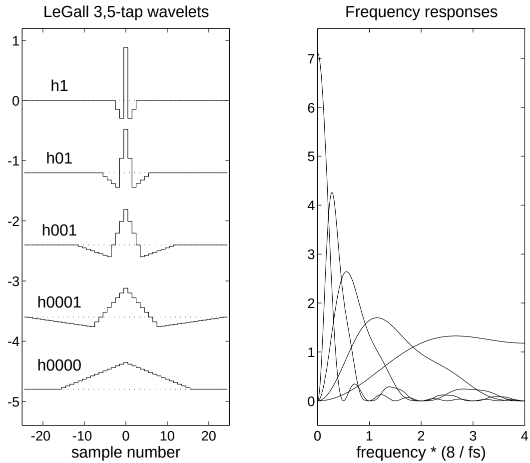
# LeGall 3,5

The LeGall 3,5-tap filter set. Given that name because it was first published in the context of 2-band filter banks by Didier LeGall in 1988.

$$
\begin{aligned}
H_0(z) &= \frac{1}{2}(z + 2 + z^{-1}) \\
G_0(z) &= \frac{1}{8}(z + 2 + z^{-1})(-z + 4 - z^{-1}) \\
&= \frac{1}{8}(-z^2 + 2z + 6 + 2z^{-1} - z^{-2})
\end{aligned}
$$

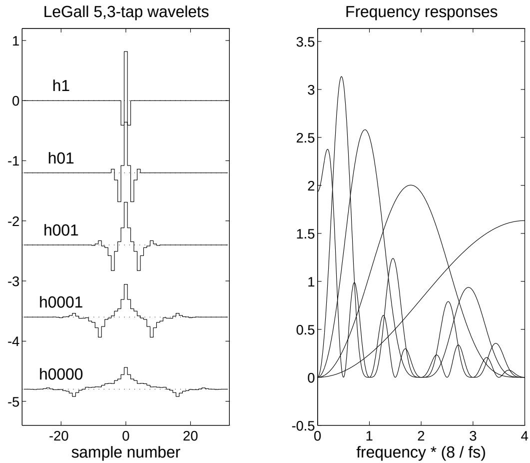
$$
\begin{aligned}
G_1(z) = z\ H_0(-z) \ &= \ \frac{1}{2}z(-z + 2 - z^{-1}) \\
H_1(z) = z^{-1}\ G_0(-z) \ &= \ \frac{1}{8}z^{-1}(-z^2 - 2z + 6 - 2z^{-1} - z^{-2})
\end{aligned}
$$

# LeGall 3,5




Impulse responses and frequency responses of the 4-level tree of LeGall 3,5-tap filters.

# LeGall 5,3 (Inverse LeGall)




Impulse and freq responses of the LeGall 5,3-tap filters. These are the reconstruction filters of the LeGall 3,5, formed from the $G_0$, $G_1$ filters.

# LeGall filters

The LeGall 3,5 filters are pretty good for image processing/analysis because of the smooth nature of the analysis filters and they are symmetric.

But the reconstruction filters are not smooth.

It turns out that you can swap the analysis and reconstruction filters around and use $G_{\mathbf{0}}$, $G_{\mathbf{1}}$ as analysis filters and $H_{\mathbf{0}}$,$H_{\mathbf{1}}$ as reconstruction filters. This is called the LeGall 5,3.

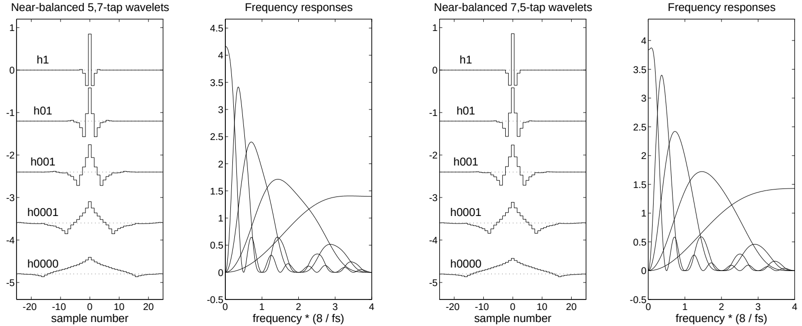
# Unbalanced System

Unbalance between analysis and reconstruction filters is also termed a bi-orthogonal transformation.

It is often regarded as being undesirable, particularly as it prevents the filtering process from being represented as an orthogonal transformation of the input signal.

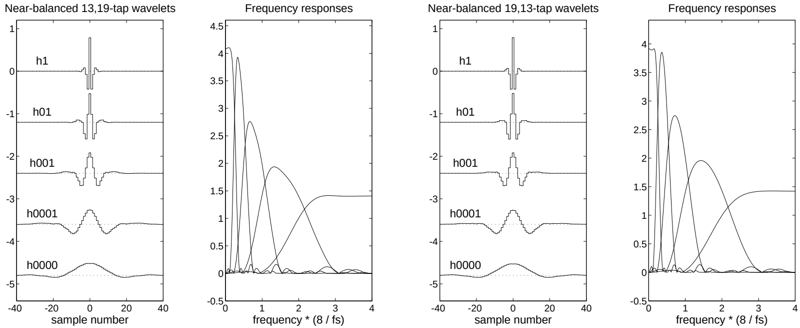
We can obtain more balanced system by using longer filters.

# 5,7 filters




near-balanced 5,7 filters. Left: analysis, Right: reconstruction.

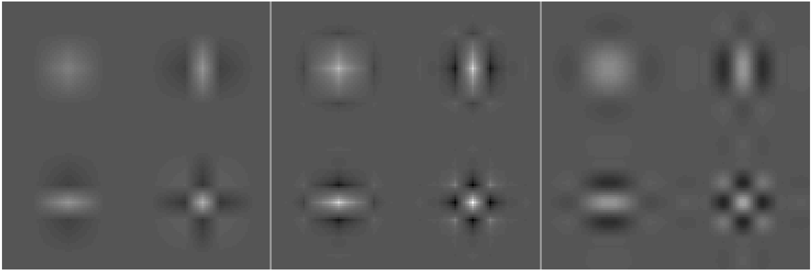
# 13,19 filters




near-balanced 13,19 filters. Left: analysis, Right: reconstruction.

# 2D Impulse responses of the separable filters

Legall 3,5-tap, and near balaced 5,7-tap and 13,19-tap 2-D wavelets at level 4



# Coding with Wavelets

# Coding with Wavelets: Examples

Haar with $Q_{step} = 30$
RMS: 5.24
Entropy: 0.845 bits/pixel

Legall 5,3 with $Q_{step} = 30$
RMS: 4.66
Entropy: 0.752 bits/pixel

# Coding with Wavelets: Examples

CDF 9,7 with $Q_{step} = 30$
RMS: 4.97
Entropy: 0.741 bits/pixel

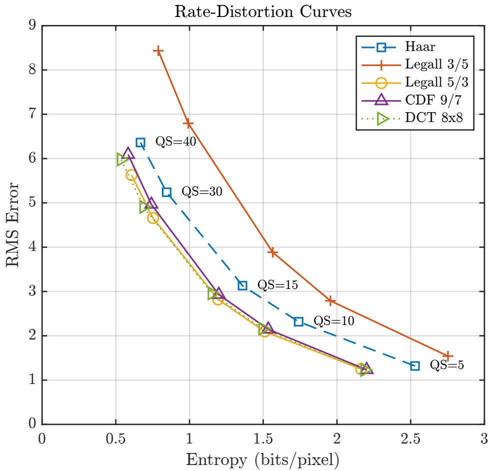
Legall 5,3 with $Q_{step} = 30$
RMS: 4.66
Entropy: 0.752 bits/pixel

# Coding with Wavelets: Examples

Legall 3,5 with $Q_{step} = 30$
RMS: 6.79
Entropy: 0.991 bits/pixel

Legall 5,3 with $Q_{step} = 30$
RMS: 4.66
Entropy: 0.752 bits/pixel

# Rate Distortion Curve for kodim19



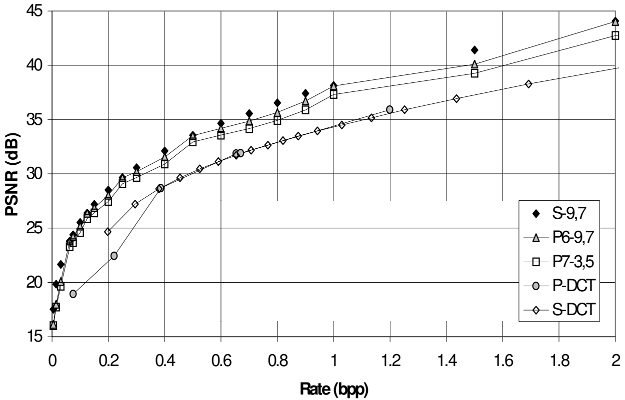

RMS error vs. entropy for some wavelets on our kodim19 image. There is not much here in between the DCT/legall 3,5 and CDF 9,7.

# Rate Distortion Curve for kodim19

Now, this is the RD graph for a single example. Also, we only used a single $Q_{step}$, and, as for JPEG, we probably should have a different $Q_{step}$ per subband. Also, we need to take into account the entropy coding as well!

So, in next slide is a better RD-curve comparison, where the quality/bitrates measured on actual bitstreams of JPEG and JPEG 2000 have been reported for a range of images. Results show that the Wavelet transform (JPEG 2000) gives us average quality gains of +3 dB to +4 dB when compared to the DCT (JPEG).

# Rate Distortion Curve: baseline JPEG vs JPEG 2000




JPEG (for two schemes of DCT) vs JPEG2000 (using 9/7 and 3/5 filters). From *Marcellin et al., "An overview of JPEG2000"*. Instead of the RMS error, we use here the PNSR. Higher PSNR means better visual quality. We can see that on a larger pool of images, the wavelet transforms 9/7 and 3/5 perform better than the DCT.

# Shift-invariance

# Shift-invariance

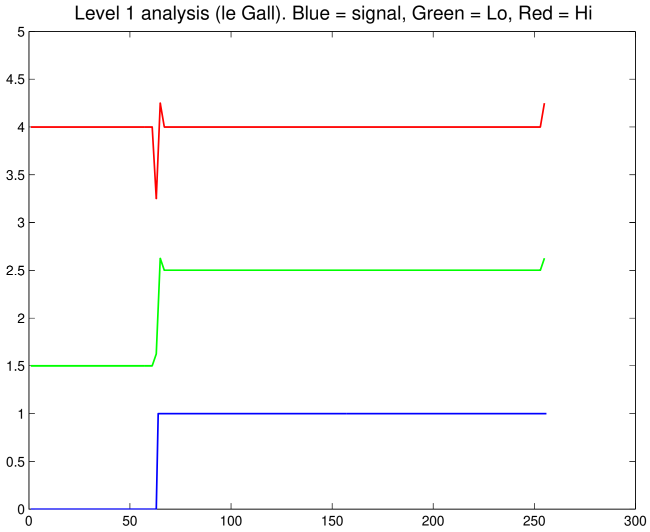
The price for decimation is aliasing.

Remember that the high-pass filters are downsampled. It works because the Perfect reconstruction condition makes sure that aliasing is cancelled by the end of the reconstruction.

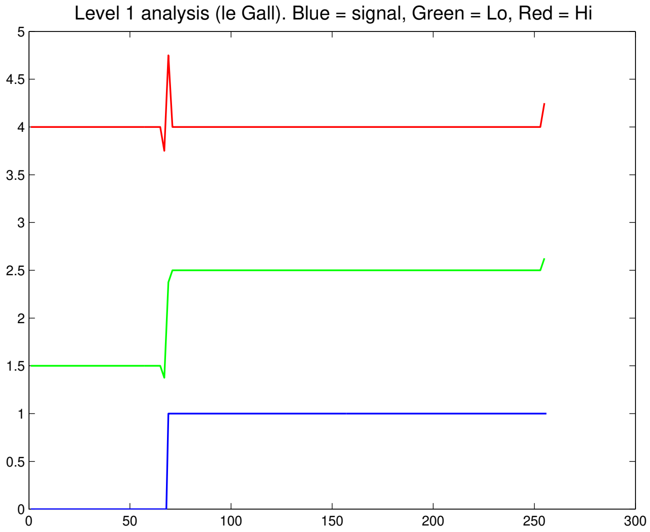
However the coefficients are aliased.

This means that DWT is shift variant! A small shift of the signal greatly perturbs the coefficient pattern at discontinuities. This is not good for analysis tasks such as pattern recognition for instance. In practice it doesn't matter too much for compression.

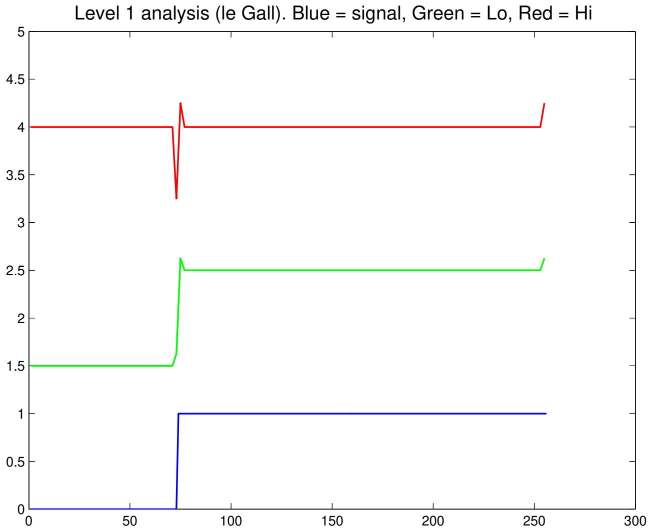
# A Shift-Variance example



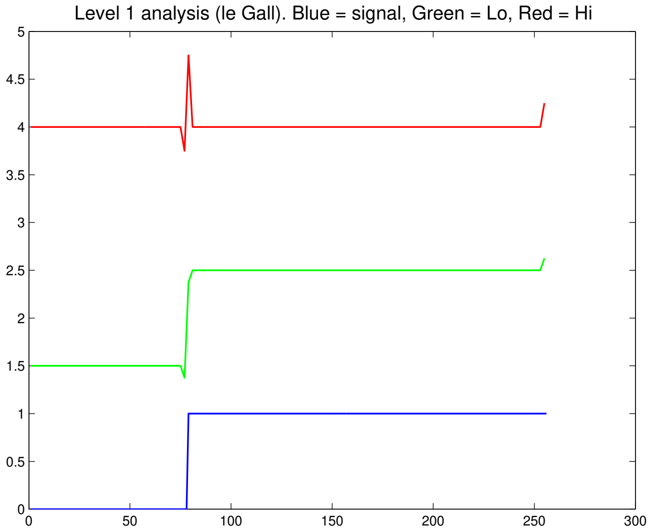
# A Shift-Variance example



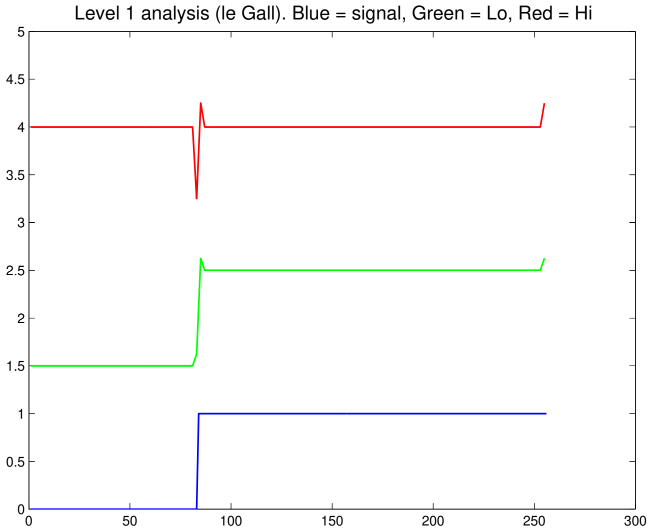
# A Shift-Variance example



# A Shift-Variance example



# A Shift-Variance example



# The solution

The solution to preserve shift-invariance is not to downsample and keep all the coefficient undecimated.

This creates a huge redundancy. As you have 4 bands per level,

$$\#\text{coeffs} = 4 \times \#\text{levels} \times \#\text{pixels}$$

OR we can use Nick Kingsbury's Dual-Tree Complex Wavelet Transform, which approximate the shift invariance at a computational cost close to normal DWT.