Image Processing: Transforms (part 2)

4C8: Digital Media Processing

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adapted from original material written by Prof. Anil Kokaram.

Transforms Overview

- Sampling Theorem
- Downsampling
- Filter Design & Examples
- The need for non-linear filters

Additional reference for this handout can be found in the book *Two-Dimensional Signal and Image Processing by Jae. S. Lim and* Applied Digital Signal Processing by Manolakis and Ingle.

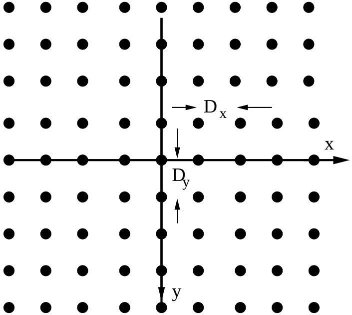
Sampling Theorem

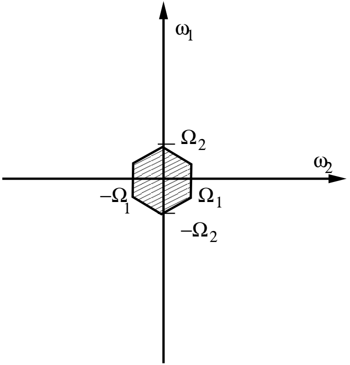
Sampling Theorem in 2D (similar arguments to 1D)

Consider a signal $f(x,y) \rightleftarrows F(\omega_1,\omega_2)$, we wish to represent $f(x,y)$ using a discrete sequence $f[m,n]$ where $f[m,n] = f(mD_x, nD_y)$. The ideal model for sampling considers multiplication of $f(x,y)$ by a grid of delta functions $s(x,y)$ where

$$s(x,y) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(x - n_1 D_x, y - n_2 D_y) \tag{1}$$

where D_x, D_y are the separations between points on the sampling grid i.e. the *period* of horizontal and vertical sampling. Sampling (or digitization) is modelled as $f_s(x,y) = f(x,y)s(x,y)$. Now we want to find out the relationship between $F(\omega_1,\omega_2)$ and $F_s(\omega_1,\omega_2)$.





Left: $s(x, y)$: each dot is a δ function coming out of the plane of the page. Right: $F(\omega_1, \omega_2)$, we assume that the spectrum of $f(x, y)$ is bandlimited between $\pm\Omega_1$ and $\pm\Omega_2$ respectively.

The derivation is similar to the 1D sampling theorem in 3C1. We start from

$$f_s(x, y) = f(x, y)s(x, y)$$

and realise that it becomes a convolution in the Fourier domain:

$$F_s(\omega_1, \omega_2) = F(\omega_1, \omega_2) \circledast S(\omega_1, \omega_2) \quad (2)$$

To compute the FT $S(\omega_1, \omega_2)$ of $s(x, y)$, it is convenient to realise that $s(x, y)$ is periodic and can be expressed as a Fourier series:

$$s(x, y) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} a_{k_1,k_2} e^{j(k_1 \omega_1^0 x + k_2 \omega_2^0 y)} \tag{3}$$

where $\omega_1^0 = \frac{2\pi}{D_x}$ and $\omega_2^0 = \frac{2\pi}{D_y}$ correspond to the fundamental frequency of $s(x, y)$ in both directions.

$$
\begin{aligned}
a_{k_1,k_2} &= \frac{1}{D_x D_y} \int_{-D_y/2}^{D_y/2} \int_{-D_x/2}^{D_x/2} s(x,y) e^{-j(k_1\omega_1^0 x + k_2 \omega_2^0 y)} dxdy \\
&= \frac{1}{D_x D_y} \int_{-D_y/2}^{D_y/2} \int_{-D_x/2}^{D_x/2} \delta(x,y) e^{-j(k_1\omega_1^0 x + k_2 \omega_2^0 y)} dxdy \\
&= \frac{1}{D_x D_y} \text{ By sifting property of the } \delta \text{ function}
\end{aligned} \tag{4}
$$

Therefore, substituting for a_{k_1,k_2} in equation 3 we can state that

$$
s(x,y) = \frac{1}{D_x D_y} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} e^{j(k_1\omega_1^0 x + k_2\omega_2^0 y)} \tag{5}
$$

and finally:

$$
S(\omega_1,\omega_2) = \frac{1}{D_x D_y} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta\left(\omega_1 - k_1\omega_1^0, \omega_2 - k_2\omega_2^0\right) \tag{6}
$$

(cont.)

$$S(\omega_1, \omega_2) = \frac{1}{D_x D_y} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta\left(\omega_1 - k_1\omega_1^0, \omega_2 - k_2\omega_2^0\right)$$

Thus, as

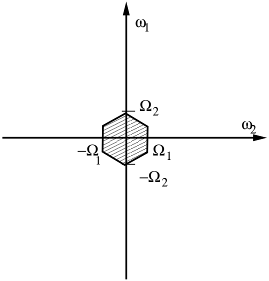
$$F_s(\omega_1, \omega_2) = F(\omega_1, \omega_2) \circledast S(\omega_1, \omega_2)$$

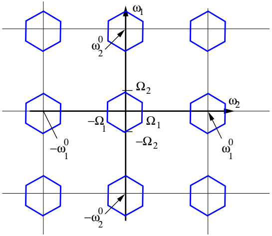
$$F_s(\omega_1, \omega_2) = \frac{1}{D_x D_y} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} F(\omega_1 - k_1\omega_1^0, \omega_2 - k_2\omega_2^0) \qquad (7)$$

Therefore sampling has the effect of scaling and replicating the spectrum.

2D Sampling Theorem

$F(\omega_1, \omega_2)$





$F_s(\omega_1, \omega_2)$

For no overlap in the spectrum to occur, $\omega_1^0 - \Omega_1 > \Omega_1$ and $\omega_2^0 - \Omega_2 > \Omega_2$. Thus we get Nyquist's theorem for 2D (it's similar to 1D).

$$\boxed{\omega_1^0 > 2\Omega_1 \quad \text{AND} \quad \omega_2^0 > 2\Omega_2} \tag{8}$$

for no aliasing to occur. ($\omega_1^0 = 2\pi/D_x$ and $\omega_2^0 = 2\pi/D_y$)

2D Sampling Theorem

Perception of aliasing will vary with the distance of the observer from the display (remember cycles/degree on eye?). So pictures which are not aliased when viewed from far away can become aliased when viewed at closer distances!

[Note: this is a crude simplification of the effects involved.]

Aliasing

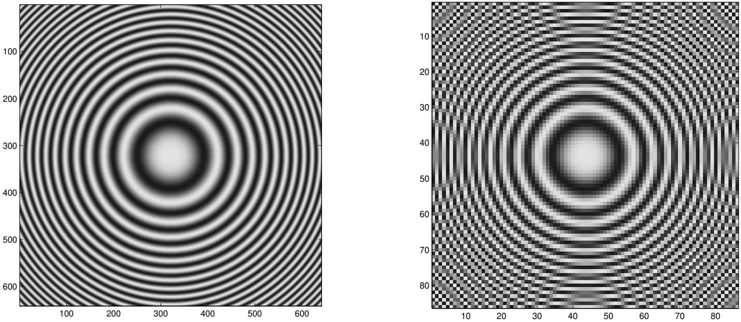
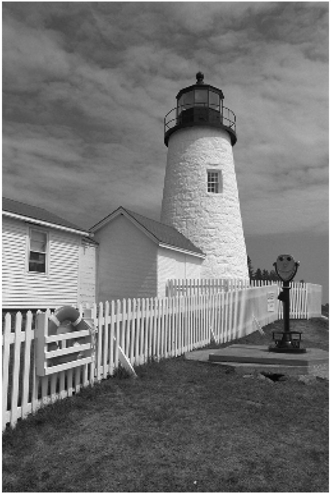


Figure: sinusoidal zone plate image $s(x, y) = \cos(a_x x^2 + a_y y^2)$, $a_x = b_x = 0.0628$

LEFT: zone plate sampled at 5× rate on RIGHT. LEFT image is ok (smooth concentric circles seen), RIGHT image shows aliasing: ghost shapes, pixelation. Effect is worse at the higher frequencies near the border of the image.

Example



Left hand image sampled at $2\times$ higher rate than right.

An Aliasing problem in archive TV and film

Resampling of TV footage recorded on film gives aliased pictures.

2D Discrete Fourier Transform

2D Discrete Fourier Transform

As expected, the 2D DFT is similar to the 1D version and is a **separable** transform.

Given an image of size $M \times N$ pixels, the 2D DFT is defined as

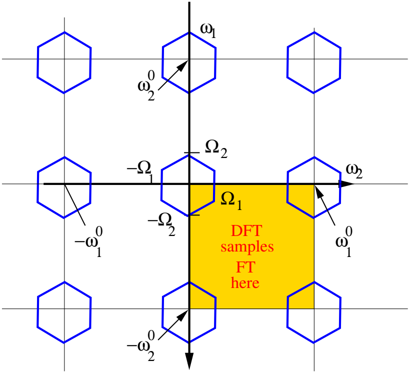
$$F[h,k] = \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m] e^{-j\left(\frac{2\pi}{N}nh + \frac{2\pi}{M}mk\right)}$$

$$f[n,m] = \frac{1}{NM} \sum_{h=0}^{N-1} \sum_{k=0}^{M-1} F[h,k] e^{j\left(\frac{2\pi}{N}nh + \frac{2\pi}{M}mk\right)} \tag{9}$$

with $0 \leq h \leq N-1, 0 \leq k \leq M-1$ and $0 \leq n \leq N-1, 0 \leq m \leq M-1$.

Fast Algorithm exists (Lim pages 163–182). Tedious to derive, read the book if you have to make devices/implementation. Note that most DSP manufacturers supply FFT libraries, TI, Analog, Intel, (Matlab : `fft, fft2`) all do. FFTW (Fastest Fourier Transform in the West), developed at MIT, good for general purpose processors.

Symmetry Properties and other practical stuff



FFT calculates the spectral components at equal intervals up till the sampling frequency. Given $N \times M$ transform: $(0,0)^{th}$ bin is the DC coeff and N^{th} bin $\equiv 2\pi/D_y$ similarly for M^{th} bin.

Symmetry Properties and other practical stuff

Need to shuffle coefficients to get zero freq at centre. See what `fftshift` does.

Symmetry about DC: the DFT is centrally symmetric $F[h,k] = F^*[-h,-k]$.

The transform can be expressed as a matrix operation on rows then columns

$$F = W^T I W$$

Downsampling

Application: Anti-Aliasing filter for factor of N downsampling

We wish to implement factor of N downsampling in an image (decrease resolution by a factor of N). That is we want to take a discrete time signal $f[m,n]$ and create a downsampled signal:

$$f_N[m,n] = f[mN,nN]$$

Application: Anti-Aliasing filter for factor of N downsampling

Recall (from sampling discussion) that the initial sampling rate is D_x, D_y pels per metre for the original sampled signal $f[m,n]$. Assuming the sampling rate is the same in both directions $D_x = D_y = D$ the sampling frequency is $[\Omega_0, \Omega_0] = [(2\pi/D), (2\pi/D)]$ radians/pixel. We want to reduce this to $[\Omega_0/N, \Omega_0/N]$ for factor of N downsampling in both directions to give new signal $f_N[m,n]$.

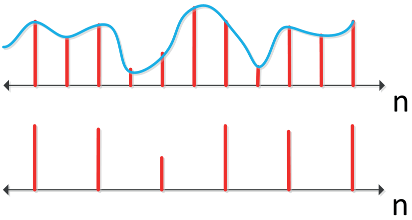


Figure: Equivalent problem in 1D. Top: an analogue signal is sampled at a sampling period of T. Bottom: the same signal downsampled by a factor of 2. Every second value is thrown away.

The spectrum of the sampled signal $f[m,n]$ contains (scaled) copies of the original spectrum $F(\omega_1,\omega_2)$ at integer multiples of $[\Omega_0,\Omega_0]$. If the sampling rate is reduced to $[\Omega_0/N,\Omega_0/N]$ then the replicas are placed closer together in the signal spectrum.

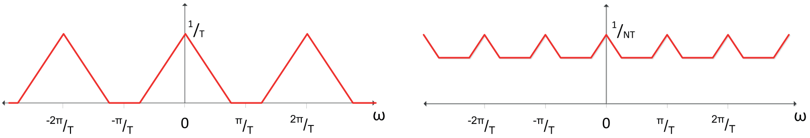
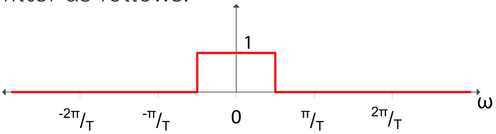


Figure: Left: notional frequency spectrum of the sampled signal. **Right**: spectrum of the downsampled signal (by a factor of 2). As the signal contains frequencies greater than $\frac{\Omega_0}{2N}=\frac{\pi}{2T}$, aliasing will occur.

Thus, to prevent aliasing, we need to need to ensure that there are no spectral components above $[\Omega_0/(2N), \Omega_0/(2N)]$, and apply a low-pass filter as follows:



...after this low-pass filter, downsampling looks like this:

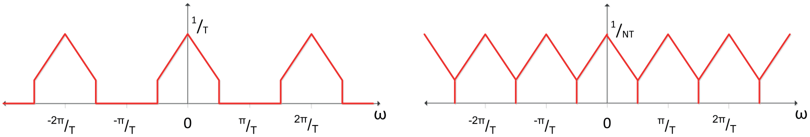


Figure: Left: spectrum of the digital signal after an anti-aliasing filter is applied. **Right**: corresponding signal spectrum after downsampling by 2.

Recall the effect of the original sampling at frequency Ω_0 which was used to obtain $f[m,n]$:

$$F_s(\omega_1,\omega_2) = \frac{1}{D^2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} F(\omega_1 - k_1\Omega_0, \omega_2 - k_2\Omega_0)$$

We want to re-sample the original signal at a higher frequency. Thus we need to re-estimate the original spectrum $F(\omega_1,\omega_2)$. We can do this by cropping the replication at $(0,0)$ whilst cutting off the other replications, and then re-scaling its values.

This can be done with a low-pass filter $H_0(\omega_1,\omega_2) \rightleftarrows h(x,y)$ with frequency response as follows:

$$\begin{aligned} H_0(\omega_1,\omega_2) &= D^2 \mathrm{rect}\left(\frac{\omega_1}{2\Omega_0}, \frac{\omega_2}{2\Omega_0}\right) \\ &= \begin{cases} D^2 & \text{for } |\omega_1| < \Omega_0 \text{ and } |\omega_2| < \Omega_0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Simply applying $F(\omega_1, \omega_2) = H_0(\omega_1, \omega_2) F_s(\omega_1, \omega_2)$ gives us the spectrum of the analogue signal.

Now we want to re-sample the signal at Ω_0/N. To avoid aliasing in the re-sampling stage, we apply an ideal 2D anti-aliasing low-pass filter $H_{\mathcal{N}}(\omega_1, \omega_2)$ with cut-off at $[\pm\Omega_0/(2N), \pm\Omega_0/(2N)]$. The frequency response is as follows:

$$
\begin{aligned}
H_{\mathcal{N}}(\omega_1, \omega_2) &= \mathrm{rect}\left(\frac{\omega_1 \mathcal{N}}{2\Omega_0}, \frac{\omega_2 \mathcal{N}}{2\Omega_0}\right) \\
&= \begin{cases} 1 & \text{for } |\omega_1| < \frac{\Omega_0}{\mathcal{N}} \text{ and } |\omega_2| < \frac{\Omega_0}{\mathcal{N}} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

Combining both effects 1) retrieving the original spectrum by cropping and rescaling and 2) low pass filtering the spectrum to avoid aliasing, we can derive our anti-aliasing filter $H(\omega_1, \omega_2)$ as follows:

$$
\begin{aligned}
H(\omega_1, \omega_2) &= H_0(\omega_1, \omega_2) H_{\mathcal{N}}(\omega_1, \omega_2) \\
&= D^2 \operatorname{rect}\left(\frac{\omega_1}{2\Omega_0}, \frac{\omega_2}{2\Omega_0}\right) \operatorname{rect}\left(\frac{\omega_1 \mathcal{N}}{2\Omega_0}, \frac{\omega_2 \mathcal{N}}{2\Omega_0}\right) \\
&= D^2 \operatorname{rect}\left(\frac{\omega_1 \mathcal{N}}{2\Omega_0}, \frac{\omega_2 \mathcal{N}}{2\Omega_0}\right) \\
&= \begin{cases} D^2 & \text{for } |\omega_1| < \frac{\Omega_0}{\mathcal{N}} \text{ and } |\omega_2| < \frac{\Omega_0}{\mathcal{N}} \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
\tag{10}
$$

The filter $H(\omega_1, \omega_2)$ is a 2D rectangular function in the frequency domain (horizontal and vertical bandwidth of $\Omega = \Omega_0/(2N)$). We've seen that in part 1. Using the Inverse 2D Fourier Transform:

$$\begin{aligned}
h(x, y) &= \frac{1}{4\pi^2} \int \int D^2 \mathrm{rect}\left(\frac{\omega_1}{2\Omega}, \frac{\omega_2}{2\Omega}\right) e^{j(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2 \\
&= \times \frac{D^2}{4\pi^2} \int_{-\Omega}^{\Omega} \int_{-\Omega}^{\Omega} e^{j(\omega_1 x + \omega_2 y)} d\omega_1 d\omega_2 \\
&= \times \frac{D^2}{4\pi^2} \left(\frac{2}{x} \sin(x\Omega)\right) \int_{-\Omega}^{\Omega} e^{j(\omega_1 y)} d\omega_2 \\
&= \times \frac{D^2}{4\pi^2} \frac{2}{x} \sin(x\Omega) \frac{2}{y} \sin(y\Omega) \\
&= \frac{D^2}{4\pi^2} {\color{blue} 2\Omega \mathrm{sinc}(x\Omega)} {\color{green} 2\Omega \mathrm{sinc}(y\Omega)}
\end{aligned} \tag{11}$$

The filter is separable!

$$h(x,y) = \frac{D^2}{4\pi^2} 2\Omega \operatorname{sinc}(x\Omega) 2\Omega \operatorname{sinc}(y\Omega)$$

But need to apply filter in digital domain. Recall samples are at $y = hD$, $x = kD$ and that $\Omega = \Omega_0/2\mathcal{N} = 2\pi/(2D\mathcal{N}) = \pi/(D\mathcal{N})$:

$$\begin{aligned} h[m,n] &= D^2 \times \frac{1}{4\pi^2} \left[\frac{2\pi}{DN} \operatorname{sinc}\left(\frac{mD\pi}{DN}\right)\right] \left[\frac{2\pi}{DN} \operatorname{sinc}\left(\frac{nD\pi}{DN}\right)\right] \\ &= \left[\frac{1}{N} \operatorname{sinc}\left(\frac{m\pi}{N}\right)\right] \left[\frac{1}{N} \operatorname{sinc}\left(\frac{n\pi}{N}\right)\right] \end{aligned}$$

Importantly the filter does not rely on the original sampling frequency Ω_0, only on the sub-sampling factor $\mathcal{N}$.

Using Windowing to create an FIR Filter

This sinc function is infinitely long and so the filter is IIR. So it is necessary to truncate the filter to a finite length of N. However, simply trowing away values of the signal for $|k| > N/2$ will result in a ripple in the passband and stopband of the filter.

So need to multiply impulse response by some data window $w[m,n]$. Use separable hamming window $w[m]w[n]$. Thus

$$h[m,n] = \left[\frac{w[m]}{N}\mathrm{sinc}\left(\frac{m\pi}{N}\right)\right]\left[\frac{w[n]}{N}\mathrm{sinc}\left(\frac{n\pi}{N}\right)\right]$$

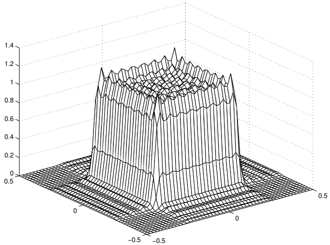
where

$$w[k] = \begin{cases} 0.54 - 0.46\cos(\frac{\pi k}{N}) & |k| < N/2 \\ 0 & \text{Otherwise} \end{cases} \tag{12}$$

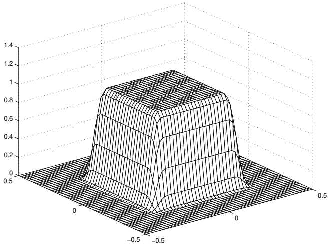
See Lim Chapter 4 for FIR Filter Design and better design method: frequency transformation pages 218–238. Also Applied Digital Signal Processing by Manolakis and Ingle has a good description of windowing and downsampling (1D).

Implementation

Gain of Frequency Response using a Rectangular Window



Gain of Frequency Response using a Hamming Window



The ripples in the passband and stopband are clearly visible when the IIR filter is simply truncated to form an FIR filter (FIR). When using the hamming window, the roll-off rate between the passband and stopband is sacrificed in order to remove the ripple.

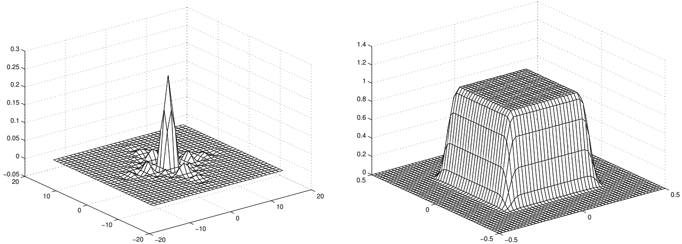
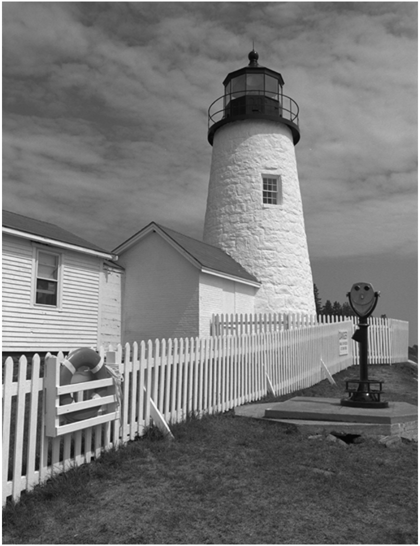


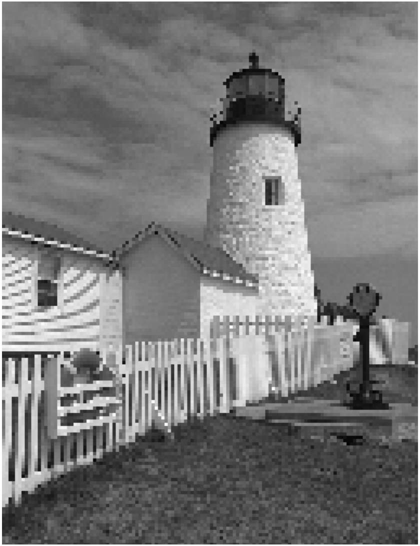
Figure: $h(x, y)$ and $|H(\omega_1, \omega_2)|$

```
N = 2;                      % subsampling factor
n = (-16:16);               % taps
h = (1/(N))*sinc(n/N);      % normalised sinc function = sinc(pi x)
w = hamming(length(n))';    % set window
h = h.*w;                   % window the impulse response
h = h/sum(sum(h));          % Make sure that DC gain of h is 1.0
h2D = h'*h;                 % Make 2D filter from 1D filters.
                            % This is the same as h2D=conv2(h',h);
```

Implementation

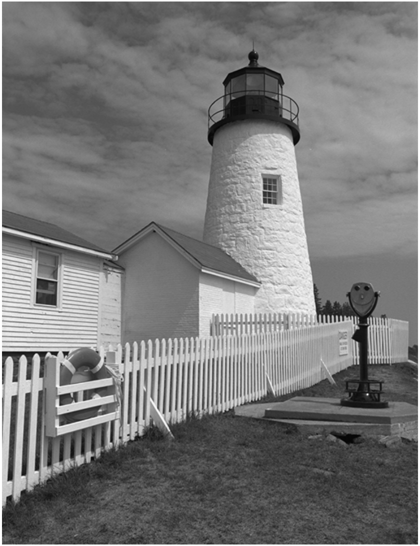


Original



subsampled $N = 2$ without anti-aliasing

Implementation



Original

subsampled $N = 2$ with anti-aliasing

Is this really necessary?

SINC filter has to be quite long to do a reasonable job. (Used 33 taps in previous example). Can cause ringing at edges, bad for image perception.

Can use a Gaussian shaped filter instead (also separable). Results acceptable for many applications.

$$h[m,n] = \exp - \left(\frac{m^2 + n^2}{2\sigma^2} \right) \tag{13}$$

where σ affects the filter bandwidth. Bigger σ implies bigger bandwidth. Need to specify the number of taps and to normalise the filter coeffs so that they sum to 1.0. `h = h/sum(sum(h))`.

we can get away with quite a short filter (15 taps).

BTW Good zooming or upsampling (needed for deinterlacing or TV→Cinema conversion) is *much* harder.

Summary

We looked at sampling of 2D analogue signals and the extension of Nyquist's theorem for 2D signals.

We also presented the 2D Discrete Fourier Transform and analysed the aliasing effect of downsampling a digital image. We designed an ideal anti-aliasing filter that needs to be applied before downsampling discrete signals.