Cash flows and discounting

LIFE INSURANCE PRODUCTS VALUATION IN R

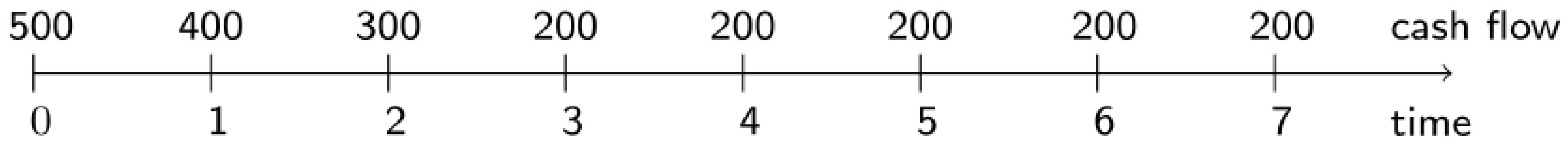
Katrien Antonio, Ph.D.
Professor, KU Leuven and University of Amsterdam

A cash flow

- Fix a capital unit and a time unit:
  - 0 is the present moment;
  - k is k time units in the future (e.g. years, months, quarters).
- Amount of money received or paid out at time k:
  - c_k
  - the cash flow at time k.

A vector of cash flows in R

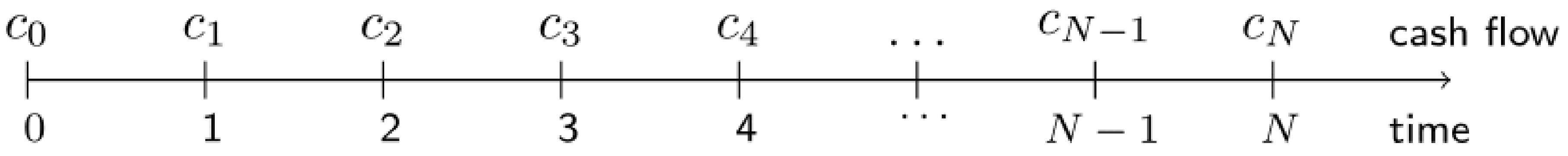
- In R:




```
# Define the cash flows
cash_flows <- c(500, 400, 300, rep(200, 5))
length(cash_flows)
```

```
8
```

- In general: for a cashflow vector $(c_0, c_1, \ldots, c_N)$:



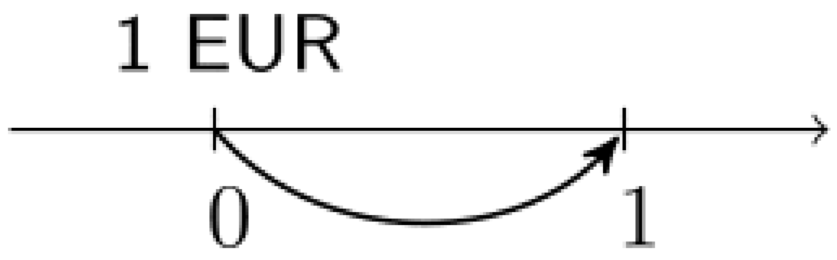
Valuation of a cash flow vector

- Crucial facts:
  - timing of cash flows matters!
  - time value of money matters!
- **Interest rate** determines growth of money.

Interest rate and discount factor

Accumulation

- i is the constant interest rate.

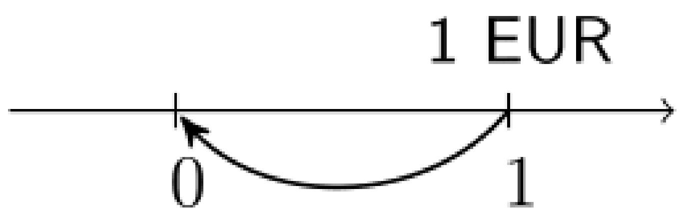



```r
i <- 0.03
1 * (1 + i)
```

```
1.03
```

Discounting

- $v = \dfrac{1}{1+i}$ the discount factor.



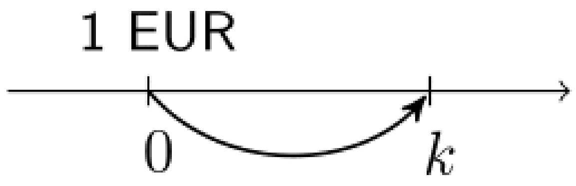

```r
v <- 1 / (1 + i)
v
```

```
0.9708738
```

From one time period to k time periods

- **Accumulation**
  - the value at time k of 1 EUR paid at time 0

$$= (1+i)^k = v^{-k}.$$

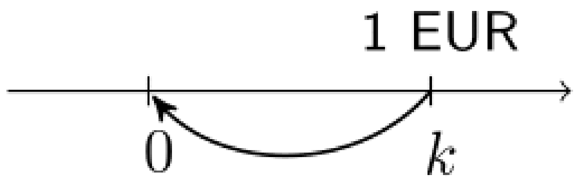



```r
i <- 0.03 ; v <- 1 / (1 + i) ; k <- 2
c((1 + i) ^ k, v  ^ -k)
```

```
1.0609 1.0609
```

- **Discounting**
  - the value at time 0 of 1 EUR paid at time k

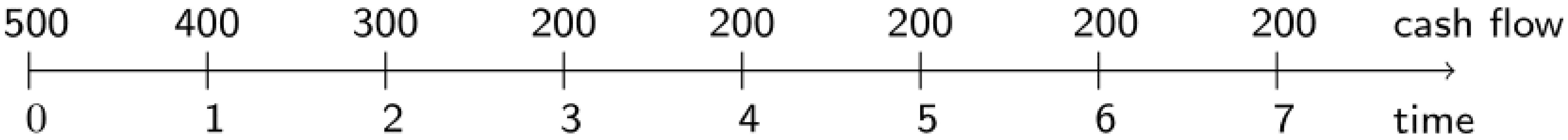
$$= (1+i)^{-k} = v^{k}.$$




```r
i <- 0.03 ; v <- 1 / (1 + i) ; k <- 2
c((1 + i) ^ -k, v  ^ k)
```

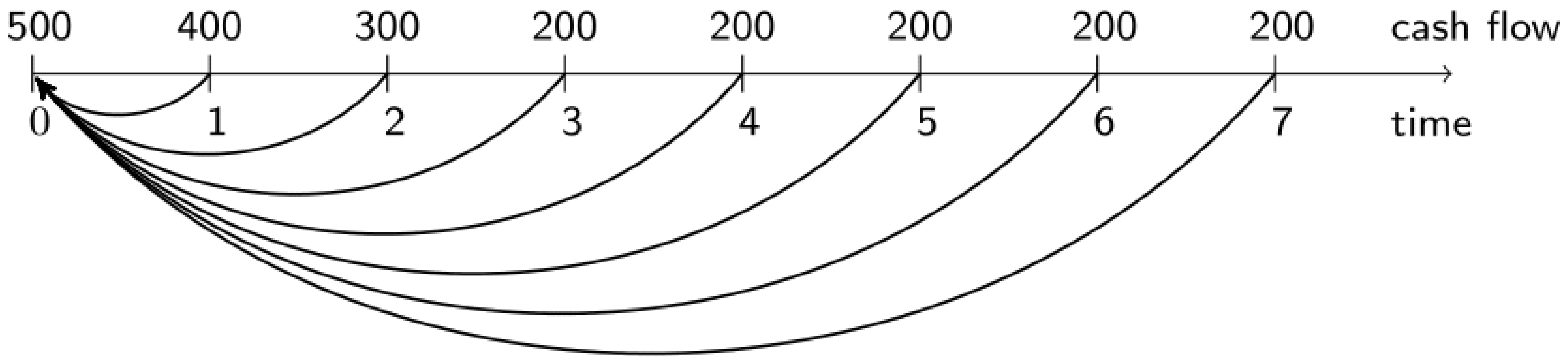
```
0.9425959 0.9425959
```

The present value of a cash flow vector



What is the value at $k = 0$ of this cash flow vector?

The present value of a cash flow vector



What is the value at $k = 0$ of this cash flow vector?

The **present value (PV)**!

The present value of a cash flow vector in R

```
# Interest rate
i <- 0.03
# Discount factor
v <- 1 / (1 + i)
# Define the discount factors
discount_factors <- v ^ (0:7)
# Cash flow vector
cash_flows <-
    c(500, 400, 300, rep(200, 5))
```

```
# Discounting cash flows
cash_flows * discount_factors
```

```
500.0000 388.3495 282.7788 183.0283
177.6974 172.5218 167.4969 162.6183
```

```
# Present value of cash flow vector
present_value <-
    sum(cash_flows * discount_factors)
present_value
```

```
[1] 2034.491
```

Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R

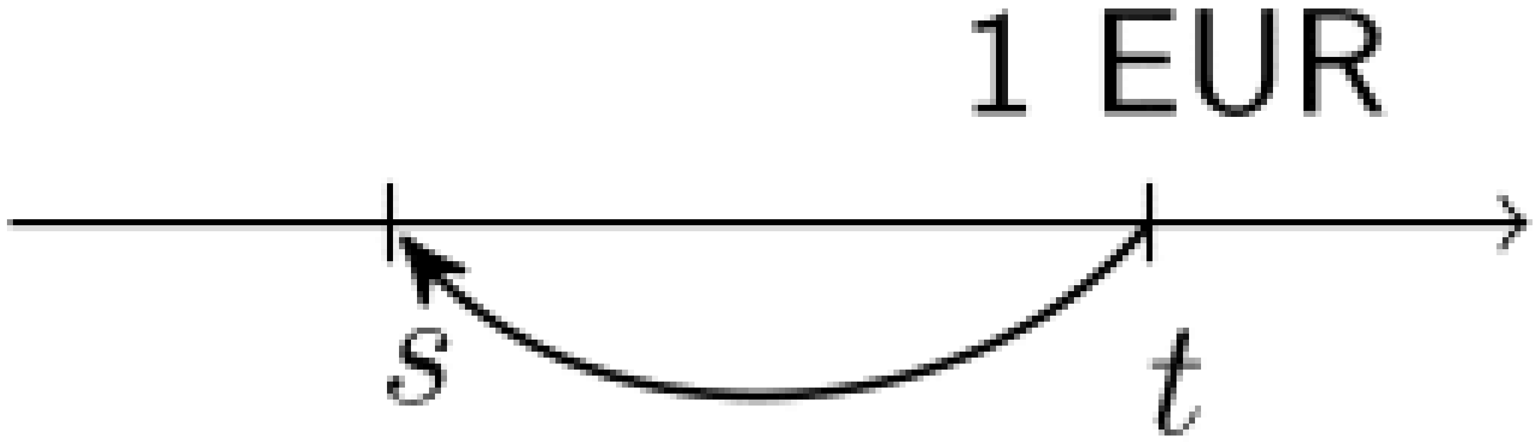
Valuation

LIFE INSURANCE PRODUCTS VALUATION IN R

Roel Verbelen, Ph.D.
Statistician, Finity Consulting

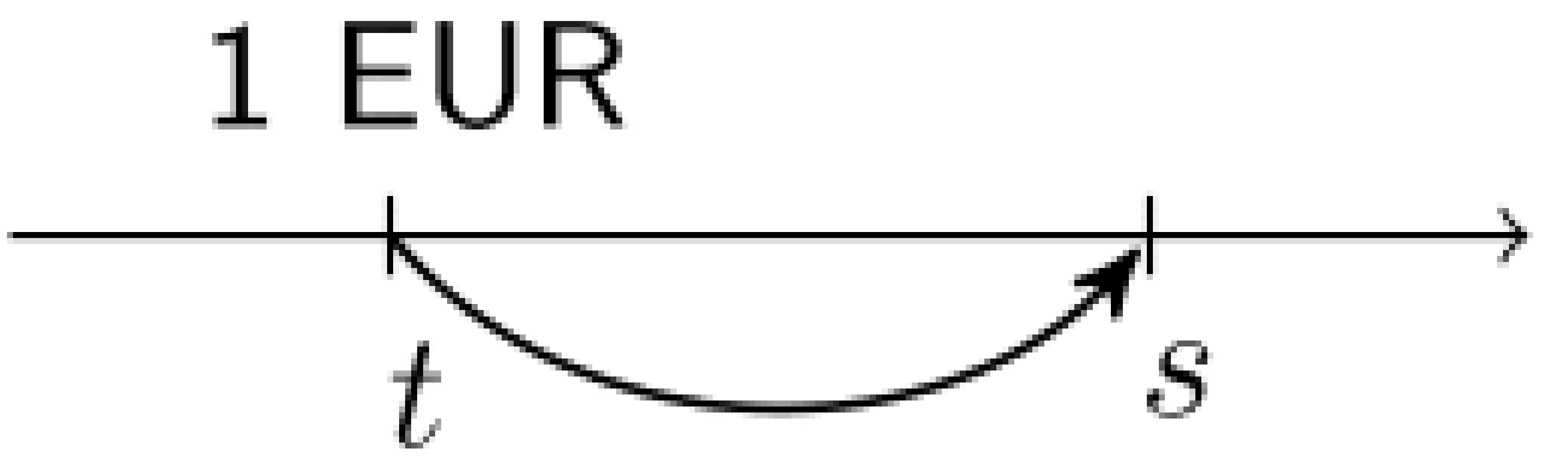
Discount factors

- Denote: $v(s, t)$ the value at time s of 1 EUR paid at time t.
- $s < t$: a discounting factor



Discount factors

- Denote: $v(s,t)$ the value at time s of 1 EUR paid at time t.
- $s > t$: an accumulation factor



Discount factors in R

```
i <- 0.03
v <- 1 / ( 1 + i)
```

With $s < t$: e.g. $s = 2$ and $t = 4$

```
s <- 2
t <- 4
```

```
# v(2, 4) = value at time 2 of 1 EUR paid at time
v ^ (t - s)
```

```
0.9425959
```

```
(1 + i) ^ - (t - s)
```

```
0.9425959
```

Discount factors in R

```r
i <- 0.03
v <- 1 / ( 1 + i)
```

With $s > t$: e.g. $s = 6$ and $t = 3$

```r
s <- 6
t <- 3
```

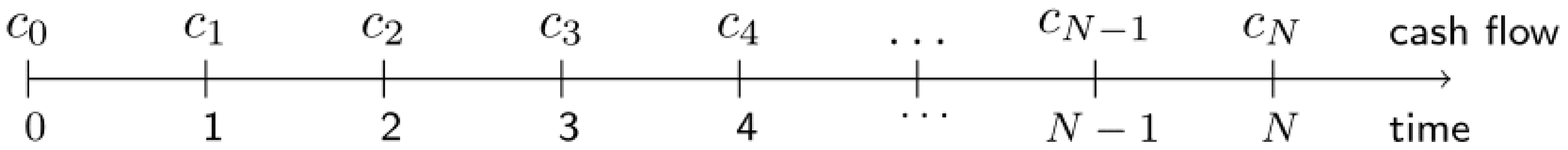
```r
# v(6, 3) = value at time 6 of
#           1 EUR paid at time 3
v ^ (t - s)
```

```
1.092727
```

```r
(1 + i) ^ - (t - s)
```

```
1.092727
```

Valuation of a cash flow vector



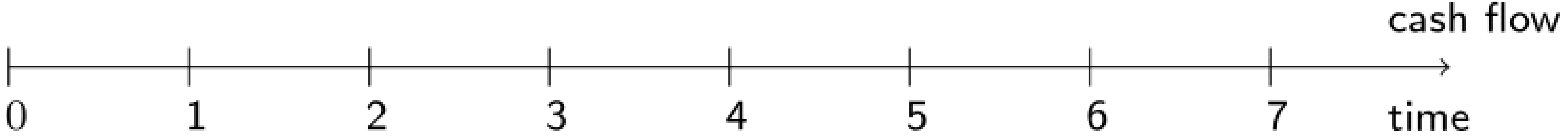
- The value at time n

$$\sum_{k=0}^{N} c_k \cdot v(n,k)$$

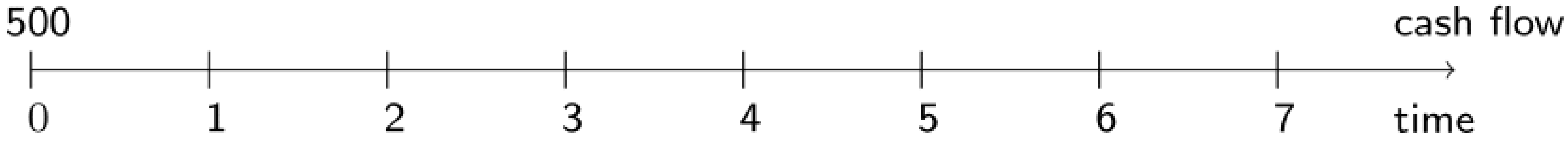
with $0 \leq n \leq N$.

- **Present** Value ($n = 0$) and **Accumulated** Value ($n = N$).

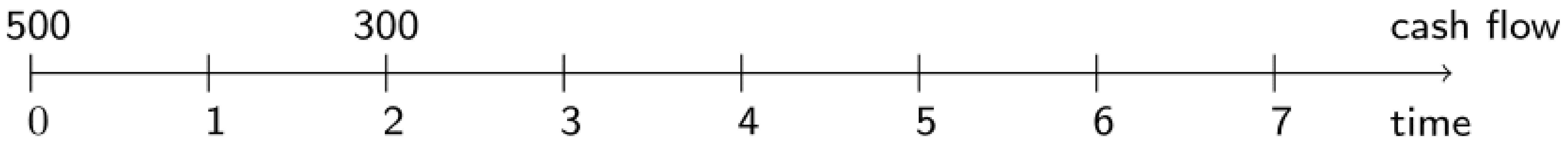
Valuation of a cash flow vector in R



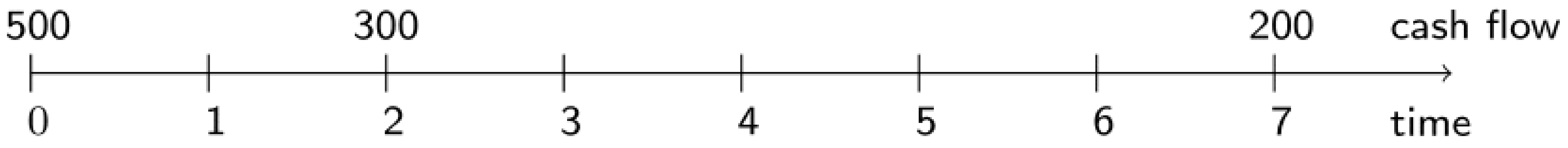
Valuation of a cash flow vector in R



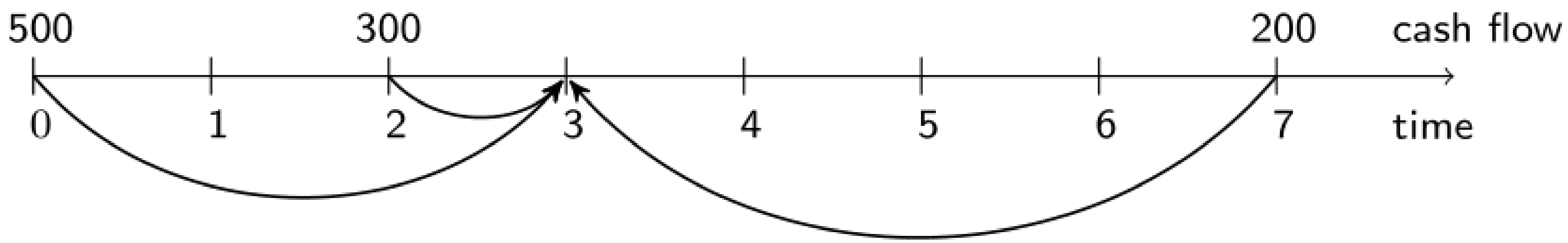
Valuation of a cash flow vector in R



Valuation of a cash flow vector in R



Valuation of a cash flow vector in R

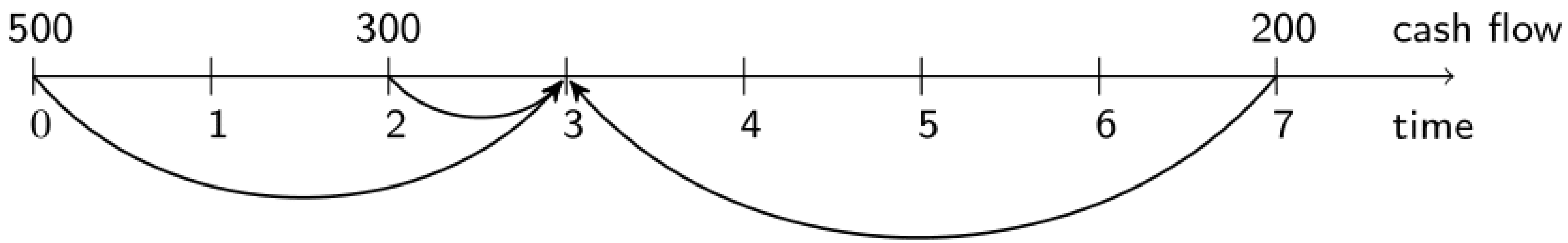



```
# Define the discount function
discount <- function(s, t, i = 0.03) {(1 + i) ^ - (t - s)}
```

```
# Calculate the value at time 3
value_3 <- 500 * discount(3, 0) + 300 * discount(3, 2) + 200 * discount(3, 7)
value_3
```

```
1033.061
```

Valuation of a cash flow vector in R




```
# Define the discount function
discount <- function(s, t, i = 0.03) {(1 + i) ^ - (t - s)}

# Define the cash flows
cash_flows <- c(500, 0, 300, rep(0, 4), 200)
```

```
# Calculate the value at time 3
sum(cash_flows * discount(3, 0:7))
```

```
1033.061
```

Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R

Actuarial equivalence

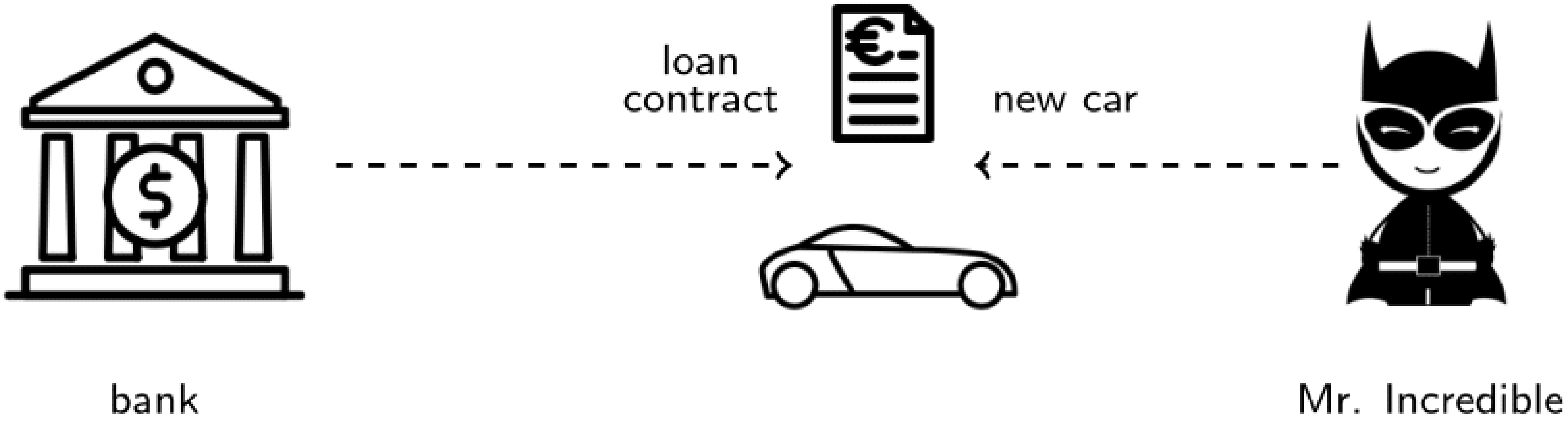
LIFE INSURANCE PRODUCTS VALUATION IN R

Katrien Antonio, Ph.D.
Professor, KU Leuven and University of Amsterdam

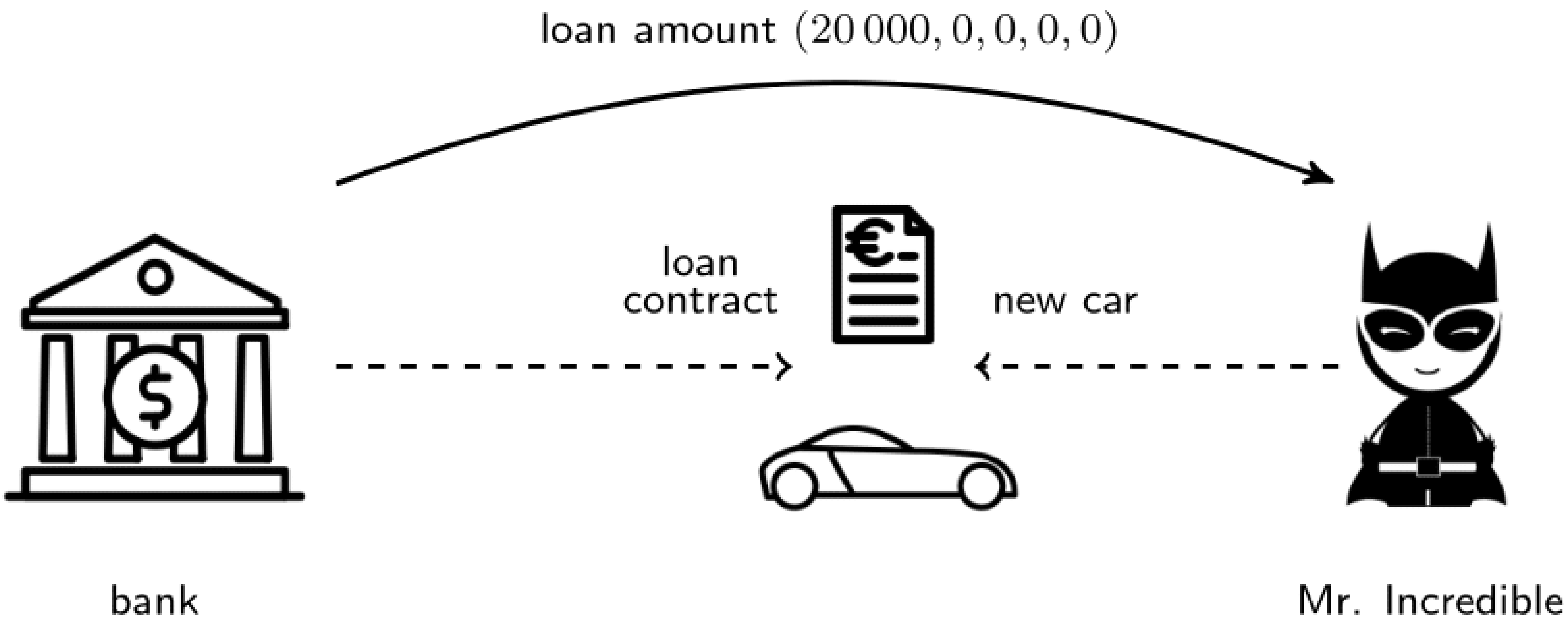
Actuarial equivalence of cash flow vectors

- **Establish an equivalence** between two cash flow vectors.
- Examples:
  - *mortgage*: capital borrowed from the bank, and the series of mortgage payments;
  - *insurance product*: benefits covered by the insurance, and the series of premium payments.

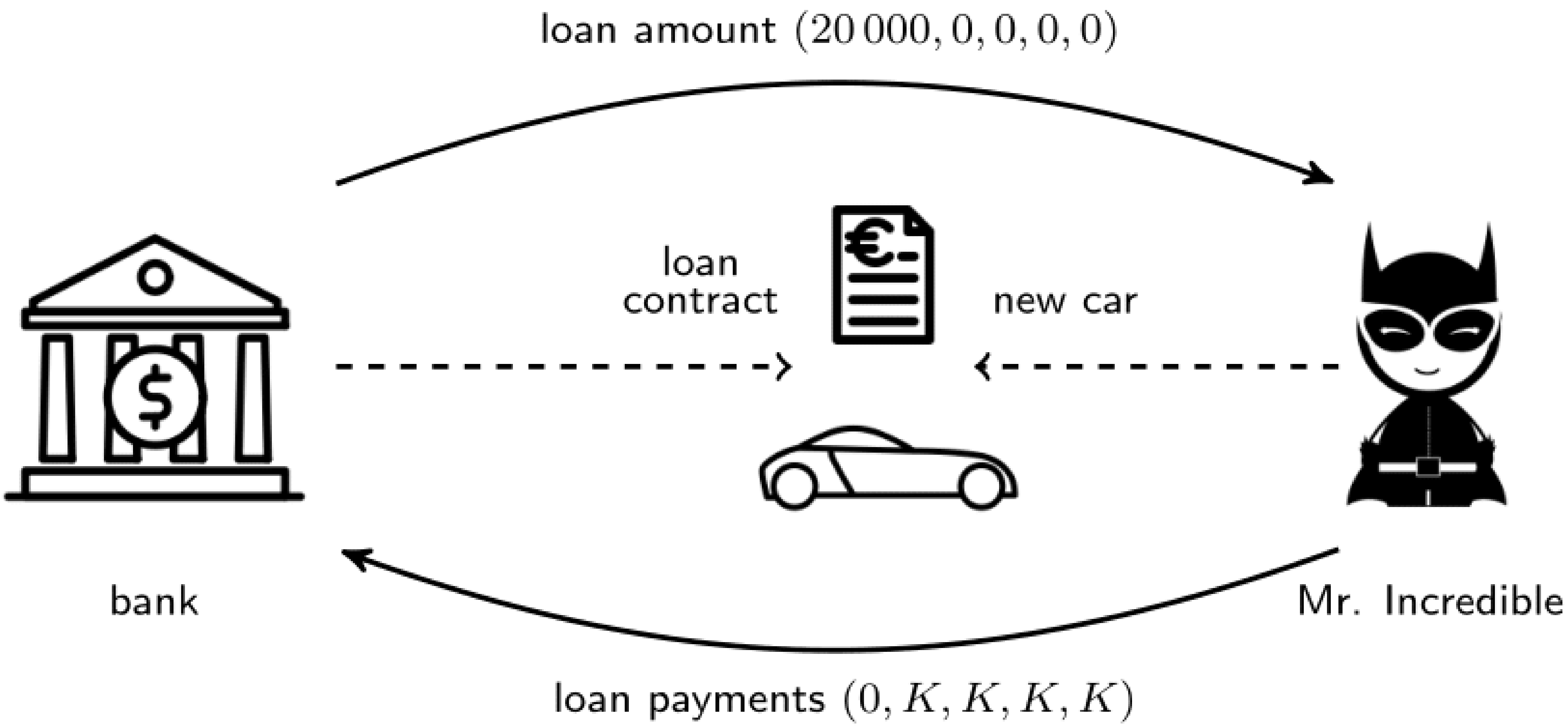
Mr. Incredible's new car



Mr. Incredible's new car



Mr. Incredible's new car



Mr. Incredible's new car

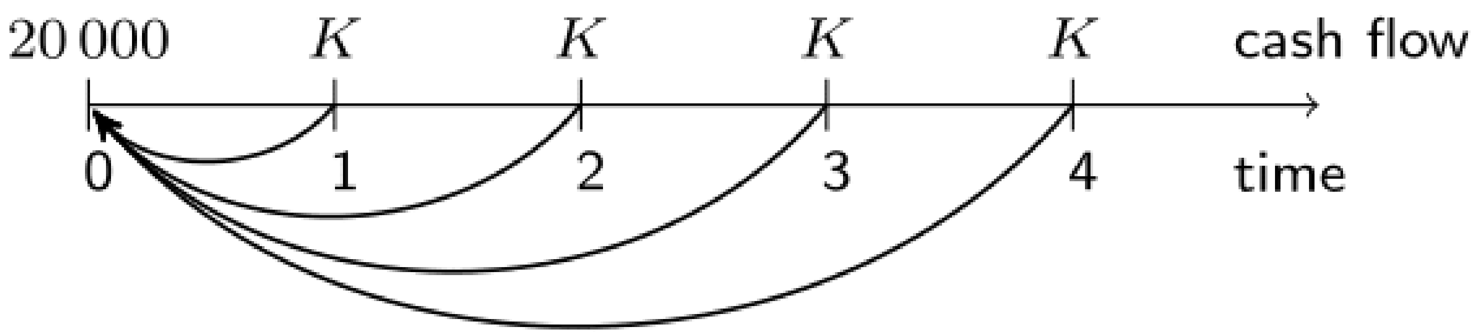
- Car is worth $20\ 000$ EUR;
- Mr. Incredible's loan payment vector is $(0, K, K, K, K)$ with Present Value:

$$\sum_{k=1}^{4} K \cdot v(0, k)$$

- Then, establish **equivalence** and solve for unknown K!

$$20\ 000 = \sum_{k=1}^{4} K \cdot v(0, k).$$

Mr. Incredible's new car in R




```
# Define the discount factors
discount_factors <- (1 + 0.03) ^ - (0:4)
# Define the vector with the payments
payments <- c(0, rep(1, 4))
# Calculate the present value of the payments
PV_payment <- sum(payments * discount_factors)
# Calculate the yearly payment
20000 / PV_payment
```

```
5380.541
```

Let's practice!

LIFE INSURANCE PRODUCTS VALUATION IN R

Change of period and term structure

LIFE INSURANCE PRODUCTS VALUATION IN R

Roel Verbelen, Ph.D.
Statistician, Finity Consulting

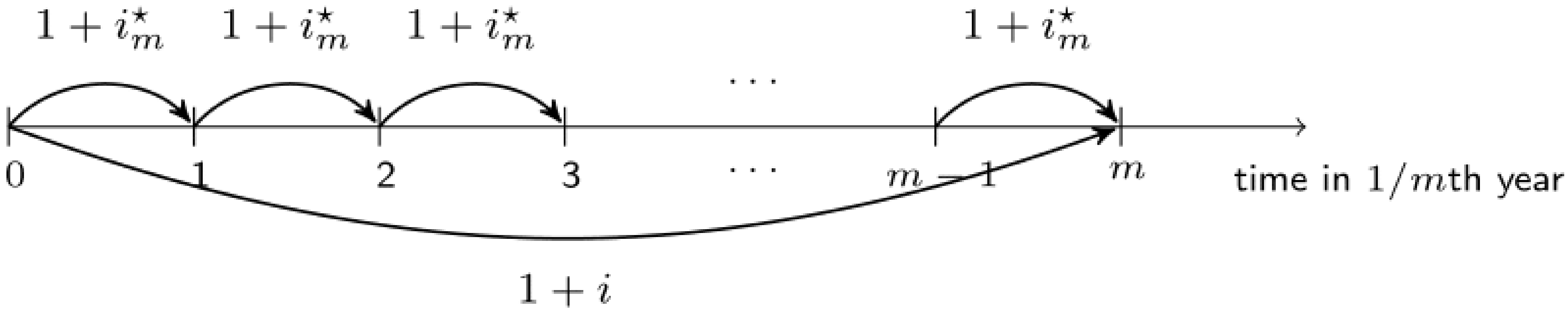
Moving away from constant, yearly interest

Two questions:

1. How to deal with interest rates when applying a **change of period** (e.g. from years to months)?
2. How to go from constant interest rate to a **rate that changes over time**?

From yearly to mth-ly interest rates

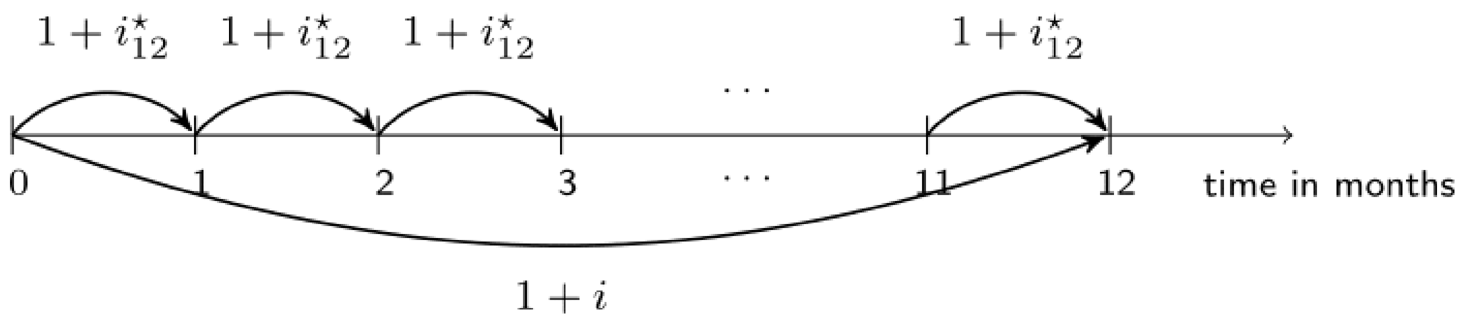
- Yearly interest rate i.
- How to derive $i^\star_m$ the rate applicable to a period of $1/m$th year?



Then:

$$1+i = (1+i^\star_m)^m \quad \Leftrightarrow \quad i^\star_m = (1+i)^{1/m} - 1.$$

From yearly to mth-ly interest rates in R




```
# Yearly interest rate
i <- 0.03
# Calculate the monthly interest rate
(monthly_interest <- (1 + i) ^ (1 / 12) - 1)
```

```
0.00246627
```

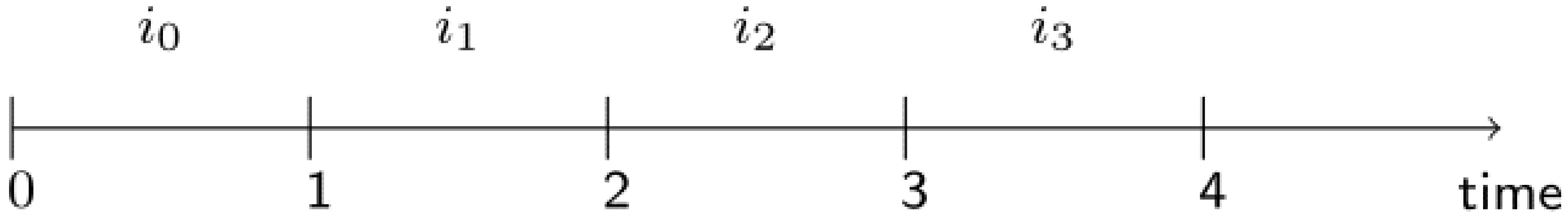
```
# From monthly to yearly interest rate
(1 + monthly_interest) ^ 12 - 1
```

```
0.03
```

Non-constant interest rates

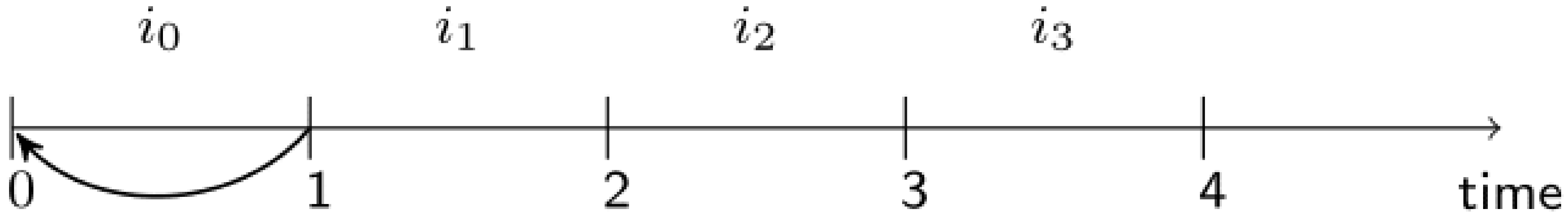
- Observations:
  - interest rates are not necessarily constant;
  - the **term structure of interest rates** or yield curve.
- Incorporate this in our notation and framework!

Non-constant interest rates



$v(0,0) = 1$

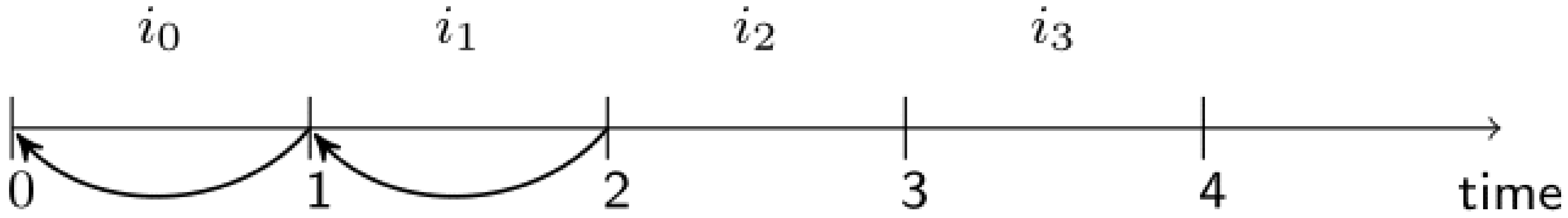
Non-constant interest rates



$$v(0,0) = 1$$

$$v(0,1) = (1 + i_0)^{-1}$$

Non-constant interest rates

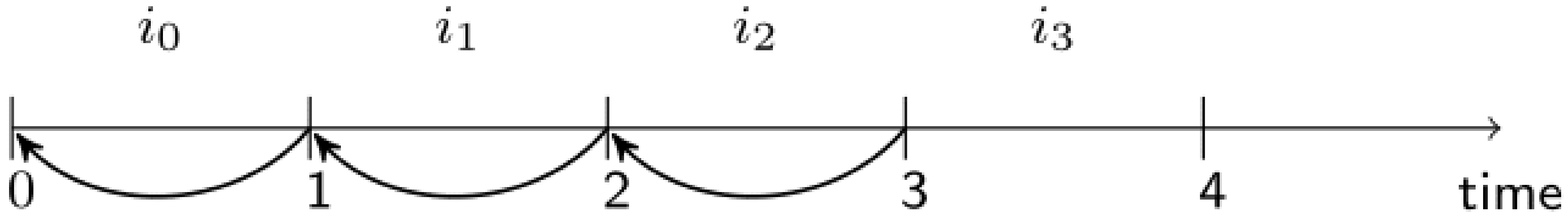


$v(0,0) = 1$

$v(0,1) = (1 + i_0)^{-1}$

$v(0,2) = (1 + i_0)^{-1} \cdot (1 + i_1)^{-1}$

Non-constant interest rates



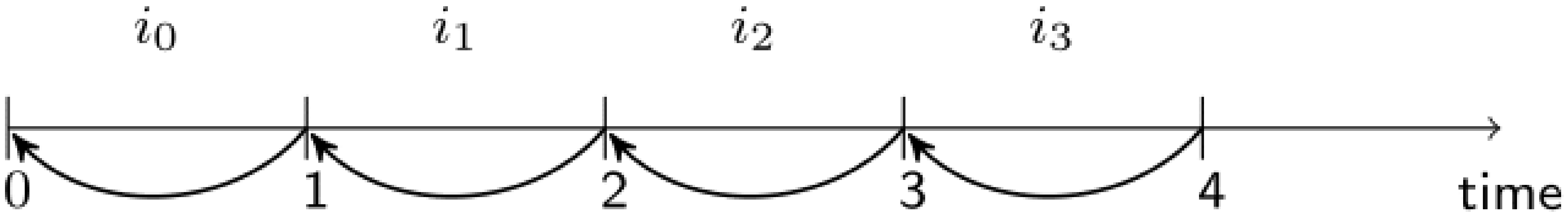
$$v(0,0) = 1$$

$$v(0,1) = (1+i_0)^{-1}$$

$$v(0,2) = (1+i_0)^{-1} \cdot (1+i_1)^{-1}$$

$$v(0,3) = (1+i_0)^{-1} \cdot (1+i_1)^{-1} \cdot (1+i_2)^{-1}$$

Non-constant interest rates

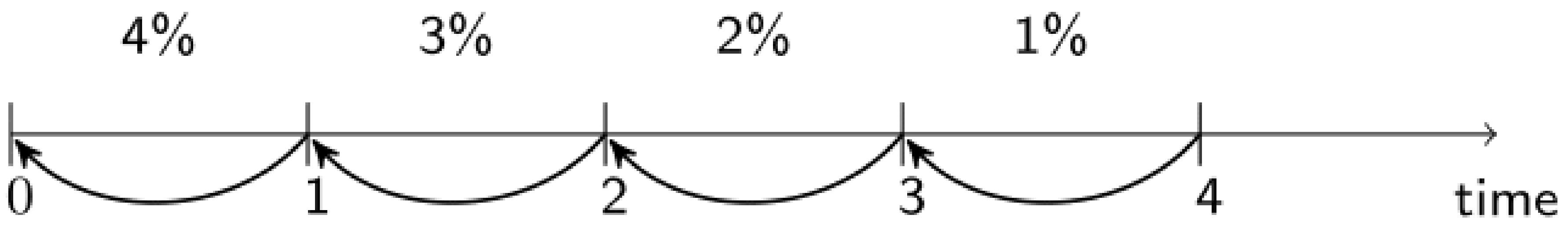


$$v(0,0) = 1$$

$$v(0,1) = (1 + i_0)^{-1}$$

$$v(0,2) = (1 + i_0)^{-1} \cdot (1 + i_1)^{-1}$$

$$v(0,3) = (1 + i_0)^{-1} \cdot (1 + i_1)^{-1} \cdot (1 + i_2)^{-1}$$




```
# Define the vector containing the interest rates
interest <- c(0.04, 0.03, 0.02, 0.01)
```

```
# Define the vector containing the inverse of 1 plus the interest rate
yearly_discount_factors <- (1 + interest) ^ - 1
```

```
# Define the discount factors to time 0 using cumprod()
discount_factors <- c(1 , cumprod(yearly_discount_factors))
discount_factors
```

```
1.0000000 0.9615385 0.9335325 0.9152279 0.9061663
```

Let's practice!

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