Unveiling Complexity: Higher-Order Modelling of Swayed Structure Branches

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Introduction

- We'll explore how nonlinear dynamics influences the behavior of trees and other branched structures in response to wind-induced motion.
- Trees sway in complex ways due to wind forces, impacting their growth, stability.
- This presentation delves into theoretical modeling, simulations, and analysis of these dynamic systems, focusing on their nonlinear aspects.
- We'll discuss objectives, methodology, analysis, and conclusions, providing insights into swaying branched structures.

Model

- Chimney Model

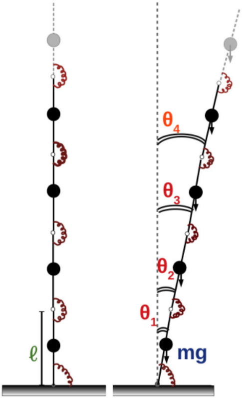

Figure: 1 Schematic diagram of the Chimney model.(Thesis)

- Extension to N-Element Systems

Model

- Equation of motion for the system with N-segments.

$$
\begin{aligned}
&\ell^2\ddot{\theta}_p M_{p,N} + \ell^2 \sum_{j=p+1}^{N} \left(\ddot{\theta}_j \cos(\theta_j-\theta_p) - \dot{\theta}_j^{\,2} \sin(\theta_j-\theta_p)\right)\widetilde{M}_{j,N} \\
&+\ell^2\sum_{i=1}^{p-1}\left(\ddot{\theta}_i\cos(\theta_p-\theta_i)+\dot{\theta}_i^{\,2}\sin(\theta_p-\theta_i)\right)\widetilde{M}_{p,N} - g\ell\sin\theta_p\widetilde{M}_{p,N} \\
&+k_p(\theta_p-\theta_{p-1})\left(1+\alpha_p(\theta_p-\theta_{p-1})^2\right) \\
&-k_{p+1}(\theta_{p+1}-\theta_p)\left(1+\alpha_{p+1}(\theta_{p+1}-\theta_p)^2\right) \\
&+\frac{1}{2}b\ell^2\left(\frac{\dot{\theta}_p}{2} + 2\sum_{i=p+2}^{N}\sum_{k=p+1}^{i-1}\cos(\theta_k-\theta_p)\dot{\theta}_k + 2\sum_{i=p+1}^{N}\sum_{j=1}^{p-1}\cos(\theta_p-\theta_j)\dot{\theta}_j\right. \\
&\left.+\sum_{j=1}^{p-1}\cos(\theta_j-\theta_p)\dot{\theta}_j + \sum_{i=p+1}^{N}\left(\cos(\theta_p-\theta_i)\dot{\theta}_i+2\dot{\theta}_p\right)\right) = 0
\end{aligned}
$$

Figure: 2(Ref:Thesis)

- Extension to N-Element Systems

Single Element System

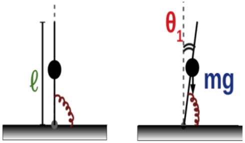

Figure: 3(Ref:Thesis)

System Dynamics Equation

$$\dot{\theta}_1 = 2g\sin(\theta_1) - 4k_1\theta_1(1+\alpha_1\theta_1^2) + 2f\cos(\omega t) - b\dot{\theta}_1$$

- This equation has gravitational instability
- Where $2f\cos(\omega t)$ is identified as the driving force, which affects the system dynamics significantly.
- An equation containing both of these nonlinear forces, the gravitational term and the cubic nonlinearity in the restoring force

Phase Potrait

- N=1,Single Element System

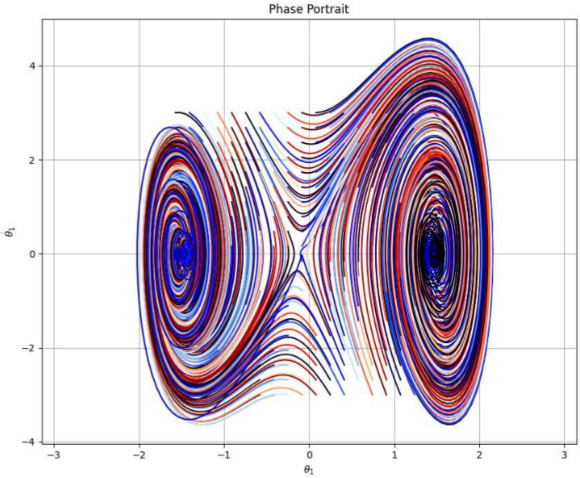

Figure: a) for f=1 ,one saddle and 2 attractors are present

Phase Potrait

- N=1,Single Element System

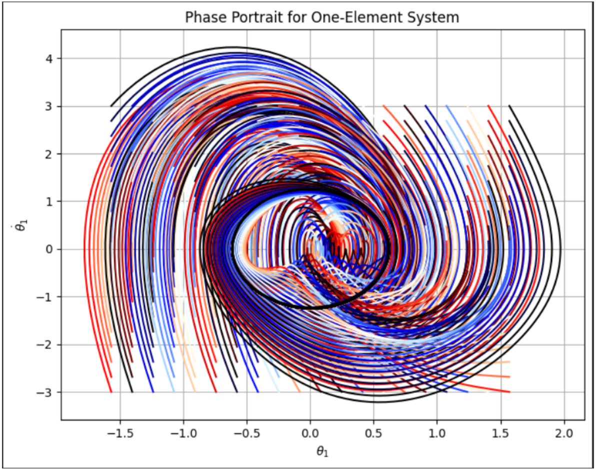

Figure: b) for f=5 ,

Two-Element System

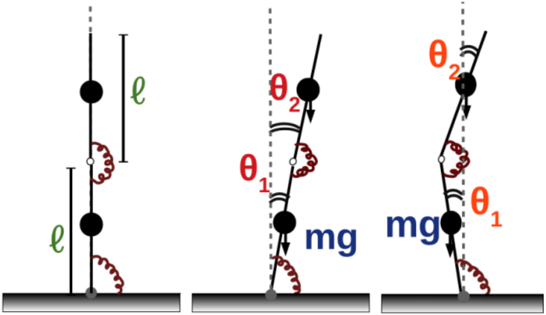

Figure: 4(Ref:Thesis)

Extension for N=2

System Dynamics Equations

$$
\begin{aligned}
\ddot{\theta}_1 = \frac{1}{1-\frac{4}{5}\cos(\theta_2-\theta_1)^2}\Bigg(&\frac{4}{5}\dot{\theta}_1^2\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1)\\
&-\frac{4}{5}g\sin(\theta_2)\cos(\theta_2-\theta_1)+\frac{8}{5}k_2(\theta_2-\theta_1)\cos(\theta_2-\theta_1)(1+\alpha_2(\theta_2-\theta_1)^2)\\
&+\frac{4}{5}b\dot{\theta}_1\cos(\theta_2-\theta_1)^2-\frac{4}{5}f\cos(\omega t)\cos(\theta_2-\theta_1)+\frac{2}{5}\dot{\theta}_2^2\sin(\theta_2-\theta_1)\\
&+\frac{6}{5}g\sin(\theta_1)-\frac{4}{5}k_1\theta_1(1+\alpha_1\theta_1^2)+\frac{4}{5}k_2(\theta_2-\theta_1)(1+\alpha_2(\theta_2-\theta_1)^2)\\
&-b\dot{\theta}_1+\frac{2}{5}f\cos(\omega t)\Bigg),
\end{aligned}
$$

Extension for N=2

System Dynamics Equations

$$
\begin{aligned}
\ddot{\theta}_2 = \frac{1}{1-\frac{4}{5}\cos(\theta_2-\theta_1)^2}\Bigg(&-\frac{4}{5}\dot{\theta}_2^2\sin(\theta_2-\theta_1)\cos(\theta_2-\theta_1) \\
&-\frac{12}{5}g\sin(\theta_1)\cos(\theta_2-\theta_1)+\frac{8}{5}k_1\theta_1\cos(\theta_2-\theta_1)(1+\alpha_1\theta_1^2) \\
&-\frac{8}{5}k_2(\theta_2-\theta_1)\cos(\theta_2-\theta_1)(1+\alpha_2(\theta_2-\theta_1)^2)+\frac{4}{5}b\dot{\theta}_2\cos(\theta_2-\theta_1)^2 \\
&-\frac{4}{5}f\cos(\omega t)\cos(\theta_2-\theta_1)-2\dot{\theta}_1^2\sin(\theta_2-\theta_1)+2g\sin(\theta_2) \\
&-4k_2(\theta_2-\theta_1)(1+\alpha_2(\theta_2-\theta_1)^2)-b\dot{\theta}_2+2f\cos(\omega t)\Bigg).
\end{aligned}
$$

Phase Potrait

- N=2,Two Element System

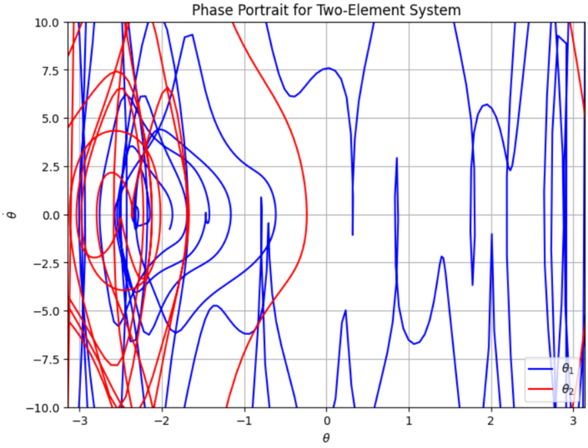

Figure: a) for f=10

Y-Segment System

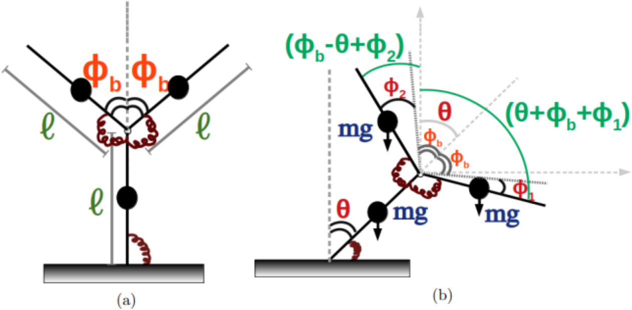

Figure: 5(Ref:Thesis)

Y-Segment System

System Dynamics Equations

$$\ddot{\theta} = \left\{ \frac{1}{\frac{9}{4}ml^2 - ml^2\cos^2(\phi_b + \phi_1(t)) - ml^2\cos^2(\phi_b + \phi_2(t))} \right\}.$$

$$\left\{ \dot{\theta}(t)\dot{\phi}_1(t)ml^2\sin(\phi_b + \phi_1(t)) + \dot{\theta}(t)\dot{\phi}_2(t)ml^2\sin(\phi_b + \phi_2(t)) + \right.$$

$$\frac{1}{2}ml^2\dot{\theta}^2(t)\left(\sin(\phi_b + \phi_1(t)) - \sin(\phi_b + \phi_2(t))\right) + 2\sin(\phi_b + \phi_1(t))\cos(\phi_b + \phi_1(t)) -$$

$$2\sin(\phi_b + \phi_2(t))\cos(\phi_b + \phi_2(t)) + k_2(\phi_1(t) + \phi_2(t)) + k_2\alpha_2(\phi_1^3(t) + \phi_2^3(t)) -$$

$$\dot{\theta}(t)\left(\frac{9}{4}bl^2 - bl^2\cos^2(\phi_b + \phi_2(t))\right) + (Q_{\text{drive1}} - Q_{\text{drive2}} + Q_{\text{drive3}}) +$$

$$mgl\left(\sin(\phi_b + \phi_2(t)) - \theta(t)\cos(\phi_b + \phi_2(t)) - \sin(\phi_b + \phi_1(t)) - \theta(t)\cos(\phi_b + \phi_1(t))\right) +$$

$$2k_2\left(\phi_1(t)\cos(\phi_b + \phi_1(t)) - \phi_2(t)\cos(\phi_b + \phi_2(t))\right) +$$

$$2k_2\alpha_2\left(\phi_1^3(t)\cos(\phi_b + \phi_1(t)) - \phi_2^3(t)\cos(\phi_b + \phi_2(t))\right) -$$

$$2(Q_{\text{drive2}}\cos(\phi_b + \phi_1(t)) - Q_{\text{drive3}}\cos(\phi_b + \phi_2(t))) +$$

$$\frac{1}{2}\dot{\phi}_1^2(t)ml^2\sin(\phi_b + \phi_1(t)) - \frac{1}{2}\dot{\phi}_2^2(t)ml^2\sin(\phi_b + \phi_2(t)) +$$

$$\left. \frac{5}{2}mgl\sin(\theta(t)) - k_1\theta(t) - k_1\alpha_1\theta^3(t) \right\}$$

Y-Segment System

System Dynamics Equations

$$
\begin{aligned}
\ddot{\phi}_1 =& \frac{1}{\frac{1}{4}ml^2}\left\{-\left(\frac{1}{4}ml^2+\frac{1}{2}ml^2\cos(\phi_b+\phi_2(t))\right)\right.\\
&/\left(\frac{9}{4}ml^2-ml^2\cos^2(\phi_b+\phi_1(t))-ml^2\cos^2(\phi_b+\phi_2(t))\right)\\
&\times\left[\dot{\theta}(t)\dot{\phi}_1(t)ml^2\sin(\phi_b+\phi_1(t))+\dot{\theta}(t)\dot{\phi}_2(t)ml^2\sin(\phi_b+\phi_2(t))\right.\\
&+\frac{1}{2}ml^2\dot{\theta}^2(t)\left(\sin(\phi_b+\phi_1(t))-\sin(\phi_b+\phi_2(t))\right)\\
&+2\sin(\phi_b+\phi_1(t))\cos(\phi_b+\phi_1(t))-2\sin(\phi_b+\phi_2(t))\cos(\phi_b+\phi_2(t))\\
&+k_2(\phi_1(t)-\phi_2(t))+k_2\alpha_2(\phi_1^3(t)-\phi_2^3(t))\\
&-\dot{\theta}(t)\left(\frac{9}{4}bl^2-bl^2\cos^2(\phi_b+\phi_1(t))\right)+(Q_{\text{drive1}}-Q_{\text{drive2}}+Q_{\text{drive3}})\\
&+mgl(\sin(\phi_b+\phi_2(t))-\theta(t)\cos(\phi_b+\phi_2(t))-\sin(\phi_b+\phi_1(t))-\theta(t)\cos(\phi_b+\phi_1(t)))\\
&+2k_2(\phi_1(t)\cos(\phi_b+\phi_1(t))-\phi_2(t)\cos(\phi_b+\phi_2(t)))\\
&+2k_2\alpha_2(\phi_1^3(t)\cos(\phi_b+\phi_1(t))-\phi_2^3(t)\cos(\phi_b+\phi_2(t)))\\
&-2(Q_{\text{drive2}}\cos(\phi_b+\phi_1(t))-Q_{\text{drive3}}\cos(\phi_b+\phi_2(t)))\\
&+\frac{1}{2}\dot{\phi}_1^2(t)ml^2\sin(\phi_b+\phi_1(t))-\frac{1}{2}\dot{\phi}_2^2(t)ml^2\sin(\phi_b+\phi_2(t))\\
&\left.\left.+\frac{5}{2}mgl\sin(\theta(t))-k_1\theta(t)-k_1\alpha_1\theta^3(t)\right]\right\}
\end{aligned}
$$

Y-Segment System

System Dynamics Equations

$$
\begin{aligned}
\ddot{\phi}_2 = & \frac{1}{\frac{1}{4}ml^2}\Bigg\{\Bigg[\left(\frac{1}{4}ml^2 + \frac{1}{2}ml^2\cos(\phi_b+\phi_2(t))\right) \\
& / \left(\frac{9}{4}ml^2 - ml^2\cos^2(\phi_b+\phi_1(t)) - ml^2\cos^2(\phi_b+\phi_2(t))\right)\Bigg] \\
& \times \Big[\dot{\theta}(t)\dot{\phi}_1(t)ml^2\sin(\phi_b+\phi_1(t)) + \dot{\theta}(t)\dot{\phi}_2(t)ml^2\sin(\phi_b+\phi_2(t)) \\
& + \frac{1}{2}ml^2\dot{\theta}^2(t)\left(\sin(\phi_b+\phi_1(t)) - \sin(\phi_b+\phi_2(t))\right) \\
& + 2\sin(\phi_b+\phi_1(t))\cos(\phi_b+\phi_1(t)) - 2\sin(\phi_b+\phi_2(t))\cos(\phi_b+\phi_2(t)) \\
& + k_2(\phi_1(t) - \phi_2(t)) + k_2\alpha_2(\phi_1^3(t) - \phi_2^3(t)) \\
& - \dot{\theta}(t)\left(\frac{9}{4}bl^2 - bl^2\cos^2(\phi_b+\phi_1(t))\right) + (Q_{\text{drive1}} - Q_{\text{drive2}} + Q_{\text{drive3}}) \\
& + mgl\left(\sin(\phi_b+\phi_2(t)) - \theta(t)\cos(\phi_b+\phi_2(t)) - \sin(\phi_b+\phi_1(t)) - \theta(t)\cos(\phi_b+\phi_1(t))\right) \\
& + 2k_2\left(\phi_1(t)\cos(\phi_b+\phi_1(t)) - \phi_2(t)\cos(\phi_b+\phi_2(t))\right) \\
& + 2k_2\alpha_2\left(\phi_1^3(t)\cos(\phi_b+\phi_1(t)) - \phi_2^3(t)\cos(\phi_b+\phi_2(t))\right) \\
& - 2\left(Q_{\text{drive2}}\cos(\phi_b+\phi_1(t)) - Q_{\text{drive3}}\cos(\phi_b+\phi_2(t))\right) \\
& + \frac{1}{2}\dot{\phi}_1^2(t)ml^2\sin(\phi_b+\phi_1(t)) - \frac{1}{2}\dot{\phi}_2^2(t)ml^2\sin(\phi_b+\phi_2(t)) \\
& + \frac{5}{2}mgl\sin(\theta(t)) - k_1\theta_1(t) - k_1\alpha_1\theta_1^3(t)\Big]\Bigg\}
\end{aligned}
$$

Phase Potrait

- y segment

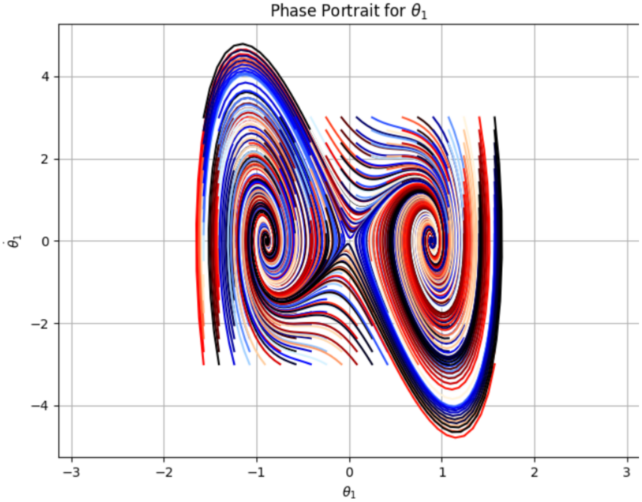

Figure: a)

Phase Potrait

- Y Segment

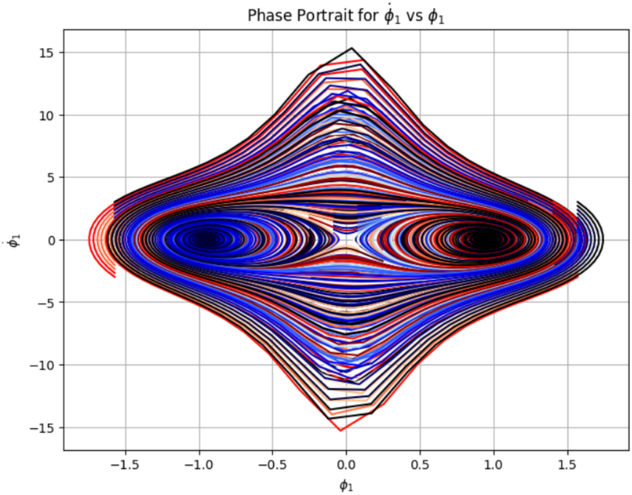

Figure: b)

Phase Potrait

- Y Segment

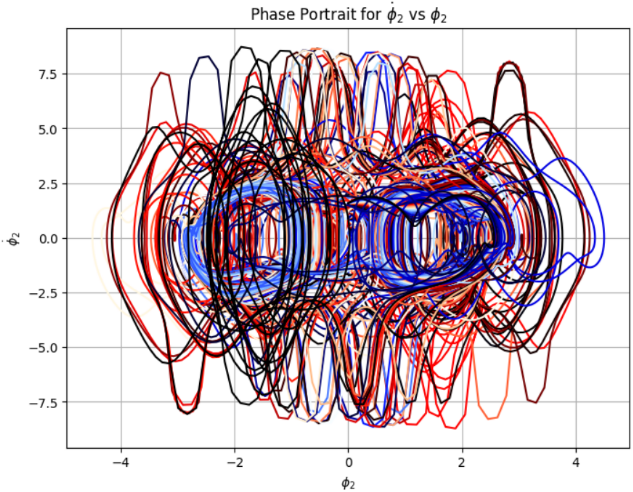

Figure: c)

Analysis

- The phase portrait for theta in the Y-branch system exhibits two spiral structures and a saddle point.
- The phase portrait for ϕ_1 displays a structure with stable spirals, indicating damped, stable oscillations bounded within specific regions, reflecting the system's energy dissipation and stability under nonlinear influences.
- The phase portrait for ϕ_2 features horizontally aligned spirals, suggesting complex, possibly chaotic interactions between coupled oscillatory modes, highlighting the system's sensitivity to initial conditions and parameter changes.

Y-Segment System code

```python
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint

# Define the differential equations for theta1
def nldpf_eq_theta(y, t, b, g, k1, alpha1, k2, alpha2, phib, Qdrive1, Qdrive2, Qdrive3):
    theta1, theta1_dot, _, _, _, _ = y
    d_theta1_dt = theta1_dot
    d_phi1_dt = 0
    d_phi2_dt = 0

    # Equations for theta1_dot
    d_theta1_dot_dt = (
        (1 / (9/4 * m * l**2 - m * l**2 * np.cos(phib) - m * l**2 * np.cos(phib)))) * (
            theta1_dot * 0 * m * l**2 * np.sin(phib) +
            theta1_dot * 0 * m * l**2 * np.sin(phib) +
            (1/2) * m * l**2 * theta1_dot**2 * (np.sin(phib) - np.sin(phib)) +
            2 * np.sin(phib) * np.cos(phib) -
            2 * np.sin(phib) * np.cos(phib) +
            k2 * (0 - 0) +
            k2 * alpha2 * (0**3 - 0**3) -
            theta1_dot * (9 * b * l**2 / 4 - b * l**2 * np.cos(phib)) +
            (Qdrive1 - Qdrive2 + Qdrive3) +
            m * g * l * (np.sin(phib) - theta1 * np.cos(phib) - np.sin(phib) - theta1 * np.cos(phib)) +
            2 * k2 * (0 * np.cos(phib) - 0 * np.cos(phib)) +
            2 * k2 * alpha2 * (0**3 * np.cos(phib) - 0**3 * np.cos(phib)) -
            2 * (Qdrive2 * np.cos(phib) - Qdrive3 * np.cos(phib)) +
            (1/2) * 0**2 * m * l**2 * np.sin(phib) -
            (1/2) * 0**2 * m * l**2 * np.sin(phib) +
            (5/2) * m * g * l * np.sin(theta1) -
            k1 * theta1 -
            k1 * alpha1 * theta1**3
        )

    return [d_theta1_dt, d_theta1_dot_dt, d_phi1_dt, 0, 0, 0]

# Parameters
m = 1  # Mass
g = 9.8  # Gravitational acceleration
l = 1  # Length of the pendulum
k1 = 1  # Spring constant 1
alpha1 = 1  # Nonlinear coefficient 1
k2 = 1  # Spring constant 2
alpha2 = 1  # Nonlinear coefficient 2
phib = 0  # Phase shift
Qdrive1 = 0  # External force 1
Qdrive2 = 0  # External force 2
Qdrive3 = 0  # External force 3
b=0.5

# Define the range of initial conditions for theta1 and its derivative
theta1_range = np.linspace(-np.pi/2, np.pi/2, 20)
theta1_dot_range = np.linspace(-3, 3, 20)

# Time points for integration
t = np.linspace(0, 20, 1000)

# Plot the phase portrait for theta1
plt.figure(figsize=(8, 6))
color_map = plt.cm.flag(np.linspace(0, 1, len(theta1_range) * len(theta1_dot_range)))

idx = 0
for theta1_0 in theta1_range:
    for theta1_dot_0 in theta1_dot_range:
        y0 = [theta1_0, theta1_dot_0, 0, 0, 0, 0]  # Initial conditions for phi1, phi2 and their derivatives are set to 0
        solution = odeint(nldpf_eq_theta, y0, t, args=(b, g, k1, alpha1, k2, alpha2, phib, Qdrive1, Qdrive2, Qdrive3))
        theta1, theta1_dot, _, _, _, _ = solution.T
        plt.plot(theta1, theta1_dot, color=color_map[idx], alpha=1)
        idx += 1

plt.xlabel(r'$\theta_1$')
plt.ylabel(r'$\dot{\theta}_1$')
plt.title('Phase Portrait for $\\theta_1$')
plt.grid(True)
```

Conclusion:

- The analysis into the dynamics of the N=1 and N=2 systems highlights a profound evolution in behavior as the complexity of the system increases.

- For N=1, the phase portrait primarily displayed stable, periodic behaviors, which are predictable and manageable, suggesting a system that can reliably return to equilibrium after disturbances.

- In contrast, the N=2 system unveiled a more intricate dynamical framework, where the interplay between components introduced a spectrum of behaviors from stable to chaotic.

- The exploration of the Y-branch configuration observed alongside the stable and chaotic patterns in θ, ϕ_1, ϕ_2 and highlights the complex interdependencies and sensitivity to initial conditions within the system.

The End