# Homework: Lambda Calculus

Due-date: Feb 13 at 11:59pm.
Submit online on Canvas (**Format: pdf**).

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*Homework must be individual's original work. Collaborations and discussions of any form with any students or other faculty members are not allowed. If you have any questions and/or concerns, post them on Piazza and/or ask 342 instructor and TAs.*

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## Learning Outcomes

- Application of knowledge of computing and mathematics
- Ability to understand the implications of mathematical formalisms in computer science

## Questions

1. Compute the following (apply reductions till the expression cannot be reduced any further)

   (a) $((\lambda x.(x\ \ x)\ \ \lambda y.y)\ \ \lambda y.y)$

   (b) $((\lambda x.\lambda y.(x\ \ (y\ \ y))\ \ \lambda a.a)\ \ b)$

   (c) $((\lambda x.(x\ \ x)\ \ \lambda y.(y\ \ x))\ \ z)$

   (d) $(\lambda g.(g\ \ \lambda x.\lambda y.x)\ \ ((\lambda a.\lambda b.\lambda h.((h\ \ a)\ \ b)\ \ z_1)\ \ z_2))$

   (e) $((\lambda t.\lambda y.(t\ \ y)\ \ \lambda n.\lambda f.\lambda x.(f\ \ ((n\ \ f)\ \ x)))\ \ \lambda g.\lambda z.(g\ \ (g\ \ z)))$

   (15pts)

2. Given the following lambda expressions and corresponding interpretations:

   - The interpretation of $\lambda f.\lambda x.x$ is natural number 0 (zero). The interpretation of $\lambda f \lambda x.(f\ \ (f\ \ (...\ \ x)))$, with $n$ applications of $f$ on $x$, is the natural number $n > 0$.
   - The interpretation of $\lambda n.\lambda f.\lambda x.(f\ \ ((n\ \ f)\ \ x))$ is a successor function $succ$ for natural numbers, where $n$ is the formal parameter corresponding to the number whose successor is computed.
   - The interpretation of $\lambda m.\lambda n.((m\ \ succ)\ \ n)$ is the addition function $add$ for two natural numbers, where $m$ and $n$ are the formal parameters corresponding to the numbers whose sum is computed.
   - The interpretation of $\lambda m.\lambda n.((m\ \ (add\ \ n))\ \ zero)$ is the multiplication function $mul$ for two natural numbers, where $m$ and $n$ are the formal parameters corresponding to the numbers whose product is computed.
   - The interpretation of $\lambda x.\lambda y.x$ is propositional constant $true$.
   - The interpretation of $\lambda x.\lambda y.y$ is propositional constant $false$.
   - The interpretation of $\lambda a.\lambda b.\lambda h.((h\ \ a)\ \ b)$ is a *pair* of entities $a$ and $b$ on which some function $h$ can be applied. We will refer to this function as $Pair$. The first or second element of the pair $((Pair\ \ z_1)\ \ z_2)$ can be obtained by applying on it the functions $\lambda g.(g\ \ \lambda a.\lambda b.a)$ (referred to as $fst$) and $\lambda g.(g\ \ \lambda a.\lambda b.b)$ (referred to as $sec$), respectively. That is. $(fst\ \ ((Pair\ \ z_1)\ \ z_2)) = z_1$ and $(sec\ \ ((Pair\ \ z_1)\ \ z_2)) = z_2$.

- The interpretation of a pair $((Pair\ \ m)\ \ n)$ where $m$ and $n$ are natural numbers is a signed number whose valuation is difference between $m$ and $n$ (i.e., $m - n$). For instance, $((Pair\ \ \lambda f.\lambda x.x)\ \ \lambda f.\lambda x.(f\ \ x))$ represents a signed number $-1$.

Identify the mathematical/logical interpretation for the following expressions. Justify your answer. (In all these problems, apply the functions on some actual arguments and examine the results; does the result correspond to some interpretation that you already know about—basic arithmetic or logical operations. We have done similar problems, when we identified the interpretation of functions representing addition and multiplication of naturals, and negation, conjunction and disjuction of propositions.).

(a) $\lambda x.((x\ \ false)\ \ true)$, where $x$ is the formal parameter corresponding to **propositional constants**.

(b) $\lambda n.((n\ \ \lambda p.((p\ \ false)\ \ true))\ \ false)$, where $n$ is the formal parameter corresponding to **natural numbers**.

(c) $\lambda m.\lambda n.((m\ \ (mul\ \ n))\ \ (succ\ \ zero))$, where $m$ and $n$ are formal parameters corresponding to **natural numbers**.

(d) $\lambda p.((Pair\ \ (sec\ \ p))\ \ (fst\ \ p))$, where $p$ is the formal parameter correspond to some **signed number**.

(e) $\lambda p_1.\lambda p_2.((Pair\ \ ((add\ \ (fst\ \ p_1))\ \ (sec\ \ p_2)))\ \ ((add\ \ (sec\ \ p_1))\ \ (fst\ \ p_2)))$, where $p_1$ and $p_2$ are formal paratemeters corresponding to some **signed numbers**.

(25pts)

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*Notes: For any lambda expression, whenever you see an application of the form $(e_1\ \ e_2)$, where $e_1$ is a lambda abstraction and $e_2$ is a "large" expression, use some variable (e.g., $\varphi$) to represent the large expression. Perform the $\beta$-reduction in the application then expand the expression $e_2$, if necessary. Carefully consider the "(...)"-matching.*

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