1. Yes, it's possible.

SPEC !AF(reset) to verify the requirement.

$\Rightarrow$ specification !AF(reset) is false

CTL counterexample:

-> State : 1.1 <-

reset = false

-> State : 1.2 <-

reset = false

-> State : 1.3 <-

reset = false

-> State : 1.4 <-

reset = true

2. $\forall z \subseteq S: f(z) = \overline{g_1(z)}$

$f(z) = S - g_1(z)$

$g_1(z) = S - f(z)$ (this function was mentioned in lecture)

we have g_1 is not monotonically non-decreasing

$\boxed{2.(b)}$ $\forall z \subseteq S: f(z) = \overline{g_2(\overline{z})}$

$f(z) = \overline{g_2(S-z)}$

$f(z) = S - g_2(S-z)$

Based on the definition of monotone function: $f: P(S) \to P(S)$ is a monotone function if $\forall x, y\ \ x \subseteq y \Rightarrow f(x) \subseteq f(y)$

it also satisfies that there is a least and a greatest fixed point

$g_2^n(\{\}) = lfp,\ g_2^n(S) = gfp$

therefore, g_2 is monotonically non-decreasing.

$\boxed{2.(c)}$ (c) Prove $lfp(f) \equiv \overline{gfp(g_2)}$

by Tarski and Knaster theorem, $lfp = f^n(\{\})$, $gfp(g_2) = g_2^n(S)$

$lfp(f) = f^n(\{\})$

$f(\{\}) = S - g_2(S)$

$f(f(\{\})) = S - g_2(S-(S-g_2(S)))$

$= S - g_2(g_2(S))$

$f(f(f(\{\}))) = S - g_2(S-(S-g_2(g_2(S))))$

$= S - g_2(g_2(g_2(S)))$

$\vdots$

$f^n(\{\}) = S - g_2(g_2 \cdots g_2(S)) \cdots)$

$gfp(g_2) = g_2^n(S)$

$= g_2(g_2 \cdots (g_2(S)) \cdots)$

then, we have

$\overline{gfp(g_2)} = S - g_2(g_2 \cdots (g_2(S)) \cdots)$

So, they are equivalent.

3, (a) Prove $[\![AG\varphi]\!]$ is the greatest fixed point k_φ.

claim 1: $[\![AG\varphi]\!]$ is the fixed point of k_φ

$$[\![AG\varphi]\!] = [\![\varphi]\!] \cap R_\forall(Z)$$

$$[\![AG\varphi]\!] = k_\varphi([\![AG\varphi]\!])$$

$$[\![AG\varphi]\!] = \{s \mid \forall \pi \in Path(s)\ \forall i \geq 0\ \forall_i \in [\![\varphi]\!]\}$$

$$= \{s \mid \forall \pi \in Path(s)\ \pi[0] \in [\![\varphi]\!] \wedge \forall i \geq 1\ \pi[i] \in [\![\varphi]\!]\}$$

$$= \{s \mid \forall \pi \in Path(s)\ \pi[0] \in [\![\varphi]\!]\} \cap \{s \mid \forall \pi \in Path(s)\ \forall i \geq 1\ \pi[i] \in [\![\varphi]\!]\}$$

$$= [\![\varphi]\!] \cap \{s \mid \forall \pi \in Path(s)\ \pi[i] \in [\![AG\varphi]\!]\}$$

$$= [\![\varphi]\!] \cap R_\forall([\![AG\varphi]\!]) = k_\varphi([\![AG\varphi]\!]) \quad \Rightarrow fp$$

claim 2: $[\![AG\varphi]\!]$ is the greatest fixed point of k_φ among all fixed points

Assume $Z = k_\varphi(Z)$ such $Z \notin [\![AG\varphi]\!]$

$\Rightarrow$ ~~$\nexists s_0$~~ $\exists s_0$ $s_0 \in Z$ and $s_0 \notin [\![AG\varphi]\!]$

$\exists s_0. (s_0 \in [\![\varphi]\!]$ and $s_0 \in R_\forall(Z))$ and $(s_0 \notin [\![\varphi]\!]$ and $s_0 \notin R_\forall([\![AG\varphi]\!]))$

We can see these "and"s. there's no "or" in this formula

also, we have $s_0 \in [\![\varphi]\!]$ and $s_0 \notin [\![\varphi]\!]$

it's contradictory. $\Rightarrow [\![AG\varphi]\!]$ is the greatest fixed point

(b) we already know $[\![AFb]\!]$ is the least fixed point. (proved in the lecture)

~~then we have $AFb = [\![b]\!] \cup R_\forall([\![AFb]\!])$~~

by Tarski and Knaster theorem, $f^n(\{\})$ is the least fp of f.

$I_b(\{\}) = [\![b]\!] = \{s_2, s_3, s_4\}$

$$I_b(I_b(\{\})) = [\![b]\!] \cup R_\forall(I_b(\{\}))$$

$$= \{s_2, s_3, s_4\} \cup R_\forall(\{s_2, s_3, s_4\})$$

$$= \{s_2, s_3, s_4\} \cup \{s_1, s_2, s_4\}$$

$$= \{s_1, s_2, s_3, s_4\}$$

$I_b(I_b(I_b(\{\})))$

$= \{\cancel{\ldots}\ [\![b]\!] \cup R_\forall(\{s_1, s_2, s_3, s_4\})$

$= \{s_2, s_3, s_4\} \cup \{s_0, s_1, s_2, s_3, s_4\}$

$= \{s_0, s_1, s_2, s_3, s_4\}$

no need to do $I_b(I_b(I_b(I_b(\{\}))))$ due to same result.

this is the ~~test~~ least fixed point of I_b

$\Rightarrow [\![AFb]\!] = \{s_0, s_1, s_2, s_3, s_4\}$

i. AG(AF(b)) $\Rightarrow [\![AG(AF(b))]\!] = \{s_0, s_1, s_2, s_3, s_4\}$ (by the computation of $AG\varphi$, see below)

we proved $[\![AG\varphi]\!]$ is the greatest fixed point K_φ in 3(a)

by Knaster-Tarski theorem, $f^n(S)$ is the greatest fixed point.

$K_b(S) = [\![b]\!] \cap R_\forall(S)$

$= [\![b]\!] = \{s_2, s_3, s_4\}$

$K_b(K_b(S)) = [\![b]\!] \cap R_\forall(\{s_2, s_3, s_4\})$

$= \{s_2, s_3, s_4\} \cap \{s_1, s_2, s_4\}$

$= \{s_2, s_4\}$

$K_b(K_b(K_b(S))) = [\![b]\!] \cap R_\forall(\{s_2, s_4\})$

$= \{s_2, s_3, s_4\} \cap \{s_1, s_4\}$

$= \{s_4\}$

$K_b(K_b(K_b(K_b(S))))$ is not necessary because of the same result: $\{s_4\}$

the greatest fixed point $AG\varphi = \{s_4\}$

$[\![AF(AG(b))]\!] = [\![AF(\{s_4\})]\!]$

we can use same way that we did in computation of $AF\varphi$

ii. AF(AG(b)) then we have $AF(AG(b)) = \{s_4\}$