Introduction to data science & artificial intelligence (INF7100)

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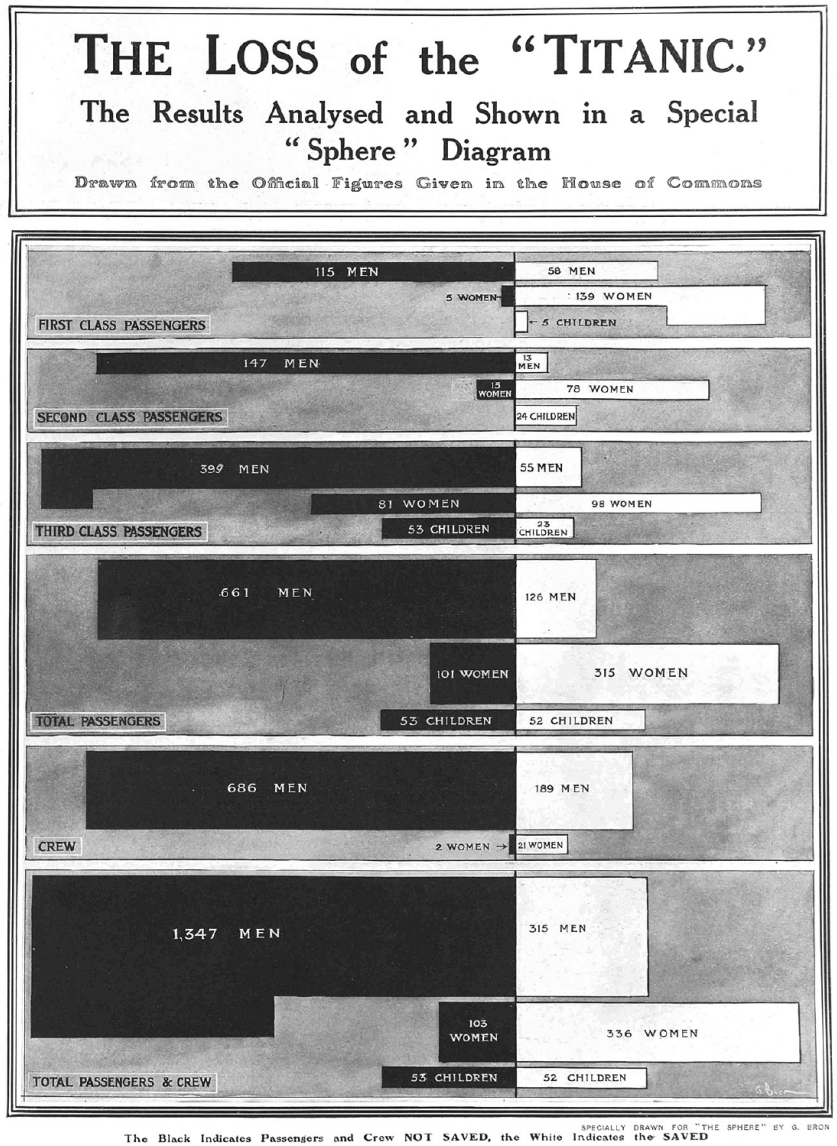
#351 Classification

été 2020

Titanic, who survived?

$y \in \{0, 1\}$, see
freakonometrics.hypotheses.org



Fisher's Discriminent Analysis

Suppose that we only want to predict the class, $\widehat{y} \in \{0, 1\}$ (or more generally $\widehat{y} \in \{a_1, a_2, \cdots, a_J\}$)

$$m^\star(\boldsymbol{x}) = \underset{y \in \{0,1\}}{\text{argmin}} \{\mathbb{P}[Y = y | \boldsymbol{X} = \boldsymbol{x}]\}$$

i.e.

$$m^\star(\boldsymbol{x}) = \underset{y \in \{0,1\}}{\text{argmin}} \left\{ \frac{\mathbb{P}[\boldsymbol{X} = \boldsymbol{x} | Y = y]}{\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}]} \right\}$$

(where $\mathbb{P}[\boldsymbol{X} = \boldsymbol{x}]$ is $f(\boldsymbol{x})$ is the continuous case).
If y takes two values – i.e. $\{0, 1\}$

$$m^\star(\boldsymbol{x}) = \begin{cases} 1 \text{ if } \mathbb{E}(Y | \boldsymbol{X} = \boldsymbol{x}) > \frac{1}{2} \\ 0 \text{ otherwise} \end{cases}$$

Fisher's Discriminent Analysis

The set

$$\mathcal{D}_S = \left\{ \boldsymbol{x} : \ \mathbb{E}(Y|\boldsymbol{X} = \boldsymbol{x}) = \frac{1}{2} \right\}$$

is called decision border.

Assume that $\boldsymbol{X}|Y = 0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$ and $\boldsymbol{X}|Y = 1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$, then, if r_y^2 is Manalahobis distance from $\boldsymbol{x}$ to $\boldsymbol{\mu}_y$

$$r_y^2 = [\boldsymbol{x} - \boldsymbol{\mu}_y]^\top \boldsymbol{\Sigma}_y^{-1} [\boldsymbol{x} - \boldsymbol{\mu}_y] \text{ for } y \in \{0, 1\},$$

$$m^\star(\boldsymbol{x}) = \begin{cases} 1 \text{ if } r_1^2 < r_0^2 + 2\log \dfrac{\mathbb{P}(Y = 1)}{\mathbb{P}(Y = 0)} + \log \dfrac{|\boldsymbol{\Sigma}_0|}{|\boldsymbol{\Sigma}_1|} \\ 0 \text{ otherwise} \end{cases}$$

Fisher's Discriminent Analysis

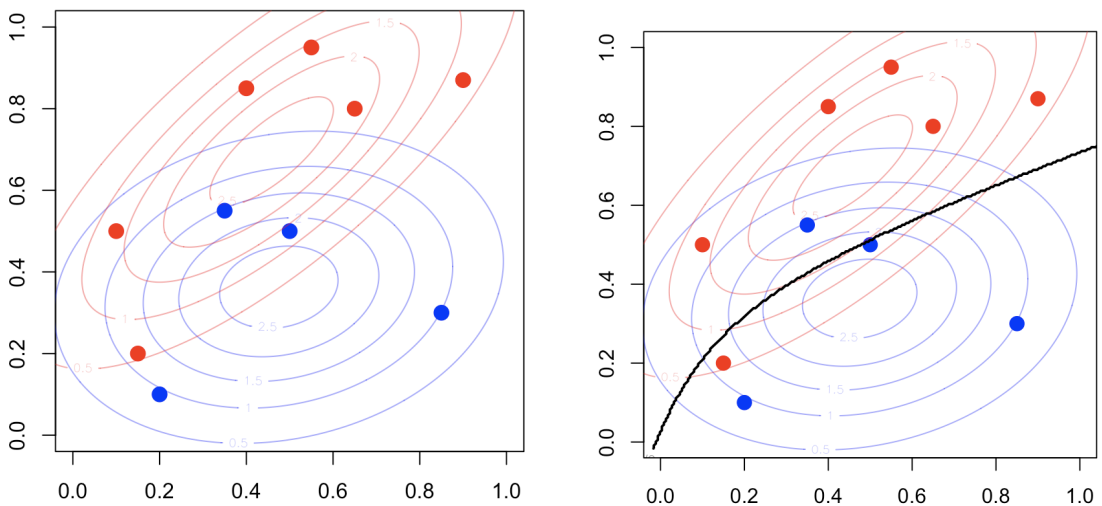
Let δ_y be defined (for $y \in \{0, 1\}$) as

$$\delta_y(\boldsymbol{x}) = -\frac{1}{2}\log|\boldsymbol{\Sigma}_y| - \frac{1}{2}[\boldsymbol{x} - \boldsymbol{\mu}_y]^\top \boldsymbol{\Sigma}_y^{-1}[\boldsymbol{x} - \boldsymbol{\mu}_y] + \log \mathbb{P}(Y = y)$$

so that the decision frontier is

$$\{\boldsymbol{x} \text{ such that } \delta_0(\boldsymbol{x}) = \delta_1(\boldsymbol{x})\}$$

which is quadratic in $\boldsymbol{x}$

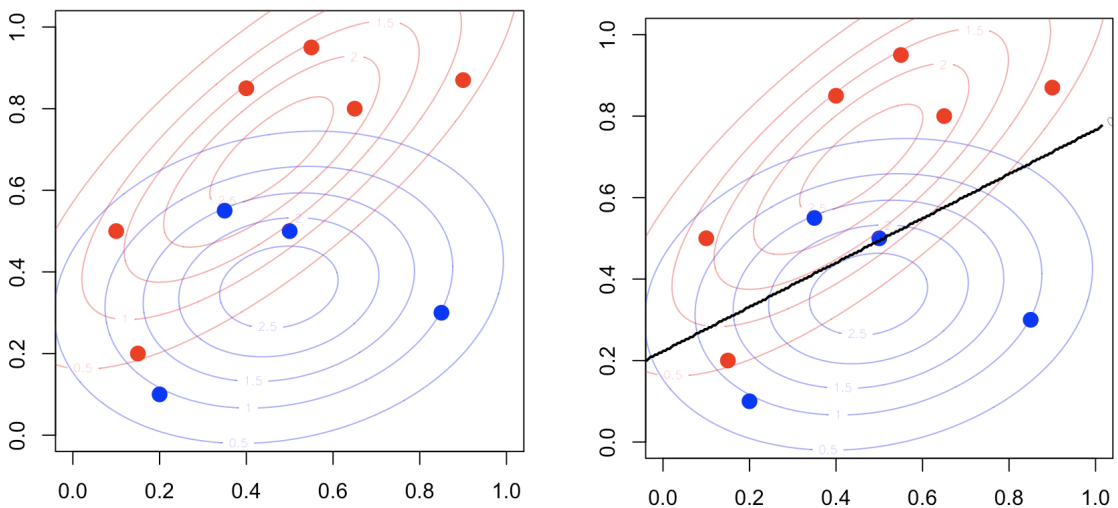


Fisher's Discriminent Analysis

Fisher (1936) added the assumption $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_1$. Then

$$\delta_y(\boldsymbol{x}) = \boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_y - \frac{1}{2} \boldsymbol{\mu}_y^\top \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}_y + \log \mathbb{P}(Y = y)$$

and the decision border is linear in $\boldsymbol{x}$



Fisher's Discriminent Analysis

If $\boldsymbol{X}|Y=0 \sim \mathcal{N}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma})$ and $\boldsymbol{X}|Y=1 \sim \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma})$ then

$$\log \frac{\mathbb{P}(Y=1|\boldsymbol{X}=\boldsymbol{x})}{\mathbb{P}(Y=0|\boldsymbol{X}=\boldsymbol{x})}$$

is equal to

$$\boldsymbol{x}^\top \boldsymbol{\Sigma}^{-1}[\boldsymbol{\mu}_y] - \frac{1}{2}[\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0]^\top \boldsymbol{\Sigma}^{-1}[\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0] + \log \frac{\mathbb{P}(Y=1)}{\mathbb{P}(Y=0)}$$

which is linear in $\boldsymbol{x}$, i.e.

$$\log \frac{\mathbb{P}(Y=1|\boldsymbol{X}=\boldsymbol{x})}{\mathbb{P}(Y=0|\boldsymbol{X}=\boldsymbol{x})} = \boldsymbol{x}^\top \boldsymbol{\beta}$$

which will be close to the linear regression...

Logistic Regression

$$p_i = \mathbb{E}(Y_i | \boldsymbol{X}_i = \boldsymbol{x}_i) \in [0,1] \quad \neq \boldsymbol{x}_i^\top \boldsymbol{\beta}$$

$\longrightarrow$ use the odds ratio

$$\text{odds}_i = \frac{\mathbb{P}[Y_i = 1]}{\mathbb{P}[Y_i = 0]} = \frac{p_i}{1 - p_i} \in [0, \infty].$$

i.e. if we take the logarithm

$$\log(\text{odds}_i) = \log\left(\frac{p_i}{1 - p_i}\right) \in \mathbb{R}.$$

That transformation is called logit,

$$\text{logit}(p_i) = \log\left(\frac{p_i}{1 - p_i}\right) = \boldsymbol{x}_i^\top \boldsymbol{\beta}$$

or

$$p_i = \text{logit}^{-1}(\boldsymbol{x}_i^\top \boldsymbol{\beta}) = \frac{\exp[\boldsymbol{x}_i^\top \boldsymbol{\beta}]}{1 + \exp[\boldsymbol{x}_i^\top \boldsymbol{\beta}]}.$$

Logistic Regression

Data : $\{(\boldsymbol{x}_i, y_i) = (x_{1,i}, x_{2,i}, y_i), i = 1, \cdots, n\}$

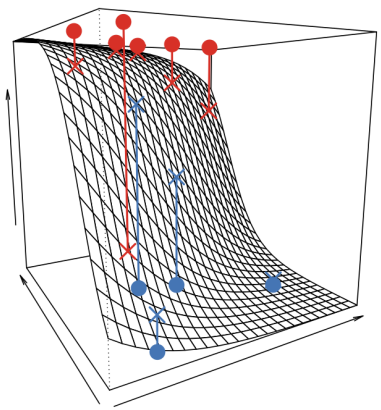
$$\mathbb{P}(Y = 1|\boldsymbol{X} = \boldsymbol{x}) = \frac{\exp[\boldsymbol{x}^\top \boldsymbol{\beta}]}{1 + \exp[\boldsymbol{x}^\top \boldsymbol{\beta}]}$$

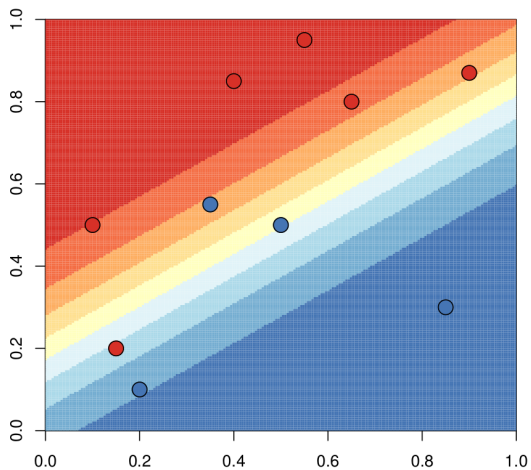
Inference using maximum likelihood techniques

$$\widehat{\boldsymbol{\beta}} = \text{argmin} \left\{ \sum_{i=1}^{n} \log[\mathbb{P}(Y = y_i|\boldsymbol{X} = \boldsymbol{x}_i)] \right\}$$

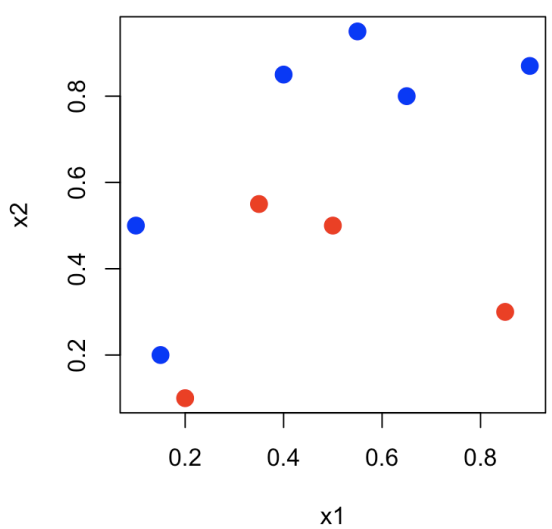
and the score model is then

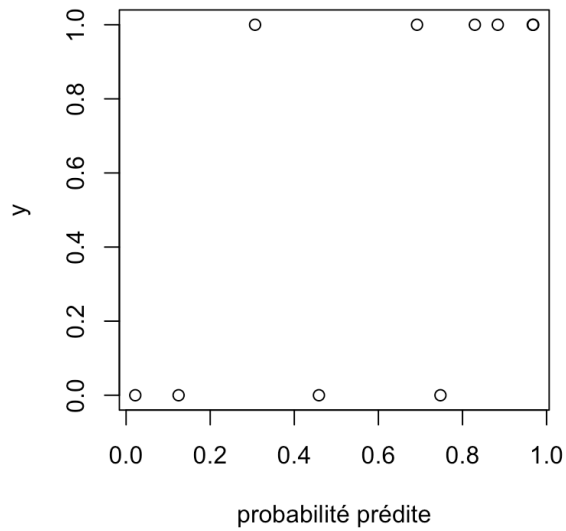
$$s(\boldsymbol{x}) = \frac{\exp[\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}]}{1 + \exp[\boldsymbol{x}^\top \widehat{\boldsymbol{\beta}}]}$$

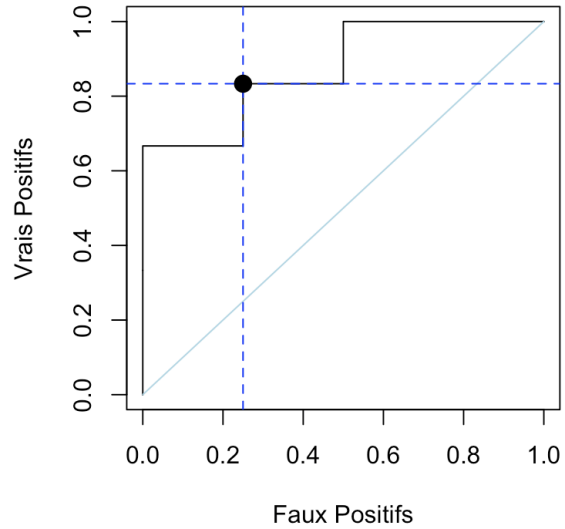


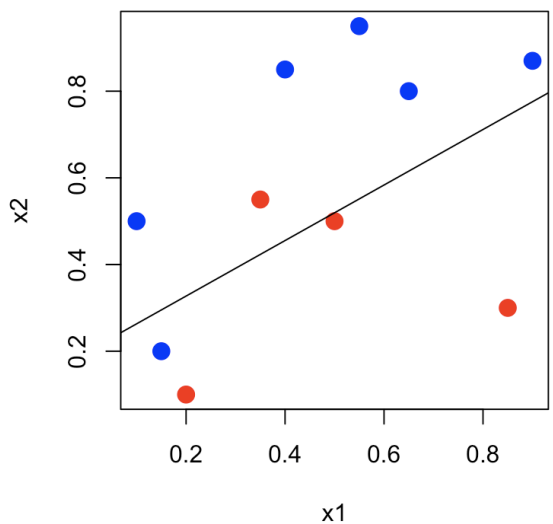


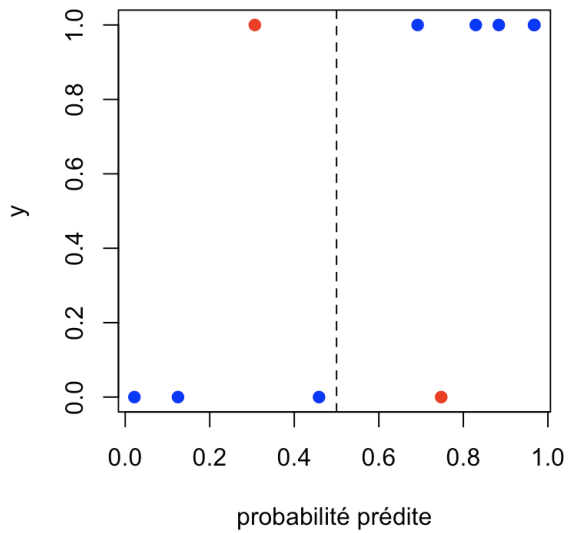
ROC Curve

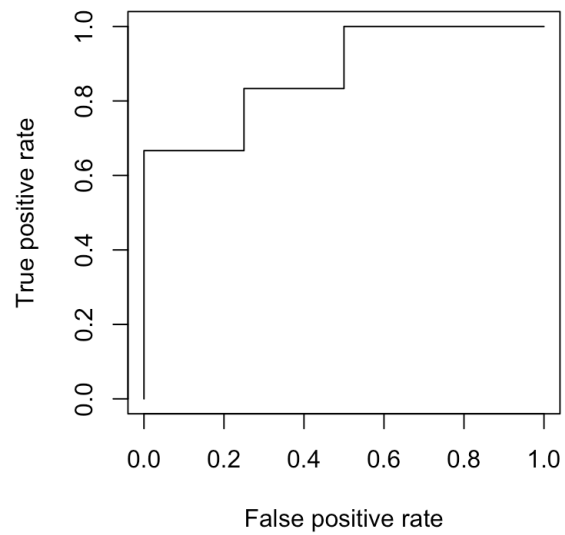

x2
0.2
0.4
0.6
0.8
x1
0.2
0.4
0.6
0.8


y
0.0
0.2
0.4
0.6
0.8
1.0
probabilité prédite
0.0
0.2
0.4
0.6
0.8
1.0


Vrais Positifs
0.0
0.2
0.4
0.6
0.8
1.0
Faux Positifs
0.0
0.2
0.4
0.6
0.8
1.0


x2
0.2
0.4
0.6
0.8
x1
0.2
0.4
0.6
0.8


y
0.0
0.2
0.4
0.6
0.8
1.0
probabilité prédite
0.0
0.2
0.4
0.6
0.8
1.0


True positive rate
0.0
0.2
0.4
0.6
0.8
1.0
False positive rate
0.0
0.2
0.4
0.6
0.8
1.0

ROC Curve

$$FPR = \frac{\mathbb{P}[y = 0, \widehat{y} = 1]}{\mathbb{P}[y = 0]} \text{ et } TPR = \frac{\mathbb{P}[y = 1, \widehat{y} = 1]}{\mathbb{P}[y = 1]}$$