Introduction to data science & artificial intelligence (INF7100)

Arthur Charpentier

#422 Linear Algebra

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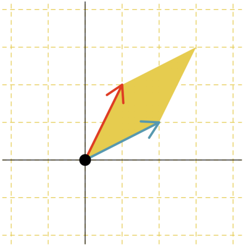
Determinant of square matrices (2×2)

Let $\boldsymbol{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, then

$$\det(\boldsymbol{M}) = |\boldsymbol{M}| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$



Note that $|\det(\boldsymbol{M})|$ is the area of the parallelogram, $\vec{\boldsymbol{u}} = \begin{pmatrix} a \\ b \end{pmatrix}$ and $\vec{\boldsymbol{v}} = \begin{pmatrix} c \\ d \end{pmatrix}$

Thus, $\det(\boldsymbol{M}) = 0$ if and only if $\vec{\boldsymbol{u}} /\!/ \vec{\boldsymbol{v}}$
Let θ denote the angle $(\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}})$,

$$\cos(\theta) = \frac{\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle}{\|\vec{\boldsymbol{u}}\| \cdot \|\vec{\boldsymbol{v}}\|} \text{ and } \sin(\theta) = \frac{\det(M)}{\|\vec{\boldsymbol{u}}\| \cdot \|\vec{\boldsymbol{v}}\|}, \text{ where } M = (\vec{\boldsymbol{u}}\ \vec{\boldsymbol{v}})$$

Inverse of square matrices (2×2)

If $\det(\boldsymbol{M}) \neq 0$, then $\boldsymbol{M}$ has an inverse, denoted $\boldsymbol{M}^{-1}$,

$$\boldsymbol{M}\boldsymbol{M}^{-1} = \boldsymbol{M}^{-1}\boldsymbol{M} = \mathbb{I}$$

Example: $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ then $\boldsymbol{M}^{-1} = \dfrac{1}{4}\begin{pmatrix} 0 & -1 \\ 2 & 1 \end{pmatrix}$

Important for linear systems: find $\boldsymbol{x}$ such that $\boldsymbol{M}\boldsymbol{x} = \boldsymbol{a}$,
then the unique solution is $\boldsymbol{x} = \boldsymbol{M}^{-1}\boldsymbol{a}$

Example: $\boldsymbol{\Sigma} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ then $\boldsymbol{\Sigma}^{-1} = \dfrac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$, if $|\rho| < 1$.
(here $\boldsymbol{\Sigma}$ is a variance matrix)

A projection matrix cannot be inverted

Eigenvalues and Eigenvectors of square matrices

Let $\boldsymbol{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, $\vec{\boldsymbol{u}} = \begin{pmatrix} x \\ y \end{pmatrix}$ and λ such that $\boldsymbol{M}\vec{\boldsymbol{u}} = \lambda\vec{\boldsymbol{u}}$,

$\vec{\boldsymbol{u}}$ is the eigenvector associated with eigenvalue λ.

$$\begin{pmatrix} a-\lambda & c \\ b & d-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ implies } \det\begin{pmatrix} a-\lambda & c \\ b & d-\lambda \end{pmatrix} = 0.$$

i.e. solve (in λ) $\det(M - \lambda\mathbb{I}) = 0$.

This can be extended in higher dimension

If $\det(\boldsymbol{M}) = 0$ then $\lambda = 0$ is an eigenvalue.

If $\boldsymbol{M}$ is symmetric, eigenvectors are orthogonal.

$\boldsymbol{M}$ is said to be positive if all eigenvalues are positive, and then $\boldsymbol{z}^\top\boldsymbol{M}\boldsymbol{z} \geq 0$, $\forall\boldsymbol{z}$.

Example: Let $\boldsymbol{X}$ be a $n \times m$ matrix, then $\boldsymbol{X}^\top\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{X}^\top$ are symmetric and positive matrices.
(think of $\boldsymbol{X}^\top\boldsymbol{X}$ as a covariance matrix).

Quadratic Forms

Consider $\boldsymbol{M} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$,

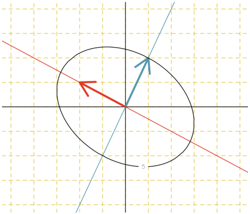
and function $\boldsymbol{z} \mapsto \boldsymbol{z}^\top \boldsymbol{M} \boldsymbol{z}$, i.e.

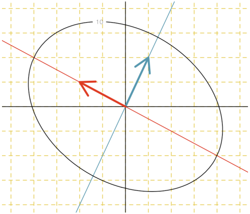
$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

or $ax^2 + 2bxy + cy^2$ is a quadratic form.

If $\det(\boldsymbol{M}) > 0$, points $\boldsymbol{z} = (x, y)$ such that $\boldsymbol{z}^\top \boldsymbol{M} \boldsymbol{z} = \gamma$, for some $\gamma > 0$, are on an ellipse (centered on $\boldsymbol{0}$)

Let $\lambda_1 \geq \lambda_2 > 0$ denote the eigenvalues of $\boldsymbol{M}$ and $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$ denote the eigenvectors.





Quadratic Forms

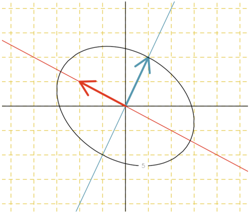
On the picture, $\boldsymbol{M} = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{pmatrix}$

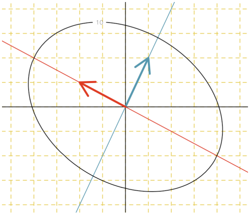
in Python

```
> import numpy as np
> from numpy import linalg
> M = np.array([[.6,.2], [.2, .9]])
> l, v = linalg.eig(M)
> print(l)
[ 0.5   1. ]
> print(v)
[[-0.89442719 -0.4472136 ]
 [ 0.4472136  -0.89442719]]
```

i.e. $\lambda_1 = 1/2$ and $\lambda_2 = 1$, and

$$\vec{\boldsymbol{v}}_1 = \begin{pmatrix} -2\sqrt{5} \\ \sqrt{5} \end{pmatrix} = \sqrt{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \ \vec{\boldsymbol{v}}_2 = \sqrt{5}\begin{pmatrix} -1 \\ -2 \end{pmatrix}$$





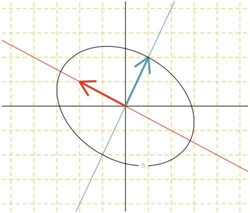
Quadratic Forms

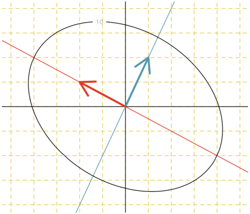
On the picture, $\boldsymbol{M} = \begin{pmatrix} 0.6 & 0.2 \\ 0.2 & 0.9 \end{pmatrix}$

in R

```
> M=matrix(c(.6,.2,.2,.9),2,2)
> eigen(M)
eigen() decomposition
$values
[1] 1.0 0.5
$vectors
           [,1]       [,2]
[1,] 0.4472136 -0.8944272
[2,] 0.8944272  0.4472136
```

Note that $\|\vec{\boldsymbol{v}}_1\| = \|\vec{\boldsymbol{v}}_2\| = 1$ and $\vec{\boldsymbol{v}}_1 \perp \vec{\boldsymbol{v}}_2$





Spectral & Singular Value Decomposition

Let $\boldsymbol{M}$ denote a symmetric $d \times d$, with eigenvalues $\lambda_1, \cdots, \lambda_d$ and eigenvectors $\vec{\boldsymbol{u}}_1, \cdots, \vec{\boldsymbol{u}}_d$ with $\|\vec{\boldsymbol{u}}_j\| = 1$,

$$\boldsymbol{M} = \underbrace{\begin{pmatrix} | & | & & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_d \\ | & | & & | \end{pmatrix}}_{d\times d} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_d \end{pmatrix}}_{d\times d} \underbrace{\begin{pmatrix} -\boldsymbol{u}_1^\top- \\ -\boldsymbol{u}_2^\top- \\ \vdots \\ -\boldsymbol{u}_d^\top- \end{pmatrix}}_{d\times d}$$

i.e. $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^\top$.

Let $\boldsymbol{M}$ denote $p \times q$, with $p \leq q$, we can write

$$\boldsymbol{M} = \underbrace{\begin{pmatrix} | & | & & | \\ \boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_p \\ | & | & & | \end{pmatrix}}_{p\times p} \underbrace{\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_p \end{pmatrix}}_{p\times p} \underbrace{\begin{pmatrix} -\boldsymbol{v}_1^\top- \\ -\boldsymbol{v}_2^\top- \\ \vdots \\ -\boldsymbol{v}_p^\top- \end{pmatrix}}_{p\times q}$$

i.e. $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^\top$, with $\boldsymbol{u} \in \mathbb{R}^p$ and $\boldsymbol{v} \in \mathbb{R}^q$

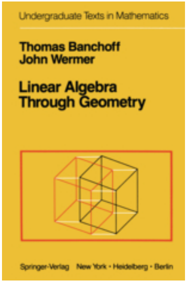
Approximations

Consider $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^\top$ et $\tilde{\boldsymbol{M}} = \boldsymbol{U}\tilde{\boldsymbol{\Lambda}}\boldsymbol{V}^\top$ where $\tilde{\boldsymbol{\Lambda}} = \text{diag}(\lambda_1\lambda_2, \boldsymbol{0})$.

$$\begin{pmatrix} 91 & 107 & 136 & 4 \\ 91 & 82 & 86 & 81 \\ 31 & 33 & -52 & 300 \\ 105 & 86 & 97 & 28 \\ 84 & 80 & 78 & 76 \\ 96 & 97 & 124 & 15 \end{pmatrix}^\top \simeq \begin{pmatrix} 103.5 & 101.4 & 130.5 & 2.2 \\ 87.9 & 85.7 & 85.6 & 80.9 \\ 33.5 & 31.7 & -53.0 & 299.7 \\ 91.1 & 89.1 & 105.6 & 30.8 \\ 82.0 & 80.0 & 79.6 & 76.5 \\ 99.3 & 97.2 & 121.3 & 14.1 \end{pmatrix}^\top$$

The old dataset $\boldsymbol{M}$ is $n = 6$ obsevations, in dimension 4
The new dataset $\tilde{\boldsymbol{M}}$ is $n = 6$ obsevations, in (real) dimension 2
(see principal components #271)

References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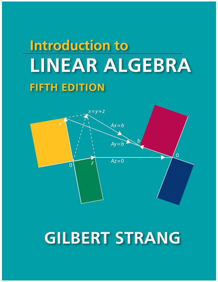
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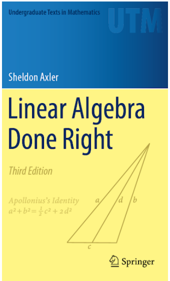
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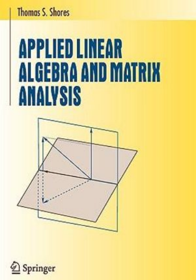
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