STT 1000 - STATISTIQVES

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Likelihood / Vraisemblance

Assume that $\{x_1, x_2, \cdots, x_n\}$ are obtained from i.i.d. random variables $X_1, X_2, \cdots, X_n$, with identical distribution F_θ, and density f_θ.

$$\mathcal{L}(\theta) = f_\theta(\boldsymbol{x}) = f_\theta(x_1, \cdots, x_n) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$\log \mathcal{L}(\theta) = \sum_{i=1}^{n} \log f_\theta(x_i)$$

$\widehat{\theta}$ is a maximum likelihood estimator of parameter θ if

$$\widehat{\theta} \in \text{argmax}\{\mathcal{L}(\theta)\} = \text{argmax}\{\log \mathcal{L}(\theta)\}$$

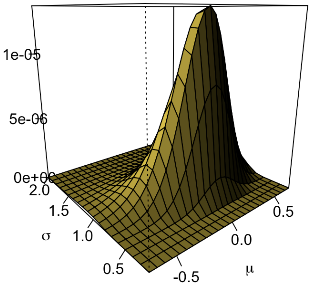
Likelihood / Vraisemblance

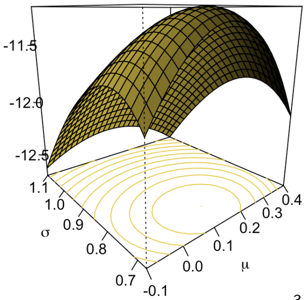
Given some sample $\{x_1, \cdots, x_n\}$
from a $\mathcal{N}(\mu, \sigma^2)$ distribution,

$$\mathcal{L}(\mu, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{n/2}} \exp\left(-\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2}\right)$$

$$\log \mathcal{L}(\mu, \sigma^2) = -\sum_{i=1}^{n} \frac{(x_i - \mu)^2}{\sigma^2} - \frac{n}{2} \log(2\pi\sigma^2)$$

Here $\boldsymbol{\theta} = (\mu, \sigma^2) \in \mathbb{R} \times (0, \infty)$.





Likelihood / Vraisemblance

The first order condition (also called likelihood equations) is

$$\left.\frac{\partial \log\left(\mathcal{L}(\theta; x_1, \cdots, x_n)\right)}{\partial \theta}\right|_{\theta=\widehat{\theta}} = 0$$

Second order condition is

$$\left.\frac{\partial^2 \log\left(\mathcal{L}(\theta; x_1, \cdots, x_n)\right)}{\partial \theta}\right|_{\theta=\widehat{\theta}} < 0$$

Example: if $X \sim \mathcal{P}(\lambda)$,

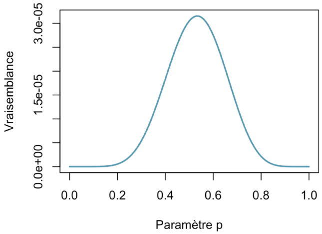
$$\log \mathcal{L}(\lambda; x_1, \cdots, x_n) = \sum_{i=1}^{n} \log\left(e^{-\lambda}\frac{\lambda^{x_i}}{x_i!}\right) = -n\lambda + n\overline{x}\log(\lambda) - \log\left(\prod_{i=1}^{n} x_i!\right)$$

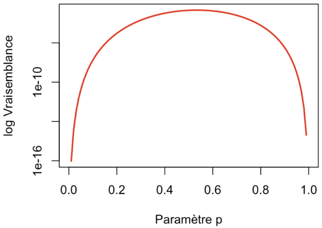
$$\frac{\partial \log\left(\mathcal{L}(\lambda; x_1, \cdots, x_n)\right)}{\partial \lambda} = -n + \frac{n\overline{x}}{\lambda}, \text{ so } \widehat{\lambda} = \overline{x}.$$

Vraisemblance, cas $\mathcal{B}(p)$

- ce que nous dit la pratique

```
> n=15
> set.seed(1)
> x=sample(0:1,size = n,prob = c(.4,.6),replace=TRUE)
> vraisemblance = function(p) prod(dbinom(x,size = 1,
   prob = p))
> vect_p = seq(0,1,by=0.01)
> plot(vect_p,Vectorize(vraisemblance)(vect_p))
```





Likelihood / Vraisemblance

Score

La fonction $S : x \mapsto \dfrac{d}{d\theta}\log f_\theta(x)$ est appelée fonction score.

If $X \sim f_\theta$, $\mathbb{E}\left(\dfrac{d}{d\theta}\log f_\theta(X)\right) = \mathbb{E}\left(S(X)\right) = 0$

Example: if $X \sim \mathcal{P}(\lambda)$,

$$\frac{d\log f_\lambda(X)}{d\lambda} = -1 + \frac{X}{\lambda},\ \text{so } \mathbb{E}\left(\frac{d}{d\lambda}\log f_\lambda(X)\right) = -1 + \frac{\mathbb{E}(X)}{\lambda} = 0$$

Example: if $X \sim \mathcal{B}(p)$,

$$\frac{d\log f_\lambda(X)}{d\lambda} = \frac{X}{p} - \frac{1-X}{1-p}, \mathbb{E}\left(\frac{d}{d\lambda}\log f_\lambda(X)\right) = \frac{\mathbb{E}(X)}{p} - \frac{1-\mathbb{E}(X)}{1-p} = 0$$

Likelihood / Vraisemblance

Si $x_1, \cdots, x_n$ est tiré suivant f_θ, $\mathbb{E}\left(\frac{d}{d\theta}\log\mathcal{L}(\boldsymbol{X})\right) = 0$

Example: if $X_i \sim \mathcal{P}(\lambda)$,

$$\frac{d\log\mathcal{L}}{d\lambda} = -n + \frac{\sum_{i=1}^n x_i}{\lambda}, \mathbb{E}\left(\frac{d}{d\lambda}\log\mathcal{L}\right) = -n + \frac{\sum_{i=1}^n \mathbb{E}(X_i)}{\lambda} = 0$$

Example: if $X_i \sim \mathcal{B}(p)$,

$$\frac{d\log\mathcal{L}}{d\lambda} = \frac{\sum_{i=1}^n X_i}{p} - \frac{n - \sum_{i=1}^n X_i}{1-p},$$

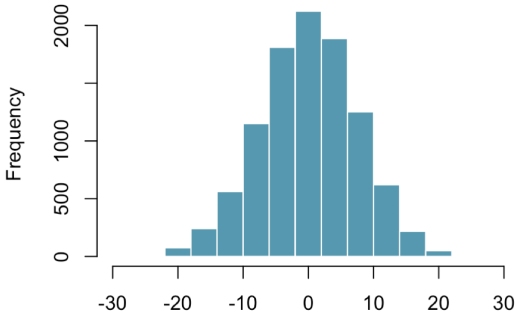
$$\mathbb{E}\left(\frac{d}{d\lambda}\log\mathcal{L}\right) = \frac{\sum_{i=1}^n \mathbb{E}(X_i)}{p} - \frac{n - \sum_{i=1}^n \mathbb{E}(X_i)}{1-p} = 0$$

Likelihood / Vraisemblance

```
> pente=rep(NA,1e4)
> for(s in 1:1e4){
+   x=sample(0:1, n,prob = c(.4,.6),replace=TRUE)
+   dlogL = function(p) sum(x)/p-(sum(1-x))/(1-p)
+   pente[s] = dlogL(0.6) }
> hist(pente)
```

La distribution empirique de $\dfrac{d}{d\theta}\log\mathcal{L}(\boldsymbol{X})$ est

```
> mean(pente)
[1] -0.03458333
> var(pente)
[1] 63.51731
> 1/var(pente)
[1] 0.01574374
> .4*.6/15
[1] 0.016
```



Information de Fisher

Information de Fisher

Fisher information associated with a density f_θ, with $\theta \in \mathbb{R}$ is

$$I(\theta) = \mathbb{E}\left(\frac{d}{d\theta}\log f_\theta(X)\right)^2 \text{ where } X \text{ has distribution } f_\theta,$$

$$I(\theta) = \text{Var}\left(\frac{d}{d\theta}\log f_\theta(X)\right) = -\mathbb{E}\left(\frac{d^2}{d\theta^2}\log f_\theta(X)\right).$$

For a sample of size n,

$$I_n(\theta) = \mathbb{E}\left(\frac{\partial}{\partial\theta}\log \mathcal{L}(\theta, X_1, \cdots, X_n)\right)^2 = nI(\theta)$$

Information de Fisher

en effet, posons $U = \dfrac{\partial}{\partial\theta} \ln f(X;\theta) = \dfrac{\frac{\partial}{\partial\theta} f(X;\theta)}{f(X;\theta)}$

$$E_\theta(U) = \int_{\mathbb{R}} \frac{\frac{\partial}{\partial\theta} f(x;\theta)}{f(x;\theta)} f(x;\theta)dx = \int_{\mathbb{R}} \frac{\partial}{\partial\theta} f(x;\theta)dx = \frac{\partial}{\partial\theta} \underbrace{\int_{\mathbb{R}} f(x;\theta)dx}_{=1} = 0$$

$$\begin{aligned}\frac{\partial^2}{\partial\theta^2} \ln f(x;\theta) &= \frac{\frac{\partial^2}{\partial\theta^2} f(x;\theta)\cdot f(x;\theta) - \left[\frac{\partial}{\partial\theta} f(x;\theta)\right]^2}{[f(x;\theta]^2} \\ &= \frac{\frac{\partial^2}{\partial\theta^2} f(x;\theta)}{f(x;\theta)} - \left[\frac{\frac{\partial}{\partial\theta} f(x;\theta}{f(x;\theta)}\right]^2\end{aligned}$$

Information de Fisher

donc

$$E_\theta\left[\frac{\partial^2}{\partial\theta^2}\ln f(X;\theta)\right]=E_\theta\left[\frac{\frac{\partial^2}{\partial\theta^2}f(X;\theta)}{f(X;\theta)}\right]-E_\theta\left[U^2\right]$$

or

$$E_\theta\left[\frac{\frac{\partial^2}{\partial\theta^2}f(X;\theta)}{f(X;\theta)}\right]=\int_{\mathbb{R}}\frac{\partial^2}{\partial\theta^2}f(x;\theta)dx=\frac{\partial^2}{\partial\theta^2}\int_{\mathbb{R}}f(x;\theta)dx=0$$

donc

$$\text{Var}\left(\frac{\partial}{\partial\theta}\log f_\theta(X)\right)=-\mathbb{E}\left(\frac{\partial^2}{\partial\theta^2}\log f_\theta(X)\right).$$

Likelihood / Vraisemblance

Example: if X has a Poisson distribution $\mathcal{P}(\theta)$,

$$\log f_\theta(x) = -\theta + x\log\theta - \log(x!) \text{ and } \frac{d^2}{d\theta^2}\log f_\theta(x) = -\frac{x}{\theta^2}$$

$$I(\theta) = -\mathbb{E}\left(\frac{d^2}{d\theta^2}\log f_\theta(X)\right) = -\mathbb{E}\left(-\frac{X}{\theta^2}\right) = \frac{1}{\theta}$$

Example: if X has a binomial distribution $\mathcal{B}(1,\theta)$,

$$I(\theta) = -\mathbb{E}\left(\frac{d^2}{d\theta^2}\log f_\theta(X)\right) = -\mathbb{E}\left(-\frac{X}{\theta^2} + \frac{1-X}{(1-\theta)^2} +\right) = \frac{1}{\theta(1-\theta)}$$

Cramér-Rao

Borne de Cramér-Rao

If $\widehat{\theta}$ is an unbiased estimator of θ, then

$$\text{Var}(\widehat{\theta}) \geq \frac{1}{nI(\theta)}$$

If that bound is attained, the estimator is said to be efficient.
An unbiased estimator $\widehat{\theta}$ is said to be optimal if it has the lowest variance among all unbiased estimators, see bias, minimum variance unbiased estimator

Likelihood / Vraisemblance

$$\text{if } \boldsymbol{\theta} \in \mathbb{R}^k, \left.\frac{\partial \log(\mathcal{L}(\boldsymbol{\theta}; x_1, \cdots, x_n))}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = \mathbf{0}$$

Second order condition is

$$\text{if } \boldsymbol{\theta} \in \mathbb{R}^k, \left.\frac{\partial^2 \log(\mathcal{L}(\boldsymbol{\theta}; x_1, \cdots, x_n))}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}'}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} \text{ is definite negative}$$

If $\boldsymbol{\theta} \in \mathbb{R}^k$, then Fisher information is the $k \times k$ matrix $I = [I_{i,j}]$ with

$$I_{i,j} = \mathbb{E}\left(\frac{\partial}{\partial \theta_i} \log f_{\boldsymbol{\theta}}(X) \frac{\partial}{\partial \theta_j} \log f_{\boldsymbol{\theta}}(X)\right).$$

i.e.

$$I(\theta) = \mathbb{E}\left[\left(\frac{d}{d\boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(X)\right)\left(\frac{d}{d\boldsymbol{\theta}} \log f_{\boldsymbol{\theta}}(X)\right)^{\top}\right]$$

$$I(\theta) = -\mathbb{E}\left(\frac{d^2}{d\boldsymbol{\theta} d\boldsymbol{\theta}^{\top}} \log f_{\boldsymbol{\theta}}(X)\right)$$

Likelihood / Vraisemblance

For a Gaussian distribution $\mathcal{N}(\theta, \sigma^2)$, $I(\theta) = \dfrac{1}{\sigma^2}$

For a Gaussian distribution $\mathcal{N}(\mu, \theta)$, $I(\theta) = \dfrac{1}{2\theta^2}$

For a Gaussian distribution $\mathcal{N}(\boldsymbol{\theta})$, $I(\boldsymbol{\theta}) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}$

Cramér-Rao bound is $\dfrac{1}{n} I^{-1} = \dfrac{1}{n} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix}$

Likelihood / Vraisemblance

En effet, la log-densité de la loi normale $\mathcal{N}(\mu, v)$ est

$$\log f(x;\mu,v) = -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log v - \frac{1}{2v}(x-\mu)^2$$

Les dérivées premières sont

$$\frac{\partial}{\partial\mu}\ln f(x;\mu,v) = \frac{1}{v}(x-\mu) \text{ et } \frac{\partial}{\partial v}\ln f(x;\mu,v) = -\frac{1}{2v} + \frac{(x-\mu)^2}{2v^2}$$

$$E\left[\frac{1}{v^2}(X-\mu)^2\right] = \frac{1}{v^2}v = \frac{1}{v} = \frac{1}{\sigma^2}$$

$$| \; E\left[\left(-\frac{1}{2v} + \frac{(X-\mu)^2}{2v^2}\right)^2\right] = E\left[\frac{1}{4v^2} - \frac{(X-\mu)^2}{2v^3} + \frac{(X-\mu)^4}{4v^4}\right]$$

$$= \frac{1}{4v^2} - \frac{v}{2v^3} + \frac{3v^2}{4v^4} = \frac{1}{2v^2} = \frac{1}{2\sigma^4}$$

car $E\left[(X-\mu)^4\right] = 3\sigma^4$.

Likelihood / Vraisemblance

Enfin

$$E\left[\frac{1}{v}(X-\mu)\left(-\frac{1}{2v}+\frac{(X-\mu)^2}{2v^2}\right)\right]=0$$

d'où la matrice d'information

$$I\left(\mu,\sigma^2\right)=\begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{pmatrix}$$

Likelihood / Vraisemblance

Let $\{x_1, \cdots, x_n\}$ be a sample with distribution $f_{\boldsymbol{\theta}}$, where $\boldsymbol{\theta} \in \boldsymbol{\Theta}$. The maximum likelihood estimator $\widehat{\boldsymbol{\theta}}_n$ of $\boldsymbol{\theta}$ is

$$\widehat{\boldsymbol{\theta}}_n \in \text{argmax}\{\mathcal{L}(\boldsymbol{\theta}; x_1, \cdots, x_n), \boldsymbol{\theta} \in \boldsymbol{\Theta}\}.$$

Propriétés asymptotiques de l'EMV

Under some technical assumptions $\widehat{\boldsymbol{\theta}}_n$ converges almost surely towards $\boldsymbol{\theta}$, $\widehat{\boldsymbol{\theta}}_n \overset{a.s.}{\to} \boldsymbol{\theta}$, as $n \to \infty$.
Under some technical assumptions $\widehat{\boldsymbol{\theta}}_n$ is asymptotically efficient,

$$\sqrt{n}(\widehat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \overset{\mathcal{L}}{\to} \mathcal{N}(0, I^{-1}(\boldsymbol{\theta})).$$

See maximum likelihood estimation

Consider some Poisson model,

```
> set.seed(1)
> (y=rpois(10,3))
[1] 2 2 3 5 2 5 6 4 3 1
> mean(y)
[1] 3.3
> NLogL = function(lambda) -sum(log(dpois(y,lambda)))
> optim(fn = NLogL,par = 1)
$par
[1] 3.3

$value
[1] 18.59581
```

Calcul numérique du maximum de vraisemblance

Consider a sample $\boldsymbol{X}=(X_1,\cdots,X_n)$ i.id. from $F_{\boldsymbol{\theta}}$. Let

$$S_{n,\boldsymbol{\theta}}(\boldsymbol{x})=\frac{\partial \log \mathcal{L}(\boldsymbol{\theta};\boldsymbol{x})}{\partial \boldsymbol{\theta}}=\sum_{i=1}^{n} S_{1,\boldsymbol{\theta}}(x_i)$$

denote the score function. Then $S_{n,\boldsymbol{\theta}}(\boldsymbol{X})$ is a random vector. Then

$$\mathbb{E}[S_{n,\boldsymbol{\theta}}(\boldsymbol{X})]=\boldsymbol{0}$$

while

$$\text{Var}[S_{n,\boldsymbol{\theta}}(\boldsymbol{X})]=I_n(\boldsymbol{\theta})=\mathbb{E}\left(\frac{\partial}{\partial \boldsymbol{\theta}} S_{n,\boldsymbol{\theta}}(\boldsymbol{X})\right).$$

$$\frac{S_{n,\theta}(\boldsymbol{X})}{n}\overset{a.s.}{\to}0 \quad \text{and} \quad \frac{S_{n,\boldsymbol{\theta}}(\boldsymbol{X})}{\sqrt{n}}\overset{\mathcal{L}}{\to}\mathcal{N}(0,I(\boldsymbol{\theta})).$$

Calcul numérique du maximum de vraisemblance

If θ is univariate, use Taylor approximation of S_n in the neighbourhood of θ_0 (the true value)

$$S_n(x) = S_n(\theta_0) + (x - \theta_0)S_n'(y) \text{ for some } y \in [x, \theta_0]$$

Set $x = \widehat{\theta}_n$, then

$$S_n(\widehat{\theta}_n) = 0 = S_n(\theta_0) + (\widehat{\theta}_n - \theta_0)S_n'(y) \text{ for some } y \in [\theta_0, \widehat{\theta}_n]$$

Hence, $\widehat{\theta}_n = \theta_0 - \dfrac{S_n(\theta_0)}{S_n'(y)}$ for $y \in [\theta_0, \widehat{\theta}_n]$.

Algorithme de Newton Raphson

$$\widehat{\theta}_n^{(i+1)} = \widehat{\theta}_n^{(i)} - \frac{S_n(\widehat{\theta}_n^{(i)})}{S_n'(\widehat{\theta}_n^{(i)})},$$

from some starting value $\widehat{\theta}_n^{(0)}$ (hopefully well chosen).

Calcul numérique du maximum de vraisemblance

Newton-Raphson:

$$\widehat{\theta}_n^{(i+1)} = \widehat{\theta}_n^{(i)} - \frac{S_n(\widehat{\theta}_n^{(i)})}{S_n'(\widehat{\theta}_n^{(i)})},$$

where

$$S_n'(\widehat{\theta}_n^{(i)}) \sim \frac{S_n(\widehat{\theta}_n^{(i)} + h) - S_n(\widehat{\theta}_n^{(i)} - h)}{2h}$$

from some starting value $\widehat{\theta}_n^{(0)}$ (hopefully well chosen), and some small $h > 0$.

Score de Fisher

$$\widehat{\theta}_n^{(i+1)} = \widehat{\theta}_n^{(i)} + \frac{S_n(\widehat{\theta}_n^{(i)})}{nI(\widehat{\theta}_n^{(i)})},$$

from some starting value $\widehat{\theta}_n^{(0)}$ (hopefully well chosen).

Calcul numérique du maximum de vraisemblance

Consider some Poisson model, $S_1(\theta) = -1 + \dfrac{x}{\theta}$

```
> Sn = function(lambda) sum(-1+y/lambda)
> h = 1e-7
> dSn = function(lambda) (Sn(lambda+h)-Sn(lambda-h))
    /(2*h)
> L = rep(NA,10)
> L[1] = 1
> for(i in 1:9){
+   L[i+1] = L[i] - Sn(L[i])/dSn(L[i])
+ }
> L
[1] 1.000 1.697 2.521 3.116 3.290 3.300 3.300 3.300
```

Calcul numérique du maximum de vraisemblance

Consider some Poisson model, with Fisher information $I(\theta) = \dfrac{1}{\theta}$

```
> I = function(lambda) 1/lambda
> L = rep(NA,10)
> L[1] = 1
> for(i in 1:9){
+   L[i+1] = L[i] - Sn(L[i])/(length(y)*I(L[i]))
+ }
> L
[1] 1.0 3.3 3.3 3.3 3.3 3.3 3.3 3.3 3.3 3.3
```