

STT 1000 - STATISTIQUES

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Average, mean, median, mode, etc

Moyenne (empirique) / empirical mean / average

Given a sample $\mathbf{y} = \{y_1, \dots, y_n\}$, the **average** is $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$

```
1 > import statistics
2 > y = [1, 2, 3, 4, 5, 6]
3 > print(statistics.mean(y))
4 3.5
```

```
1 > y = 1:6
2 > mean(y)
3 [1] 3.5
```

Average, mean, median, mode, etc

On peut montrer que \bar{y} est solution de $\bar{y} = \operatorname{argmin}_{m \in \mathbb{R}} \left\{ \sum_{i=1}^n (y_i - m)^2 \right\}$

Preuve: soit $\mathbf{y} = \{y_1, \dots, y_n\}$, posons

$$g(m) = \sum_{i=1}^n (y_i - m)^2$$

$$\frac{\partial g(m)}{\partial m} = \frac{\partial}{\partial m} \sum_{i=1}^n (y_i - m)^2 = \sum_{i=1}^n \frac{\partial}{\partial m} (y_i - m)^2 = \sum_{i=1}^n -2(y_i - m)$$

Condition du premier ordre $\left. \frac{\partial g(m)}{\partial m} \right|_{m=m^*} = 0,$

$$\sum_{i=1}^n (y_i - m^*) = 0 \text{ si et seulement si } \sum_{i=1}^n y_i = nm^*$$

i.e. $y^* = \bar{y}$.

Average, mean, median, mode, etc

Espérance mathématique

The average (mean) is the empirical version of the expected value of a random variable,

$$\mathbb{E}(X) = \sum_x x\mathbb{P}[X = x] \text{ if } \sum_x |x|\mathbb{P}[X = x] < \infty$$

$$\mathbb{E}(X) = \int xf(x)dx \text{ if } \int |x|f(x)dx < \infty$$

Example: a coin has *heads* with probability p . Let $x = \mathbf{1}(\text{heads})$,

$$\mathbb{E}(X) = 1 \cdot p + 0 \cdot (1 - p) = p$$

Example: if X is uniform over $[0, 1]$, $f(x) = \mathbf{1}_{[0,1]}(x)$

$$\mathbb{E}(X) = \int_0^1 x \mathbf{1} dx = \frac{1}{2}$$

Average, mean, median, mode, etc

Linéarité de l'espérance mathématique

For all X and Y such that $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ exist

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}[Y], \quad \forall a, b \in \mathbb{R}.$$

$$\mathbb{E}(X_1 + \cdots + X_k) = \mathbb{E}(X_1) + \cdots \mathbb{E}(X_k), \quad \forall X_1, \cdots, X_k$$

Example: toss n coins, of bias p , X is the number of heads

$$\mathbb{E}(X) = \mathbb{E}(X_1 + \cdots + X_n) = \mathbb{E}(X_1) + \cdots \mathbb{E}(X_n) = np$$

Average, mean, median, mode, etc

$$\mathbb{E}(X) = \sum_x x \mathbb{P}[X = x] \text{ ou } \int x f(x) dx$$

$$\mathbb{E}(\psi(X)) = \sum_x \psi(x) \mathbb{P}[X = x] \text{ ou } \int \psi(x) f(x) dx$$

Exemple: pour une loi $\mathcal{N}(0, 1)$, que vaut $\mathbb{E}[\cos[X]]$?

$$\mathbb{E}[\cos[X]] = \int_{-\infty}^{+\infty} \cos(x) \varphi(x) dx$$

```
1 > f = function(x) cos(x)*dnorm(x,0,1)
2 > integrate(f,-Inf,Inf)
3 0.6065307 with absolute error < 7.2e-08
4 > log(integrate(f,-Inf,Inf)$value)
5 [1] -0.5
```

Nonlinear transformation & Jensen Inequality

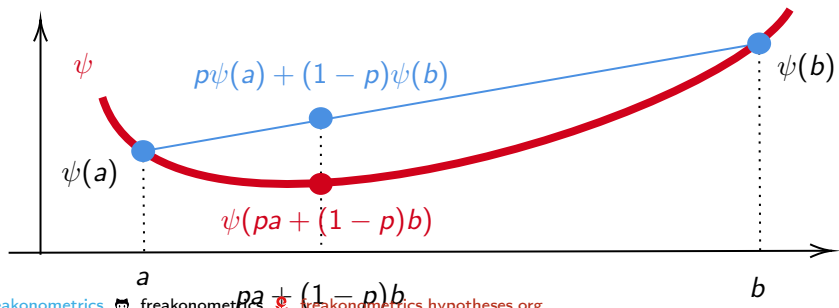
Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathbb{E}(\psi(X)) = \sum_x \psi(x) \mathbb{P}[X = x] \text{ or } \int \psi(x) f(x) dx \neq \psi(\mathbb{E}(X))$$

Example if X takes values in $\{a, b\}$, with probability p and $1 - p$,

$$\mathbb{E}(\psi(X)) = \psi(a)p + \psi(b)(1 - p)$$

If ψ is a **convex** function, $\mathbb{E}(\psi(X)) \geq \psi(\mathbb{E}(X))$



Nonlinear transformation & Jensen Inequality

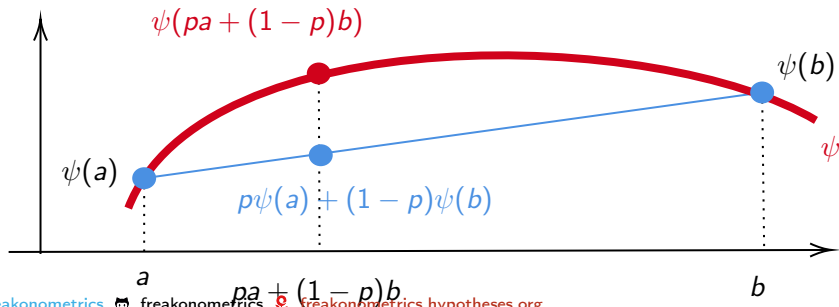
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$$\mathbb{E}(\psi(X)) = \psi(a)p + \psi(b)(1 - p)$$

If ψ is a **concave** function, $\mathbb{E}(\psi(X)) \leq \psi(\mathbb{E}(X))$



St Petersburg's Paradox

As we will see (**law of large numbers**) if x_i are realizations of random variables X_i (with identical expected value μ), $\bar{x} \rightarrow \mu$ as $n \rightarrow \infty$.

A fair coin is tossed at each stage. The initial stake begins at 2 dollars and is doubled every time heads appears. The first time tails appears, the game ends and the player wins whatever is in the pot. Let X denote the gain.

$$\mathbb{E}(X) = \frac{1}{2} \cdot 2 + \frac{1}{4} \cdot 4 + \frac{1}{8} \cdot 8 + \frac{1}{16} \cdot 16 + \dots = 1 + 1 + 1 + 1 + \dots = +\infty$$

the expected value is infinite (but the average always exists)

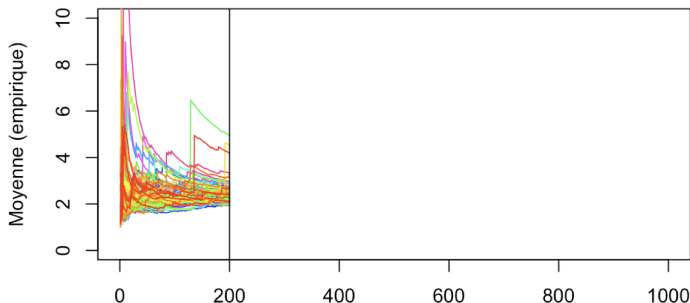
Average, mean, median, mode, etc

Example Loi de Pareto, $F(x) = 1 - x^{-\alpha}$ pour $x \geq 1$

La densité est $f(x) = \frac{\alpha}{x^{\alpha+1}}$ et l'espérance

$$\mathbb{E}[X] = \int_{x_m}^{\infty} \frac{x\alpha}{x^{\alpha+1}} dx = \begin{cases} \frac{\alpha}{\alpha-1} & \text{si } \alpha > 1 \\ \infty & \text{si } \alpha \leq 1 \end{cases}$$

mais $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ est toujours fini ! Exemple pour $\alpha = 1.7$



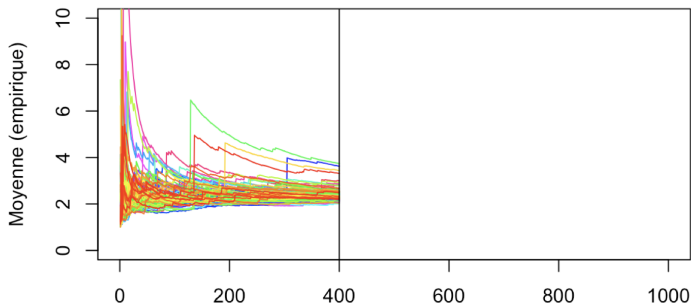
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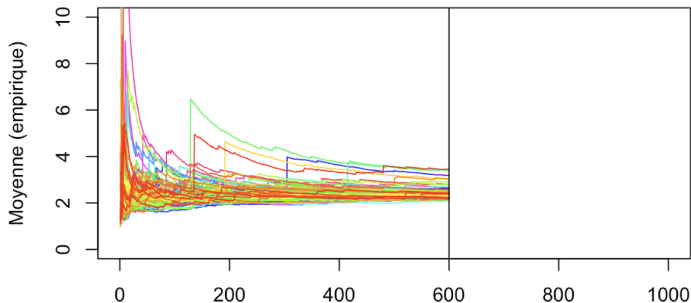
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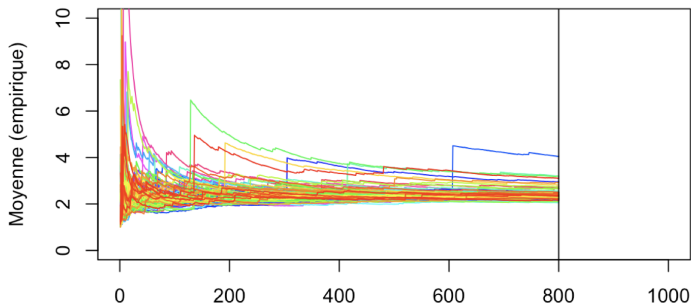
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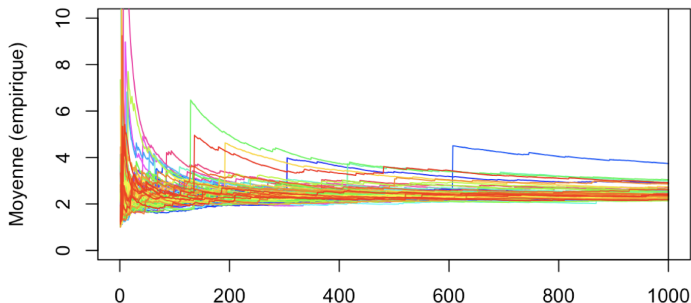
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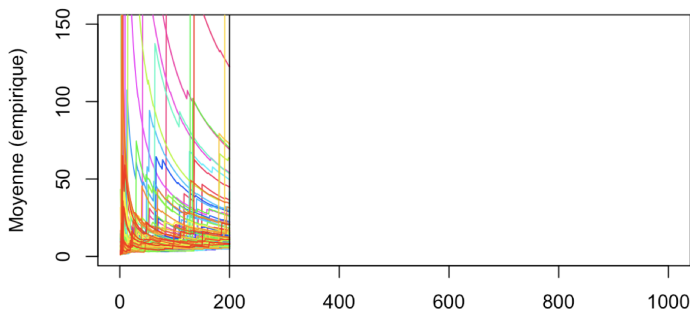
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mais $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ est toujours fini ! Exemple pour $\alpha = 0.9$



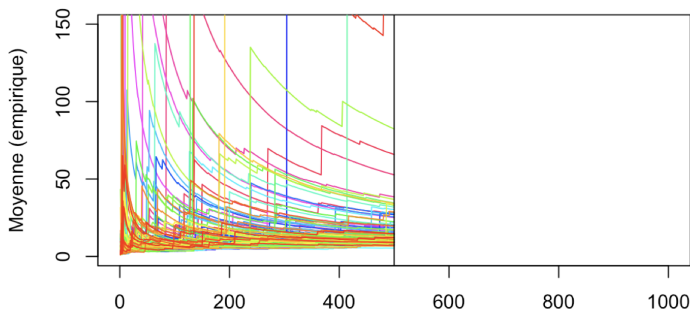
Average, mean, median, mode, etc

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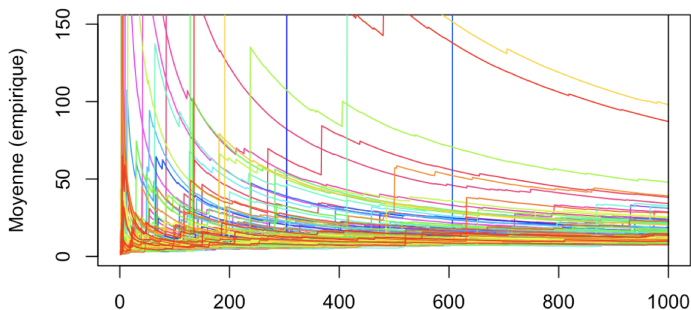
Average, mean, median, mode, etc

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Average, mean, median, mode, quantiles etc

Quantile

Pour une loi F ,

$$Q(p) = \inf \{x \in \mathbb{R} : p \leq F(x)\}$$

Le quantile est la seule fonction telle que

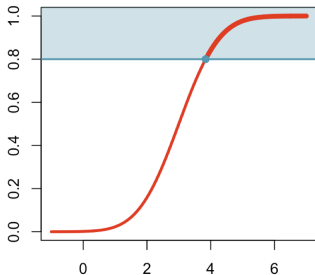
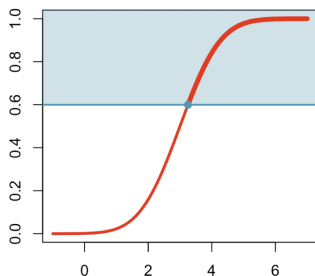
$$Q(p) \leq x \text{ si et seulement si } p \leq F(x)$$

Si F est continue et strictement croissante

$$Q(p) = F^{-1}(p).$$

Q est l'inverse à gauche: presque sûrement

$$Q(F(X)) \stackrel{p.s.}{=} X$$



Average, mean, median, mode, quantiles etc

Intégrale de la fonction quantile

Si l'espérance d'une variable de loi F existe,

$$\int_0^1 Q(p) dp = \int_{-\infty}^{\infty} xf(x) dx = \mathbb{E}[X]$$

En effet, par changement de variable $p = F(x)$,
 $dp = F'(x)dx = f(x)dx$,

$$\int_0^1 F^{-1}(p) dp = \int_{-\infty}^{\infty} xf(x) dx = \mathbb{E}[X]$$

On peut aussi écrire

$$\mathbb{E}[X] = \mathbb{E}[F^{-1}(U)] = \int_0^1 F^{-1}(u) du$$

où U suit une loi uniforme.

Average, mean, median, mode, quantiles etc

Exemple: pour une loi $\mathcal{N}(\mu, 1)$,

```
1 > mu = 7.3
2 > f = function(x) x*dnorm(x,mu,1)
3 > integrate(f,-Inf,Inf)
4 7.3 with absolute error < 3.3e-06
5 > g = function(p) qnorm(p,mu,1)
6 > integrate(g,0,1)
7 7.3 with absolute error < 8.1e-14
```

Définir les quantiles empiriques est plus compliqué

```
1 > ?quantile
2 > quantile(0:10,.95,type=7)
3 95%
4 9.5
5 > quantile(0:10,.95,type=3)
6 95%
7 9
```

Average, mean, median, mode, quantiles etc

Quantile empirique (1)

Notons $\{x_{(i)}\}$ une version ordonnée de $\{x_i\}$, $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. Le quantile empirique de niveau $p \in (0, 1)$ est

$$\hat{q}_p = (1 - f)x_{(k)} + fx_{(k+1)}$$

où $k = \lceil np \rceil$ et $f = n\alpha - \lfloor np \rfloor$.

Quantile empirique (2)

Étant donné $\{x_i\}$, si \hat{F} est la fonction de répartition empirique associée, on peut poser

$$\tilde{q}_p = \hat{F}^{-1}(p) = x_{(k)} \text{ où } k = \lceil np \rceil$$

Average, mean, median, mode, etc

The average is very sensitive to outliers and extremal values.

Médiane

Notons $\{x_{(i)}\}$ une version ordonnée de $\{x_i\}$, $x_{(1)} \leq x_{(2)} \leq \dots \leq x_{(n)}$. On appelle **médiane** $Q(1/2)$, et sa version empirique est

$$md(x) = \begin{cases} x_{((n+1)/2)} & \text{si } n \text{ pair} \\ \frac{1}{2}(x_{(n/2)} + x_{(n/2+1)}) & \text{si } n \text{ impair} \end{cases}$$

(50% observations are smaller/larger)

Note that $md(x) \in \operatorname{argmin}_{m \in \mathbb{R}} \left\{ \sum_{i=1}^n |x_i - m| \right\}$

Average & Paradoxes

See also Will Rogers phenomenon,

"When the Okies left Oklahoma and moved to California, they raised the average intelligence level in both states"

$$\{1, 2, 3, 4, 5\} \{6, 7, 8, 9, 10\}$$

$$\{1, 2, 3, 4, 5, 6\} \{7, 8, 9, 10\}$$

See The Will Rogers phenomenon. Stage migration and new diagnostic techniques as a source of misleading statistics for survival in cancer for real implications

Variance

Given a sample $\mathbf{x} = \{x_1, \dots, x_n\}$, $s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

Note that $s^2 = \min_{m \in \mathbb{R}} \left\{ \frac{1}{n-1} \sum_{i=1}^n (x_i - m)^2 \right\}$

```
1 > import statistics
2 > x = [1, 2, 3, 4, 5, 6]
3 > statistics.variance(x)
4 3.5
5 > print(statistics.stdev(x))
6 1.8708286933869707
```

```
1 > x = 1:6
2 > var(x)
3 [1] 3.5
4 > sd(x)
5 [1] 1.870829
```

$s = \sqrt{s^2}$ is `stdev(x)` (standard deviation)

Dispersion, variance, standard deviation

Variance empirique

Étant donné un échantillon x_1, \dots, x_n , on appelle variance

$$\text{empirique } s^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$$

Note C'est celle calculée par la plupart des logiciels

```
1 > x = 1:6
2 > var(x)
3 [1] 3.5
4 > sum((x-mean(x))^2)/5
5 [1] 3.5
6 > sum((x-mean(x))^2)/6
7 [1] 2.916667
```

C'est la version empirique de la variance ...

Dispersion, variance, standard deviation

Variance

On appelle variance d'une variable aléatoire

$$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

(à condition que $\mathbb{E}[X^2] < \infty$).

Covariance

On appelle covariance d'un couple de variable aléatoire

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

(à condition que $\mathbb{E}[X^2] < \infty$ et $\mathbb{E}[Y^2] < \infty$).

Dispersion, variance, standard deviation

Covariance empirique / sample covariance

Étant donné un échantillon apparié $(x_1, y_1), \dots, (x_n, y_n)$,

$$cv = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

Corrélation

On appelle corrélation d'un couple de variable aléatoire

$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

(à condition que $\mathbb{E}[X^2] < \infty$ et $\mathbb{E}[Y^2] < \infty$).

Dispersion, variance, standard deviation

Si X et Y sont indépendantes, $\text{Cov}(X, Y) = 0$.

Non linéarité de la variance

For all X and Y such that $\mathbb{E}[X^2]$ and $\mathbb{E}[Y^2]$ exist

$$\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}[Y] + 2ab \text{Cov}(X, Y), \quad \forall a, b \in \mathbb{R}.$$

Example: toss a coin of bias p , with outcome $X \in \{0, 1\}$,

$$\mathbb{E}(X) = p, \quad \mathbb{E}(X^2) = p, \quad \text{Var}(X) = p - p^2 = p(1 - p).$$

$$\text{Var}(aX + b) = a^2 \text{Var}(X), \quad \forall a, b \in \mathbb{R}, X$$

$\text{Var}(X_1 + \dots + X_k) = \text{Var}(X_1) + \dots + \text{Var}(X_k)$, if X_i 's are not correlated

See symmetric random walk, $X_i \in \{-1, +1\}$, $X = X_1 + \dots + X_n$,
then

$$\mathbb{E}(X) = 0, \quad \text{Var}(X) = n \text{ and } \text{stdev}(X) = \sqrt{n}$$

Dispersion, variance, standard deviation

Note Même si $\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$,

$$\frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n-1} \sum_{i=1}^n x_i^2 - \bar{x}^2 \neq \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2$$

```
1 > x = 1:6
2 > var(x)
3 [1] 3.5
4 > mean(x^2) - mean(x)^2
5 [1] 2.916667
6 > (6-1)/6*var(x)
7 [1] 2.916667
```

Dispersion, variance, standard deviation

Example: The outcome of a (fair) six-sided die has expected value

$$\mathbb{E}[Y] = \sum_{i=1}^6 \frac{1}{6}i = \frac{1+2+3+4+5+6}{6} = \frac{21}{6} = \frac{7}{2}$$

and variance

$$\text{Var}[Y] = \frac{1}{5} \sum_{i=1}^6 \left(i - \frac{7}{2}\right)^2 = \frac{1}{5} \left[\left(\frac{2-7}{2}\right)^2 + \dots + \left(\frac{12-7}{2}\right)^2 \right] = \frac{7}{2}$$

```
1 > x = 1:6
2 > var(x)
3 [1] 3.5
4 > sd(x)
5 [1] 1.870829
```

Dispersion, variance, standard deviation

Écart-type (*standard deviation*)

L'écart-type empirique est $s = \sqrt{s^2} = \sqrt{\frac{1}{n-1} \sum_{i=1}^n [x_i - \bar{x}]^2}$

Variance empirique (2)

On peut définir $\tilde{s}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$

Moments & Generating Function

1. Define the moment generating function, $M(t) = \mathbb{E}[e^{tX}]$,

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx$$

One can prove that

$$\frac{d^k M(t)}{dt^k} = \frac{d^k}{dt^k} \int_{-\infty}^{+\infty} e^{tx} f(x) dx = \int_{-\infty}^{+\infty} \frac{d^k}{dt^k} e^{tx} f(x) dx$$

$$\text{and } \left. \frac{d^k M(t)}{dt^k} \right|_{t=0} = \mathbb{E}[X^k], \forall k \in \mathbb{N}.$$

Example: si $X \sim \mathcal{E}(\lambda)$, $f(x) = \lambda e^{-\lambda x} \mathbf{1}_{\mathbb{R}_+}(x)$, pour $t < \lambda$

$$M(t) = \mathbb{E}[e^{tX}] = \int_0^{\infty} \lambda e^{-\lambda x} e^{tx} dx = \left[-\frac{\lambda}{\lambda - t} e^{-(\lambda - t)x} \right]_0^{\infty} = \frac{\lambda}{\lambda - t}$$

$$\frac{d^k M(t)}{dt^k} = \frac{\lambda k!}{(\lambda - t)^{k+1}} \text{ et } \mathbb{E}[X^k] = \frac{k!}{\lambda^k}$$

Moments & Generating Function

2. Define the moment generating function, $M(t) = \mathbb{E}[e^{tX}]$,

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x) dx = \int_{-\infty}^{+\infty} \left(1 + tx + \frac{t^2 x^2}{2} + \dots \right) f(x) dx$$

$$M(t) = \int_{-\infty}^{+\infty} 1 f(x) dx + \int_{-\infty}^{+\infty} tx f(x) dx + \int_{-\infty}^{+\infty} \frac{t^2 x^2}{2} f(x) dx + \dots$$

$$M(t) = \underbrace{\int_{-\infty}^{+\infty} f(x) dx}_{=1} + t \underbrace{\int_{-\infty}^{+\infty} x f(x) dx}_{=\mathbb{E}[X]} + \frac{t^2}{2} \underbrace{\int_{-\infty}^{+\infty} \frac{t^2 x^2}{2} f(x) dx}_{=\mathbb{E}[X^2]} + \dots$$

$$M(t) = 1 + t\mathbb{E}X + \frac{t^2}{2}\mathbb{E}X^2 + \frac{t^3}{2 \cdot 3}\mathbb{E}X^3 + \dots$$

so that $\left. \frac{d^k M(t)}{dt^k} \right|_{t=0} = \mathbb{E}[X^k], \forall k \in \mathbb{N}.$

Moments & Generating Function

Example: $X \sim \mathcal{P}(\lambda)$,

1. Espérance,

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \mathbb{P}(X = k) = \sum_{k=1}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.\end{aligned}$$

Note La fonction génératrice des moments s'écrit

$$\begin{aligned}M(t) &= \sum_{k=0}^{\infty} e^{tk} \mathbb{P}(X = k) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.\end{aligned}$$

Moments & Generating Function

$$\frac{dM(t)}{dt} = \lambda e^{\lambda(-1+e^t)+t} \text{ et } \left. \frac{dM(t)}{dt} \right|_{t=0} = \lambda$$

2. Variance,

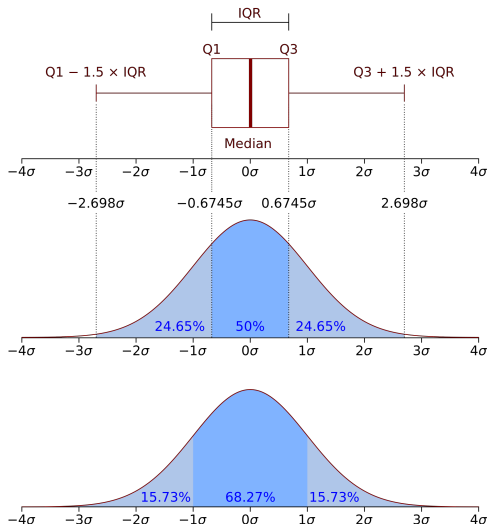
$$\begin{aligned} V(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_{k=1}^{\infty} k^2 \mathbb{P}(X = k) - \lambda^2 = \sum_{k=1}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} - \lambda^2 \\ &= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{k \lambda^{k-1}}{(k-1)!} - \lambda^2 = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{d}{d\lambda} \frac{\lambda^k}{(k-1)!} - \lambda^2 \\ &= \lambda e^{-\lambda} \frac{d}{d\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} - \lambda^2 = \lambda e^{-\lambda} \frac{d}{d\lambda} \left[\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \right] - \lambda^2 \\ &= \lambda e^{-\lambda} \frac{d}{d\lambda} [\lambda e^{\lambda}] - \lambda^2 = \lambda e^{-\lambda} (\lambda + 1) e^{\lambda} - \lambda^2 \\ &= \lambda(\lambda + 1) - \lambda^2 = \lambda. \end{aligned}$$

Intervalle interquartile

Interquartile range (IQR)

$$\text{IQR} = Q(3/4) - Q(1/4)$$

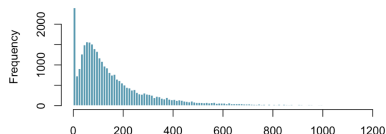
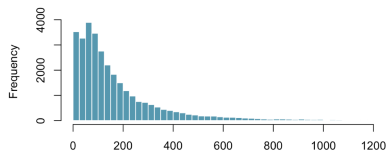
cf [wikipedia](#)



Statistiques descriptives

31,492 appels au service à la clientèle d'une banque

```
1 > hist(bankcall$Time[bankcall$Time<1200] , breaks=seq  
    (0,1200,by=10))
```



```
1 > mean(bankcall$Time/60)  
2 [1] 3.143204
```

soit 3 minutes et 8 secondes pour la moyenne

```
1 > median(bankcall$Time/60)  
2 [1] 1.916667
```

soit 1 minute et 55 secondes pour la médiane (115 sec.)

Statistiques descriptives

La moyenne tronquée (**trimmed mean** en anglais) à α (5% ou 10%) est la moyenne du jeu de données obtenu en supprimant une proportion α des plus petites valeurs et α plus grandes valeurs :

```
1 > mean(bankcall$Time/60, trim=.1)
2 [1] 2.357288
```

```
1 > mean(bankcall$Time/60)
2 [1] 3.143204
```

```
1 > quantile(bankcall$Time)
2   0%   25%   50%   75%  100%
3    1   57  115  225 28739
```

Calculs Formels

Exemple: $X \sim \mathcal{N}(1, 3^2)$ et $Y \sim \mathcal{E}(2)$, indépendantes. Que vaut

$$E_1 = \mathbb{E}[(X^2 - 1)Y] ? \quad E_1 = \frac{9}{2}$$

Exemple: $\mathbb{P}[X \leq x] = \frac{x^2 - 2x + 2}{2}$ sur $[1, 2]$, 0 avant et 1 après.

que vaut $E_2 = \mathbb{E}[X]$?

Exemple: $\mathbb{P}[X \leq x] = \frac{x}{8}$ sur $[0, 2)$, $\frac{x^2}{16}$ sur $[2, 4)$, 0 avant et 1

après. Que vaut $E_3 = \text{Var}[X - 1] + 1$? $E_3 = \frac{311}{144}$

Exemple:

$$f_X(x) = \begin{cases} 2(3 - 2x)/5 & \text{pour } 0 \leq x \leq 1 \\ 2(2 - x)/5 & \text{pour } 1 \leq x \leq 2 \\ 0 & \text{sinon} \end{cases}$$

Que vaut m_1 la médiane de X ? $m_1 = \frac{1}{2}$

Calculs Numériques

Exemple: $X \sim \mathcal{P}(10)$, $p_1 = \mathbb{P}[X > 20]$? $p_1 \approx 0.001588$

Exemple: $X \sim \mathcal{B}(.1, 50)$, $p_2 = \mathbb{P}[X \leq \mathbb{E}(X)]$? $p_2 \approx 0.6161$

Exemple: $X \sim \mathcal{N}(10, 5)$, $p_3 = \mathbb{P}[X \leq 0]$? $p_3 \approx 0.02275$

Exemple: $X \sim \text{Beta}(2, 3)$ et $Y \sim \text{Beta}(5, 4)$, indépendantes, $p_4 = \mathbb{P}[X_1 + X_2 \leq 1]$? $p_4 \approx 0.5757$

Exemple: $X \sim \mathcal{N}(10, 5)$, q_1 tel que $\mathbb{P}[X > q_1] = 20\%$?
 $q_1 \approx 14.208$

Exemple: $X \sim \text{Beta}(2, 3)$ et $Y \sim \text{Beta}(5, 4)$, indépendantes, q_2 tel que $\mathbb{P}[X_1 + X_2 \leq q_2] = 50\%$? $q_2 \approx 0.9487$

Exemple: $X \sim \mathcal{N}(0, 1)$ et $Y \sim \mathcal{N}(0, 1)$, indépendantes, q_3 tel que $\mathbb{P}\left[\frac{X_1}{X_2^2} > q_3\right] = 10\%$? $q_3 \approx 10.4079$

Exemple: $U_1, \dots, U_4 \sim \mathcal{U}_{[0,1]}$, indépendantes, q_4 tel que $\mathbb{P}[X_1 + X_2 + X_3 + X_4 \leq q_4] = 10\%$? $q_4 \approx 1.2465$