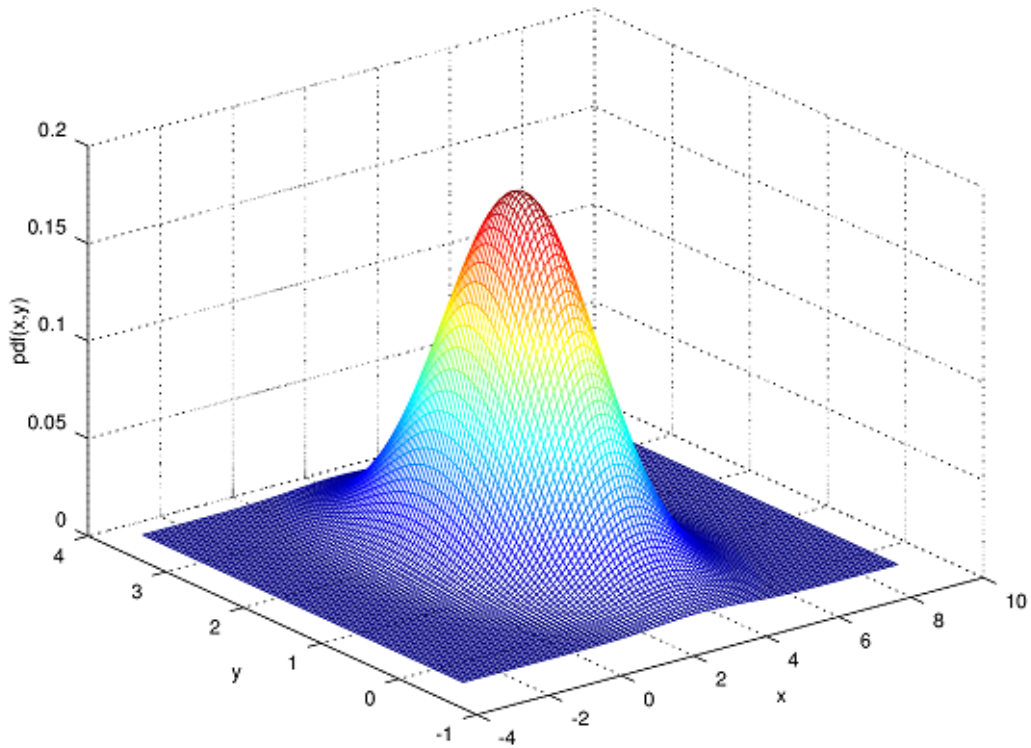
STT 1000 - STATISTIQVES

ARThVR ChARPENTIER

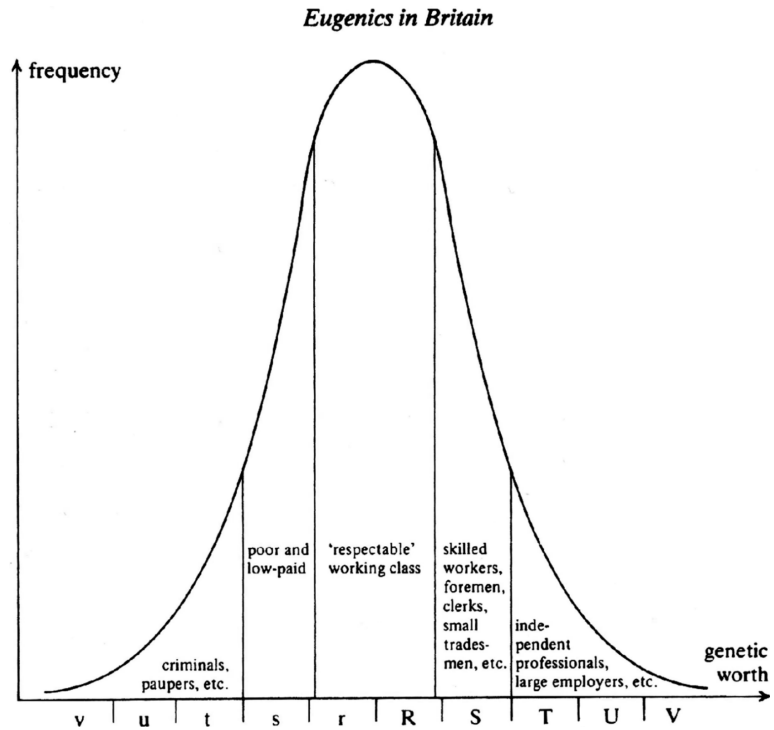
Karl Friedrich Gauss & the Gaussian distribution



Gaussian distribution

Legendre and Gauss (or Gauß) introduced the distribution as a *law of errors*...

Quetelet's average man
Galton's view of British social structure (picture Eugenics in Britain)



Galton needed to revolutionize this branch of mathematics, error theory and the use of the Gauss distribution as a distribution of errors from a mean value. A new statistical paradigm was needed, The Structure of Scientific Revolutions, Kuhn 1970.

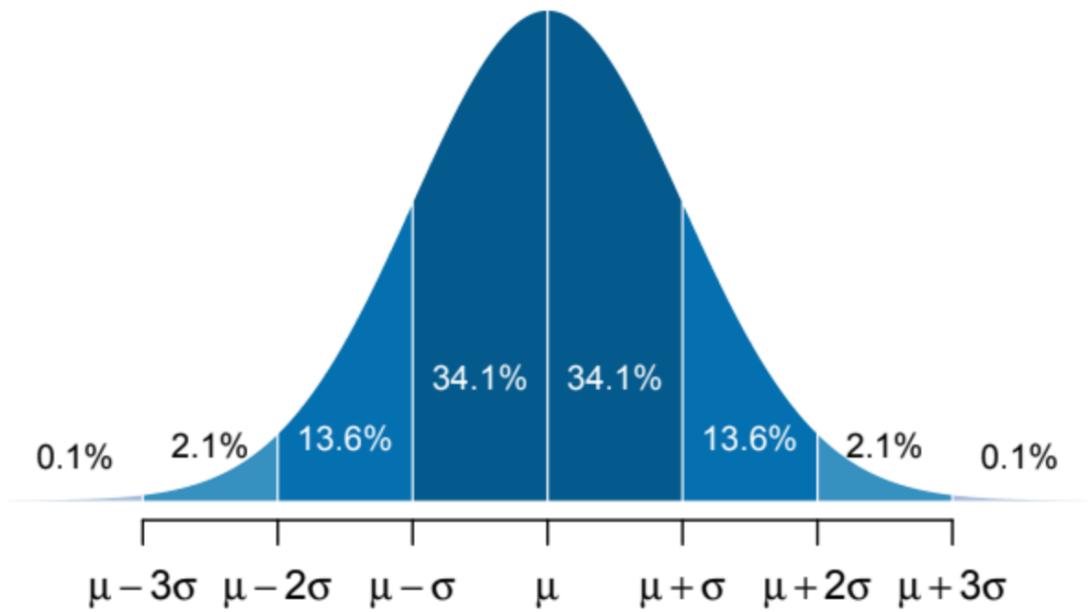
Gaussian distribution

Loi normale / Gaussienne $\mathcal{N}(\mu, \sigma^2)$

$X \sim \mathcal{N}(\mu, \sigma^2)$, with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

$\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$ (or σ is the standard deviation)



Gaussian Tables

In many applications we should solve

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left[-\frac{x^2}{2}\right] dx = p$$

no simple analytical formula...
Need for a standard normal table

Hence $\Phi(1.64) = 95\%$
and $\Phi(1.96) = 97.5\%$,
$\Phi^{-1}(0.975) = 1.96$
$\Phi^{-1}(0.025) = -1.96$

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> qnorm(.95)
[1] 1.644854
> qnorm(.975)
[1] 1.959964
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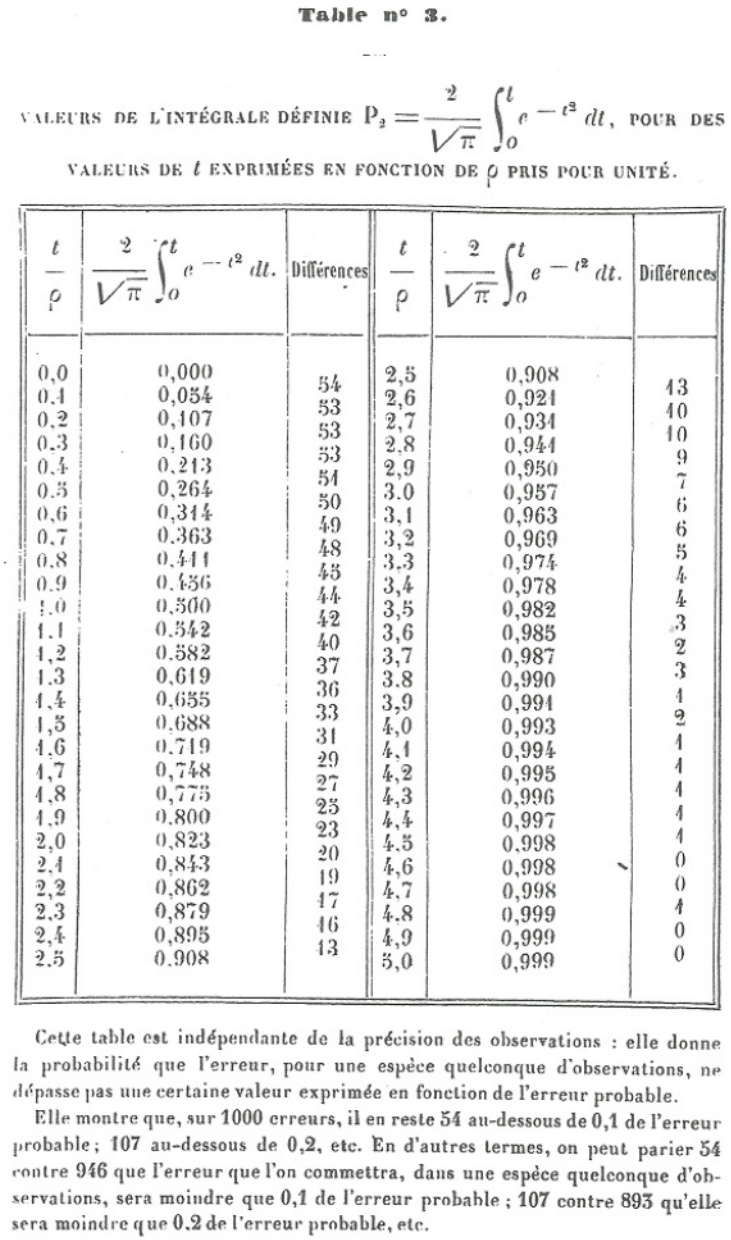


Table n° 3.

VALEURS DE L'INTÉGRALE DÉFINIE $P_t = \frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$, POUR DES VALEURS DE t EXPRIMÉES EN FONCTION DE ρ PRIS POUR UNITÉ.

$\frac{t}{\rho}$	$\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$.	Différences	$\frac{t}{\rho}$	$\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt$.	Différences
0,0	0,000	54	2,5	0,908	13
0,1	0,054	53	2,6	0,921	10
0,2	0,107	53	2,7	0,931	10
0,3	0,160	53	2,8	0,941	9
0,4	0,213	51	2,9	0,950	7
0,5	0,264	50	3,0	0,957	6
0,6	0,314	49	3,1	0,963	6
0,7	0,363	48	3,2	0,969	5
0,8	0,411	45	3,3	0,974	4
0,9	0,456	44	3,4	0,978	4
1,0	0,500	42	3,5	0,982	3
1,1	0,542	40	3,6	0,985	2
1,2	0,582	37	3,7	0,987	3
1,3	0,619	36	3,8	0,990	1
1,4	0,655	33	3,9	0,991	2
1,5	0,688	31	4,0	0,993	1
1,6	0,719	29	4,1	0,994	1
1,7	0,748	27	4,2	0,995	1
1,8	0,775	25	4,3	0,996	1
1,9	0,800	23	4,4	0,997	1
2,0	0,823	20	4,5	0,998	1
2,1	0,843	19	4,6	0,998	0
2,2	0,862	17	4,7	0,998	0
2,3	0,879	16	4,8	0,999	1
2,4	0,895	13	4,9	0,999	0
2,5	0,908		5,0	0,999	0

Cette table est indépendante de la précision des observations : elle donne la probabilité que l'erreur, pour une espèce quelconque d'observations, ne dépasse pas une certaine valeur exprimée en fonction de l'erreur probable.

Elle montre que, sur 1000 erreurs, il en reste 54 au-dessous de 0,1 de l'erreur probable ; 107 au-dessous de 0,2, etc. En d'autres termes, on peut parier 54 contre 946 que l'erreur que l'on commettra, dans une espèce quelconque d'observations, sera moindre que 0,1 de l'erreur probable ; 107 contre 893 qu'elle sera moindre que 0,2 de l'erreur probable, etc.

Central Limit Theorem

Let $X_i \sim \mathcal{B}(p)$,

$$\mathbb{P}(X_i = 0) = 1 - p \text{ and } \mathbb{P}(X_i = 1) = p.$$

then $X = X_1 + \cdots + X_n \sim \mathcal{B}(n, p)$ (binomial distribution), for $k = 0, 1, \cdots, n$,

$$\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \ \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

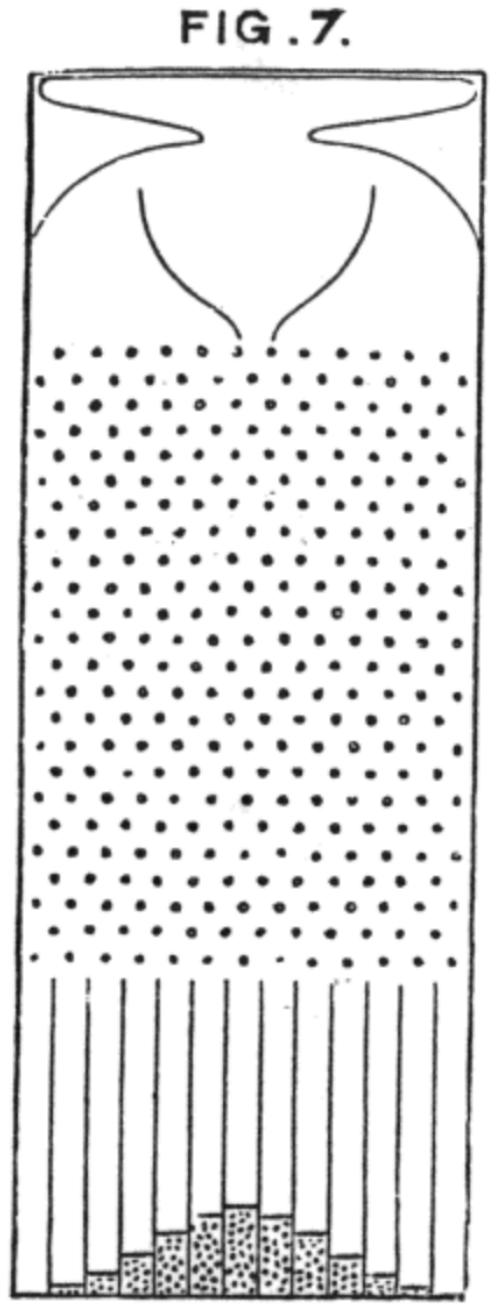
then, when n is large enough

$$X \simeq \mathcal{N}(np, np(1-p))$$

or

$$\overline{X} = \frac{X}{n} \simeq \mathcal{N}\left(p, \frac{p(1-p)}{n}\right)$$

(picture Quincunx, or Galton's box)



Central Limit Theorem

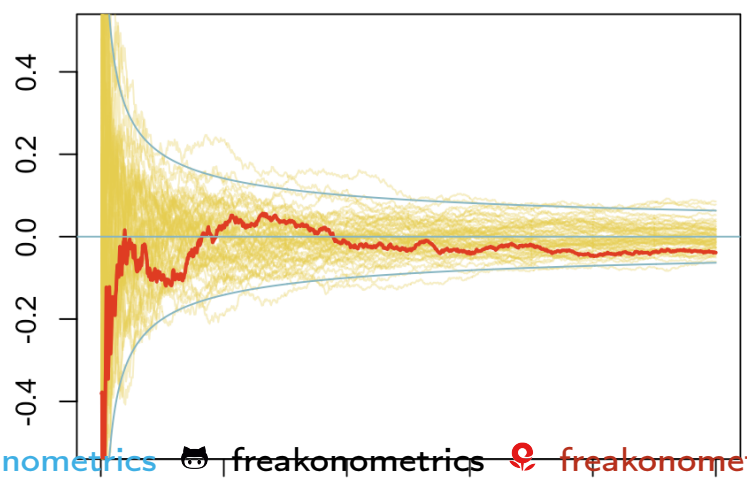
If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ are independent,

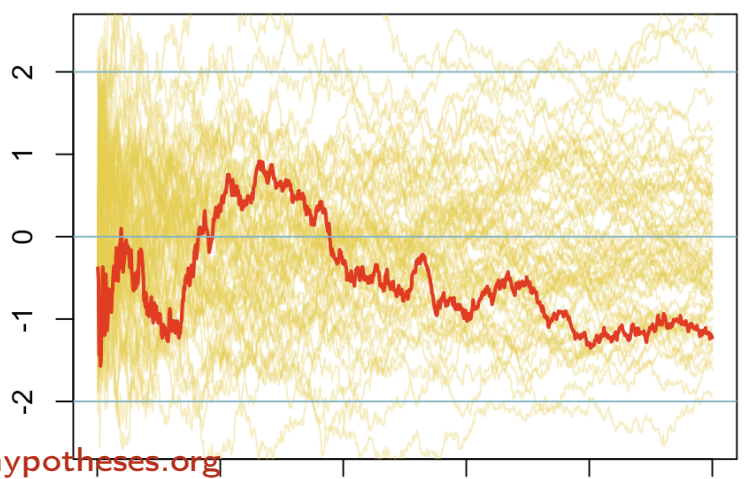
$$X_1 + X_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Central Limit Theorem

Suppose $\{X_1, \ldots, X_n, \ldots\}$ is a sequence of i.i.d. random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2 < \infty$, then, if $\overline{X}_n = X_1 + \ldots + X_n$ as n goes to infinity, $\sqrt{n}(\overline{X}_n - \mu)$ converges toward a $\mathcal{N}(0, \sigma^2)$ distribution

$$\sqrt{n}\left(\overline{X}_n - \mu\right) \rightarrow \mathcal{N}\left(0, \sigma^2\right).$$





Gaussian (multivariate) distribution

$\boldsymbol{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with density

$$f_{\mathbf{X}}(x_1, \ldots, x_d) = \frac{1}{\sqrt{(2\pi)^d |\boldsymbol{\Sigma}|}} \exp\left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right)$$

where $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and $\text{Var}(\boldsymbol{X}) = \boldsymbol{\Sigma}$.

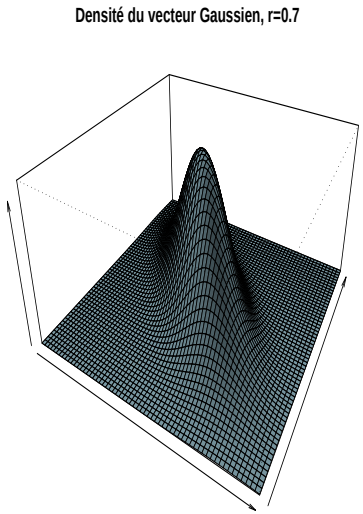
Estimates are $\overline{\boldsymbol{x}} = \frac{1}{n} \sum_{i=1}^{n} \boldsymbol{x}_i$ and $\widehat{\boldsymbol{\Sigma}} = \frac{1}{n} \sum_{i=1}^{n} (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top$

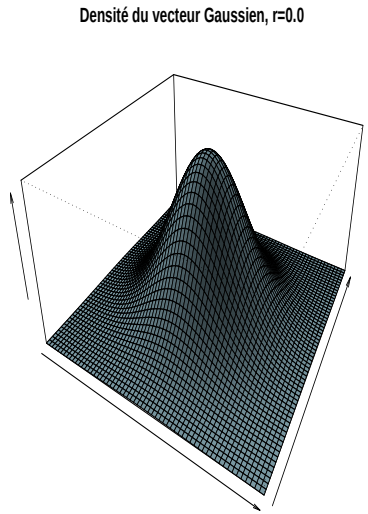
In dimension 2, $f(x, y)$ is proportional to

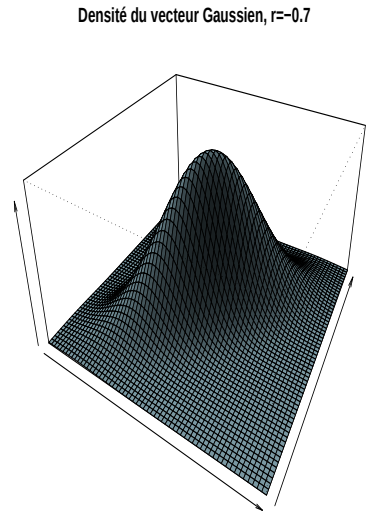
$$\exp\left(-\frac{1}{2(1-\rho^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} \right] \right)$$

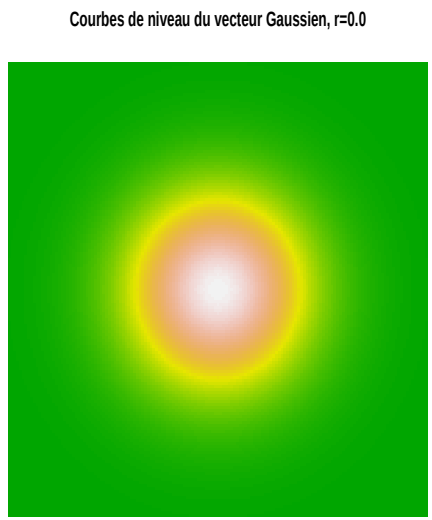
levels curves (isodensities) are ellipses.

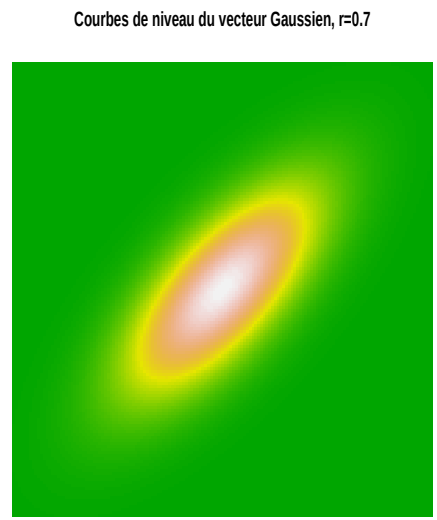
Gaussian (multivariate) distribution











Random Vectors

Soit $\boldsymbol{X}$ un vecteur aléatoire de dimension d

- L'espérance de $\boldsymbol{X}$, notée $\mathbb{E}(\boldsymbol{X})$ est définie (si elle existe) par le vecteur de dimension d $\mathbb{E}(\boldsymbol{X}) = (\mathbb{E}(\boldsymbol{X}_1), \cdots, \mathbb{E}(\boldsymbol{X}_d))^\top$.
- La matrice de covariance (appelée aussi matrice de variance-covariance de $\boldsymbol{X}$) est définie (si elle existe) par la matrice de taille (d, d)

$$\text{Var}(\boldsymbol{X}) = \mathbb{E}\left((\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))(\boldsymbol{X} - \mathbb{E}(\boldsymbol{X}))^\top\right).$$

Ainsi le terme ij de cette matrice représente la covariance entre X_i et X_j,

$$\text{Cov}(X_i, X_j) = [(X_i - (X_i))(X_j - (X_j))].$$

Random Vectors

Soit $\boldsymbol{X}$ un vecteur aléatoire de dimension d, de moyenne $\boldsymbol{\mu}$ et de matrice de covariance $\boldsymbol{\Sigma}$.
Soient $\boldsymbol{A}$ et $\boldsymbol{B}$ deux matrices réeeles de taille (d, p) et (d, q) et enfin soit $\boldsymbol{a} \in \mathbb{R}^p$ alors

- $\text{Var}(\boldsymbol{X}) = \mathbb{E}\left((\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right) = \mathbb{E}(\boldsymbol{X}\boldsymbol{X}^\top) - \boldsymbol{\mu}\boldsymbol{\mu}^\top$.
- $\mathbb{E}\left(\boldsymbol{A}^\top\boldsymbol{X} + \boldsymbol{a}\right) = \boldsymbol{A}^\top\boldsymbol{\mu} + \boldsymbol{a}$.
- $\text{Var}\left(\boldsymbol{A}^\top\boldsymbol{X} + \boldsymbol{a}\right) = \boldsymbol{A}^\top\boldsymbol{\Sigma}\boldsymbol{A}$.
- $\text{Cov}\left(\boldsymbol{A}^\top\boldsymbol{X}, \boldsymbol{B}^\top\boldsymbol{X}\right) = \boldsymbol{A}^\top\boldsymbol{\Sigma}\boldsymbol{B}$.

The Gaussian Distribution

A Gaussian variable, with distribution $\mathcal{N}(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$, has density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right), \text{ for all } x \in \mathbb{R}.$$

Then $\mathbb{E}(X) = \mu$ and $\text{Var}(X) = \sigma^2$.
Observe that if $Z \sim \mathcal{N}(0,1)$, $X = \mu + \sigma Z \sim \mathcal{N}(\mu, \sigma^2)$.
The Gaussian **vector** $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$: $\boldsymbol{X} = (X_1, ..., X_n)$ is a Gaussian vector with mean $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and covariance matrix $\text{Var}(\boldsymbol{X}) = \boldsymbol{\Sigma} = \mathbb{E}\left((\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^\top\right)$ non-degenerated ($\boldsymbol{\Sigma}$ is invertible) if its density is

$$f(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \boldsymbol{\Sigma}}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right), \ \boldsymbol{x} \in \mathbb{R}^n,$$

see multivariate Gaussian distribution

The Gaussian Distribution

If $\boldsymbol{X}$ is a Gaussian vector, then for any i, X_i has a (univariate) Gaussian distribution, but its converse it not necessarily true.

Let $\boldsymbol{X} = (X_1, ..., X_n)$ be a random vector with mean $\mathbb{E}(\boldsymbol{X}) = \boldsymbol{\mu}$ and with covariance matrix $\boldsymbol{\Sigma}$, if $\boldsymbol{A}$ is a $k \times n$ matrix, and $\boldsymbol{b} \in \mathbb{R}^k$, then $\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{b}$ is a Gaussian vector $\mathbb{R}^k$, with distribution $\mathcal{N}\left(\boldsymbol{A}\boldsymbol{\mu} + \boldsymbol{b}, \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}^{\mathsf{T}}\right)$.

Observe that if (X_1, X_2) is a Gaussian vector, X_1 and X_2 are independent if and only if

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}\left((X_1 - \mathbb{E}(X_1))(X_2 - \mathbb{E}(X_2))\right) = 0.$$

The Gaussian Distribution

Let $\boldsymbol{Z} = (Y, X) \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_Y \\ \mu_X \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} \Sigma_{Y,Y} & \Sigma_{Y,X} \\ \Sigma_{X,Y} & \Sigma_{X,X} \end{pmatrix}$$

Then

$$Y|X = x \sim \mathcal{N}(\mu_{Y|x}, \sigma^2_{Y|x}) \text{ where } \begin{cases} \mu_{Y|x} = \mu_Y + \Sigma_{Y,X}\Sigma^{-1}_{X,X}(x - \mu_X) \\ \sigma^2_{Y|x} = \Sigma_{Y,Y} - \Sigma_{Y,X}\Sigma^{-1}_{X,X}\Sigma_{X,Y} \end{cases}$$

Hence, $\mathbb{E}[Y|X = x] = \mu_{Y|x}$ is linear in x, with slope

$$\text{Corr}(X, Y)\sqrt{\Sigma_{Y,Y}\Sigma^{-1}_{X,X}}$$

and $\text{Var}[Y|X = x] = \sigma^2_{Y|x}$ is constant
xw(furthermore $\text{Var}[Y|X = x] \leq \text{Var}[Y]$)

Chi-Square Distribution

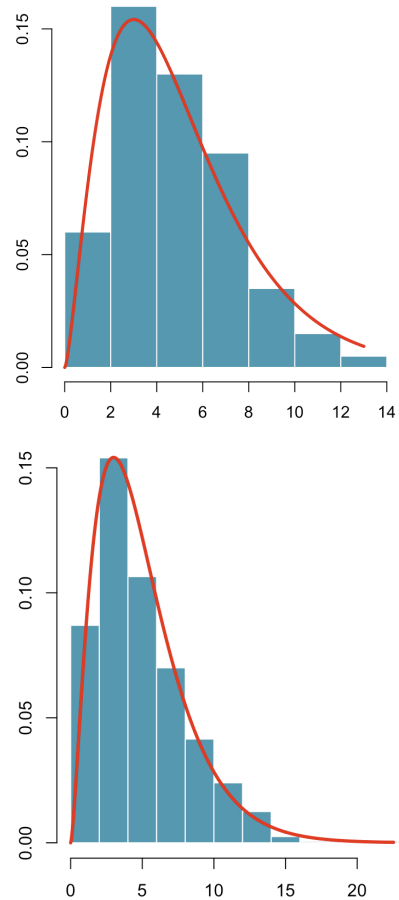
The chi-squared distribution $\chi^2(\nu)$, with $\nu \in \mathbb{N}^*$ has density

$$x \mapsto \frac{(1/2)^{\nu/2}}{\Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \text{ where } x \in [0; +\infty),$$

where Γ denotes the Gamma function ($\Gamma(n+1) = n!$). Observe that $\mathbb{E}(X) = \nu$ et $\text{Var}(X) = 2\nu$. ν are the degrees of freedom, see chi-squared distribution

If $X_1, \cdots, X_\nu \sim \mathcal{N}(0,1)$ are independent variables, then $Y = \sum_{i=1}^{\nu} X_i^2 \sim \chi^2(\nu)$, when $\nu \in \mathbb{N}$.

This is a particular case of the Gamma distribution, $X \sim \mathcal{G}\left(\frac{k}{2}, \frac{1}{2}\right)$, see see gamma distribution



Student's t Distribution

The Student's-t distribution $\mathcal{S}t(\nu)$, has density

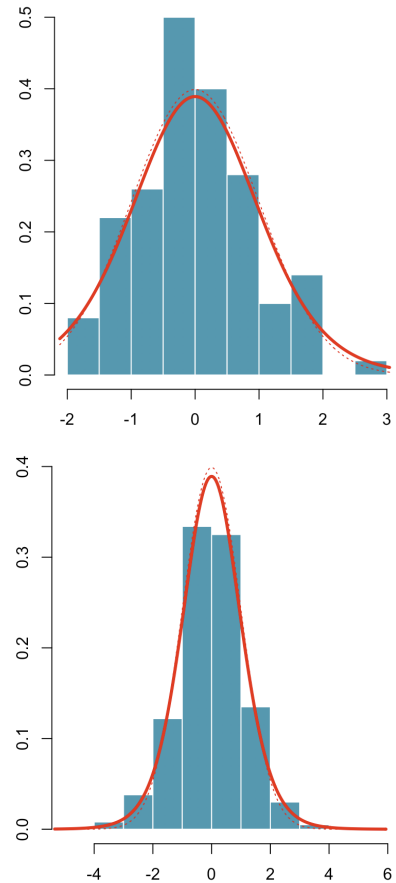
$$f(t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\,\Gamma(\frac{\nu}{2})}\left(1+\frac{t^2}{\nu}\right)^{-(\frac{\nu+1}{2})},$$

Observe that

$$\mathbb{E}(X) = 0 \text{ and } \text{Var}(X) = \frac{\nu}{\nu-2} \text{ when } \nu > 2.$$

If $X \sim \mathcal{N}(0,1)$ and $Y \sim \chi^2(\nu)$ are independents, then

$$T = \frac{X}{\sqrt{Y/\nu}} \sim \mathcal{S}t(\nu).$$



see Student's t

Student's t Distribution

Let $X_1, \cdots, X_n$ be $\mathcal{N}(\mu, \sigma^2)$ independent random variables. Let

$$\overline{X}_n = \frac{X_1 + \cdots + X_n}{n} \text{ and } {S_n}^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(X_i - \overline{X}_n\right)^2.$$

Then $\frac{(n-1)S_n^2}{\sigma^2}$ has a $\chi^2(n-1)$ distribution, and furthermore

$$T = \sqrt{n} \frac{\overline{X}_n - \mu}{S_n} \sim \mathcal{S}t(n-1).$$

Fisher's F Distribution

The Fisher distribution $\mathcal{F}(d_1, d_2)$, has density

$$x \mapsto \frac{1}{x\,\mathrm{B}(d_1/2, d_2/2)} \left(\frac{d_1\,x}{d_1\,x + d_2}\right)^{d_1/2} \left(1 - \frac{d_1\,x}{d_1\,x + d_2}\right)^{d_2/2}$$

for $x \geq 0$ and $d_1, d_2 \in \mathbb{N}$, where B denotes the Beta function.

$\mathbb{E}(X) = \dfrac{d_2}{d_2 - 2}$ when $d_2 > 2$ and $\mathrm{Var}(X) = \dfrac{2\,d_2^2\,(d_1 + d_2 - 2)}{d_1(d_2 - 2)^2(d_2 - 4)}$ when $d_2 > 4$.

If $X \sim \mathcal{F}(\nu_1, \nu_2)$, then $\dfrac{1}{X} \sim \mathcal{F}(\nu_2, \nu_1)$.

If $X_1 \sim \chi^2(\nu_1)$ and $X_2 \sim \chi^2(\nu_2)$ are independent

$$Y = \frac{X_1/\nu_1}{X_2/\nu_2} \sim \mathcal{F}(\nu_1, \nu_2)$$

see Fisher's $\mathcal{F}$ on wikipedia

Fisher's F Distribution

On peut montrer que si $X \sim \mathcal{S}td(\nu)$, alors $X^2 \sim \mathcal{F}(1,\nu)$. Ou dit autrement si F_{1-p} est le quantile de niveau $1-p$ de la loi $\mathcal{F}(1,\nu)$, $F_{1-p} = {t_{1-p/2}}^2$ où t_{1-p} est le quantile de niveau $1-p$ de la loi $\mathcal{S}td(\nu)$.
La loi $\mathcal{F}(1,\nu)$ a pour densité

$$f(u) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \nu^{\nu/2} u^{-1/2} (\nu+u)^{-(\nu+1)/2} \text{ sur } \mathbb{R}_+$$

$$f(u) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} u^{-1/2} \left(1+\frac{u}{\nu}\right)^{-(\nu+1)/2} \text{ sur } \mathbb{R}_+$$

aussi

$$\int_0^{F_{1-p}} f(u)du = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)} \int_0^{F_{1-p}} u^{-1/2} \left(1+\frac{u}{\nu}\right)^{-(\nu+1)/2} du = 1-p$$

Fisher's F Distribution

Faisons le changement de variable, $t = \sqrt{u}$,

$$2\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\Gamma\left(\frac{\nu}{2}\right)}\int_0^{\sqrt{F_{1-p}}}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2}dt = 1-p$$

on reconnaît une intégrale associée à la loi de Student.
Si $T \sim \mathcal{S}td(\nu)$, on a écrit $\mathbb{P}(T \in [0, \sqrt{F_{1-p}}])$,

$$2\mathbb{P}(T \in [0, \sqrt{F_{1-p}}]) = 1-p \text{ i.e. } \frac{1-p}{2} = \mathbb{P}(T \leq \sqrt{F_{1-p}}]) - \underbrace{\mathbb{P}[T \leq 0]}_{=1/2}$$

$$\mathbb{P}(T \leq \sqrt{F_{1-p}}]) = 1-\frac{p}{2} \text{ mais on sait que } \mathbb{P}(T \leq t_{1-p/2}]) = 1-\frac{p}{2}$$

donc $F_{1-p} = t_{1-p/2}^2$.

```
> qf(.95,1,10)
[1] 4.964603
> qt(.975,10)^2
[1] 4.964603
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