Ex 5 :

$$f_\theta(x) = \frac{1+\theta x}{2} \mathbb{1}_{[-1,1]}(x)$$

1) $$E_\theta(X) = \int_{-1}^{1} \frac{x+\theta x^2}{2} dx$$

$$= \left. x^2 + \frac{x^3}{2\times 3}\theta \right]_{-1}^{1} = \frac{\theta}{6} + \frac{\theta}{6} = \boxed{\frac{\theta}{3}}$$

$$\bar{x} = \frac{\hat{\theta}}{3} \Rightarrow \hat{\theta} = 3\bar{x}$$

$$E(\hat{\theta}) = 3E(\bar{x}) = 3\frac{\sum E(x_i)}{n}$$

$$= \frac{3 \cdot n(\theta/3)}{n} = \theta$$

sans biais.

$$Var(\hat{\theta}) = Var(3\bar{x})$$

$$= 9\, var(\bar{x})$$

$$= \frac{9}{n^2} var\left(\sum x_i\right) = \frac{9}{n^2} \sum var(x_i)$$

$$= \frac{9}{n^2}\sum\left(\int_{-1}^{1} \frac{t^2(1+\theta t)}{2} dt - \frac{\theta^2}{9}\right)$$

$$= \frac{9}{n^2} \sum \left(\left[\frac{t^3}{6} + \frac{\theta t^4}{4}\right]_{-1}^{1} - \frac{\theta^2}{9}\right)$$

$$= \frac{9}{n^2} \left(\frac{n}{3} - \frac{n\theta^2}{9}\right)$$

$$= \left(\frac{3}{n} - \frac{\theta^2}{n}\right)$$

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Ex 1

1) L'esperance de X ; f_θ

$$f_\theta(x) = \theta(1+\theta)x^{\theta-1}(1-x)\mathbb{1}_{[0,1]}(x)$$

$$E[X] = \int_0^1 x f(x)\,dx$$

$$= \int_0^1 \theta(1+\theta)x^{\theta}(1-x)\,dx$$

$$= \theta(\theta+1)\left[\frac{x^{\theta+1}}{\theta+1} - \frac{x^{\theta+2}}{\theta+2}\right]_0^1$$

$$= \theta(\theta+1)\left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right]$$

$$= \theta - \frac{\theta(\theta+1)}{\theta+2} = \frac{\theta^2+2\theta-\theta^2-\theta}{(\theta+2)}$$

$$= \frac{\theta}{\theta+2}$$

De telle sort que la methode de momment est $\hat{\theta}$
solution des momment de $\hat{\theta}$;
solution de $\bar{x} = \frac{\hat{\theta}}{\hat{\theta}+2}$

$\Rightarrow \hat{\theta} = \bar{x}(\hat{\theta}+2)$

$\Rightarrow \hat{\theta} = \bar{x}\hat{\theta} + 2\bar{x} \Rightarrow (1-\bar{x})\hat{\theta} = 2\bar{x}$

$\hat{\theta} = \frac{2\bar{x}}{1-\bar{x}}$

Commençant par calculer la formule de Fisher

$$\ln(f(x,\theta)) = \ln(\theta) + \ln(1+\theta) + (\theta-1)\ln(x) + \ln(1-x)$$

Soit

$$\frac{\partial}{\partial x}(\ln f(x,\theta)) = \frac{1}{\theta} + \frac{1}{1+\theta} - \ln(x)$$

De telle sorte que

$$\frac{\partial}{\partial\theta}\operatorname{Log} f_\theta(x) = -\frac{1}{\theta^2} - \frac{1}{(1+\theta)^2}$$

et donc si

$X \sim f_\theta$;

$$I(\theta) = -n\,E_\theta\left[\frac{\partial \operatorname{Log} f_\theta(x)}{\partial\theta^2}\right]$$

$$= \left(\frac{\theta^2+(\theta+1)^2}{\theta^2(\theta+1)^2}\right)n$$

La borne de Cramer Rao est içi

$$\frac{(\theta+1)^2\theta^2}{n[\theta^2+(\theta+1)^2]}$$

Et si on regarde la difference entre Var $(\hat{\theta})$ et cette borne on obtient

$$\frac{\theta(\theta+2)^2}{2n(\theta+3)} - \frac{(\theta+1)^2\theta^2}{n(\theta^2+(\theta+1)^2)}$$

$$= \frac{\theta(\theta^2+4+4\theta)(\theta^2+(\theta+1)^2) - 2\theta^2(\theta+1)^2(\theta+3)}{2n(\theta+3)(\theta^2+(\theta+1)^2)}$$

$$= \frac{\theta[2\theta^4-2\theta^4+(2+8-6-4)\theta^3+(8+1+2-2-6)\theta^2+(4+8-6)\theta+4}{2n(\theta+3)(\theta^2+(\theta+1)^2)}$$

$$= \frac{3\theta^2+6\theta+4}{2n(\theta+3)(\theta^2+(\theta+1)^2)} > 0$$

Tous les termes du ratio etant positive cette estimateur n'est pas efficace.

Ex 2:

$\theta \in (0,1)$; $a, b \in \mathbb{R}_+$

$$X_i = \begin{cases} U_i \sim \mathcal{U}([0,a]) & \text{avec prob } \theta \\ V_i \sim \mathcal{U}([0,b]) & \text{avec prob } 1-\theta \end{cases}$$

N nbre d'observation de X_i ; $0 < X_i < a$

1) $N = \sum \mathbb{1}(0 < X_i < a)$

Posons $Y = \mathbb{1}(0 < X < a)$

cōē les variables X_i sont independent les variables Y_i le sont egalement

Et cōē on a des variables Y_i sont des variables distribués suivant une Loi Bernouilli, N soit une Loi binomiale

plus precisement $N \sim \mathcal{B}(n, p = P(Y=1))$

or içi

$$p = P(X \leqslant a) = \underbrace{P(X \leqslant a \mid X = U_1)}_{P(U_1 \leqslant a)} \cdot \underbrace{P(X = U_1)}_{\theta}$$

$$+ \underbrace{P(X \leqslant a \mid X = U_2)}_{P(U_2 \leqslant a) = a/b} \cdot \underbrace{P(X = U_2)}_{1-\theta}$$

Soit $p = \theta + \frac{a}{b}(1-\theta) = \frac{b-a}{b}\theta + \frac{1}{b}$

fct linéaire de θ

Pour l'estimateur du maximum de vraisemblance de θ on peut utiliser une propri´et´e que nous avons vu lorsqu'on reparam´etrise un mod`ele statistique

Si $\hat{p}$ est l'estimateur du maximum de vraisemblance de p,

et si θ = g(p) alors l'estimateur du maximum de vraisemblance de θ est $\hat{\theta} = g(\hat{p})$.
Ici, on utilise le fait que l'estimation du maximum de vraisemblance de la probabilit´e dans un mod`ele binomial est la proportion.

Autrement dit, $\hat{p}$ *est la proportion d'observations inf´erieure* a, *i.e.*

$$\hat{p} = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(x_i < a)$$

Et à partir de là on en deduit l'estimateur de maximum de vraisemblance de θ puisque $\hat{\theta}$ est solution de :

$$\frac{b-a}{b}\hat{\theta} + \frac{1}{b} = \frac{1}{n}\sum \mathbb{1}(x_i < a)$$

$$\hat{\theta} = \left(\frac{1}{n}\sum \mathbb{1}(x_i < a) - \frac{1}{b}\right)\frac{b}{b-a}$$

Ex3. Soit m, median de $f(x)$

$$\int_0^m f_\theta(x)\,dx = 0.5$$

$$\int_0^m \frac{2x}{\theta^2}\,dx = 0.5$$

$$\left.\frac{x^2}{\theta^2}\right]_0^m = 0.5 \Rightarrow m^2 = 0.5\,\theta^2$$
$$\Rightarrow m = \sqrt{\frac{1}{2}}\,\theta$$

$$\mathcal{L}(x_1,\dots,x_n,\theta) = \begin{cases} \frac{2^n}{\theta^{2n}} \prod x_i & \text{si } \max(x_i) \geq \theta \\ 0 & \text{si } \max(x_i) < \theta \end{cases}$$

$$= \frac{2^n}{\theta^{2n}} \prod x_i \,\mathbb{1}(\max(x_i) \geq \theta)$$

Alors $\theta = \max(x_i)$

$$m = \sqrt{\frac{1}{2}}\,\max(x_i)$$

Ex4:

La surface du cercle S:

$$S = \pi \left(\frac{d}{2}\right)^2 \quad ; \quad d = d_0 + \varepsilon \; ;$$
$$\varepsilon \sim N(0, \sigma^2)$$

$d - d_0 \sim N(0, \sigma^2)$

$d \sim N(d_0, \sigma^2)$; un estimateur de d_0, σ,

D'après la méthode des max de vrais.

estimateur de d_0 ; $\hat{d}_0 = \frac{\sum d_i}{n}$

$$\hat{\sigma}_0^2 = \frac{1}{n} \sum_{i=1}^{n} (d_i - \hat{d}_0)^2$$

$$S = \left(\frac{d_0}{2}\right)^2 \pi \; ;$$

$$\hat{S} = \frac{\pi}{4} \cdot \left[\hat{d}_0^2 + \hat{\sigma}_0^2 - \frac{1}{n-1} \sum (d_i - d_0) \right)$$

$$E(\hat{S}) = \frac{\pi}{4} \left[\frac{E(\hat{d}_0^2)}{n^2} + E(\hat{\sigma}_0^2) - E(S^2) \right] \text{ with; } S^2 = \frac{\sum (d_i - d_0)^2}{n}$$

$$= \frac{\pi}{4} \left[\frac{Var(\sum d_i)}{n^2} + \left(\frac{E(\sum d_i)}{n}\right)^2 + \frac{(n-1)}{n} E(S^2) - E(S^2) \right.$$

$$= \frac{\pi}{4} \left[\frac{\sigma^2}{n} + \frac{n^2 d_0^2}{n^2} + \frac{n-1}{n} \sigma^2 - \sigma^2 \right]$$

$$= \frac{\pi}{4} d_0^2$$

Ex 6

$$f_\theta(x) = \frac{1}{2}\exp\left(-|\theta - x|\right)$$

$$E(X) = \int_{-\infty}^{\theta} \frac{x}{2} \exp(-\theta + x)\,dx + \int_{\theta}^{+\infty} \frac{x}{2} \exp(\theta - x)\,dx$$

$$= \left[\frac{x}{2}\exp(-\theta + x)\right]_{-\infty}^{\theta} - \frac{1}{2}\int_{-\infty}^{\theta} \exp(-\theta + x)\,dx + \left[-\frac{x}{2}\exp(+\theta - x)\right]_{\theta}^{\infty} + \frac{1}{2}\int_{\theta}^{\infty} \exp(\theta - x)\,dx$$

$$= \frac{\theta}{2} - \frac{1}{2}\left[\exp(-\theta + x)\right]_{-\infty}^{\theta} + \frac{\theta}{2} - \frac{1}{2}\left[\exp(\theta - x)\right]_{\theta}^{+\infty}$$

$$= \theta - \frac{1}{2}(1 - 1)$$

$$= \theta$$

$$\hat{\theta}^{EMV} = \bar{x}$$

$$E(\hat{\theta}) = E(\bar{x})$$

$$= \frac{1}{n}\sum E(x_i) = \frac{1}{n} n\theta = \theta$$

$$Var(\hat{\theta}) = Var(\bar{x})$$

$$= Var(\bar{x})$$

$$= \frac{Var}{n^2}\left(\sum x_i\right)$$

$$= \frac{\sum Var(x_i)}{n^2}$$

$$= \frac{1}{n^2}\sum_i \left(E(X_i^2) - (E(X_i))^2\right)$$

$$= \frac{1}{n^2}\sum_i \left(\int_{-\infty}^{\infty} \frac{x^2}{2}\exp(-|\theta - x|)\,dx - \theta^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{\infty}\left(\int_{-\infty}^{\theta}\frac{x^2}{2}\exp(x-\theta)\,dx + \int_{\theta}^{+\infty}\frac{x^2}{2}\exp(\theta - x)\,dx - \theta^2\right)$$

$$= \frac{1}{n^2}\sum_{i=1}^{\infty}\left(\left[\frac{x^2}{2}\exp(x-\theta)\right]_{-\infty}^{\theta} - \int_{-\infty}^{\theta} x\exp(x-\theta)\,dx\right.$$

$$+\left[-\frac{x^2}{2}\exp(\theta - x)\right]_{\theta}^{+\infty} - \int_{\theta}^{+\infty} x(-1)\exp(\theta - x)\,dx$$

$$= \frac{1}{n^2}\left[\sum_{i=1}^{n}\left(\frac{\theta^2}{2} - \int_{-\infty}^{\theta} x\exp(x-\theta)\,dx\right) + \frac{\theta^2}{2} + \int_{\theta}^{+\infty} x\exp(\theta - x)\,dx\right) - \theta^2\right]$$

$$= \frac{1}{n^2}\left(\sum_i \theta^2 - \left(\left[x\exp(x-\theta)\right]_{-\infty}^{\theta} - \int_{-\infty}^{\theta}\exp(x-\theta)\,dx\right) + \left[(-x)\exp(\theta - x)\right]_{\theta}^{\infty} + \int_{\theta}^{+\infty}\exp(\theta - x)\,dx - \theta^2\right)$$

$$= \frac{1}{n^2}\left(\sum_i (-\theta) + 1 + \theta + 1\right.$$

$$= \frac{2n}{n^2} = \frac{2}{n}$$

3) $f_\theta(x) = \frac{1}{2} \exp(-|\theta - x|)$

$$\mathcal{L}(x_1, \dots, x_n, \theta) = \frac{1}{2^n} \exp\left(\Sigma - |\theta - x|\right)$$

$$\log \mathcal{L}(x_1, \dots, x_n, \theta) = n\ln(2) - \sum_{i=1}^{n} |\theta - x|$$

$$= -n\ln(2) - \sum \sqrt{(\theta - x)^2}$$

$$\frac{\partial \ell}{\partial} = -\sum \frac{\partial \sqrt{(\theta - x_i)^2}}{\partial \theta}$$

$$= -\sum \frac{\partial}{\partial \theta}\left((\theta - x_i)^2\right)^{\frac{1}{2}}$$

$$= -\sum \frac{1}{2}\left((\theta - x_i)^2\right)^{-\frac{1}{2}} \cdot \left((\theta - x_i)^2\right)'$$

$$= -\sum \frac{1}{2} \frac{1}{\sqrt{(\theta - x_i)^2}} \cdot 2(\theta - x_i)$$

$$= -\sum \frac{1}{2} \frac{\theta - x_i}{|\theta - x_i|} = 0$$

$$= -\sum_{i=1}^{n} \frac{1}{2}\left(+1_{[-\infty, \theta]}(x_i) - 1_{[\theta, +\infty]}[\theta < x_i]\right)$$

$$= 0$$

$$\text{si } \sum_{i=1}^{n}\left(1[x_i < \theta) - 1[\theta < x_i]\right)$$

$$\Rightarrow \sum_{i=1}^{n} 1(x_i < \theta) = \sum_{i=1}^{n} 1(\theta < x_i)$$

$$\hat{\theta} = \text{median}(X)$$

Ex 7

$$f_\mu(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{((\ln x)-\mu)^2}{2\sigma^2}\right)$$

$$\mathcal{L} = \prod f_\mu(x_i) = \frac{1}{\sigma^n(\sqrt{2\pi})^n} \cdot \frac{1}{\prod_{i=1} x_i} \exp\left(-\sum\frac{((\ln x_i)-\mu)^2}{2\sigma^2}\right)$$

$$\ln \mathcal{L} = -n\ln(\sigma) - \frac{n}{2}\ln(2\pi) - \sum x_i - \frac{1}{2\sigma^2}\sum(\ln(x_i)-\mu)^2$$

$$\frac{\partial}{\partial\mu}\ell = +\frac{2}{2\sigma^2}\left(\sum \ln(x_i) - n\mu\right) = 0$$

$$\Rightarrow \hat{\mu} = \frac{\sum \ln(x_i)}{n}$$

$$\Rightarrow E(\hat{\mu}) = \frac{\sum E(\ln(x_i))}{n}$$

$$= \frac{n\mu}{n} = \mu$$

Ex 9.

$$F_\theta(x) = \left(1+\frac{1}{x^2}\right)^{-\theta} \mathbb{1}(x>0)$$

$$= \begin{cases} \left(1+\frac{1}{x^2}\right)^{-\theta} ; & x>0 \\ 0 ; & x<0 \end{cases}$$

$$f_\theta(x) = \begin{cases} (+\theta)\left(1+\frac{1}{x^2}\right)^{-\theta-1}\left(+\frac{2}{x^3}\right) & x>0 \\ 0 & x<0 \end{cases}$$

$$\prod f_\theta(x) = (+\theta)\prod_{i=1}^{n}\left(+\frac{2}{x^3}\right)\left(\prod_{i=1}^{n}\left(1+\frac{1}{x_i^2}\right)\right)^{-(\theta+1)}$$

$$\ell = n\ln(+\theta) + \sum \ln\left(+\frac{2}{(x_i)^3}\right) - (\theta+1)\sum\ln\left(1+\frac{1}{x_i^2}\right)$$

$\hat{\theta}^{EMV}$ est solution of $\frac{\partial \ell}{\partial\theta}\Big|_{\hat{\theta}_{EMV}} = 0$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\ln\left(1+\frac{1}{x_i^2}\right)$$

for $\theta = \hat{\theta}^{EMV}$; $\frac{\partial \ell}{\partial\theta}\Big|_{\theta=\hat{\theta}_{EMV}} = 0$

$$\Rightarrow \frac{n}{\hat{\theta}^{EMV}} = \sum_{i=1}^{n}\ln\left(1+\frac{1}{x_i^2}\right) \Rightarrow \hat{\theta}^{EMV} = \left(\frac{\sum_{i=1}^{n}\ln(1+1/x_i^2)}{n}\right)^{-1}$$

$$I(\theta) = E\left[\left(\frac{\partial \log f(X,\theta)}{\partial\theta}\right)^2 \Big| \theta\right]$$

$$= -n\,E\left(\frac{\partial^2 \log f(X,\theta)}{\partial\theta^2}\right)$$

$$\log f(X,\theta) = \log\left(+\theta\left(1+\frac{1}{x^2}\right)^{-\theta-1}\left(+\frac{2}{x^3}\right)\right)$$

$$\log f(X,\theta) = \ln(+\theta) - (\theta+1)\ln(1+1/x^2) + \ln\left(+\frac{2}{x^3}\right)$$

$$\frac{\partial}{\partial\theta}\log f(X,\theta) = \frac{1}{\theta} - \ln(1+1/x^2)$$

$$\frac{\partial}{\partial\theta^2}\log(f(X,\theta)) = -\frac{1}{\theta^2}$$

$$I(\theta) = \frac{n}{\theta^2}$$

Ex11:

$\mathcal{U} \sim U[-\theta, +\theta]$

1) $E(U_{1:n}) = E[\min(U_i)]$

$F_{U_{1:n}}(x) = P(\min(U_i) \leqslant x)$

$= P(\cup \{U_k \leqslant x\})$

$= 1 - P(\cap \{U_k > x\})$

$= 1 - \prod P(U_k > x)$

$= 1 - \left(\frac{\theta - x}{2\theta}\right)^n$

$f_{U_{1:n}}(x) = +n \frac{(\theta - x)^{n-1}}{(2\theta)^n}$

$E[U_{1:n}] = +\int_{-\theta}^{\theta} \frac{n x (\theta - x)^{n-1}}{(2\theta)^n} dx$

$= -\left[\frac{n \cancel{x} (\theta - x)^n}{(2\theta)^n n}\right]_{-\theta}^{\theta} + \int_{-\theta}^{\theta} \frac{n}{(2\theta)^n} \cdot \frac{(\theta - x)^n}{n} dx$

$= -\left[\frac{n\theta}{(2\theta)^n} \cdot \frac{(2\theta)^n}{n}\right] - \left[\frac{(\theta - x)^{n+1}}{(2\theta)^n (n+1)}\right]_{-\theta}^{\theta}$

$= -(\theta) + \frac{(2\theta)^{n+1}}{(2\theta)^n (n+1)}$

$= -\theta + \frac{2\theta}{n+1}$.

$\underline{U_{n:n}}$:

$$P(U_{n,n} < x) = \left(P(U_1 < x)\right)^n$$

$$= \left(\int_{-\theta}^{x} \frac{1}{2\theta} dx\right)^n = \left(\frac{x+\theta}{2\theta}\right)^n$$

$$f_{U_{n,n}}(x) = n\,\frac{(x+\theta)^{n-1}}{(2\theta)^n}$$

$$E_{U_{n,n}} = \int_{-\theta}^{\theta} \frac{n x\,(x+\theta)^{n-1}}{(2\theta)^n}$$

$$= \left[\frac{x}{(2\theta)^n}(x+\theta)^n\right]_{-\theta}^{\theta} - \int_{-\theta}^{\theta} \frac{1}{(2\theta)^n}(x+\theta)^n dx$$

$$= \frac{\theta(2\theta)^n}{(2\theta)^n} - \left[\frac{(x+\theta)^{n+1}}{(2\theta)^n(n+1)}\right]_{0}^{\theta}$$

$$= \theta - \frac{2\theta}{n+1} = \frac{(n+1)\theta - 2\theta}{n+1} = \frac{(n-1)\theta}{n+1}$$

2) $P(|U_i| < x) = P(-x < U_i < x)$ si $x>0$

$$= \int_{-x}^{x} \frac{1}{2\theta} dt = \frac{x}{\theta} \quad \text{si } x>0$$

$P(|U_i| < x) = 0$ si $x<0$

$$f_{|U_i|}(x) = \frac{1}{\theta} 1(0<x<\theta)$$

$$P(\max |U_i| < x) = \prod P(|U_i| < x)$$

$$= \left(P(|U_i| < x)\right)^n$$

$$= \left(\int_0^x \frac{1}{\theta} d\theta\right)^n = \left(\frac{x}{\theta}\right)^n$$

$$f_{\max|U_i|}(x) = n\,\frac{x^{n-1}}{\theta^n}$$

$$E(\max|U_i|) = \int_0^\theta \frac{n x^n}{\theta^n} dx$$

$$= \frac{1}{\theta^n} \theta^{n+1} \cdot \frac{n}{n+1} = \frac{\theta n}{n+1}$$

4) $$\mathcal{L}(u_1, \ldots, u_n, \theta) = \prod_{i=1}^n f_\theta(x_i)$$

$$= \prod_{i=1}^n \frac{1}{2\theta} 1_{[-\theta,\theta]}(x_i)$$

$$= \begin{cases} 0 \text{ sinon} \\ \frac{1}{(2\theta)^n} \; ; \max(|x_i|) < \theta \end{cases}$$

La fct $\theta \to \left(\frac{1}{2\theta}\right)^n$ est decroissanu en α pour $\theta > 0$; donc L prend son maximum en $\theta = \max|x_i|$, donc l'estimateur de vraisemblance s'ecrit:

$$\hat{\theta}^{MV} = \max|X_i|$$

5) $$E(\hat{\theta}^{MV}) = \theta \cdot \frac{n}{n+1} \neq \theta$$

6) $$Var(U_{[\theta;\theta]}) = E(U^2) - (E(U))^2$$

$$= \int_{-\theta}^{\theta} \frac{u^2}{2\theta} du$$

$$= \left[\frac{u^3}{3 \cdot 2\theta}\right]_{-\theta}^{\theta}$$

$$= \left(\frac{\theta^3}{6\theta} + \frac{\theta^3}{6\theta}\right)$$

$$= \frac{\theta^3}{3}$$