Ex 1:

$n_{11}$ : 1ère res. 2ème reu $i=1$
$n_{12}$ : 1ère re 2ème ech $i=2$
$n_{21}$ : 1ère ech 2ème res. $i=3$
$n_{22}$ : 1ère ech 2ème ech $i=4$

$H_0$ : $p_{1.} = p_{.1}$

$H_1$ : $p_{1.} \neq p_{.1}$

1) $N_i$ represente le nombre d'individu dans la classe $i$

$H_0$ : $p_{21} = p_{12}$ ; $p_{22} = p_{2.} - p_{21} = (1 - p_{1.}) - p_{21}$
$= (1 - p_{11} - p_{12}) - p_{21}$
$= 1 - p_{11} - 2p_{21}$

with $n_{22} = n - n_{11} - n_{12} - n_{21}$

$L(p_{11}, p_{12}, p_{21}, p_{22}) = P(N_1 = n_{11}, N_2 = n_{12}, N_3 = n_{21}, N_4 = n_{22} \quad p_{.1} = p_{1.})$
$= P(N_1 = n_{11}, N_2 = n_{12}, N_3 = n_{21}, N_4 = n_{22} / p_{21} = p_{12})$

$$= \frac{n!}{n_{11}! n_{12}! n_{21}! n_{22}!} \times p_{11}^{n_{11}} \cdot p_{12}^{n_{12}} \cdot p_{12}^{n_{21}} \cdot p_{22}^{n_{22}}$$

$$= \frac{n!}{n_{11}! n_{12}! n_{21}! n_{22}!} p_{11}^{n_1} p_{12}^{(n_{12}+n_{21})} (1 - p_{11} - 2p_{12})^{n_{22}}$$

2) Maximisons

$$\ln L(p_{11}, p_{12}) = cte + n_{11} \ln p_{11} + (n_{12} + n_{21}) \ln p_{12} + n_{22} \ln(1 - p_{11} - 2p_{12})$$

En annulant les 2 dérivée partielles % $p_{11}$, $p_{12}$

$$\begin{cases} \frac{\partial \ln L}{\partial p_{11}}(p_{11}, p_{12}) = \frac{n_{11}}{p_{11}} - \frac{n_{22}}{1 - p_{11} - 2p_{12}} \\ \frac{\partial \ln L}{\partial p_{12}}(p_{11}, p_{12}) = \frac{(n_{12} + n_{21})}{p_{12}} - \frac{2n_{22}}{1 - p_{11} - 2p_{12}} \end{cases}$$

En annulan. on optient

$$\frac{n_{11}}{p_{11}} = \frac{n_{22}}{1 - p_{11} - 2p_{12}} = \frac{n_{12} + n_{21}}{2p_{12}} \overset{?}{=} \frac{n}{1}$$

preuve : $n - \frac{n_{11}}{p_{11}} = ?$

On a

$$n_{11} \times p_{22} = n_{22} \times p_{11}$$

$$n_{11} \times 2p_{12} = (n_{12} + n_{21}) p_{11}$$

$$\frac{n}{1} = \frac{n_{11} + n_{12} + n_{21} + n_{22}}{p_{11} + 2p_{12} + p_{22}}$$

$$= \frac{p_{11} \cdot \frac{n_{11}}{p_{11}} + n_{11} \times \frac{2p_{21}}{p_{11}} + n_{11} \cdot \frac{p_{22}}{p_{11}}}{p_{11} + 2p_{12} + p_{22}}$$

$$= \frac{\frac{n_{11}}{p_{11}} (p_{11} + 2p_{21} + p_{22})}{(p_{11} + 2p_{21} + p_{22})} = \frac{n_{11}}{p_{11}}$$

d'où les estimation de MV sous $H_0$:

$$\hat{p}_{11}^{EMV} = \frac{n_{11}}{n.} ; \quad \hat{p}_{12}^{EMV} = \frac{n_{12} + n_{21}}{2n} = \hat{p}_{21}$$

$$\hat{p}_{22}^{EMV} = \frac{n_{22}}{n}$$

3. La frequence attendue sous $H_0$ st resp:

$n\hat{p}_{11} = n_{11}$ , $n\hat{p}_{22} = n_{22}$

$n\hat{p}_{12} = \frac{n_{12}+n_{21}}{2} = \hat{p}_{21} n$

D'où :

$$Q = \sum_{j=1}^{4} \frac{(n_j - n\hat{p}_j)^2}{n\hat{p}_j}$$

$$= \frac{(n_{11} - n\hat{p}_{11})^2}{n\hat{p}_{11}} + \frac{(n_{12} - n\hat{p}_{12})^2}{n\hat{p}_{12}} + \frac{(n_{21} - n\hat{p}_{21})}{n\hat{p}_{21}} + \frac{(n_{22} - n\hat{p}_{22})}{n\hat{p}_{22}}$$

$$= \frac{(n_{11}-n_{11})^2}{n_{11}} + \frac{\left(n_{12} - \frac{n_{12}+n_{21}}{2}\right)^2}{\frac{n_{12}+n_{21}}{2}} + \frac{\left(n_{21} - \frac{n_{21}+n_{12}}{2}\right)^2}{\frac{n_{12}+n_{21}}{2}} + \frac{(n_{22}-n_{22})^2}{n\hat{p}_{22}}$$

$$Q = \frac{0}{n_{11}} + \frac{\left(n_{12} - \frac{n_{12}+n_{21}}{2}\right)^2}{\frac{n_{12}+n_{21}}{2}} + \frac{\left(n_{21} - \frac{n_{12}+n_{21}}{2}\right)^2}{\frac{n_{12}+n_{21}}{2}} + \frac{0}{n_{22}}$$

$$= \frac{2}{n_{12}+n_{21}} \times \left( \frac{(n_{12}-n_{21})^2}{4} + \frac{(n_{21}-n_{12})^2}{4} \right)$$

$$= \frac{(n_{12}-n_{21})^2}{n_{12}+n_{21}}$$

Le nombre de categorie est 4 et il y a 2 paramètre à estimer $H_0$, $p_{11}$ et $p_{12}$ donc selon le theorme de Cramer, les degré de liberté st $4-1-2=1$

Ex2c

X rep. le nombre de dés

$X \sim Bin.\left(3, \frac{1}{6}\right)$

$x = \{0, 1, 2, 3\}$

$p = \{(1-p)^3; \; p(1-p)^2 \times 3; \; p^2(1-p) \times 3; \; p^3\}$

$\Leftrightarrow$

$H_0$ : 
$$p_0 = (1-p)^3$$
$$p_1 = 3(1-p)^2 \times p$$
$$p_2 = 3p^2(1-p)$$
$$p_3 = p^3$$

$p = \frac{1}{6}$

$$Q = \frac{\left(48 - 100 \times \left(\frac{5}{6}\right)^3\right)^2}{100 \times (1-p)^3} + \frac{\left(34 - 100 \times 3 \times p(1-p)^2\right)^2}{100 \times 3 \times p(1-p)^2} + \frac{\left(15 - 100 \times 3 \times p^2(1-p)\right)^2}{100 \times 3p^2(1-p)} + \frac{\left(3 - 100 \times p^3\right)}{100 \times p^3}$$
$$\simeq 39.41$$

$Q \sim \chi^2(k-1) = \chi^2(3)$

$q_{1-5\%, 3} = 7.81$

$Q_{obs} > 7.81 \Rightarrow$

Ho est rejetté.