STT 1000 - STATISTIQVES

ARTHVR CHARPENTIER

Loi de Bernoulli

$$f(x) = \mathbb{P}(X = x) = \begin{cases} p & \text{si } x = 1, \\ 1 - p = q & \text{si } x = 0, \\ 0 & \text{sinon.} \end{cases}$$

$$\mathbb{P}(X = x) = p^x(1 - p)^{1-x} = p^x q^{1-x}, x \in \{0, 1\}$$

$$\mathbb{E}[X] = p \text{ et } \mathrm{Var}[X] = pq$$

```
> sample_y = c(1, 1, 1, 0, 1, 1, 0, 0, 1, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1,
      0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1, 1, 1, 1, 0, 0,
      1, 1, 0, 0, 0, 1, 1, 1, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1,
      0, 1, 0, 1)
> n
[1] 76
```

Méthode des moments

```
> mean(sample_y)
[1] 0.6842105
```

Maximum de Vraisemblance (analytique)

Étant donné un échantillon $\{x_1, \cdots, x_n\}$ de loi $\mathcal{B}(p)$,

$$\mathcal{L}(p; x_1, \cdots, x_n) = \mathbb{P}\big[X_1 = x_1, \cdots, X_n = x_n\big] = \prod_{i=1}^{n} \mathbb{P}[X_i = x_i]$$

$$\mathcal{L}(p; x_1, \cdots, x_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{s_n}(1-p)^{n-s_n},\ s_n = \sum_{i=1}^{n} x_i = n\overline{x}$$

$$\log \mathcal{L}(p; x_1, \cdots, x_n) = n\overline{x}\log(p) + n(1-\overline{x})\log(1-p)$$

$$\frac{\partial \log \mathcal{L}(p; x_1, \cdots, x_n)}{\partial p} = n\overline{x}\frac{1}{p} + n(1-\overline{x})\frac{-1}{1-p} = \frac{n\overline{x}[1-p] - n(1-\overline{x}p)}{p(1-p)}$$

de telle sorte que la dérivée s'annule si le numérateur s'annule

$$\left.\frac{\partial \log \mathcal{L}(p; x_1, \cdots, x_n)}{\partial p}\right|_{p=\widehat{p}} = 0 \iff n\overline{x}[1-\widehat{p}] - n(1-\overline{x}\widehat{p}) = 0$$

i.e. $\widehat{p} = \overline{x}$

Maximum de Vraisemblance (numérique)

```
> logL = function(p){
  sum(log(dbinom(sample_y,size=1,prob=p)))
}
> logL(.4)
[1] -59.90693
> logL(.7)
[1] -47.44244
> vect_p = seq(.01,.99,by=.01)
> vect_L = Vectorize(logL)(vect_p)
> optim(par = .5,fn = function(p) -logL(p))
$par
[1] 0.6841797

$value
[1] 47.39777
> optimize(f =  function(p) -logL(p), interval = c(0,1))
$minimum
[1] 0.6842177

$objective
[1] 47.39777
```

Supposons que P est ici une variable aléatoire, avec

$$(X|P=p) \sim \mathcal{B}(p),\ f(x|p) = p^x(1-p)^(1-x), x \in \{0,1\}$$

et un a-priori de loi Beta pour P

$$P \sim \mathcal{B}(a,b),\ \pi(p) = \frac{p^a(1-p)^b}{B(a,b)},\ p \in [0,1]$$

Alors

$$(P|\boldsymbol{X}=\boldsymbol{x}) \sim \mathcal{B}(a+n, b+n\overline{x}),\ \pi(p|\boldsymbol{x}) = \frac{p^a(1-p)^b}{B(a,b)},\ p \in [0,1]$$