Liste 5:

Ex1:

$$MSE(\hat{\theta}) = Biais(\hat{\theta})^2 + Var(\hat{\theta})$$

$$MSE(\hat{\theta}) = E((\theta - \hat{\theta})^2)$$

$$= E(\hat{\theta} - E(\hat{\theta}) + Biais(\hat{\theta}))^2$$

Démonstration [masquer]

Rappelons d'abord que $\text{Biais}(\hat{\theta}) \overset{\text{def}}{=} \mathbb{E}(\hat{\theta}) - \theta$ et $\mathbb{E}(\hat{\theta})$ sont des constantes, ce qui permet d'utiliser la linéarité de l'espérance : $\mathbb{E}(c_1 X + c_2) = c_1 \mathbb{E}(X) + c_2$.

$$\begin{aligned}
\text{MSE}(\hat{\theta}) \overset{\text{def}}{=} \mathbb{E}\left[(\hat{\theta} - \theta)^2\right] &= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \text{Biais}(\hat{\theta})\right)^2\right] \\
&= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)^2 + 2\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)\text{Biais}(\hat{\theta}) + \text{Biais}(\hat{\theta})^2\right] \\
&= \mathbb{E}\left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)^2\right] + 2\mathbb{E}\left(\hat{\theta} - \mathbb{E}(\hat{\theta})\right)\text{Biais}(\hat{\theta}) + \text{Biais}(\hat{\theta})^2 \\
&= \text{Var}(\hat{\theta}) + 2\left(\mathbb{E}(\hat{\theta}) - \mathbb{E}(\hat{\theta})\right)\text{Biais}(\hat{\theta}) + \text{Biais}(\hat{\theta})^2 \\
&= \text{Var}(\hat{\theta}) + \text{Biais}(\hat{\theta})^2
\end{aligned}$$

$$E(y) = 1/\lambda$$

$$V(Y) = 1/\lambda^2$$

$$E(\hat{\theta}) = E\left(\frac{1}{n} \sum y_i\right) = \frac{1}{n} \cdot n E(y_1) = 1/\lambda$$

$$\left(\text{Biais}(\hat{\theta})\right)^2 = E(\hat{\theta} - \theta)^2 = \left(\frac{1}{\lambda} - \frac{1}{\lambda}\right)^2 = 0$$

$$V(\hat{\theta}) = \left(\frac{1}{n}\right)^2 \cdot \sum \text{Var}(Y_i)$$

$$= \frac{1}{n^2} \cdot n \frac{1}{\lambda^2} = \frac{1}{n} \cdot \frac{1}{\lambda^2}$$

$$\text{MSE}(\hat{\theta}) = \frac{1}{n} \frac{1}{\lambda^2} = \frac{1}{n} \theta^2$$

$$E(\tilde{\theta}) = E\left(\frac{1}{n-1} \Sigma(Y_i)\right)$$

$$= \frac{1}{n+1} \Sigma E(Y_i) = \frac{n}{n+1}\left(\frac{1}{\lambda}\right)$$

$$\left(\text{Biais}(\tilde{\theta})\right)^2 = \left(\frac{n}{n+1} - 1\right)^2 \theta^2$$

$$= \left(\frac{1}{n+1}\right)^2 \theta^2$$

$$\text{Var}(\tilde{\theta}) = \left(\frac{1}{n+1}\right)^2 \sum \text{Var}(y_i)$$

$$= \frac{n}{(n+1)^2} \theta^2$$

$$\text{EQM}(\tilde{\theta}) = \frac{n+1}{(n+1)^2} \theta^2 = \frac{1}{n+1} \theta^2$$

Ex2 :

$\{y_1, \ldots, y_n\}$ iid

$Y_i \sim N(\mu, \sigma^2)$

1) Comme l'estimateur de maximum de vraisemblance est asymptotique normal

on peut construire un interval de confiance telle qu'il contient le parametre avec un prob $1-\alpha$

$$C_n = \left[\hat{\mu}_n \pm \phi^{-1}(1-\alpha/2)\ \hat{\sigma}_{\hat{\theta}_n}\right]$$

Donc

Longueur. $2\phi^{-1}(1-\alpha/2)\ \hat{\sigma}_{\hat{\theta}_n}$

$$\hat{\sigma}_{\hat{\theta}_n}^2 = Var(\hat{\theta}) = Var\left(\frac{\sum x_i}{n}\right)$$
$$= \frac{1}{n^2}\sum Var(x_i) = \frac{1}{n}\sigma^2$$

$$\hat{\sigma}_{\hat{\theta}_n} = \frac{1}{\sqrt{n}} \sigma$$

$$\phi^{-1}\left(1-\frac{\alpha}{2}\right) = 1.96$$

$$\hat{\sigma}^2_{ML} = \frac{1}{n} \sum (x_i - \bar{x})$$

$$\sum (y_i - \bar{y})^2 \sim \sigma^2 \chi^2_{n-1}$$

$$\sum \left(\frac{y_i - \bar{y}}{\sigma}\right)^2 \sim \chi^2_{n-1} \Rightarrow n \frac{\hat{\sigma}^2_{ML}}{\sigma^2} \sim \chi^2_n$$

$$\frac{n \hat{\sigma}^2_{ML}}{\sigma^2} \longrightarrow N(n, 2n)$$

$$\frac{\frac{n \sigma^2_{ML}}{\sigma^2} - n}{\sqrt{2n}} \sim N(0,1)$$

$$n - u_{\alpha/2}\sqrt{2n} < \frac{n \sigma^2_{ML}}{\sigma^2} < n + u_{\alpha/2}\sqrt{2n}$$

$$\frac{n\hat{\sigma}^2}{n + u_{\alpha/2}\sqrt{2}} < \sigma^2 < \frac{n\hat{\sigma}^2}{n - u_{\alpha/2}\sqrt{2n}}$$

Ex 3 : $y_1, \dots, y_n \sim \mathcal{U}[0,\theta] \;/\; \theta > 0$

1) $\hat{\theta}^{MLE} = \max(y_i) = y_{n:n}$

2) Fdr : $F_{y_{n:n}}(x) = P(y_{n:n} < x)$

$$= P(y_1 < x, \dots, y_n < x)$$
$$= \prod P(y_i < x)$$
$$= \left(P(y_1 < x\right)^n = \left(1(0<x<\theta) \cdot \frac{x}{\theta}\right)^n$$
$$= 1(0<x<\theta)\, \frac{x^n}{\theta^n}$$

3) Intervalle de confiance unilatérale à gauche de niveau $1-\alpha$ pour le paramètre θ est un intervalle aléatoire $[\hat{a}(y); +\infty]$

$P(\hat{\theta} \in [\hat{a}(y); +\infty] = 1-\alpha$

sachant que $Y_{n:n}$ est définie sur $[0;\theta]$

Then

$P(\hat{\theta} \in [\hat{a}(y); +\infty]) = P(\hat{\theta} \in [\hat{a}(y); \theta]$

$$= P(\hat{a}(y) < U_{n:n} < \theta)$$
$$= 1 - \alpha$$

$$\int_{\hat{a}}^{\theta} \frac{n x^{n-1}}{\theta^n}\, dx = \left[\frac{x^n}{\theta^n}\right]_{\hat{a}}^{\theta}$$
$$= \frac{\theta^n}{\theta^n} - \frac{\hat{a}^n}{\theta^n} = 1 - \alpha$$
$$= 1 - \frac{\hat{a}^n}{\hat{\theta}^n} = 1 - \alpha$$

$$\hat{a} = \theta \cdot (\alpha)^{1/n}$$

$$\theta \alpha^{1/n} < U_{n:n} < \theta$$
$$\Rightarrow \alpha^{1/n} < \frac{U_{n:n}}{\theta} < 1$$
$$1 < \frac{\theta}{U_{n:n}} < \alpha^{-1/n}$$
$$U_{n:n} < \theta < \alpha^{-1/n} U_{n:n}$$

Ex 4:

$\mu = 614$

$\sigma = 26$

$n = 158$

$X_i \sim N(614, 26)$

$\hat{\mu} = \bar{X}$ est. de μ

$IC = [\mu \pm \sigma \cdot \varphi_{\alpha/2} \cdot \sqrt{n}]$

$IC = [614 \pm 26 \times 1.96 \times \sqrt{158}]$

$= [-26.55 ; 1254.55]$

$S = \sum_{i=1}^{158} \mu$

$IC_S = [-26.55 \times 158 ; 1254.55 \times 158]$

Ex5: $f(x,n,p) = \binom{x+n-1}{x} p^n (1-p)^x$

$$\ln f(x,n,p) = \ln \binom{x+n-1}{x} + n\ln p + x\ln(1-p)$$

$$\frac{\partial}{\partial p} \ln f = \frac{n}{p} - \frac{x}{1-p} = 0$$

$$\frac{n(1-p) - xp}{p(p-1)} = 0$$

$$n - np + xp = 0$$

$$\Rightarrow n - (n+x)p = 0$$

$$p = n/n+x$$

Binomial

$$g(x,n,p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$\ln g = \ln \binom{n}{x} + x\ln p + (n-x)\ln(1-p)$$

$$\frac{\partial}{\partial p} \ln g = \frac{x}{p} - \frac{(n-x)}{1-p}$$

$$\frac{\partial}{\partial p} \ln g \Big|_{p=\hat{p}^{MLE}} = 0$$

$$\frac{x(1-\hat{p}) - \hat{p}(n-x)}{p(1-\hat{p})} = 0$$

$$x - x\hat{p} - \hat{p}n + x\hat{p} = 0$$

$$x - \hat{p}n = 0$$

$$\hat{p} = \frac{x}{n}$$

Ex 6:

$P \sim N(\mu, 0.5)$

$\hat{\mu} = 3.6$

$n = 49$

1) $IC = \left[\hat{\mu} \pm \frac{0.5 \times 1.96}{\sqrt{49}}\right.$

$= \left[3.5 \pm \frac{0.5 \cdot 1.96}{\sqrt{49}}\right]$

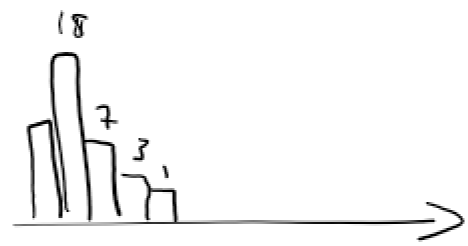
2) $L = \left[\frac{2 \times 0.5 \times u_{\alpha/2}}{\sqrt{49}}\right] = 0.1$

$u_{\alpha/2} = 0.1 \times 7 = 0.7$

$1 - \frac{\alpha}{2} = 0.75$

$\frac{\alpha}{2} = 0.25$

$\alpha = 0.5$



Ex7:

$$\overline{X} = \frac{\sum x_i}{n}$$

$\{x_1, \ldots, x_n\}$ iid de Loi $P(\lambda)$

$$\left[\overline{X} \pm u_{\alpha/2} \frac{\sqrt{\overline{X}}}{\sqrt{n}}\right]$$

$$= \left[0.9 \pm (1.96) \frac{\sqrt{0.9}}{\sqrt{50}}\right]$$

$$= [0.637 ; 1.16]$$

Ex8:

$X_1 \sim Ber(p, pq)$

$X_1, \ldots, X_n$

$n = 1000$

$$\hat{p} = \frac{45}{400} = 0.1125$$

$$\left[\hat{p} - u_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})} ; \hat{p} + u_{\alpha/2}\sqrt{\hat{p}(1-\hat{p})}\right]$$

$$\left[0.1125 - 0.95\sqrt{\frac{45}{400}\left(\frac{355}{400}\right)} ; 0.1125 + 0.95\sqrt{\frac{45 \times 325}{400 \times 400}}\right]$$

$$C_3 = \left[10000 \times 0.1125 \times 0.95\sqrt{\frac{45}{400}\left(\frac{355}{400}\right)} ; 1000 \times 0.1125 \times 0.95\sqrt{\frac{45 \times 325}{(400)^2}}\right]$$

Ex9) 1/ S suit Loi binomiale negative de paramètre n et p

$$g(x,n,p) = \binom{x+n-1}{x} p^n (1-p)^x$$

$X_i \sim$ Loi geom

$$f(x,p) = p(1-p)^x$$

<u>Initialization:</u>

<u>n = 1</u> : $g(x,p) = \binom{x}{x} p^1 (1-p)^x = p(1-p)^x$

$\sum_{i=1}^{n} X_i = X_1$

$f(x,p) = p(1-p)^x$

<u>Then</u>

$g(x,p,1) = f(x,p)$

<u>Recurrence:</u>

Consider that

$\sum_{i=1}^{n} X_i$ suit une Loi binomial neg de paramètre n et p

<u>proof that</u> :

$$P\left(\sum_{i=1}^{n+1} X_i = x\right) = \binom{x+n}{x} p^{n+1} (1-p)^x = \frac{(x+n)!}{x!\,n!} p^{n+1} (1-p)^x$$

$$P\left(\sum_{i=1}^{n+1} X_i = x\right) = \sum_{j=0}^{x} P\left(\sum_{i=1}^{n} X_i = j\right) P(X_{n+1} = x-j)$$

$$= \sum_{j=0}^{x} \binom{j+n-1}{j} p^n (1-p)^j \cdot p (1-p)^{x-j}$$

$$= \left[\binom{n-1}{0} + \dots + \binom{x+n-1}{x}\right] p^{n+1} (1-p)^x$$

$$= \sum_{j=0}^{x} \binom{n-1+j}{j} p^{n+1} (1-p)^x$$

$$= \binom{(n-1)+x+1}{x} p^{n+1} (1-p)^x$$

$$= \binom{x+n}{x} p^{n+1} (1-p)^x$$

<u>with</u>: $\sum_{j=0}^{x} \binom{n-1+j}{j} = \binom{(n-1)+x+1}{x}$

2) $H_0 : p_0 = 1/3$ vs $H_1 | p_1 = 2/3$

$p_0 < p_1$

$$L(x_1 .. x_n, p) = p^n (1-p)^{\sum_{i=1}^n x_i}$$

$$\frac{L(x_1, ..., x_n, p_1)}{L(x_1 .. x_n, p_0)} = \frac{p_1^n (1-p_1)^{\sum x_i}}{p_0^n (1-p_0)^{\sum x_i}} = h_{p_0, p_1}(\Sigma(x))$$

with $h_{p_0 p_1}(x)$ est strict decroiss.

Le modèle est donc à rapport de vraisemblance strict decroissant en $T(x) = \sum x_i$

l'espérance de X est égal à $1/p$

donc si l'on estiment cette esp. par x,

un test de H_0 contre H_1 à par exemple pour region critique :

Region de rejet: R_0

$$R_0 = \left\{ \left(\frac{p_1}{p_0}\right)^n \cdot \left(\frac{1-p_1}{1-p_0}\right)^{\sum x_i} > k \right\}$$

$$R_0 = \left\{ n \ln\left(\frac{p_1}{p_0}\right) + \sum x_i \ln\left(\frac{1-p_1}{1-p_0}\right) > \ln(k) \right\}$$

$$R_0 = \left\{ \sum x_i < \left(\ln k - n \ln\left(\frac{p_1}{p_0}\right)\right) \times \left(\ln\left(\frac{1-p_1}{1-p_0}\right)\right)^{-1} < 0 \right\}$$

3) $P(\sum x_i < k) \Leftrightarrow P(S < k) = 5\% \; ; n = 4$

$P(S=0) = \left(\frac{1}{3}\right)^4 = 0.01$

$P(S=1) = \left(\frac{1}{3}\right)^4 \times 4 \times \left(\frac{2}{3}\right) = 0.032$

$P(S<2) = 0.045 < 5\%$

$\Longrightarrow k = 2$

$$\Rightarrow \left(\ln \ell - \ln\left(\left(\frac{p_1}{p_0}\right)^4\right)\right) = k \ln\left(\frac{p_0}{p_1}\right)$$

$$\Rightarrow \ln \ell = \ln\left[\left(\frac{p_0}{p_1}\right)^2 \times \left(\frac{p_1}{p_0}\right)^4\right)$$

$$\Rightarrow \ell = \frac{p_1^2}{p_0^2} = 4$$

<u>Puissance</u> :

$$P(R_0 / H_1) = P\left(\sum x_i < 2 \mid \theta = \theta_1 = \frac{2}{3}\right)$$

$$= 0.68$$

$$\Rightarrow P(S=0|\theta_1) = \left(\frac{2}{3}\right)^4$$

$$P(S=1|\theta_1) = \left(\frac{2}{3}\right)^4 \cdot \left(\frac{1}{3}\right) 4$$

$$P(S < 2 | \theta_1) = 0.46$$

Ex15 : $X_1, \dots, X_n$ iid ; $X_i \sim \exp(\theta)$

1) Distribution de La premiere ampoule ne meurt

$X_{1:n} = \min\{X_1, \dots, X_n\}$

$P(X_{1:n} < x) = P(\min(X_1, \dots, X_n) < x)$

$= 1 - P(\min(X_1, \dots, X_n > x)$

$= 1 - \prod P(X_i > x)$

$= 1 - \prod(1 - P(X_i < x))$

$= 1 - \prod(1 - (1 - e^{-\lambda x})$

$= 1 - \prod(e^{-\lambda x}) = 1 - e^{-\frac{n}{\theta}x}$

$X_{1:n} \sim \exp\left(\frac{\theta}{n}\right)$

$E(X_{1:n}) = \frac{\theta}{n}$

Distribution de La derniere ampoule ne meurt

$X_{n:n} = \max\{X_1, \dots, X_n\}$

$P(X_{1:n} < x) = P(\max(X_1, \dots, X_n) < x)$

$= \prod_{i=1}^{n}\left(1 - e^{-\frac{1}{\theta}x}\right)$

$= \left(1 - e^{-\frac{1}{\theta}x}\right)^n$

$f(x, \theta) = n\left(\frac{1}{\theta}e^{-\frac{1}{\theta}x}\right)\left(1 - e^{-\frac{1}{\theta}x}\right)^{n-1}$

$E(X_{[n:n]}) = \int_0^{+\infty} n\frac{x}{\theta}e^{-\frac{1}{\theta}x}\left(1 - e^{-\frac{1}{\theta}x}\right)^{n-1} dx$

$\int x \; x \left(\frac{n}{\theta}xe^{-\frac{1}{\theta}x}\right)\left(1 - e^{-\frac{1}{\theta}x}\right)^{n-1}$

$= \left[x\left(1 - e^{-\frac{1}{\theta}x}\right)^n\right]_0^{+\infty} - \int_0^{+\infty}\left(1 - e^{-\frac{1}{\theta}x}\right)^n dx$

2) $L(\theta, X) = \prod f(x_i, \theta)$

$= \left(\frac{1}{\theta}\right)^n e^{-\frac{1}{\theta}\sum x_i}$

$= \lambda^n e^{-\lambda \sum x_i}$

3) $\ln L(\theta, X)$

$= n \ln \lambda - (\Sigma x_i)\lambda$

$$\frac{\partial}{\partial \lambda} \ln L(\lambda, X) = \frac{n}{\lambda} - \Sigma(x_i)$$

$$\left.\frac{\partial}{\partial \lambda} \ln(\lambda, X)\right|_{\lambda = \hat{\lambda}^{EMV}} = 0$$

$$\Rightarrow \frac{n}{\hat{\lambda}_{EMV}} - \Sigma X_i = 0$$

$$\Rightarrow \hat{\lambda} = \frac{1}{\bar{x}} \Rightarrow \hat{\theta}^{EMV} = \bar{x}$$

4) $Var(\bar{x}) = Var\left(\frac{\Sigma X_i}{n}\right)$

$$= \frac{1}{n^2} \Sigma Var(X_i)$$

$$= \frac{1}{n}\theta^2$$

5) $\theta_1 > \theta_0 \Rightarrow \frac{1}{\theta_1} < \frac{1}{\theta_0}$

$H_0 : \theta = \theta_0$

$H_1 = \theta = \theta_1$

$$LR = \frac{L(X, \theta_0)}{L(X, \theta_1)} = \left(\frac{\theta_0}{\theta_1}\right)^{-n} \times \exp\left[\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \Sigma X_i\right]$$

<u>Intuitivement :</u>

<u>si</u> Les donnés soutient H_1

<u>**Alors**</u> La fct de vraisemblance $L(X, \theta_1)$ devrait être petit

<u>Donc</u> Le Rapport de vraisemblance est élevé

Ainsi nous rejetons l'hypothèse nulle

si le rapport de vraisemblance est grand

$LR \geq k$ ou k est cte telque

$P(LR \geq k) = \alpha$

sous l'hypothèse nulle $(\theta = \theta_0)$

6) TCL

$$\overline{X}_n \longrightarrow N\left(\theta, \frac{\theta^2}{n}\right)$$

$$S_n \longrightarrow N(n\theta, n\theta^2)$$

7) $$\frac{\overline{X}_n - \theta}{\theta/\sqrt{n}} \longrightarrow N(0,1)$$

$$P(LR \geq k) =$$

$$LR \geq k$$

$$\Leftrightarrow$$

$$\exp\left[\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \times \Sigma X_i\right] \geq k\left(\frac{\theta_0}{\theta_1}\right)^n$$

$$\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \Sigma X_i \geq \ln\left(k\left(\frac{\theta_0}{\theta_1}\right)^n\right)$$

$$\Sigma X_i \leq \ln\left[k\left(\frac{\theta_0}{\theta_1}\right)^n\right] \times \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^{-1}$$

$$\Sigma X_i \leq a$$

Under H_0:

$$\frac{\Sigma X_i - n\theta_0}{\theta\sqrt{n}} \longrightarrow N(0,1)$$

$$P(LR \geq k) = \alpha$$

$$P(\Sigma X_i \leq a) = \alpha$$

$$P\left(\frac{\Sigma X_i - n\theta_0}{\sqrt{n}\,\theta_0} \leq \frac{a - n\theta_0}{\sqrt{n}\,\theta_0}\right) = \alpha$$

$$\frac{a - n\theta_0}{\sqrt{n}\,\theta_0} = -1.64$$

$$a = \left(-1.64 \times (\sqrt{n}\,\theta_0)\right) + n\theta_0$$

$$a = \ln\left[k\left(\frac{\theta_0}{\theta_1}\right)^n\right] \times \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^{-1}$$

$$= a \times \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) = \ln\left(k\left(\frac{\theta_0}{\theta_1}\right)^n\right)$$

$$k\left(\frac{\theta_0}{\theta_1}\right)^n = \exp(b)$$

$$k = (\exp(b)) \times \left(\frac{\theta_1}{\theta_0}\right)^n$$

$$k = 25.19$$

8) $P(LR \geqslant k / H_1)$

$P(\sum X_i \leqslant a / H_1)$

$P\left(\frac{\sum X_i - \theta_1 n}{\sqrt{n}\,\theta_1} \leqslant \frac{a - \theta_1 n}{\sqrt{n}\,\theta_1}\right) = \beta$

$F\left(\frac{a-\theta_1 n}{\sqrt{n}\,\theta_1}\right) = \beta$

9)

Ex10:

$H_0 : \theta = \theta_0$

$H_1 : \theta > \theta_0$

$\alpha = 1\%$

$\theta_0 = 4\%$

Test statistic

$Z = \frac{\hat{\theta} - \theta_0}{\sqrt{\frac{\theta_0(1-\theta_0)}{n}}}$

$n = 1243$

$\sigma_{\bar{X}_n} = \sqrt{\frac{\theta_0(1-\theta_0)}{n}}$

$z = \frac{\frac{72}{1243} - 0.04}{\sigma_{\bar{X}}} = 3.224$

$P(z > 3.224) = 1 - P(Z < 3.224)$

$= 0.0006$

As the value of $p < 0.01$

We reject the null hyp.

La puissance du test est:

$P = 1 - \beta$

$\beta = P(X < k);\ X \sim B(1243, \theta)$

EXII

$\bar{X}_1 = 325\$$

$\sigma_1 = 26\$$

$\bar{X}_2 = 338$

$\sigma_2 = 28$

$H_0 : \bar{X}_1 - \bar{X}_2 = 0$

$H_1 : \bar{X}_1 - \bar{X}_2 > 0$

$$T\text{-value} = \frac{\bar{X}_2 - \bar{X}_1}{\sqrt{\frac{Var\,1}{n_1} + \frac{Var\,2}{n_2}}}$$

$$= 1.82$$

$$\text{Degree of Freedom} = \frac{\left(\frac{\sigma_1^4}{n_1} + \frac{\sigma_2^2}{n_2}\right)^2}{\frac{\left(\frac{\sigma_1^4}{n_1}\right)^2}{n_1 - 1} + \frac{\left(\frac{\sigma_2^4}{n_2}\right)^2}{n_2 - 1}} = 57.63$$

c'est reduit à 57

$\alpha = 5\%$

value = 2.00

Comparons cette valeur contre la valeur calculé :

1.87 indique de H_0 est accépté

Ex 12:

$H_0: p_A = p_B$

$$p_A' - p_B' \sim N\left(0, \sqrt{p_c(1-p_c)\left(\frac{1}{n_A} + \frac{1}{n_B}\right)}\right)$$

$$p_c = \frac{X_A + X_B}{n_A + n_B}$$

test statistic

$$Z = \frac{(p_A' - p_B') - (p_A - p_B)}{\sqrt{p_c(1-p_c)\left(\frac{2}{n}\right)}}$$

Estimation propde.

$A: p_A' = 0.2$

$B: p_B' = 0.4$

$p_A' - p_B' = 0.2$

$$p_c = \frac{p_A' + p_B'}{400} = 0.3$$

$$\frac{1}{2} p \text{ value} = P(P_A' - P_B' < p_A' - p_B')$$

$$p\text{-value} = 2P(P_A' - P_B' < p_A' - p_B')$$

$$P\left(\frac{P_A' - P_A'}{\left(p_c(1-p_c)\left(\frac{1}{2n}\right)\right)} \leq \frac{0.2}{\left(0.3(0.7)\frac{1}{400}\right)}\right) = 1.22 \times 10^{-5}$$

$\alpha = 0.01$

p-value $< \alpha$ accept H_0

Ex 13 :

$p_A = 510/980$

$p_B = 905/1030$

$$p_c = \frac{\frac{510}{980} + \frac{605}{1030}}{980 + 1030}$$

$$\text{p-value} = P\left(\frac{p_A' - p_B'}{\left(p_c(1-p_c)\left(\frac{1}{n_A+n_B}\right)\right)^{1/2}} \leq \frac{p_A - p_B}{\left(p_c(1-p_c)\left(\frac{1}{n_A+n_B}\right)\right)^{1/2}}\right)$$

$$= 4.15 \times 10^{-14} < \alpha$$

Ex14 :

G : 400

D : 900

$$p_G = \frac{190}{400}$$

$$p_D = \frac{300}{900}$$

$$p_C = \frac{\frac{190}{400} + \frac{300}{900}}{1\ 200}$$

p-value $< \alpha$

Ex 16.

$H_0 : \theta = \theta_0$

$H_1 : \theta = \theta_1$

$$LR = \frac{L(X,\theta_0)}{L(X,\theta_1)}$$

$$= \left(\frac{\theta_0}{\theta_1}\right)^{-n} \exp\left[\left[\frac{1}{\theta_1} - \frac{1}{\theta_0}\right] \Sigma X_i\right]$$

If $\theta_1 > \theta_0$, $\frac{1}{\theta_0} > \frac{1}{\theta_1}$

Intuitivement:

Si Les données soutient H_1

Alors La fit de vraisemblance $L(X,\theta_1)$ devrait être petit

Donc Le Rapport de vraisemblance est élevé

Ainsi nous rejetons l'hypothèse nulle

si le rapport de vraisemblance est grand

$LR \geq k$ ou k est cte telque

$P(LR \geq k) = \alpha$)

sous l'hypothèse nulle $(\theta = \theta_0)$

$P(LR \geq k) =$

$LR \geq k$

$\Leftrightarrow$

$$\exp\left[\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \times \Sigma X_i\right] \geq k \left(\frac{\theta_0}{\theta_1}\right)^n$$

Si $\theta_0 < \theta_1$

$$\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right) \Sigma X_i \geq \ln\left(k\left(\frac{\theta_0}{\theta_1}\right)^n\right)$$

$$\Sigma X_i \leq \ln\left(k\left(\frac{\theta_0}{\theta_1}\right)^n\right) \times \left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)^{-1}$$

$$\Sigma X_i \leq a$$

Puissance $(\theta_1 < \theta_0)$:

$P(LR \geq k / H_1)$

$P(\Sigma X_i \leq a / H_1)$

$$P\left(\frac{\Sigma X_i - \theta_1 n}{\sqrt{n}\,\theta_1} \leq \frac{a - \theta_1 n}{\sqrt{n}\,\theta_1}\right) = \beta$$

$$F\left(\frac{a - \theta_1 n}{\sqrt{n}\,\theta_1}\right) = \beta$$