Big Data for Economics # 3

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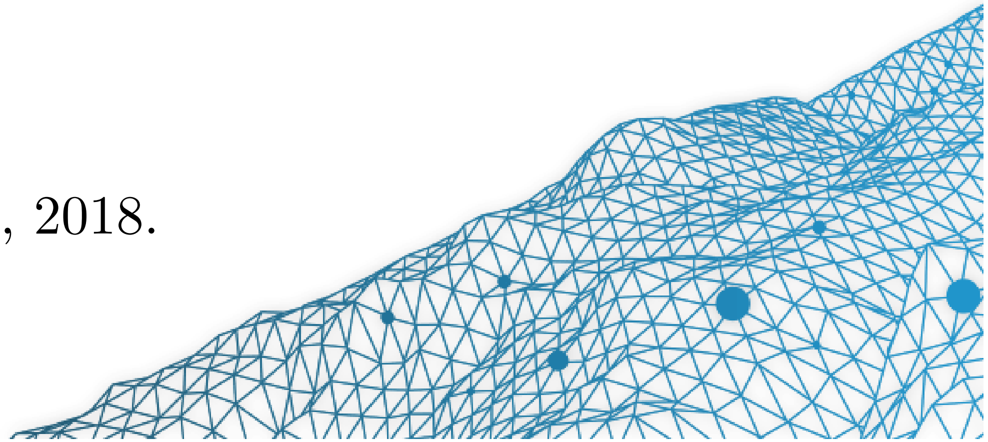
https://github.com/freakonometrics/ub



UB School of Economics Summer School, 2018.

#3 Loss Functions : from OLS to Quantile Regression

References

Motivation

Machado & Mata (2005). Counterfactual decomposition of changes in wage distributions using quantile regression, JAE.

References

Givord & d'Haultfœuillle (2013) La régression quantile en pratique, INSEE

Koenker & Bassett (1978) Regression Quantiles, Econometrica.

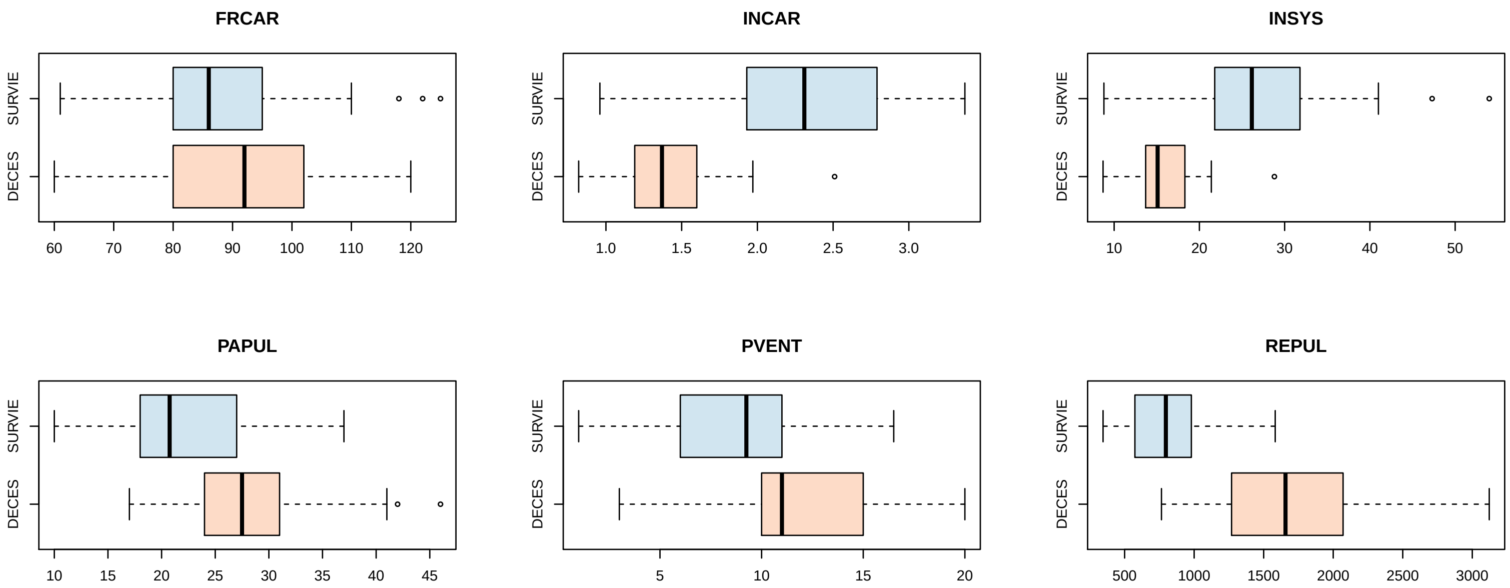
Koenker (2005). Quantile Regression. Cambridge University Press.

Newey & Powell (1987) Asymmetric Least Squares Estimation and Testing, Econometrica.

Quantiles

Let Y denote a random variable with cumulative distribution function F, $F(y) = \mathbb{P}[Y \leq y]$. The quantile is

$$Q(u) = \inf\left\{x \in \mathbb{R}, F(x) > u\right\}.$$



Box plot, from Tukey (1977, Exploratory Data Analysis).

Defining halfspace depth

Given $\boldsymbol{y} \in \mathbb{R}^d$, and a direction $\boldsymbol{u} \in \mathbb{R}^d$, define the closed half space

$$H_{\boldsymbol{y},\boldsymbol{u}} = \{\boldsymbol{x} \in \mathbb{R}^d \text{ such that } \boldsymbol{u}'\boldsymbol{x} \leq \boldsymbol{u}'\boldsymbol{y}\}$$

and define depth at point $\boldsymbol{y}$ by

$$\text{depth}(\boldsymbol{y}) = \inf_{\boldsymbol{u},\boldsymbol{u}\neq\boldsymbol{0}} \{\mathbb{P}(H_{\boldsymbol{y},\boldsymbol{u}})\}$$

i.e. the smallest probability of a closed half space containing $\boldsymbol{y}$.

The empirical version is (see Tukey (1975))

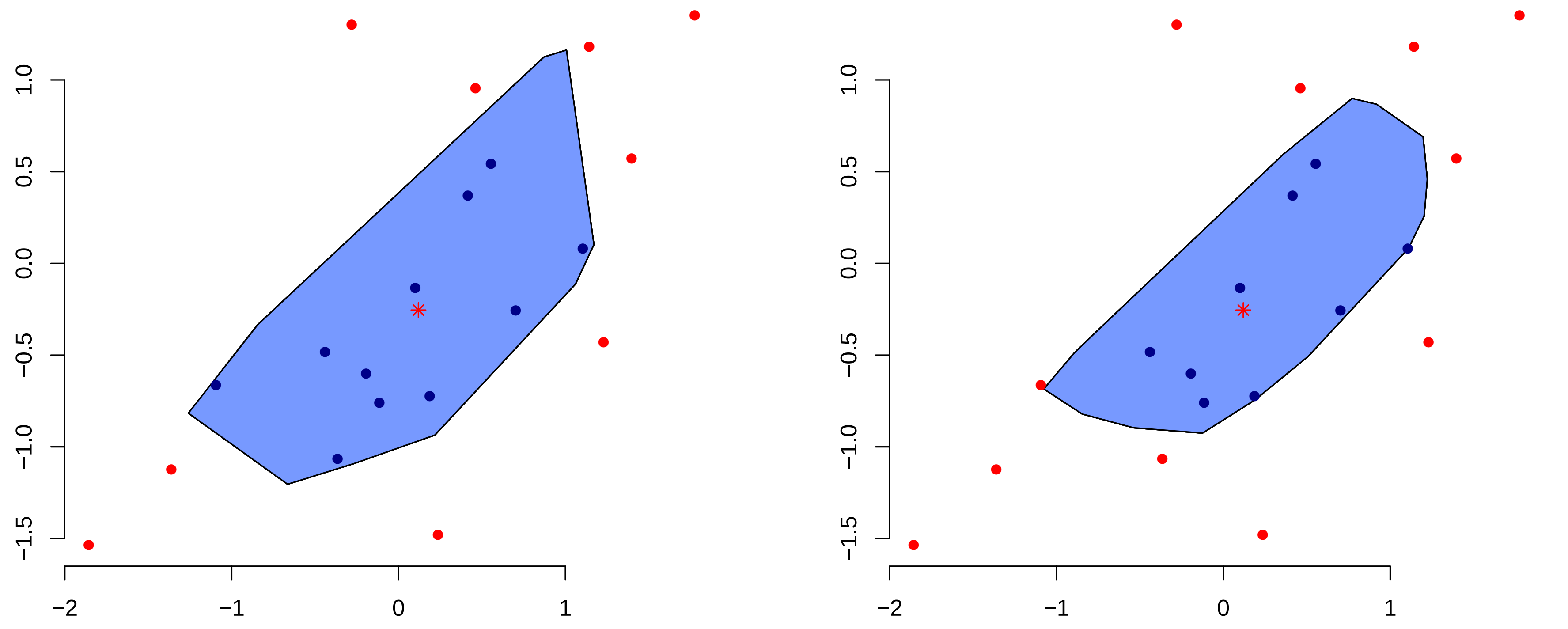
$$\text{depth}(\boldsymbol{y}) = \min_{\boldsymbol{u},\boldsymbol{u}\neq\boldsymbol{0}} \left\{\frac{1}{n}\sum_{i=1}^n \mathbf{1}(\boldsymbol{X}_i \in H_{\boldsymbol{y},\boldsymbol{u}})\right\}$$

For $\alpha > 0.5$, define the depth set as

$$D_\alpha = \{\boldsymbol{y} \in \mathbb{R} \in \mathbb{R}^d \text{ such that } \geq 1 - \alpha\}.$$

The empirical version is can be related to the bagplot, Rousseeuw *et al.*, 1999.

Empirical sets extremely sentive to the algorithm



where the blue set is the empirical estimation for D_α, $\alpha = 0.5$.

The bagplot tool

The depth function introduced here is the multivariate extension of standard univariate depth measures, e.g.

$$\text{depth}(x) = \min\{F(x), 1 - F(x^-)\}$$

which satisfies $\text{depth}(Q_\alpha) = \min\{\alpha, 1 - \alpha\}$. But one can also consider

$$\text{depth}(x) = 2 \cdot F(x) \cdot [1 - F(x^-)] \text{ or } \text{depth}(x) = 1 - \left|\frac{1}{2} - F(x)\right|.$$

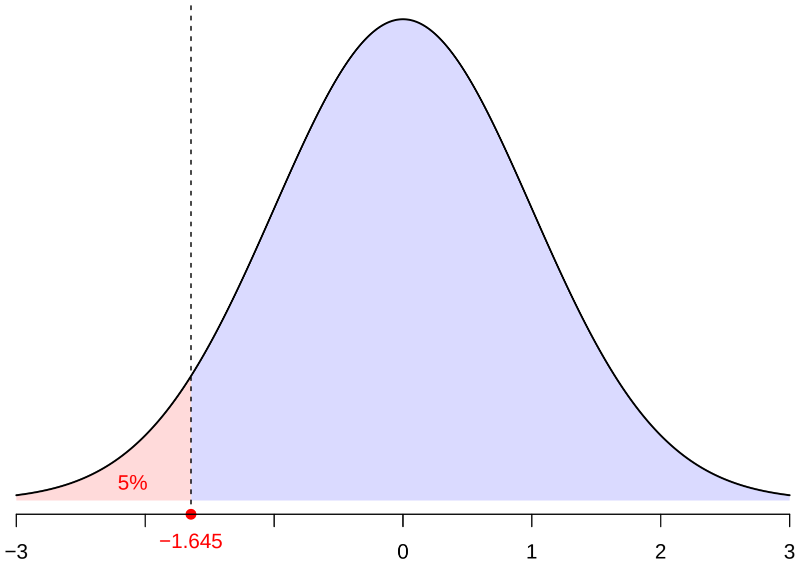
Possible extensions to functional bagplot. Consider a set of functions $f_i(x)$, $i = 1, \cdots, n$, such that

$$f_i(x) = \mu(x) + \sum_{k=1}^{n-1} z_{i,k}\varphi_k(x)$$

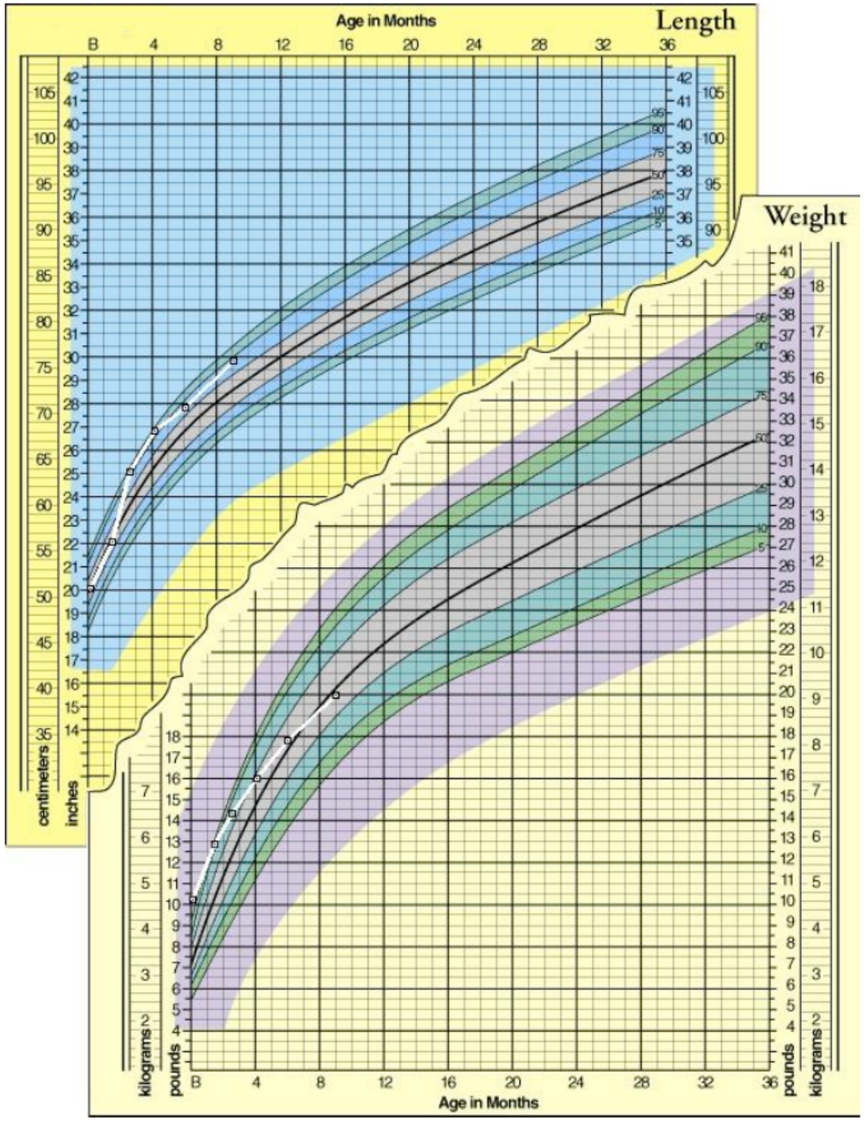
(i.e. principal component decomposition) where $\varphi_k(\cdot)$ represents the eigenfunctions. Rousseeuw et al., 1999 considered bivariate depth on the first two scores, $\boldsymbol{x}_i = (z_{i,1}, z_{i,2})$. See Ferraty & Vieu (2006).

Quantiles and Quantile Regressions

Quantiles are important quantities in many areas (inequalities, risk, health, sports, etc).



Quantiles of the $\mathcal{N}(0,1)$ distribution.



A First Model for Conditional Quantiles

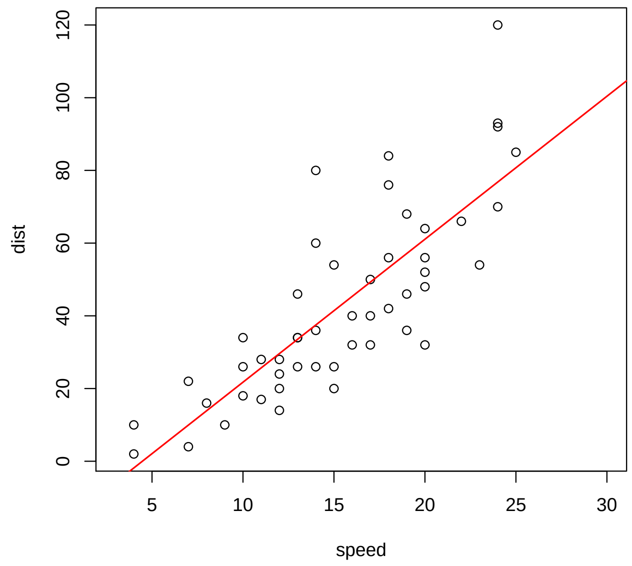
Consider a location model, $y = \beta_0 + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta} + \varepsilon$ i.e.

$$\mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}] = \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}$$

then one can consider

$$Q(\tau|\boldsymbol{X} = \boldsymbol{x}) = \beta_0 + Q_\varepsilon(\tau) + \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}$$

where $Q_\varepsilon(\cdot)$ is the quantile function of the residuals.



OLS Regression, ℓ_2 norm and Expected Value

Let $\boldsymbol{y} \in \mathbb{R}^d$, $\overline{y} = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} \big[\underbrace{y_i - m}_{\varepsilon_i}\big]^2 \right\}$. It is the empirical version of

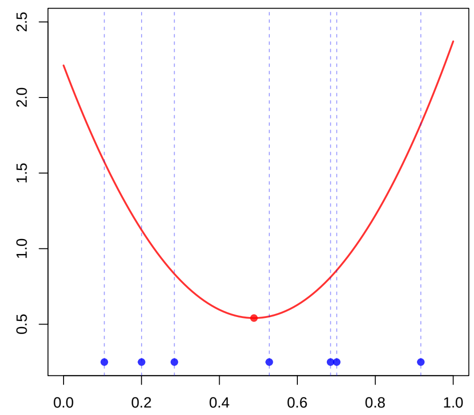
$$\mathbb{E}[Y] = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \int \big[\underbrace{y - m}_{\varepsilon}\big]^2 dF(y) \right\} = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \mathbb{E}\big[\|\underbrace{Y - m}_{\varepsilon}\|_{\ell_2}\big] \right\}$$

where Y is a random variable.

Thus, $\underset{m(\cdot):\mathbb{R}^k\to\mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} \big[\underbrace{y_i - m(\boldsymbol{x}_i)}_{\varepsilon_i}\big]^2 \right\}$ is the empirical version of $\mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}]$.

See Legendre (1805) Nouvelles méthodes pour la détermination des orbites des comètes and Gauβ (1809) Theoria motus corporum coelestium in sectionibus conicis solem ambientium.

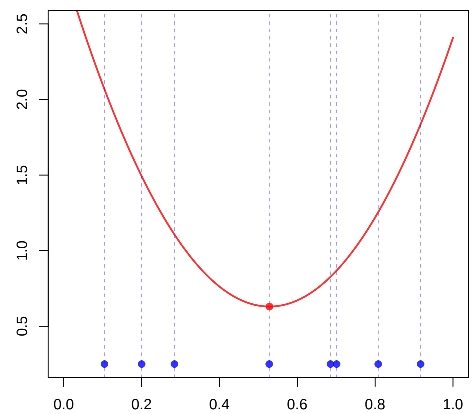
OLS Regression, ℓ_2 norm and Expected Value



Sketch of proof: (1) Let $h(x) = \sum_{i=1}^{d}(x-y_i)^2$, then

$$h'(x) = \sum_{i=1}^{d} 2(x-y_i)$$

and the FOC yields $x = \dfrac{1}{n}\sum_{i=1}^{d} y_i = \overline{y}$.



(2) If Y is continuous, let $h(x) = \int_{\mathbb{R}}(x-y)f(y)dy$ and

$$h'(x) = \frac{\partial}{\partial x}\int_{\mathbb{R}}(x-y)^2 f(y)dy = \int_{\mathbb{R}}\frac{\partial}{\partial x}(x-y)^2 f(y)dy$$

i.e. $x = \int_{\mathbb{R}} x f(y)dy = \int_{\mathbb{R}} y f(y)dy = \mathbb{E}[Y]$

Median Regression, ℓ_1 norm and Median

Let $\boldsymbol{y} \in \mathbb{R}^d$, $\text{median}[\boldsymbol{y}] \in \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} \big|\underbrace{y_i - m}_{\varepsilon_i}\big| \right\}$. It is the empirical version of

$$\text{median}[Y] \in \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \int \big|\underbrace{y - m}_{\varepsilon}\big| dF(y) \right\} = \underset{m\in\mathbb{R}}{\text{argmin}} \left\{ \mathbb{E}\big[\|\underbrace{Y - m}_{\varepsilon}\|_{\ell_1}\big] \right\}$$

where Y is a random variable, $\mathbb{P}[Y \leq \text{median}[Y]] \geq \frac{1}{2}$ and $\mathbb{P}[Y \geq \text{median}[Y]] \geq \frac{1}{2}$.

$\underset{m(\cdot):\mathbb{R}^k\to\mathbb{R}}{\text{argmin}} \left\{ \sum_{i=1}^{n} \frac{1}{n} \big|\underbrace{y_i - m(\boldsymbol{x}_i)}_{\varepsilon_i}\big| \right\}$ is the empirical version of $\text{median}[Y|\boldsymbol{X} = \boldsymbol{x}]$.

See Boscovich (1757) De Litteraria expeditione per pontificiam ditionem ad dimetiendos duos meridiani and Laplace (1793) Sur quelques points du système du monde.

Median Regression, ℓ_1 norm and Median

Sketch of proof: (1) Let $h(x) = \displaystyle\sum_{i=1}^{d} |x - y_i|$

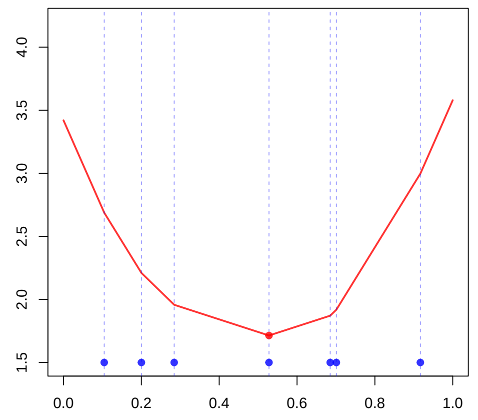
(2) If F is absolutely continuous, $dF(x) = f(x)dx$, and the median m is solution of $\displaystyle\int_{-\infty}^{m} f(x)dx = \frac{1}{2}$.

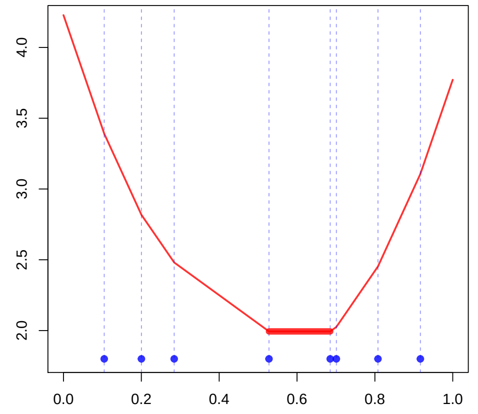
Set $h(y) = \displaystyle\int_{-\infty}^{+\infty} |x - y| f(x)dx$

$$= \int_{-\infty}^{y} (-x + y) f(x)dx + \int_{y}^{+\infty} (x - y) f(x)dx$$

Then $h'(y) = \displaystyle\int_{-\infty}^{y} f(x)dx - \int_{y}^{+\infty} f(x)dx$, and FOC yields

$$\int_{-\infty}^{y} f(x)dx = \int_{y}^{+\infty} f(x)dx = 1 - \int_{-\infty}^{y} f(x)dx = \frac{1}{2}$$





Bayesian Statistics and Loss Functions

In statistics, consider some functional ℓ seen as a distance between θ and $\widehat{\theta}$, e.g.

squared loss : $\ell_2(\widehat{\theta},\theta)=(\widehat{\theta}-\theta)^2$

absolute loss : $\ell_1(\widehat{\theta},\theta)=|\widehat{\theta}-\theta|$

zero/one loss $\ell_{\circ|}(\widehat{\theta},\theta)=\mathbf{1}(|\widehat{\theta}-\theta|>\varepsilon)$ for some $\varepsilon>0$

Define risk as the expected loss,

$$R(\widehat{\theta},\theta)=\int \ell(\widehat{\theta}(\boldsymbol{y}),\theta)\,\underbrace{f(\boldsymbol{y}|\theta)}_{\mathcal{L}(\theta,\boldsymbol{y})}\,d\boldsymbol{y}$$

(where the average is over the sample space).

Bayesian Statistics and Loss Functions

Bayes'Rule: Minimize average risk

$$\mathcal{R}(\widehat{\theta}) = \int R(\widehat{\theta}, \theta)\pi(\theta)d\theta$$

and set

$$\widehat{\theta} = \text{argmin}\left\{\mathcal{R}(\widehat{\theta})\right\}$$

Hence

$$\widehat{\theta} = \text{argmin}\left\{\int_\Theta \int_{\mathbb{R}^n} \ell(\widehat{\theta}(\boldsymbol{y}), \theta) f(\boldsymbol{y}|\theta)d\boldsymbol{y}\pi(\theta)d\theta\right\}$$

$$\widehat{\theta} = \text{argmin}\left\{\int_{\mathbb{R}^n} \int_\Theta \ell(\widehat{\theta}(\boldsymbol{y}), \theta)\pi(\theta|\boldsymbol{y})d\theta f(\boldsymbol{y})d\boldsymbol{y}\right\}$$

Bayesian Statistics and Loss Functions

If $\ell = \ell_2$, $\widehat{\theta}$ is the posterior mean, $\widehat{\theta} = \mathbb{E}[\theta|\boldsymbol{y}]$: to solve

$$\text{argmin}\left\{\int_\Theta (\widehat{\theta}(\boldsymbol{y}), \theta)^2 \pi(\theta|\boldsymbol{y})d\theta\right\}$$

consider the first order condition

$$2\int (\widehat{\theta}(\boldsymbol{y}), \theta)\pi(\theta|\boldsymbol{y})d\theta = 0$$

i.e. $\widehat{\theta}\displaystyle\int \pi(\theta|\boldsymbol{y})d\theta = \int \theta\pi(\theta|\boldsymbol{y})d\theta$

If $\ell = \ell_1$, $\widehat{\theta}$ is the posterior median, $\widehat{\theta} = \text{median}[\theta|\boldsymbol{y}]$

If $\ell = \ell_{\circ|}$, $\widehat{\theta}$ is the posterior mode

Bayesian Statistics and Loss Functions

Application : head/tails Bernoulli with uniform prior

$$f(y) = \theta^y(1-\theta)^{1-y}, \text{ where } y \in \{0,1\}$$

and $\pi(\theta) = \mathbf{1}(\theta \in [0,1])$. Then likelihood is

$$\mathcal{L}(\theta, \boldsymbol{y}) = f(\boldsymbol{y}|\theta) = \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

From Baye's Theorem,

$$\pi(\theta|\boldsymbol{y}) \propto \pi(\theta) \cdot f(\boldsymbol{y}|\theta) \propto \theta^{\sum y_i}(1-\theta)^{n-\sum y_i}$$

which is a Beta distribution. Recall that $\mathcal{B}(\alpha, \beta)$

$$g(u|\alpha,\beta) = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)} \text{ on } [0,1]$$

Bayesian Statistics and Loss Functions

and $\mathbb{E}[U]=\dfrac{\alpha}{\alpha+\beta}$, $\text{Var}[U]=\dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, while

$$\text{median}[U]\sim\frac{3\alpha-1}{3\alpha+3\beta-2}\text{ and mode}[U]=\frac{\alpha-1}{\alpha+\beta-2}\text{ if }\alpha,\beta>1.$$

Here, posterior distribution is $\mathcal{B}(n\overline{y}+1,n(1-\overline{y})+1)$.

Here with ℓ_2, $\widehat{\theta}_2=\mathbb{E}[\theta|\boldsymbol{y}]=\dfrac{n\overline{y}+1}{n+2}$ while with $\ell_{\circ|}$, $\widehat{\theta}=\text{mode}[\theta|\boldsymbol{y}]=\overline{y}$ (which is also the maximum likelihood estimator).

With a Beta $\mathcal{B}(a,b)$ prior, posterior distribution is $\mathcal{B}(n\overline{y}+a+1,n(1-\overline{y})+b+1)$.

Here with ℓ_2, $\widehat{\theta}_2=\mathbb{E}[\theta|\boldsymbol{y}]=\dfrac{n\overline{y}+a}{n+a+b}$ while with $\ell_{\circ|}$,
$\widehat{\theta}=\text{mode}[\theta|\boldsymbol{y}]=\dfrac{n\overline{y}+a-1}{n+a+b-2}$ (which is no longer the maximum likelihood estimator).

Bayesian Statistics and Loss Functions

Example: Gaussian case (with known variance), $Y_i \sim \mathcal{N}(\mu, \sigma_0^2)$. Consider Gaussian prior, $\mu \sim \mathcal{N}(m, s^2)$. We are uncertain here about the value of μ. Then, the posterior distribution for μ is a Gaussian distribution $\mu|\boldsymbol{y} \sim \mathcal{N}(m_{\boldsymbol{y}}, s_{\boldsymbol{y}}^2)$ where

$$m_{\boldsymbol{y}} = s_{\boldsymbol{y}}^2 \left(\frac{m}{s^2} + \frac{n\overline{y}}{\sigma_0^2} \right) \text{ and } s_{\boldsymbol{y}}^2 = \left(\frac{1}{s^2} + \frac{n}{\sigma_0^2} \right)^{-1}$$

Here $m_{\boldsymbol{y}}$ is Bayes estimator of μ under loss functions ℓ_1, ℓ_2 and $\ell_{\circ|}$.

When σ_0^2 is no longer known, but is just a nuisance parameter, the natural approach is to consider a joint posterior density for $\boldsymbol{\theta} = (\mu, \sigma^2)$ and then marginalize by integrating out the nuisance parameters

OLS vs. Median Regression (Least Absolute Deviation)

Consider some linear model, $y_i = \beta_0 + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta} + \varepsilon_i$,and define

$$(\widehat{\beta}_0^{\text{ols}}, \widehat{\boldsymbol{\beta}}^{\text{ols}}) = \text{argmin}\left\{\sum_{i=1}^n \left(y_i - \beta_0 - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}\right)^2\right\}$$

$$(\widehat{\beta}_0^{\text{lad}}, \widehat{\boldsymbol{\beta}}^{\text{lad}}) = \text{argmin}\left\{\sum_{i=1}^n \left|y_i - \beta_0 - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}\right|\right\}$$

Assume that $\varepsilon|\boldsymbol{X}$ has a symmetric distribution, $\mathbb{E}[\varepsilon|\boldsymbol{X}] = \text{median}[\varepsilon|\boldsymbol{X}] = 0$, then $(\widehat{\beta}_0^{\text{ols}}, \widehat{\boldsymbol{\beta}}^{\text{ols}})$ and $(\widehat{\beta}_0^{\text{lad}}, \widehat{\boldsymbol{\beta}}^{\text{lad}})$ are consistent estimators of $(\beta_0, \boldsymbol{\beta})$.

Assume that $\varepsilon|\boldsymbol{X}$ does not have a symmetric distribution, but $\mathbb{E}[\varepsilon|\boldsymbol{X}] = 0$, then $\widehat{\boldsymbol{\beta}}^{\text{ols}}$ and $\widehat{\boldsymbol{\beta}}^{\text{lad}}$ are consistent estimators of the slopes $\boldsymbol{\beta}$.

If $\text{median}[\varepsilon|\boldsymbol{X}] = \gamma$, then $\widehat{\beta}_0^{\text{lad}}$ converges to $\beta_0 + \gamma$.

OLS vs. Median Regression

Median regression is stable by monotonic transformation. If

$$\log[y_i] = \beta_0 + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta} + \varepsilon_i \text{ with } \text{median}[\varepsilon|\boldsymbol{X}] = 0,$$

then

$$\text{median}[Y|\boldsymbol{X} = \boldsymbol{x}] = \exp\big(\text{median}[\log(Y)|\boldsymbol{X} = \boldsymbol{x}]\big) = \exp\big(\beta_0 + \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}\big)$$

while

$$\mathbb{E}[Y|\boldsymbol{X} = \boldsymbol{x}] \neq \exp\big(\mathbb{E}[\log(Y)|\boldsymbol{X} = \boldsymbol{x}]\big) \;\; (= \exp\big(\mathbb{E}[\log(Y)|\boldsymbol{X} = \boldsymbol{x}]\big) \cdot [\exp(\varepsilon)|\boldsymbol{X} = \boldsymbol{x}]$$

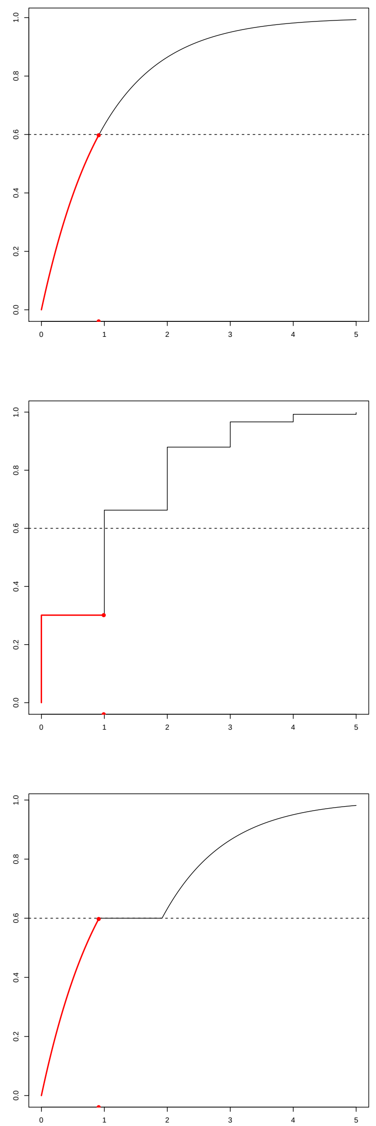
```
> ols <- lm(y~x, data=df)
> library(quantreg)
> lad <- rq(y~x, data=df, tau=.5)
```

Notations

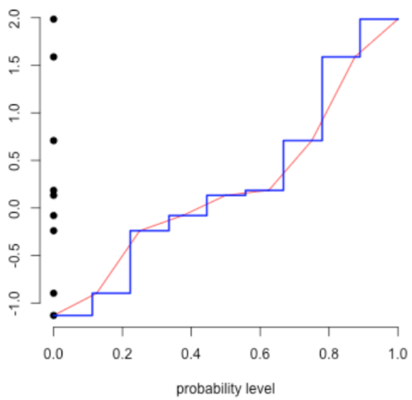
Cumulative distribution function $F_Y(y) = \mathbb{P}[Y \leq y]$.
Quantile function $Q_X(u) = \inf\{y \in \mathbb{R} : F_Y(y) \geq u\}$,
also noted $Q_X(u) = F_X^{-1}u$.

One can consider $Q_X(u) = \sup\{y \in \mathbb{R} : F_Y(y) < u\}$

For any increasing transformation t, $Q_{t(Y)}(\tau) = t\big(Q_Y(\tau)\big)$
$F(y|\boldsymbol{x}) = \mathbb{P}[Y \leq y|\boldsymbol{X} = \boldsymbol{x}]$
$Q_{Y|\boldsymbol{x}}(u) = F^{-1}(u|\boldsymbol{x})$



Empirical Quantile




```
> set.seed(2); X=rnorm(9)
> Q=quantile(X,u,type=1)
```

$\alpha X_{[np]:n} + (1-\alpha) X_{[np]+1:n}$

where

$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$

Inverse of empirical distribution function

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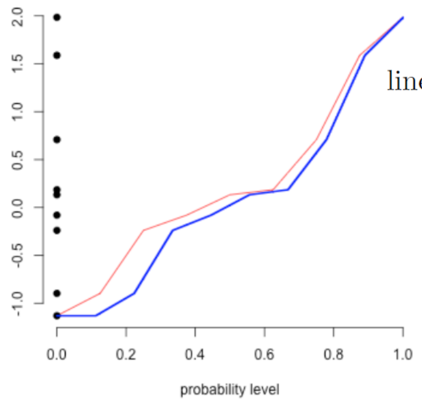


```
> set.seed(2); X=rnorm(9)
> Q=quantile(X,u,type=4)
```

linear interpolation between points

$\{(p(k), X_{k:n}); k = 1, \cdots, n\}$

$p(k) = \frac{k}{n}$

Linear interpolation of the empirical cdf

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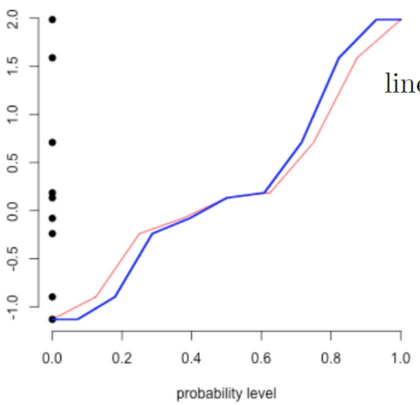


```
> set.seed(2); X=rnorm(9)
> Q=quantile(X,u,type=6)
```

linear interpolation between points

$\{(p(k), X_{k:n}); k = 1, \cdots, n\}$

$p(k) = \frac{k}{n+1}$

Minitab SPSS

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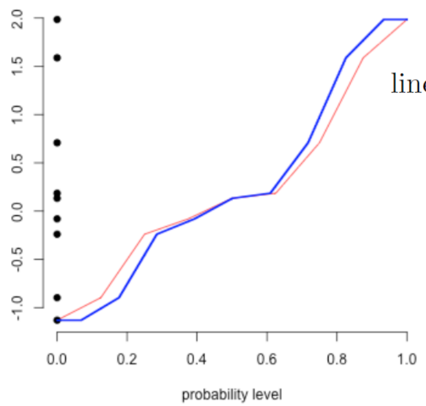


```
> set.seed(2); X=rnorm(9)
> Q=quantile(X,u,type=9)
```

linear interpolation between points

$\{(p(k), X_{k:n}); k = 1, \cdots, n\}$

$p(k) = \frac{k-3/8}{n+1/4}$

Approximately median-unbiased (when Gaussian)

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Quantiles and optimization : numerical aspects

Consider the case of the median. Consider a sample $\{y_1, \cdots, y_n\}$.

To compute the median, solve $\min_{\mu}\left\{\sum_{i=1}^n |y_i-\mu|\right\}$ which can be solved using linear programming techniques.

More precisely, this problem is equivalent to $\min_{\mu,\boldsymbol{a},\boldsymbol{b}}\left\{\sum_{i=1}^n a_i+b_i\right\}$ with $a_i, b_i \geq 0$ and $y_i-\mu=a_i-b_i$, $\forall i=1,\cdots,n$.

Consider a sample obtained from a lognormal distribution

```
n = 101
set.seed(1)
y = rlnorm(n)
median(y)
[1] 1.077415
```

Quantiles and optimization : numerical aspects

Here, one can use a standard optimization routine

```
md=Vectorize(function(m) sum(abs(y-m)))
optim(mean(y),md)
$par
[1] 1.077416
```

or a linear programing technique : use the matrix form, with $3n$ constraints, and $2n+1$ parameters,

```
library(lpSolve)
A1 = cbind(diag(2*n),0)
A2 = cbind(diag(n), -diag(n), 1)
r = lp("min", c(rep(1,2*n),0),
rbind(A1, A2),c(rep("&gt;=", 2*n), rep("=", n)), c(rep(0,2*n), y))
tail(r$solution,1)
[1] 1.077415
```

Quantiles and optimization : numerical aspects

More generally, consider here some quantile,

```
tau = .3
quantile(x,tau)
      30%
0.6741586
```

The linear program is now $\min_{\mu,\boldsymbol{a},\boldsymbol{b}} \left\{ \sum_{i=1}^{n} \tau a_i + (1-\tau) b_i \right\}$ with $a_i, b_i \geq 0$ and $y_i - \mu = a_i - b_i$, $\forall i = 1, \cdots, n$.

```
A1 = cbind(diag(2*n),0)
A2 = cbind(diag(n), -diag(n), 1)
r = lp("min", c(rep(tau,n),rep(1-tau,n),0),
rbind(A1, A2),c(rep("&gt;=", 2*n), rep("=", n)), c(rep(0,2*n), y))
tail(r$solution,1)
[1] 0.6741586
```

Quantile regression ?

In OLS regression, we try to evaluate $\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}]=\displaystyle\int_{\mathbb{R}} ydF_{Y|\boldsymbol{X}=\boldsymbol{x}}(y)$

In quantile regression, we try to evaluate

$$Q_u(Y|\boldsymbol{X}=\boldsymbol{x})=\inf\left\{y: F_{Y|\boldsymbol{X}=\boldsymbol{x}}(y)\geq u\right\}$$

as introduced in Newey & Powell (1987) Asymmetric Least Squares Estimation and Testing.

Li & Racine (2007) Nonparametric Econometrics: Theory and Practice suggested

$$\widehat{Q}_u(Y|\boldsymbol{X}=\boldsymbol{x})=\inf\left\{y: \widehat{F}_{Y|\boldsymbol{X}=\boldsymbol{x}}(y)\geq u\right\}$$

where $\widehat{F}_{Y|\boldsymbol{X}=\boldsymbol{x}}(y)$ can be some kernel-based estimator.

Quantiles and Expectiles

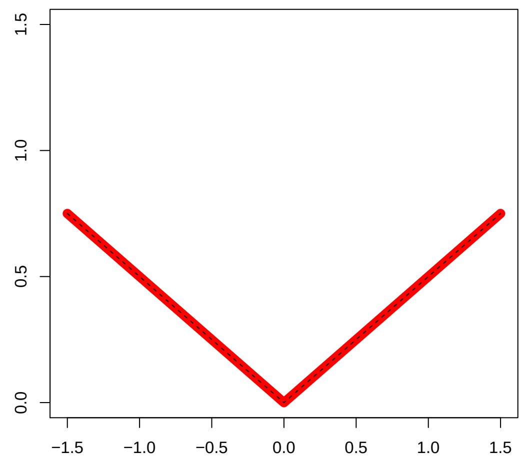
Consider the following risk functions

$$\mathcal{R}^{\mathsf{q}}_{\tau}(u) = u \cdot \big(\tau - \mathbf{1}(u < 0)\big), \;\; \tau \in [0,1]$$

with $\mathcal{R}^{\mathsf{q}}_{1/2}(u) \propto |u| = \|u\|_{\ell_1}$, and

$$\mathcal{R}^{\mathsf{e}}_{\tau}(u) = u^2 \cdot \big(\tau - \mathbf{1}(u < 0)\big), \;\; \tau \in [0,1]$$

with $\mathcal{R}^{\mathsf{e}}_{1/2}(u) \propto u^2 = \|u\|^2_{\ell_2}$.

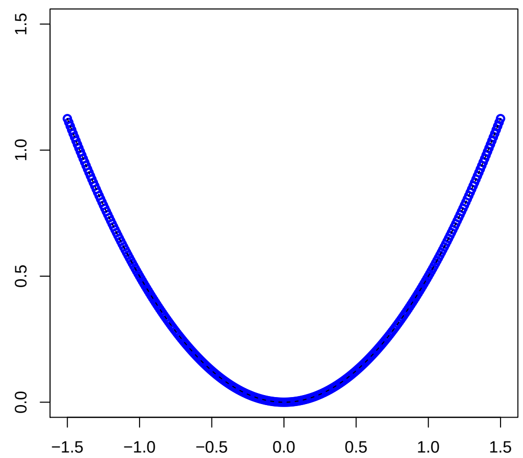


$$Q_Y(\tau) = \underset{m}{\text{argmin}}\big\{\mathbb{E}\big(\mathcal{R}^{\mathsf{q}}_{\tau}(Y - m)\big)\big\}$$

which is the median when $\tau = 1/2$,

$$E_Y(\tau) = \underset{m}{\text{argmin}}\big\{\mathbb{E}\big(\mathcal{R}^{\mathsf{e}}_{\tau}(X - m)\big)\big\}$$

which is the expected value when $\tau = 1/2$.



Quantiles and Expectiles

One can also write

$$\text{quantile: argmin}\left\{\sum_{i=1}^n \omega_\tau^{\mathsf{q}}(\varepsilon_i)\Big|\underbrace{y_i - q_i}_{\varepsilon_i}\Big|\right\} \text{ where } \omega_\tau^{\mathsf{q}}(\epsilon) = \begin{cases} 1-\tau \text{ if } \epsilon \leq 0 \\ \tau \text{ if } \epsilon > 0 \end{cases}$$

$$\text{expectile: argmin}\left\{\sum_{i=1}^n \omega_\tau^{\mathsf{e}}(\varepsilon_i)\Big(\underbrace{y_i - q_i}_{\varepsilon_i}\Big)^2\right\} \text{ where } \omega_\tau^{\mathsf{e}}(\epsilon) = \begin{cases} 1-\tau \text{ if } \epsilon \leq 0 \\ \tau \text{ if } \epsilon > 0 \end{cases}$$

Expectiles are unique, not quantiles...

Quantiles satisfy $\mathbb{E}[\text{sign}(Y - Q_Y(\tau))] = 0$

Expectiles satisfy $\tau\mathbb{E}\big[(Y - e_Y(\tau))_+\big] = (1-\tau)\mathbb{E}\big[(Y - e_Y(\tau))_-\big]$

(those are actually the first order conditions of the optimization problem).

Quantiles and M-Estimators

There are connections with M-estimators, as introduced in Serfling (1980) Approximation Theorems of Mathematical Statistics, chapter 7.

For any function $h(\cdot,\cdot)$, the M-functional is the solution β of

$$\int h(y,\beta)dF_Y(y) = 0$$

, and the M-estimator is the solution of

$$\int h(y,\beta)d\widehat{F}_n(y) = \frac{1}{n}\sum_{i=1}^n h(y_i,\beta) = 0$$

Hence, if $h(y,\beta) = y-\beta$, $\beta = \mathbb{E}[Y]$ and $\widehat{\beta} = \overline{y}$.

And if $h(y,\beta) = \mathbf{1}(y<\beta)-\tau$, with $\tau\in(0,1)$, then $\beta = F_Y^{-1}(\tau)$.

Quantiles, Maximal Correlation and Hardy-Littlewood-Polya

If $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$, then $\displaystyle\sum_{i=1}^n x_i y_i \geq \sum_{i=1}^n x_i y_{\sigma(i)}$, $\forall \sigma \in \mathcal{S}_n$, and $\boldsymbol{x}$ and $\boldsymbol{y}$ are said to be comonotonic.

The continuous version is that X and Y are comonotonic if

$$\mathbb{E}[XY] \geq \mathbb{E}[X\tilde{Y}] \text{ where } \tilde{Y} \stackrel{\mathcal{L}}{=} Y,$$

One can prove that

$$Y = Q_Y(F_X(X)) = \underset{\tilde{Y} \sim F_Y}{\text{argmax}}\{\mathbb{E}[X\tilde{Y}]\}$$

Expectiles as Quantiles

For every $Y \in L^1$, $\tau \mapsto e_Y(\tau)$ is continuous, and striclty increasing

if Y is absolutely continuous, $\dfrac{\partial e_Y(\tau)}{\partial \tau} = \dfrac{\mathbb{E}[|X - e_Y(\tau)|]}{(1-\tau)F_Y(e_Y(\tau)) + \tau(1 - F_Y(e_Y(\tau)))}$

if $X \leq Y$, then $e_X(\tau) \leq e_Y(\tau)$ $\forall \tau \in (0,1)$

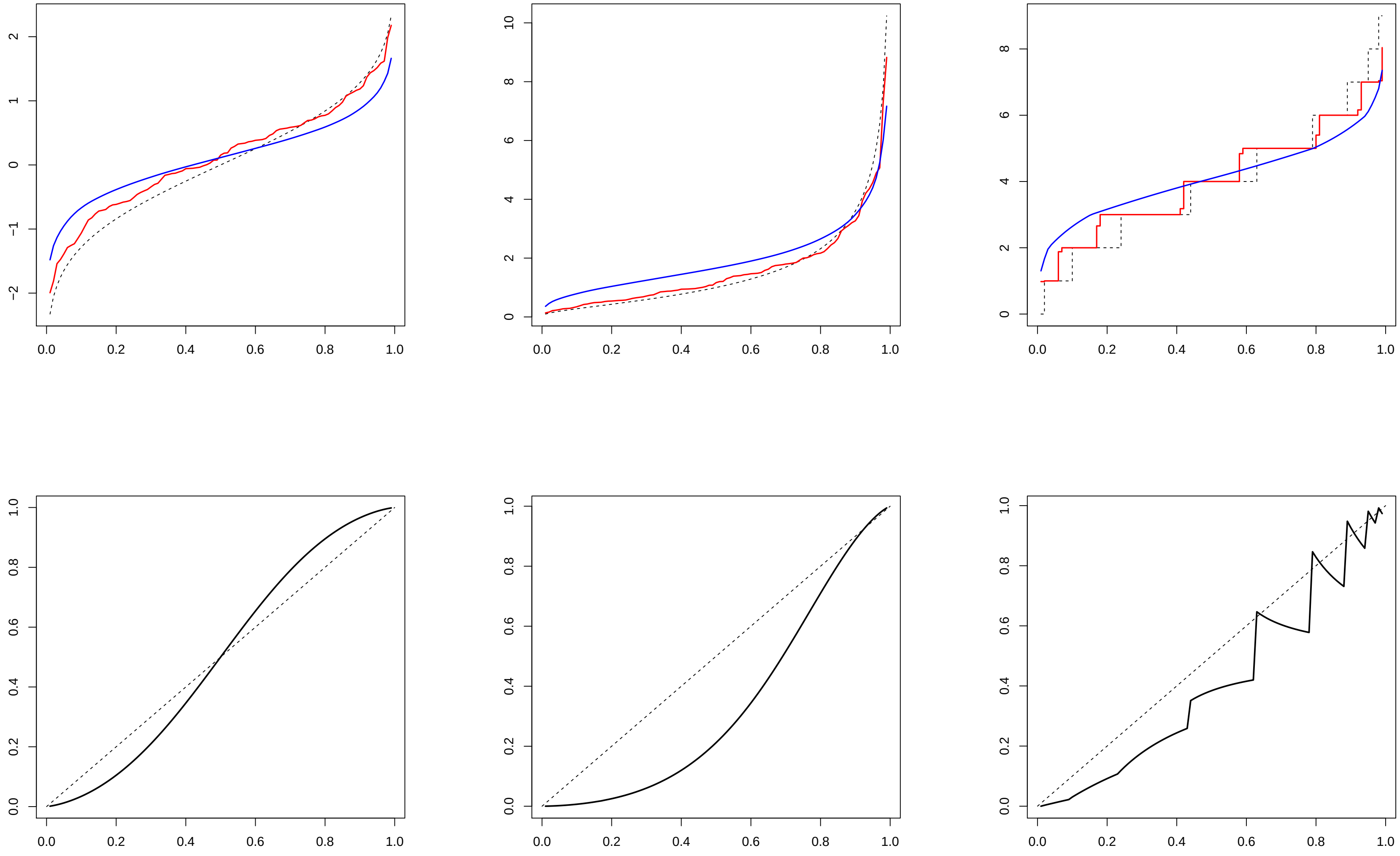
"Expectiles have properties that are similar to quantiles" Newey & Powell (1987) Asymmetric Least Squares Estimation and Testing. The reason is that expectiles of a distribution F are quantiles a distribution G which is related to F, see Jones (1994) Expectiles and M-quantiles are quantiles: let

$$G(t) = \frac{P(t) - tF(t)}{2[P(t) - tF(t)] + t - \mu} \text{ where } P(s) = \int_{-\infty}^{s} ydF(y).$$

The expectiles of F are the quantiles of G.

```
> x <- rnorm(99)
> library(expectreg)
> e <- expectile(x, probs = seq(0, 1, 0.1))
```

Expectiles as Quantiles



Elicitable Measures

"elicitable" means "being a minimizer of a suitable expected score"

T is an elicatable function if there exits a scoring function $S:\mathbb{R}\times\mathbb{R}\rightarrow[0,\infty)$ such that

$$T(Y)=\underset{x\in\mathbb{R}}{\text{argmin}}\left\{\int_{\mathbb{R}}S(x,y)dF(y)\right\}=\underset{x\in\mathbb{R}}{\text{argmin}}\left\{\mathbb{E}\big[S(x,Y)\big]\text{ where }Y\sim F.\right\}$$

see Gneiting (2011) Making and evaluating point forecasts.

Example: mean, $T(Y)=\mathbb{E}[Y]$ is elicited by $S(x,y)=\|x-y\|_{\ell_2}^2$

Example: median, $T(Y)=\text{median}[Y]$ is elicited by $S(x,y)=\|x-y\|_{\ell_1}$

Example: quantile, $T(Y)=Q_Y(\tau)$ is elicited by
$S(x,y)=\tau(y-x)_++(1-\tau)(y-x)_-$

Example: expectile, $T(Y)=E_Y(\tau)$ is elicited by
$S(x,y)=\tau(y-x)_+^2+(1-\tau)(y-x)_-^2$

Elicitable Measures

Remark: all functionals are not necessarily elicitable, see Osband (1985)
Providing incentives for better cost forecasting

The variance is not elicitable

The elicitability property implies a property which is known as convexity of the level sets with respect to mixtures (also called Betweenness property) : if two lotteries F, and G are equivalent, then any mixture of the two lotteries is also equivalent with F and G.

Empirical Quantiles

Consider some i.id. sample $\{y_1, \cdots, y_n\}$ with distribution F. Set

$$Q_\tau = \text{argmin}\left\{\mathbb{E}\big[\mathcal{R}^{\mathfrak{q}}_\tau(Y-q)\big]\right\} \text{ where } Y\sim F \text{ and } \widehat{Q}_\tau \in \text{argmin}\left\{\sum_{i=1}^n \mathcal{R}^{\mathfrak{q}}_\tau(y_i-q)\right\}$$

Then as $n\rightarrow\infty$

$$\sqrt{n}\big(\widehat{Q}_\tau - Q_\tau\big) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, \frac{\tau(1-\tau)}{f^2(Q_\tau)}\right)$$

Sketch of the proof: $y_i = Q_\tau + \varepsilon_i$, set $h_n(q) = \dfrac{1}{n}\displaystyle\sum_{i=1}^n \big(\mathbf{1}(y_i<q)-\tau\big)$, which is a non-decreasing function, with

$$\mathbb{E}\left[Q_\tau + \frac{u}{\sqrt{n}}\right] = F_Y\left(Q_\tau + \frac{u}{\sqrt{n}}\right) \sim f_Y(Q_\tau)\frac{u}{\sqrt{n}}$$

$$\text{Var}\left[Q_\tau + \frac{u}{\sqrt{n}}\right] \sim \frac{F_Y(Q_\tau)[1-F_Y(Q_\tau)]}{n} = \frac{\tau(1-\tau)}{n}.$$

Empirical Expectiles

Consider some i.id. sample $\{y_1, \cdots, y_n\}$ with distribution F. Set

$$\mu_\tau = \text{argmin}\left\{\mathbb{E}\left[\mathcal{R}^{\mathsf{e}}_\tau(Y-m)\right]\right\} \text{ where } Y \sim F \text{ and } \widehat{\mu}_\tau = \text{argmin}\left\{\sum_{i=1}^n \mathcal{R}^{\mathsf{e}}_\tau(y_i - m)\right\}$$

Then as $n \to \infty$

$$\sqrt{n}\big(\widehat{\mu}_\tau - \mu_\tau\big) \xrightarrow{\mathcal{L}} \mathcal{N}\left(0, s^2\right)$$

for some s^2, if $\text{Var}[Y] < \infty$. Define the identification function

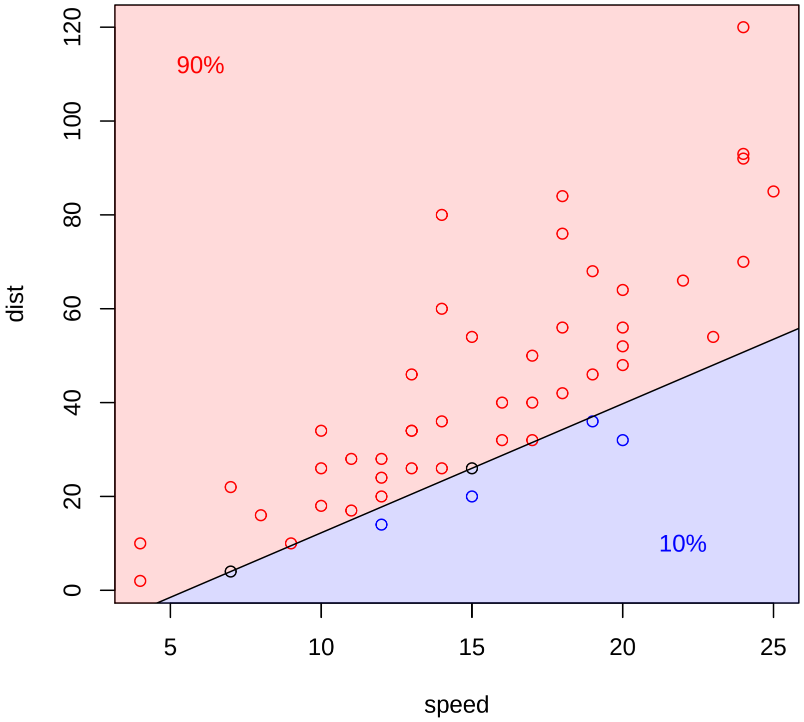
$$\mathcal{I}_\tau(x,y) = \tau(y-x)_+ + (1-\tau)(y-x)_- \qquad \text{(elicitable score for quantiles)}$$

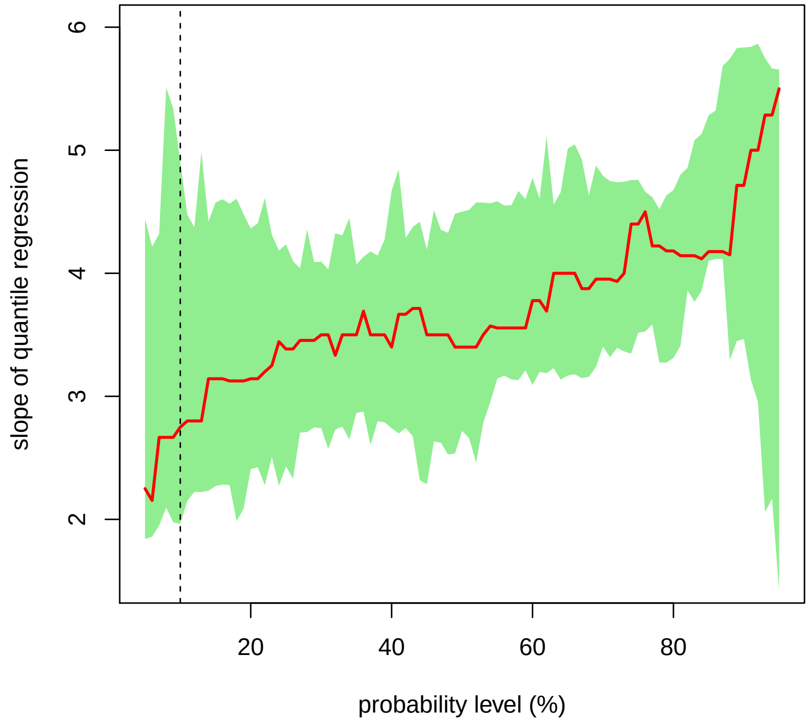
so that μ_τ is solution of $\mathbb{E}\big[\mathcal{I}(\mu_\tau, Y)\big] = 0$. Then

$$s^2 = \frac{\mathbb{E}[\mathcal{I}(\mu_\tau, Y)^2]}{(\tau[1-F(\mu_\tau)] + [1-\tau]F(\mu_\tau))^2}.$$

Quantile Regression

We want to solve, here, $\min\left\{\sum_{i=1}^{n}\mathcal{R}_\tau^{\mathsf{q}}(y_i-\boldsymbol{x}_i^{\mathsf{T}}\beta)\right\}$





$$y_i=\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta}+\varepsilon_i \text{ so that } \widehat{Q}_{y|\boldsymbol{x}}(\tau)=\boldsymbol{x}^{\mathsf{T}}\widehat{\boldsymbol{\beta}}+F_\varepsilon^{-1}(\tau)$$

Geometric Properties of the Quantile Regression

Observe that the median regression will always have two supporting observations.

Start with some regression line, $y_i = \beta_0 + \beta_1 x_i$

Consider small translations $y_i = (\beta_0 \pm \epsilon) + \beta_1 x_i$

We minimize $\displaystyle\sum_{i=1}^{n} \big| y_i - (\beta_0 + \beta_1 x_i) \big|$

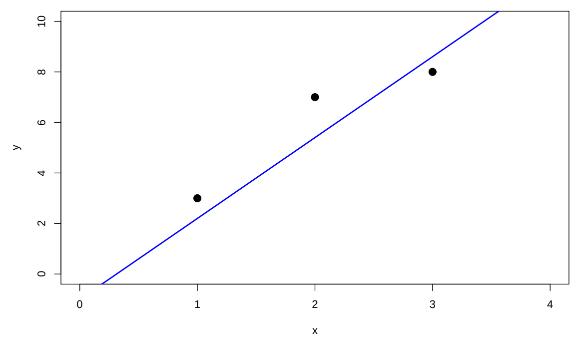
From line blue, a shift up decrease the sum by ϵ

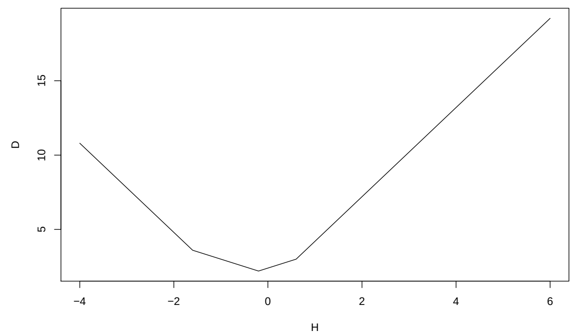
until we meet point on the left

an additional shift up will increase the sum

We will necessarily pass through one point

(observe that the sum is piecwise linear in ϵ)





Geometric Properties of the Quantile Regression

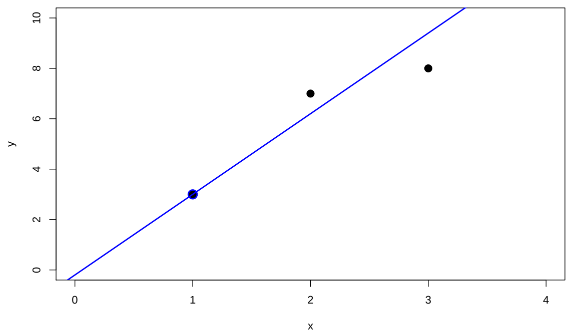
Consider now rotations of the line around the support point

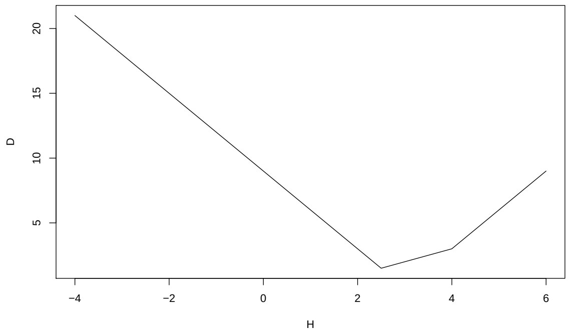
If we rotate up, we increase the sum of absolute difference (large impact on the point on the right)

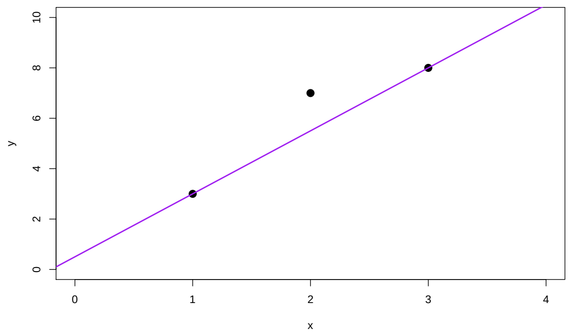
If we rotate down, we decrease the sum, until we reach the point on the right

Thus, the median regression will always have two supportting observations.

```
> library(quantreg)
> fit <- rq(dist~speed, data=cars, tau=.5)
> which(predict(fit)==cars$dist)
 1 21 46
 1 21 46
```







Numerical Aspects

To illustrate numerical computations, use

```
base=read.table("http://freakonometrics.free.fr/rent98_00.txt",header
    =TRUE)
```

The linear program for the quantile regression is now

$$\min_{\mu,\boldsymbol{a},\boldsymbol{b}}\left\{\sum_{i=1}^{n}\tau a_i+(1-\tau)b_i\right\}$$

with $a_i,b_i\geq 0$ and $y_i-[\beta_0^\tau+\beta_1^\tau x_i]=a_i-b_i$, $\forall i=1,\cdots,n$.

```
require(lpSolve)
tau = .3
n=nrow(base)
X = cbind( 1, base$area)
y = base$rent_euro
A1 = cbind(diag(2*n), 0,0)
A2 = cbind(diag(n), -diag(n), X)
```

Numerical Aspects

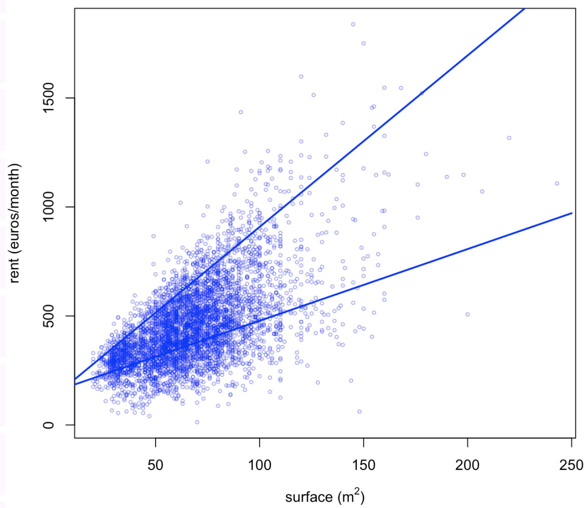
```
r = lp("min",
        c(rep(tau,n), rep(1-tau,n),0,0), rbind(A1, A2),
        c(rep("&gt;=", 2*n), rep("=", n)), c(rep(0,2*n), y))
tail(r$solution,2)
[1] 148.946864   3.289674
```

see

```
library(quantreg)
rq(rent_euro~area, tau=tau, data=base)
Coefficients:
(Intercept)        area
 148.946864    3.289674
```

Numerical Aspects

```
plot(base$area,base$rent_euro,xlab=expression(paste("surface (",m^2,")")),
     ylab="rent (euros/month)",col=rgb(0,0,1,.4),cex=.5)
sf=0:250
yr=r$solution[2*n+1]+r$solution[2*n+2]*sf
lines(sf,yr,lwd=2,col="blue")
tau = .9
r = lp("min",c(rep(tau,n), rep(1-tau,n),0,0), rbind(A1, A2),c(rep(">=", 2*n), rep("=", n)), c(rep(0,2*n), y))
yr=r$solution[2*n+1]+r$solution[2*n+2]*sf
lines(sf,yr,lwd=2,col="blue")
```



Numerical Aspects

For multiple regression, we should consider some trick (R function assumes all variables are nonnegative)

```
 tau = 0.3
n = nrow(base)
X = cbind(1, base$area, base$yearc)
y = base$rent_euro
r = lp("min",
c(rep(tau, n), rep(1 - tau, n), rep(0, 2 * 3)),
cbind(diag(n), -diag(n), X, -X),
rep("=", n),
y)
beta = tail(r$solution, 6)
beta = beta[1:3] - beta[3 + 1:3]
beta
[1]  -5542.503252     3.978135     2.887234
```

Numerical Aspects

which is consistant with

```
library(quantreg)
rq(rent_euro~area+yearc, tau=tau, data=base)
Coefficients:
 (Intercept)          area        yearc
-5542.503252     3.978135     2.887234
```

Distributional Aspects

OLS are equivalent to MLE when $Y - m(\boldsymbol{x}) \sim \mathcal{N}(0, \sigma^2)$, with density

$$g(\epsilon) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{\epsilon^2}{2\sigma^2}\right)$$

Quantile regression is equivalent to Maximum Likelihood Estimation when $Y - m(\boldsymbol{x})$ has an asymmetric Laplace distribution

$$g(\epsilon) = \frac{\sqrt{2}}{\sigma} \frac{\kappa}{1+\kappa^2} \exp\left(-\frac{\sqrt{2}\kappa^{\mathbf{1}(\epsilon>0)}}{\sigma\kappa^{\mathbf{1}(\epsilon<0)}}|\epsilon|\right)$$

Quantile Regression and Iterative Least Squares

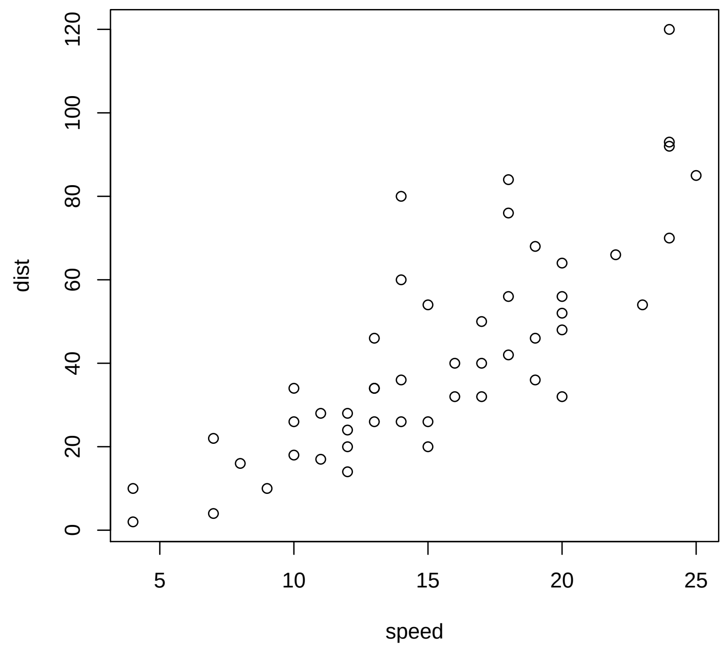
start with some $\boldsymbol{\beta}^{(0)}$ e.g. $\boldsymbol{\beta}^{\mathsf{ols}}$

at stage k :

let $\varepsilon_i^{(k)} = y_i - \boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta}^{(k-1)}$

define weights $\omega_i^{(k)} = \mathcal{R}'_\tau(\varepsilon_i^{(k)})$

compute weighted least square to estimate $\boldsymbol{\beta}^{(k)}$



One can also consider a smooth approximation of $\mathcal{R}^{\mathsf{q}}_\tau(\cdot)$, and then use Newton-Raphson.

Optimization Algorithm

Primal problem is

$$\min_{\boldsymbol{\beta},\boldsymbol{u},\boldsymbol{v}} \left\{\tau\mathbf{1}^\top\boldsymbol{u}+(1-\tau)\mathbf{1}^\top\boldsymbol{v}\right\} \text{ s.t. } \boldsymbol{y}=\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{u}-\boldsymbol{v},\text{ with } \boldsymbol{u},\boldsymbol{v}\in\mathbb{R}_+^n$$

and the dual version is

$$\max_{\boldsymbol{d}} \left\{\boldsymbol{y}^\top\boldsymbol{d}\right\} \text{ s.t. } \boldsymbol{X}^\top\boldsymbol{d}=(1-\tau)\boldsymbol{X}^\top\mathbf{1} \text{ with } \boldsymbol{d}\in[0,1]^n$$

Koenker & D'Orey (1994) A Remark on Algorithm AS 229: Computing Dual Regression Quantiles and Regression Rank Scores suggest to use the simplex method (default method in R)

Portnoy & Koenker (1997) The Gaussian hare and the Laplacian tortoise suggest to use the interior point method »

Interior Point Method

See Vanderbei *et al.* (1986) A modification of Karmarkar's linear programming algorithm for a presentation of the algorithm, Potra & Wright (2000) Interior-point methods for a general survey, and and Meketon (1986) Least absolute value regression for an application of the algorithm in the context of median regression.

Running time is of order $n^{1+\delta}k^3$ for some $\delta > 0$ and $k = \dim(\boldsymbol{\beta})$ (it is $(n+k)k^2$ for OLS, see wikipedia).

Quantile Regression Estimators

OLS estimator $\widehat{\boldsymbol{\beta}}^{\text{ols}}$ is solution of

$$\widehat{\boldsymbol{\beta}}^{\text{ols}} = \text{argmin}\left\{\mathbb{E}\Big[\big(\mathbb{E}[Y|\boldsymbol{X}=\boldsymbol{x}] - \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}\big)^2\Big]\right\}$$

and Angrist, Chernozhukov & Fernandez-Val (2006) Quantile Regression under Misspecification proved that

$$\widehat{\boldsymbol{\beta}}_\tau = \text{argmin}\left\{\mathbb{E}\Big[\omega_\tau(\boldsymbol{\beta})\big(Q_\tau[Y|\boldsymbol{X}=\boldsymbol{x}] - \boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}\big)^2\Big]\right\}$$

(under weak conditions) where

$$\omega_\tau(\boldsymbol{\beta}) = \int_0^1 (1-u) f_{y|\boldsymbol{x}}(u\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta} + (1-u)Q_\tau[Y|\boldsymbol{X}=\boldsymbol{x}])du$$

$\widehat{\boldsymbol{\beta}}_\tau$ is the best weighted mean square approximation of the tru quantile function, where the weights depend on an average of the conditional density of Y over $\boldsymbol{x}^\mathsf{T}\boldsymbol{\beta}$ and the true quantile regression function.

Assumptions to get Consistency of Quantile Regression Estimators

As always, we need some assumptions to have consistency of estimators.

- observations $(Y_i, \boldsymbol{X}_i)$ must (conditionnaly) i.id.
- regressors must have a bounded second moment, $\mathbb{E}\big[\|\boldsymbol{X}_i\|^2\big] < \infty$
- error terms ε are continuously distributed given $\boldsymbol{X}_i$, centered in the sense that their median should be 0,

$$\int_{-\infty}^{0} f_\varepsilon(\epsilon)d\epsilon = \frac{1}{2}.$$

- "local identification" property : $\big[f_\varepsilon(0)\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\big]$ is positive definite

Quantile Regression Estimators

Under those weak conditions, $\widehat{\boldsymbol{\beta}}_\tau$ is asymptotically normal:

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_\tau - \boldsymbol{\beta}_\tau) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \tau(1-\tau)D_\tau^{-1}\Omega_x D_\tau^{-1}),$$

where

$$D_\tau = \mathbb{E}\big[f_\varepsilon(0)\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\big] \text{ and } \Omega_x = \mathbb{E}\big[\boldsymbol{X}^\mathsf{T}\boldsymbol{X}\big].$$

hence, the asymptotic variance of $\widehat{\boldsymbol{\beta}}$ is

$$\widehat{\text{Var}}\big[\widehat{\boldsymbol{\beta}}_\tau\big] = \frac{\tau(1-\tau)}{[\widehat{f}_\varepsilon(0)]^2}\left(\frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i^\mathsf{T}\boldsymbol{x}_i\right)^{-1}$$

where $\widehat{f}_\varepsilon(0)$ is estimated using (e.g.) an histogram, as suggested in Powell (1991) Estimation of monotonic regression models under quantile restrictions, since

$$D_\tau = \lim_{h\downarrow 0}\mathbb{E}\left(\frac{\mathbf{1}(|\varepsilon|\leq h)}{2h}\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\right) \sim \frac{1}{2nh}\sum_{i=1}^n \mathbf{1}(|\varepsilon_i|\leq h)\boldsymbol{x}_i\boldsymbol{x}_i^\mathsf{T} = \widehat{D}_\tau.$$

Quantile Regression Estimators

There is no first order condition, in the sense $\partial V_n(\boldsymbol{\beta},\tau)/\partial\boldsymbol{\beta} = \mathbf{0}$ where

$$V_n(\boldsymbol{\beta},\tau) = \sum_{i=1}^{n} \mathcal{R}_\tau^{\mathsf{q}}(y_i - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta})$$

There is an asymptotic first order condition,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \boldsymbol{x}_i\psi_\tau(y_i - \boldsymbol{x}_i^\mathsf{T}\boldsymbol{\beta}) = \mathcal{O}(1),\ \text{ as } n \to \infty,$$

where $\psi_\tau(\cdot) = \mathbf{1}(\cdot < 0) - \tau$, see Huber (1967) The behavior of maximum likelihood estimates under nonstandard conditions.

One can also define a Wald test, a Likelihood Ratio test, etc.

Quantile Regression Estimators

Then the confidence interval of level $1-\alpha$ is then

$$\left[\widehat{\beta}_\tau \pm z_{1-\alpha/2}\sqrt{\widehat{\mathrm{Var}}\big[\widehat{\boldsymbol{\beta}}_\tau\big]}\right]$$

An alternative is to use a boostrap strategy (see #2)

- generate a sample $(y_i^{(b)},\boldsymbol{x}_i^{(b)})$ from $(y_i,\boldsymbol{x}_i)$
- estimate $\boldsymbol{\beta}_\tau^{(b)}$ by
$$\widehat{\boldsymbol{\beta}}_\tau^{(b)} = \text{argmin}\left\{\mathcal{R}_\tau^{\mathsf{q}}\left(y_i^{(b)} - \boldsymbol{x}_i^{(b)\mathsf{T}}\boldsymbol{\beta}\right)\right\}$$
- set $\widehat{\mathrm{Var}}^\star\big[\widehat{\boldsymbol{\beta}}_\tau\big] = \dfrac{1}{B}\displaystyle\sum_{b=1}^{B}\big(\widehat{\boldsymbol{\beta}}_\tau^{(b)} - \widehat{\boldsymbol{\beta}}_\tau\big)^2$

For confidence intervals, we can either use Gaussian-type confidence intervals, or empirical quantiles from bootstrap estimates.

Quantile Regression Estimators

If $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_m)$, one can prove that

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_{\boldsymbol{\tau}} - \boldsymbol{\beta}_{\boldsymbol{\tau}}) \xrightarrow{\mathcal{L}} \mathcal{N}(\mathbf{0}, \Sigma_{\boldsymbol{\tau}}),$$

where $\Sigma_{\boldsymbol{\tau}}$ is a block matrix, with

$$\Sigma_{\tau_i,\tau_j} = (\min\{\tau_i, \tau_j\} - \tau_i\tau_j)D_{\tau_i}^{-1}\Omega_x D_{\tau_j}^{-1}$$

see Kocherginsky *et al.* (2005) Practical Confidence Intervals for Regression Quantiles for more details.

Quantile Regression: Transformations

Scale equivariance

For any $a > 0$ and $\tau \in [0, 1]$

$$\hat{\boldsymbol{\beta}}_\tau(aY, \boldsymbol{X}) = a\hat{\boldsymbol{\beta}}_\tau(Y, \boldsymbol{X}) \text{ and } \hat{\boldsymbol{\beta}}_\tau(-aY, \boldsymbol{X}) = -a\hat{\boldsymbol{\beta}}_{1-\tau}(Y, \boldsymbol{X})$$

Equivariance to reparameterization of design

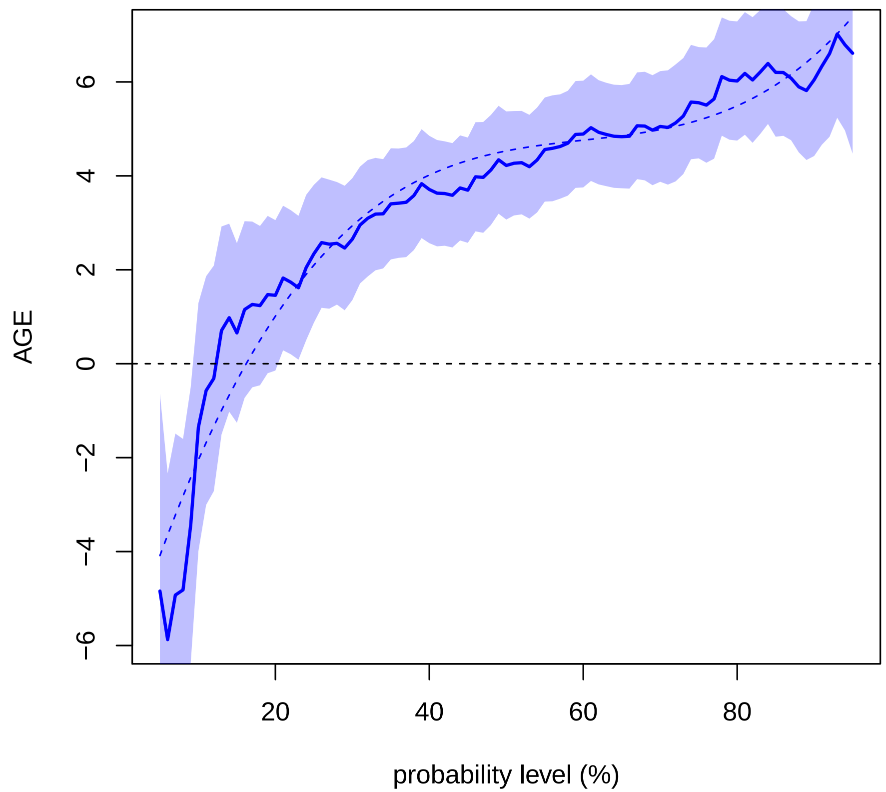
Let $\boldsymbol{A}$ be any $p \times p$ nonsingular matrix and $\tau \in [0, 1]$

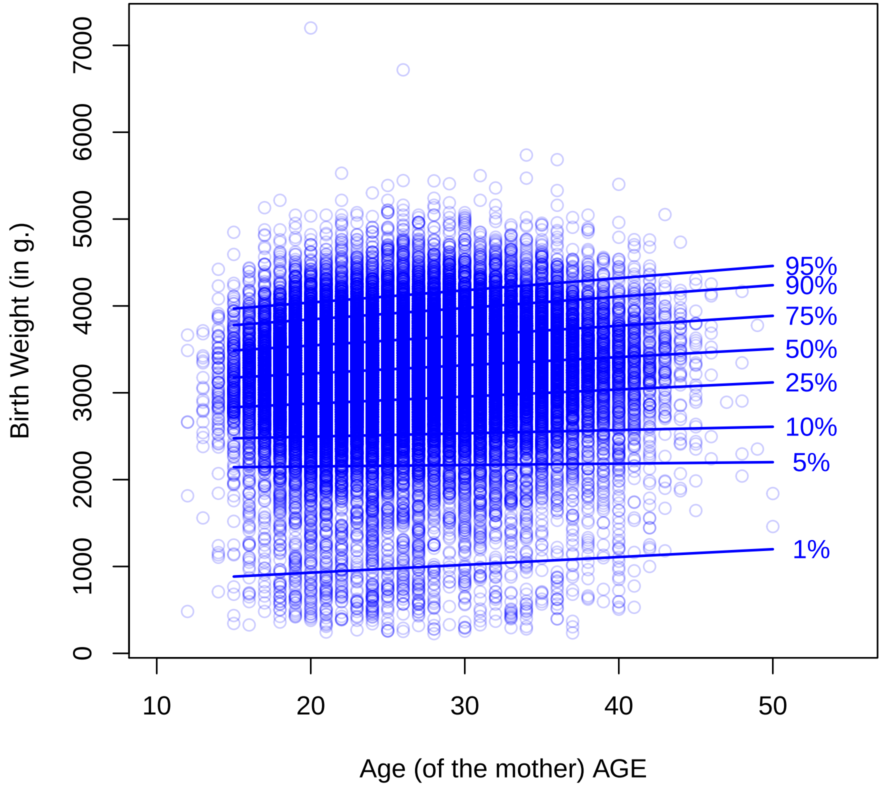
$$\hat{\boldsymbol{\beta}}_\tau(Y, \boldsymbol{X}\boldsymbol{A}) = \boldsymbol{A}^{-1}\hat{\boldsymbol{\beta}}_\tau(Y, \boldsymbol{X})$$

Visualization, $\tau \mapsto \widehat{\beta}_\tau$

See Abreveya (2001) The effects of demographics and maternal behavior...

```
> base=read.table("http://freakonometrics.free.fr/natality2005.txt")
```



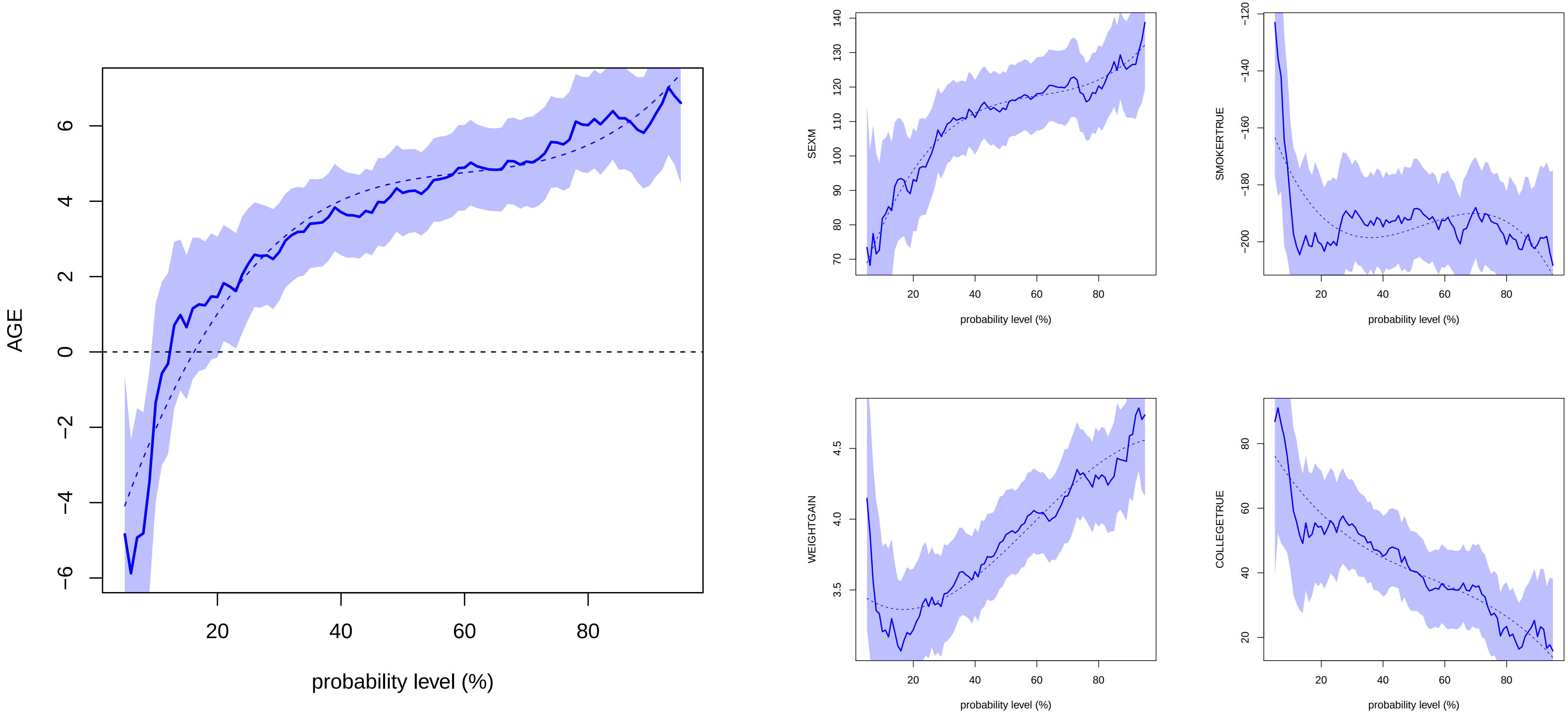


Visualization, $\tau \mapsto \widehat{\beta}_\tau$

```
> base=read.table("http://freakonometrics.free.fr/natality2005.txt",
    header=TRUE,sep=";")
> u=seq(.05,.95,by=.01)
> library(quantreg)
>  coefstd=function(u) summary(rq(WEIGHT~SEX+SMOKER+WEIGHTGAIN+
    BIRTHRECORD+AGE+ BLACKM+ BLACKF+COLLEGE,data=sbase,tau=u))$
    coefficients[,2]
> coefest=function(u) summary(rq(WEIGHT~SEX+SMOKER+WEIGHTGAIN+
    BIRTHRECORD+AGE+ BLACKM+ BLACKF+COLLEGE,data=sbase,tau=u))$
    coefficients[,1]
CS=Vectorize(coefstd)(u)
CE=Vectorize(coefest)(u)
```

Visualization, $\tau \mapsto \widehat{\beta}_\tau$

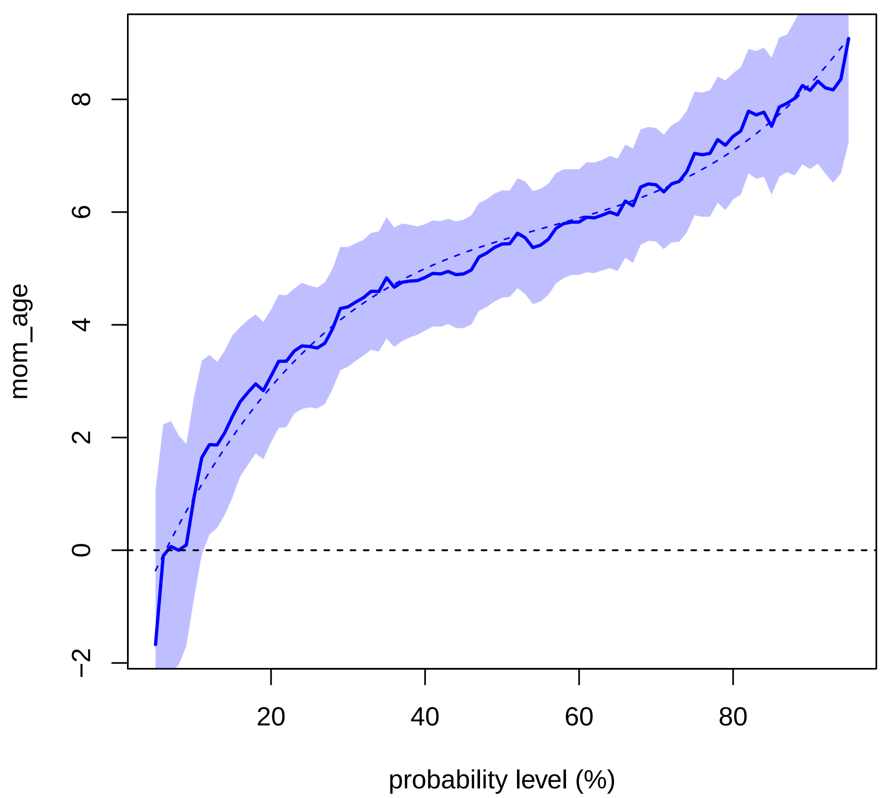
See Abreveya (2001) The effects of demographics and maternal behavior on the distribution of birth outcomes

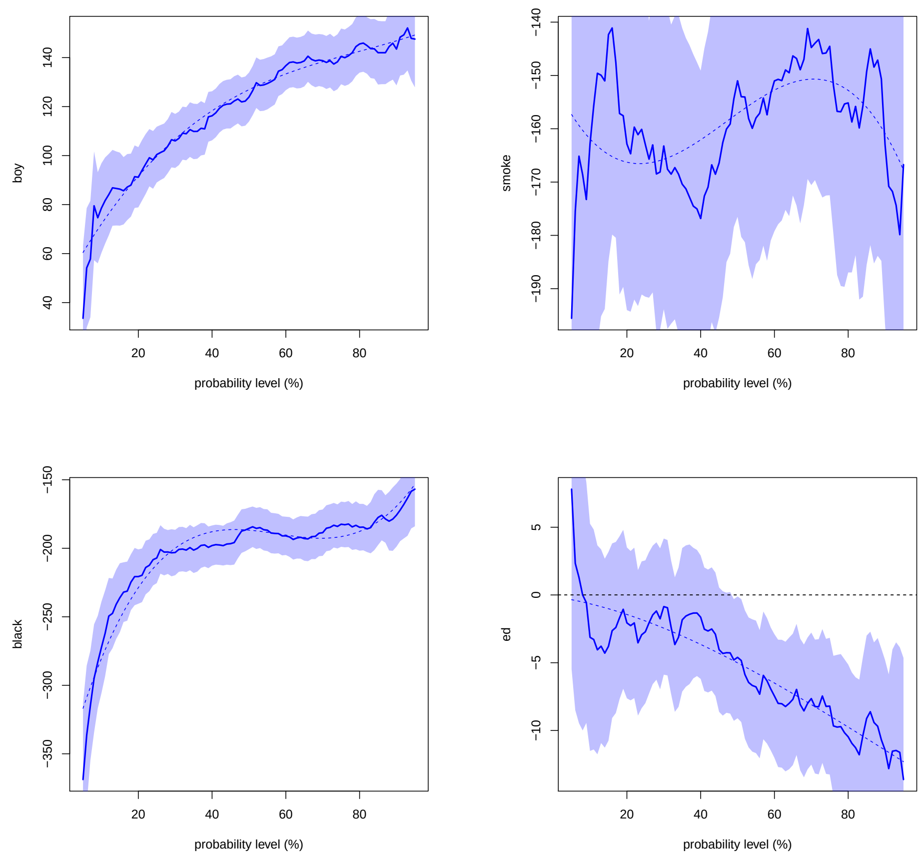


Visualization, $\tau \mapsto \widehat{\beta}_\tau$

See Abreveya (2001) The effects of demographics and maternal behavior...

```
> base=read.table("http://freakonometrics.free.fr/BWeight.csv")
```

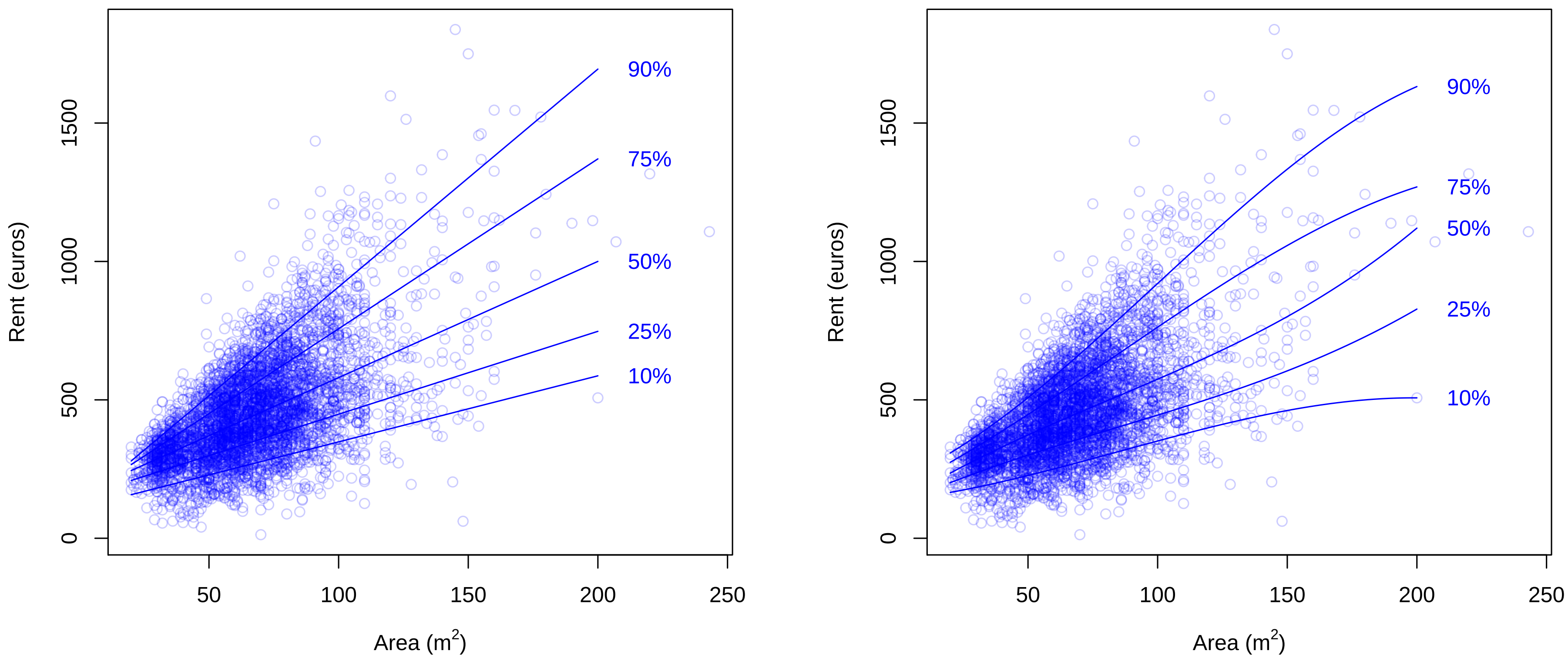




Quantile Regression, with Non-Linear Effects

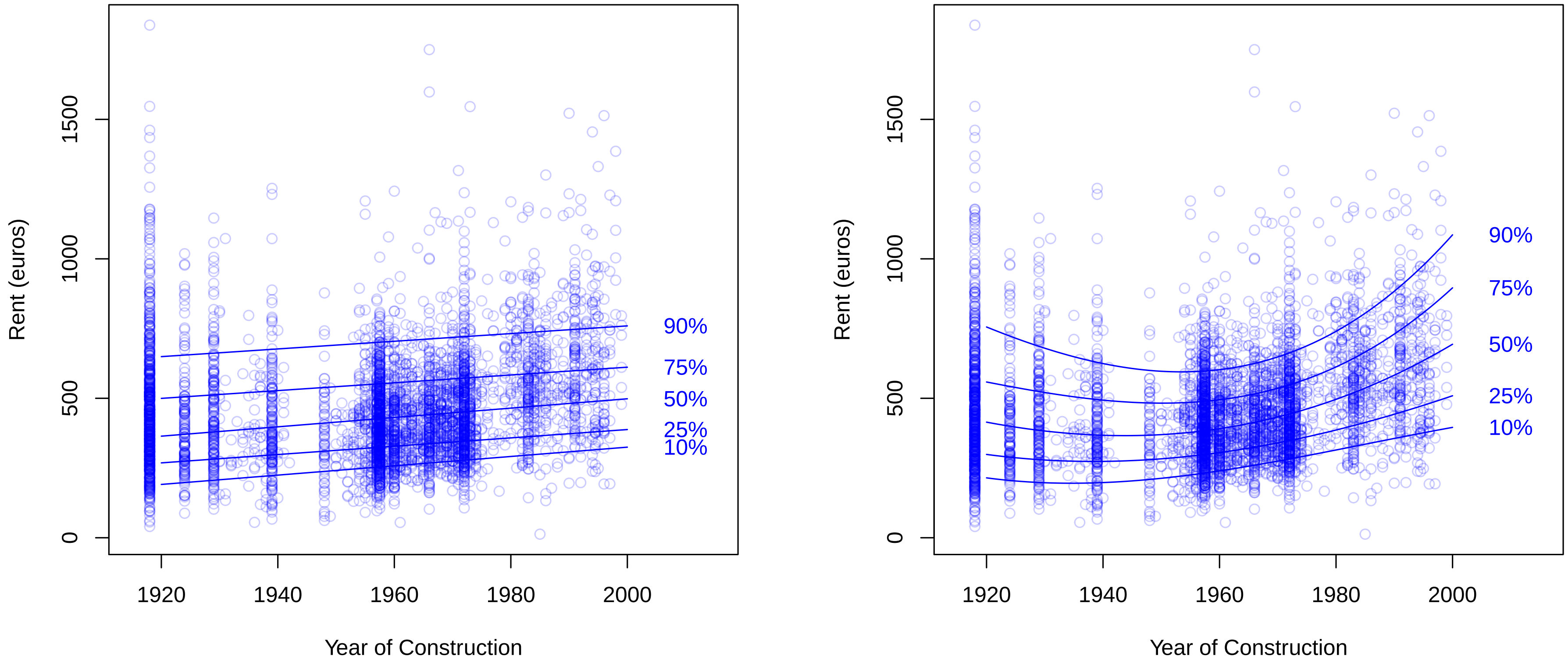
Rents in Munich, as a function of the area, from Fahrmeir *et al.* (2013)
Regression: Models, Methods and Applications

```
> base=read.table("http://freakonometrics.free.fr/rent98_00.txt")
```



Quantile Regression, with Non-Linear Effects

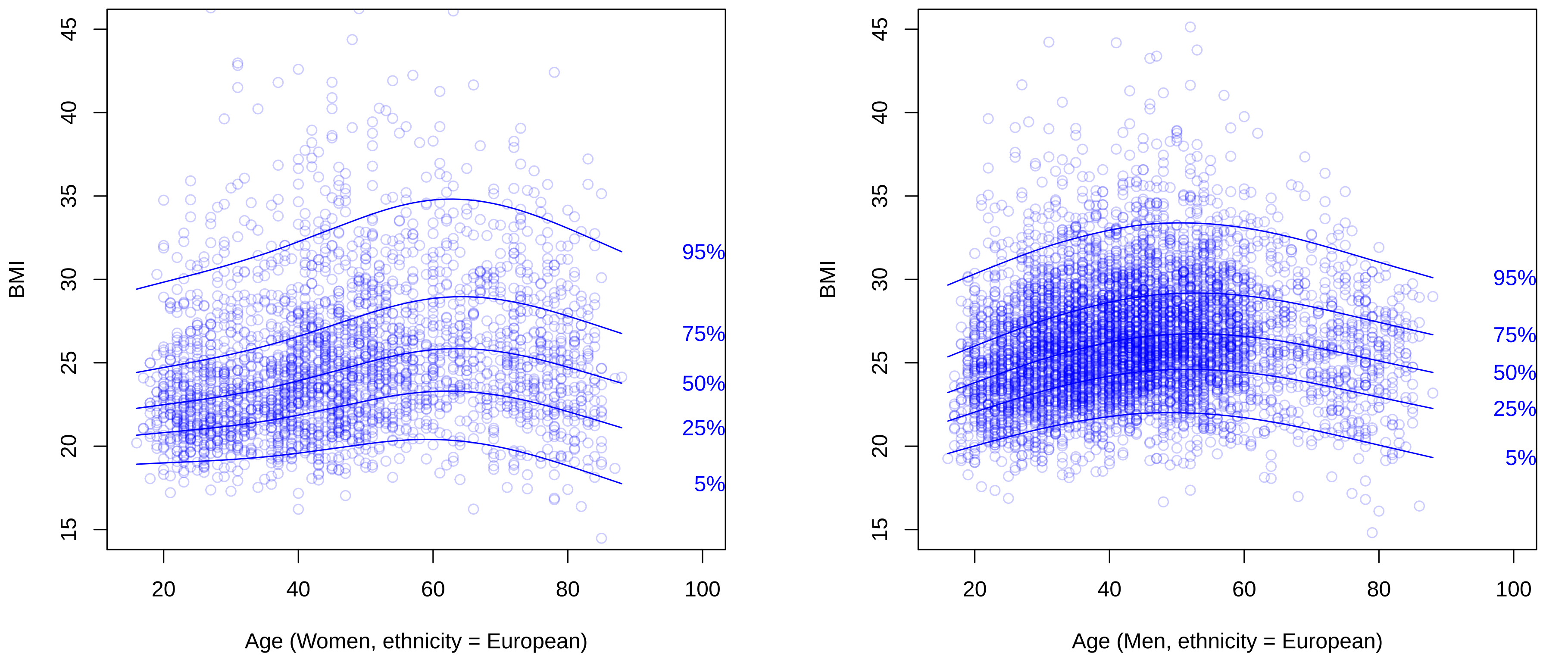
Rents in Munich, as a function of the year of construction, from Fahrmeir *et al.* (2013) Regression: Models, Methods and Applications



Quantile Regression, with Non-Linear Effects

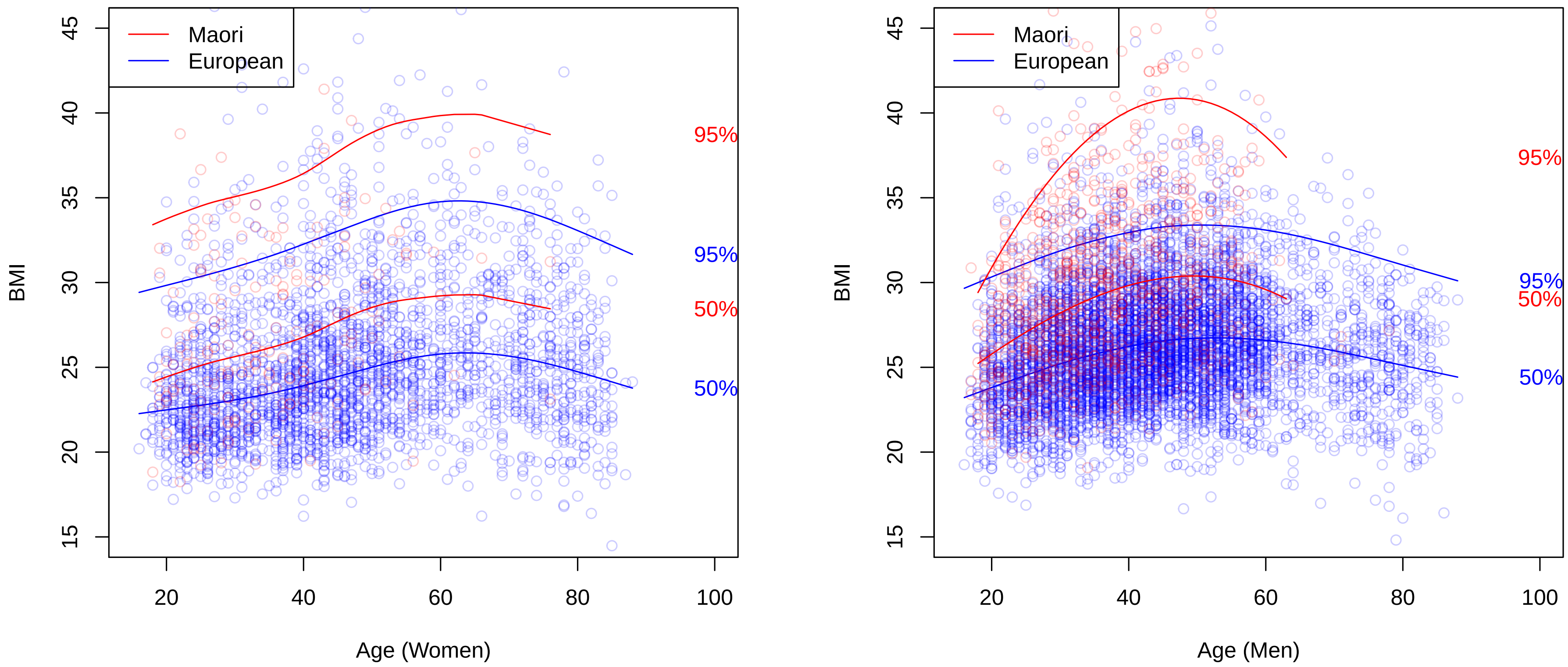
BMI as a function of the age, in New-Zealand, from Yee (2015) Vector Generalized Linear and Additive Models, for Women and Men

```
> library(VGAMdata); data(xs.nz)
```



Quantile Regression, with Non-Linear Effects

BMI as a function of the age, in New-Zealand, from Yee (2015) Vector Generalized Linear and Additive Models, for Women and Men



Quantile Regression, with Non-Linear Effects

One can consider some local polynomial quantile regression, e.g.

$$\min\left\{\sum_{i=1}^{n}\omega_i(\boldsymbol{x})\mathcal{R}_\tau^{\mathsf{q}}\big(y_i-\beta_0-(\boldsymbol{x}_i-\boldsymbol{x})^\mathsf{T}\boldsymbol{\beta}_1\big)\right\}$$

for some weights $\omega_i(\boldsymbol{x})=H^{-1}K(H^{-1}(\boldsymbol{x}_i-\boldsymbol{x}))$, see Fan, Hu & Truong (1994) Robust Non-Parametric Function Estimation.

Asymmetric Maximum Likelihood Estimation

Introduced by Efron (1991) Regression percentiles using asymmetric squared error loss. Consider a linear model, $y_i = \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i$. Let

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} Q_\omega(y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}), \text{ where } Q_\omega(\epsilon) = \begin{cases} \epsilon^2 \text{ if } \epsilon \leq 0 \\ w\epsilon^2 \text{ if } \epsilon > 0 \end{cases} \quad \text{where } w = \frac{\omega}{1-\omega}$$

One might consider $\omega_\alpha = 1 + \dfrac{z_\alpha}{\varphi(z_\alpha) + (1-\alpha)z_\alpha}$ where $z_\alpha = \Phi^{-1}(\alpha)$.

Efron (1992) Poisson overdispersion estimates based on the method of asymmetric maximum likelihood introduced asymmetric maximum likelihood (AML) estimation, considering

$$S(\boldsymbol{\beta}) = \sum_{i=1}^{n} Q_\omega(y_i - \boldsymbol{x}_i^\top \boldsymbol{\beta}), \text{ where } Q_\omega(\epsilon) = \begin{cases} D(y_i, \boldsymbol{x}_i^\top \boldsymbol{\beta}) \text{ if } y_i \leq \boldsymbol{x}_i^\top \boldsymbol{\beta} \\ wD(y_i, \boldsymbol{x}_i^\top \boldsymbol{\beta}) \text{ if } y_i > \boldsymbol{x}_i^\top \boldsymbol{\beta} \end{cases}$$

where $D(\cdot,\cdot)$ is the deviance. Estimation is based on Newton-Raphson (gradient descent).

Noncrossing Solutions

See Bondell *et al.* (2010) Non-crossing quantile regression curve estimation.

Consider probabilities $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_q)$ with $0 < \tau_1 < \cdots < \tau_q < 1$.

Use parallelism : add constraints in the optimization problem, such that

$$\boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}_{\tau_j} \geq \boldsymbol{x}_i^\top \widehat{\boldsymbol{\beta}}_{\tau_{j-1}} \quad \forall i \in \{1, \cdots, n\}, j \in \{2, \cdots, q\}.$$

Quantile Regression on Panel Data

In the context of panel data, consider some fixed effect, α_i so that

$$y_{i,t} = \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta}_\tau + \alpha_i + \varepsilon_{i,t} \text{ where } Q_\tau(\varepsilon_{i,t}|\boldsymbol{X}_i) = 0$$

Canay (2011) A simple approach to quantile regression for panel data suggests an estimator in two steps,

- use a standard OLS fixed-effect model $y_{i,t} = \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta} + \alpha_i + u_{i,t}$, i.e. consider a within transformation, and derive the fixed effect estimate $\widehat{\boldsymbol{\beta}}$

$$(y_{i,t} - \overline{y}_i) = \big(\boldsymbol{x}_{i,t} - \overline{\boldsymbol{x}}_{i,t}\big)^{\mathsf{T}}\boldsymbol{\beta} + (u_{i,t} - \overline{u}_i)$$

- estimate fixed effects as $\widehat{\alpha}_i = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T}\big(y_{i,t} - \boldsymbol{x}_{i,t}^{\mathsf{T}}\widehat{\boldsymbol{\beta}}\big)$
- finally, run a standard quantile regression of $y_{i,t} - \widehat{\alpha}_i$ on $\boldsymbol{x}_{i,t}$'s.

See rqpd package.

Quantile Regression with Fixed Effects (QRFE)

In a panel linear regression model, $y_{i,t} = \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta} + u_i + \varepsilon_{i,t}$,

where u is an unobserved individual specific effect.

In a fixed effects models, u is treated as a parameter. Quantile Regression is

$$\min_{\boldsymbol{\beta},\boldsymbol{u}} \left\{ \sum_{i,t} \mathcal{R}_{\alpha}^{\mathsf{q}}(y_{i,t} - [\boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta} + u_i]) \right\}$$

Consider Penalized QRFE, as in Koenker & Bilias (2001) Quantile regression for duration data,

$$\min_{\boldsymbol{\beta}_1,\cdots,\boldsymbol{\beta}_\kappa,\boldsymbol{u}} \left\{ \sum_{k,i,t} \omega_k \mathcal{R}_{\alpha_k}^{\mathsf{q}}(y_{i,t} - [\boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta}_k + u_i]) + \lambda \sum_i |u_i| \right\}$$

where ω_k is a relative weight associated with quantile of level α_k.

Quantile Regression with Random Effects (QRRE)

Assume here that $y_{i,t} = \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta} + \underbrace{u_i + \varepsilon_{i,t}}_{=\eta_{i,t}}$.

Quantile Regression Random Effect (QRRE) yields solving

$$\min_{\boldsymbol{\beta}}\left\{\sum_{i,t}\mathcal{R}_{\alpha}^{\mathsf{q}}(y_{i,t}-\boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta})\right\}$$

which is a weighted asymmetric least square deviation estimator.

Let $\Sigma = [\sigma_{s,t}(\alpha)]$ denote the matrix

$$\sigma_{ts}(\alpha) = \begin{cases} \alpha(1-\alpha) & \text{if } t = s \\ \mathbb{E}[\mathbf{1}\{\varepsilon_{it}(\alpha) < 0, \varepsilon_{is}(\alpha) < 0\}] - \alpha^2 & \text{if } t \neq s \end{cases}$$

If $(nT)^{-1}\boldsymbol{X}^{\mathsf{T}}\{\mathbb{I}_n \otimes \Sigma_{T\times T}(\alpha)\}\boldsymbol{X} \to \boldsymbol{D}_0$ as $n\to\infty$ and $(nT)^{-1}\boldsymbol{X}^{\mathsf{T}}\Omega_f\boldsymbol{X} = \boldsymbol{D}_1$, then

$$\sqrt{nT}\left(\widehat{\boldsymbol{\beta}}^{Q}(\alpha)-\boldsymbol{\beta}^{Q}(\alpha)\right)\xrightarrow{\mathcal{L}}\mathcal{N}\left(0,\boldsymbol{D}_1^{-1}\boldsymbol{D}_0\boldsymbol{D}_1^{-1}\right).$$

Quantile Treatment Effects

Doksum (1974) Empirical Probability Plots and Statistical Inference for Nonlinear Models introduced QTE - Quantile Treatement Effect - when a person might have two Y's : either Y_0 (without treatment, $D=0$) or Y_1 (with treatement, $D=1$),

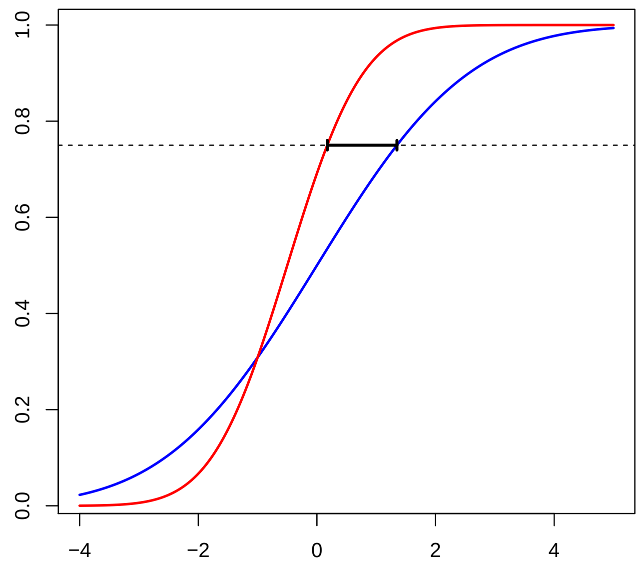
$$\delta_\tau = Q_{Y_1}(\tau) - Q_{Y_0}(\tau)$$

which can be studied on the context of covariates.

Run a quantile regression of y on $(d, \boldsymbol{x})$,

$$y = \beta_0 + \delta d + \boldsymbol{x}_i^\top \boldsymbol{\beta} + \varepsilon_i \ \ : \text{ shifting effect}$$

$$y = \beta_0 + \boldsymbol{x}_i^\top \big[\boldsymbol{\beta} + \delta d\big] + \varepsilon_i \ \ : \text{ scaling effect}$$



Quantile Regression for Time Series

Consider some GARCH(1,1) financial time series,

$$y_t = \sigma_t \varepsilon_t \text{ where } \sigma_t = \alpha_0 + \alpha_1 \cdot |y_{t-1}| + \beta_1 \sigma_{t-1}.$$

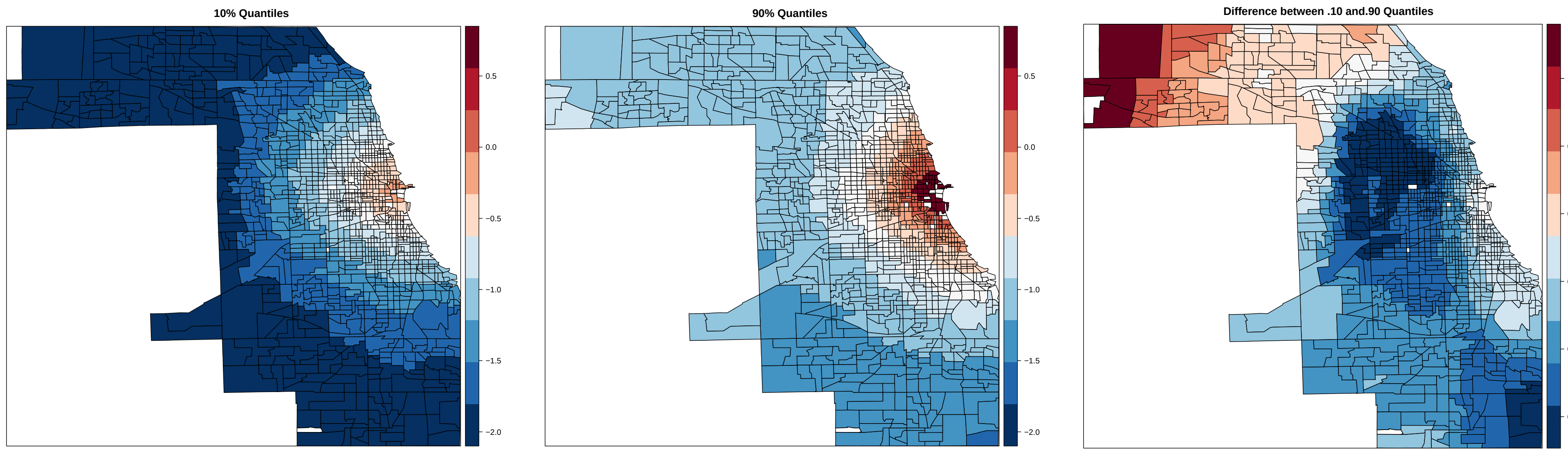
The quantile function conditional on the past - $\mathcal{F}_{t-1} = \underline{Y}_{t-1}$ - is

$$Q_{y|\mathcal{F}_{t-1}}(\tau) = \underbrace{\alpha_0 F_\varepsilon^{-1}(\tau)}_{\tilde{\alpha}_0} + \underbrace{\alpha_1 F_\varepsilon^{-1}(\tau)}_{\tilde{\alpha}_1} \cdot |y_{t-1}| + \beta_1 Q_{y|\mathcal{F}_{t-2}}(\tau)$$

i.e. the conditional quantile has a GARCH(1,1) form, see Conditional Autoregressive Value-at-Risk, see Manganelli & Engle (2004) CAViaR: Conditional Autoregressive Value at Risk by Regression Quantiles

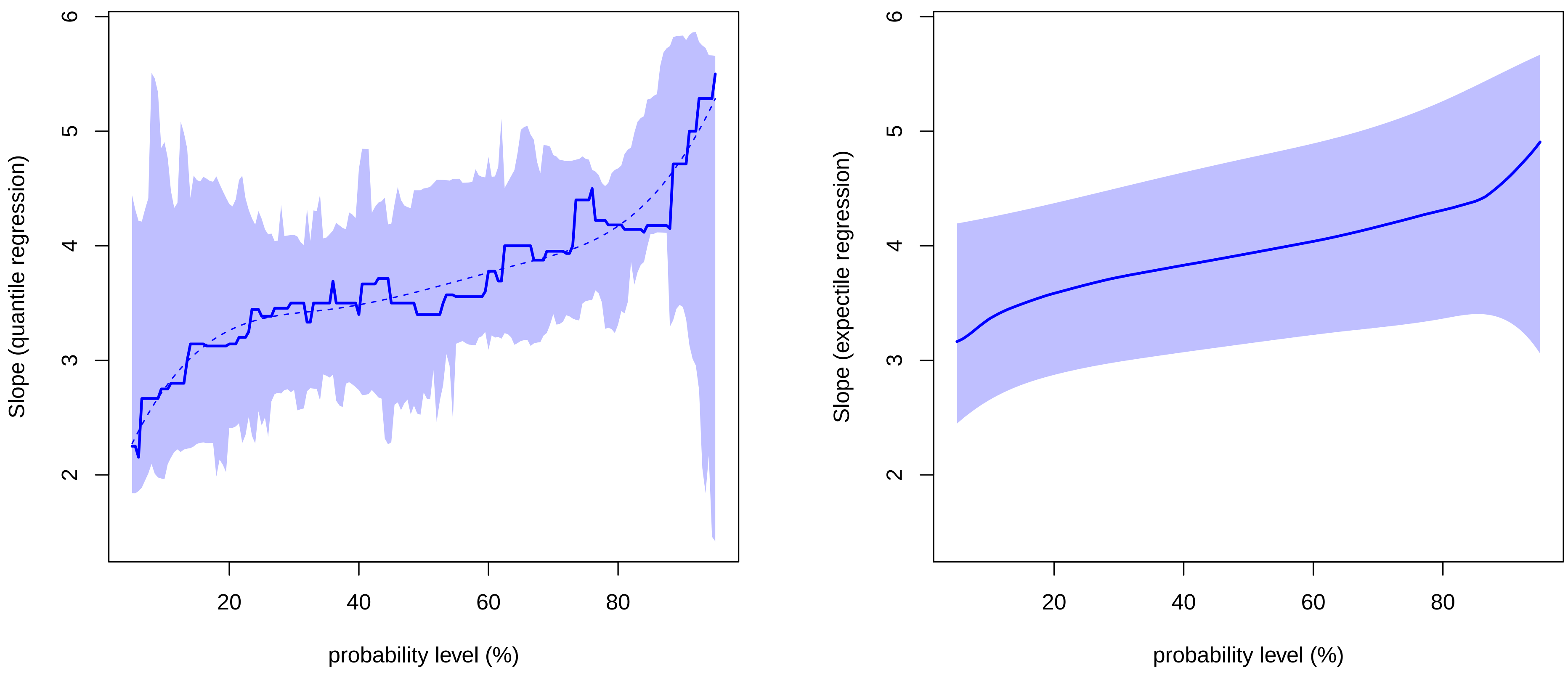
Quantile Regression for Spatial Data

```
> library(McSpatial)
> data(cookdata)
> fit <- qregcpar(LNFAR~DCBD, nonpar=~LATITUDE+LONGITUDE, taumat=c
    (.10,.90), kern="bisq", window=.30, distance="LATLONG", data=
    cookdata)
```



Expectile Regression

Quantile regression vs. Expectile regression, on the same dataset (cars)



see Koenker (2014) Living Beyond our Means for a comparison quantiles-expectiles

Expectile Regression

Solve here $\min_{\boldsymbol{\beta}}\left\{\sum_{i=1}^{n}\mathcal{R}_\tau^{\mathsf{e}}(y_i-\boldsymbol{x}_i^{\mathsf{T}}\boldsymbol{\beta})\right\}$ where $\mathcal{R}_\tau^{\mathsf{e}}(u)=u^2\cdot\big(\tau-\mathbf{1}(u<0)\big)$

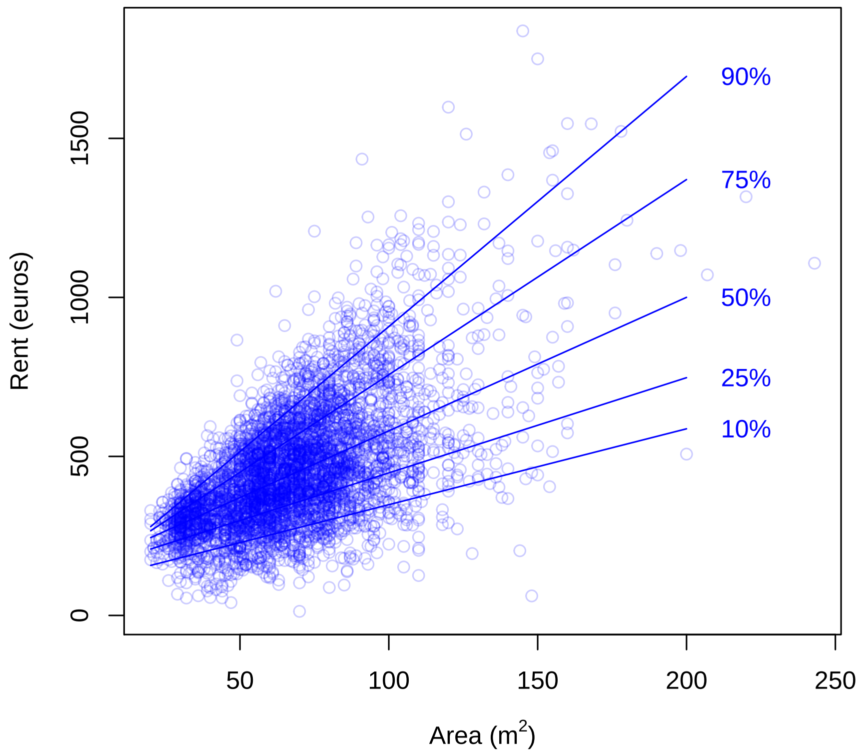
"*this estimator can be interpreted as a maximum likelihood estimator when the disturbances arise from a normal distribution with unequal weight placed on positive and negative disturbances*" Aigner, Amemiya & Poirier (1976) Formulation and Estimation of Stochastic Frontier Production Function Models.

See Holzmann & Klar (2016) Expectile Asymptotics for statistical properties.

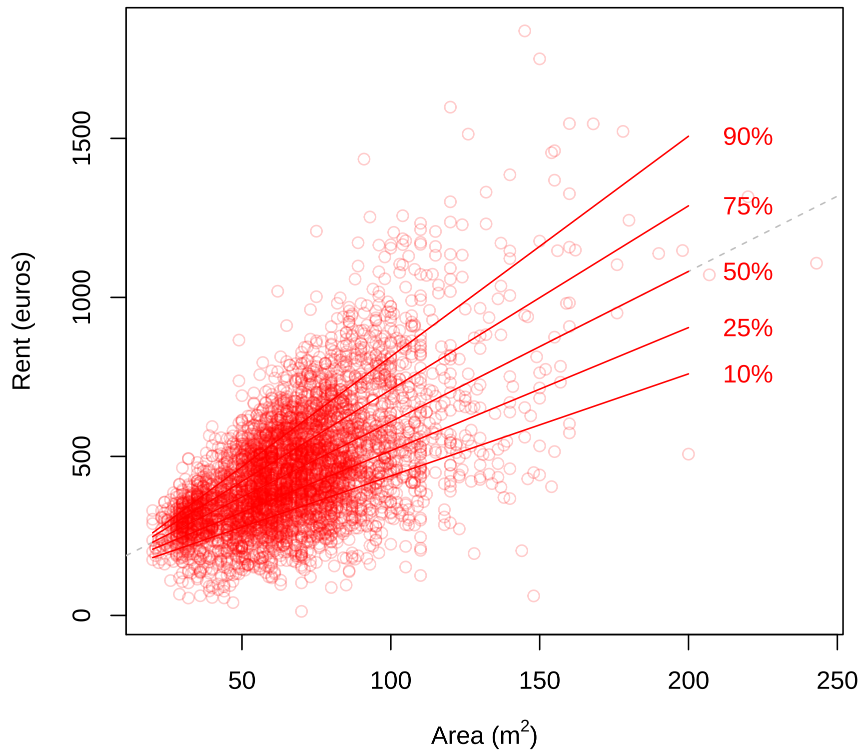
Expectiles can (also) be related to Breckling & Chambers (1988) M-Quantiles.

Comparison quantile regression and expectile regression, see Schulze-Waltrup *et al.* (2014) Expectile and quantile regression - David and Goliath?

Expectile Regression, with Linear Effects



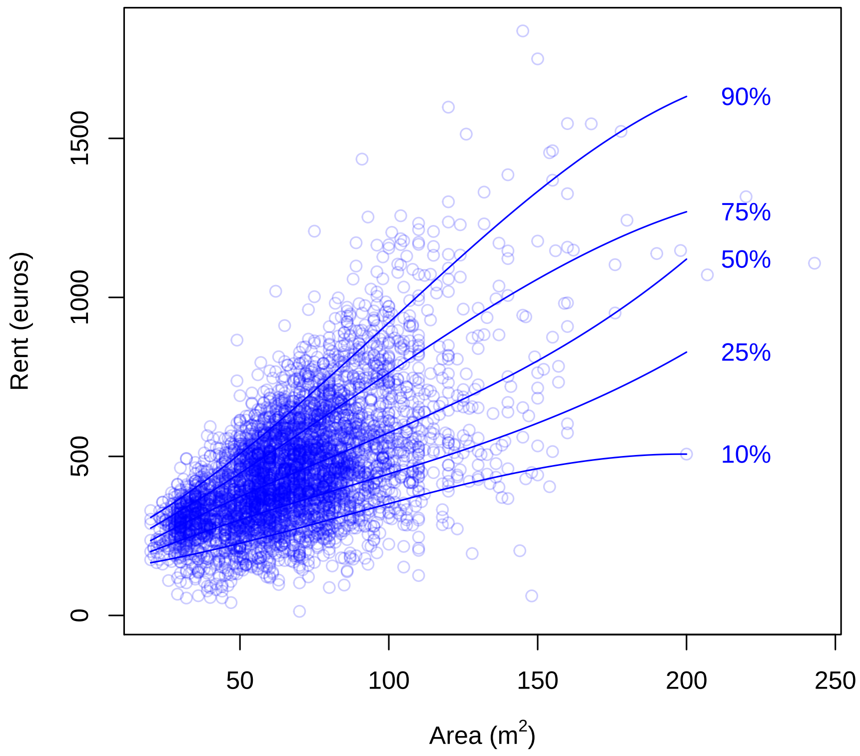
Quantile Regressions



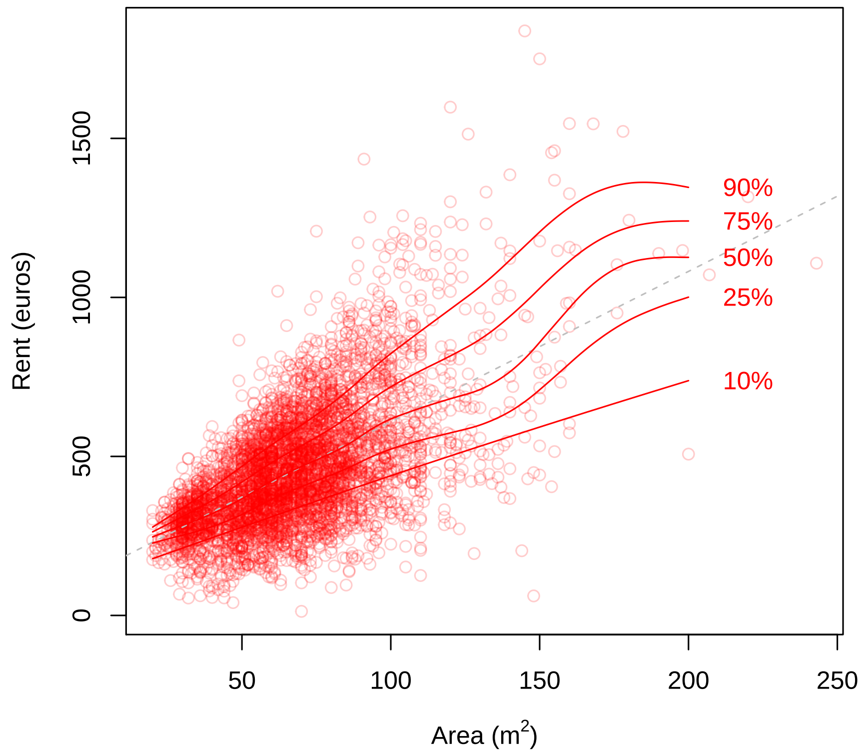
Expectile Regressions

Expectile Regression, with Non-Linear Effects

See Zhang (1994) Nonparametric regression expectiles



Quantile Regressions



Expectile Regressions

Expectile Regression, with Linear Effects

```
> library(expectreg)
> coefstd=function(u) summary(expectreg.ls(WEIGHT~SEX+SMOKER+
    WEIGHTGAIN+BIRTHRECORD+AGE+ BLACKM+ BLACKF+COLLEGE,data=sbase,
    expectiles=u,ci = TRUE))[,2]
> coefest=function(u) summary(expectreg.ls(WEIGHT~SEX+SMOKER+
    WEIGHTGAIN+BIRTHRECORD+AGE+ BLACKM+ BLACKF+COLLEGE,data=sbase,
    expectiles=u,ci = TRUE))[,1]
> CS=Vectorize(coefstd)(u)
> CE=Vectorize(coefest)(u)
```

Expectile Regression, with Random Effects (ERRE)

Quantile Regression Random Effect (QRRE) yields solving

$$\min_{\boldsymbol{\beta}} \left\{ \sum_{i,t} \mathcal{R}^{\mathsf{e}}_{\alpha}(y_{i,t} - \boldsymbol{x}_{i,t}^{\mathsf{T}}\boldsymbol{\beta}) \right\}$$

One can prove that

$$\widehat{\boldsymbol{\beta}}^{\mathsf{e}}(\tau) = \left(\sum_{i=1}^{n} \sum_{t=1}^{T} \widehat{\omega}_{i,t}(\tau) \boldsymbol{x}_{it} \boldsymbol{x}_{it}^{\mathsf{T}} \right)^{-1} \left(\sum_{i=1}^{n} \sum_{t=1}^{T} \widehat{\omega}_{i,t}(\tau) \boldsymbol{x}_{it} y_{it} \right),$$

where $\widehat{\omega}_{it}(\tau) = \left| \tau - \mathbf{1}(y_{it} < \boldsymbol{x}_{it}^{\mathsf{T}} \widehat{\boldsymbol{\beta}}^{\mathsf{e}}(\tau)) \right|$.

Expectile Regression with Random Effects (ERRE)

If $W = \text{diag}(\omega_{11}(\tau), \ldots \omega_{nT}(\tau))$, set

$$\overline{W} = \mathbb{E}(W), H = \boldsymbol{X}^\mathsf{T}\overline{W}\boldsymbol{X} \text{ and } \Sigma = \boldsymbol{X}^\mathsf{T}\mathbb{E}(W\varepsilon\varepsilon^\mathsf{T}W)\boldsymbol{X}.$$

and then

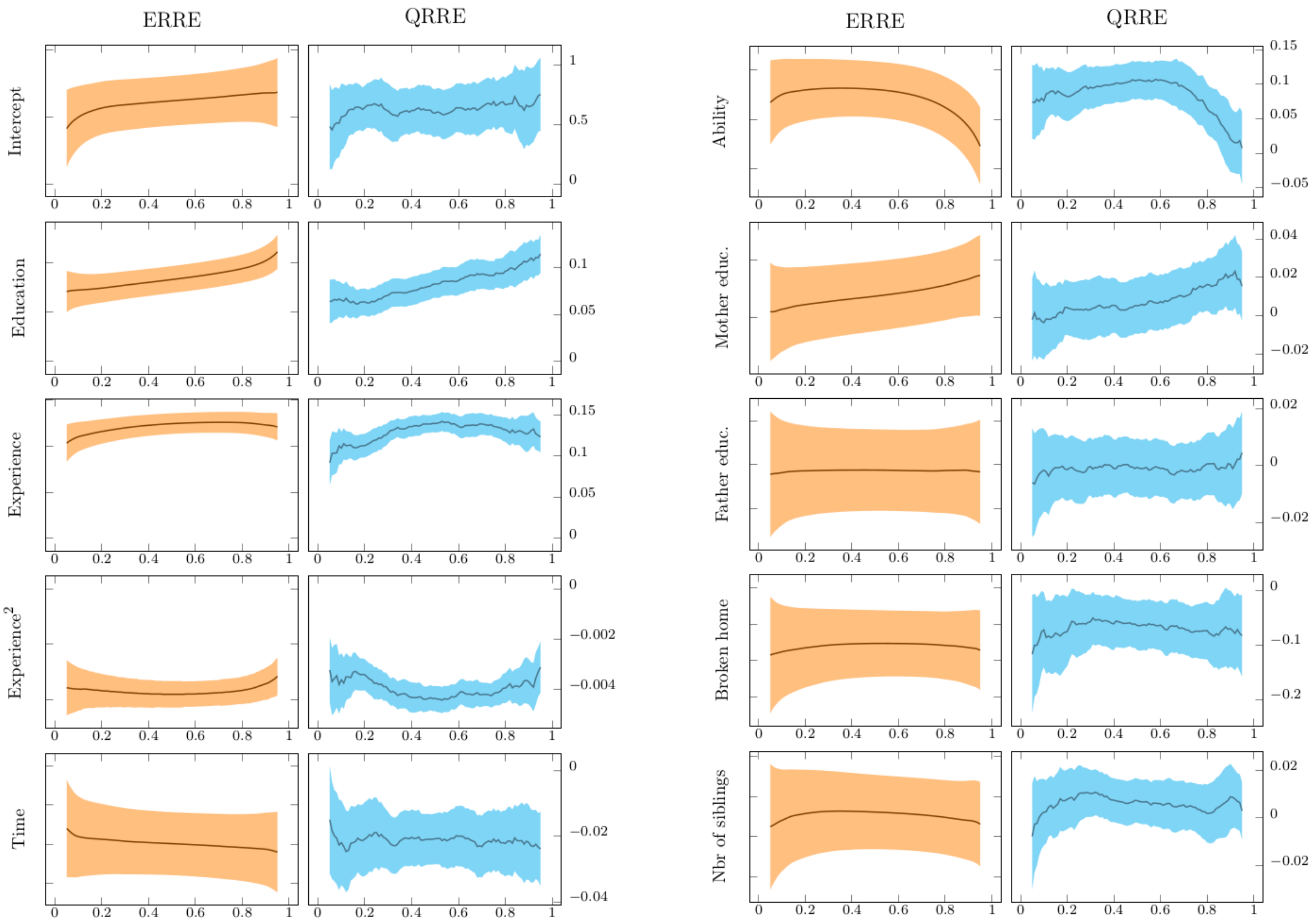
$$\sqrt{nT}\{\widehat{\boldsymbol{\beta}}^{\mathsf{e}}(\tau) - \boldsymbol{\beta}^{\mathsf{e}}(\tau)\} \xrightarrow{\mathcal{L}} \mathcal{N}(0, H^{-1}\Sigma H^{-1}),$$

see Barry *et al.* (2016) Quantile and Expectile Regression for random effects model.

See, for expectile regressions, with R,

```
> library(expectreg)
> fit <- expectreg.ls(rent_euro ~ area, data=munich, expectiles=.75)
> fit <- expectreg.ls(rent_euro ~ rb(area,"pspline"), data=munich,
    expectiles=.75)
```

Application to Real Data



Extensions

The mean of Y is $\nu(F_Y) = \displaystyle\int_{-\infty}^{+\infty} y dF_Y(y)$

The quantile of level τ for Y is $\nu_\tau(F_Y) = F_Y^{-1}(\tau)$

More generaly, consider some functional $\nu(F)$ (Gini or Theil index, entropy, etc), see Foresi & Peracchi (1995) The Conditional Distribution of Excess Returns

Can we estimate $\nu(F_{Y|\boldsymbol{x}})$?

Firpo *et al.* (2009) Unconditional Quantile Regressions suggested to use influence function regression

Machado & Mata (2005) Counterfactual decomposition of changes in wage distributions and Chernozhukov *et al.* (2013) Inference on counterfactual distributions suggested indirect distribution function.

Influence function of index $\nu(F)$ at y is

$$IF(y,\nu,F) = \lim_{\varepsilon\downarrow 0} \frac{\nu((1-\epsilon)F+\epsilon\delta_y)-\nu(F)}{\epsilon}$$