# 12. ω-regular languages

Computer-Aided Verification


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# Overview

- Recap (safety properties & NFAs) + examples
- $\omega$-regular languages/properties
- Nondeterministic Büchi automata (NBAs)

- See [BK08] Sections 4.3–4.4, 5.2

# Recap

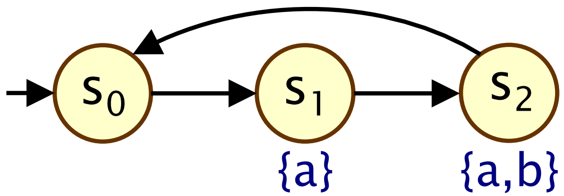
- Model checking regular safety property $P_{safe}$ on LTS $M$
  - 1. find NFA $\mathcal{A}$ representing the bad prefixes of $P_{safe}$
  - 2. build LTS-NFA product $M \otimes \mathcal{A}$
  - 3. check no "accept" state is reachable in $M \otimes \mathcal{A}$

$$\boxed{M \models P_{safe} \quad \Leftrightarrow \quad M \otimes \mathcal{A} \models \Box\neg \text{accept}}$$

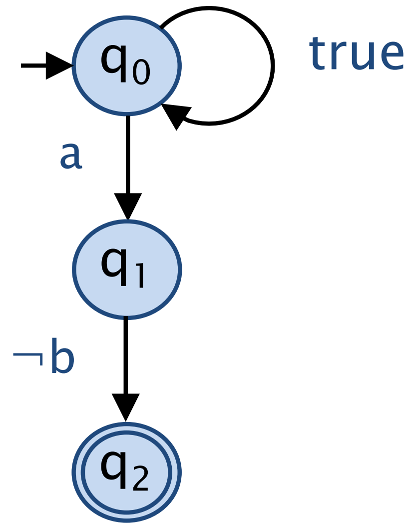
# Example

- Model check $\Box(a \rightarrow \bigcirc b)$ on LTS M

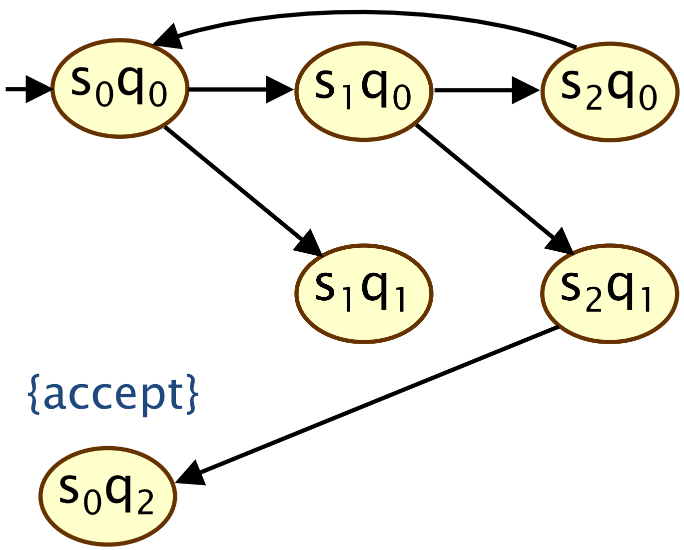
LTS M




NFA $\mathcal{A}$




Product M ⊗ $\mathcal{A}$

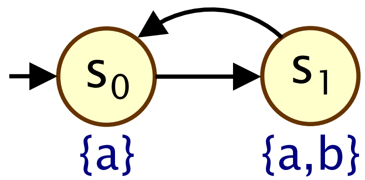



$M \not\models \Box(a \rightarrow \bigcirc b)$

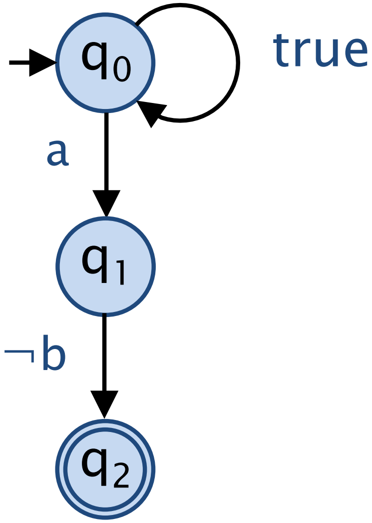
# Example

- Model check $\Box(a \rightarrow \bigcirc b)$ on LTS M

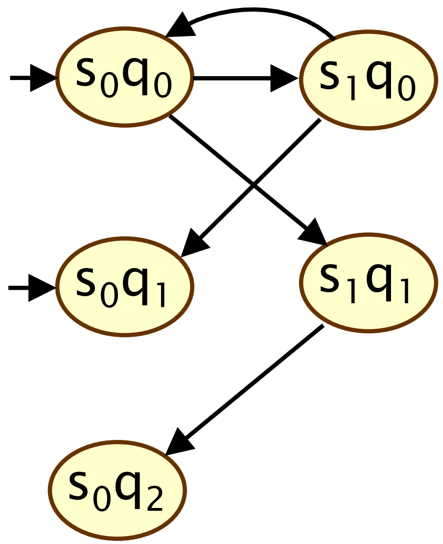
LTS M




NFA $\mathcal{A}$




Product $M \otimes \mathcal{A}$



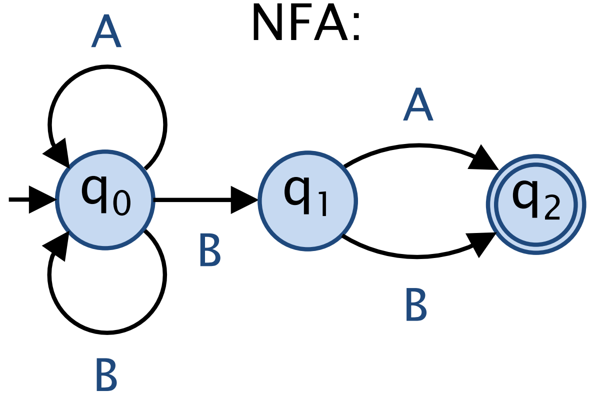

$M \not\models \Box(a \rightarrow \bigcirc b)$

# Beyond regular languages

- So far: regular safety properties (e.g. in LTL)
  - ("bad behaviour happens in finite time")
- What about other properties (e.g. in LTL)?
  - liveness, e.g. "for every request, an ack eventually follows"
  - fairness, e.g. "every enabled process is scheduled infinitely often"
- Regular languages:
  - e.g. "penultimate symbol is B"
- This lecture:
  - ω-regular languages/expressions
  - nondeterministic Büchi automata

Regexp: $(A+B)^*B(A+B)$

NFA:



# ω–regular expressions

- A regular expression $E$ over alphabet $\Sigma$ takes the form:
  - $E ::= \varnothing \mid \varepsilon \mid A \mid E + E \mid E.E \mid E^*$ (where $A \in \Sigma$)
- An ω–regular expression over $\Sigma$ takes the form:
  - $G = E_1.(F_1)^\omega + E_2.(F_2)^\omega + \ldots + E_n.(F_n)^\omega$
  - where $E_i$ and $F_i$ are regular expressions with $\varepsilon \notin \mathcal{L}(F_i)$
- Example: $(A+B+C)^*(B+C)^\omega$ for $\Sigma = \{ A, B, C \}$
- $\mathcal{L}_\omega(G) \subseteq \Sigma^\omega$ is the language of an ω–regular expression $G$
  - $\mathcal{L}_\omega(G) = \mathcal{L}(E_1).\mathcal{L}(F_1)^\omega \cup \mathcal{L}(E_2).\mathcal{L}(F_2)^\omega + \ldots + \mathcal{L}(E_n).\mathcal{L}(F_n)^\omega$
  - where $\mathcal{L}(E)$ is the language of regular expression $E$
  - and $\mathcal{L}(E)^\omega = \{ w^\omega \mid w \in \mathcal{L}(E) \}$

# $\omega$–regular languages/properties

- Language $\mathcal{L} \subseteq \Sigma^\omega$ is an $\omega$–regular language if
  - $\mathcal{L} = \mathcal{L}_\omega(G)$ for some $\omega$–regular expression G

- $P \subseteq (2^{AP})^\omega$ is an $\omega$–regular property
  - if P is an $\omega$–regular language over $2^{AP}$

- Example (for AP = {wait,crit})
  - e.g. $((\neg crit)^* crit)^\omega$
  - "crit is true infinitely often"

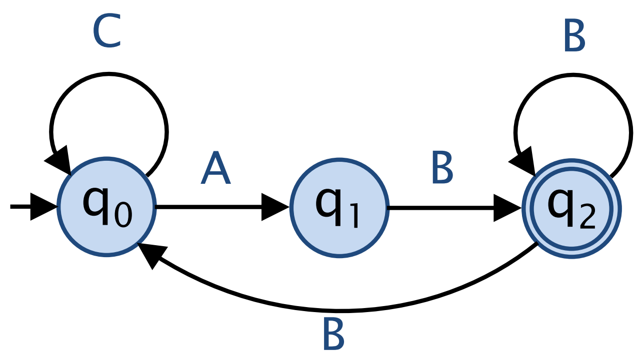
crit – shorthand for {{crit},{wait,crit}}

$\neg$crit – shorthand for {$\varnothing$,{wait}}

- Note:
  - any regular safety property is an $\omega$–regular property
  - all linear–time properties seen so far are $\omega$–regular
  - any LTL formula corresponds to an $\omega$–regular property

# Nondeterministic Büchi automata

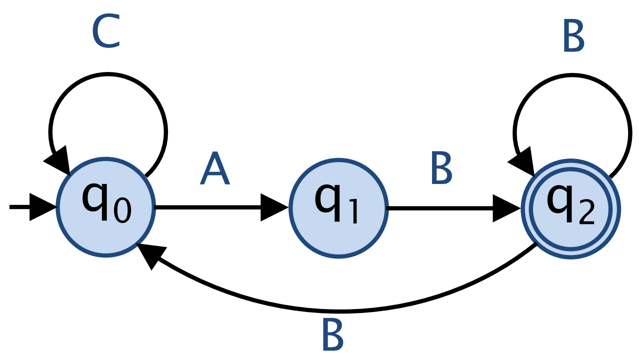
- A nondeterministic Büchi automaton (NBA) is:
  - a tuple $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$
- where:
  - $Q$ is a finite set of states
  - $\Sigma$ is an alphabet
  - $\delta : Q \times \Sigma \rightarrow 2^Q$ is a transition function
  - $Q_0 \subseteq Q$ is a set of initial states
  - $F \subseteq Q$ is a set of "accept" states




- i.e. NBAs are identical, syntactically, to NFAs
  - the difference is the acceptance condition…
  - "accept" states need to be visited infinitely often

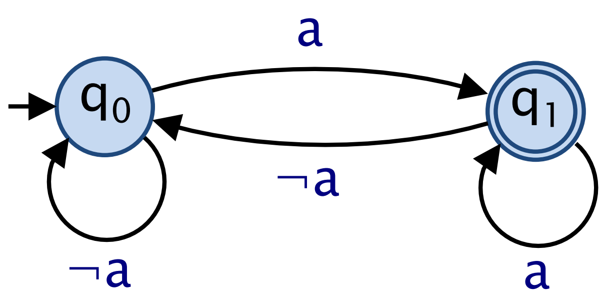
# Language of an NBA

- A run of NBA $\mathcal{A}$ on an infinite word $w = A_0A_1A_2\ldots$ is:
  - a sequence of automata states $q_0q_1q_2\ldots$ such that:
  - $q_0 \in Q_0$ and $q_i \xrightarrow{A_i} q_{i+1}$ for all $i \geq 0$
- An accepting run is a run with $q_i \in F$ for infinitely many i
- The language of $\mathcal{A}$, denoted $\mathcal{L}_\omega(\mathcal{A})$ is:
  - the set of all (infinite) words accepted by $\mathcal{A}$



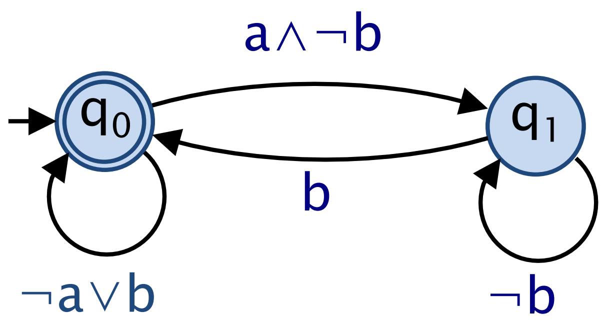
# Example

- "infinitely often a" – $\Box\Diamond a$



# Example

- "b always follows a" – $\square(a \rightarrow \Diamond b)$



# Nondeterministic Büchi automata

- Represent ω-regular languages
  - same expressivity as ω-regular expressions
- Can be built systematically from ω-regular expressions
- Are an example of ω-automata
  - there are many others: Rabin, Streett, Muller, …
- Are closed under intersection
- Are closed under complementation
- Are more expressive than deterministic Büchi automata
  - unlike for finite automata

# Example

- "eventually always a" – $\Diamond\Box a$