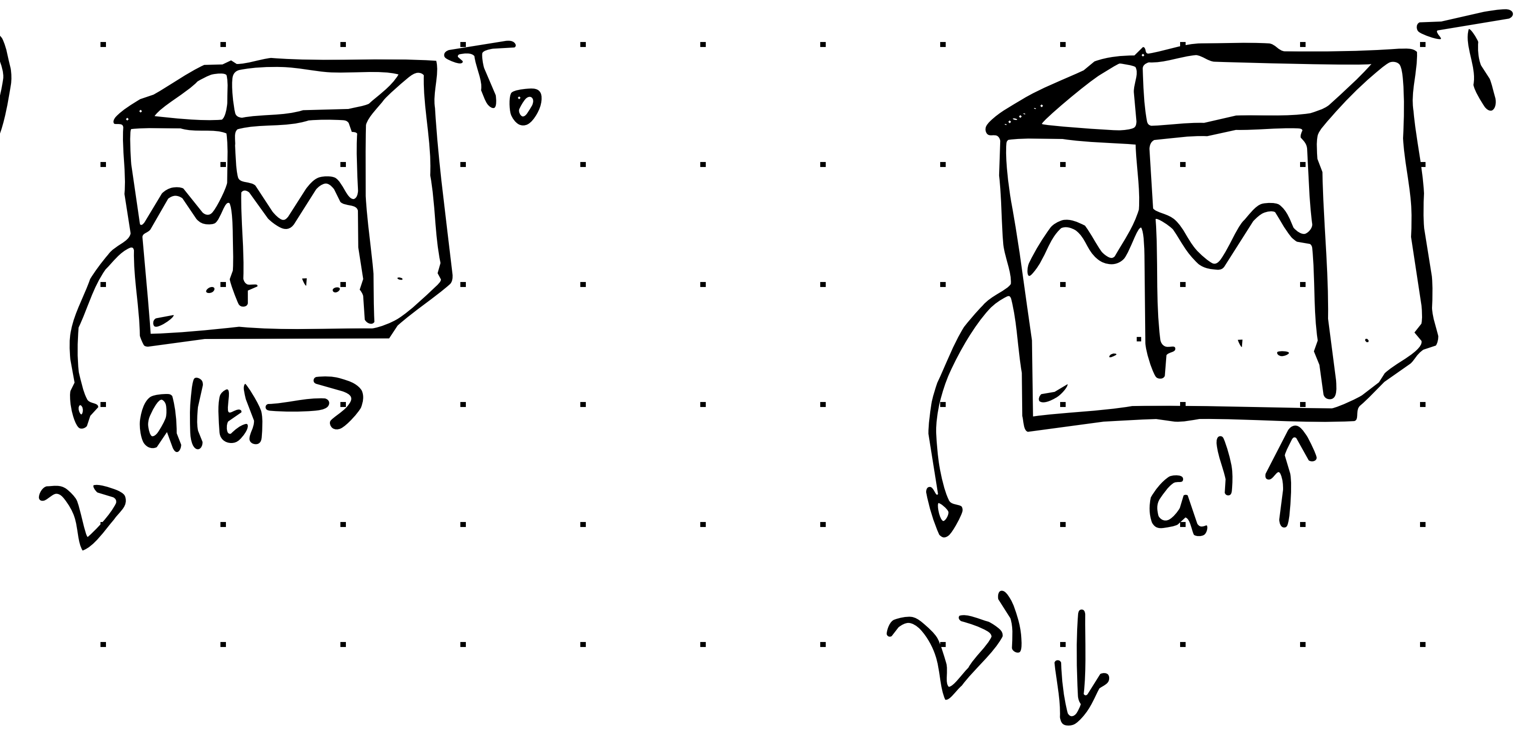
1) Redshifting CMB vs. Non-relativistic Gas

(a)



for a photon $p = \frac{h}{\lambda} = \frac{E}{c} = \frac{h\nu}{c} \propto a^{-1} \Rightarrow \nu \propto a^{-1}$

$E = pc$ since $p = \frac{h}{\lambda}$, $p \propto \frac{1}{L}$ since $\lambda \propto L$

total Energy of photon gas in conserved adiabatic:

$$u(\nu,T)\,d\nu = \frac{8\pi h}{c^3} \frac{\nu^3}{e^{h\nu/kT} - 1} d\nu$$

$a \propto \frac{1}{\nu}$ so when $a \to a'$, $\nu \to \nu'$

$\frac{\nu'}{a'} = \frac{\nu}{a}$

$$M(\nu,T)\,d\nu = \frac{8\pi h}{c^3} \frac{1}{e^{h\nu/kT} - 1} \left(\frac{a}{a'}\right)^3 d\nu$$

since $\nu \propto a^{-1}$

as the box stretches, the frequency decreases (wavelength increases).

as the box contracts, the frequency increases (wavelength decreases)

shape is retained

$\frac{h\nu}{kT} = $ constant

we can also look at the wavelength.

$$B(\lambda,T)\,d\lambda = \frac{2hc^2}{\lambda^5} \frac{d\lambda}{e^{\frac{hc}{k\lambda T}} - 1} \left(\frac{\lambda'}{\lambda}\right)^3$$

wavelengths are stretched by a

$\lambda' = a\lambda$

the shape is unaltered if $\lambda T = $ constant, so

$\cancel{\lambda} T_0 = a \cancel{\lambda} T'$

so $T \propto \frac{1}{a} \Leftarrow \therefore T = \frac{T_0}{a}$

(b) non relativistic ideal gas

$V = a\,E$

$p \propto a^{-1}$

$p = mv$ $\quad p' = \frac{p}{a}$

$v = \frac{p}{m}$

- for an adiabatic process for ideal gas $T \propto \frac{1}{a^2}$

$$f(v)\, d^3v = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_BT}\right) d^3v$$

$$E = \frac{p^2}{2m} = \frac{m^2 v^2}{2m}$$

integrating over a solid angle: $f(v) = \left[\frac{m}{2\pi k_B T}\right]^{3/2} 4\pi v^2 \exp\left(\frac{-mv^2}{2k_BT}\right)$

· it does not retain it's shape. $f(p')dp' = 4\pi \left(\frac{p'^2}{(2\pi m k_B T')^{3/2}}\right) e^{\frac{-p'^2}{2mk_BT'}} dp'$

$$f_E(E) = 2\sqrt{\frac{E}{\pi}} \left[\frac{1}{k_BT}\right]^{3/2} \exp\left(\frac{-E}{k_BT}\right)$$

and for ideal gas for adiabatic expansion

(c) for non-relativistic ideal gas

$E = \frac{3}{2} N k_B T$

$E \propto \frac{1}{a^2}$ $\qquad c = \nu\lambda$

for photon gas $E = h\nu$
$= h\frac{c}{\lambda}$

$E \propto a^{-1}$

since $E_k \propto p^2 \propto \frac{1}{a^2}$

and $\Delta U = -W = -p\Delta V$

in a normal adiabatic expansion,

$E \propto -p\,\Delta V$

* the total energy decreases w/ more than just "$p\Delta V$"

② CMB Temperature Dipole

show $T' = \dfrac{T}{\gamma(1-\beta\cos\theta')}$

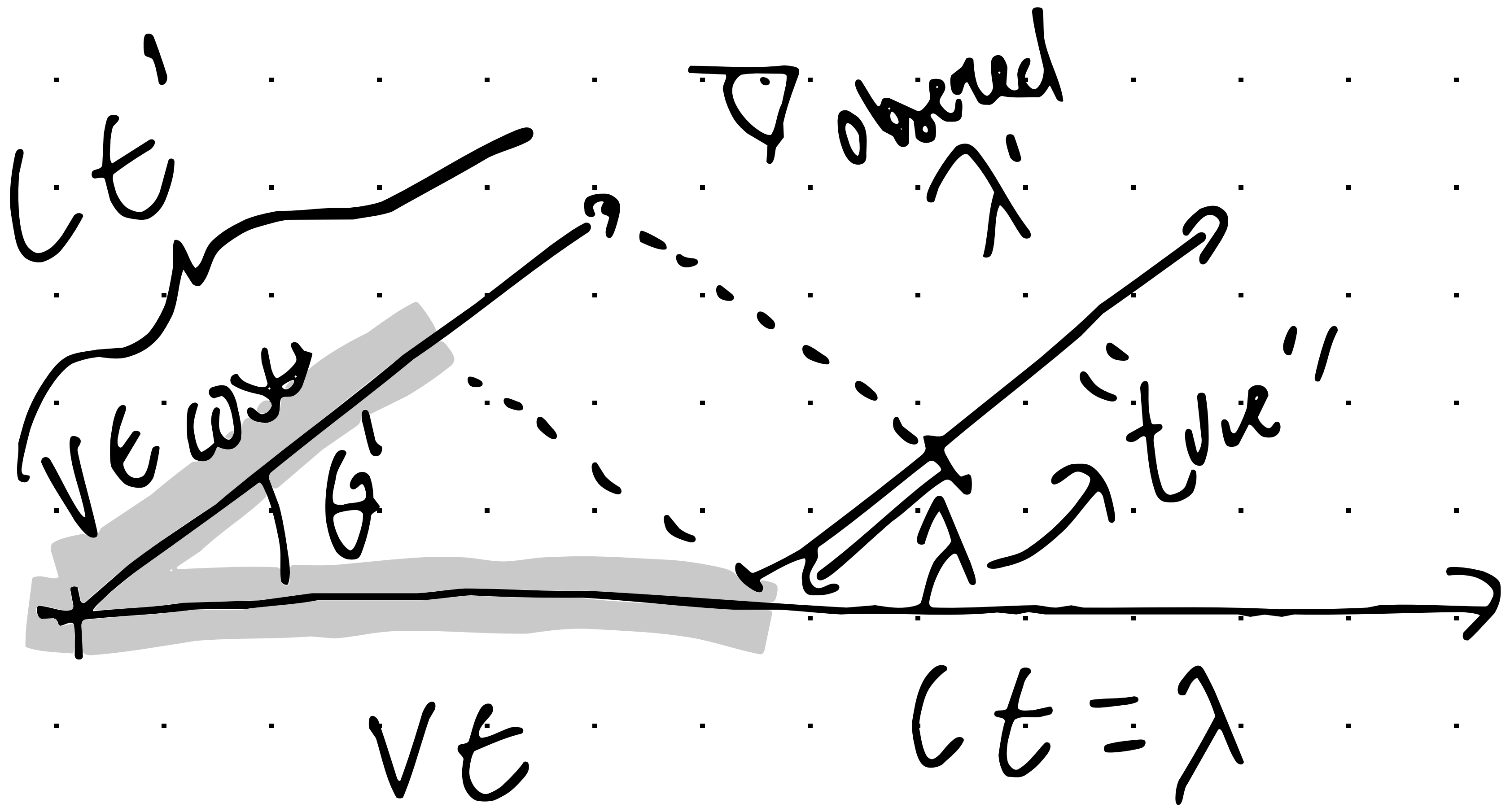
ⓐ

$$\beta \equiv \frac{v}{c}$$

$$\gamma \equiv (1-\beta^2)^{-1/2}$$

doppler boost: $\lambda' = \lambda - v\dfrac{\lambda}{c}$

$$t' = \gamma t$$



since $c = f\lambda$, $f = \dfrac{c}{\lambda}$

$$f' = \frac{f}{\gamma\left(1-\frac{v}{c}\right)} = \frac{f}{\gamma(1-\beta)}$$

$$\lambda' = ct' - vt\cos\theta'$$

$$= \gamma t(c - v\cos\theta')$$

$$= \gamma\frac{\lambda}{c}(c - v\cos\theta')$$

$$\lambda' = \gamma\lambda(1-\beta\cos\theta')$$

via weins law: $T \propto \dfrac{1}{\lambda}$

$$\frac{1}{T'} = \frac{\gamma(1-\beta\cos\theta')}{T}$$

$$T' = \frac{T}{\gamma(1-\beta\cos\theta')}$$

(b) $v = 368$ km/s

rest frame of galaxies coincides with that of the cmb $\sigma_z = 0.1$ $\langle z \rangle = 0.5$

$$n_{gal}(z) = \frac{1}{0.1\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{z-0.5}{0.1}\right)^2\right]$$

$$SNR \sim \frac{N_{gals}}{\sqrt{N_{gals}}}$$

dipole induced shift in $\langle z \rangle$?

$$T' = \frac{T}{\gamma(1+\beta\cos\theta')}$$

~ 1

$\gamma = (1-\beta^2)^{-1/2}$ $\beta = \frac{368}{10^5}$

$\sim 10^{-3}$

$T' \sim \underbrace{0.99}_{10^{-2}\text{ precision.}} T$

to achieve a precision of $\sim 10^{-2}$

and given that the expectation value is

$z = 0.5 \pm 0.1$

we want an $SNR \sim 100$

so $(100)^2 = \frac{N_{gals}^2}{N_{gals}}$

$N_{gals} \sim 10^4$

for $SNR \sim 100$

③ Rotation curves for sphere less Disk

$M = 10^{12} M_\odot$, 50 kpc cutoff radius

Case 1: infinitely thin disk

$$\Sigma(R) = \Sigma_0 \exp\left(\frac{-R}{R_d}\right)$$

$R_d = 8.5$ kpc

for a infinitely thin disk

→ to find the central surface density Σ_0

$$M = \int_0^{R_{max}} \Sigma(R)\, \underbrace{2\pi R}_{\to \text{thin ring}}\, dR$$

$$M = \Sigma_0\, 2\pi \int_0^{R_{max}} \exp\left(\frac{-R}{R_d}\right) R\, dR$$

$$10^{12} = \Sigma_0\, 2\pi \underbrace{\left[-R_d(R+R_d)\, e^{-R/R_d}\right|_0^{50\,\text{kpc}}}$$

where $R_d = 8$ kpc ~ 63 kpc^2

$$\frac{V^2}{R} = \frac{d\Phi}{dR} \to \text{we need this}$$

→ for a thin disk, Binney & Tremain determined that

$$\Phi(R) = -\pi G \Sigma_0 R\left[I_0(y) K_1(y) - I_1(y) K_0(y)\right]$$

where $y \equiv \frac{R}{2R_d}$

Σ_0 → central surface density.

where I & K are the modified Bessel functions

$$\Sigma_0 = \frac{10^{12} M_\odot}{2\pi\, 63\ \text{kpc}^2} \sim 2.5 \times 10^3 \frac{M_\odot}{pc^2}$$

Accordingly

$$V_{disk}(R) = \left[R\frac{d\Phi}{dR}\right]^{1/2}$$

$$V_{disk}(R) = \left\{4\pi G \Sigma_0 R_d\, y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y)\right]\right\}^{1/2}$$

Case 2 $\quad \frac{V^2(R)}{R} = \frac{G\,M(<R)}{R^2}$ $\qquad M(<R) = \int_0^R \Sigma(R)\, 2\pi R\, dR$

$V_{sphere}(R) = \sqrt{\frac{G\,M(<R)}{R}}$ where $M(<R) = \int_0^R \Sigma_0 \exp\left(\frac{-R}{R_d}\right) 2\pi R\, dR$

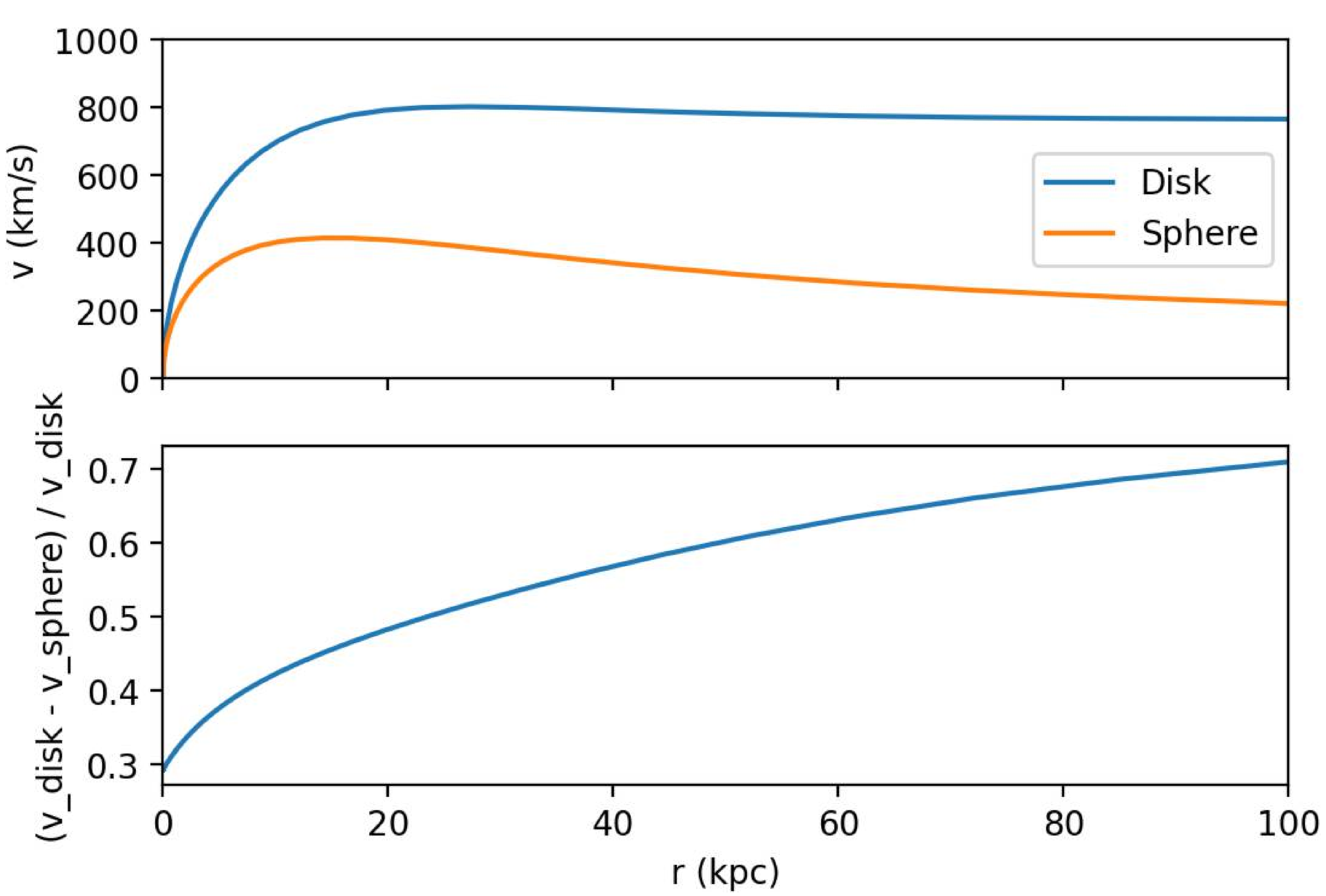
$= 2\pi\Sigma_0 \left[R_d^2 - (R_d R + R_d^2)\, e^{-R/R_d}\right]$

where, taking from last question:

$\Sigma_0 \sim 2.5 \times 10^3 \frac{M_\odot}{pc^2} \quad R_d \sim 8.5 \times 10^3\, pc$

is this what they mean?

- We see that for the disk, it's flatter and v_{circ} is higher.
- the relative error grows as $R \uparrow$



$M_{disk}(R) = \cancel{\Sigma_0 2\pi}\left[-R_d(R + R_d)\, \cancel{e^{-R/R_d}}\right]$

$M_{sphere}(R) = \cancel{\Sigma_0 2\pi}\left[R_d^2 - (R_d R + R_d^2)\, \cancel{e^{-R/R_d}}\right]$

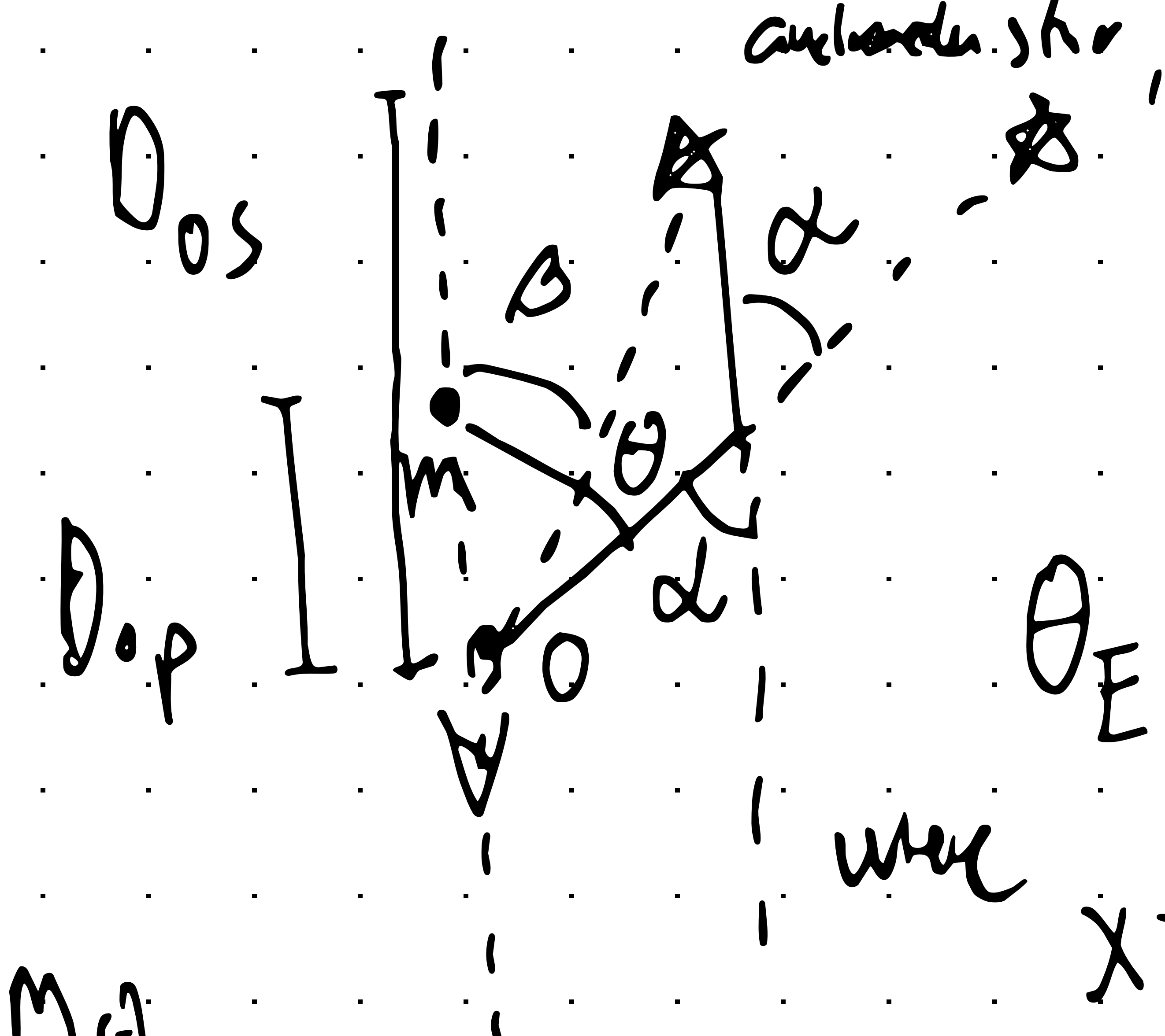
$\frac{M_{disk}}{M_{sphere}} = \frac{-R_d R - R_d^2}{\cancel{R_d^2} - R_d R - \cancel{R_d^2}} = \frac{R_d R - R_d^2}{-R_d R}$

(4) Lensing Machos

suppose all DM are Machos. $\quad d_{Andromeda-MW} = 700\,kpc$

DM sphere, $V_{circ} = 220\,km/s$ at its surface

→ calculate the probability that a star in the Andromeda galaxy will be lensed by one other Macho.
& lensing timescale



let:

$M_{tot} = 10^{12} M_\odot$

macho mass $m \sim 1\, M_\odot$

distance observer → source $D_{os} \sim 700\,kpc$

distance observer → plane $D_{op} \sim 50\,kpc$

· we know from lecture that the lense area or einstein ring is θ_E

$$\theta_E = 0.007 \left(\frac{M_{Lense}}{0.5}\right)^{1/2} \left(\frac{D_{op}}{4\,kpc}\right)^{-1/2} \left(\frac{1-x}{0.5}\right)^{1/2}$$

where $x = \frac{D_{op}}{D_{os}} \simeq 0.07 \quad m_{Lense} = 1\, M_\odot$

$$\theta_E \sim 0.0038'' (60)(60)\left(\frac{\pi}{180}\right) \sim 2 \times 10^{-8}\ \text{rads}$$

$$\rightarrow p\,(\text{star getting lensed}) \sim \frac{\Delta\Omega}{4\pi}$$

$\Delta\Omega$ → total lensing area

10^{12} → # machos

$$\Rightarrow \Delta\Omega \sim 10^{12}\, \pi\, (2 \cdot 10^{-8})^2 \sim 0.001$$

$$p = \frac{0.001}{4\pi} \sim 8 \times 10^{-5} \sim 10^{-6}$$

→ typical lensing time scale

from lecture, we know

$$t_{lense} = \frac{\theta_E D_{op}}{V_{macho}}$$

D_{op} → plane - observer distance

V_{macho} → 220 km/s

here we assume

that the macho's movement

is non-negligible while

the star's movement is negligible

→ andromeda stars

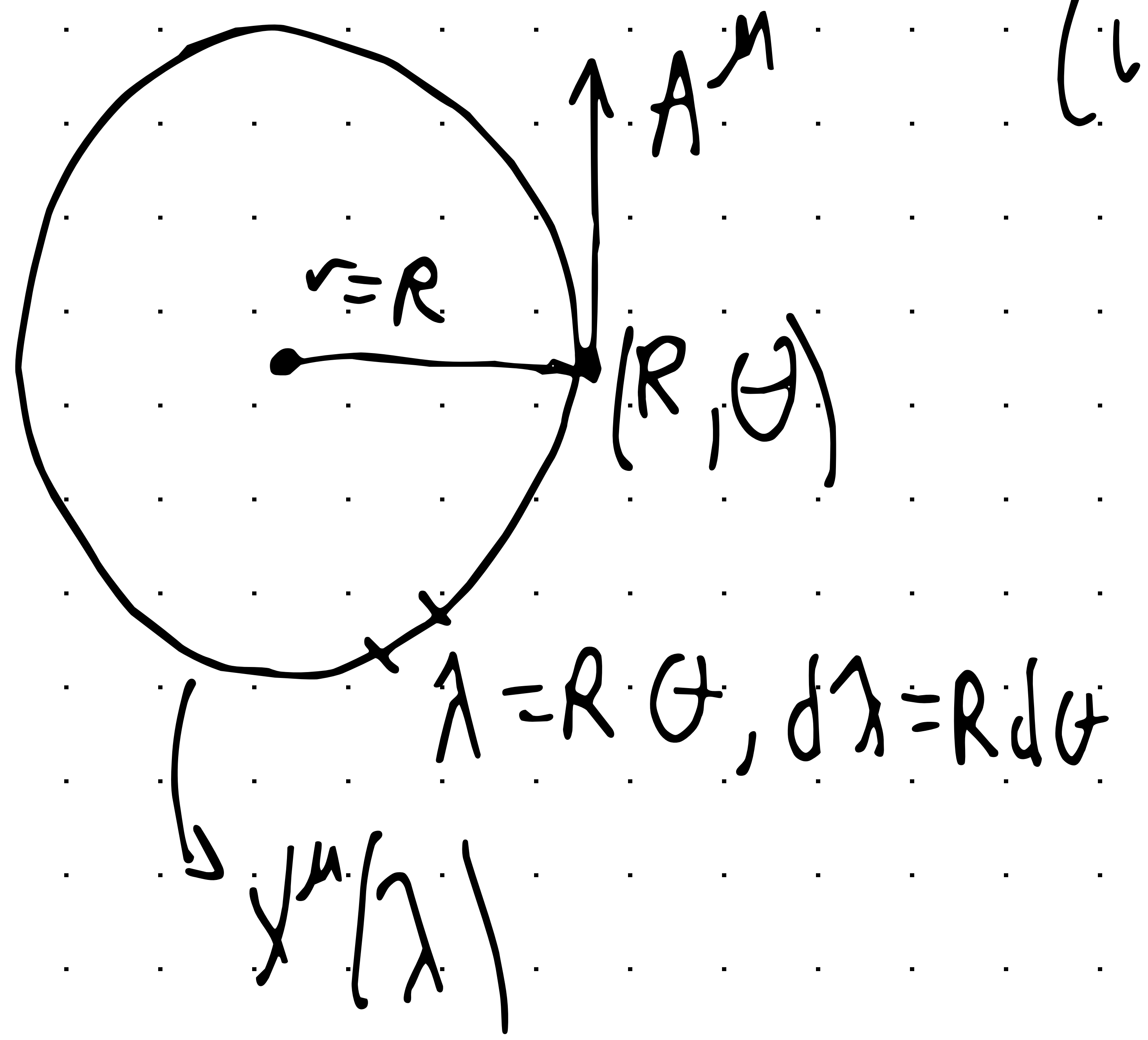
where $\theta_E D_{op} = 2.85\,AU \left(\frac{M}{0.5M_\odot}\right)^{1/2} \left(\frac{D_{op}}{4kp}\right)^{1/2} \left(\frac{x(1-x)}{0.25}\right)^{1/2}$ derived in lecture.

$\cong 7.2AU \sim 10^9$ km

$$t_{lense} \sim \frac{10^9}{220} \sim 4 \times 10^6 \text{ s}$$

~ 52 days

(5) Transformation of Contravariant and Covariant Vectors



(i) use $A^\mu = \dfrac{dx^\mu}{d\lambda}$ to find the components A^r and A^θ in polar coordinates

- recall, a contravariant A^μ (upper index) represents the direction & magnitude.

$\Rightarrow$ under transformations, they change inversely to the basis vectors

in polar coordinates: $A^r = \dfrac{dr}{d\lambda} = 0$

- since $r=R$ is constant for all points in a circle.
- tangent vector not pointing radial direction. or inward or outward.

as you move along the circle, θ changes

$\theta = \dfrac{\lambda}{R}$, $A^\theta = \dfrac{d\theta}{d\lambda} = \dfrac{1}{R}$

in polar coordinates $\Rightarrow A^\mu = (A^r, A^\theta) = \left(0, \dfrac{1}{R}\right)$

Covariant: change in response to deformation of surface

Contravariant: maintain form, but components change as the coordinates deform.

(ii) the transformation matrix is the jacobian

polar $\rightarrow$ Cartesian $\dfrac{\partial x'^\mu}{\partial x^\nu}$?

we know that Cartesian $\rightarrow$ polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

(iii) transform the vector to find (A^x, A^y)

$$\begin{pmatrix} A^x \\ A^y \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} A^r \\ A^\theta \end{pmatrix} = \begin{matrix} A^r\cos\theta - A^\theta r\sin\theta \\ A^r\sin\theta + A^\theta r\cos\theta \end{matrix}$$

where $r=R$, we check the tangent vector.

$$(A^r, A^\theta) = \left(0, \frac{1}{R}\right)$$

$$\begin{cases} A^x = 0\cdot\cos\theta - \frac{1}{R}\cdot R\sin\theta = -\sin\theta \\ A^y = 0\sin\theta + \frac{1}{R} R\cos\theta = \cos\theta \end{cases}$$

$$(A^x, A^y) = (-\sin\theta, \cos\theta)$$

(iv) the length $\|A\| = \sqrt{\underbrace{g_{\mu\nu}A^\mu}_{\text{contravariant}} A^\nu}$ where the metric tensor $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the Cartesian coordinates:

$$\|A\| = \sqrt{g_{xx}(A^x)^2 + g_{yy}(A^y)^2} \quad \text{where } g_{xx}=1,\ g_{yy}=1$$

$$= \sqrt{(A^x)^2 + (A^y)^2}$$

$$= \sqrt{(-\sin\theta)^2 + (\cos\theta)^2} = \sqrt{\sin^2 + \cos^2\theta} = \sqrt{1} =$$

in polar:

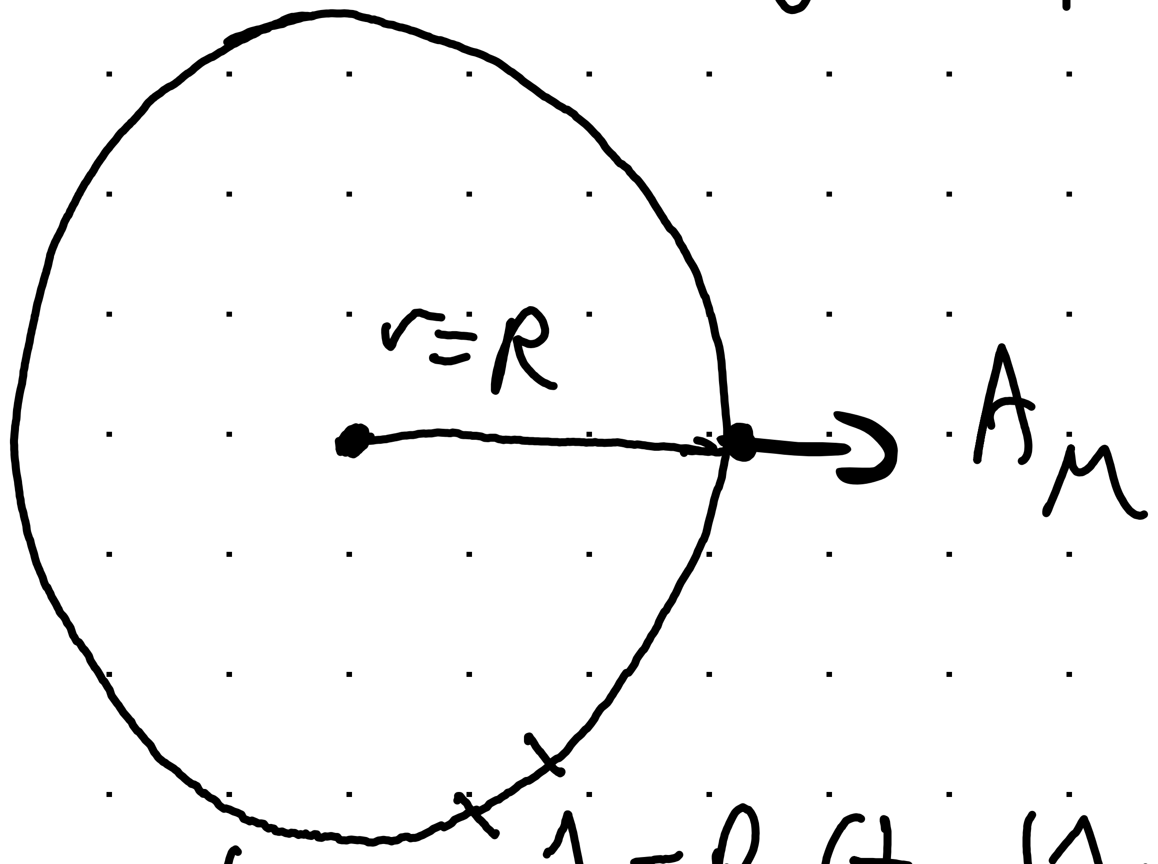
$$\|A\| = \sqrt{g_{rr}(A^r)^2 + g_{\theta\theta}(A^\theta)^2} \qquad g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$= \sqrt{1(A^r)^2 + r^2(A^\theta)^2}$$

$$A^\mu = (A^r, A^\theta) = \left(0, \frac{1}{R}\right), \text{ so } \|A\| = \sqrt{0 + r^2\left(\frac{1}{R}\right)^2}, \ r=R$$

for polar $\|A\| = \sqrt{R^2 \frac{1}{R^2}} = \sqrt{1} = 1$ ✓

Covariant vector.



$\lambda = R\theta,\ d\lambda = R\,d\theta$

$\phi(x^\mu) = R$

$\rightarrow$ $A_\mu = \dfrac{\partial \phi}{\partial x^\mu}$

· in polar coordinates (r, θ),
the surface is parametrized by
the function $\phi(r,\theta) = r - R$

$\rightarrow$ circle with radius R, $r = R$

$\rightarrow$ to find covariant components $A_\mu = (A_r, A_\theta)$

$$A_r = \frac{\partial \phi}{\partial r} = \frac{\partial (r-R)}{\partial r} = 1$$

$$A_\theta = \frac{\partial \phi}{\partial \theta} = \frac{\partial (r-R)}{\partial \theta} = 0$$

$$(A^r, A^\theta) = (1, 0)$$

$\rightarrow$ the transformation matrix, we need the inverse jacobian.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\frac{\partial x^\mu}{\partial x^{\mu'}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\dfrac{\sin\theta}{r} & \dfrac{\cos\theta}{r} \end{pmatrix}$$

$\to (A^r, A^\theta) \to (A^x, A^y)$ $(A^r, A^\theta) = (1, 0)$

$$\begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} \begin{pmatrix} A^r \\ A^\theta \end{pmatrix} \to$$

$$A_x = A_r \cos\theta + A_\theta \sin\theta = \cos\theta + 0 \sin\theta = \cos\theta$$

$$A_y = -A_r \frac{\sin\theta}{r} + A_\theta \frac{\cos\theta}{r} = \frac{-\sin\theta}{r} \cdot 1 + 0 = \frac{-\sin\theta}{r}$$

cartesian $A_{\mu'} = \left(\cos\theta, \frac{-\sin\theta}{r}\right)$

$\to$ length in cartesian

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$||A|| = \sqrt{g^{\mu\nu} A_\mu A_\nu}$$

$g_{xx} = 1$ $g_{yy} = 1$

$$\Rightarrow \sqrt{1 \cdot (\cos\theta)^2 + 1 \cdot \left(\frac{-\sin\theta}{r}\right)^2}$$

$$||A|| = \sqrt{\cos^2\theta + \frac{\sin^2\theta}{r^2}}$$

note, for $r = R = 1$, $||A|| = 1$

$\to$ length in polar $g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

$$||A|| = \sqrt{g^{\mu\nu} A_\mu A_\nu}$$

$$= \sqrt{g^{rr} A_r^2 + g^{\theta\theta} A_\theta^2}$$

$$= \sqrt{1 \cdot 1^2 + r^2 \cdot 0^2} = \sqrt{1} = 1$$

So in polar $||A|| = 1$

note, in cartesian, it has an is what restriction, which is met when $R = 1$