Question 1 Geodesic Equation - Flat 2D Space

→ in polar coordinates (r, θ), the metric is given by

$$ds^2 = dr^2 + r^2 d\theta^2$$

(a) show that when computing the Christoffel symbols that the two geodesic equations are

$$\frac{d^2 r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 \quad \text{and} \quad \frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = 0$$

recall the metric tensor

$$g_{\mu\nu} = \begin{pmatrix} g_{rr} & g_{r\theta} \\ g_{\theta r} & g_{\theta\theta} \end{pmatrix} = \begin{array}{c} r \\ \theta \end{array} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

the Christoffel symbols are:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}}\right)$$

→ we need to basically solve for

$$\Gamma^{r}_{\mu\nu} = \begin{bmatrix} \Gamma^{r}_{rr} & \Gamma^{r}_{r\theta} \\ \Gamma^{r}_{\theta r} & \Gamma^{r}_{\theta\theta} \end{bmatrix} \quad \text{and} \quad \Gamma^{\theta}_{\mu\nu} = \begin{bmatrix} \Gamma^{\theta}_{rr} & \Gamma^{\theta}_{r\theta} \\ \Gamma^{\theta}_{\theta r} & \Gamma^{\theta}_{\theta\theta} \end{bmatrix}$$

→ note, only g_{rr} and $g_{\theta\theta}$ are non-0.

restating the general formula: $\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{rr}}{\partial r} + \frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r} \right) = \frac{1}{2}(1)\left[0+0-0\right] = 0$$

$$\underbrace{\Gamma^r_{r\theta} = \Gamma^r_{\theta r}}_{\text{symmetry}} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{rr}}{\partial \theta} - \frac{\partial g_{r\theta}}{\partial r} \right) = 0$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr} \left(\frac{\partial g_{r\theta}}{\partial \theta} + \frac{\partial g_{r\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial r} \right) = \frac{1}{2}(1)(-2r) = \boxed{-r}$$

$\rightarrow$ now for $\Gamma^\theta_{\mu\nu}$, $g^{\theta r} = 0$, $g_{r\theta} = 0$, $g^{rr} = 1$, $g^{\theta\theta} = \frac{1}{r^2} = r^{-2}$ $\qquad g_{\theta\theta} = r^2$

$$\Gamma^\theta_{rr} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{r\theta}}{\partial r} + \frac{\partial g_{r\theta}}{\partial r} - \frac{\partial g_{rr}}{\partial \theta} \right) = \frac{1}{2}\left(\frac{1}{r^2}\right) \cdot (0+0-0) = 0$$

$$\Gamma^\theta_{r\theta} = \frac{1}{2} g^{\theta\theta} \left(\cancel{\frac{\partial g_{\theta r}}{\partial \theta}} + \frac{\partial g_{\theta\theta}}{\partial r} - \cancel{\frac{\partial g_{r\theta}}{\partial \theta}} \right) = \frac{1}{2}\left(\frac{1}{r^2}\right)(2r) = \boxed{\frac{1}{r}} = \Gamma^\theta_{\theta r}$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta} \left(\frac{\partial g_{\theta\theta}}{\partial \theta} + \frac{\partial g_{\theta\theta}}{\partial \theta} - \frac{\partial g_{\theta\theta}}{\partial \theta} \right) = \frac{1}{2}\left(\frac{1}{r^2}\right)(0) = 0$$

$$\Rightarrow \Gamma^r_{\mu\nu} = \begin{bmatrix} \Gamma^r_{rr} & \Gamma^r_{r\theta} \\ \Gamma^r_{\theta r} & \Gamma^r_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -r \end{bmatrix} \quad \text{and} \quad \Gamma^\theta_{\mu\nu} = \begin{bmatrix} \Gamma^\theta_{rr} & \Gamma^\theta_{r\theta} \\ \Gamma^\theta_{\theta r} & \Gamma^\theta_{\theta\theta} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{r} \\ \frac{1}{r} & 0 \end{bmatrix}$$

$\Rightarrow$ on to the geodesic equation: $\mu, \nu = \theta, r$

$$\frac{d^2 r}{ds^2} + \Gamma^r_{\mu\nu} \frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$$

$$\frac{d^2 r}{ds^2} + \cancel{\Gamma^r_{rr}\left(\frac{dr}{ds}\right)\left(\frac{dr}{ds}\right) + \Gamma^r_{r\theta}\left(\frac{dr}{ds}\right)\frac{d\theta}{ds}} \rightarrow 0 + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{ds}\right)\left(\frac{d\theta}{ds}\right) = 0$$

$$\frac{d^2 r}{ds^2} + \Gamma^r_{\theta\theta}\left(\frac{d\theta}{ds}\right)^2 = 0 \implies \frac{d^2 r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 \checkmark$$

$\Rightarrow$ now for $\frac{d^2\theta}{ds^2} + \Gamma^\theta_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0$

$$\frac{d^2\theta}{ds^2} + \cancel{\Gamma^\theta_{rr}\left(\frac{dr}{ds}\right)\left(\frac{dr}{ds}\right)} \rightarrow 0 + \Gamma^\theta_{r\theta}\frac{dr}{ds}\left(\frac{d\theta}{ds}\right) + \cancel{\Gamma^\theta_{\theta\theta}}\rightarrow 0 \left(\frac{d\theta}{ds}\right)\frac{d\theta}{ds} = 0$$

$$\frac{d^2\theta}{ds^2} + \frac{1}{r}\frac{dr}{ds}\frac{d\theta}{ds} = 0 \implies r\frac{d^2\theta}{ds^2} + \frac{dr}{ds}\frac{d\theta}{ds}$$

recall:

for $\frac{d}{ds}(uv) = v\frac{du}{ds} + v\frac{du}{ds}$

$$\implies \frac{d}{ds}\left(r\frac{d\theta}{ds}\right) = 0$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0 \checkmark$$

(b)

using angular equation

$$\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = 0 \quad \text{means}$$

$$r^2 \frac{d\theta}{ds} = \text{some constant } l$$

$$\frac{d\theta}{ds} = \frac{l}{r^2}$$

→ we then use the chain rule

$$\frac{d\theta}{dr} = \frac{d\theta}{ds}\frac{ds}{dr}$$

$$\left(\frac{d\theta}{dr}\right)^2 = \left[\frac{l}{r^2}\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}\right]^2$$

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{l^2}{r^4}\left[1+r^2\left(\frac{d\theta}{dr}\right)^2\right]$$

$$r^4\left(\frac{d\theta}{dr}\right)^2 = l^2 + l^2 r^2\left(\frac{d\theta}{dr}\right)^2$$

$$\left(\frac{d\theta}{dr}\right)^2\left[r^4 - l^2 r^2\right] = l^2$$

using radial equation

$$\frac{d^2r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 = r\left(\frac{l}{r^2}\right)^2$$

$$\frac{d^2r}{ds^2} = \frac{l^2}{r^3}$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

$$ds^2 = \left(1 + r^2\frac{d\theta^2}{dr^2}\right)dr^2$$

$$ds = dr\sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

$$\frac{ds}{dr} = \sqrt{1+r^2\left(\frac{d\theta}{dr}\right)^2}$$

$$\left(\frac{d\theta}{dr}\right)^2 = \frac{l^2}{r^2(r^2-l^2)}$$

$$\int d\theta = \int \frac{l}{r\left[r^2-l^2\right]^{1/2}}\,dr$$

wolfram alpha...

$$\theta + \theta_0 = -\arcsin\left(\frac{l}{r}\right)$$

$$\sin(\theta_0 - \theta) = \frac{l}{r}$$

$$r = \frac{l}{\sin(\theta_0 - \theta)} \Rightarrow r = \frac{l}{\cos(\theta - \theta_0)}$$

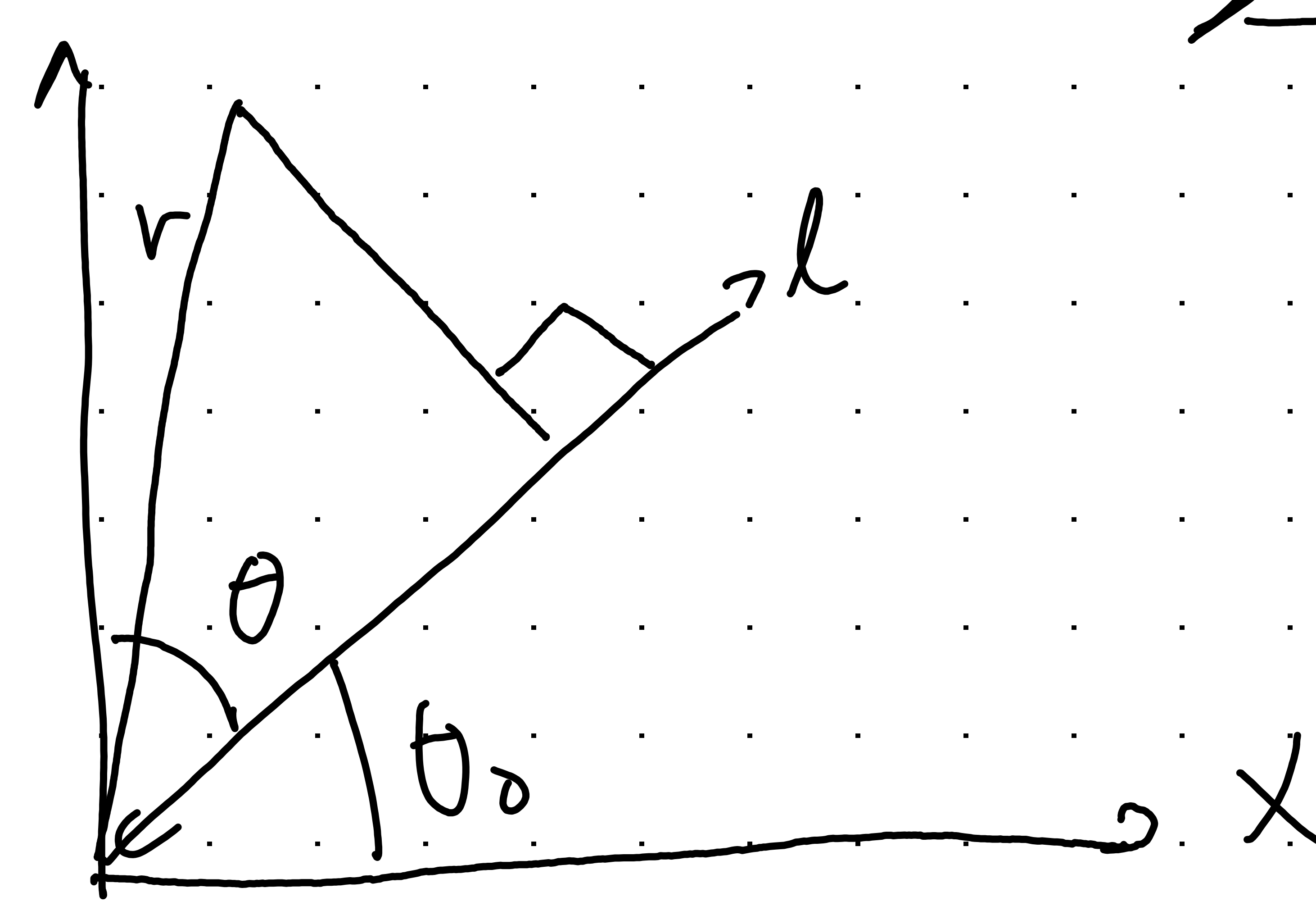
trig identity.

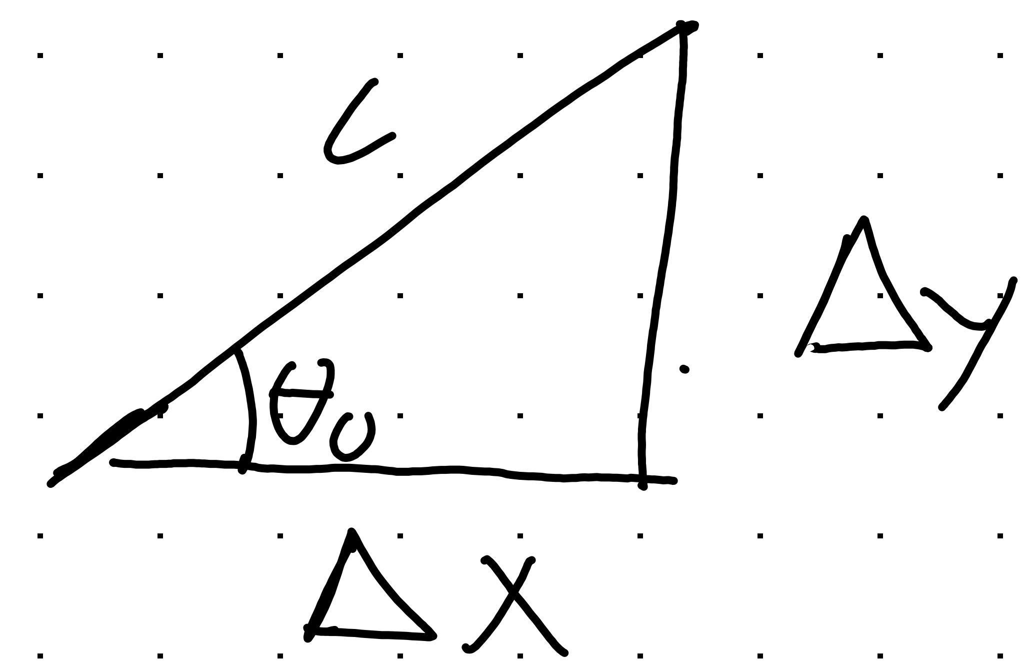
✓

c.) $r = \dfrac{l}{\cos(\theta - \theta_0)}$ $\qquad x = r\cos\theta$

$\qquad\qquad\qquad\qquad\quad\; y = r\sin\theta$

$$l = r\cos(\theta - \theta_0)$$
$$= r\left[\cos(\theta)\cos\theta_0 + \sin\theta\sin\theta_0\right]$$





$$l = x\cos\theta_0 + y\sin\theta_0$$

$$y = -\frac{\cos\theta_0}{\sin\theta_0}x + \frac{l}{\sin\theta_0}$$

$$y = \frac{-1}{\tan\theta_0}x + \frac{l}{\sin\theta_0}$$

$$y = \tan(\theta_0 - 90^\circ)x + \frac{l}{\sin\theta_0}$$

$$y = m\,x + b \checkmark$$

$$\tan(x - 90^\circ) = \frac{-1}{\tan\theta_0}$$

$$\tan\theta_0 = \frac{\Delta y}{\Delta x} = m$$

$$\sin\theta_0 = \frac{\Delta y}{l}$$

$$\cos\theta_0 = \frac{\Delta x}{l}$$

Problem 2 Geodesic Equation

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

recall the metric tensor for 2D sphere in polar coordinates

$$g_{ij} = \begin{array}{c} \theta \\ \phi \end{array} \begin{bmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{bmatrix} \qquad g^{ij} = \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2\sin^2\theta} \end{bmatrix}$$

$$g_{\theta\theta} = R^2 \qquad g_{\phi\phi} = R^2\sin^2\theta \qquad g^{\theta\theta} = \frac{1}{a^2} \qquad g^{\phi\phi} = \frac{1}{a^2\sin^2\theta}$$

$$g_{\theta\phi} = g_{\phi\theta} = 0$$

recall the christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(\frac{\partial g_{lj}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right)$$

we know

$g^{\theta\phi} = g^{\phi\theta} = 0$, so we can skip

$$\frac{\partial g_{\theta\theta}}{\partial\theta} = \frac{\partial(a^2)}{\partial\theta} = 0, \quad \frac{\partial g_{\theta\theta}}{\partial\phi} = 0$$

$$\Gamma^\theta_{\theta\theta} = \frac{1}{2} g^{\theta\theta}\left(\frac{\partial g_{\theta\theta}}{\partial\theta} + \frac{\partial g_{\theta\theta}}{\partial\theta} - \frac{\partial g_{\theta\theta}}{\partial\theta}\right) = 0$$

$$\Gamma^\theta_{\theta\phi} = \Gamma^\theta_{\phi\theta} = \frac{1}{2} g^{\theta\theta}\left(\frac{\partial g_{\theta\phi}}{\partial\theta} + \frac{\partial g_{\theta\theta}}{\partial\phi} - \frac{\partial g_{\phi\theta}}{\partial\theta}\right) = 0$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta}\left(\cancel{\frac{\partial g_{\theta\phi}}{\partial\phi}} + \cancel{\frac{\partial g_{\phi\theta}}{\partial\phi}} - \frac{\partial g_{\phi\phi}}{\partial\theta}\right) = -\frac{1}{2}\frac{1}{a^2}(2a^2\sin\theta\cos\theta) = -\sin\theta\cos\theta$$

$$\Gamma^\theta_{mn} = \begin{bmatrix} 0 & 0 \\ 0 & \sin\theta\cos\theta \end{bmatrix}$$

$$g^{\phi\phi} = \frac{1}{a^2 \sin^2\theta}$$

$$\Gamma^{\phi}_{\phi\phi} = \frac{1}{2} g^{\phi\phi}\left(\frac{\partial g_{\phi\phi}}{\partial \phi} + \frac{\partial g_{\phi\phi}}{\partial \phi} - \frac{\partial g_{\phi\phi}}{\partial \phi}\right) = 0$$

$$\frac{\partial g_{\phi\phi}}{\partial \theta} = 2a^2 \sin\theta \cos\theta$$

$$\Gamma^{\phi}_{\theta\theta} = \frac{1}{2} g^{\phi\phi}\left(\cancel{\frac{\partial g_{\phi\theta}}{\partial \theta}} + \cancel{\frac{\partial g_{\theta\phi}}{\partial \theta}} - \cancel{\frac{\partial g_{\theta\theta}}{\partial \phi}}\right) = 0$$

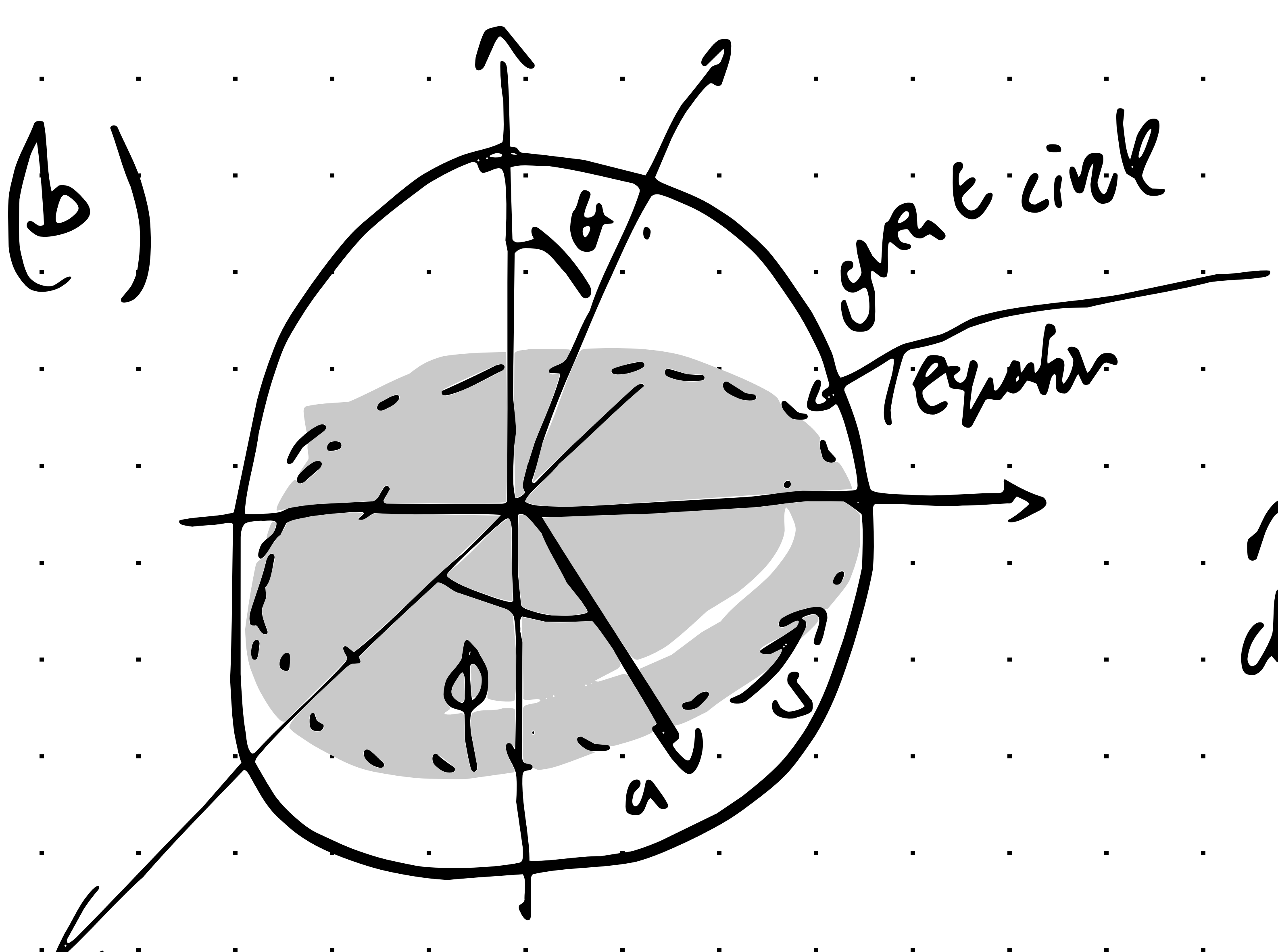
$$\frac{\partial g_{\phi\phi}}{\partial \phi} = 0$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \frac{1}{2} g^{\phi\phi}\left(\cancel{\frac{\partial g_{\theta\phi}}{\partial \phi}} + \frac{\partial g_{\phi\phi}}{\partial \theta} - \cancel{\frac{\partial g_{\phi\theta}}{\partial \phi}}\right)$$

$$= \frac{1}{\cancel{2}} \frac{1}{a^2 \sin^2\theta}\left(\cancel{2}a^2 \sin\theta\cos\theta\right) = \frac{\cos\theta}{\sin\theta}$$

$$\Gamma^{\phi}_{\mu\nu} = \begin{bmatrix} 0 & \cot\theta \\ \cot\theta & 0 \end{bmatrix}$$

(b)



$\theta = \frac{\pi}{2}$ has the equation $\phi = \frac{s}{a}$

radius a
distance s

→ the geodesic equation for θ is

$$\frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{\phi\phi}\left(\frac{d\phi}{ds}\right)^2 = 0$$

$$\frac{d^2\theta}{ds^2} - \sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2 = 0, \quad \theta = \frac{\pi}{2}$$

$$\frac{d^2\theta}{ds^2} = 0, \text{ so } \frac{\pi}{2} \text{ is valid} ✓$$

→ geodesic equation for ϕ ↳

$$\frac{d^2\phi}{ds^2} + \Gamma^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$$

$$\frac{d^2\phi}{ds^2} + \cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \quad \theta = \frac{\pi}{2}$$

north pole is such that $\theta = 0$

$$ds^2 = a^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

Problem 3 Tensor Algebra and Einstein's Equations

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$

Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

→ here is the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{bmatrix}$$

$$g^{\mu\nu} = \begin{array}{cc} & \begin{array}{cccc} t & r & \theta & \phi \end{array} \\ & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-(1-kr^2)}{a^2} & 0 & 0 \\ 0 & 0 & \frac{-1}{a^2r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{a^2r^2\sin^2\theta} \end{bmatrix} & \begin{array}{c} t \\ r \\ \theta \\ \phi \end{array} \end{array}$$

→ The Riemann curvature tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu}$$

→ is solved by solving the Christoffel symbol

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)$$

Note:

- g^{tt}, g^{rr}, $g^{\theta\theta}$, $g^{\phi\phi}$ are relevant everything else is not
- $a = a(t)$ is a function of time

$$\Gamma^t_{tt} = \frac{1}{2} g^{tt}\left(\cancel{\frac{\partial g_{tt}}{\partial t} + \frac{\partial g_{tt}}{\partial t}} - \frac{\partial g_{tt}}{\partial t}\right)^{\to 0} = 0$$

$$\Gamma^t_{rr} = \frac{1}{2} g^{tt}\left(\cancel{\frac{\partial g_{rt}}{\partial r}}^{0} + \cancel{\frac{\partial g_{rt}}{\partial r}}^{0} - \frac{\partial g_{rr}}{\partial t}\right) = \frac{1}{2}\,1\left[\frac{\partial}{\partial t}\left(\frac{-a^2}{1-kr^2}\right)\right] = \frac{1}{2}\left(\frac{2a\dot{a}}{1-kr^2}\right) = \frac{a\dot{a}}{1-kr^2}$$

$$\Gamma^t_{\theta\theta} = \frac{1}{2} g^{tt}\left(\cancel{\frac{\partial g_{\theta t}}{\partial \theta}} + \cancel{\frac{\partial g_{\theta t}}{\partial \theta}} - \frac{\partial g_{\theta\theta}}{\partial t}\right) = \frac{1}{2} g^{tt}\left[-\frac{\partial}{\partial t}\left(-a^2r^2\right)\right] = \frac{1}{2}(1)\left[2a\dot{a}r^2\right] = a\dot{a}r^2$$

$$\Gamma^t_{\phi\phi} = \frac{1}{2} g^{tt}\left(\cancel{\frac{\partial g_{\phi t}}{\partial \phi}} + \cancel{\frac{\partial g_{\phi t}}{\partial \phi}} - \frac{\partial g_{\phi\phi}}{\partial t}\right) = \frac{1}{2}(1)\left[\frac{\partial}{\partial t} -a^2r^2\sin^2\theta\right] = \frac{1}{2}\left[2a\dot{a}r^2\sin^2\theta\right] = a\dot{a}r^2\sin^2\theta$$

$$\Gamma^r_{tt} = \frac{1}{2} g^{rr}\left(\frac{\partial g_{tr}}{\partial t} + \frac{\partial g_{tr}}{\partial t} - \frac{\partial g_{tt}}{\partial r}\right) = \frac{1}{2} g^{rr}(0) = 0$$

$$\Gamma^r_{rr} = \frac{1}{2} g^{rr}\left(\frac{\partial g_{rr}}{\partial r} + \cancel{\frac{\partial g_{rr}}{\partial r} - \frac{\partial g_{rr}}{\partial r}}\right) = \frac{1}{2}\left(\frac{-(1-kr^2)}{\cancel{a^2}}\right)\left[\frac{\partial}{\partial r}\frac{-\cancel{a^2}}{1-kr^2}\right] = \frac{1}{2}(1-kr^2)\frac{2kr}{(kr^2-1)^2} = \frac{kr}{1-kr^2}$$

$$\Gamma^r_{\theta\theta} = \frac{1}{2} g^{rr}\left(\cancel{\frac{\partial g_{\theta r}}{\partial \theta}} + \cancel{\frac{\partial g_{r\theta}}{\partial \theta}} - \frac{\partial g_{\theta\theta}}{\partial r}\right) = \frac{1}{2}\left(\frac{1-kr^2}{-a^2}\right)(2a^2r) = -r(1-kr^2)$$

$$\Gamma^r_{\phi\phi} = \frac{1}{2} g^{rr}\left(\cancel{\frac{\partial g_{\phi r}}{\partial \phi}} + \cancel{\frac{\partial g_{r\phi}}{\partial \phi}} - \frac{\partial g_{\phi\phi}}{\partial r}\right) = \frac{1}{2}\left(\frac{1-kr^2}{-a^2}\right)(2a^2r\sin^2\theta) = -r(1-kr^2)\sin^2\theta$$

→ Note, $\Gamma^i_{\alpha\beta}$ is non-0 if one of the indices is spatial and the other is time.

$$\Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\frac{\dot{a}}{a}$$

→ and the derivatives are non-0 only if $\Gamma^0_{\alpha\beta} = -\frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial t}$

are spacial indices $i,j = 1,2,3$

$= r\theta\phi$

or

from symmetry we expect these to be the same

$$\Gamma^r_{tr} = \Gamma^r_{rt} = \frac{1}{2} g^{rr}\left\{ \frac{\partial g_{rr}}{\partial t} - \cancel{\frac{\partial g_{rt}}{\partial r}} - \cancel{\frac{\partial g_{tr}}{\partial r}} \right\} = \frac{1}{2}\,\frac{-a^2}{\cancel{1-kr^2}}\,\frac{\partial}{\partial t}\,\frac{-(\cancel{1-kr^2})}{a^2} = \boxed{\dot{a}/a}$$

$$\Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} = \frac{1}{2} g^{\theta\theta}\left\{ \frac{\partial g_{\theta\theta}}{\partial t} \right\} = \frac{1}{2}\,\frac{-1}{a^2 r^2}\left(-\dot{a}^2 r^2\right) = \boxed{\frac{\dot{a}}{a}}$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{2} g^{\theta\theta}\left\{ \frac{\partial g_{\theta\theta}}{\partial r} + 0 - 0 \right\} = \frac{1}{2}\left(\frac{1}{-a^2 r^2}\right)\left(-a^2 r\right) \to \boxed{\frac{1}{r}}$$

$$\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta}\left\{ \cancel{\frac{\partial g_{\phi\theta}}{\partial \phi}} + \cancel{\frac{\partial g_{\theta\phi}}{\partial \phi}} - \frac{\partial g_{\phi\phi}}{\partial \theta} \right\} = \frac{1}{2}\left(\frac{-1}{a^2 r^2}\right)\left(a^2 r^2\, 2\sin\theta\cos\theta\right) = \boxed{-\sin\theta\cos\theta}$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{2} g^{\phi\phi}\left\{ \frac{\partial g_{\phi\phi}}{\partial r} + 0 - 0 \right\} = \frac{1}{2}\left(\frac{1}{-a^2 r^2 \sin^2\theta}\right)\left(-2a^2 r \sin^2\theta\right) = \boxed{\frac{1}{r}}$$

$$\Gamma^\phi_{\phi t} = \Gamma^\phi_{t\phi} = \frac{1}{2} g^{\phi\phi}\left\{ \frac{\partial g_{\phi\phi}}{\partial t} \right\} = \frac{1}{2}\left(\frac{-1}{a^2 r^2 \sin^2\theta}\right)\left(-\dot{a}^2 r^2 \sin^2\theta\right) = \boxed{\frac{\dot{a}}{a}}$$

$$\Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \frac{1}{2} g^{\phi\phi}\left\{ \frac{\partial g_{\phi\phi}}{\partial \theta} + \cancel{\frac{\partial g_{\phi\theta}}{\partial \phi}} - \cancel{\frac{\partial g_{\theta\phi}}{\partial \phi}} \right\} = \cancel{\frac{1}{2}}\left(\frac{1}{-a^2 r^2 \sin^2\theta}\right)\left(-a^2 r^2 \cancel{2} \sin\theta\cos\theta\right) = \boxed{\cot\theta}$$

Onto the <u>Ricci tensor</u>, which takes the form

0 1 2 3
$t\ r\ \theta\ \phi$

$$R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\lambda}_{\sigma\lambda} - \Gamma^{\sigma}_{\mu\lambda}\Gamma^{\lambda}_{\sigma\nu}$$

$$R_{tt} = -\partial_t \Gamma^{r}_{tr} - \partial_t \Gamma^{\theta}_{t\theta} - \partial_t \Gamma^{\phi}_{t\phi} - \Gamma^{r}_{tr}\Gamma^{\theta}_{\theta t} - \Gamma^{\phi}_{t\phi} - \Gamma^{\phi}_{\phi t}$$

$$= -3\frac{d}{dt}\left(\frac{\dot a}{a}\right) - 3\left(\frac{\dot a}{a}\right)^2 = -3\left(\frac{\ddot a a}{a^2} + \frac{\dot a^2}{a^2}\right) - 3\frac{\dot a^2}{a^2} = -3\frac{\ddot a}{a} + \cancel{\frac{3\dot a^2}{a^2}} - 3\frac{\dot a^2}{a^2} = \boxed{-3\frac{\ddot a}{a}}$$

$$R_{rr} = \partial_t \Gamma^{t}_{rr} - \partial_r \Gamma^{\theta}_{r\theta} - \partial_r \Gamma^{\phi}_{r\phi} + \Gamma^{t}_{rr}\Gamma^{\theta}_{t\theta} + \Gamma^{t}_{rr}\Gamma^{\phi}_{t\phi} - \Gamma^{r}_{rt}\Gamma^{t}_{rr} + \Gamma^{r}_{rr}\Gamma^{\theta}_{r\theta} + \Gamma^{r}_{rr}\Gamma^{\phi}_{r\phi} - \Gamma^{\theta}_{r\theta}\Gamma^{\theta}_{\theta r}$$
$$- \Gamma^{\phi}_{r\phi}\Gamma^{\phi}_{\phi r}$$

$$= \partial_t\left(\frac{a\dot a}{1-kr^2}\right) - \partial_r\left(\frac{1}{r}\right) - \partial_r\left(\frac{1}{r}\right) + \frac{a\dot a}{1-kr^2}\frac{\dot a}{a} + \cancel{\left(\frac{a\dot a}{1-kr^2}\right)\frac{\dot a}{a} - \left(\frac{a\dot a}{1-kr^2}\right)\left(\frac{\dot a}{a}\right)} + \frac{kr}{1-kr^2}\frac{1}{r} + \frac{kr}{1-kr^2}\left(\frac{1}{r}\right) - \left(\frac{1}{r}\right)^2 - \left(\frac{1}{r}\right)^2$$

$$= \partial_t\left(\frac{a\dot a}{1-kr^2}\right) \cancel{- 2\partial_r\left(\frac{1}{r}\right)} + \frac{a\dot a}{1-kr^2}\frac{\dot a}{a} + \frac{2k}{1-kr^2} \cancel{- 2\left(\frac{1}{r}\right)^2}$$

$$= \frac{a\ddot a}{1-kr^2} + \frac{2\dot a^2}{1-kr^2} + \frac{2k}{1-kr^2} = \boxed{\frac{a\ddot a + 2\dot a^2 + 2k}{1-kr^2}}$$

$$R_{\theta\theta} = \partial_t \Gamma^t_{\theta\theta} + \partial_r \Gamma^r_{\theta\theta} - \partial_\theta \Gamma^\phi_{\theta\phi} + \Gamma^t_{\theta\theta}\Gamma^r_{tr} + \Gamma^t_{\theta\theta}\Gamma^\phi_{t\phi} + \underline{\Gamma^r_{\theta\theta}\Gamma^r_{rr}} + \underline{\Gamma^r_{\theta\theta}\Gamma^\phi_{r\phi}} - \underline{\Gamma^\theta_{\theta t}\Gamma^t_{\theta\theta}}$$

$$- \Gamma^\theta_{\theta r}\Gamma^r_{\theta\theta} - \Gamma^\phi_{\theta\phi}\Gamma^\phi_{\phi\theta}$$

$$= \partial_t\, a\dot{a}r^2 + \partial_r\left(-r(1-kr^2)\right) - \partial_\theta \cot\theta + \cancel{a\dot{a}r^2\left(\frac{\dot{a}}{a}\right)} + \dot{a}ar^2\left(\frac{\dot{a}}{\cancel{a}}\right) + -r\left(\cancel{1-kr^2}\right)\frac{kr}{\cancel{1-kr^2}}$$

$$+ \cancel{-r(1-kr^2)\frac{1}{r}} - \cancel{\frac{\dot{a}}{a}}\, a\dot{a}r^2 + r\left(\cancel{1-kr^2}\right)\frac{1}{r} - \cot^2(\theta)$$

$$= \partial_t(a\dot{a}r^2)\ \partial_r(-r(1-kr^2)) - \partial_\theta \cot\theta + \dot{a}^2r^2 - kr^2 - \cot^2\theta = (\ddot{a}a + 2\dot{a}^2 + 2k)r^2$$

$$R_{\phi\phi} = \partial_t \Gamma^t_{\phi\phi} + \partial_r \Gamma^r_{\phi\phi} + \partial_\theta \Gamma^\theta_{\phi\phi} + \Gamma^t_{\phi\phi}\Gamma^r_{tr} + \Gamma^t_{\phi\phi}\Gamma^\theta_{t\theta} + \Gamma^r_{\phi\phi}\Gamma^r_{rr} + \Gamma^r_{\phi\phi}\Gamma^\theta_{r\theta} - \Gamma^\phi_{\phi t}\Gamma^t_{\phi\phi} - \Gamma^\phi_{\phi r}\Gamma^r_{\phi\phi} - \Gamma^\phi_{\phi\theta}\Gamma^\theta_{\phi\phi}$$

$$= \partial_t(a\dot{a}r^2\sin^2\theta) + \partial_r(-r(1-kr^2)\sin^2\theta) - \partial_\theta(\sin\theta\cos\theta) + 2a\dot{a}r^2\sin^2\theta\,\frac{\dot{a}}{a} - r(1-kr^2)\sin^2\theta\left(\frac{ka}{1-kr^2}\right)$$

$$-r(1-kr^2)\sin^2\theta\left(\frac{1}{r}\right) - \frac{\dot{a}}{a}\,a\dot{a}r\sin^2\theta + \frac{1}{r}\,r(1-kr^2)\sin^2\theta + \cot\theta\sin\theta\cos\theta$$

wolfram alpha

$$= (\ddot{a}a + 2\dot{a}^2 + 2k)r^2\sin^2\theta$$

0 1 2 3

$t\ r\ \theta\ \phi$

Ricci Scalar $\quad R = g^{\mu\nu} R_{\mu\nu}$

$$
\begin{array}{c}
\begin{array}{cccc} t & r & \theta & \phi \end{array} \\
g^{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-(1-kr^2)}{a^2} & 0 & 0 \\ 0 & 0 & \frac{-1}{a^2 r^2} & 0 \\ 0 & 0 & 0 & \frac{-1}{a^2 r^2 \sin^2\theta} \end{bmatrix}
\end{array}
\begin{array}{c} t \\ r \\ \theta \\ \phi \end{array}
$$

$$R = R_{tt} g^{tt} + R_{rr} g^{rr} + R_{\theta\theta} g^{\theta\theta} + R_{\phi\phi} g^{\phi\phi}$$

$$= -3\frac{\ddot{a}}{a} + \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1-kr^2}\left(\frac{1-kr^2}{-a^2}\right) - \frac{(\ddot{a}a + 2\dot{a}^2 + 2k)r^2}{a^2 r^2}$$

$$- \frac{(\ddot{a}a + 2\dot{a}^2 + 2k) r^2 \sin^2\theta}{a^2 r^2 \sin^2\theta} \Rightarrow R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

expanding universe

Einstein tensor $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

$$G_{tt} = R_{tt} - \frac{1}{2} R g_{tt} = -3\frac{\ddot{a}}{a} - \frac{1}{2}(-6)\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)$$

contributes positively.

$$G_{rr} = R_{rr} - \frac{1}{2} R g_{rr} = \frac{(a\ddot{a} + 2\dot{a} + 2k)}{1-kr^2} + \frac{3}{a^2}(a\ddot{a} + \dot{a}^2 + k)\left(-\frac{a^2}{1-kr^2}\right) = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\left(\frac{-a^2}{1-kr^2}\right)$$

$$G_{\theta\theta} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k) + \frac{3}{a^2}(a\ddot{a} + \dot{a}^2 + k)(-a^2 r^2) = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)(-a^2 r^2)$$

$$G_{\phi\phi} = r^2 \sin^2\theta(a\ddot{a} + 2\dot{a}^2 + 2k) - \frac{3}{a^2}(a\ddot{a} + \dot{a}^2 + k) = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)(-a^2 r^2 \sin^2\theta)$$

where the 3 spatial can be written as $G_{ii} = \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) g_{ii}$

→ diagonal stress tensor $T^{\mu}_{\ \nu} = (\rho, -p, -p, -p)$

0-0 and i-i components of einstein's equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$T^{\mu\nu} = \begin{bmatrix} \rho & & & 0 \\ & -p & & \\ & & -p & \\ 0 & & & -p \end{bmatrix}$$

→ for time

$$G_{tt} = 8\pi G T_{tt} \Rightarrow 3\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) = 8\pi G \rho$$

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8}{3}\pi G \rho \checkmark$$

→ for spatial

$$G_{ii} = 2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\ g_{ii}$$

$g_{ii} T^{i}_{\ i}$

$$\Rightarrow \left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\cancel{g_{ii}} = 8\pi G T_{ii}$$

where $T^{i}_{\ i} = \begin{bmatrix} -p & 0 & \\ 0 & -p & \\ & & -p \end{bmatrix}$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G p \checkmark$$

Problem 4: Metric w/ positive curvature by Embedding

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

compute this distance when it is restricted to the surface of a 3 sphere.

$$x^2 + y^2 + z^2 + w^2 = R^2 \quad , R = \text{const.}$$

-> show after transforming to spherical polar coordinates, you recover FRW

$$ds^2 = \underbrace{(1-kr^2)^{-1} dr^2}_{dr^2\left(\frac{1}{1-kr^2}\right)} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

-> to derive the metric on the surface we set

$$w = \sqrt{R^2 \underbrace{- x^2 - y^2 - z^2}_{\equiv r^2}} \quad \text{such that } dw = \frac{-r\,dr}{\sqrt{R^2 - r^2}}$$

-> from above here, no need to integrate.
we can say

$$ds^2 = dx^2 + dy^2 + dz^2 + \frac{r^2\,dr^2}{R^2 - r^2}$$

$$= dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{r^2\,dr^2}{R^2 - r^2}$$

$$= r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dr^2\left(\frac{\cancel{r^2}}{\cancel{R^2}\left(1 - \frac{r^2}{R^2}\right)}\right) \checkmark$$

$$= dr^2\left(\frac{1}{1 - kr^2}\right) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$k \equiv \frac{1}{R}$$

Problem 5: Friedman equation

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

$\rho = \frac{\rho_0}{a^3}$ $\rho_0 \rightarrow$ present day mass density

present $t = t_0$ $\quad a_0 \equiv a(t_0) = 1$

(a) show that $a(\theta) = \frac{4\pi G\rho_0}{3k}(1-\cos\theta)$

and $t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta)$ is a solution.

by the chain rule $\frac{da}{dt} = \frac{da}{d\theta}\frac{d\theta}{dt} = \left(\frac{\cancel{4\pi G\rho_0}}{3k}\sin\theta\right)\frac{3k^{3/2}}{\cancel{4\pi G\rho_0}(1-\cos\theta)}$

$= k^{1/2}\frac{\sin\theta}{1-\cos\theta}$

$$\frac{da}{d\theta} = \frac{4\pi G\rho_0}{3k}\sin\theta$$

$$\frac{dt}{d\theta} = \frac{4\pi G\rho_0}{3k^{3/2}}(1-\cos\theta)$$

$$\frac{d\theta}{dt} = \frac{3k^{3/2}}{4\pi G\rho_0(1-\cos\theta)}$$

$$\frac{\dot a^2}{a^2} = \frac{k\frac{\sin^2\theta}{(1-\cos\theta)^2}}{\left(\frac{4\pi G\rho_0}{3k}(1-\cos\theta)\right)^2}$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{k^3\sin^2\theta}{\left(\frac{4\pi G\rho_0}{3}\right)^2(1-\cos\theta)^4}$$

-> now on the right hand side.

using $\rho = \frac{\rho_0}{a^3} = \rho_0 \left(\frac{3k}{4\pi G \rho_0}\right)^3 \frac{1}{(1-\cos\theta)^3}$

RHS

$$\frac{8\pi G}{3}\rho - \frac{k}{a^2} = \frac{8\pi G}{3}\rho_0 \left(\frac{3K}{4\pi G \rho_0}\right)^3 \frac{1}{(1-\cos\theta)^3} - \left(\frac{3K}{4\pi G\rho_0}\right)^2 \frac{k}{(1-\cos\theta)^2}$$

$$= \frac{k^3}{\left(\frac{4\pi G\rho_0}{3}\right)^2} \frac{1}{(1-\cos\theta)^2} \left\{ \frac{8\pi G}{3}\rho_0 \left(\frac{3k}{4\pi G\rho_0}\right) \frac{1}{1-\cos\theta} - 1 \right\}$$

$$= \frac{k^3 \sin^2\theta}{\left(\frac{4\pi G \rho_0}{3}\right)^2 (1-\cos\theta)^4} = LHS$$

it's a solution.

(b) plotting, $k=1$, $G=1$, $c=1$