GR6014 Fall 2024

Cosmology, Black Holes and Gravity *Homework 1 due Monday, Oct. 7*

Problem 1: Redshifting of CMB vs. Non-Relativistic Gas

a) Consider a box, filled with a gas of photons with a thermal (blackbody) spectrum at initial temperature T_0, whose overall (linear) size evolves with time as $a(t)$. Use the fact (which we will derive soon) that the momentum p of a photon scales with the scale factor as $p \propto a^{-1}$ to show that the spectrum always retains its blackbody shape during any expansion or contraction of the box. How does the temperature of the photons depend on a?

b) Consider the same box, but filled with a non–relativistic ideal gas, initially in thermal equilibrium, so that their velocities are given by the Maxwell-Boltzmann distribution:

$$f(v)d^3v = \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left(-\frac{mv^2}{2k_B T}\right) d^3v \tag{1}$$

Assume again that the momentum p of the particles scale with the scale factor as $p \propto a^{-1}$. Does the distribution of momenta retain its thermal equilibrium shape during the expansion/contraction ? If so, how does the temperature depend on a?

c) How does the total energy content of the box evolve with a in the above two cases? Is the increase/decrease in the total energy fully accounted for by the "$p\Delta V$" work done by the gas?

Problem 2: CMB Temperature Dipole

a) The cosmic microwave background (CMB) is an isotropic blackbody only in one particular Lorentz frame. We detect a large dipole anisotropy in the CMB temperature (after the known motion of the Earth around the Sun is corrected for), which is thought to arise from the *peculiar velocity* of the Solar System relative to the frame in which the CMB is at rest. Show that an observer, moving at velocity v with respect to the frame in which the CMB is isotropic with temperature T, observes a blackbody spectrum with temperature, T':

$$T' = \frac{T}{\gamma(1 - \beta\cos\theta')}, \tag{2}$$

where $\beta \equiv v/c$, $\gamma \equiv (1-\beta^2)^{-1/2}$ and θ' is the angle between v and the direction of a particular measurement.(Hint: you will need the relativistic Doppler boost formulae.)

b) The above peculiar velocity is measured to be $v = 368$ km/s. This motion of the solar system should also manifest itself in a large galaxy survey, provided that the rest–frame of galaxies coincides with that of the CMB. For simplicity, assume that the effective number density of galaxies probed by a survey is Gaussian distributed in redshift with an expectation value of $\langle z \rangle = 0.5$ and standard deviation $\sigma_z = 0.1$. Also assume that we can measure each galaxy's redshift to infinite precision. Roughly how many galaxies do we need to observe in order to detect the dipole–induced shift in $\langle z \rangle$?

Problem 3: Rotation Curves from a Sphere versus a Thin Disk

An estimate of the mass of a galaxy can be derived from the galaxy's *rotation curve*: the orbital velocity of stars and gas around the center of the galaxy as a function of distance from the center. In this problem, we study the effect of a flattened distribution of matter on the rotation curve.

Consider a galaxy with a total mass $M = 10^{12} M_\odot$ extending out to a radius of 50 kpc (with the density zero beyond this radius).

- In case 1, assume that all mass is condensed into an infinitely thin disk, with the surface mass density $\Sigma(R) = \Sigma_0 \mathrm{exp}(-\mathrm{R}/\mathrm{R_d})$, where $R_d = 8.5$ kpc. Calculate the rotation curve for this disk. (Hint: the integrals are difficult and you will need to consult a table of integrals. Alternatively, you may use *Mathematica* or a similar symbolic package, or present a numerical solution).
- In case 2, assume that the mass has a spherically symmetric distribution, such that the mass $M(< R)$ enclosed inside a sphere of radius R, is the same as the mass enclosed inside a circle of radius R for the exponential disk above. Calculate the rotation curve for this spherically symmetric mass distribution.
- Compare and discuss the results: what is the fractional difference of the velocities in the two cases (as a function of R)? What is the uncertainty of the inferred mass for a given observed velocity?

Problem 4: Lensing by MACHOs

Suppose all dark matter in our Galaxy consists of MAssive Compact Halo Objects (MACHOs). For simplicity, assume that the dark halo is a homogeneous sphere, and it has a circular velocity of 220 km/s at its surface. Calculate the probability that a star in the Andromeda Galaxy will be lensed by one of these MACHOs. Estimate the typical lensing timescale. Order of magnitude estimates are sufficient for this problem, and feel free to make your own approximations for this purpose - but make sure you state these approximations clearly. (Note: the distance to Andromeda is 700kpc.)

Problem 5: Transformation of Contravariant and Covariant Vectors

Consider a circle with radius $r = R$ in two-dimensional Euclidean space as a parameterized curve $x^\mu(\lambda)$ (with the parameter λ measuring the distance along the circle). The tangent to this curve at the position (r, θ) is a contravariant vector A^μ. (i) Use $A^\mu = dx^\mu/d\lambda$ to find the components A^r and A^θ in polar coordinates, (ii) find the transformation matrix $\partial x'^\mu/\partial x^\nu$ to the (primed) Cartesian coordinates, (iii) transform the vector to find the components A^x and A^y in Cartesian coordinates, and check that you got the right answer. (iv) Also check explicitly that you find the same length for the vector, $g_{\mu\nu}A^\mu A^\nu$, in both coordinates. Repeat (i)-(iv) for the covariant vector, defined as the vector that is perpendicular to the circle at the position (r, θ) (i.e., for step (i), use $\phi(x^\mu) = \mathrm{const} = R$ to describe a parameterized surface, and $A_\mu = \partial\phi/\partial x^\mu$ to find the components of the vector).