Cosmology, Black Holes and Gravity *Homework 2 due Wed, Oct. 23*

Problem 1: Geodesic Equation - Flat 2D Space

A simple illustration of the geodesic equation is to compute the geodesics in flat, 2D space, in polar coördinates. These, of course, are straight lines.

The metric in polar coördinates is given by

$$ds^2 = dr^2 + r^2 d\theta^2. \tag{1}$$

(a) Compute the Christoffel symbols and show that the two geodesic equations, if you adopt the distance s itself as the parameter along a curve, take the following form:

$$\frac{d^2r}{ds^2} = r\left(\frac{d\theta}{ds}\right)^2 \tag{2}$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0. \tag{3}$$

(b) Show that integrating these equations gives you a straight line. In particular, instead of explicitly solving for r and θ as functions of s, you may directly find the shape of the resulting curve by eliminating s and solving for $r = r(\theta)$. (Hint: the easiest way to do this is to solve for $\theta = \theta(r)$ first.) Show that this gives the solution $r = \ell/\cos(\theta - \theta_0)$, where ℓ and θ_0 are constants.

(c) Substitute $x = r\cos\theta$ and $y = r\sin\theta$ to convert the answer to Cartesian coördinates, and show that this is indeed the equation for a straight line.

Problem 2: Geodesic Equation - Curved 2D Space

The metric on the surface of a two-dimensional sphere can be written in polar coordinates as

$$dS^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2)\ , \tag{4}$$

where a is the radius of the sphere, θ is a radial coordinate, and ϕ is the polar angle.

(a) Calculate the Christoffel symbols by hand.

(b) Show that a great circle is a solution of the geodesic equation. (Hint: Make use of the freedom to orient the coordinates so the equation of a great circle is simple.)

Problem 3: Tensor Algebra and Einstein's Equations

The Friedmann-Robertson-Walker (FRW) metric is the only metric consistent with the cosmological principle, i.e. with a homogeneous and isotropic universe. In units where $c = 1$, the FRW metric can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]\ , \tag{5}$$

where $k = 0$ or ± 1. Calculate the non-zero components of the Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}. \tag{6}$$

Be sure to take advantage of symmetries of the Christoffel symbols and of the Riemann tensor to save yourself time. You may use *Mathematica* or a similar symbolic software package, but be sure to include your code. *Note that a tensor index is raised or lowered by multiplying with the metric. For example:* $T_{\mu\nu} = g_{\mu\lambda} T^{\lambda}_{\nu}$ *and likewise* $G_{\mu\nu} = g_{\mu\lambda} G^{\lambda}_{\nu}$.

Show that with the diagonal stress–energy tensor $T^{\mu}_{\nu} = (\rho, -p, -p, -p)$ of an ideal fluid, the $0-0$ and $i-i$ components of the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ reduce to

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{7}$$

and

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = -8\pi G \rho, \tag{8}$$

known as the Friedmann equation and the acceleration equation.

Problem 4: Metric with Positive Curvature by Embedding

One can derive the spatial part of the FRW metric with positive curvature by embedding a 3-sphere in flat, 4D Euclidean space. Show this explicitly, by considering the distance in Cartesian coordinates, $ds^2 = dx^2 + dy^2 + dz^2 + dw^2$, and computing it when it is restricted to the surface of a 3-sphere, $x^2 + y^2 + z^2 + w^2 = R^2$, where R=constant. Show that after a transformation to spherical polar coördinates, you recover the spatial part of the FRW metric $ds^2 = (1 - kr^2)^{-1} dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)$. How is k related to R ?

Problem 5: Friedmann Equation for $k > 0$

The full Friedmann equation in general can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \tag{9}$$

Consider the case when the Universe contains only non-relativistic matter, so that the energy density $\rho = \rho_0/a^3$. Note that we have set the speed of light $c = 1$, and ρ_0 is the present–day mass density of the universe (more generally, the subscript $_0$ on any parameter refers to the value of the parameter at the present epoch; hence t_0 is the present age of the Universe, and a_0 is the present–day scale factor). According to this convention, at the present day, $t = t_0$, we have $a_0 \equiv a(t_0) = 1$.

(a) Demonstrate that the following parametric solution

$$a(\theta) = \frac{4\pi G \rho_0}{3k}(1 - \cos\theta)$$

$$t(\theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\theta - \sin\theta)$$

solves this equation. Here θ is a parameter that runs from $0 \leq \theta \leq 2\pi$.

(b) Make a plot of both a and t as functions of the parameter θ. With the help of these, plot the evolution of the scale factor $a(t)$.