GR6014 Fall 2024

Cosmology, Black Holes and Gravity *Homework 2 due Wed, Nov. 20*

Problem 1: Relativistic Equilibrium Thermodynamics

As mentioned in class, the phase space occupation function is

$$f(|\,\vec{p}\,|) = \frac{1}{\exp(\frac{E-\mu}{kT}) \pm 1}, \tag{1}$$

where + and - signs refer to Fermi-Dirac and Bose-Einstein distributions, and where $E^2 = p^2c^2 + m^2c^4$ is the energy. (The chemical potential μ is normally small in cosmology, but we have included it for completeness here). This expression can be used to directly find the number density n of a species in equilibrium, by

$$n = \frac{g}{(2\pi\hbar)^3} \int f(|\,\vec{p}\,|) d^3p, \tag{2}$$

where g is the degeneracy factor (for internal degrees of freedom). Show that n is given by simple expressions in the following limiting cases:
a) Relativistic limit ($kT \gg mc^2$ and $kT \gg \mu$):

$$n = \frac{\zeta(3)}{\pi^2} g \left(\frac{kT}{\hbar c}\right)^3 \tag{3}$$

for bosons, and 3/4 times the above expression for fermions. For extra credit, show that if you include the next order term in μ/kT, the above formula acquires a factor of $(1 + A\mu/kT)$, and find A.
b) Non-relativistic limit ($kT \ll mc^2$):

$$n = g \left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \exp(-\frac{mc^2 - \mu}{kT}). \tag{4}$$

Problem 2: Recession Velocity vs. Redshift

The recessional velocity $v_{\rm rec}$ of a galaxy due to the expansion on the universe today can be interpreted as the rate of change in its proper distance, $v_{\rm rec} = \dot{a}r$, where r is the fixed comoving coordinate of the galaxy. Calculate the coordinate distance r to a galaxy at redshift z, and plot $v_{\rm rec}$ as a function of z. Show that for the nearby universe, you recover Hubble's law, but that the recession velocity can exceed the speed of light. What is the redshift at which this happens? Assume a flat Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km/s/Mpc.

Problem 3: Measuring the Change in Redshift

Since the universe is expanding, the redshift of a source at a fixed comoving coordinate (such as a galaxy or a quasar), will change with time. Sensitive spectrometers (developed for the purpose of finding extrasolar planets via their radial velocities) can measure the Doppler shifts corresponding to velocities of ~ 3 m/s in nearby (Galactic) stars. How long would you need to wait before we can observe a change in the redshift of a $z = 2.0$, and $z = 10$ quasar (the current record holder) by this amount? Assume a flat Universe with $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$ and $H_0 = 70$ km/s/Mpc, and ignore the peculiar velocity of both the quasar and of the detector on Earth.
(Hint: this problem is a more subtle than it sounds. Recall that the observed redshift of a photon is given by the *ratio* of scale factors at the time of the emission and the detection of the photon: $1 + z \equiv a(t_{\rm obs})/a(t_{\rm em})$. So far, when we discussed redshift, set have $a(t_{\rm obs}) = a(t_0) = 1$, since we detect the photon at the present time. However, in this problem, you must allow the scale factor at the detector to increase, as well, not just at the time of the emission. You may still assume that the time elapsed at the detector is much smaller than the Hubble time H_0^{-1}.)

Problem 4: The Age of the Universe in ΛCDM

Calculate the age of a flat ($k = 0$), two–component universe with non–relativistic matter, $\Omega_m \neq 0$, and a cosmological constant, $\Omega_\Lambda = 1 - \Omega_m$, as a function of redshift. Express your answer in terms of Ω_Λ and the current value of the Hubble parameter, H_0. How does the present age $t_0 = t(z = 0)$ behave as $\Omega_\Lambda \to 1$? Make a sketch of your results (t_0 vs. Ω_Λ). The are claims that some globular cluster stars have an age of $\sim$ 15 Gyr. What constraint would such an old age put on Ω_Λ in a flat universe? (you may assume that $H_0 = 70$ km/s/Mpc).

Problem 5: Seeing Around the Universe.

A photon is emitted at the time of the Big Bang in a universe that contains only non-relativistic matter, and has $k > 0$. Show that the photon travels precisely all the way around the universe by the time of the "Big Crunch".
(a) Recall that the spatial part of the metric can be written as

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[dr^2 + R_0^2 \sin^2(r/R_0)(d\theta^2 + \sin^2\theta d\phi) \right] \tag{5}$$

for a positively curved space-time, describing a sphere of radius R_0, where the radius of curvature R_0 is related to k by $k = c^2/R_0^2$. What is the physical (i.e. as opposed to comoving) circumference of the universe at fixed cosmic time t?
(b) Choose the coordinate system such that the photon is emitted at $t = 0$ at the origin $r = 0$, and travels along the geodesic with $\theta = \phi = 0$. Write down the expression for its physical distance from the origin at fixed cosmic time t as an integral over t. Divide the expression you found by your answer to (a), to obtain the *fraction* $f(t)$ of the circumference covered by the photon by time t.
(c) Use the parametric solution given in part (a) of the previous problem to convert the integral over t in (b) to an integral over θ, and find the function $f(\theta)$. Show that $f(\theta = \pi) = 1/2$ and $f(\theta = 2\pi) = 1$.