Q1 Relativistic Equilibrium Thermodynamics

$$f(|\vec{p}|) = \frac{1}{\exp\left(\frac{E-\mu}{kT}\right) \pm 1}$$

(+) Fermi-Dirac

(−) Bose Einstein distribution

$$n = \frac{g}{(2\pi\hbar)^3} \int f(|\vec{p}|)\, d^3 p$$

$E^2 = p^2c^2 + m^2c^4$

$E_{total} = E_{rest} + E_{kinetic}$

$E = \sqrt{c^2(p^2 + m^2c^2)}$ for some g degeneracy factor

a.) what is $kT \gg mc^2$, which is the rest energy

$kT \gg \mu$, chemical potential

$E^2 = (mc^2)^2 + (pc)^2$, $E_{kinetic} = mc^2\left\{\left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2} - 1\right\}$

$E = \left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2} mc^2$ for $p \gg mc$

$E_k \sim pc$

for the relativistic limit, $E \sim |\vec{p}|c$ $\Rightarrow f(|\vec{p}|) \sim \dfrac{1}{\exp\left(\frac{|\vec{p}|c - \mu}{kT}\right) \pm 1}$

integrating over a sphere $d^3p = 4\pi |\vec{p}|^2 d|\vec{p}|$

let $x = \dfrac{|\vec{p}|c}{kT}$ $\quad d|\vec{p}| = \dfrac{kT}{c} dx$

$|\vec{p}| = \dfrac{xkT}{c}$, $|\vec{p}|^2 = \dfrac{x^2k^2T^2}{c^2}$

$$n = \int_0^\infty g\frac{4\pi|\vec{p}|^2}{(2\pi\hbar)^3} \frac{1}{\exp\left(\frac{|\vec{p}|c - \mu}{kT}\right) \pm 1} d|\vec{p}|$$

$$\Rightarrow n = g\frac{4\pi(kT)^3}{2\pi\hbar^3 c^3} \int_0^\infty \frac{x^2\, dx}{\exp\left(x - \frac{\mu}{kT}\right) \pm 1}$$

for $kT \gg \mu$, $\exp\left(x - \frac{\mu}{kT}\right) \sim \exp(x)$

$$\Rightarrow \ldots \int_0^\infty \frac{x^2\, dx}{\exp(x) + 1} \quad \text{fermions}$$

$$\ldots \int_0^\infty \frac{x^2\, dx}{\exp(x) - 1} \quad \text{Bosons}$$

we know (mathematica)

$$g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3} \times \int_0^\infty \frac{x^2\,dx}{\exp(x)-1} = 2\zeta(3) = \frac{g4\pi(kT)^3}{8\pi^3\hbar^3} \times \zeta(3) = \frac{\zeta(3)}{\pi^2} g\left(\frac{kT}{\hbar c}\right)^3 \checkmark$$

↳ bosons

$$\int_0^\infty \frac{x^2\,dx}{\exp(x)+1} = \frac{3}{2}\zeta(3) = g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3}\frac{3}{2}\zeta(3)$$

↳ fermions

$$= \frac{3}{4}\frac{\zeta(3)}{\pi^2} g\left(\frac{kT}{\hbar c}\right)^3$$

extra credit: $n = g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3}\int_0^\infty \frac{x^2\,dx}{\exp(x-\frac{\mu}{kT})\pm 1}$

$$\exp\left(x-\frac{\mu}{kT}\right) = \exp(x)\cdot\exp\left(\frac{-\mu}{kT}\right)$$

if we expand $\exp\left(\frac{-\mu}{kT}\right) \sim 1-\frac{\mu}{kT}$

$$\frac{1}{\exp(x-\frac{\mu}{kT})\pm 1} \sim \frac{1}{\exp(x)\pm 1} + \frac{\mu/kT}{[\exp(x)\pm 1]^2}$$

$$\Rightarrow n = g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3}\int_0^\infty x^2\left[\frac{1}{\exp(x)\pm 1} + \frac{\mu/kT}{[\exp(x)\pm 1]^2}\right]dx$$

$$= \cdots \int_0^\infty \frac{x^2\,dx}{\exp(x)\pm 1} + \frac{\mu}{kT}\int_0^\infty \frac{x^2\,dx}{[\exp(x)\pm 1]^2}$$

we already know $\int_0^\infty \frac{x^2\,dx}{\exp(x)-1} = 2\zeta(3)$ $\int_0^\infty \frac{x^2\,dx}{\exp(x)+1} = \frac{3}{4}\zeta(3)$

since $\frac{1}{[\exp(x)\pm 1]^2} = \frac{-d}{dx}\left(\frac{1}{\exp(x)\pm 1}\right) \Rightarrow \int_0^\infty \frac{x^2\,dx}{[\exp(x)\pm 1]^2} = \int_0^\infty \frac{x\,dx}{\exp(x)\pm 1} = \begin{cases} 2\zeta(2) & \text{bosons} \\ \frac{3}{2}\zeta(2) & \text{fermions} \end{cases}$

$\zeta(2) = \frac{\pi^2}{6}$

$$\Rightarrow g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3}\left[2\zeta(3) + 2\frac{\mu}{kT}\zeta(2)\right] \text{ for } (-1)$$

$$n \sim g\frac{4\pi(kT)^3}{(2\pi\hbar c)^3}\left[\frac{3}{4}\zeta(3) + \frac{3}{2}\frac{\mu}{kT}\zeta(2)\right] \text{ for } (+1)$$

0th order

$$n \sim n_0\left(1 + \frac{A\mu}{kT}\right)$$ such that for bosons $A = \frac{\zeta(2)}{\zeta(3)} \approx \frac{\pi^2/6}{1.2} \sim 1.64$

Fermions $A = \frac{\zeta(2)}{\zeta(3)} \cdot 2 \sim 3.28$

(b) show that for non-relativistic limit $(kT \ll mc^2)$

$$n = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{mc^2-\mu}{kT}\right)$$

$$n = g\int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\exp\left(\frac{E-\mu}{kT}\right) \pm 1}$$

for non-relativistic, $p \ll mc$

$$E_{total} = E_{rest} + E_k$$
$$= mc^2 + mc^2\left\{\left[1 + \left(\frac{p}{mc}\right)^2\right]^{1/2} - 1\right\}$$
$$= mc^2 + mc^2\left[\cancel{1} + \frac{1}{2}\frac{p^2}{m^2c^2} - 1\right]$$
$$= mc^2 + m\cancel{c^2}\left[\frac{1}{2}\frac{p^2}{m^2\cancel{c^2}}\right]$$

$$n = g\int \frac{d^3p}{(2\pi\hbar)^3} \frac{1}{\exp\left(\frac{mc^2 + \frac{p^2}{2m} - \mu}{kT}\right) \pm 1}$$

$mc^2 \gg kT$, $\exp\left(-\frac{mc^2-\mu}{kT}\right)$ dominates

$$E_{total} = mc^2 + \frac{p^2}{2m}$$

$$n = g\exp\left[\frac{\mu - mc^2}{kT}\right] \underbrace{\int \frac{d^3\vec{p}}{(2\pi\hbar)^3} \exp\left[\frac{-p^2}{2mkT}\right]}$$

$$\int \exp\left(\frac{-|p|^2}{2mkT}\right) d^3p = 4\pi(2mkT)^{3/2}\int_0^\infty \sqrt{x}\, e^{-x}\, dx$$

$$d^3p = 4\pi|\vec{p}|^2 d|\vec{p}|$$

let $x = \frac{\vec{p}^2}{2mkT}$, $|\vec{p}|^2 = 2mkT\,x$

$$d|\vec{p}| = \sqrt{mkT}\, dx$$

$$\int_0^\infty \sqrt{x}\, e^{-x}\, dx = \Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \int \exp\left(-\frac{\vec{p}^2}{2mkT}\right) d^3p = (2\pi mkT)^{3/2}$$

$$\Rightarrow n = g\,e^{\frac{\mu - mc^2}{kT}} \frac{(2\pi mkT)^{3/2}}{(2\pi\hbar)^3} = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{mc^2-\mu}{kT}\right)$$

Problem 2: Recession Velocity vs. Redshift

recessional velocity of galaxy due to the expansion of the universe $V_{rec} = \dot{a} r$

$\dot{a}$: expansion rate

r: fixed comoving coordinate of the galaxy

→ calculate the coordinate distance r to a galaxy at z

→ plot $V_{recessional}(z)$

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz}{E(z)}$$

$$E(z) = \left[\Omega_{\gamma,0}(1+z)^4 + \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_\Lambda \right]^{1/2}$$

$$E(z) = \sqrt{\Omega_m (1+z)^3 + \Omega_\Lambda}$$

let $\Omega_m = 0.3$

$\Omega_\Lambda = 0.7$

$H_0 = 70\ \mathrm{km/s/Mpc}$

assuming radiation $\Omega_{\gamma,0} \sim 0$

$\Omega_{k,0} = 0$ flat by definition.

$$\Rightarrow r(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{\left[\Omega_m (1+z')^3 + \Omega_\Lambda \right]^{1/2}}$$

→ solving numerically

$\Rightarrow V_{rec} = \dot{a} r$, today at t_0

$H(t_0) = \dot{a} = H_0 = 70\ \mathrm{km/s/Mpc}$

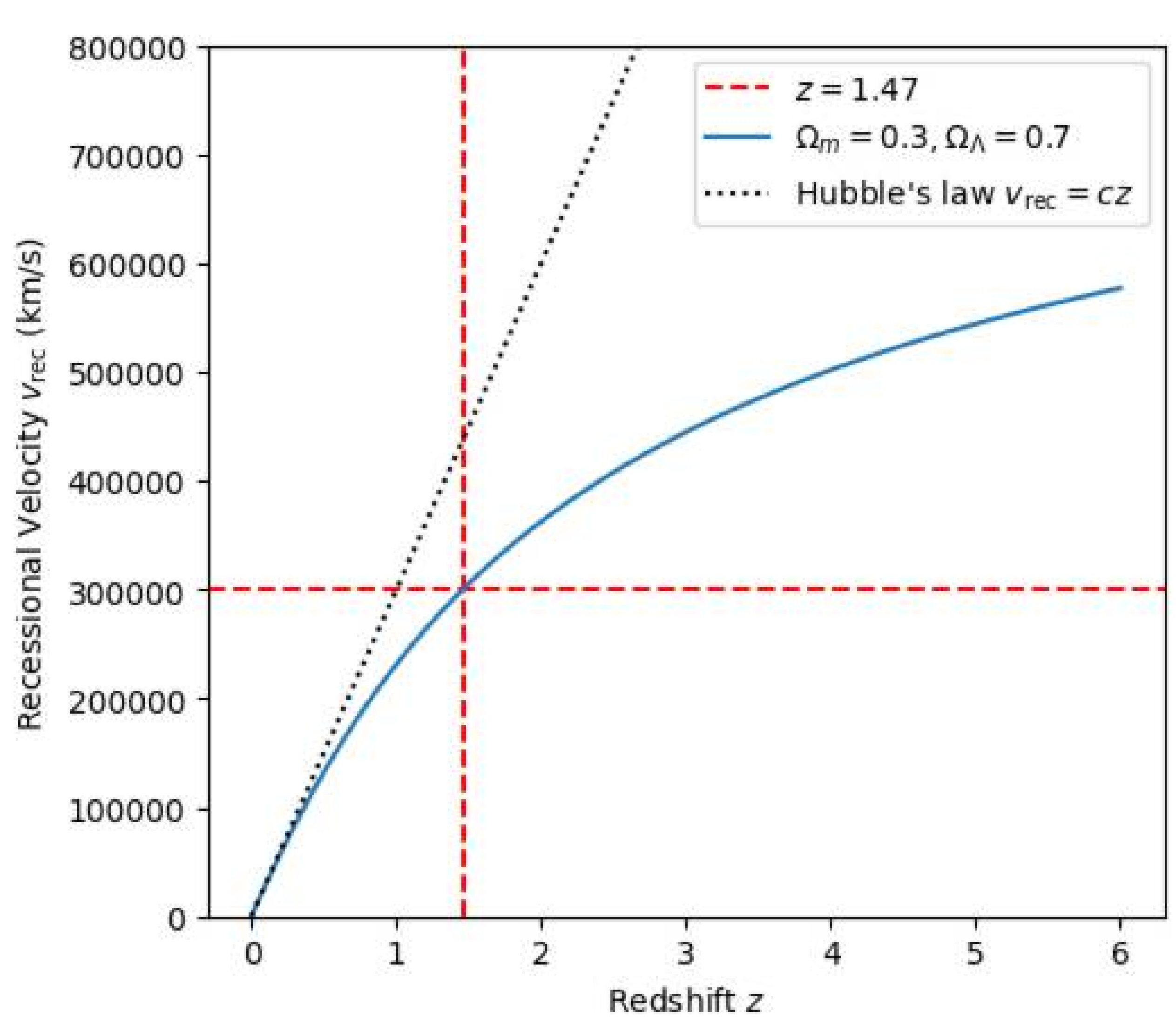
- numerically, the recessional velocity

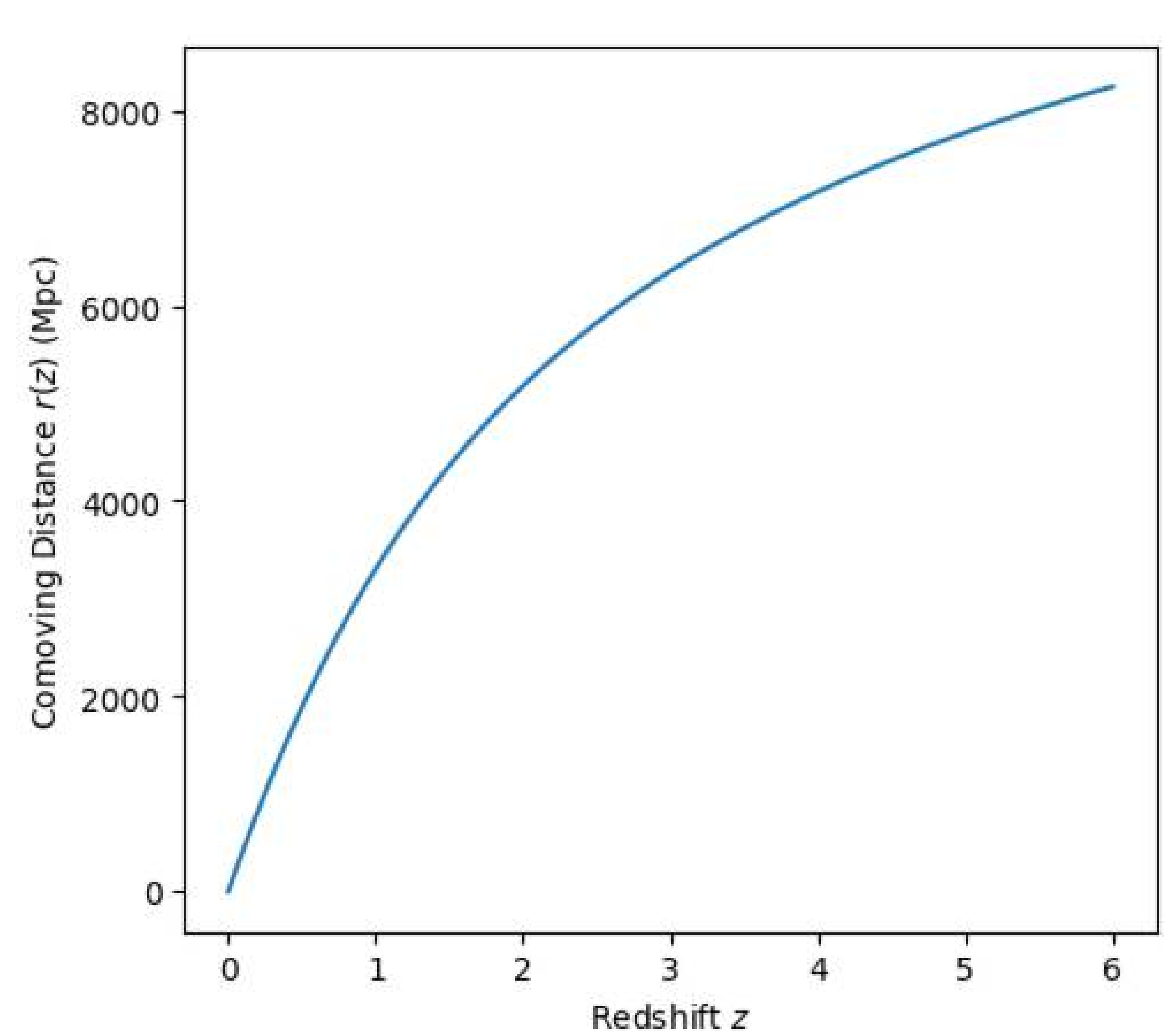
$V_{rec} = H_0\, r(z) > c$

$z \sim 1.47$ for $V_{rec} > c$, for $z \ll 1$,

$$r(z) \sim \frac{cz}{H_0}$$

such that $V_{rec} = H_0 \frac{cz}{H_0} = cz$





Problem 3: Measuring the change in Redshift

the z of a source at ~ fixed comoving coordinate will change

spectrometers can measure redshifts corresponding to $V \sim 3m/s$

→ how long would you need to observe a change in redshift of a $z = 2.0$

$z = 10$ quasar

$\Omega_m = 0.3$

$\Omega_\Lambda = 0.7$

$H_0 = 70$ km/s/Mpc

$$1+z = \frac{a(t_{obs})}{a(t_{em})} = \frac{a(t_1)}{a(t_2)}$$

$$H(t) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

$$\frac{d}{dt_{obs}}(1+z) = \frac{d}{dt_{obs}}\left(\frac{a(t_{obs})}{a(t_{em})}\right)$$

$$= \frac{\dot a(t_{obs})}{a(t_{em})} - \frac{a(t_{obs})}{a(t_{em})^2}\cdot \dot a(t_{em})\frac{dt_{em}}{dt_{obs}}$$

$$\dot a \equiv \frac{da}{dt}, \quad H(t) \equiv \frac{\dot a(t)}{a(t)}$$

$$\frac{d}{dt_{obs}}(1+z) = (1+z)\left[H(t_{obs}) - \cancel{H}(t_{em})\frac{dt_{em}}{dt_{obs}}\right]$$

by $v \approx 3$ m/s

$$\Delta z \sim \frac{\Delta v}{c} \Rightarrow \frac{3\ m/s}{3\times10^8\ m/s} = 10^{-8}$$

change in redshift $\Delta z \sim 10^{-8}$

$$\frac{dt_{em}}{dt_{obs}} \quad \frac{1}{1+z}$$

$$\frac{d}{dt_{obs}}(1+z) = (1+z)\left[H(t_{obs}) - \frac{H(t_{em})}{1+z}\right]$$

$$\frac{dz}{dt_{obs}} = \frac{d}{dt_{obs}}(1+z) = (1+z)H(t_{obs}) - H(t_{em})$$

or if observed now t_0 :

$$\frac{dz}{dt_{obs}} = (1+z)H_0 - H(t)$$

$$\frac{dz}{dt} = (1+z)H(t_{obs}) - H(t_{em}) \xrightarrow[\text{now}]{\text{observe}} \frac{dz}{dt} = (1+z)H_0 - H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$$

for an object at $z=2$

$$\frac{dz}{dt} = H_0\left[1+z-\sqrt{\Omega_m(1+z)^3+\Omega_\Lambda}\right]$$
$$= 70\left[3-\sqrt{0.3(27)+0.7}\right]$$
$$= 70[3-2.96] = 2.1 \text{ km/s/Mpc}$$

convert

$$\Rightarrow \frac{dz}{dt} = \frac{2.1\times10^{3}}{\underbrace{3\times10^{22}}_{1\text{ mpc}}} = 7\times10^{-20}\ s^{-1}$$

, recall $\Delta z \sim 10^{-8}\ s^{-1}$

$$\Delta t = \frac{\Delta z}{\left|\frac{dz}{dt}\right|} = \frac{10^{-8}}{7\times10^{-20}} \cong 1.4\times10^{11} \text{ seconds} \sim \frac{10^{11}}{3\times10^{7}} \sim 5000 \text{ years}$$

for a $z=2$ object.

$\to$ Now, $z=10$

$$\frac{dz}{dt} = 70\left[11-\sqrt{0.3(11)^3+0.7}\right] = -630 \text{ km/s/Mpc}$$

$$\frac{dz}{dt} = \frac{-630\times10^{3}}{3.086\times10^{22}} = -2.0\times10^{-17} \Rightarrow$$

$$\Delta t = \frac{10^{-8}}{2\times10^{-17}}$$

$$\Delta t = \frac{4.9\times10^{8}\ s}{3\times10^{7}\ s/\text{year}} \cong 16 \text{ years}$$

Problem 4: The age of the universe in ΛCDM

flat, two component universe with non relativistic matter

$\Omega_m \neq 0$

$\Omega_\Lambda = 1 - \Omega_m$

$\Omega_m = 1 - \Omega_\Lambda$

$H_0 = 70\ \text{km/s/Mpc}$

calculate age(z)

express answer in terms of Ω_Λ and H_0

how does $t_0 = t(z=1)$ behave as $\Omega_\Lambda \to 1$

sketch t_0 vs. Ω_Λ

→ for a flat ($k=0$) universe, $H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda}$

if $\Omega_\Lambda = 1 - \Omega_m$

$$H(z) = H_0\sqrt{\Omega_m(1+z)^3 + 1 - \Omega_m}$$

$$t(z) = \frac{1}{H_0}\int_z^\infty \frac{dz'}{(1+z')\sqrt{\Omega_m(1+z')^3 + 1 - \Omega_m}}$$

$a = \frac{1}{1+z}$ $\quad dz = \frac{-da}{a^2}$

$\frac{da}{dz} = \frac{-1}{(1+z)^2}$

$\frac{a^3}{a^{3/2}}$

$$= \frac{1}{H_0}\int_0^{\frac{1}{1+z}} \frac{1}{a\sqrt{\Omega_m a^{-3} + 1 - \Omega_m}}\left(\frac{-da}{a^2}\right)$$

at $z=0, a=1$ $\quad$ at $z=\infty, a=0$

$z=z, a=\frac{1}{1+z}$

$$= \frac{1}{H_0}\int_0^{\frac{1}{1+z}} \frac{da}{a^3\sqrt{\Omega_m a^{-3} + 1 - \Omega_m}} \Rightarrow t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a^{3/2}\sqrt{\Omega_m + \Omega_\Lambda a^3}}$$

the general expression which we can numerically integrate is

$$t(z) = \frac{1}{H_0}\int_0^{1/1+z} \frac{da}{a^{3/2}\sqrt{\Omega_m + (1-\Omega_m)a^3}}$$

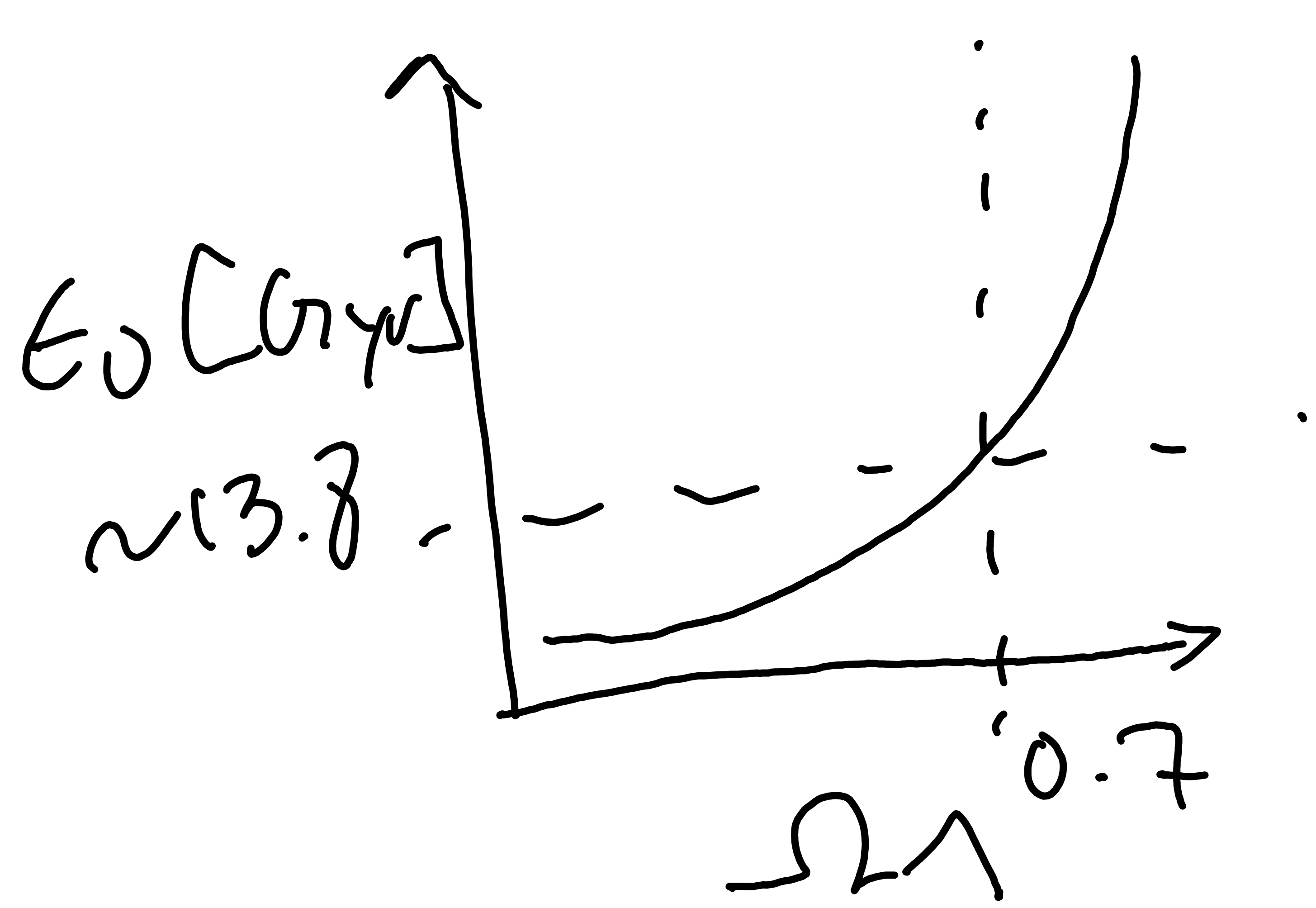
$$= \frac{1}{H_0}\int_0^{1/1+z} \frac{da}{a^{3/2}\left[(1-\Omega_\Lambda) + \Omega_\Lambda a^3\right]^{1/2}}$$

→ How does t_0 behave as $\Omega_\Lambda \to 1$ for various values of Ω_Λ

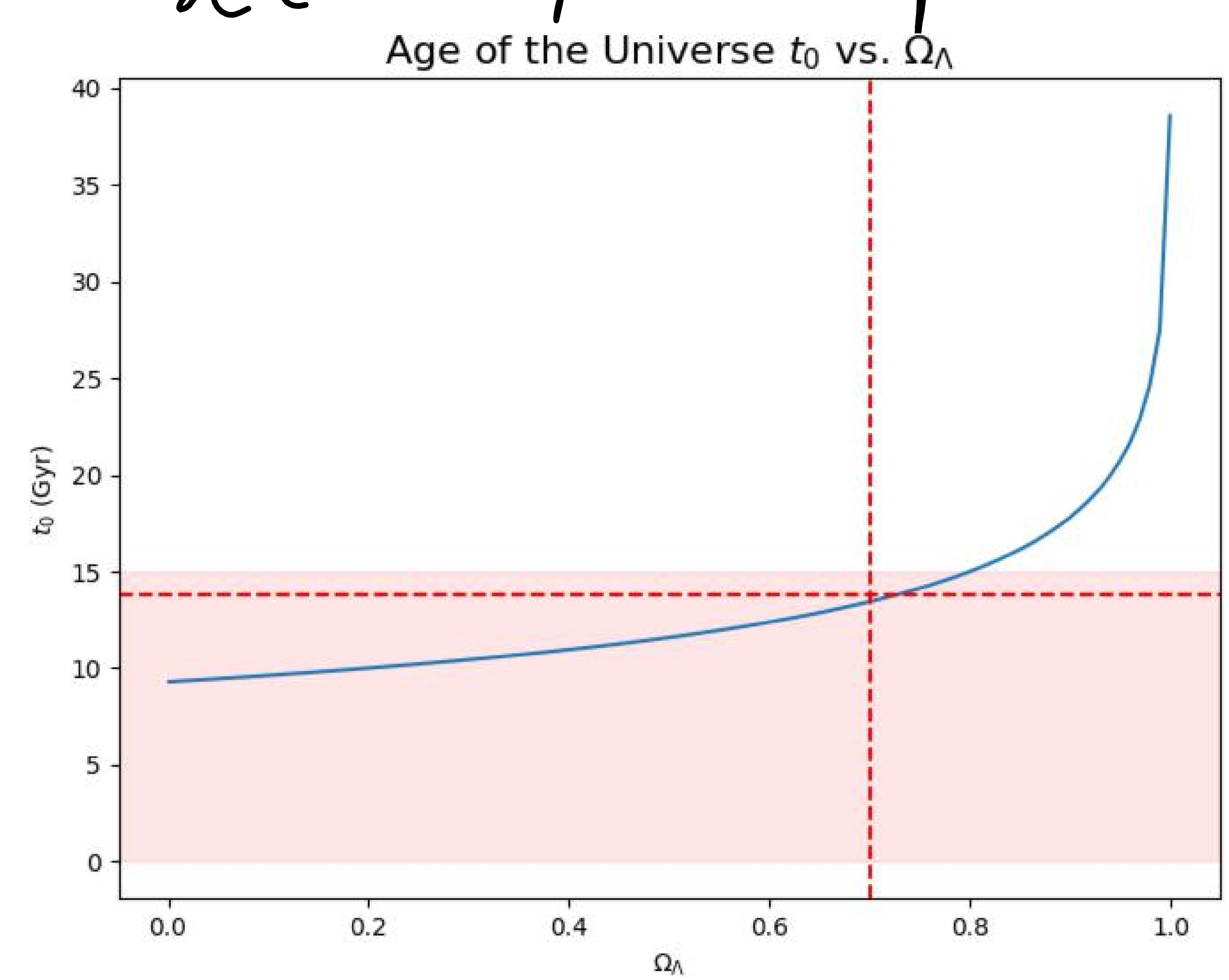
$$t_0 = \frac{1}{H_0}\int_0^1 \frac{da}{a^{3/2}\sqrt{(1-\Omega_\Lambda)+\Omega_\Lambda a^3}} \quad \Omega_\Lambda \to \quad t_0 \sim \frac{1}{H_0}$$

in a flat universe $t_0 \lesssim 15$ Gyr implies values of $\Omega_\Lambda \lesssim 0.8$

$$t_0 \propto \frac{1}{H_0}\frac{1}{\sqrt{\Omega_\Lambda}} \quad \text{as}$$



see the plot computed numerically:

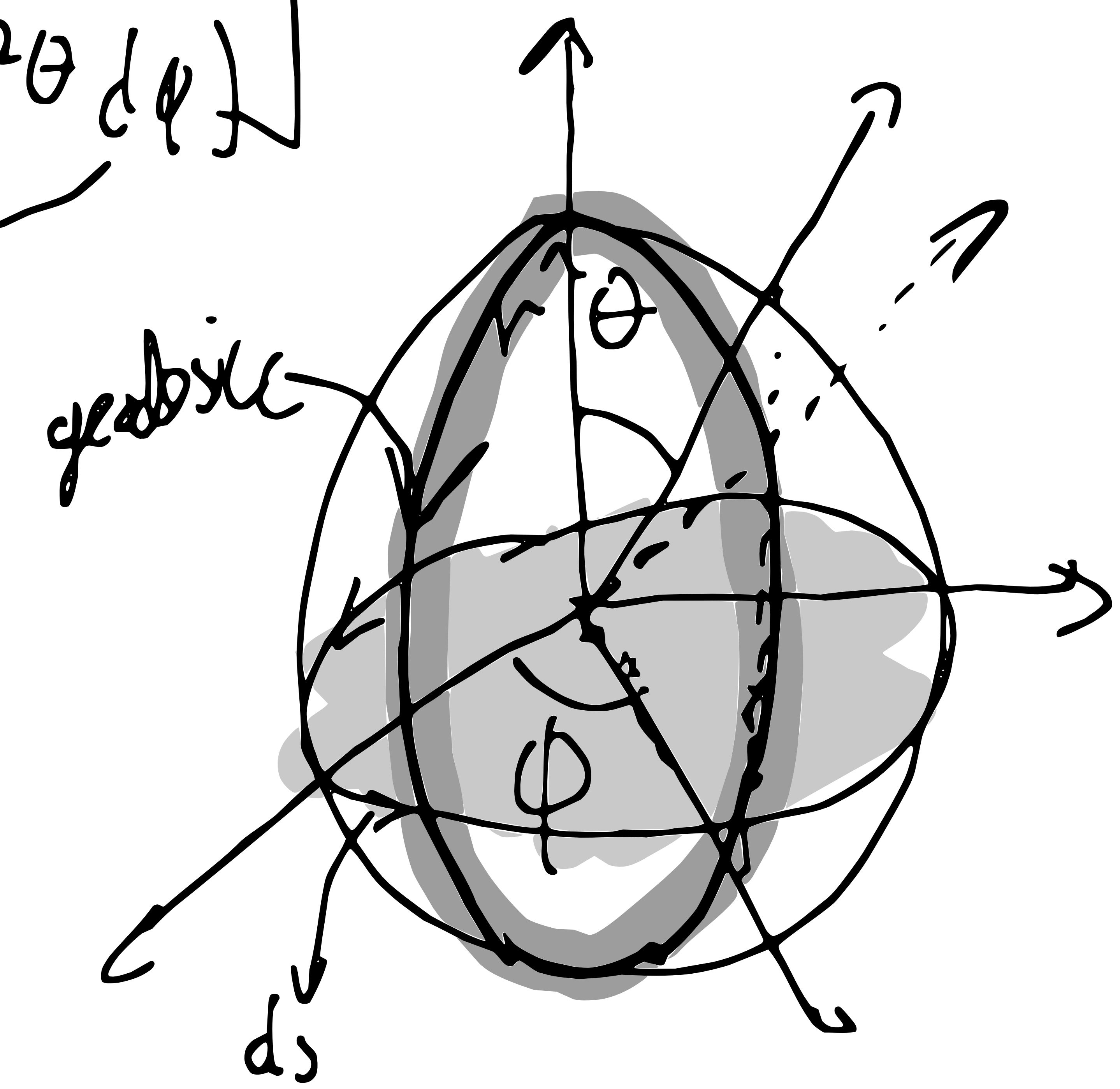


Problem 5: Seeing Around the Universe

relativistic matter, $k>0$, show that a photon travels precisely all the way around the universe by the time of the big crunch.

(a) $ds^2 = -c^2dt^2 + \underbrace{a(t)^2}_{\text{scaling}}\left[dr^2 + R_0^2\sin^2\left(\frac{r}{R_0}\right)\underbrace{(d\theta^2 + \sin^2\theta\, d\phi)}_{\text{angular part}}\right]$

comoving distance along the spatial slice



$k = \frac{c^2}{R_0^2}$ physical circumference at fixed cosmic time.

Circumference of a great circle $C = 2\pi r$, $\theta = \frac{\pi}{2}$, $r = R_0$, $dr = 0$

→ so for spatial part, $dt = 0$

$\sin\left(\frac{\pi}{2}\right) = 1$

$ds^2 = a(t)^2\left[\underbrace{dr^2}_{0} + R_0^2\underbrace{\sin^2\left(\frac{r}{R_0}\right)}_{\sin(1)}(\cancel{d\theta^2}\,^{0} + \cancel{\sin^2\theta}\, d\phi)\right]$

physical circumference

$C_{phys}(t) = \int_0^{2\pi} a(t)\, R_0 \sin(1)\, d\phi$

$C_{phys}(t) = 2\pi\, \underbrace{a(t)}_{\text{scale factor at time } t}\, R_0 \sin(1)$

$R_0 = \frac{c}{\sqrt{k}}$ curvature radius

(b) $t=0$, $r=0$, $\theta = \phi = 0$

$d\theta = 0$ $d\phi = 0$, $r = 0$

$ds^2 = -c^2dt^2 + a(t)^2dr^2$

$0 = -c^2dt^2 + a(t)^2dr^2$

$c\,dt = a(t)\,dr$

$dr = \frac{c}{a(t)}dt$

integrating from $t=0$ $(r=0)$ to a time t

$$r(t) = \int_0^t \frac{c}{a(t')}\, dt'$$

$$D_{phys}(t) = a(t)\, r(t) = a(t) \int_0^t \frac{c}{a(t')}\, dt'$$

$$f(t) = \frac{D_{phys}(t)}{C_{phys}(t)} = \frac{a(t) \int_0^t \frac{c}{a(t')}\, dt'}{2\pi a(t) R_0 \sin(1)}$$

$$= \frac{\int_0^t \frac{c}{a(t')}\, dt'}{2\pi R_0 \sin(1)}$$

(c) $a(\theta) = \frac{4\pi G \rho_0}{3k}(1-\cos\theta)$

$t(\theta) = \frac{4\pi G \rho_0}{3k^{3/2}}(\theta - \sin\theta)$

$$\frac{dt}{d\theta} = \frac{d}{d\theta}\left[\frac{4\pi G \rho_0}{3k^{3/2}}(\theta - \sin\theta)\right]$$

$$= \frac{4\pi G \rho_0}{3k^{3/2}}(1-\cos\theta)$$

$$dt = \frac{4\pi G \rho_0}{3k^{3/2}}(1-\cos\theta)\, d\theta$$

$$r(t) = \int_0^t \frac{c}{a(t')}\,dt' \Rightarrow r(\theta) = \int_0^\theta \frac{c}{a(\theta')} \cdot \frac{4\pi G \rho_0}{3k^{3/2}} (1-\cos\theta')\,d\theta'$$

$\rightarrow$ since $a(\theta) = \frac{4\pi G \rho_0}{3k} \cdot (1-\cos\theta)$, $\quad \frac{1}{a(\theta)} = \frac{3k}{4\pi G \rho_0} \cdot \frac{1}{1-\cos\theta}$

$$r(t) = \int_0^\theta \frac{c}{1-\cos\theta'} \frac{3k}{4\pi G \rho_0} \cdot \frac{4\pi G \rho_0}{3k^{3/2}} (1-\cos\theta')\,d\theta'$$

$$= \frac{c}{k^{1/2}} \int_0^\theta d\theta'$$

$$f(\theta) = \frac{D_{phys}(\theta)}{C_{phys}(\theta)} = \frac{r(t)}{2\pi R_0} = \frac{\frac{c}{k^{1/2}} \int_0^\theta d\theta'}{2\pi \frac{c}{k^{1/2}}} = \frac{\int_0^\theta d\theta'}{2\pi}$$

$$f(\theta) = \frac{\theta}{2\pi} \quad \therefore \quad f(\theta = \pi) = \frac{1}{2}$$

$$f(\theta = 2\pi) = \underline{1}$$