# **Cosmology, Black Holes and Gravity** *Homework 4 due Mon, Dec. 9*

## Problem 1 – Reionization: Gunn–Peterson Bound on Neutral Hydrogen

Consider a photon emitted with wavelength $\lambda_\alpha = 1200\text{Å}$ by a quasar at $z = 3$. Assume that all the hydrogen in the Universe is in neutral form, and calculate the probability $\tau_\alpha$ that this photon is absorbed on its way to us. (Hint: you have to integrate over redshift, similar to the calculation of the electron scattering optical depth from reionization, which I discussed in class). Assume, for simplicity, a flat $\Omega_m = 1$ universe, with $H_0 = 72$ km/s/Mpc, $\Omega_b = 0.04$, and a hydrogen mass fraction $Y_H = 0.76$. Feel free to either look up the Lyman $\alpha$ absorption cross-section (for a temperature of $10^4$K appropriate for photoionized gas); or, for simplicity, you may approximate it as a function of wavelength by a narrow "top–hat" with $\sigma_\alpha = 6 \times 10^{-14}$ cm$^2$ for photons with wavelengths $\lambda_\alpha = 1215 \pm 0.05\text{Å}$ and $\sigma_\alpha = 0$ outside this range. Observations show that $\tau_\alpha < 0.05$. What does this imply for the fraction of hydrogen that can be in neutral form at $z \approx 3$ ?

## Problem 2 – Recombination: Hydrogen vs. Helium Recombination

As shown in class, hydrogen recombined at a redshift of approximately $z = 1200$, with recombination starting already at $z \sim 1600$.

**a)** What was the ratio of the number density of hydrogen–ionizing photons (i.e. photons with energies $E \geq$13.6eV) to the number density of hydrogen atoms at $z = 1200$ ? Explain why this ratio does not have to equal unity at recombination.

**b)** What was the ratio of the number density of helium–ionizing photons (i.e. photons with energies $E \geq$24.6eV) to the number density of helium atoms at $z = 1600$ ? Does your answer imply that helium recombines before or after hydrogen?

For this problem, assume the following: at the present day, baryons contribute a fraction $\Omega_b = 0.04$ of the critical density, of which $Y_{\rm H} = 76\%$ by mass is hydrogen, and $Y_{\rm He} = 24\%$ is helium. Assume a Hubble constant of $H_0$ =70 km/s/Mpc. The temperature of the cosmic microwave background (CMB) today is $T_0 = 2.725$K.

## Problem 3 – BBNS: Gamow's Estimate for the Present–Day CMB Temperature

The photo–dissociation threshold of deuterium is 2.22 MeV, corresponding to a temperature of $T \sim 2 \times 10^{10}$K. Roughly, deuterium should form at the epoch when the CMB temperature drops below this value. At this epoch, the Universe is flat and dominated by the CMB. The condition that a trace amount of deuterium should be produced is $\langle \sigma v \rangle n_b t \sim 1$, where $\sigma$ is the cross–section for $n + p \to D + \gamma$, $v$ is the relative velocity of $n$ and $p$, $\langle \sigma v \rangle \approx 5 \times 10^{-20}$ cm$^3$ s$^{-1}$ is a velocity–average collision rate, and $t$ is the age of the Universe at deuterium formation.

**a)** Compute $t$ at the epoch when the radiation temperature is $T \sim 2 \times 10^{10}$K, and therefore obtain an estimate of the baryon density $n_b$ at the epoch of deuterium formation.

**b)** Compare your answer above to the present–day baryon density ($\Omega_b = 0.04$), derive the scale factor $a$ at deuterium formation, and use $T \propto a^{-1}$ to find the present–day CMB temperature.

**c)** Comment on why your answer might be different from the measured value of 2.7K (i.e., which of the above assumptions is suspect)?

**Problem 4: Power–Law Inflation**

Inflation is usually associated with a scalar field whose potential is a power–law ($V \propto \phi^n$), which, as we discussed, can lead to exponential inflation with $a(t) \propto \exp(Ht)$ with $H \sim$ constant. An alternative is a steeper, exponential potential, which results in a slower "power–law inflation".

**a)** Consider a scalar field $\phi$ with the potential $V(\phi) = V_0 \exp(\phi/\phi_0)$. Show that this potential, with $a(t) = a_0(t/t_0)^n$, is a solution of both the Friedmann equation (for simplicity, assume a flat universe) and the equation of motion for $\phi$ (ignore spatial gradients). Find an expression for $n$ as a function of $\phi_0$ and $V_0$.

**b)** In order to solve the horizon problem, the expansion must be "superluminal", or equivalently, the integral for the particle horizon must diverge. Show that this requires $n > 1$, and find the corresponding lower limit on $\phi_0$.