Problem 1 - Reionization: Gunn-Peterson Bound on Neutral Hydrogen

- photon emitted at $\lambda_\alpha = 1200\,\text{Å}$ at $z=3$
- assume HI all around
- what is the probability τ_α that this photon is absorbed on its way to us?

assume: $\Omega_m = 1$, flat $\Omega_\Lambda = 0$ → note $\tau_\alpha < 0.05$ from observations

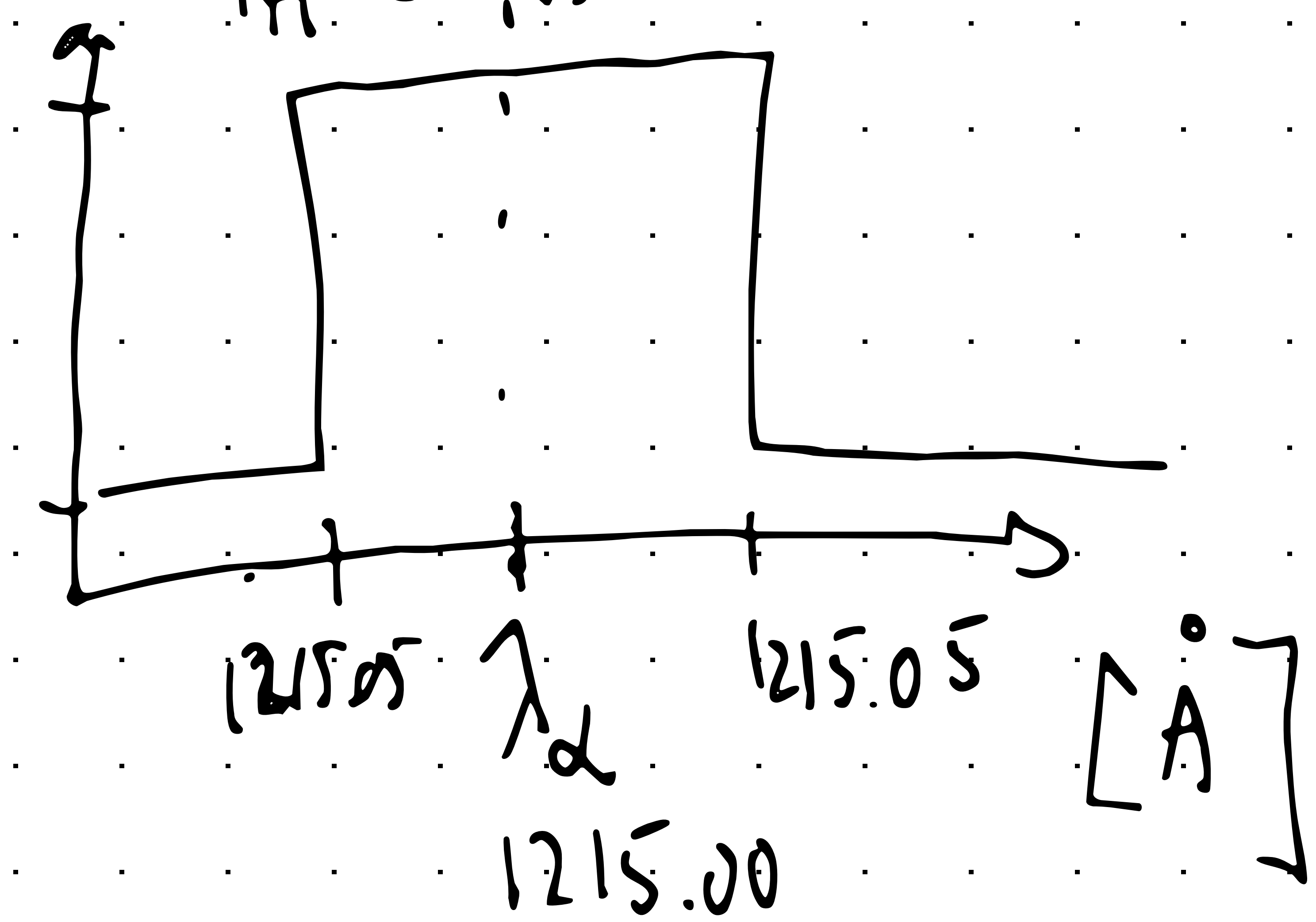
$H_0 = 72$ km/s/Mpc $\Omega_b = 0.04$

hydrogen mass fraction $Y_H = 0.76$.

σ_α 6×10^{-14}
$[cm^2]$

$\lambda_{low} = (1+z)\,1214.95$
$\lambda_{high} = (1+z)\,1215.05$ $\Big|_{z=4}$

0



Recall from Class

the surface of last scattering
→ electron scattering rate Γ
→ prob. for a single photon to scatter b/w t_{lss} and present time t_0

$$\tau_e = \int_{t_{lss}}^{t_0} \Gamma(t)\,dt = 1$$

$$\tau(z_{lss}) = \int_0^{z_{lss}} \frac{\Gamma(z)\,dz}{H(z)(1+z)}$$

where $\Gamma = n_e \sigma_T c$

$$\Gamma = n_H(z)\,\sigma_\alpha\, c$$

where $n_H(z) = n_{H,0}(1+z)^3$

$$n_{H,0} = \frac{\Omega_b \rho_{crit}}{m_H} Y_H$$

$\sigma_\alpha = 6\times10^{-14}\ cm^2$

$$\rho_{crit} = \frac{3H_0^2}{8\pi G}$$

$m_H = 1.67\times10^{-24}\,g$

$H(z) = H_0(1+z)^{3/2}$ since $\Omega_m = 1$

where $n_H(z) = \frac{\Omega_b Y_H}{m_H}\left(\frac{3H_0^2}{8\pi G}\right)(1+z)^3$

$$\int \frac{\Gamma(z)\, dz}{H_0 (1+z)^{3/2} (1+z)} = \int_0^3 \frac{n_H(z)\, \sigma_\alpha\, c\, dz}{H_0 (1+z)^{3/2} (1+z)}$$

$$= \int_0^3 \frac{\frac{\Omega_b}{m_H} Y_H\, \rho_{crit}\, (1+z)^{6/2}\, \sigma_\alpha\, c\, dz}{H_0 (1+z)^{3/2} (1+z)}$$

$$= \frac{\Omega_b}{m_H} \frac{Y_H}{H_0} \frac{3H_0^2}{8\pi G} \sigma_\alpha c \int_0^3 (1+z)^{1/2}\, dz$$

$$= \frac{\Omega_b}{m_H} \frac{Y_H H_0 c\, \sigma_\alpha}{8\pi G} \frac{2}{3}\left[4^{3/2} - 1\right]$$

$$= \frac{0.04(0.76)(H_0)(c)(6\times10^{-14})}{1.67\times10^{-24}(8\pi)\, 6.67\times10^{-8}} \left[4.66\right]$$

$Y_H = 0.76$

$\Omega_b = 0.04$

$\sigma_\alpha = 6\times10^{-14}\ cm^2$

$c = 3\times10^{10}\ cm/s$

$H_0 = 72\ km/s/Mpc$
$\simeq 2.4\times10^{-15}\ cm/s$

$G = 6.67\times10^{-8}\ \frac{cm^3}{g s^2}$

$m_H = 1.67\times10^{-24}\ g$

$$\tau_\alpha \sim 2\times10^8$$ in a matter dominated universe.

τ_α means the photo will <u>likely</u> interact; however, since we can see it, it means it is likely not absorbed. Hence H mostly exists as H^+ or ionized state at $z=3$, which means the photon just scatters.

Problem 2 - Recombination H vs. He

we know that hydrogen recombined at $z \sim 1200$, starting at $z \sim 1600$

assuming $\Omega_b = 0.04$

$Y_H = 76\%$ and $H_0 = 70\ \text{km/s/Mpc}$

$Y_{He} = 24\%$ $T_0 = 2.725\ \text{K}$

a.) what was $\frac{n_\gamma}{n_H}$ where n_γ is the number-density of photons with $E \geq 13.6\ \text{eV}$

$$n_\gamma = 0.243 \left(\frac{kT}{\hbar c}\right)^3$$

$$T_{CMB} = T_0 (1+z)$$

from

$$\frac{1-X}{X^2} = 3.84\, \eta \left(\frac{kT}{m_e c^2}\right)^{3/2} \exp\left(\frac{Q}{kT}\right)$$

Q → 13.6 eV

$$\eta = \frac{n_b}{n_\gamma} = \frac{n_p}{X n_\gamma}$$ $\boxed{\text{lecture 20}}$

$$n_\gamma \sim 0.243 \left(\frac{kT}{\hbar c}\right)^3$$

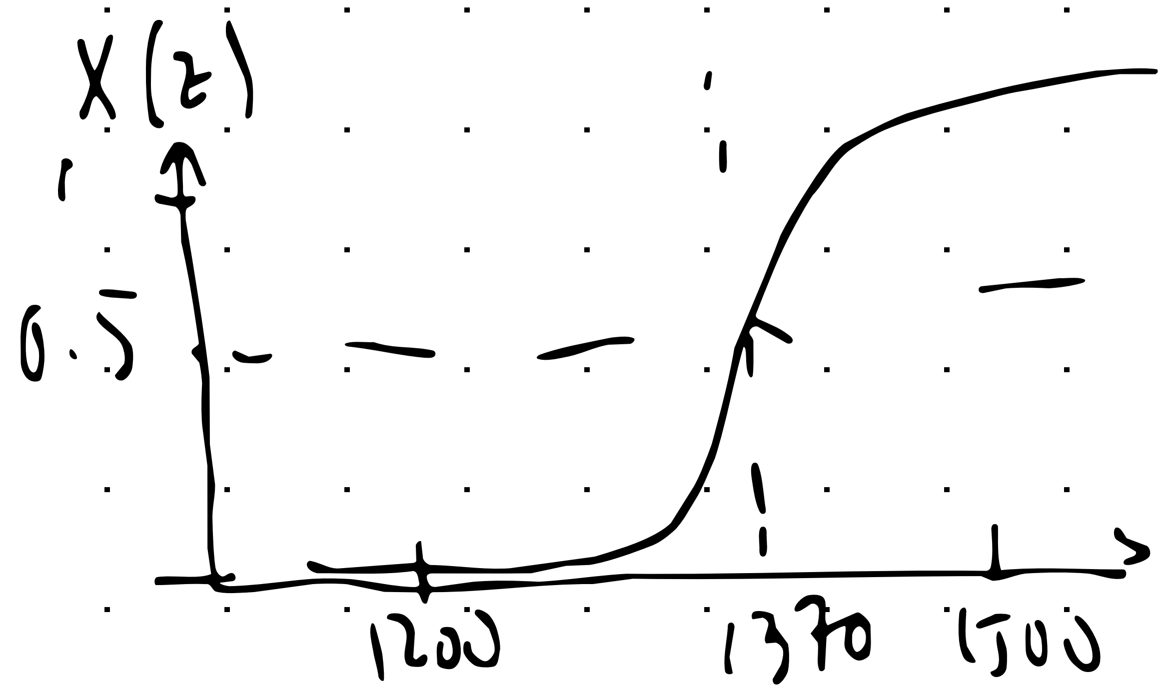
$$n_\gamma \sim 7 \times 10^{11}\ \text{cm}^{-3}$$

$$n_H = \frac{\Omega_H \rho_b}{m_H} (1+z)^3$$

baryon density

$$\rho_b = \Omega_b \rho_{crit} = 0.04 \left(\frac{3H_0^2}{8\pi G}\right) \sim 10^{-29}\ \text{g/cm}^3, \quad \Omega_H = 0.04\,(Y_H)$$

$$n_H = \frac{\Omega_b (Y_H)(10^{-29})}{m_H} (1+z)^3 \Big|_{z=1200} = 362\ \text{cm}^{-3}$$



$$\eta = \frac{n_H}{n_\gamma} \sim 10^{-9}$$, note "η" need not be 1 since recombination involves electrons binding to H^+ due to the universe expanding and the mean free path increasing. Direct recombination to the H ground state are insufficient, H atoms generally form at higher energy state, transition to lower energy state then release photons.

From the 2p state after recombining, Ly-α photon is released to get to ground state, this emitted photon gets reabsorbed by another Hydrogen atom in its ground state.

(b) $z = 1600$

$E \geq 24.6\,eV$

$T_{CMB} = T_0 (1+z)$

$$n_{He} = \frac{\Omega_B (Y_{He})(10^{-29})(1+z)^3}{m_{He}}$$

$$= \frac{0.04(0.24)\,10^{-29}(1601)^3}{4(1.67\times10^{-24})} \sim 60\ cm^{-3}$$

Here we can't use the n_γ derived in class since we are dealing with He-ionization so we use from text, eq 6.9, Peebles

$$n(\omega)\,d\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2\,d\omega}{e^{\hbar\omega/kT}-1} \qquad \text{since } E=\hbar\omega \quad \omega=\frac{E}{\hbar} \quad d\omega = \frac{dE}{\hbar}$$

$$n(E)\,dE = \frac{1}{\pi^2 c^3} \frac{\left(\frac{E}{\hbar}\right)^2}{\exp\left(\frac{E}{kT}\right)-1} \frac{dE}{\hbar} \Rightarrow \frac{E^2}{\pi^2 \hbar^3 c^3} \frac{dE}{\exp\left(\frac{E}{kT}\right)-1}$$

$$n(E) = \frac{1}{\pi^2 \hbar^3 c^3} \int_{24.6eV}^{\infty} \frac{E^2}{\exp\left(\frac{E}{kT}\right)-1}\,dE$$

let $x \equiv \dfrac{E}{kT}$ $\Rightarrow$ $$n_\gamma = \frac{(kT)^3}{\pi^2 \hbar^3 c^3} \int_{x_0}^{\infty} \frac{x^2\,dx}{e^x-1} \sim 3.46\times10^{16}\,(2.41)$$

$E = xkT$

$dE = kT\,dx$ $2.725(1601)\,K$ $8\times10^{16}\ cm^{-3}$

$$x_0 = \frac{24.6\,eV}{(8.6\times10^{-5}\,\frac{eV}{K})\,T_{CMB}} = 66$$

$$\Rightarrow \eta = \frac{n_{He}}{n_\gamma} = \frac{60}{8\times10^{16}} \sim 7\times10^{-16}$$

helium recombines, before, H^+

Problem 3 - BBNS : Gamow's estimate for the present-day CMB Temperature

deuterium 2.22 MeV photo dissociation energy

$T \sim 2 \times 10^{10}$ K, so below this, deuterium should form

universe is flat and dominated by CMB

a) freeze amount of deuterium: $\langle \sigma v \rangle n_b t \sim 1$

<u>in a radiation dominated universe.</u>

energy density $u = aT^4$

$$u = 7.56 \times 10^{-16} \frac{J}{m^3 K^4} \left(2 \times 10^{10} K\right)^4$$

$$u = 1.2 \times 10^{26} \frac{J}{m^3}$$

source: "radiation energy density" hyperphysics.phy-astr.gsu.edu

$$\frac{u}{c^2} = \frac{3H^2}{8\pi G}$$

$$H = \left[\frac{8\pi G u}{3c^2}\right]^{1/2} = 0.864 \ s^{-1}$$

for a radiation dominated univ (wikipedia)

$a \propto t^{1/2}$

$$\text{so} \quad t = \frac{1}{2H} = \frac{1}{2(0.864)} \cong 0.6 \text{ seconds}$$

σ is the crossection for $n + p \to D + \gamma$

v relative velocity of n and p

$$\langle \sigma v \rangle \sim 5 \times 10^{-20} \frac{cm^3}{s}$$

t is age of the universe.

$$n_b \sim \frac{1}{\langle \sigma v \rangle t} \sim \frac{1}{5 \times 10^{-20} (0.6)}$$

$$n_b \cong 3 \times 10^{19} \ cm^{-3}$$

b.) $n_b \sim 3 \times 10^{19}\ cm^{-3}$ at deuterium formation.

$$n_{Baryon,0} = \frac{3H_0^2}{8\pi G}\frac{\Omega_b}{m_p} \sim 2.329 \times 10^{-7}\ cm^{-3}$$

$H_0 = 72\ km/s/Mpc$

$$\frac{n_{b,\text{deuterium form}}}{n_{,\text{Baryon},0}} \sim \frac{3 \times 10^{19}}{2 \times 10^{-7}} \sim 1.5 \times 10^{26} \text{ times}$$

$a \propto t^{1/2}$ because

$$\frac{\dot{a}}{a^2} = \frac{8\pi G}{3} \cdot \frac{\rho_{rad,0}}{a^4}$$

$$\frac{da}{dt} = \sqrt{\frac{8\pi G}{3}}\ \frac{1}{a}$$

$$\int da = \int \sqrt{\frac{8\pi G}{3}}\ \frac{1}{a}\ dt$$

$$a(t) = \left[\frac{32}{3}\pi G \rho_{rad,0}\right]^{1/4} t^{1/2}$$

$$a(t) \sim t^{1/2}$$

So, given $a \propto t^{1/2}$ and $t \sim 0.6\ s$ from previous problem

since $T \propto a^{-1}$

$$T_{now} = T_{then}\left(\frac{t}{t_{now}}\right)^{1/2}$$

previous part

$$T_{now} = 2 \times 10^{10}\left(\frac{0.6}{3.15 \times 10^7 \times 10^{10}}\right)^{1/2} K$$

$T_{now} \sim 28K$

c) it is different from 2.7K since the radiation-dominated assumption does not hold up at later epochs, e.g. it switches to a matter dominated universe, causing $a \propto t^{2/3}$.

Problem 4: Power law inflation

inflation is a scalar field whose potential is a power-law $V \propto \phi^n$

can lead to exponential inflation $a(t) \propto \exp(Ht)$
$H \sim$ const.

a) scalar field ϕ
potential $V(\phi) = V_0 \exp(\phi/\phi_0)$
with $a(t) = a_0 \left(\frac{t}{t_0}\right)^n$

$V'(\phi) = \frac{1}{\phi_0} V_0 e^{\phi/\phi_0}$

steeper potential, exponential, that can slow it down.
equation of motion.

Freidmann Eq.

$H^2 = \frac{8\pi G}{3}\rho$

$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$

$\left(\frac{n}{t}\right)^2 = \frac{8\pi G}{3}\rho$

where $\rho = \rho_{kin} + \rho_{potential}$

$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$

$\left(\frac{n}{t}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\phi}^2 + V_0 \exp\left(\frac{\phi}{\phi_0}\right)\right] \rightarrow H^2 = \frac{1}{3}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$

Equation of motion

$\ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\phi} + V'(\phi) = 0 \Rightarrow \ddot{\phi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\phi} \quad \frac{1}{\phi_0} V_0 e^{\phi/\phi_0}$

we were told to do the following:
(1) flat universe
(2) ignore spatial gradients $\dot{\phi} = \ddot{\phi} \ll V(\phi)$

$3H^2 = \frac{V_0}{\phi_0}\exp\left(\phi/\phi_0\right)$

$$\dot{\phi} = \frac{-1}{\phi_0} H \qquad \dot{\phi} = -\frac{\sqrt{3V(\phi)}}{3\phi_0}$$

$$H = \frac{\dot{a}}{a} \Rightarrow \frac{d\phi}{\cancel{dt}} = \frac{-1}{\phi_0} \int \frac{da}{dt} \frac{1}{a} \Rightarrow \int_{\phi_0}^{\phi} d\phi = \frac{-1}{\phi_0} \int_{a_0}^{a} da \frac{1}{a}$$

$$\phi - \phi_0 = \frac{1}{\phi_0} \left[\ln(a) - \ln(a_0) \right]$$

$$a(\phi) = a_0 e^{(\phi - \phi_0)/\phi_0}$$

$$\frac{d\phi}{dt} = -\frac{\sqrt{3}}{3\phi_0} \sqrt{V_0 \exp(\phi/\phi_0)}$$

$$\frac{d\phi}{dt} = \frac{-\sqrt{3V_0}}{3\phi_0} \exp\left(\frac{\phi}{2\phi_0}\right)$$

$$\int \exp\left(-\frac{\phi}{2\phi_0}\right) d\phi = -\frac{\sqrt{3V_0}}{3\phi_0} dt$$

$$-2\phi_0 e^{-\phi/2\phi_0} = -\frac{\sqrt{3V_0}}{3\phi_0} t$$

$$e^{-\phi/2\phi_0} = \frac{\sqrt{3V_0}}{6\phi_0^2} t$$

$$\phi = 2\phi_0 \left[\ln\left(\frac{\sqrt{3V_0}}{6\phi_0^2}\right) + \ln(t) \right]$$

it's a solution ✓

$a(t) = a_0 \left(\frac{t}{t_0}\right)^n$

$H = \frac{\dot a}{a} = \frac{n}{t}$

$\phi(t) = \phi_0 \ln\left(\frac{t}{t_0}\right)$

$\dot\phi = \frac{\phi_0}{t}$

$\ddot\phi = \frac{-\phi_0}{t^2}$

scalar field eq of motion

$$\frac{-\phi_0}{t^2} + 3\frac{n}{t}\left(\frac{\phi_0}{t}\right) + \frac{V_0}{\phi_0}\exp\left(\frac{\phi}{\phi_0}\right) = 0$$

$$\frac{\phi_0}{t^2} + \frac{3n\phi}{t^2} + \frac{V_0}{\phi_0}\exp\left(\frac{\phi}{\phi_0}\right) = 0$$

$$\frac{\phi_0}{t^2}(3n-1) + \frac{V_0}{\phi_0 t_0} t \exp\left(\frac{\phi_i}{t_0}\right) = 0$$

$$\left(\frac{n}{t}\right)^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot\phi^2 + V_0 \exp\left(\frac{\phi}{\phi_0}\right)\right]$$

$3n - 1 = 0$

$$n^2 = \frac{8\pi G \phi_0^2 V_0}{3}$$

$$n = \sqrt{\frac{8\pi G \phi_0^2 V_0}{3}}$$

b.) in order for it to diverge

horizon $\eta(t) = \int \frac{d}{a(t)}$

$a(t) = a_0\left(\frac{t}{t_0}\right)^n$

$$\eta(t) = \int \frac{dt}{a_0\left(\frac{t}{t_0}\right)^n} = \frac{t^n}{a_0}\int t^{-n}\,dt$$

$$\eta = \frac{t_0^n}{a_0} \cdot t^{1-n}/1-n \text{ for } n \neq 1$$

such that $n > 1$

so....

$$n = \frac{8\pi G \phi_0^2 V_0}{3}$$

$$n > 1$$

$$\frac{8\pi G \phi_0^2 V_0}{3} > 1$$

$$\phi_0^2 > \frac{3}{8\pi G V_0}$$

$$\phi_0 > \sqrt{\frac{3}{8\pi G V_0}}$$