09/05/24

I. cosmological principle

homogeneous and isotropic

Galaxy counts

- follow $N(>f) \propto f^{-3/2}$, historical argument

- Statistics $\delta N \langle N \rangle$ vs L

  monotonically decreasing with L

  $L \sim 30\ M_{pc}$ $\delta N \langle N \rangle \sim 0.5$ (

Anisotropy

- cosmic microwave Background
- matter power spectrum

power spectrum fourier transform of ...

II. Expansion Kinematics

linear expansion $v = H_0 d$

$\gtrsim$ Gpc, non linear

HST 73.8 ± 1 km/s/Mpc (cepheids, Type Ia SNe)

CMB 67.4 ± 0.5 km/s/Mpc

III expansion dynamics

- galaxies recede from us
  - ↳ would imply centered the Universe

uniform expansion of universe

↳ $\frac{1}{H_0} = 14$ Gyr

↳ solves Olber's paradox (redshift, finite age)

Inconsistent with "perfect" cosmological principle

- inspired steady state model

requires $\frac{d\rho}{dt} = 3H_0\rho = 6\times10^{-28} \sim \frac{1\ \text{proton}}{m^3}$

IV Hot big bang

2.725 K Black body planck spectrum today.

very hot, blackbody uniform,

cool adiabatically, preserves the spectrum,

but cools and temperature drops

V. Inflation

Brief super-luminal expansion $t < 10^{-34}$ sec

solves some major problems.

- flatness problems
- horizon problems
- monopole problem

Flat universe with critical density

$$\Omega_{total} = 1.02 \pm 0.02$$

Scale invariant power spectrum of initial density fluctuations

$$n = \frac{d \ln P}{d \ln k} = 0.968 \pm 0.006 \text{ (Planck)}$$

VI. The Dark Sector

non linear expansion within general relativity

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Pillars of standard Model

→ expansion history

$$\mu = m - M = -2.5 \log\left(\frac{F}{F_{10}}\right)$$

$$= 5\left[\log d_{pc} - 1\right]$$

$$\log(d_{pc}) = \frac{5+\mu}{5}$$

spect uslder redshift

$$\frac{\lambda_{obs}}{\lambda_{em}} = \frac{v}{c} = 1+z$$

- $Ly\alpha \sim 1250$ Å, gets stretched.

$Ly\beta$

$Ly\gamma$

$e^{-\tau}$, column density.

Helium abundance:

- measured in stellar spectra

Lithium abundance:

- measured in stellar spectra
- Li is depleted

Deuterium abundance

- Destroyed easily in stars; must look for gas that has never cycled through a star.

quasar absorption lines:

- low-density gas
- far back in time
- extra neutron makes electron slightly more tightly bound.

Microwave Background Radiation

$$y = \int \sigma_T n_e \frac{kT}{m_e c^2} \, dl \leq 1.5 \times 10^{-5} \quad (\text{COBE})$$

$$u = a_B T^4 = 4.8 \times 10^{-34} \text{ g/cm}^3$$

$$n_\gamma = 420 \text{ cm}^{-3}$$

$$\langle h\nu \rangle = 6.3 \times 10^{-4} \text{ eV}$$

CMB Anisotropies

· Angular fluctuation power: $\frac{\delta T}{T}(\vec{\theta}) = \sum_{l,m \in [-l,l]} a_{lm} Y_{lm}(\vec{\theta})$

$$C_l = \left\langle |a_{lm}|^2 \right\rangle_m$$ m is not physical

<u>Dipole</u> ($l=1$, $m=0$) observer moving at $\vec{V}$ can get the temperature

$$T(\vec{\theta}) = T_0\left(1 + \frac{V}{c}\cos\theta\right)$$

· monopole: 2.72548 ± 0.00057 K

subtract from CMB, you can get monopole

$$V_{sun} - V_{CMB} = 369.82 \pm 0.11 \text{ km/s}$$

$$V_{sun} - V_{LG} = 306 \pm 18 \text{ km/s}$$

$l = 0$ is just fluctuations

$l = 1$ is caused by our movement

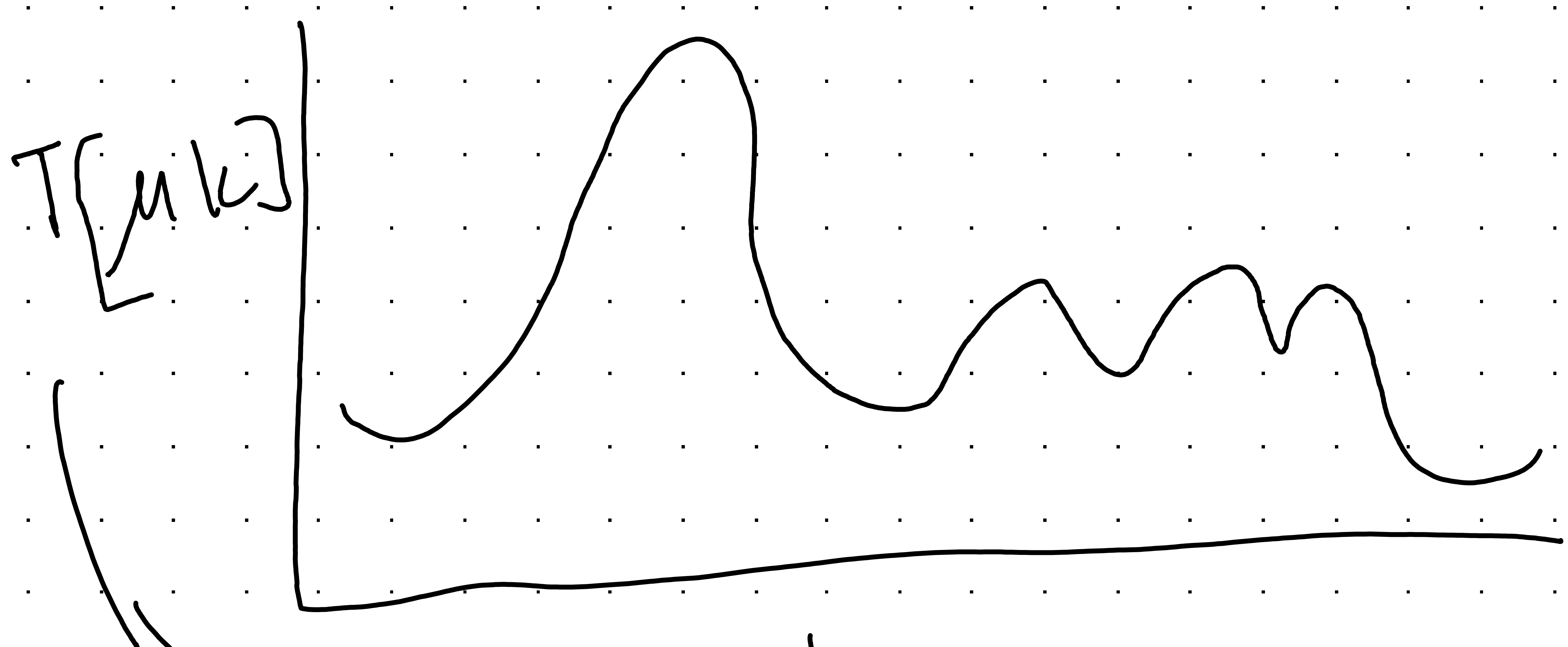
$l \geq 2$ is locally determined.

CMB Anisotropies

ℓ : spherical harmonics index.

1 degree in the scale

high $\ell \Rightarrow$ small scales



$$\left[\frac{\ell(\ell+1)C_\ell}{2\pi}\right]^{1/2}$$

Polarization.

oscillations in various ℓ

Olber's Paradox

Simplest model
- → universe infinitely large
- → infinitely old
- → uniformly filled with stars

→ Finite age and volume.

→ frequency redshift.

Galaxy Distributions

$\sim 10^8$ galaxies out 1 Gpc

$$\langle \delta(\vec{k})\,\delta(\vec{k}')\rangle = (2\pi)^3\,\delta_D(\vec{k}-\vec{k}')\,P(k)$$

- we weigh galaxy clusters by looking at strong lensing.

- weak lensing, tangential stretching

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- Supermassive BH
  - now thought to reside at the centres of all galaxies

- Luminosity function

$$\frac{d\Phi}{dL} = \frac{\Phi_*}{L_*}\left(\frac{L}{L_*}\right)^{\alpha} \exp\left(\frac{-L}{L_*}\right)$$

$\alpha \sim -1$

total light : $\rho_L = \Gamma(\alpha+2)\,\Phi_* L_*$
is finite

total number : diverges → lower cutoff

- Spiral Galaxies (Tully-Fisher)

$$V_{circ} = 220\left(\frac{L}{L_*}\right)^{1/4} \text{km/s}$$

- Elliptical Galaxies (Faber Jackson)

$$\sigma_{LOS} = 16\left(\frac{L}{L_*}\right)^{1/4} \text{km/s}$$

$$V_{circ} = 2^{1/2}\sigma_{LOS} = 220\left(\frac{L}{L_*}\right)^{1/4} \text{km/s}$$

Andromeda

-> Nearest large galaxy

-> spectrum is blue-shifted

· Virgo super cluster

~ Groups + clusters

~30 Mpc diameter, flat (pancake) structure.

~ total mass ~1.5 × 10^{15} $M_\odot$

JWST Deep Field

$z = 14.32$ galaxy

$z = 10.1$ quasar

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· Dark Matter

(1) in solar neighborhood

- anomalies in solar neighborhood

· Oort stellar disk in solar neighborhood is very thin.

height ~ few 100 pc
distance ~ 8 kpc

$$\left.\begin{matrix}\text{height} \sim \text{few } 100\text{ pc} \\ \text{distance} \sim 8\text{ kpc}\end{matrix}\right\} \frac{h}{r} \lesssim 0.01$$

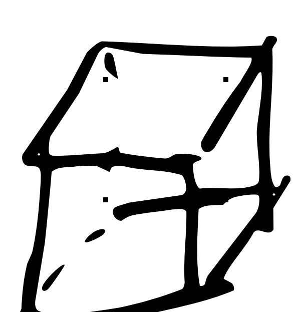
→ Jeans in 2D:

(i) $$\frac{-\partial \Phi}{\partial z} = \frac{1}{\rho_*} \frac{\partial}{\partial z}\left(\rho_* \overbrace{\sigma_z^2}^{\sim \text{ temperature.}}\right)$$

velocity dispersion of stars in z direction.

Φ = grav potential

ρ_*: 3D stellar (mass) density



$P\,dz$

dA

$$\left.\begin{matrix} F = dA\left(dz \dfrac{dP}{\partial z}\right) \\ m = \rho\, dV \end{matrix}\right\} a = \frac{F}{m} = \frac{1}{\rho}\frac{dP}{dz}$$

(ii) Poisson Eq: $$\frac{\partial^2 \Phi}{\partial z^2} = 4\pi G \rho_{tot}$$

(i) + (iii)

$$\frac{\partial}{\partial z}\left[\frac{1}{\rho_*}\frac{\partial}{\partial z}\left(\rho_* \sigma_z^2\right)\right] = -4\pi G \rho_{Tot}$$

ρ_{tot} follows from observations / measurements of $\rho_*(z)$; σ_z^2

Oort finds $\rho_{tot}(R = 8\,kpc, z = 0) = 0.15\ \frac{M_\odot}{pc^3}$

→ ρ_{tot} is noisy because of 3 numerical derivatives

→ eliminate 1 derivative

$$\Sigma_{tot}(z) = \int_{-z}^{z} \rho(z')\,dz' = \frac{-1}{2\pi G \rho_*}\frac{\partial}{\partial z}\left(\rho_* \sigma_z^2\right)$$

Oort: $\Sigma = 90\ \frac{M_\odot}{pc^2}$

Bahcall (1984):

	$\rho\ [M_\odot pc^{-3}]$	$\Sigma\ [M_\odot\ pc^{-2}]$
F-stars	0.14	40 (200 pc)
K-giants	0.21	75 (700 pc)

Intuition: $\frac{1}{2} m v^2 \sim \frac{G M m}{r} \rightarrow v^2 \sim \frac{G M}{r}$

annulus: $M \sim \Sigma\, \pi r^2 \rightarrow v^2 \sim 2\pi G\, \Sigma r$

$$\sigma_z^2 \sim 2\pi G \Sigma h$$

$$\sigma_z^2 \sim 2\pi G \rho h^2$$

Compare to light

<u>Aside</u>: mass-to-light ratios

$\frac{M_\odot}{L_\odot} \sim 1$ for Sun, $\frac{M}{L} \simeq 10^{-3}\left(\frac{M_\odot}{L_\odot}\right)$ for bright O-stars

$\frac{M}{L} \simeq 10^{3}\left(\frac{M_\odot}{L_\odot}\right)$ for M-dwarfs (dim)

$\langle \frac{M}{L} \rangle$ - average over stellar IMF very sensitive to slope.

- For solar neighborhood, "Salpeter IMF" $\frac{dN}{dm} \propto M^{-2.3}$

between 0.1 - 100 $M_\odot$

$\Rightarrow \langle \frac{M}{L} \rangle \simeq 4 \left(\frac{M_\odot}{L_\odot}\right)$ typical to stellar population $= 4$

$\Rightarrow \rho_* = \langle \frac{M}{L} \rangle j_*$ → luminosity density.

- for our solar neighborhood:

ρ_* (visible stars) $= 0.044\ M_\odot\, pc^{-3}$

ρ_* (WDs) $= 0.028\ M_\odot\, pc^{-3}$

ρ_{gas} (gas ISM) $= 0.042\ M_\odot\, pc^{-3}$

$\rho_{baryon} \sim 0.114\ M_\odot\, pc^{-3}$

Total bary density in our solar neighborhood.

$$\rho_b \simeq 0.114\ M_\odot\ pc^{-3}$$

the competing observed accounts for <u>more mass</u> <u>than baryons</u>.

- <u>factor of 2</u> discrepancy in our solar neighborhood.

(2) Solar circle — orbit of solar system around center of MW.

Mestel Disk

$$\Sigma(R) = \Sigma_0 \left(\frac{R_0}{R}\right)$$ for axisymmetric potential / distribution.

$$V_c^2 = R\,\frac{\partial \Phi}{\partial R}$$

$$V_c^2 = \frac{GM(R)}{R} \quad \text{with } M(R) = 2\pi \int_0^R \Sigma(R')\,R'\,dR'$$

[HW]

- Bessel function
- calculate potential
- doesn't really matter

$$\Sigma = \frac{V_c^2}{2\pi G R_0}$$ distance of sun from center of MW, $R_0 \sim 8.5$ kpc

V_c, circular velocity of sun.

$V_c \sim 230$ km/s

$$\boxed{\Sigma = 210\ \frac{M_\odot}{pc^2}}$$ factor 2–3 discrepancy

(3) DM on galactic scales

1970's Vera Rubin, Kent Ford

"Rotation curves" of disk galaxies $V_c(R)$

M 31 = $V_c(R)$ to 24 kpc (hot gas emission lines)

$V_c(R)$ to 35 kpc (HI 21 cm lines)

CF: surface brightness profile.

$$I(R) = I(0) \exp\left(\frac{-R}{R_s}\right)$$ "Scale Radius" ~ 5 kpc.

Measurement of $V_c(R)$:

$$V_{los}(R) = V_{com} + V_c(R) \sin i$$

i is inclination.

V_{los}: line of sight velocity (measured)

V_{com}: center of mass vel of galaxy (fitted)

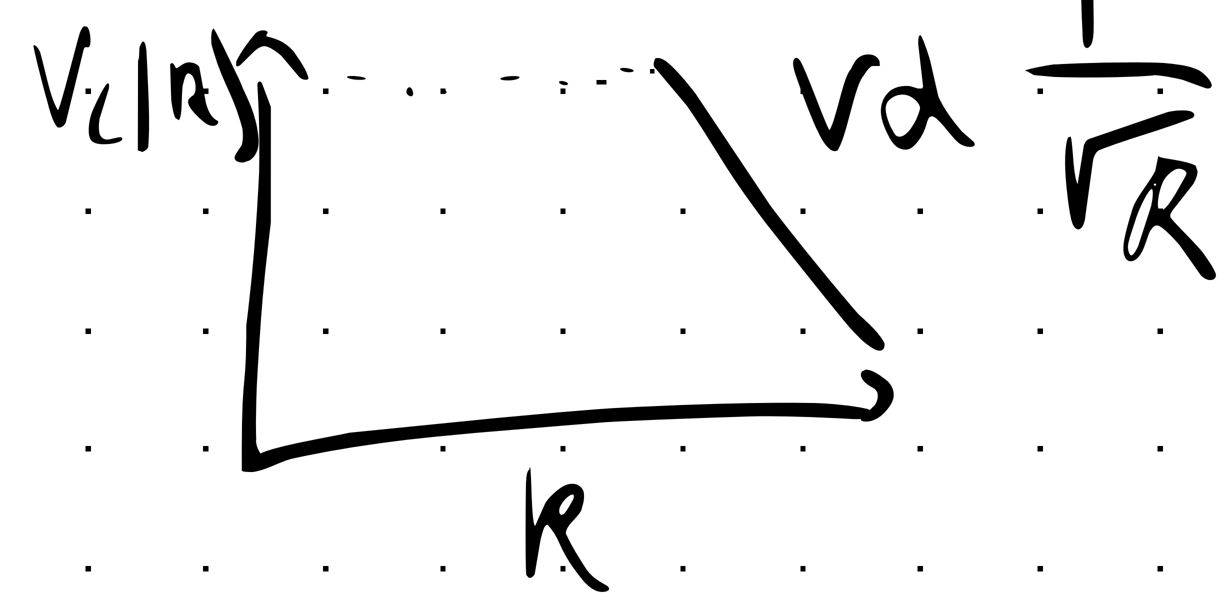
$$\sin i = \sqrt{1 - \left(\frac{b}{a}\right)^2}$$

$\frac{b}{a}$: axis ratio. (i.e. circular, inclined disk)

$$V_c(R) = \frac{V_{los}(R) - V_{com}}{\sqrt{1 - b^2/a^2}}$$

- Far from galaxy centre $(R \gg R_s)$ we expect keplerian drop-off

$$V_c(R) = \sqrt{\frac{G M(<R)}{R}}$$



Keplerian dropoffs not seen instead $V_c(R) \sim const$

$$M(<R) = \frac{V^2 R}{G}$$

$$V \sim const \rightarrow M(<R) \propto R$$

$$\rightarrow \rho(R) \propto \frac{1}{R^2}$$

"singular isothermal sphere"

In the MW

$$M(<R) = 10^{11} M_\odot \left(\frac{V_c}{230\ km/s}\right)^2 \left(\frac{R}{8.5\ kpc}\right)$$

gas measurements to ~20 kpc

globular clusters and satellites (LMC/SMC) ~75 kpc

$R_{halo} \gtrsim 75\ kpc$

$M_{MW} \gtrsim 10^{12} M_\odot$

Light (luminous) stars in MW imply

$$\frac{M}{L} \sim 64 \left(\frac{R_{mass}}{100\ kpc}\right)$$

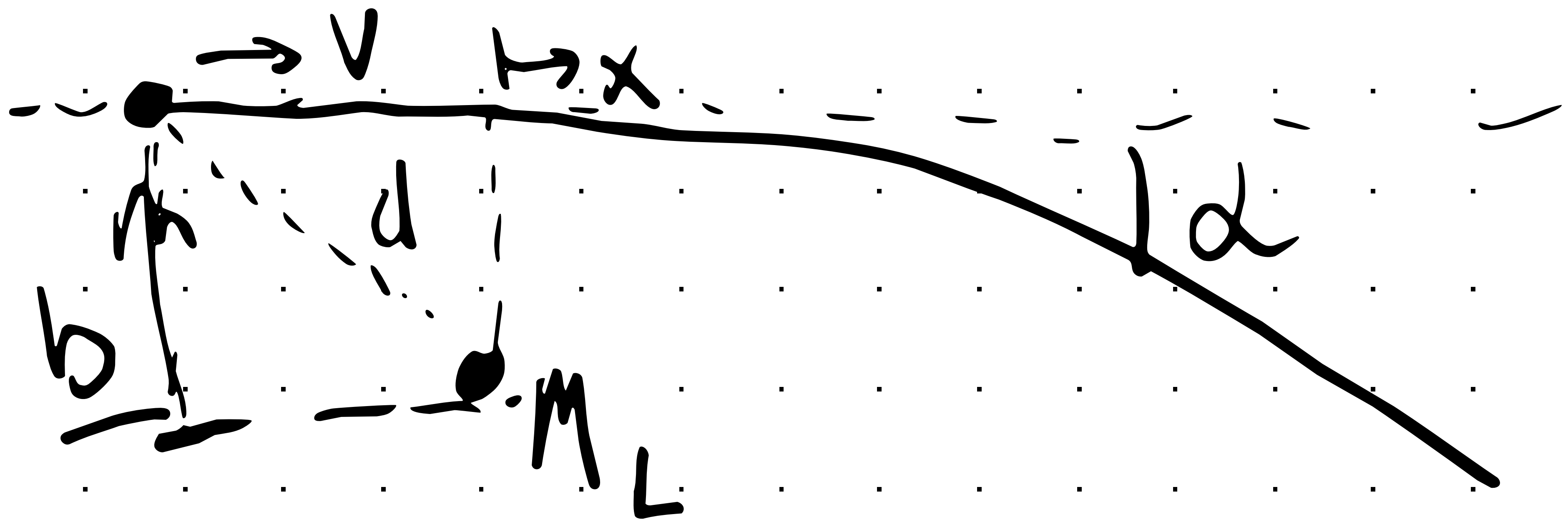
→ roughly spherical distribution of DM

NB: other ways to find DM on galactic scales:
-> stabilizes disk to bar formation.

-> not puff up the disk due to satellite crossings.

(4) DM in galactic halos and clusters of galaxies

GRAVITATIONAL LENSING, Newton's version.



b = impact parameter / closest approach of unperturbed trajectory

α = deflection angle

x = coordinate

v = (nearly) constant

Newtonian analog to grav. lensing
"impulse approximation"

$$F = \frac{G M_L m}{d^2} \qquad a = \frac{F}{m} = \frac{G M_L}{d^2}$$

acceleration:

$$a = \frac{G M_L}{(b^2 + x^2)} \qquad a_\perp = \frac{G M_L}{(b^2+x^2)} \frac{b}{(b^2+x^2)^{1/2}}$$

$$a_{\parallel} = \frac{GM_L}{(b^2+x^2)} \frac{x}{(b^2+x^2)^{1/2}}$$

impulse approximation: $\alpha \ll 1$, $\Delta v_{\parallel} \simeq 0$

only interested in $\Delta v_{\perp}$

$$\Delta v_{\perp} = \int_{-\infty}^{\infty} a_{\perp}\, dt = GM_L \int_{-\infty}^{\infty} \frac{b\, dt}{(b^2+v^2t^2)^{3/2}} \qquad x = vt$$

$$= \frac{2GM_L}{bv}$$

$$\sin\alpha \simeq \alpha = \frac{\Delta v_t}{v} = \frac{2GM_L}{bv^2}$$

$$\boxed{\alpha = \frac{2GM_L}{bv^2}}$$ deflection angle ($\alpha \ll 1$)

In reality in GR

$$\alpha = \frac{4GM_L}{bc^2}$$ valid for $b \gg R_s$

$$R_s = \frac{2GM_L}{c^2}$$

schwarzschild radius

or $\alpha \ll 1$

$$\alpha = \frac{2R_s}{b}$$

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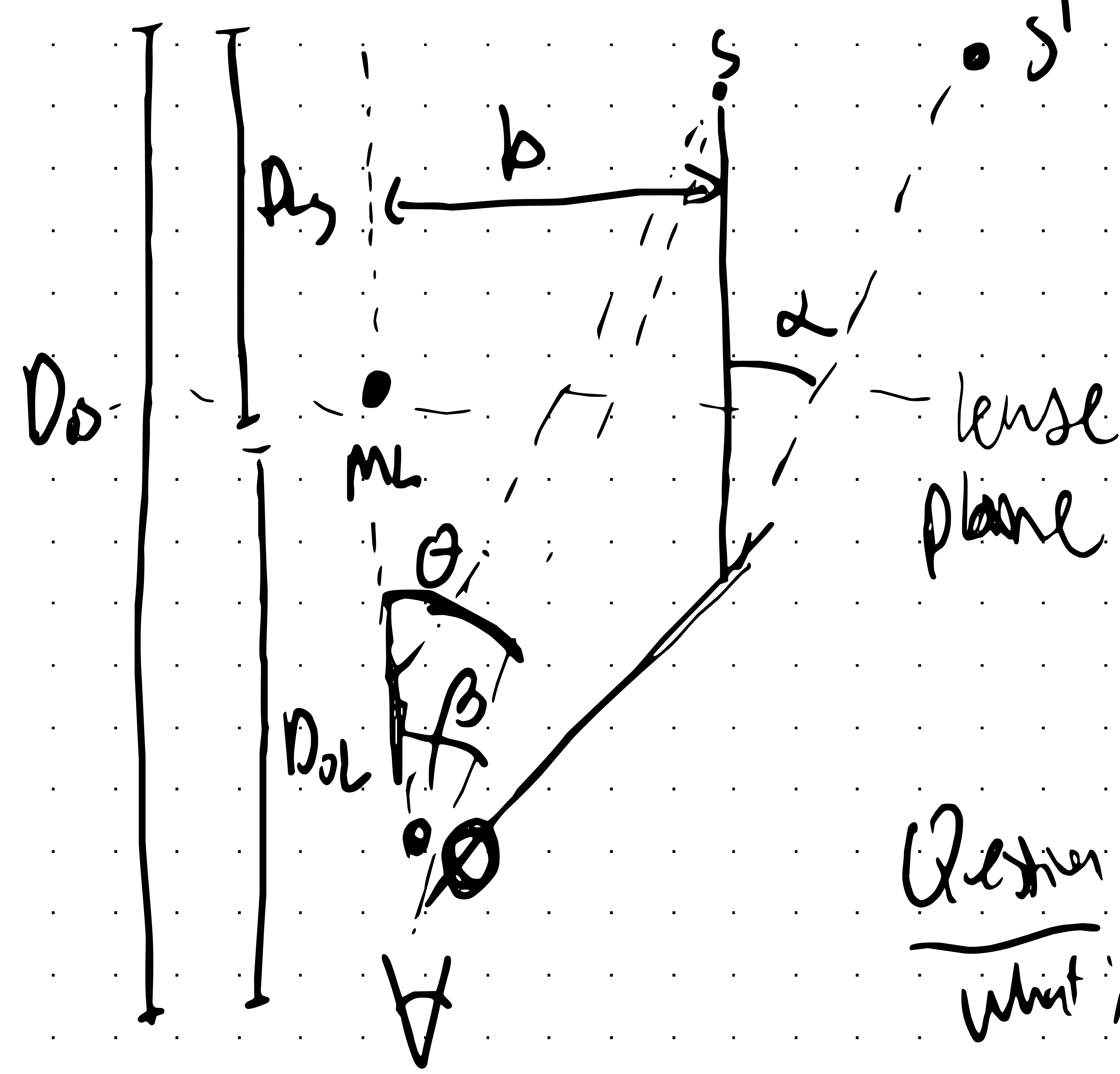
Q: what do we see, if look at light source placed behind a (point-like) lense?



b impact parameter.

α deflection angle.

β true angular position

θ apparent position.

Question:

what is $\theta = \theta(\beta)$

Assume: $\alpha, \beta, \theta \ll 1$

$$\beta D_{OS} = \theta D_{OS} - \alpha D_{LS}$$

$$\beta = \theta - \alpha \frac{D_{LS}}{D_{OS}}$$

$$\beta = \theta - \frac{\alpha}{\theta} \frac{D_{LS}}{D_{OS}} \frac{b}{D_{OL}}$$

use $\alpha = \frac{2R_s}{b}$

$$\beta = \theta - \frac{1}{\theta} \frac{2R_s D_{LS}}{D_{OS} D_{OL}}$$

einstein Radius $\theta_E = \left(\frac{2 R_s D_{LS}}{D_{OS} D_{OL}} \right)^{1/2}$

depend only on the mass of the lensing object.

"lens equation"

$$\beta = \theta - \frac{\theta_E^2}{\theta} \quad \text{"lens equation"}$$

quadratic eqn. form, two roots

$$\theta_\pm = \frac{1}{2}\left(\beta \pm \sqrt{4\theta_E^2 + \beta^2} \right)$$

two apparent images on either side of the lens

→ perfect allignment $\beta = 0 \rightarrow \theta = \pm \theta_E$

really a circle, since no preferred orientation

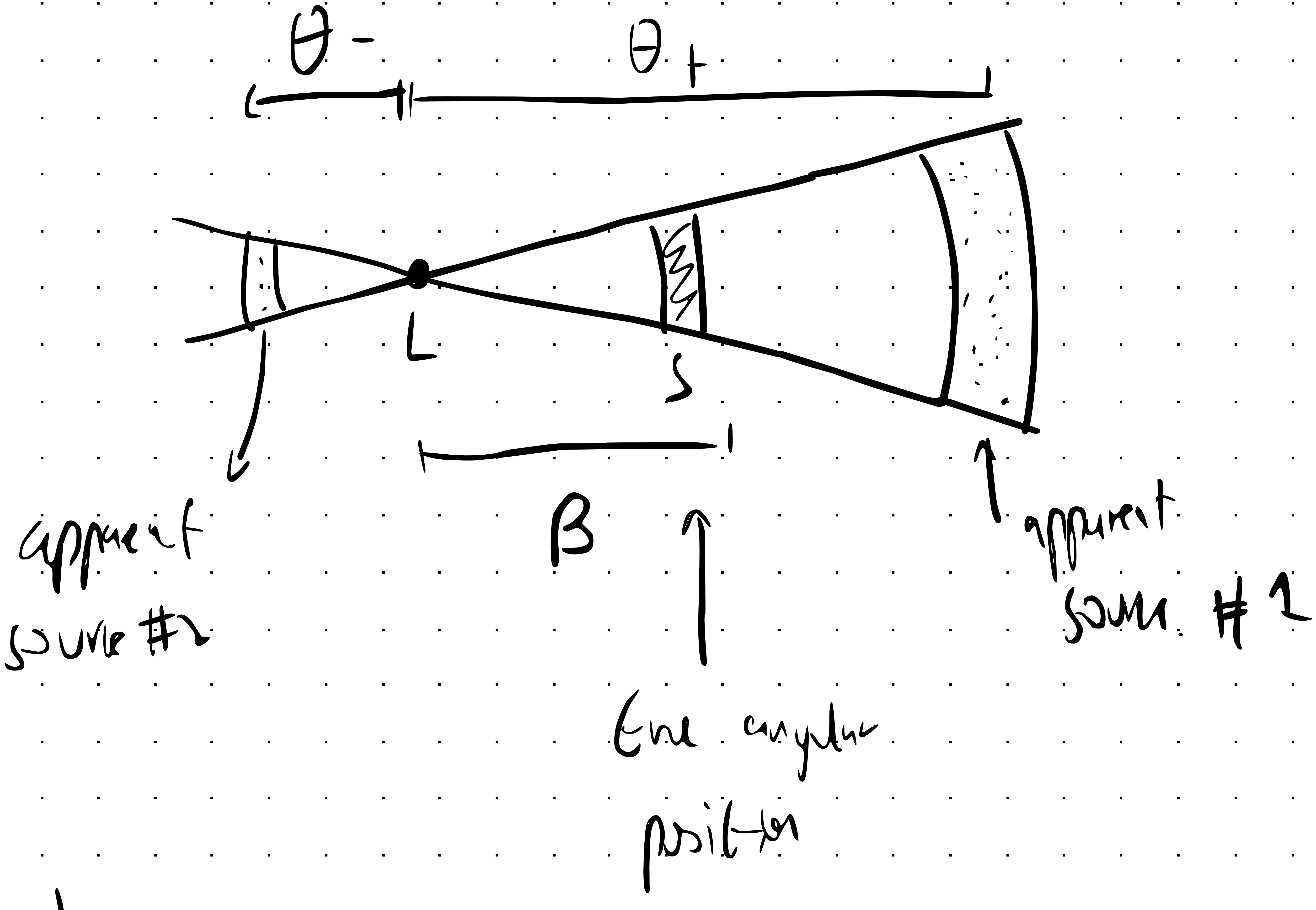
"einstein ring" in the sky

circle on the sky with angular radius θ_E

Magnification

replace point source by a finite source size

in plane of lens + obs + source system



→ lensing conserves surface brightness (no photons are destroyed)

→ magnification is proportional to image's (angular) area → solid angle.

$\mu \equiv \frac{F_{obs}}{F_{\text{unlensed flux}}}$

$\mu_{\pm} = \frac{\theta_{\pm}\, d\theta_{\pm}}{\beta\, d\beta}$

$\theta_{\pm} = \theta_{\pm}(\beta)$

$\frac{d\theta_{\pm}}{d\beta} = \ldots$

$\mu = \mu_+ + \mu_- = \ldots = \frac{u^2+2}{u\sqrt{u^2+4}}$

e.g. $u = 1$

$\mu = \frac{3}{\sqrt{5}} \sim 1.34$

34% brightening

$u = \frac{\beta}{\theta_E}$

$u \to 0 \ (\beta \to 0)$

$\mu \to \frac{1}{u}$

$0 \leq x \leq 1$

$\theta_E = 0.7\,\text{mas}$

$\times \left(\frac{M_L}{0.5 M_\odot}\right)^{1/2}$

$\times \left(\frac{D_{OL}}{4\,\text{kpc}}\right)^{-1/2}$

$\times \left(\frac{1-x}{0.5}\right)^{1/2}$

Characteristic numbers

$x = \frac{D_{OL}}{D_{OS}}$

$$\theta_E D_{OS} = 5.7\,AU \left(\frac{m}{0.5\,M_\odot}\right)^{1/2} \left(\frac{D_{OL}}{4\,kpc}\right)^{1/2} \left(\frac{x(1-x)}{0.25}\right)^{1/2}$$

$$\theta_E D_{OL} = 2.85 \left\{ \text{———} \,\|\, \text{———} \right\}$$

$$t_E = \frac{\theta_E D_{OS}}{V_S} = 24.7\,\text{days} \left(\frac{m}{0.5\,M_\odot}\right)^{1/2} \left(\frac{D_{OL}}{4\,kpc}\right)^{1/2} \cdot \left[\frac{x(1-x)}{0.25}\right]^{1/2} \cdot \left[\frac{V}{200\,km/s}\right]^{-1}$$

$V_S \rightarrow$ moving source

probability for a star in a random place on the sky to be magnified by a factor of order unity]

$$p \sim \frac{\Delta\Omega}{4\pi} \propto \text{total area of Einstein circles}$$

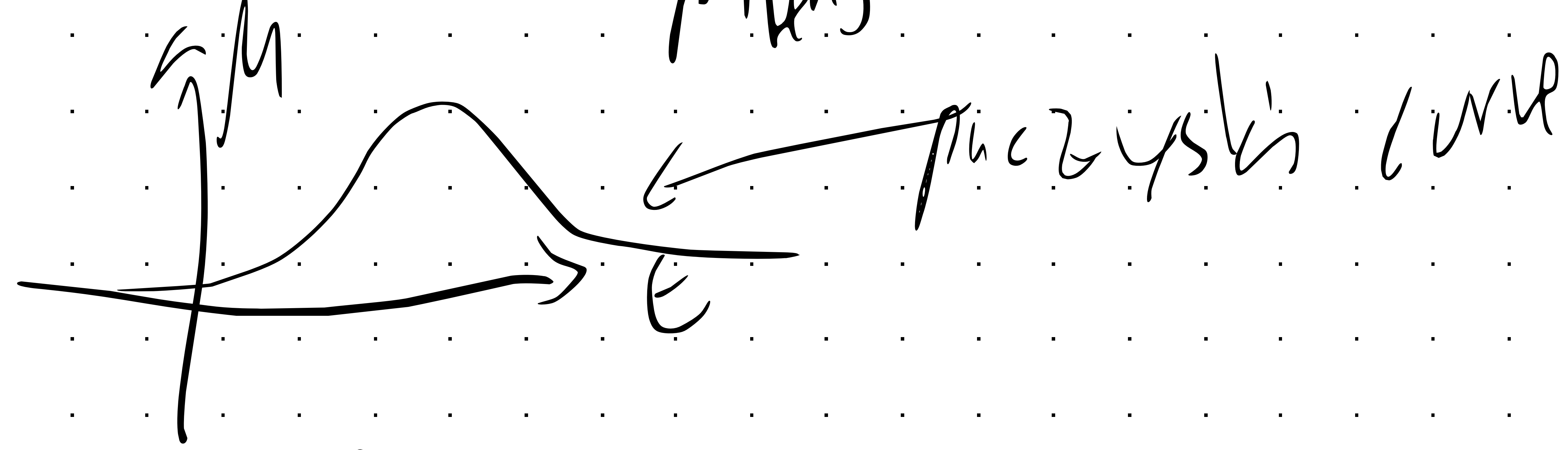
$4\pi \leftarrow$ full sky

$$p \sim \frac{10^{12}\,\text{lenses}}{4\pi} * \left(\frac{\pi\theta_E^2}{4\pi}\right) \sim 10^{-6}$$

need to monitor $\sim 10^6$ stars for weeks to find 1 lensing event.

if halo is entirely composed of solar-mass dim objects, Macho's = massive compact halo objects

then $N_{lens} \sim \frac{M_{halo}}{M_{lens}} \sim 10^{12}$



→ But from the observations: at most 10% of DM in halo of MW is in Machos.

(5) Clusters of galaxies (1 Mpc–10 Mpc)

(i) lensing

$$\theta_E = 1' \left(\frac{M}{10^{13} M_\odot}\right)^{1/2} \left(\frac{D_{OL}}{1\,\text{Gpc}}\right)^{-1/2} \quad x = \frac{1}{2}$$

↳ 1 arcminute.

→ typical $M/L \sim 400-800 \left(\frac{M_\odot}{L_\odot}\right)$

(ii) virial thm

apply to the velocities of galaxies.

$$2KE = -PE$$

$$\frac{1}{2} M \langle \sigma^2 \rangle = -\frac{1}{2} \underbrace{\alpha}_{\sim 1,\text{ depending on density profile.}} \frac{GM^2}{r_{half}}$$

$$E_{pot} = \int_0^\infty dr \frac{4\pi r^2 \rho(r)}{r} GM(<r)$$

$$M = \frac{\langle \sigma^2 \rangle r_h}{2G}$$

assume: 1) $\langle \sigma_t^2 \rangle = \langle \sigma_r^2 \rangle$ velocity dispersion is isotropic

2) $\frac{M}{L}$ constant, indep of r

measure surface density of galaxy light + v_{LOS}^2's

COMA cluster; 1973 Zwicky

$$\sigma^2_{LOS} = \langle \sigma^2 \rangle = 880 \text{ km/s}$$

$$\langle \sigma_{3D} \rangle \sim 3\sigma^2_{LOS}$$

$$= 2500 \text{ km/s}$$

$$r_h(\text{light}) = r_h(\text{mass}) = 1.5 \text{ Mpc.}$$

$$\alpha = 0.5$$

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→ General Relativity

(1) Newton's theory

(i) motion

1) bodies at rest remain at rest (or move straight) unless acted on by a force.

2) $\vec{F} = m\vec{a}$

3) is bod A body exerts a force same but opposite sign.

09/25/24

- We wanted a coordinate space that are coordinate invariant.

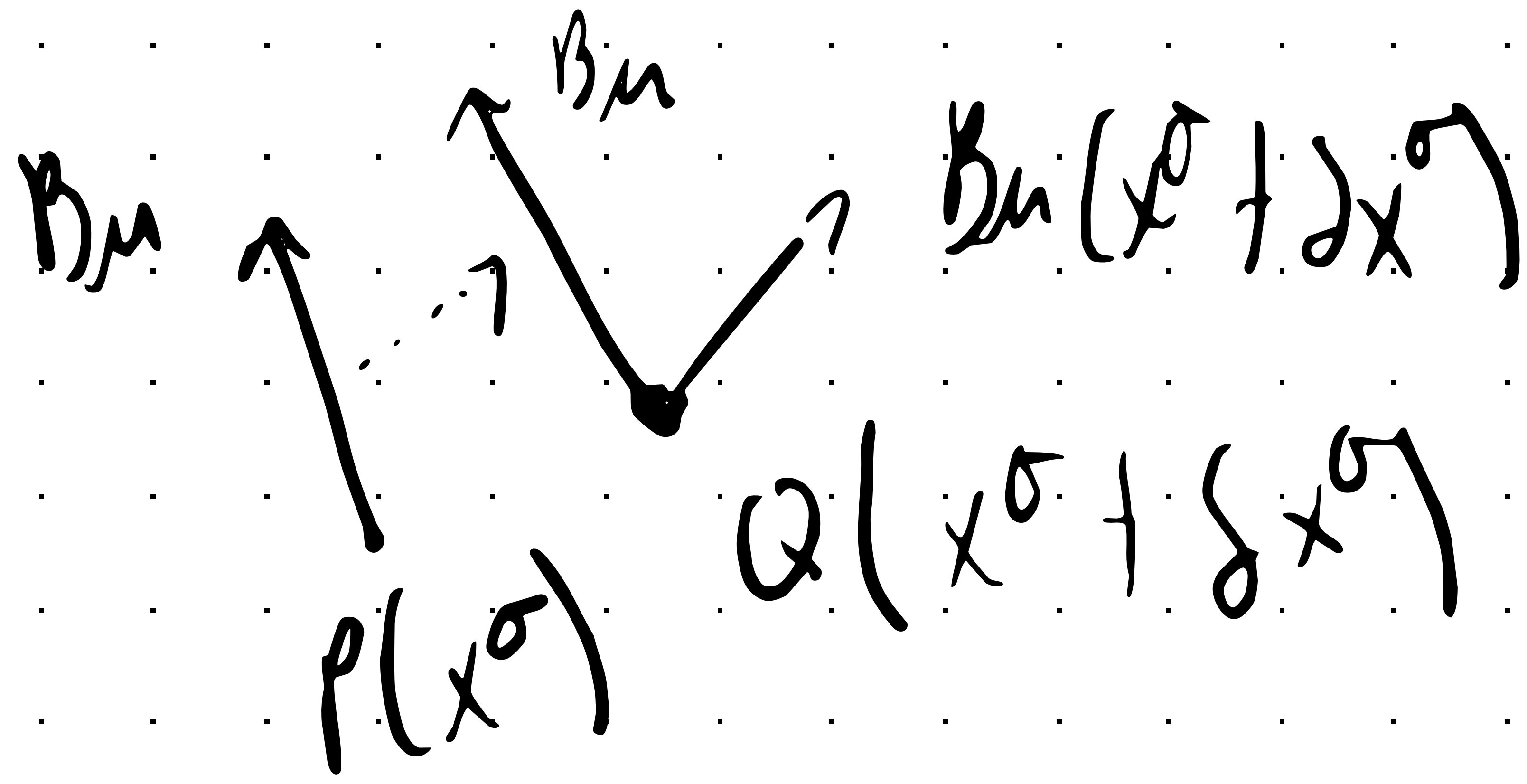
Derivatives

coordinate transformation coefficient depend on coordinates value.

→ spoils tensorial nature of this naive derivative

- What went wrong?

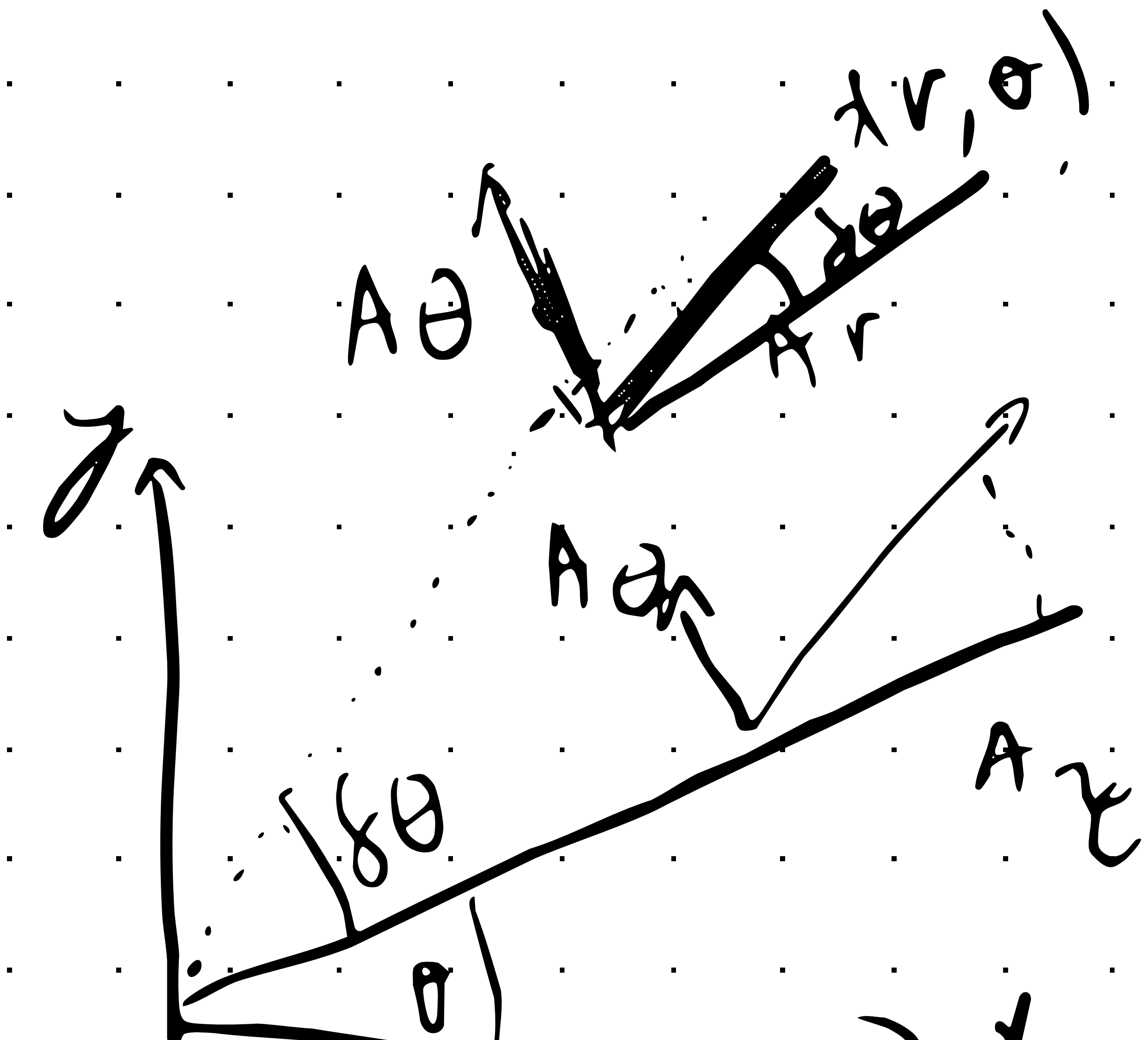
$$\frac{\partial B_\mu}{\partial x^\sigma} = \lim_{\delta x^\sigma \to} \frac{B_\mu(x^\sigma + \delta x^\sigma + B_\mu(x^\sigma)}{\delta x^\sigma}$$



Flat space

We tried to subtract two vector defined at two different points.

parallel transport of a vector.



Components change.

e.g. flat space in polar coords

$$A_r' = A_r \cos(\delta\theta) + A_\theta \sin(\delta\theta)$$

$$A_\theta' = -A_r \sin(\delta\theta) + A \cos(\delta\theta)$$

$$A_r' = A_r + \delta\theta\, A_\theta + O(\delta\theta^2)$$

$$A_\theta' = -\delta\theta A_r + A_\theta$$

- Changing components of A_μ is linear in displacement δx^σ and also in original components A_μ.

$$\delta A_r = (\delta r \;\; \delta\theta) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}$$

$$\delta A_\theta = (\delta r \;\; \delta\theta) \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \begin{pmatrix} A_r \\ A_\theta \end{pmatrix}$$

- More generally, change in some B_μ is

$$\delta B_\mu = \Gamma^{\nu}_{\mu\sigma} B_\nu \delta x^\sigma$$

Γ = Christoffel connection
affine connection

Levi-Civita symbol

Riemann connection

→ depends on the coordinate system (even in flat space)
- depends on spacetime curvature.
→ describes parallel transport of vectors.
→ can be used to construct covariant derivative.
→ Γ is <u>not</u> a tensor.

· Covariant derivative

"True change" in a vector under small displacement.

$$B_\mu(x^\sigma + \delta x^\sigma) - [B_\mu(x^\sigma) + \delta B_\mu] =$$

$$= \frac{\partial B_\mu}{\partial x^\nu}\delta x^\nu + B_\mu(x^\sigma) - B_\mu(x^\sigma) - \delta B_\mu$$

$$= \left[\frac{\partial B_\mu}{\partial x^\nu} - \Gamma^\alpha_{\mu\nu} B_\alpha\right]\delta x^\nu$$

· Define a new derivative:

$$B_{\mu;\nu} = \nabla_\nu B_\mu = \text{Covariant derivative}$$

$$B_{\mu;\nu} = \frac{\partial B_\mu}{\partial x^\nu} - \Gamma^\sigma_{\mu\nu} B_\sigma$$

Can show that Γ must transform as

$$\Gamma'^\sigma_{\mu\nu}$$

$g_{\mu\nu}$ - metric defines shape of metric

Reimman geometry

(1) $g_{\mu\nu;\sigma} = 0$ (equivalence principle)

(2) $\Gamma^{\sigma}_{\mu\gamma} = \Gamma^{\sigma}_{\gamma\mu}$ christopher symbol is symmetric (torsion-free condition)

$B_{\mu;\nu}$ to transform as a rank-2 tensor. ☒

Under these conditions, $\Gamma = \Gamma(g, g')$

solving for Γ as a function of metric

$$\text{(i)}\quad g_{\mu\nu;\sigma} = \frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \overset{(1)}{\cancel{\Gamma^{\tau}_{\mu\sigma} g_{\tau\nu}}} - \overset{(2)}{\cancel{\Gamma^{\tau}_{\nu\sigma} g_{\mu\tau}}} = 0$$

$$\text{(ii)}\quad g_{\sigma\mu;\nu} = \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \overset{(1)}{\cancel{\Gamma^{\tau}_{\sigma\nu} g_{\tau\mu}}} - \Gamma^{\tau}_{\mu\nu} g_{\sigma\tau} = 0$$

$$\text{(iii)}\quad g_{\nu\sigma;\mu} = \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \Gamma^{\tau}_{\nu\mu} g_{\tau\sigma} - \overset{(2)}{\cancel{\Gamma^{\tau}_{\sigma\mu} g_{\nu\tau}}} = 0$$

(i) - (ii) - (iii)

$$\frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\sigma\mu}}{\partial x^\nu} + 2\Gamma^{\tau}_{\mu\nu} g_{\tau\sigma} = 0 \quad / \; g^{\lambda\nu}$$

$$\Gamma^{\lambda}_{\mu\gamma} = \frac{1}{2} g^{\lambda\sigma} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^{\mu}} + \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} \right\}$$

- covariant derivative of a contravariant vector.

- covariant derivative of a covariant vector?

- $A^{\nu} = g^{\mu\nu} A_{\mu}$
- $A_{\nu} = g_{\mu\nu} A^{\mu}$
- $B_{\mu} A^{\mu} =$ scalar, inner product.
- $A_{\nu} A^{\nu} = (g_{\mu\nu} A^{\mu}) A^{\nu} =$ "length" of a vector

$(B_{\mu} A^{\mu})_{;\nu} =$ vector, since $B_{\mu} A^{\mu}$ is a scalar.

$\rightarrow (B_{\mu} A^{\mu})_{;\nu} = (B_{\mu} A^{\mu})_{,\nu}$

$B_{\mu;\nu} A^{\mu} + B_{\mu} A^{\mu}_{;\nu} = B_{\mu,\nu} A^{\mu} + B_{\mu} A^{\mu}_{,\nu}$

B_{μ}

10/02/24

Riemann tensor

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\sigma\mu\nu} V^\sigma$$

$$R^\alpha{}_{\sigma\mu\nu} = \frac{\partial \Gamma^\alpha_{\nu\sigma}}{\partial x^\mu} - \frac{\partial \Gamma^\alpha_{\mu\sigma}}{\partial x^\nu} + \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^\mu} - \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\sigma\mu}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right\}$$

Properties of Riemann Tensor

→ R is given in terms of Γ and $\partial \Gamma$

→ in GR/Riemannian geometry $\Gamma = \Gamma(g, \partial g)$

so R is a function of $g, \partial g, \partial^2 g$

→ even though $[\nabla_\mu, \nabla_\nu]$ is an operator,

its effect on a vector field is simple linear multiplication

(if we had not assumed $\Gamma^\lambda_{\mu\gamma} = \Gamma^\lambda_{\gamma\mu}$ (torsion free condition)

then we'd find

$$[\nabla_\mu, \nabla_\nu] V^\alpha = R^\alpha{}_{\sigma\mu\nu} V^\sigma - T^\lambda_{\mu\nu} \nabla_\lambda V^\alpha$$

$$T_{\mu\nu}{}^{\lambda} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}$$ – "torsion tensor"

→ can show $R^{\alpha}{}_{\sigma\mu\nu}$ is a tensor (rank 4 tensor)

→ Define $R_{\alpha\sigma\mu\nu} = g_{\alpha\lambda} R^{\lambda}{}_{\sigma\mu\nu}$

in locally inertial frame, at some point P

$$R_{\rho\mu\sigma\tau} = \frac{1}{2}\left(\tilde{g}_{\mu\sigma,\rho\tau} + \tilde{g}_{\rho\tau,\mu\sigma} - \tilde{g}_{\mu\tau,\rho\sigma} - \tilde{g}_{\rho\sigma,\mu\tau}\right)$$

in terms of metric in LIF at this point

→ entirely composed of 2^{nd} derivatives of g

→ many symmetry properties follow

$$R^{\alpha}{}_{\sigma\mu\nu} = -R^{\alpha}{}_{\sigma\nu\mu}$$ symmetric and antisymmetric

$$R_{\alpha\sigma\mu\nu} = -R_{\sigma\alpha\mu\nu}$$ Bianchi identity

$$R_{\alpha\sigma\mu\nu} = R_{\mu\nu\alpha\sigma}$$ LIF = local inertial frame

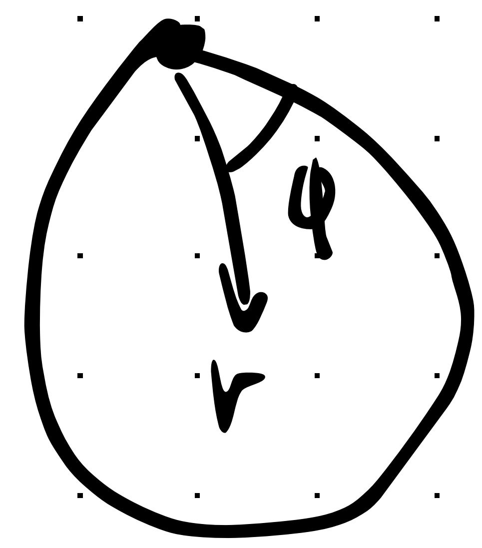
→ How many independent components?

# of independent components $= \dfrac{n^2(n^2-1)}{12}$ in n dimensions

n	#	
1	0	→ no curvature possible
2	1	→ single number
3	6	→
4	20	→

Example: 2d surface of a sphere (S^2, or "two-sphere")

$$ds^2 = R^2\left(d\varphi^2 + \sin\right) \qquad r = R\theta$$



$$ds^2 = dr^2 + R^2\sin^2\left(\frac{r}{R}\right)d\varphi^2$$

$$ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

take derivatives w.r.t. θ and φ

What's R^{α} in this coo system? (...)

$$\Gamma^{\theta}_{\phi\phi} = \sin\theta\cos\theta$$

$$\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

all other Γ's are 0

$$R^{\theta}{}_{\varphi\theta\varphi} = \sin^2\theta$$

$$R_{\theta\varphi\theta\varphi} = R^2\sin^2\theta$$

Ricci tensors - contractions of Riemann tensor

Ricci tensor $R_{\mu\sigma} = g^{\rho\tau}R_{\rho\mu\sigma\tau}$ (rank-2)

Ricci Scalar $R = g^{\mu\sigma}R_{\mu\sigma}$

$$R_{\theta\theta} = g^{\phi\tau}R_{\phi\theta\varphi\tau} = 1$$

$$R_{\theta\varphi} = R_{\phi\theta} = 0$$

$$R_{\varphi\varphi} = \sin^2\theta$$

$$R = g^{\theta\tau}R_{\theta\tau} + g^{\varphi\varphi}R_{\varphi\varphi}$$

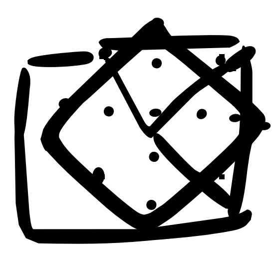
$$= \frac{2}{R^2}$$

constant positive curvature

Einstein tensor

$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, has special property no divergence

$\Rightarrow \nabla^{\mu} G_{\mu\nu} = 0$, show yourself



rank 2 tensor.

Modify Newton's Theories

1.) Motion $\vec{F} = m\vec{a}$ $\quad \vec{a} = -\vec{\nabla}\phi$ $\quad \phi$ = gravitational potential

2.) gravity $\nabla^2 \phi = 4\pi G \rho$

1.) geodesic Equation

describe motion of a free-falling body

"free fall" in flat space, in inertial frame of reference, in cartesian coordinates

$$\frac{d^2 x^{\mu}}{d\lambda^2} = 0$$

, equation for a line in flat space

λ scalar parameter along path, not true in polar coordinates.

generalize concept of straight line, to

(i) arbitrary coordinate [even in flat space]

(ii) curved spaces $\quad$ def 1 path that minimizes dist. b/w two points.

def #2, path such that its tangent vector, parallel-transported along path remains unchanged.

"cheat" #3: minimal coupling

Minimal coupling:

Step 1: take any law of physics valid in flat space, in inertial coordinate systems.

Step 2: write this law as an equation involving PDE's

Step 3: take the PDE's and make this transition

$$\eta_{\alpha\beta} \to g_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\alpha} \to \nabla_\alpha$$

Step 4 we are done - postulate that new equations are valid in all coo systems and in curved spacetimes

Example: equation of motion for a free falling particle?

$$\frac{d^2 x^\mu}{d\lambda^2} = 0$$

vector

not a vector replace this by ∇_ν

vector

$$\frac{d^2 x^\mu}{d\lambda^2} = \frac{dx^\nu}{d\lambda} \frac{\partial}{\partial x^\nu} \frac{dx^\mu}{d\lambda} = 0$$

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0$$

geodesic

def #2 : "directional derivative" along the path

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda} \frac{\partial}{\partial x^\mu}$$

covariant / tensorial version,

$$\frac{D}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu$$

in general, requiring a tensor to be constant along the curve just means

$$\frac{DT}{d\lambda} = \frac{dx^\mu}{d\lambda} \nabla_\mu T = 0$$

example

$$T = \frac{dx^\mu}{d\lambda} \rightarrow \text{geodesic equation}$$

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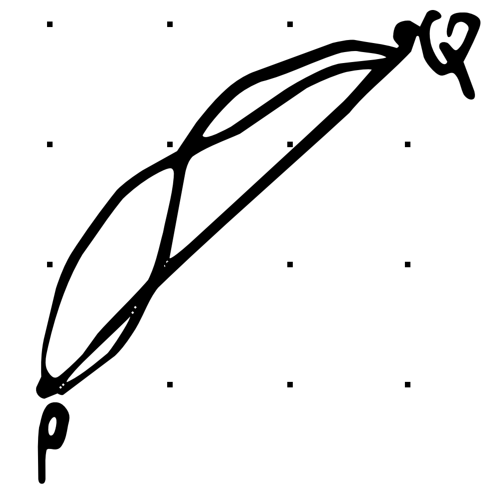
def 1 shortest distance



$$\delta \int_P^Q ds = 0$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$ds = \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$$

$$L^2 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}$$

$$\delta \int_P^Q L \, d\lambda = \int_P^Q (\delta L) \, d\lambda$$

$$L = \left(g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2}$$

$$\delta L = \frac{1}{2} \frac{1}{L} \left\{ \mathrm{I} + \mathrm{II} + \mathrm{III} \right\}$$

(I) $\dfrac{\partial g_{\mu\nu}}{\partial x^\sigma} (\delta x^\sigma) \dfrac{dx^\mu}{d\lambda} \dfrac{dx^\nu}{d\lambda}$

(II) $g_{\mu\nu} \, \delta\left[\dfrac{dx^\mu}{d\lambda}\right] \left[\dfrac{dx^\nu}{d\lambda}\right]$

$$\frac{d(\delta x^\mu)}{d\lambda} = \text{"}\frac{dv}{ds}\text{"} \quad \text{"}v\text{"} = \delta x^\sigma$$

(III) —— " —— $\mu \leftrightarrow \nu$

Let's choose scalar $\lambda = s$ (proper distance measured along path)

$$ds = L \, d\lambda \quad L = 1$$

$$\int_P^Q \delta L \, ds = ?$$

integrate by parts (II & III)

$$\int u \frac{dv}{ds} ds = \cancel{uv}^{\,0} - \int v \, du$$

$$\int_P^Q \left[g_{\mu\sigma} \frac{d^2 x^\mu}{ds^2} + \frac{1}{2} \left\{ \frac{\partial g_{\nu\sigma}}{\partial x^\mu} + \frac{\partial g_{\mu\sigma}}{\partial x^\nu} - \frac{\partial g_{\mu\nu}}{\partial x^\sigma} \right\} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \right] \delta x^\sigma \, ds = 0$$

integral must be 0 for arbitrary δx^σ

$\rightarrow$ integrand $[\quad] = 0$

$$[\quad] \, g^{\tau\sigma} = \frac{d^2 x^\tau}{ds^2} + \Gamma^\tau_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

geodesic equation

<u>Weak Newtonian limit</u>

(i) particle is travelling slowly $v \ll c$, $dx/dt \ll c$

(ii) space almost flat: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $|h| \ll 1$ when written in locally inertial coords

$\eta_{\mu\nu}$ ↳ minkowski metric.

(iii) metric nearly static

$$\frac{\partial}{\partial t} g_{\mu\nu} = \partial_0 g_{\mu\nu} \simeq 0 \qquad \left[x^0 = t \text{ - notation}\right]$$

x^i = three spatial coo's

(i) slower $\frac{dx^i}{d\tau} << \frac{dt}{d\tau}$ $\qquad \frac{dx^i}{dt} << 1$

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left(\frac{dt}{d\tau}\right)^2 = 0$$

(iii) static: $\partial_0 g_{\mu\nu} \sim 0$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\lambda}\left[\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}\right]$$

$$\Gamma^\mu_{00} = -\frac{1}{2} g^{\mu\lambda} \partial_\lambda g_{00}$$

weak curvature $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Gamma_{00}^\mu = -\frac{1}{2} \eta^{\mu\nu} \partial_\lambda h_{00}$$

geodesic eqn: $\frac{d^2 x^\mu}{d\tau^2} = \frac{1}{2} \eta^{\mu\lambda} \partial_\lambda h_{00} \left(\frac{dt}{d\tau}\right)^2 = 0$

$\partial_0 h_{00} \to 0$ (condition ii)

$\mu = 0$ component: $\frac{d^2 t}{d\tau^2} = 0 \to \frac{dt}{d\tau} = \text{const}$

$\mu = i$ components: $\frac{d^2 x^i}{dt^2} = \frac{1}{2} \partial_i h_{00}$ $\qquad g_{00} = -(1 + 2\phi)$ → metric perturbation plays this role.

$h_0 = 2\phi \longrightarrow$ Newtonian Potential

$\vec{a} = -\vec{\nabla}\phi$

Find a replacement for poisson's equation.

$\nabla^2 \phi = 4\pi G\rho$, looking for a tensorial replacement of this should involve 2^{nd} derivatives of the metric (should be a tensor, rank-2)

RHS should also be a rank-2 tensor, generalize concept of mass density.

$T_{\mu\nu}$ = energy-momentum tensor, stress-energy tensor.

Energy-momentum tensor

1) consider a single particle in minkowski metric $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ $c = 1$

trajectory $x^\mu(\lambda)$ → world line of particle

$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$ → + space-like. → 0 null geodesic → − timelike

e.g. massive particle. $d\tau^2 = -ds^2$ proper time.

Trajectory: $\tau = \int d\lambda \left(\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right)^{1/2} = 0$

Four-velocity "velocity through space-time"

$u^\mu = \frac{dx^\mu}{d\tau}$ $\eta_{\mu\nu} u^\mu u^\nu = -1$ speed through spacetime is always -1

the rest frame of particle: $u^\mu = (1, 0, 0, 0)$

Four-momentum $p^\mu = \underbrace{mc^2}_{\text{rest mass of particle.}} u^\mu$

$p^\mu = m u^\mu$

→ rest frame of particle.

$p^\mu = (m, 0, 0, 0)$

→ in a moving frame (x-axis)

$p^\mu = (\gamma m, v\gamma m, 0, 0)$

→ after lorentz transformation

$\gamma = \frac{1}{\sqrt{1 - v^2}}$ lorentz factor.

* in the limit of $v \ll 1$

$p^0 = m + \frac{1}{2}mv^2$ ← energy

$p^i = mv$ ← momentum

length of momentum 4-vector.

$p_\mu p^\mu = \eta_{\mu\nu} p^\mu p^\nu = -m^2$

$$-(p^0)^2 + |\vec{p}|^2 = -m^2$$

$$[E^2 = (mc^2)^2 + (pc)^2]$$

$$E = p^0 = \sqrt{m^2 + |\vec{p}|^2}$$

...

Eg Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$$f^\mu = q u^\lambda F_\lambda{}^\mu$$

Lorentz 4 force

EM field tensor (antisymmetric) composed of $\vec{E}, \vec{B}$ components

10/09/24

T^{ij} = "flux of p^i in j Direction"
= shear/stress

(Simplest Example: "Dust"

collection of particles at rest w.r.t to each other.

→ rest mass m

$u^\mu(x^\nu)$ – 4-velocity field (constant)

$N^\mu = n u^\mu$ = number-flux 4 vector
↳ # density of particles in the rest-frame.

N^μ = 4 vector by construction
N^0 = coo-dependent # density
N^i = coo dependent flux of particles in the i-direction.

rest frame $\rho = \underbrace{mn}$ = (rest frame) mass density.
↳ n = 0^{th} component of a four-vector N^μ
m = 0^{th} component of a four-vector p^μ ↳ momentum four-vector.

Concatenate N^μ and p^μ to form a rank 2 tensor

$$T^{\mu\nu} = N^\mu p^\nu = m n u^\mu u^\nu = \rho u^\mu u^\nu$$ → tensor by construction. ✓
→ no pressure (in rest frame) ✓

Example (next simplest): ideal fluid

→ ρ = mass density $\quad p$ = isotropic $\quad$ no shear/stress

$$T^{\mu\nu}_{dust} = \begin{pmatrix} \rho & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} \qquad T^{\mu\nu}_{\text{ideal fluid}} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

* How can we construct a tensor that looks like this in the rest frame?

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu} \quad \text{(Special relativity)} \qquad \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

$$= (\text{in rest frame}) = \begin{pmatrix} \rho+p & & & 0 \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix} + \begin{pmatrix} -p & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

SR → GR

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$$ ← energy momentum tensor

what does this mean (physically)?

Let's require : $\partial_\mu T^{\mu\nu} = 0$ ← no divergence

→ 4 equations for $\nu = 0,1,2,3$

→ Newtonian limit $(v \ll 1)\,(p \ll \rho)$

→ Minkowski metric

$$\left[\quad\right]$$

mass conservation

↳ component $\parallel$ to u^μ (1 eqn): $\underbrace{\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$

momentum conservation

→ components $\perp$ to u^μ (3 eqn): $\underbrace{\partial_t \vec{v}} + \underbrace{(\vec{v}\cdot\vec{\nabla})\vec{v}} = \frac{-1}{\rho}\vec{\nabla} p$

Euler's $F = ma$ — flux in and out of the box

[Euler equations in Newtonian fluid mechanics]

$\nabla^2 \phi = 4\pi G \rho$ ↳ $T^{\mu\nu}$

??

→ $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$

→ has contraction of Riemann Tensor
→ right # of indices
→ symmetric $\mu \leftrightarrow \nu$
→ contains 2^{nd} derivatives of $g_{\mu\nu}$

LHS, want rank-2 tensor, involve 2^{nd} derivatives of $g^{\mu\nu}$

→ $g_{\mu\nu;\lambda\chi}$ → not good, $g_{\mu\nu;\chi} = 0$
[equivalence principle]

→ Use $G_{\mu\nu}$ = Einstein tensor.

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$

→ $T = g_{\mu\nu} T^{\mu\nu}$ → $T_{;\nu} = 0$ → T = constant

→ this tensor is special, has zero divergence: $G_{\mu\nu}{}^{;\mu} = 0$ [can show this]

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}$$

Fix constant k by requiring Newtonian limit to yield $\nabla^2\phi = 4\pi G\rho$

Newtonian Limit

(i) weak field: weak ~ small curvature.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| << 1$$

(ii) slow speeds $v << 1$

(iii) nearly static $\frac{\partial}{\partial t} \to 0$

(iv) p is small, pressure $\to 0$ $\left(p << \rho ;\ v < 1 \right)$

non-relativistic "special"

RHS: $T^{\mu\nu} = (\rho + \cancel{p}) u^\mu u^\nu + \cancel{p g^{\mu\nu}}$

(iv)

$T^{\mu\nu} = \rho u^\mu u^\nu$

in rest frame of fluid $u^\mu = (1,0,0,0)$

$T_{00} = \rho$ all other components are zero

$$\underbrace{g^{\mu\nu} R_{\mu\nu}}_{R} \underbrace{- \frac{1}{2} R \underbrace{g_{\mu\nu} g^{\mu\nu}}_{4}}_{-2R} = \underbrace{K g^{\mu\nu} T_{\mu\nu}}_{T} \Rightarrow R = -KT$$

→ Rewrite Einstein's eqns:

$$R_{\mu\nu} = K \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$$

only have 0 0 component.

$R_{00} = \frac{1}{2} K \rho$

$$T = g^{00} T_{00} = -\rho$$

LHS

$$R_{00} = R^\lambda{}_{0\lambda 0} = R^i{}_{0i0}$$

$\left(R^0{}_{000} = \text{by the antisymmetry property.} \right)$

Reimman tensor.

(iii) $\quad O(h^2) \quad O(h^2)$

$$R^i{}_{0j0} = \partial_j \Gamma^i_{00} - \cancel{\partial_0 \Gamma^i_{j0}} + \cancel{\Gamma^i_{j\lambda} \Gamma^\lambda_{00}} - \cancel{\Gamma^i_{0\lambda} \Gamma^\lambda_{j0}}$$

recall $\Gamma \sim g(\partial g + \partial g - \partial g)$

$O(u)$

$R_{00} = R^{i}_{\ 0i0} = \partial_i \Gamma^{i}_{00}$

$= \partial_i \left[\underbrace{\tfrac{1}{2} g^{i\lambda}}_{g \to \eta,\ \text{diagonal}} \left(\cancel{\partial_0 g_{\lambda 0}} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00} \right) \right]$ (iii)

∂_i $\quad$ $g \to h$

$R_{00} = -\frac{1}{2} \delta_{ij} \partial_i \partial_j h_{00} = \boxed{-\frac{1}{2} \nabla^2 h_{00} = R_{00}}$ LHS

RHS $= \frac{1}{2} K \rho$

$\to \nabla^2 h_{00} = -K\rho$

For Newtonian limit of geodesic equation, we identified $h_{00} = -2\phi$, choose $K = 8\pi G$

$$\boxed{\to \nabla^2 \phi = 4\pi G \rho}$$ ✓

Einstein's eqns: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$

or $R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right)$

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volume integral.

$d^4 x' = \left\| \frac{\partial x'^{\mu}}{\partial x^{\nu}} \right\| d^4 x = \sqrt{-g}\, d^4 x$

going from 3D cartesian to polar coo's in 3D.

$\int_{vol} dx\, dy\, dz = \int dr\, d\theta\, d\phi$

$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r\cos\theta$

$\det[J] = [r^2 \sin\theta]$

$J = \text{matrix} \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix}$

$dx\, dy\, dz = r^2 \sin\theta\, dr\, d\theta\, d\phi$

vol space $\underbrace{\int_0^r r^2 dr}_{\frac{r^3}{3}} \underbrace{\int_0^\pi \sin\theta d\theta}_{2} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi} = \frac{4\pi}{3} r^3$

going to sted

Einstein's equations, set variation of action S to zero w.r.t. to changes in metric $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$: $\delta S = \delta \int \sqrt{-g}\, R\, d^4x = 0$

a lot of math integration by parts, etc. $\Longrightarrow$ $-\int \delta g_{\mu\nu} \left[\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}_{} \right] \sqrt{-g}\, d^4x = 0$

↳ einstein's equations in a vacuum.

properties of Einstein's equations

- 10 second-order partial differential equations for the 10 components of $g_{\mu\nu}$
- 4 constraints eqns for $\nabla^\mu T_{\mu\nu} = \nabla^\mu G_{\mu\nu} = 0$

  6 independent PDE's for 6 physical degrees of freedom

  ↳ 4 functions involving a coord transformation

$$x'^\mu = x'^\mu (x^\nu)$$

vacuum: $R_{\mu\nu} = 0$ (still non-trivial)

$T(g_{\mu\nu})$ is a function of the metric → non linear cannot add two solutions together like: $\nabla^2(\varphi_1 + \varphi_2) = 4\pi G(\rho_1 + \rho_2)$

[if φ_1, ρ_1 and φ_2, ρ_2 are solutions, so is $\varphi_1 + \varphi_2, \rho_2 + \rho_1$]

generally, solved numerically

Analytic solutions in rare cases, with high degree of symmetry:

(1) point mass → Schwarzchild / kerr metrics,

(2) cosmology → solution required to obey cosmological principle.

Freidmann-Robertson-Walker (FRW) metric

- Cosmological principle: universe is homogeneous and isotropic

We must be able to write the metric as 3D surfaces at fixed t:

$$ds^2 = dt^2 + \underbrace{R^2(t)}_{\text{overall "scale factor" (depends on time, not } x^i)} g_{ij}\, dx^i dx^j$$

→ include isotropy: $g_{ij} = A(r)\,dr^2 + \underbrace{r^2(d\theta^2 + \sin^2\theta\, d^2\phi)}_{\text{isotropy}}$

isotropic (A is function of r only)

$$g_{0i} = 0$$

because $\left.\begin{array}{l}\Delta x^\mu = (0, \Delta x^i) \\ \Delta y^\mu = (\Delta \varphi, 0)\end{array}\right\} \Delta X \cdot \Delta y = \underbrace{g_{ij}\,\Delta x^i \Delta y^j}$ in any coordinate system.

Ricci scalar R should be constant, independent of (r, θ, ϕ) in 3D

Most general form of metric:

$$ds^2 = -dt^2 + R^2(t)\left\{ \frac{dr}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta)\, d^2\phi \right\}$$

k = arbitrary constant

$R(t)$ = arbitrary function of time "scale factor"

$R(t)$ dimensionless (by choice), then $[r]$ = length

$$[k] = \frac{1}{\text{Length}^2}$$

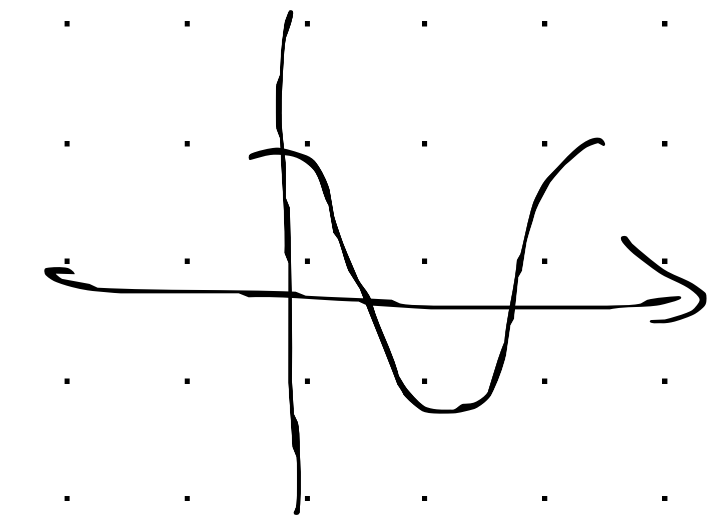
$R(t) \to a(t)$

$\to k = \dfrac{K}{R_0^2} \to K = 0, \pm 1$

R_0 → curvature scale $[R_0]$ = length alternate coo system (for spatial metric): $ds^2 = -dt^2 + a(t)^2 d\epsilon^2$

...

spatial part of metric $R^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin^2 x & 0 \\ 0 & 0 & \sin^2 x \sin^2\theta \end{bmatrix}$



$\det[g] = R^6 \sin^4 x \sin^2\theta$

$\sqrt{g} = R^3 \sin^2 x \sin\theta$

volume $= R^3 \underbrace{\int_0^{\pi} dx \sin^2 x}_{\frac{\pi}{2}} \underbrace{\int_0^{\pi} \sin\theta \, d\theta}_{2} \underbrace{\int_0^{2\pi} d\phi}_{2\pi} = 2\pi^2 R^2$

evolution of scale factor in FRW metric:

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right\}$$

$a(t)$ = dimensionless

$K = \frac{k}{R_0^2}$ $k = 0, \pm 1$

we can obtain $a(t)$ from einstein's eqns,

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)$$

LHS: plug $g_{\mu\nu}$ into $R_{\mu\nu}$ ...

<u>RHS</u> ideal fluid $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$

contract $T = g^{\mu\nu} T_{\mu\nu} = -\rho + 3p$

LHS = RHS, 4 eqns

0- component: $-3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p)$

i-i component: $\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p)$

$(\ldots)$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$; $\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}(\rho + 3p)$

friedmann Eqn. ; Acceleration Eqn

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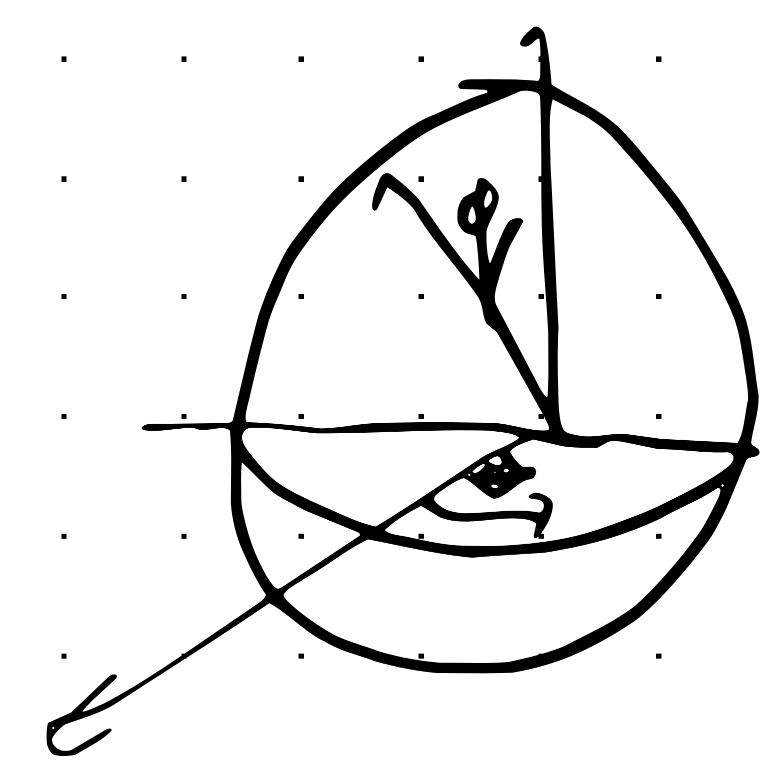
$g_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}$ $\quad ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2$



$\Gamma^\rho_{ij} = \frac{1}{2} g^{\rho m}\left[\partial_i g_{jm} + \partial_j g_{mi} - \partial_m g_{ij}\right]$

christoffel symbol

because of curvature $DA^\mu \neq \partial_\nu A^\mu + \Gamma$

$R^\theta{}_{\phi\theta\phi} = -R^\phi{}_{\theta\phi\theta}$

Reimann: $R^\rho{}_{\mu\nu\sigma}$
↳ Ricci
↳ Ricci Scalar

$R^\rho{}_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$

$R^\theta{}_{\phi\theta\phi} = \partial_\theta\Gamma^\theta_{\phi\phi} - \partial_\phi\Gamma^\theta_{\theta\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi}$

<u>Note</u> from HW: $R_{\phi\theta} = g_{\theta\phi} R^\theta{}_{\phi\theta\phi}$

$R_{\sigma\nu} = \partial_\alpha\Gamma^\alpha_{\nu\sigma} - \partial_\nu\Gamma^\alpha_{\alpha\sigma} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\sigma}$

for 2D case

general formula for riemann tensor

$\partial_0\Gamma^0_{\nu\sigma} + \partial_1\Gamma^1_{\nu\sigma}$

$\alpha = 0, \lambda = 0, 1$

$\Gamma^0_{00}\Gamma^0_{\nu\sigma} + \Gamma^0_{01}\Gamma^1_{\nu\sigma}$

$\alpha = 1$: $+\Gamma^1_{10}\Gamma^0_{\nu\sigma} + \Gamma^1_{11}\Gamma^1_{\nu\sigma}$

going back to the $g_{ij} = \begin{bmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{bmatrix}$ metric

the christoffel symbol: $\Gamma^\theta_{\phi\phi} = \frac{1}{2} g^{\theta\theta}\left(\partial_\phi g_{\theta\phi} + \partial_\phi g_{\theta\phi} + \partial_\theta g_{\phi\phi}\right)$

$\frac{1}{2}\cancel{g^{\theta\phi}}(\dots)$

0

$= \frac{g^{\theta\theta}}{2}\left(\partial_\theta g_{\phi\phi}\right) = \frac{1}{2}\left[\partial_\theta\left(R^2\sin^2\theta\right)\right] = -\sin\theta\cos\theta$

HW2 P#3

Ricci: $R_{\sigma\nu} = \partial_\alpha \Gamma^\alpha_{\nu\sigma} - \partial_\nu \Gamma^\alpha_{\alpha\sigma} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\alpha_{\nu\lambda}\Gamma^\lambda_{\alpha\sigma}$

Ricci: $R_{ij} \to$ time-time R_{00} → we probably did this in TA in this office hours.

time-space R_{0i}

(t, r, θ, ϕ)
$\{0, 1, 2, 3\}$

space-space: $i=j$ R_{ii} $i \neq j$ R_{ij}

P#3 $ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{bmatrix} \quad g^{ij} = \begin{bmatrix} 1 & & & \\ & -\frac{(1-kr^2)}{a^2} & & \\ & & -\frac{1}{a^2r^2} & \\ & & & -\frac{1}{a^2r^2\sin^2\theta} \end{bmatrix}$$

$$R_{\underset{ij}{00}} = \partial_\alpha \Gamma^\alpha_{00} - \partial_0 \Gamma^\alpha_{\alpha 0} + \Gamma^\alpha_{\alpha\lambda}\Gamma^\lambda_{00} - \Gamma^\alpha_{0\lambda}\Gamma^\lambda_{0\alpha}$$

$\Gamma^{0,1,2,3}_{0,0}$

$$\Gamma^{0,1,2,3}_{0(0,1,2,3)} = \Gamma^{(0,1,2,3)}_{(0,1,2,3)0}$$

$$\Gamma^0_{\underset{ij}{00}} = \frac{g^{0m}}{2}\left[\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}\right]$$

$$\Gamma^0_{ij} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{bmatrix}$$

make sure that this is non-zero in the first place

$$\Gamma^0_{00} = \frac{g^{0m}}{2}\left[\underset{\text{time}}{\partial_0} g_{0m} + \partial_0 g_{m0} + \partial_m g_{00}\right]$$

$$m=0$$
$$= \frac{g^{00}}{2}\left[\partial_0 g_{00}\right]$$
$$\frac{\partial}{\partial t}(1) = 0$$

$$\Gamma^0_{ij} \rightarrow \mu \rightarrow (t, r, \theta, \phi)$$
$$(0, 1, 2, 3)$$

$i \neq j$
$\rho = 0$

$$\Gamma^\rho_{ij} = \Gamma^0_{ij} = \frac{g^{00}}{2}\left(\cancel{\partial_i g_{0j}} + \cancel{\partial_j g_{i0}} - \cancel{\partial_0 g_{ij}}\right)$$

when $i \neq j$
$$= 0$$

$g_{0j} \neq 0 \quad j = 0$

$$\Gamma^0_{ii} = \Gamma^0_{11} \quad \Gamma^0_{22} \quad \Gamma^0_{33}$$

problem 1a

$$ds^2 = dr^2 + r^2 d\theta^2$$

(1) $\frac{ds^2}{ds^2} = 1 = \frac{dr^2}{ds^2} + r^2 \frac{d\theta^2}{ds^2} \longrightarrow \frac{d\theta}{dr}$ ← get this so you can integrate. $\rightarrow \theta(r)$; $\frac{d\theta}{dr} = \frac{d\theta}{ds}\frac{ds}{dr}$

(2) $\frac{d}{ds}\left(r^2 \frac{d\theta}{ds}\right) = 0 \rightarrow r^2 \frac{d\theta}{ds} = \text{Const.} = \text{"}\ell\text{"}$; $\left(\frac{ds}{ds}\right)^2 = \left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\theta}{ds}\right)^2$

$$= 1 \qquad \ell / r^2$$

(3) $\frac{d\theta}{ds}(r)$

Problem 5 can put in arbitrary time units

$$t_0 = \left(\frac{3}{8\pi G \rho_0}\right)^{1/2}$$

Problem 4:

→ no need to integrate over θ

$$ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + dw^2$$

$$\underbrace{x^2 + y^2 + z^2}_{r^2} + w^2 = R^2$$

$\hookrightarrow w(r, R)$

$\hookrightarrow dw(dr, R)$

10/21/2024

<u>cosmological Redshift</u> -
property of FRW metric

$$ds^2 = -c^2 dt^2 + a^2(t)\left\{dr^2 + S_k^2(r)\, d\Omega^2\right\} \qquad S_k = \begin{cases} R_0 \sin(r/R_0) & k=+1 \\ r & k=0 \\ R_0 \sinh(r/R_0) & k=-1 \end{cases}$$

- consider photon propagating in this metric, $ds^2=0$ (null geodesic)
- consider a photon emitted in some distant galaxy located at fixed (comoving) coordinate r at time t_e propagating radially (along r coordinate only, $d\Omega=0$) to us
  we detect photon at $r=0$ at t present time $t_0 > t_e$.

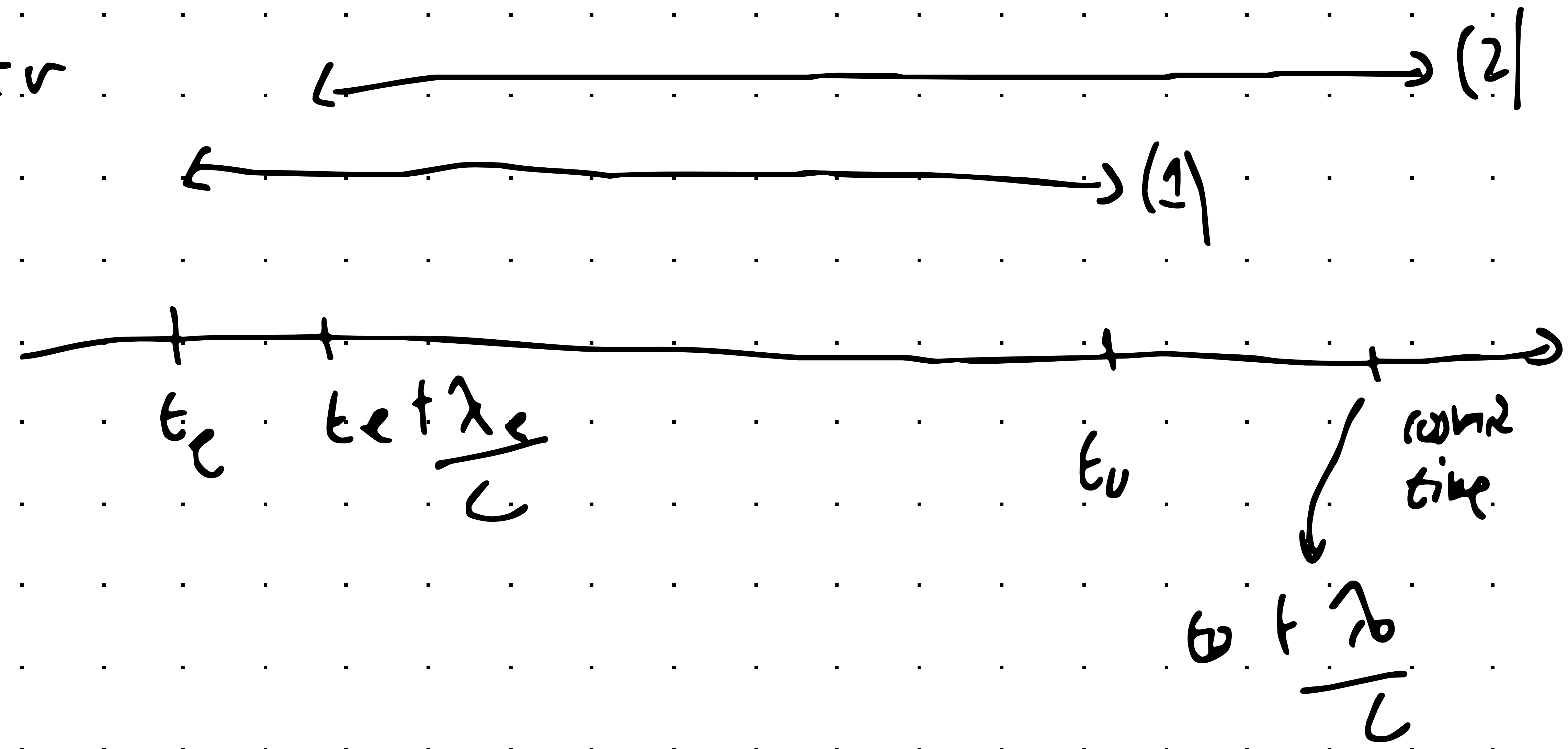
- $ds=0 \Rightarrow \dfrac{c\,dt}{a(t)} = dr$

$$c\int_{t_e}^{t_0} \frac{dt}{a(t)} = \int_0^r dr = r \longrightarrow \text{maximum of one wavelength}$$

$$(1)\quad c\int_{t_e+dt_e}^{t_0+dt_0} \frac{dt}{a(t)} = r \longrightarrow \text{next maximum (next photon) emitted } dt_e \text{ later.}$$

$$(2)\quad c\int_{t_e+\frac{\lambda_e}{c}}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)} = r$$

(1) - (2):



$$\int_{t_e}^{t_e+\frac{\lambda_e}{c}} \frac{dt}{a(t)} = \int_{t_0}^{t_0+\frac{\lambda_0}{c}} \frac{dt}{a(t)}$$

universe evolves slowly $\left(\frac{a}{\dot a} \ggg \frac{\lambda_0}{c} ; \frac{\lambda_e}{c}\right)$

$$\frac{1}{a(t_e)}\frac{\lambda_e}{c} = \frac{1}{a(t_0)}\frac{\lambda_0}{c} \longrightarrow \frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = (1+z) \longrightarrow \text{cosmological redshift}$$

Comments: (i) not a doppler shift

(ii) logic is not general, holds for any time interval as long as $ds = 0$ and $\frac{\dot a}{a}$ is slow

Alternative derivation:

geodesic eqn. $\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\sigma} \frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

u^μ = four-velocity $= \frac{dx^\mu}{d\tau}$

$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\lambda\sigma} u^\lambda u^\sigma = 0$

$\mu = 0$ (time component): $\frac{du^0}{d\tau} + \Gamma^0_{\lambda\sigma} u^\lambda u^\sigma = 0$

in FRW metric $= \Gamma^0_{0\sigma} = \Gamma^0_{\lambda 0} = 0$

$\Gamma^0_{ij} = \frac{\dot a}{a} h_{ij}$ h_{ij} = spatial part of $g_{\mu\nu}$

$\frac{du^0}{d\tau} + \frac{\dot a}{a} h_{ij} u^i u^j = 0$

$u^\mu = (u^0, u^i) = (\gamma, \gamma u^i) = (u^0, \vec u)$

$h_{ij} u^i u^j = |\vec u|^2$

$\frac{du^0}{d\tau} + \frac{\dot a}{a} |\vec u|^2 = 0$

Recall

$|u^0|^2 - |\vec u|^2 = 1$ for four-velocity.

$\Rightarrow u^0 du^0 = |\vec u| d|\vec u|$ $\Rightarrow \frac{1}{u^0}\frac{d|\vec u|}{d\tau} + \frac{\dot a}{a}|\vec u| = 0$

$\Rightarrow du^0 = \frac{|\vec u|}{u^0} d|\vec u|$

$\frac{d|\vec u|}{dt} + \frac{\dot a}{a}|\vec u| = 0$

$$\frac{|\dot{\vec{u}}|}{|\vec{u}|} = -\frac{\dot{a}}{a} \Rightarrow |\vec{u}| \propto a^{-1}$$

$$|\vec{u}| = \frac{|\vec{v}|}{\sqrt{1-|\vec{v}|}} \propto a^{-1}$$

$$p^{\mu} = m u^{\mu} \longrightarrow |\vec{p}| \propto a^{-1}$$

→ "peculiar velocity" and "peculiar momentum"
decays as the universe expands.

for photons, geodesic eqn leads to the same result:

$$\lambda = \frac{h}{|\vec{p}|} \propto a$$

i-i component: $\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}$

non-relativistic

$a(t)$, $p(t)$ → ~~two~~ tracks, we need more equations

1st law of thermodynamics, $dQ = dE + p\,dV$

heat flow in/out of small region

change of energy in region

change volume.

homogeneous, isotropic universe, $dQ = 0$

$$dE + p\,dV = 0$$

$$\dot{E} + p\dot{V} = 0$$

$$E(t) = V(t)\,\varepsilon(t)$$

energy density

volume

energy

$$\boxed{\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + p) = 0}$$

Fluid equation.