To The Quantum Future - Assignment 0

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1 Oddtown & Eventown Problem

Let's consider the **eventown problem** first. Let $v_i \in \mathbb{F}_2^n$ denote a vector where $v_{ij} = 1$ if j'th person is in i'th club and 0 otherwise. Let V denote the span of $\{v_1, v_2, \cdots, v_m\}$. Consider any vector $v \in V$ formed by the linear combination of v_i, say $v_i \quad v_j$. Then, as v_i and v_j have even number of one's, and so does $v_i \cdot v_j$, we see that $v = v_i + v_j$ also has even elements, which means that V is a subspace under addition and multiplication of $\mathbb{F}_2^n$. Also $\forall v, w \in V$; From the condition that clubs share only even number of members, we have that $v \cdot w = 0 \Rightarrow V \subset V^\perp$. Thus

$$n = \text{dim}V + \text{dim}V^\perp \geq 2\text{dim}V$$

$$\text{dim}V \leq n/2 \Rightarrow |V| \leq 2^{n/2}$$

For the **oddtown problem**, the given conditions imply that $\langle v_i | v_j \rangle = 1$ if $i = j$ and 0 otherwise. Thus the v_i make a basis for a subspace of the n dimensional space $\mathbb{F}_2^n$ which implies $m \leq n$ as the number of basis for the subspace is equal to the dimension of the subspace which can not be more than the dimension of $\mathbb{F}_2^n$ itself.

2 Weaker Suffices

For any operator A in hilbert space V,

$$\langle v, Aw \rangle = \langle A^* w, v \rangle$$

Thus if A is hermitian, we easily see that $\langle x, Ax \rangle = \langle A^* x, x \rangle = \langle Ax, x \rangle$ Now to prove its converse, put $x = v + w$

$$\langle v + w, A(v + w) \rangle = \langle A(v + w), v + w$$

$$\langle w, Av \rangle + \langle v, Aw \rangle = \langle Av, w \rangle + \langle Aw, v \rangle$$

$$\langle A^* v, w \rangle + \langle A^* w, v \rangle = \langle Av, w \rangle + \langle Aw, v \rangle$$

$$\langle (A^* - A)v, w \rangle = \langle (A - A^*)w, v \rangle$$

Notice that the above two are conjugate of each other, thus$\langle (A^* - A)v, w \rangle$ is real for all v, w.

However if we multiply v by i, i.e. use vectors (iv, w) instead, the inner product gets multiplied by i, but it still remains real. Thus it must be 0 for all values.

Now setting $w = (A^* - A)v$, $||(A^* - A)v|| = 0 \forall v \Rightarrow (A^* - A) = O \Rightarrow A^* = A$ as required

3 Higher Dimensions

Consider two matrices $A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$ and $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

A has two eigenvalues $\lambda = \frac{7 \pm \sqrt{57}}{2}$ and eigenvectors $|\lambda\rangle$, and X has two eigenvalues $\mu = \pm 1$ and eigenvectors $|\mu\rangle$. Now for the tensor product $(A \otimes X)$,

$$(A \otimes X)|\lambda\rangle \otimes |\mu\rangle = A|\lambda\rangle \otimes X|\mu\rangle = \lambda\mu(|\lambda\rangle \otimes |\mu\rangle)$$

Thus eigenvalues of $(A \otimes X)$ are $\lambda\mu$

(a) Matrix $= A \otimes X \Rightarrow$ eigenvalues $= \pm\frac{7 \pm \sqrt{57}}{2}$

(b) Matrix $= X \otimes A \Rightarrow$ same as above

(c) Matrix $= A \otimes A \Rightarrow$ eigenvalues $= \left(\frac{7+\sqrt{57}}{2}\right)^2, \left(\frac{7-\sqrt{57}}{2}\right)^2, -2$

4 Algebra and Technicalities

Norm: $\sqrt{\langle x|x\rangle}$

1. Positive-definiteness: $\sqrt{\langle x|x\rangle} = 0 \Rightarrow \sum x_i^2 = 0 \Rightarrow x_i = 0 \forall i$

2. Absolute Homogenity: norm(sx) $= \sqrt{\langle sx|sx\rangle} = \sqrt{s\bar{s}\langle x|x\rangle} = |s|\sqrt{\langle x|x\rangle}$

3. Triangle Inequality:
   $\sqrt{\langle x+y|x+y\rangle} = \sqrt{\langle x|x\rangle + 2\langle x|y\rangle + \langle y|y\rangle} \leq \sqrt{|x|^2 + 2|x||y| + |y|^2} = |x| + |y|$

   where we have used $\langle x|y\rangle^2 \leq \langle x|x\rangle\langle y|y\rangle$ due to Cauchy-Shwarz Inequality.

Metric: $d(x, y) = |x - y|$

1. $d(x, x) = 0$

2. $d(x, y) = d(y, x)$

3. If $x \neq y$, then $d(x, y) > 0$

4. $d(x, y) + d(y, z) \geq d(x, z)$

We easily see that the above definition of metric satisfies all the necessary properties, thus $d(x,y) = |x-y|$ is a valid metric.
Assuming $n \geq m$ $\langle f_n|f_m\rangle = 1 + \int_0^{1/n}(1-nt)(1-mt)dt = 1 + \frac{3n-m}{6n^2}$

To prove its cauchy, we need to show that $d(f_m - f_n) < \epsilon \forall m, n \geq M$ for some M; for all epsilon.

$$|f_m - f_n| = \langle f_m - f_n|f_m - f_n\rangle = \left(1 - \frac{m}{n}\right)\frac{1}{\sqrt{3m}}$$

We can set $m > \frac{1}{\epsilon^2}$ to get $|f_m - f_n| < \epsilon$ for as small a ϵ as we want. Thus $\{f_n\}$ is Cauchy.
However $\{f_n\}$ converges to a function g which is discontinuous at $x = 0 \Rightarrow$ it is not there in our subspace of continuous real functions $\Rightarrow$ this inner product space is not complete.

Show that any finite dimensional vector space V under field $\mathbb{F}$ is a Hilbert space under any valid inner product on V for $\mathbb{F} = \mathbb{R}/\mathbb{C}$

We know that $\mathbb{F}^n$ is a complete space and thus a Hilbert space under any valid inner product. Consider a finite dimensional space V. Let it have orthonormal basis $\{v_1, v_2 \cdots, v_n\}$ and let $\{e_i, e_2, \cdots e_n\}$ denote the basis for $\mathbb{F}^n$ where $n = \dim V$. Then we can describe a transformation T such that $T(v_i) = e_i$. Clearly T is a bijection and takes V to $\mathbb{F}^n$, therefore V is isometric to $\mathbb{F}^n$.
Also $\langle T(v_i), T(v_j)\rangle = \langle e_i, e_j\rangle = \delta_{ij} = \langle v_i, v_j\rangle$. Therefore T is an isomorphism of inner product spaces and therefore any property of V (as an inner product space) is implied by the corresponding property in $\mathbb{F}^n$, in particular that it is Hilbert.

5 Hilbert-Schmidt and Vectorization

We know that

$$\langle \alpha(v_i \otimes w_i)|\beta(v_j \otimes w_j)\rangle = \alpha^*\beta\langle v_i|v_j\rangle\langle w_i|w_j\rangle$$

$$\begin{aligned}
\langle U|V\rangle_{HS} &= \langle TU|TV\rangle \\
&= \langle \sum_{i,j}\alpha_{ij}(v_i \otimes w_j)|\sum_{m,n}\beta_{mn}(v_m \otimes w_n)\rangle \\
&= \sum_{i,j,m,n}\alpha^*_{ij}\beta_{mn}\langle v_i|v_m\rangle\langle w_j|w_n\rangle \\
&= \sum_{i,j}\alpha^*_{ij}\beta_{ij}\langle v_i|v_i\rangle\langle w_j|w_j\rangle \\
&= \sum_{i,j}\alpha^*_{ij}\beta ij \\
&= tr(U^*V)
\end{aligned}$$