MOMENT GENERATING FUNCTIONS (MGF)

Definition

Let X be any random variable, The moment generating function (MGF) of X is denoted by $M_X(t)$ and is defined by

$$M_X(t) = E_X\left(e^{tx}\right)$$

If X is a discrete random variable taking values on the non-negative integers then,

$$M_X(t) = E_X\left(e^{tx}\right) = \sum_{k=0}^{\infty} e^{tk} p(k) \qquad \text{where } p(k) = p(X = k)$$

Example

Find the moment generating function for the discrete random variable X with probability distribution .

X	2	5
$p(X = x)$	0.4	0.6

Solution

$$M_X(t) = E_X\left(e^{tx}\right) = \sum_{x=0}^{\infty} e^{tx} p(k)$$
$$= 0.4e^{2t} + 0.6e^{5t}$$

MGF for a Bernoulli trial random variable.

Example

Find the MGF , $M_X(t)$ of the random variable X , which takes value 1 with probability p and value 0 with probability $q = 1 - p$.

Solution

$$M_X(t) = E_X\left(e^{tx}\right) = \sum_{k=0}^{\infty} e^{tk} p(k)$$

X	e^{tk}	$p(X = k)$	$E\left(e^{tx}\right)$
0 1	1 e^t	q p	q pe^t
Total		1	$q + pe^t$

$$M_X(t) = q + pe^t$$

MGF for a continuous random variable

For a continuous random variable X lying on the interval (a,b), the MGF is calculated as an integral using the density function.

$$M_X(t) = E\left(e^{tx}\right) = \int_a^b e^{tx} f(x)\,dx$$

Example

Let X have the density function $f(x) = e^{-x}$, where $0 \leq x \leq \infty$. Calculate the MGF of X.

Solution

$$\begin{aligned}
M_X(t) = E\left(e^{tx}\right) &= \int_0^\infty e^{tx} f(x)\,dx \\
&= \int_0^\infty e^{tx} e^{-x}\,dx \\
&= \int_0^\infty e^{-x+tx}\,dx \\
&= \int_0^\infty e^{-(1-t)x}\,dx \\
&= \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_0^\infty \\
&= \left(\frac{e^{-\infty}}{-(1-t)}\right) - \left(\frac{e^{0}}{-(1-t)}\right) \\
&= 0 + \frac{1}{1-t} \\
&= \frac{1}{1-t}
\end{aligned}$$

$\therefore \quad M_X(t) = \dfrac{1}{1-t}$ for all $t < 1$

Properties of the Moment Generating Function.

1. Moments $M_X^k(0) = E\left(X^k\right)$
   The superscript (k) means the k^{th} derivative.
2. Linear transformation.
   If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$
3. Sums of independent random variables.
   If $X_1,\ X_2, \ldots, X_n$ are independent random variable and $S = X_1 + X_2 + \ldots + X_n$, then $M_S(t) = M_{X_1}(t).\ \mathrm{M}_{\mathrm{X}_2}(t)\ .\ \ldots\ \mathrm{M}_{\mathrm{X}_n}(t)$

4. Corollary to (3). If X_1, X_2, ….., X_n are independent random variables, all with common distribution X, then

$$M_S(t) = [M_X(t)]^n$$

Proof

1. $M_X(t) = E(e^{tx})$

$$M_X^{/}(t) = \frac{d}{dt}[M_X(t)] = \frac{d}{dt}E(e^{tX}) = E\left[\frac{d}{dt}e^{tX}\right] = E(X\ e^{tX})$$

$$\therefore M_X^{/}(t) = E(X\ e^{tX})$$

Taking successive derivatives leads to $M_X^{(k)}(t) = E(X^k e^{tX})$

Hence, evaluating at $t = 0$,

$$M_X^{(k)}(0) = E(X^K.e^0) = E(X^K.1) = E(X^K)$$

2. If $Y = aX + b$, then

$$M_Y(t) = E\left(e^{(aX+b)t}\right) = E\left(e^{aXt}e^{bt}\right) = e^{bt}E\left(e^{(aX)t}\right) = e^{bt}M_X(at)$$

3.

$$M_S(t) = E\left(e^{t(X+X_{21}+\ldots+X_n)}\right)$$
$$= E(e^{tX_1}).E(e^{tX_2}).......E(e^{tX_n})$$
$$= M_{X_1}(t).M_{X_2}(t)........M_{X_n}(t)$$

Calculating Mean and Variance with MGF $M_X(t)$.

$$E(X) = \mu = M_X^{/}(0)$$

$$Var(X) = M_X^{//}(0) - (M_X^{/}(0))^2$$

Short-cut method

Let X be a random variable with MGF, $M_X(t)$.

Define $h(t) = \ln(M_X(t))$

Proof

$$M_X(t) = E(e^{tX})$$

$$M_X(0) = E(e^0) = E(1) = 1$$

$$h(t) = \ln(M_X(t))$$

$$h^{/}(t) = \frac{1}{M_X(t)} \times M_X^{/}(t) = \frac{M_X^{/}(t)}{M_X(t)} = \frac{E(X\ e^{tX})}{\mathrm{E}(\mathrm{e}^{\mathrm{tX}})}$$

$$h^{/}(0)=\frac{M_X^{/}(0)}{M_X(0)}=\frac{E\left(X\,e^0\right)}{\mathrm{E}\left(\mathrm{e}^0\right)}=\frac{E(X)}{E(1)}=\frac{E(X)}{1}=E(X)$$

$$h^{//}(t)=\frac{M_X(t).\mathrm{M}_{\mathrm{X}}^{//}(t)-\left(M_X^{/}(t)\right)^2}{\left(M_X(t)\right)^2}$$

$$h^{//}(t)=\frac{E\left(e^{tX}\right).E\left(X^2e^{tX}\right)-\left[E\left(Xe^{tX}\right)\right]^2}{\left[E\left(e^{tX}\right)\right]^2}$$

$$h^{//}(0)=\frac{E\left(e^0\right).E\left(X^2e^0\right)-\left[E\left(Xe^0\right)\right]^2}{\left[E\left(e^0\right)\right]^2}$$

$$h^{//}(0)=\frac{E(1)..E\left(X^2\right)-\left[E(X)\right]^2}{\left[E(1)\right]^2}$$

$$h^{//}(0)=\frac{1..E\left(X^2\right)-\left[E(X)\right]^2}{\left[1\right]^2}$$

$$h^{//}(0)=E\left(X^2\right)-\left[E(X)\right]^2$$

$$\therefore Var(x)=E\left(X^2\right)-\left[E(X)\right]^2$$

Example

1. For the Bernoulli trial random variable in example above use the MGF to calculate the mean and variance of X .
   Solution
   The random variable X takes value 1 with probability p and takes the value 0 with probability $q=1-p$
   $$M_X(t)=q+pe^t$$
   $$M_X^{/}(t)=pe^t$$
   $$E(X)=M_X^{/}(0)=pe^0=p$$
   $$M_X^{//}(t)=pe^t$$
   $$M_X^{//}(0)=pe^0=p$$
   $$Var(X)=M_X^{/}(0)-\left[M_X^{//}(0)\right]^2=p-p^2=p(1-p)=pq$$
2. For the continuous random variable in example (2) use the MGF to calculate mean and variance.
   Solution
   The random variable X was defined to have the density function $f(x)=e^{-x}$, where $0\le x\le\infty$

$M_X(t) = \frac{1}{1-t}$

$M_X^{/}(t) = -(1-t)^{-2}$

$M_X^{//}(t) = 2(1-t)^{-3}$

$E(X) = M_X^{/}(0) = (1-0)^{-2} = 1$

$M_X^{//}(0) = 2(1-0)^{-3} = 2$

$Var(X) = M_X^{//}(0) - \left[M_X^{/}(0)\right]^2 = 2 - 1^2 = 2 - 1 = 1$

3. Let X have density function $f(x) = 2e^{-2x}$, $0 \le x \le \infty$. Calculate the MGF of X and find the mean and variance using short-cut method.

Solution

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}(2e^{-x})dx = \int_0^\infty 2e^{-(2-t)x}dx = \left[\frac{2e^{-(2-t)x}}{-(2-t)}\right]_0^\infty = \frac{2}{2-t}$$

The MGF exists for all $t < 2$.

Let $h(t) = \ln[M_X(t)] = \ln\left[\frac{2}{2-t}\right] = \ln 2 - \ln(2-t)$

$h^{/}(t) = 0 - \frac{1}{(2-t)}(-1) = \frac{1}{2-t} = (2-t)^{-1}$

$h^{//}(t) = (2-t)^{-2}$

$E(X) = h^{/}(0) = (2-0)^{-1} = 2^{-1} = \frac{1}{2}$

$Var(X) = h^{//}(0) = (2-0)^{-2} = 2^{-2} = \frac{1}{4}$

4. Let X have density function $f(x) = \lambda e^{-\lambda x}$, $0 \le x \le \infty$. Calculate the MGF of X and find the mean and variance using short-cut method.

Solution

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx}(\lambda e^{-\lambda x})dx = \int_0^\infty \lambda e^{-(\lambda-t)x}dx = \left[\frac{\lambda e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty = \frac{\lambda}{\lambda-t}$$

The MGF exists for all $t < \lambda$.

Let $h(t) = \ln M_X(t) = \ln\left[\frac{\lambda}{\lambda-t}\right] = \ln\lambda - \ln(\lambda-t)$

$h^{/}(t) = \frac{1}{\lambda-t}$

$E(X) = h^{/}(0) = \frac{1}{\lambda-0} = \frac{1}{\lambda}$

$h^{//}(t) = \frac{1}{(\lambda-t)^2}$

$$Var(X) = h^{\prime}(0) = \frac{1}{(\lambda - 0)^2} = \frac{1}{\lambda^2}$$

5. Find the moment generating function of the exponential distribution.

$$f(x) = \begin{cases} \frac{1}{c} e^{-x/c} & ,0 \le x \le \infty \\ 0, \quad elsewhere \end{cases}$$

Where $c > 0$. Hence find its mean and standard deviation.

Solution

The moment generating function is given by

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} \frac{1}{c} e^{-x/c} dx = \frac{1}{c} \int_0^{\infty} e^{(t - 1/c)x} dx = \frac{1}{c} \left[\frac{e^{(t-1/c)x}}{t - \frac{1}{c}} \right]_0^{\infty} = \frac{1}{1 - ct}$$

Let $h(t) = \ln M_X(t) = \ln(1 - ct)^{-1} = -\ln(1 - ct)$

$$h^{'}(t) = \frac{c}{1 - ct}$$

$$E(X) = h^{\prime}(0) = \frac{c}{1 - 0} = c$$

$$h^{\prime\prime}(t) = \frac{c^2}{(1 - ct)}$$

$$Var(X) = h^{\prime\prime}(0) = \frac{c^2}{1 - 0} = c^2$$

Standard deviation $\sqrt{Var(X)} = \sqrt{c^2} = c$

Exercise

1.The continuous random variable X has density function

$$f(x) = \begin{cases} 5e^{-5x} & for \ \ x \ge 0 \\ 0, & elsewhere \end{cases}$$

(i) Compute the moment generating function for X, $M_X(t)$.

(ii) Use $M_X(t)$ to compute the expected value and variance for X.

Moment Generating Functions for the Poisson distribution

If X is Poisson with parameter λ, then the MGF for X is given by

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^x}{x!}$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$
$$= e^{-\lambda} \left[1 + \frac{e^t \lambda}{1!} + \frac{(e^t \lambda)^2}{2!} + \frac{(e^t \lambda)^3}{3!} + ... + \frac{(e^t \lambda)^k}{k!} \right]$$
$$= e^{-\lambda} e^{\lambda t}$$
$$\therefore \mathrm{M_X}(\mathrm{t}) = e^{(t-1)\lambda}$$

Moment Generating Function of a Binomial distribution

Let X be a binomial random variable with n trials and probability of success p .Then the MGF for X is given by

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p(x)$$
$$= \sum_{x=0}^{\infty} e^{tx}\, {}^nC_x p^x q^{n-x}$$
$$= \sum_{x=0}^{\infty} {}^nC_x (e^{tx} p^x) q^{n-x}$$
$$= \sum_{x=0}^{\infty} {}^nC_x (e^t p)^x q^{n-x}$$
$$= {}^nC_0 (e^t p)^0 q^n + {}^nC_1 (e^t p)^1 q^{n-1} + {}^nC_2 (e^t p)^2 q^{n-2} + ... + {}^nC_n (e^t p)^n q^{n-n}$$
$$\therefore M_X(t) = (pe^t + q)^n$$

Let $h^/(t) = \ln[M_X(t)] = \ln[pe^t + q]^n = n \ln[pe^t + q]$

$$h^/(t) = n\left(\frac{1}{pe^t + q}\right)(pe^t) = \frac{npe^t}{pe^t + q}$$

$$E(X) = h^/(0) = \frac{npe^0}{pe^0 + q} = \frac{np}{p+q} = \frac{np}{1} = np$$

$$h^{//}(t) = \frac{(pe^t + q)(npe^t) - (npe^t)(pe^t)}{(pe^t + q)^2}$$

$$Var(X) = h^{//}(0) = \frac{(pe^0 + q)(npe^0) - (npe^0)(pe^0)}{(pe^0 + q)^2}$$

$$= \frac{(p+q)(np) - np^2}{(p+q)^2}$$

$$= \frac{np^2 + npq - np^2}{(p+q)^2}$$

$$= \frac{npq}{1^2}$$

$$\therefore Var(X) = npq$$

Example

1. Let $M_X(t) = \left[\frac{1}{3}e^t + \frac{2}{3}\right]^5$. Find
   - (i) $E(X)$
   - (ii) $Var(X)$
2. Let $M_X(t) = \frac{e^t}{2 - e^t}$. Find
   - (i) $E(X)$
   - (ii) $Var(X)$
3. Let $M_X(t) = e^{2(e^t - 1)}$. Find.
   - (i) $E(X)$
   - (ii) $Var(X)$
4. Consider a random variable Z with MGF given by $M_Z(t) = (0.2 + 0.8e^t)^5$. Find the mean and standard deviation for the random variable Z.
5. Suppose a random variable V has MGF, $M_V(t) = \frac{0.3}{(1 - 0.7e^t)^4}$. Find the expected value and standard deviation for V.