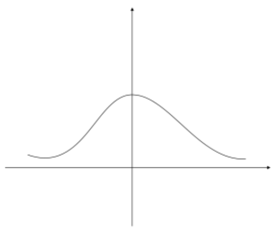
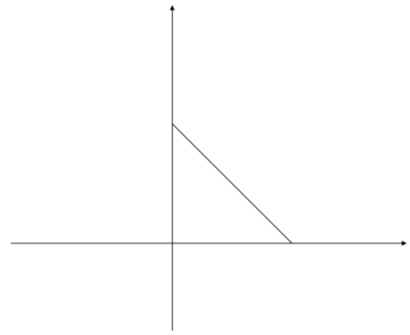
CONTINUOUS DISTRIBUTIONS

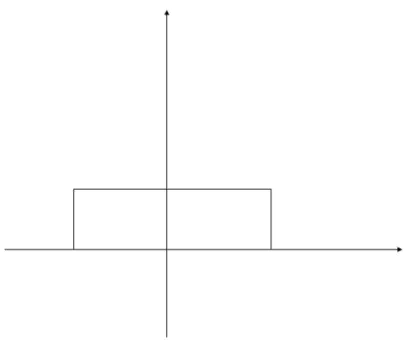
Common shapes of graph for a continuous distribution.



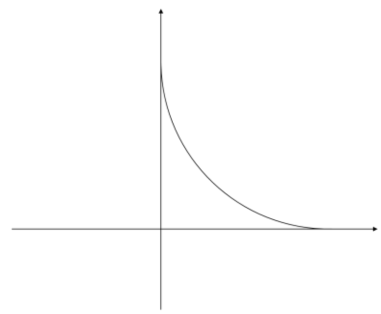
Normal



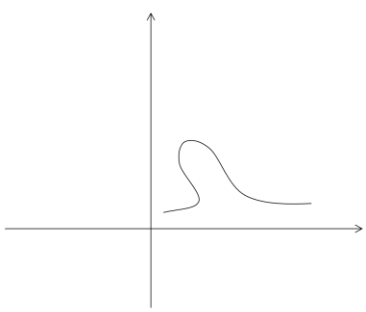
Triangular



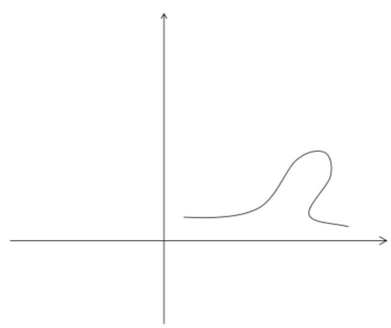
Uniform



Reverse

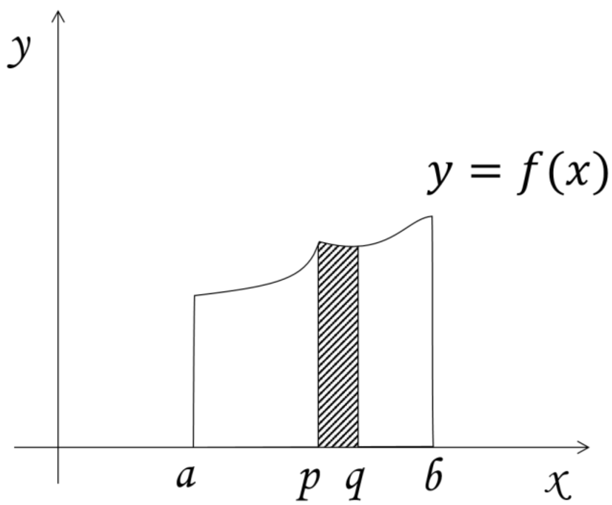


Positively Skewed



Negatively Skewed

Each continuous distribution cab be defined by a curve with equation $y = f(x)$ for a domain say $a < x < b$. $f(x)$ is known as the **Probability Density Function** (PDF). The area under the graph will give the total of the probability , i.e. 1



i.e. $\int_a^b f(x)\,dx = 1$

What is $P(p < x < q)$?

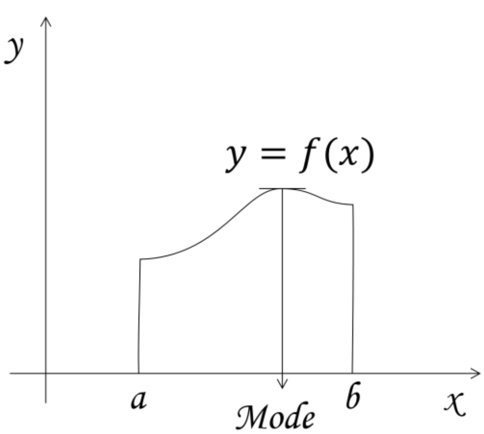
$$= \int_p^q f(x)\,dx$$

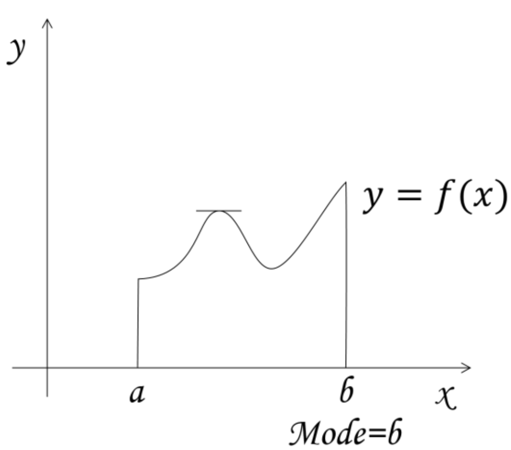
Often we write $P(p < x < q)$ where the variable X to be a continuous random variable with particular values x

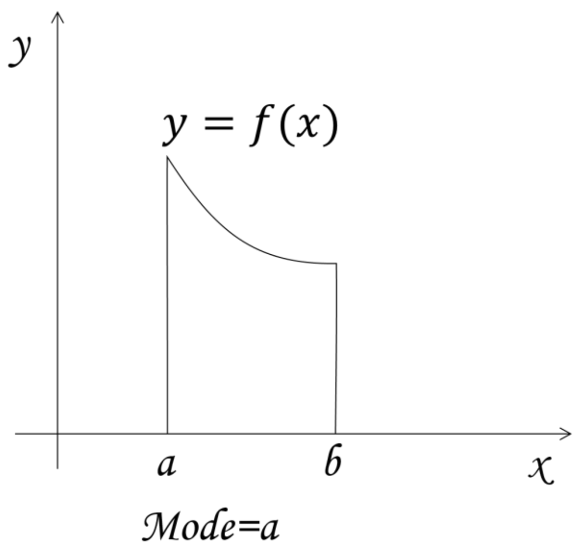
$$P(X = p) = 0$$

Measures of central tendency

Mode



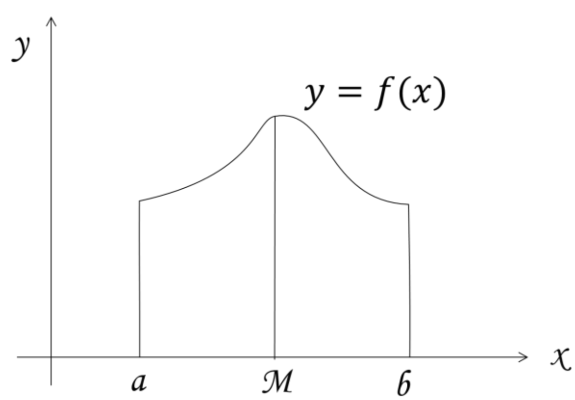




This is the value of x for which $f(x)$ is a maximum in the interval . This may a calculus maximum in the interval $a < x < b$ or it may be at one of the point of the boundary points $x = a$ or $x = b$.

i.e. $f'(x) = 0$, then solve the equation.

Median.



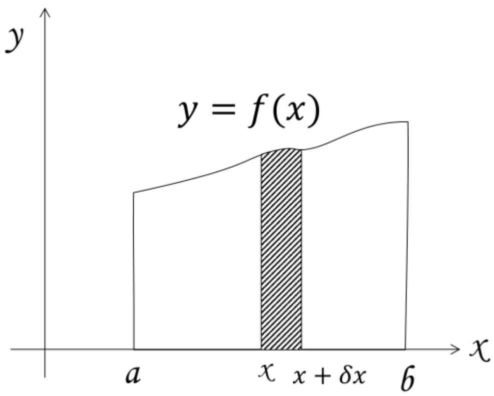
Median (M) is that value of x for which $\int_a^M f(x)\,dx = \frac{1}{2}$

Mean : μ or $E(X)$

Discrete random variable: $\mu = E(X) = \sum_{all\ x} x\,P(x)$

Continuous random variable: $\mu = E(X) = \sum_{x=a}^{x=b} x\ f(x)\,dx$

$$\to \int_a^b x\ f(x)dx \ \text{ as } \ dx \to 0$$



$P(x < X < x + \delta x) \approx f(x)\,dx$

For continuous:

$E(X) = \mu = \int_a^b x\ f(x)\,dx$ also $\int_a^b f(x)\,dx = 1$

MEASURES OF VARIABILITY

Semi-interquartile range (or Quartile Deviation)

$$= \frac{1}{2}(Q_3 - Q_1)$$

Where $\int_{a_1}^{Q_3} f(x)\,dx = \frac{3}{4} \quad or \quad 0.75$

$\int_a^{Q_1} f(x)\,dx = \frac{1}{4} \quad or \quad 0.25$

VARIANCE

Discrete:

$Var(X) = \sum_{all\ x} (x-\mu)^2$ also $Var(X) = E\left[(x-\mu)^2\right]$

Continuous: $Var(X) = \sum_{all\ x} (x-\mu)^2 f(x)\,dx \qquad \to Var(X) = \int_a^b (x-\mu)^2 f(x)\,dx$

$$Var(X) = \int_a^b (x-\mu)^2 f(x)\,dx = \int_a^b \left(x^2 - 2x\mu + \mu^2\right) \mathrm{f}(\mathrm{x})\,dx$$

$$= \int_a^b \mathrm{x}^2 f(x)dx - \int_a^b 2x\mu\ f(x)\,dx + \int_a^b \mu^2 f(x)\,dx$$

$$= \int_a^b x^2 f(x)dx - 2\mu\int_a^b x f(x)\,dx + \mu^2 \int_a^b f(x)dx$$

$$= \int_a^b x^2 f(x)\,dx - 2\mu(\mu) + \mu^2(1)$$

$$= \int_a^b x^2 f(x)dx - 2\mu^2 + \mu^2$$

$$= \int_a^b x^2 f(x)dx - \mu^2$$

$$\therefore Var(X) = \int_a^b x^2 f(x)\,dx - \mu^2 \quad or \quad E\left(x^2\right) - [E(x)]^2$$

Remark:

Let $f(x)$ be a continuous function , then $E(X) = \mu = \int_{-\infty}^{\infty} x\ f(x)\,dx$ and $Var(X) = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

NOTE: $f(x)$ is called a probability density function if

(i) $f(x) \geq 0$ for every value of x .

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) $\int_a^b f(x)\,dx = P(a < x < b)$

Examples

1. A function $f(x)$ is defined as follows

$$f(x) = \begin{cases} 0, & x < 2 \\ \frac{1}{18}(2x+3), & 2 \leq x \leq 4 \\ 0, & x > 4 \end{cases}$$

Show that it is a probability density function.

Solution

If $f(x)$ is a probability density function , then

(i) $\int_{-\infty}^{\infty} f(x)\ dx = 1$

Here $\int_2^4 \frac{1}{18}(2x+3)dx = \frac{1}{18}\left[x^2 + 3x\right]_2^4 = \frac{1}{18}[16+12-4-6] = 1$

(ii) $f(x) > 0 \quad for \quad 2 \le x \le 4$

2. The diameter of an electric cable is assumed to be continuous random variate with probability density function:

$$f(x) = \begin{cases} 6x(1-x), & 0 \le x \le 1 \\ 0, & elewhere \end{cases}$$

(i) Verify that above is a probability density function.

(ii) Find the mean and variance

Solution

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 6x(1-x)\,dx = \int_0^1 \left(6x - 6x^2\right)dx = \left[3x^2 - 2x^3\right]_0^1 = 3 - 2 = 1$$

Secondly, $f(x) > 0, \quad for \quad 0 \le x \le 1$

Hence the given function is a probability density function.

(ii) $$\mu = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 x.6x(1-x)\,dx = \int_0^1 \left(6x^2 - 6x^3\right)dx = \left[2x^3 - \frac{3}{2}x^4\right]_0^1 = 2 - \frac{3}{2} = \frac{1}{2}$$

$$Var(x) = \int_0^1 x^2 f(x)\,dx - \mu^2 = \int_0^1 x^2.6x(1-x)dx = \int_0^1 \left(6x^3 - 6x^4\right)dx - \left(\frac{1}{2}\right)^2$$

$$= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5\right]_0^1 - \frac{1}{4} = \frac{3}{2} - \frac{6}{5} - \frac{1}{4} = \frac{1}{20}$$

3. The probability density function $f(x)$ of a continuous random variable x is defined by

$$f(x) = \begin{cases} \dfrac{A}{x^3}, & 5 \le x \le 10 \\ 0, & \text{elsewhere} \end{cases}$$

Determine the value of A.

Solution

$$\int_5^{10} \frac{A}{x^3}dx = 1 \text{ ,}$$

$$\int_5^{10} A\,x^{-3}dx = \left[\frac{Ax^{-2}}{-2}\right]_5^{10} = \left[\frac{A}{-2x^2}\right]_5^{10} = \frac{A}{-2(10)^2} - \frac{A}{-2(5)^2} = \frac{A}{-200} - \frac{A}{-50} = \frac{3A}{200} = 1$$

$$\frac{3A}{200} = 1$$

$$A = \frac{200}{3}$$

4. A number of no-defective hours in a factory can be modelled by the continuous random variable X with p.d.f $f(x)$ given by

$$f(x)=\begin{cases} k(4x-x^2) & for \quad 0\le x\le 4 \\ 0, & \text{elsewhere} \end{cases}$$

Find:

(i) The value of constant k;

(ii) The expected value, $E(x)$;

(iii) The variance, $Var(x)$;

(iv) The mode of x.

(v) The median of x

(vi) $P(1\le x\le 3)$

Solution

(i) $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_0^4 k(4x-x^2)dx=k\left[2x^2-\frac{x^3}{3}\right]_0^4=k\left[32-\frac{64}{3}\right]=\frac{32}{3}k=1$$

$$\frac{32}{3}k=1$$

$$k=\frac{3}{32}$$

(ii)

$$\mu=\int_{\infty}^{\infty}x\ f(x)dx=\int_0^4 xk(4x-x^2)dx=\int_0^4\frac{3}{32}x(4x-x^2)dx=\int_0^4\left(\frac{3}{8}x^2-\frac{3}{32}x^3\right)dx$$

$$=\left[\frac{1}{8}x^3-\frac{3}{128}x^4\right]_0^4=\frac{1}{8}\times 64-\frac{3}{128}\times 256=8-6=2$$

(iii)

$$Var(x)=\int_{-\infty}^{\infty}x^2 f(x)dx$$

$$Var(x)=\int_0^4 x^2.\frac{3}{32}(4x-x^2)dx-\mu^2=\int_0^4\left(\frac{3}{8}x^3-\frac{3}{32}x^4\right)dx-2^2=\left[\frac{3}{32}x^4-\frac{3}{160}x^5\right]_0^4-4$$

$$=\frac{3}{32}\times 256-\frac{3}{160}\times 1024-4$$

$$=24-19.5-4$$

$$=0.5$$

(iv) Mode

$$f(x)=\frac{3}{32}\left(4x-x^2\right)$$

$$f(x)=\frac{3}{8}x-\frac{3}{32}x^2$$

$$f'(x)=0, \quad f'(x)=\frac{3}{8}-\frac{3}{16}x, \quad \frac{3}{8}-\frac{3}{16}x=0 \quad \therefore x=2$$

(v) Median

$$\int_0^M f(x)\,dx=\frac{1}{2}$$

$$\int_0^M\left(\frac{3}{8}x-\frac{3}{32}x^2\right)dx=\left[\frac{3}{16}x^2-\frac{1}{32}x^3\right]_0^M=\frac{3}{16}M^2-\frac{1}{32}M^3=\frac{1}{2}$$

$$or \quad \frac{1}{32}M^3-\frac{3}{16}M^2+\frac{1}{2}=0$$

$$M=5.4641\ or\ -1.4641\ \ or\ \ 2$$

(vi)

$$P(1\le x\le 3)=\int_1^3\frac{3}{32}\left(4x-x^2\right)dx=\int_1^3\left(\frac{3}{8}x-\frac{3}{32}x^2\right)dx=\left[\frac{3}{16}x^2-\frac{1}{32}x^3\right]_1^3$$

$$=\left(\frac{3}{16}\times3^2-\frac{1}{32}\times3^3\right)-\left(\frac{3}{16}\times1^2-\frac{1}{32}\times1^3\right)$$

$$=\frac{27}{16}-\frac{27}{32}-\frac{3}{16}+\frac{1}{32}$$

$$=\frac{11}{16}$$

5. A continuous random variable X is modelled by a probability density function.

$$f(x)=\begin{cases}kx, & 0<x<2\\ k, & 2<x<5\\ 0, & elsewhere\end{cases}$$

Where k is a constant.

Determine:

(i) Value of constant k;

(ii) Mean value of x;

(iii) Median

Solution

(i) $\int_{-\infty}^{\infty} f(x)\,dx=1$

$$\int_0^2 kx\,dx + \int_2^5 k\,dx = 1$$

$$\left[\frac{kx^2}{2}\right]_0^2 + [kx]_2^5 = 1$$

$$2k + 5k + 2k = 1$$

$$9k = 1$$

$$k = \frac{1}{9}$$

(ii)

$$E(x) = \mu = \int_{-\infty}^{\infty} x\ f(x)\,dx$$

$$\mu = \int_0^2 x.k\ x\,dx + \int_2^5 x.k\,dx$$

$$= \int_0^2 \frac{1}{9}x^2 dx + \int_2^5 \frac{1}{9}x\,dx$$

$$= \left[\frac{1}{27}x^3\right]_0^2 + \left[\frac{1}{18}x^2\right]_2^5$$

$$= \frac{1}{27}\times 2^3 + \frac{1}{18}\times 5^2 - \frac{1}{18}\times 2^2$$

$$= \frac{8}{27} + \frac{25}{18} - \frac{4}{18}$$

$$= 1\frac{25}{54} \quad or \quad 1.4630$$

(iii)

$$\int_0^M f(x)\,dx = \frac{1}{2}$$

$$\int_0^M \frac{1}{9}x dx + \int_M^5 \frac{1}{9}dx = \frac{1}{2}$$

$$\left[\frac{1}{18}x^2\right]_0^M + \left[\frac{1}{9}x\right]_M^5 = \frac{1}{2}$$

$$\frac{1}{18}M^2 + \frac{5}{9} - \frac{1}{9}M = \frac{1}{2}$$

$$\frac{1}{18}M^2 - \frac{1}{9}M + \frac{1}{18} = 0 \quad or \quad M^2 - 2M + 1 = 0$$

$$M = 1 \quad or \quad 1$$

$$\therefore M = 1$$

6. The length of a circle is assumed to be a continuous random variable x with a probability density function $f(x)$ defined by:

$$f(x)=\begin{cases} kx^3, & 0 \le x \le 1 \\ k, & 1 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

Determine the:

(i) Value of the constant k;

(ii) Mean;

(iii) $P\left(x \le \frac{1}{2}\right)$

Solution

(i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 kx^3 dx + \int_1^2 k\,dx = 1$$

$$\left[\frac{x^4}{4}k\right]_0^1 + [kx]_1^2 = 1$$

$$\frac{1}{4}k + 2k - k = 1$$

$$\frac{5}{4}k = 1$$

$$k = \frac{4}{5}$$

(ii)

$$\mu = \int_0^1 x.\frac{4}{5}x^3 dx + \int_1^2 x.\frac{4}{5}dx$$

$$= \int_0^1 \frac{4}{5}x^4 dx + \int_1^2 \frac{4}{5}x\,dx$$

$$= \left[\frac{4}{25}x^5\right]_0^1 + \left[\frac{4}{10}x^2\right]_1^2$$

$$= \frac{4}{25} + \frac{16}{10} - \frac{4}{10}$$

$$= 1\frac{9}{25} \quad or \quad 1.36$$

(iii)

$$P\left(x \le \frac{1}{2}\right) = \int_0^{\frac{1}{2}} \frac{4}{5}x^3 dx = \left[\frac{4}{20}x^4\right]_0^{\frac{1}{2}} = \frac{4}{20}\left(\frac{1}{2}\right)^4 = \frac{4}{320} = \frac{1}{80} \quad or \quad 0.0125$$

7. A continuous random variable t has a probability density function defined by

$$f(t)=\begin{cases} c(1-t)^2, & 1<t<3 \\ 0, & elsewhere \end{cases}$$

Determine the:

(i) Value of the constant c;

(ii) Mean;

(iii) $P(1.2 \le t \le 2.2)$

Solution

(i) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_1^3 c(1-t)^2\,dt \text{ , } \quad let\ u = 1-t \text{ , } \frac{du}{dt} = -1 \quad or \quad dt = -du$$

$$\int_1^3 cu^2(-du) = \int_1^3 -cu^2\,du = \left[\frac{-cu^3}{3}\right]_1^3 = \left[\frac{-c(1-t)^3}{3}\right]_1^3 = \left(\frac{-c(1-3)^3}{3}\right) - \left(-\frac{-c(1-1)^3}{3}\right)$$

$$= \left(\frac{-c(-2)^3}{3}\right) - \left(\frac{-c(0)^3}{3}\right) = \frac{-c(-8)}{3} - 0 = \frac{8c}{3}$$

$$\frac{8c}{3} = 1$$

$$c = \frac{3}{8}$$

(ii)

$$\mu = \int_1^3 t.\frac{3}{8}(1-t)^2\,dt = \int_1^3 \frac{3}{8}t\left(1-2t+t^2\right)dt = \int_1^3 \left(t-2t^2+t^3\right)dt$$

$$= \left[\frac{t^2}{2} - \frac{2}{3}t^3 + \frac{t^4}{4}\right]_1^3 = \left(\frac{9}{2} - \frac{54}{3} + \frac{81}{4}\right) - \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)$$

$$= \frac{27}{4} - \frac{1}{12}$$

$$= \frac{20}{3} \quad or \quad 6.667$$

(iii)

$$P(1.2 \le t \le 2.2) = \int_{1.2}^{2.2} \frac{3}{8}(1-t)^2\,dt = \int_{1.2}^{2.2} \frac{3}{8}\left(1-2t+t^2\right)dt$$

$$= \int_{1.2}^{2.2}\left(\frac{3}{8} - \frac{3}{4}t + \frac{3}{8}t^2\right)dt = \left[\frac{3}{8}t - \frac{3}{8}t^2 + \frac{1}{8}t^3\right]_{1.2}^{2.2}$$

$$= \left(\frac{3}{8}\times 2.2 - \frac{3}{8}\times 2.2^2 + \frac{1}{8}\times 2.2^3\right) - \left(\frac{3}{8}\times 1.2 - \frac{3}{8}\times 1.2^2 + \frac{1}{8}\times 1.2^3\right)$$

$$= 0.341 - 0.126$$

$$= 0.215$$

8. The continuous random variable X has expected value $E(X) = 0.6$ and density function

$$f(x) = \begin{cases} ax + bx^2 & for \quad 0 < x < 1 \\ 0, & elsewhere \end{cases}$$

(i) Find the constant a and b.

(ii) Find the $P(x \le 0.4)$.

(iii) Calculate σ_x.

Solution

(i) $\int_0^1 f(x) = 1$

$$\int_0^1 \left(ax + bx^2\right)dx = 1$$

$$\left[\frac{ax^2}{2} + \frac{bx^3}{3}\right]_0^1 = 1$$

$$\frac{a}{2} + \frac{b}{3} = 1 \quad or \quad 3a + 2b = 6.....(i)$$

$$E(x) = \int_0^1 xf(x)\,dx = 0.6$$

$$\int_0^1 x\left(ax + bx^2\right)dx = 0.6$$

$$\int_0^1 \left(ax^2 + bx^3\right)dx = 0.6$$

$$\left[\frac{ax^3}{3} + \frac{bx^4}{4}\right]_0^1 = 0.6$$

$$\frac{a}{3} + \frac{b}{4} = 0.6 \quad or \quad 4a + 3b = 7.2.......(ii)$$

Solving the two equations we get $a = 3.6$ and $b = -2.4$

(ii)

$$P(x \le 0.4) = \int_0^{0.4}\left(3.6x - 2.4x^2\right)dx = \left[1.8x^2 - 0.8x^3\right]_0^{0.4} = 1.8\times 0.4^2 - 0.8\times 0.4^3 = 0.2368$$

(iii)

$$Var(x)=\int_0^1\left(3.6x^3-2.4x^4\right)dx-0.6^2$$
$$=\left[0.9x^3-0.48x^4\right]_0^1-0.36$$
$$=0.9-0.48-0.36$$
$$=0.06$$
$$\therefore \sigma_x=\sqrt{0.06}=0.244948974$$

Exercises

1. The random variable x has the probability density function

$$f(x)=\begin{cases} kx & if \quad 0\le x\le 2 \\ 0, & elsewhere \end{cases}$$

Find k. Find x such that

(i) $\Pr(X\le x)=0.1$

(ii) $\Pr(X\le x)=0.95$

Answer [$k=\frac{1}{2}$, (i) $x=0.632$, (ii) $x=1.949$]

2. A continuous random variable X has a pdf $f(x)$ given by

$$f(x)=\begin{cases} k(3-x)(1+x) & for \quad 0\le x\le 3 \\ 0, & \text{elsewhere} \end{cases}$$

Determine:

(i) Constant k;

(ii) $E(X)$;

(iii) $Var(X)$

Answer [$k=\frac{1}{9}$, $E(X)=1.25$, $Var(X)=\frac{43}{80}$ **]**

3. A random variable X has a pdf $f(x)$ given by

$$f(x)=\begin{cases} \left[\frac{2+\sqrt{3}}{2}\right][2+x]^{-1/2} & for \quad 1\le x\le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Determine mode. **Answer** [1]

4. A random variable X has pdf $f(x)$ given by

$$f(x)=\begin{cases} k(2+x)^{-1/2} & for \quad 1\le x\le 2 \\ 0, & \text{elsewhere} \end{cases}$$

Determine:

(i) Constant k;

(ii) Median of the distribution

Answer [$k = \frac{2+\sqrt{3}}{2}$, $M = \frac{4\sqrt{3}-1}{4}$]

5. Continuous random variable has pdf $f(x)$ given by

$$f(x)=\begin{cases}\frac{3}{64}x^2(4-x) & for \quad 0 \le x \le 4 \\ 0, & \text{elsewhere}\end{cases}$$

Show that the median M is a root of equation $3M^4 - 16M^3 + 128 = 0$.

6. X is a continuous random variable with pdf $f(x)$ given by

$$f(x)=\begin{cases}\frac{x^2}{k}-\frac{kx}{18}+\frac{1}{3} & for \quad 0 \le x \le 3 \\ 0, & elsewhere\end{cases}$$

Where k is a positive constant .

(i) Determine k;

(ii) Determine the mode.

Answer [$k = 6, \quad Z = 3$]

7. Find mean, median, mode for continuous distribution whose pdf $f(x)$ given by

$$f(x)=\begin{cases}0.08(5-x) & for \quad 0 \le x \le 5 \\ 0, & elsewhere\end{cases}$$

Answer [$1\frac{2}{3}$, 1.4645]

8. Determine $E(X)$ for the distribution with pdf $f(x)$ given by

$$f(x)=\begin{cases}\frac{3}{64}x^2(4-x) & for \quad 0 \le x \le 4 \\ 0, & elsewhere\end{cases}$$

Answer [2.4]

9. For a random variable X with pdf

$$f(x)=\begin{cases}\frac{3}{2}x^{\frac{1}{2}} & for \quad 0 \le x \le 1 \\ 0, & elsewhere\end{cases}$$

Determine median M correct to three decimal places.

Answer [0.630]

10. A continuous random variable X has the probability density function $f(x)$ defined by

$$f(x)=\begin{cases}\frac{c}{3}x, & 0\le x\le 3\\ c, & 3\le x\le 4\\ 0, & otherwise\end{cases}$$

Where c is a positive constant.
Determine the;

(i) Value of c;

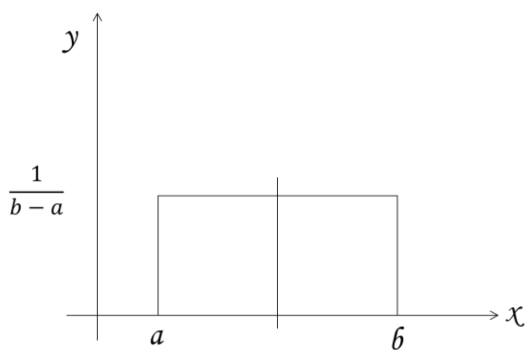
(ii) $E(5x-4)$

Answer [$c=\frac{2}{5},\ \ 9$**]**

CONTINUOUS UNIFORM DISTRIBUTION

A continuous random variable X has a uniform distribution over the interval (a,b) ,with pdf $f(x)$ given by

$$f(x)=\begin{cases}\frac{1}{b-a} & for \quad a<x<b\\ 0, & elsewhere\end{cases}$$



Check: $\int_a^b \frac{1}{b-a}\,dx=1$

$$\int_a^b \frac{1}{b-a}\,dx=\left[\frac{x}{b-a}\right]_a^b=\frac{b}{b-a}-\frac{a}{b-a}=\frac{b-a}{b-a}=1$$

By symmetry $Mean=Median=\frac{1}{2}(a+b)$

$$\mu = \int_a^b x\ f(x)\,dx = \int_a^b \frac{x}{b-a}\,dx = \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \frac{b+a}{2}$$

$$Var(x) = \int_a^b x^2 f(x)\,dx - \mu^2 = \int_a^b \frac{x^2}{b-a}dx - \left(\frac{b+a}{2}\right)^2$$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b - \frac{1}{4}(b+a)^2$$

$$= \frac{b^3}{3(b-a)} - \frac{a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{b^3-a^3}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4}$$

$$= \frac{4(b^2+ab+a^2)-3(b^2+2ab+a^2)}{12}$$

$$= \frac{4b^2+4ab+4a^2-3b^2-6ab-3a^2}{12}$$

$$= \frac{b^2-2ab+a^2}{12}$$

$$\therefore Var(x) = \frac{(b-a)^2}{12}$$

Examples

1. Assume the time of arrival is uniformly distributed on the interval from $12:00$ noon to $12:30$ p.m.
   (i) Find the density and distribution function for T.
   (ii) Find the mean and standard deviation of T .
   (iii) Joseph arrives at the bus stop at precisely noon. Calculate the probability he waits at least 15 minutes for the bus to arrive.

Solution

(i) $$f(t) = \begin{cases} \frac{1}{30} & for \quad 0 < t < 30 \\ 0, & elsewhere \end{cases}$$

(ii) $$\mu = \frac{30+0}{2} = 15 \quad \min utes \quad \text{and } \sigma = \sqrt{\frac{(30-0)^2}{12}} = \sqrt{75} = 8.66025$$

(iii) $P(t > 15) = \int_{15}^{30} \frac{1}{30}\,dt = \left[\frac{t}{30}\right]_{15}^{30} = \frac{30}{30} - \frac{15}{30} = \frac{15}{30} = \frac{1}{2}$

2. Dennis travels regularly by Kenya Airways, when he travels by plane. From past records he found that it is equally likely that his take-off time will be between 80 and 120 minutes after check in at the airport which he departures. Determine the probability that
   (i) He wants more than 105minutes for take-off after checking in.
   (ii) His waiting time for take-off after checking in will be within 1.5 standard deviation of the mean waiting time.

Solution

(i) PDF $f(t) = \frac{1}{40}$

$$P(T > 105) = \int_{105}^{120} \frac{1}{40}\,dt = \left[\frac{t}{40}\right]_{105}^{120} = \frac{120}{40} - \frac{105}{40} = \frac{15}{120} = \frac{3}{8}$$

Or Area of the rectangle $= L \times W = 15 \times \frac{1}{40} = \frac{3}{8}$

(ii)

Or $\mu = \int_{80}^{120} t \times \frac{1}{40}\,dt = \int_{80}^{120} \frac{t}{40}dt = \left[\frac{t^2}{80}\right]_{80}^{120} = \frac{120^2}{80} - \frac{80^2}{80} = \frac{14400 - 6400}{80} = 100$

Or $\mu = \frac{120 + 80}{2} = 100$

$$Var(T) = \int_{80}^{120} t^2 f(t)\,dt - \mu^2 = \int_{80}^{120} \frac{t^2}{40}\,dt - 100^2 = \left[\frac{t^3}{120}\right]_{80}^{120} - 10{,}000$$

$$= \frac{120^3}{120} - \frac{80^3}{120} - 10{,}000$$

$$= \frac{400}{3}\, or 133\frac{1}{3}$$

Standard deviation $= \sqrt{133\frac{1}{3}} = 11.547$ minutes

$P(T\ wthin 1.5\ s\tan dard\ \ deviation\ of\ \ the\ \ mean\ \ waiting\ \ time)$

$$= \int_{100-1.5\times11.547}^{100+1.5\times11.547} \frac{t}{40}\,dt = \left[\frac{t}{80}\right]_{100-1.5\times11.547}^{100+1.5\times11.547} = \frac{117.3205}{80} - \frac{82.6795}{80} = 0.866025$$

NEGATIVE EXPONENTIAL DISTRIBUTION

Has a pdf $f(x)$ given by

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & for \quad 0<x<\infty \\ 0, & elsewhere\end{cases}$$

λ is said to be the parameter of the distribution as $mean=\frac{1}{\lambda}$ and $variance=\frac{1}{\lambda^2}$.

Reverse 'J' shaped graph.

Check: $\int_0^\infty \lambda e^{-\lambda x}dx=\left[\frac{\lambda e^{-\lambda x}}{-\lambda}\right]_0^\infty=\left[-e^{-\lambda x}\right]_0^\lambda=-e^{-\infty}--e^0=0+1=1$

Cumulative Density Function (CDF) $F(x)=P(0<X<x)$

$$\begin{aligned}&=\left[-e^{-\lambda x}\right]_0^x\\&=-e^{-\lambda x}--e^0\\&=1-e^{-\lambda x}\end{aligned}$$

Mean $\mu=E(x)=\int_0^\infty \lambda xe^{-\lambda x}dx$

Applying integration by parts $\int udv=uv-\int vdu$

Let $u=\lambda x$, $\frac{du}{dx}=\lambda$ or $du=\lambda\,dx$

Let $dv=e^{-\lambda x}dx$,then on integrating both sides we get

$$\int dv=\int e^{-\lambda x}dx$$

$$v=-\frac{e^{-\lambda x}}{\lambda}$$

$$\mu = \int_0^\infty \lambda x e^{-\lambda x} dx = \lambda x\left(\frac{-\lambda e^{-\lambda x}}{\lambda}\right) - \int_0^\infty \frac{-e^{-\lambda x}}{\lambda}(\lambda dx)$$

$$= -xe^{-\lambda x} + \int_0^\infty e^{-\lambda x} dx$$

$$= \left[-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}\right]_0^\infty$$

$$= \left(-\infty e^{-\infty} - \frac{e^{-\infty}}{\lambda}\right) - \left(0 - \frac{e^0}{\lambda}\right)$$

$$= (0-0) - \left(-\frac{1}{\lambda}\right)$$

$$\therefore \mu = \frac{1}{\lambda}$$

Variance: $Var(x) = \sigma^2 = \int_0^\infty \lambda x^2 e^{-\lambda x} dx - \left(\frac{1}{\lambda}\right)^2$

Applying integration by parts $\int u dv = uv - \int v du$

Let $u = \lambda x^2$, then $\frac{du}{dx} = 2\lambda x$ or $du = 2\lambda x\, dx$

Let $dv = e^{-\lambda x} dx$,then on integrating both sides we get

$$\int dv = \int e^{-\lambda x} dx$$

$$v = -\frac{e^{-\lambda x}}{\lambda}$$

$$Var(x) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx - \frac{1}{\lambda^2}$$

$$= \lambda x^2\left(\frac{-e^{-\lambda x}}{\lambda}\right) - \int_0^\infty \left(\frac{-e^{-\lambda x}}{\lambda}\right)(2\lambda x\, dx) - \frac{1}{\lambda^2}$$

$$= -x^2 e^{-\lambda x} + \int_0^\infty 2xe^{-\lambda x} dx - \frac{1}{\lambda^2} \qquad \text{..............(1)}$$

Again applying integration by parts for $\int_0^\infty 2xe^{-\lambda x} dx$

Let $u = 2x$, then $\frac{du}{dx} = 2$ or $du = 2dx$

Let $dv = e^{-\lambda x} dx$,then on integrating both sides we get

$$\int dv = \int e^{-\lambda x} dx$$

$$v = -\frac{e^{-\lambda x}}{\lambda}$$

$$\int_0^\infty 2xe^{-\lambda x} dx = 2x\left(\frac{-e^{-\lambda x}}{\lambda}\right) - \int_0^\infty \frac{-e^{-\lambda x}}{\lambda} 2dx$$

$$= \frac{-2xe^{-\lambda x}}{\lambda} + \int_0^\infty \frac{2e^{-\lambda x}}{\lambda} dx$$

$$= \frac{-2xe^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2} \ldots\ldots\ldots\ldots(2)$$

Substituting the RHS of equation (2) into equation (1) gives

$$Var(x) = -x^2 e^{-\lambda x} + \int_0^\infty 2xe^{-\lambda x} dx - \frac{1}{\lambda^2}$$

$$= \left[-x^2 e^{-\lambda x} - \frac{2xe^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}\right]_0^\infty - \frac{1}{\lambda^2}$$

$$= (0-0-0) - \left(0-0-\frac{2}{\lambda^2}\right) - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\therefore \mathrm{Var}(\mathrm{x}) = \frac{1}{\lambda^2}$$