PROBABILITY DISTRIBUTIONS

BINOMIAL DISTRIBUTION

For n trials in each of which there are any two outcomes with constant probabilities q (for failure) and p (for success) respectively, the probability of exactly x successes is given by

$$P(x) = {}^nC_x \, p^x q^{n-x} \quad , x = 0, 1, 2, \ldots, n.$$

$P(x)$ is the coefficient of t^x in PGF

$$G(t) = \sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x} t^x$$

$$= \sum_{x=0}^{n} {}^nC_x \, q^{n-x} p^x t^x$$

$$= \sum_{x=0}^{n} {}^nC_x \, q^{n-x} (pt)^x$$

$$= (q+pt)^n$$

i.e $G(t) = (q+pt)^n$ for binomial distribution.

NB putting $t=1$

$$G(t)=\sum_{x=0}^{n} {}^{n}C_x p^x q^{n-x} t^x$$

$$=\sum_{x=0}^{n} {}^{n}C_x p^x q^{n-x} (1)^x$$

$$=\sum_{x=0}^{n} {}^{n}C_x p^x q^{n-x}$$

$$\therefore G(1)=(q+p)^n$$

Example 1

Use the PGF to show that for binomial distribution

(i) $\mu = E(x) = np$

(ii) $Var(x) = npq$

Solution

(i) $G(t)=(q+pt)^n$

$$G'(t)=n(q+pt)^{n-1}(p)$$

Putting $t=1$

$$G'(1)=n(q+p)^{n-1}p$$

$$=n(1)^{n-1}p$$

$\therefore \mu = E(x) = np$

(ii) $G(t) = (q+pt)^n$

$G'(t) = n(q+pt)^{n-1}(p)$

$G''(t) = n(n-1)(q+pt)^{n-2}(p)(p)$

$= n(n-1)(q+pt)^{n-2}p^2$

Putting $t=1$

$G'(1) = n(q+p)^{n-1}p$

$= n(1)^{n-1}p$

$= np$

$G''(1) = n(n-1)(q+p)^{n-2}p^2$

$= n(n-1)(1)^{n-2}p^2$

$= n(n-1)p^2$

$= n^2p^2 - np^2$

$\therefore Var(x) = G''(1) + G'(1) - [G'(1)]^2$

$= n^2p^2 - np^2 + np - (np)^2$

$= n^2p^2 - np^2 + np - n^2p^2$

$= np - np^2$

$= np(1-p)$

$\therefore Var(x) = npq$ Since $p+q=1$.

Assumptions of Binomial distribution

1. The no. of trials or n is finite and fixed. The performance must be repeated for a fixed number of times.
2. There must be only two possible outcomes of the event which are mutually exclusive and exhaustive. For example when we toss a coin there are only two possible outcomes Head or Tail and one of them must happen.
3. The probability of the happening of the event (or success) in any trial of the event is constant. Similarly the probability of not happening of the event (or failure) should also be constant for all trials.
4. All trials must be independent of each other. The result of any trial should not be affected by the result of a previous or subsequent trials.

Example 1

In the town of Lodwar the probability that an inhibitant catches malaria during any month is 0.2. A firm employs 15 local people.

(i) Determine the mean and variances of the number of employees catching malaria in any month.

(ii) Determine the probability that exactly 3 employees catches malaria during any month.

(iii) Determine the probability that more than six employees catches malaria during any month.

(iv) Determine the probability that at least two employees catches malaria during any month.

(v) Determine the probability that atmost three employees catataches malaria during any month.

Solution

(i) $p = 0.2$, $q = 0.8$, $n = 15$

mean $= \mu = np = 15 \times 0.2 = 3$

Variance $= npq = 15 \times 0.2 \times 0.8 = 2.4$

(ii) $P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, \dots, n$

$P(3) = {}^{15}C_{3} \times 0.2^{3} \times 0.8^{12}$

$= 0.250$

(iii) $P(\text{more than } 6) = P(x > 6) = 1 - P(x \leq 6)$

$= 1 - [P_{(0)} + P_{(1)} + P_{(2)} + P_{(3)} + P_{(4)} + P_{(5)} + P_{(6)}]$

$P_{(0)} = {}^{15}C_{0} \times 0.2^{0} \times 0.8^{15} = 0.0351844$

$P_{(1)} = {}^{15}C_{1} \times 0.2^{1} \times 0.8^{14} = 0.131941$

$P_{(2)} = {}^{15}C_{2} \times 0.2^{2} \times 0.8^{13} = 0.230897$

$P_{(3)} = {}^{15}C_{3} \times 0.2^{3} \times 0.8^{12} = 0.250139$

$P_{(4)} = {}^{15}C_{4} \times 0.2^{4} \times 0.8^{11} = 0.187604$

$P_{(5)} = {}^{15}C_{5} \times 0.2^{5} \times 0.8^{10} = 0.103182$

$P_{(6)} = {}^{15}C_{6} \times 0.2^{6} \times 0.8^{9} = 0.042993$

TOTAL $= 0.981940$

$\therefore$ Required probability $= 1 - 0.981940$

$= 0.01806.$

(iv) $P(\text{at least } 2) = P(x \geq 2) = 1 - P(x \leq 1)$

$= 1 - [P_{(0)} + P_{(1)}]$

$= 1 - 0.0351844 - 0.131941$

$= 0.8328746$

(v) P(atmost 2) = $P(X \le 2) = P_{(0)} + P_{(1)} + P_{(2)}$

$= 0.0351844 + 0.131941 + 0.230897$

$= \underline{0.3980224}$.

Example 2.

The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students

(i) none will graduate.

(ii) one will graduate

(iii) atleast one will graduate.

Solution:

(i) $n = 5,\ p = 0.4,\ q = 0.6$

$P(x) = {}^nC_x\, p^x q^{n-x},\quad x = 0, 1, 2, \ldots, n$

$P(0) = {}^5C_0 \times 0.4^0 \times 0.6^5 = 0.046.$

(ii) $P_{(1)} = {}^5C_1 \times 0.4^1 \times 0.6^4 = 0.2592$

(iii) P(atleast 1) = $P(x \ge 1) = P_{(1)} + P_{(2)} + \cdots + P_{(5)}$

or

$= 1 - P(x < 1)$

$= 1 - P(0)$

$= 1 - 0.46$

$= \underline{0.954}.$

Example 3.

The probability of a man hitting a target is $1/4$. How many times must he fire so that the probability of his hitting the target at least once is greater than $2/3$.

Solution.

$P = 1/4$, $P(x \geq 1) > 2/3$

To find $n = ?$

$P(x \geq 1) = 1 - P(0) > 2/3$

i.e $1 - {}^nC_0 (1/4)^0 (3/4)^n > 2/3$ [Hitting the target is success]

$1 - (0.75)^n > 2/3$

$-(0.75)^n > -1/3$

or

$(0.75)^n < 1/3$

$\log (0.75)^n < \log 1/3$

$n \log 0.75 < \log 1/3$

$n(-0.1249) < -0.47716$

$$n > \frac{0.47716}{0.1249}$$

$n > 3.82$

$\therefore n = \underline{\underline{4}}$

Example 4.

The expected number of non-defective bolts in a box is 8, and the variance is 1.6. Find the probability that there is only one non-defective bolt in the box.

Solution.

mean $= \mu = np$

$np = 8$ ...(i)

$npq = 1.6$ ...(ii)

Substituting $np = 8$ into equation (ii)

$8q = 1.6$

$q = 0.2$, then $p = 0.8$

But $np = 8$

$0.8n = 8 \Rightarrow n = 10$

$P(x) = {}^nC_x p^x q^{n-x}, \quad x = 0, 1, 2, \dots, n$

$P(1) = {}^{10}C_1 \times 0.8^1 \times 0.2^9$

$= 0.000004096$

Example 5:

Eight coins are thrown simultaneously. Show that the probability of obtaining at least 6 heads is 37/256.

Solution

$n = 8, \; p = \frac{1}{2}, \; q = \frac{1}{2}$

$P(x \geq 6) = P(6) + P(7) + P(8)$

$= {}^8C_6 \times (\frac{1}{2})^6 (\frac{1}{2})^2 + {}^8C_7 (\frac{1}{2})^7 (\frac{1}{2})^1 + {}^8C_8 (\frac{1}{2})^8 (\frac{1}{2})^0$

$= \frac{28}{256} + \frac{8}{256} + \frac{1}{256} = \frac{37}{256}$.

Example 6.

Assuming the probability of a male birth as $\frac{1}{2}$, find the probability that a family of 3 children will have

(i) At least one girl

(ii) two boys and one girl

(iii) At most two girls.

Solution

(i) $n = 3,\ p = \frac{1}{2},\ q = \frac{1}{2}$

$$P(x \geq 1) = 1 - P(x < 1) = 1 - P(0)$$

$$= 1 - {}^3C_0 (\tfrac{1}{2})^0 (\tfrac{1}{2})^3$$

$$= 1 - \tfrac{1}{8}$$

$$= \tfrac{7}{8}$$

(ii) P(2 boys and one girl)

$$= {}^3C_2 (\tfrac{1}{2})^2 (\tfrac{1}{2})^1 = \tfrac{3}{8}$$

(iii) P(at most two girls) $= P(x \leq 2)$

$$= P_{(0)} + P_{(1)} + P_{(2)}$$

$$= {}^3C_0 (\tfrac{1}{2})^0 (\tfrac{1}{2})^3 + {}^3C_1 (\tfrac{1}{2})^1 (\tfrac{1}{2})^2 + {}^3C_2 (\tfrac{1}{2})^2 (\tfrac{1}{2})$$

$$= \frac{1}{8} + \frac{3}{8} + \frac{3}{8}$$

$$= \tfrac{7}{8}$$

Example 7:

A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of 11 steps he is one step a way from the starting point.

Solution:

The man will take either 6 steps forward and 5 backwards or 5 steps forward and 6 backward.

Required probability

$$= {}^{11}C_6 \times (0.4)^6 (0.6)^5 + {}^{11}C_5 (0.4)^5 (0.6)^6$$

$$= 0.367873228$$

$$= \underline{\underline{0.3679}}$$

Exercises.

1. Four coins are tossed simultaneously, what is the probability of getting
   (i) 2 heads and 2 tails
   (ii) at least two heads
   (iii) at least one head

   Ans [3/8, 11/16, 15/16]

2. Three percent of a given lot of manufactured parts are defective. What is the probability that in a sample of four items none will be defective?

   Ans [0.885].

3. The incidence of occupational disease in an industry is such that the workmen have a 20% chance of suffering

from it. What is the probability that out of six workmen 4 or more will contract the disease? Ans [0.016].

4. A box contains 100 transistors, 20 of which are defective. 10 are selected for inspection. Indicate what is the probability that

(i) all 10 are defective.
(ii) all 10 are good.
(iii) at least one is defective
(iv) at most 3 are defective.

Ans: $\left(\frac{1}{5}\right)^{10}$, $\left[1-\left(\frac{1}{5}\right)^{10}\right]$, $\left[1-\left(\frac{4}{5}\right)^{10}\right]$, 0.859

5. Assuming that half of the population is vegetarian so that choice of an individual being a vegetarian is $\frac{1}{2}$. Assuming that 100 investigators take sample of 10 individuals each to see whether they are vegetarian. How many investigators would you expect to report that three people or less were vegetarian?

Ans [17]

6. The probability of failure in Physics practical examination is 20%. If 25 batches of 6 students each take the examination, in how many batches 4 or more students would pass?

Ans [23].

7. A binomially distributed random variable X has expected value, $E(x) = 27.52$ and variance, $Var(x) = 15.69$. Find the value of the two parameters n and p, where n is the sample size and p is the probability of succecess.
Ans [$\cdot$ 64, 0.57]

8. Four percent of resistors produced by machine are defective. In a random sample of 10 resistors, determine the probability that
(i) three resistors are defective.
(ii) at least two resistors are defective
(iii) at most three resistors are defective.

[0.00577, 0.05815, 0.9995574].

9. It is generally known that 60% of match boxes from a manufacturing process in a factory have exactly 40 match sticks. Based on this, determine the probability that among 12 randomly selected match boxes from the factory
(i) exactly 4 boxes will have 40 match sticks
(ii) between 2 and 4 boxes inclusive will have 40 match sticks.

(iii) at least 4 boxes will have 40 match sticks.

Ans. [0.04204   0.05699   0.98473]

POISSON DISTRIBUTION

Is a discrete probability distribution and is useful in such cases where the value of p is very small and the value of n is very large

Definition:

Is a limiting form of binomial distribution as n moves towards infinity and p moves towards zero but np or mean remains constant and finite.

Conditions of Poisson distribution:

(i) The variable is discrete.
(ii) The number of trials i.e n should be very large.
(iii) The probability of success i.e p is very small.
(iv) The probability of success in each trial is constant.
(v) np is constant and finite.

Form of Poisson distribution:

(i) The Poisson distribution is a discrete distribution.
(ii) It has a single parameter which is the mean of the distribution and

is denoted by λ.

(iii) The probability of exactly 0, 1, 2, 3, ... success is found by

$$P(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

The PGF of Poisson distribution of RV X G(t) is given by

$$G(t) = E(t^x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x t^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} (\lambda t)^x / x!$$

$$= e^{-\lambda} \left[(\lambda t)^0 / 0! + \frac{(\lambda t)^1}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \frac{(\lambda t)^3}{3!} + \cdots \right]$$

$$= e^{-\lambda} \left[e^{\lambda t} \right]$$

$$= e^{-\lambda + \lambda t}$$

$$= e^{\lambda t - \lambda}$$

$$G(t) = e^{\lambda (t-1)}$$

Differentiating $G(t)$ w.r.t 't'

$G'(t) = \lambda e^{\lambda(t-1)}$

$G''(t) = \lambda^2 e^{\lambda(t-1)}$

Putting $t = 1$

$G'(1) = \lambda e^{\lambda(1-1)}$

$= \lambda e^{\lambda(0)}$

$= \lambda e^{0}$

$= \lambda$

$\therefore E(X) = \mu = G'(1) = \lambda$

Putting $t = 1$ in $G''(t)$

$G''(1) = \lambda^2 e^{\lambda(1-1)}$

$= \lambda^2 e^{\lambda(0)}$

$= \lambda^2 e^{0}$

$= \lambda^2.$

$Var(x) = G''(1) + G'(1) - [G'(1)]^2$

$= \lambda^2 + \lambda - [\lambda]^2$

$= \lambda^2 + \lambda - \lambda^2$

$= \lambda.$

NB: In a Poisson distribution mean is equal to variance.
i.e mean $= \lambda = np$ = variance.