Example 1.

Given a probability distribution of X as below.

x	0	1	2	3
$P(X=x)$	$1/8$	$1/4$	$3/8$	$1/4$

(i) find the mean

(ii) find the standard deviation of X.

Solution

(i) $E(X) = \mu = \sum_{x=0}^{3} x\, p(X=x)$

$= \left(0 \times \frac{1}{8}\right) + \left(1 \times \frac{1}{4}\right) + \left(2 \times \frac{3}{8}\right) + \left(3 \times \frac{1}{4}\right)$

$= 0 + \frac{1}{4} + \frac{3}{4} + \frac{3}{4}$

$= 1\frac{3}{4}$ or 1.75

(ii) $Var(x) = E(x^2) - \mu^2$

$= \sum_{x=0}^{3} x^2 p(X=x) - 1.75^2$

$= \left(0^2 \times \frac{1}{8}\right) + \left(1^2 \times \frac{1}{4}\right) + \left(2^2 \times \frac{3}{8}\right) + \left(3^2 \times \frac{1}{4}\right) - 1.75^2$

$= 0 + \frac{1}{4} + \frac{3}{2} + \frac{9}{4} - 1.75^2$

$= 4 - 1.75^2$

$\sigma = \sqrt{var(x)} = \sqrt{0.9375} = 0.968246.$

Example 2.

The probability distribution of a r.v X is as shown below

x	0	1	2
$P(X=x)$	$1/6$	$1/2$	$1/3$

(a) (i) find the mean

(ii) find the standard deviation

(b) When $y = 12x + 6$, hence find

(i) the mean of y

(ii) the standard deviation.

Solution.

(a) (i) $E(X) = \mu = \sum_{x=0}^{2} x p(X=x)$

$= (0 \times \frac{1}{6}) + (1 \times \frac{1}{2}) + (2 \times \frac{1}{3})$

$= 0 + \frac{1}{2} + 2/3$

$= 7/6$

(ii) $Var(x) = \sum_{x=0}^{2} x^2 p(X=x) - \mu^2$

$= (0^2 \times \frac{1}{6}) + (1^2 \times \frac{1}{2}) + (2^2 \times \frac{1}{3}) - (7/6)^2$

$= 0 + \frac{1}{2} + 4/3 - (7/6)^2$

$= 17/36$

$\sigma = \sqrt{Var(x)} = \sqrt{17/36} = 0.6872.$

(b) (i) $E(Y) = 12E(X) + 6$

$= 12\left(7/6\right) + 6$

$= 14 + 6$

$= 20$

(ii) $Var(Y) = Var(12X + 6)$

$= 12^2 \, Var(X)$

$= 144 \times \frac{17}{36}$

$= 68$

Example3.

Let X be a r.v with $P(X=1) = 0.2$, $P(X=2) = 0.3$, and $P(X=3) = 0.5$. What is the expected value and standard deviation of

(i) X

(ii) $Y = 5X - 10$

Solution

(i) $E(X) = \mu = \sum_{x=1}^{3} x\,p(X=x)$

$= (1\times 0.2) + (2\times 0.3) + (3\times 0.5)$

$= 0.2 + 0.6 + 1.5$

$= 2.3$

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \sum_{x=1}^{3} x^2 p(X=x) - \mu^2$$
$$= (1^2 \times 0.2) + (2^2 \times 0.3) + (3^2 \times 0.5) - 2.3^2$$
$$= 0.2 + 1.2 + 4.5 - 5.29$$
$$= 0.61$$

(ii) $E(Y) = E(5X - 10)$
$$= 5E(X) - 10$$
$$= \cancel{5(2.3) - 10}$$
$$= 5(2.3) - 10$$
$$= 1.5$$

$$Var(Y) = Var(5X - 10)$$
$$= 5^2 Var(X)$$
$$= 25(0.61)$$
$$= 15.25$$

$$\therefore \sigma = \sqrt{Var(Y)} = \sqrt{15.25} = 3.90512$$

Exercises.

1. Suppose X has a probability mass function given by

x	2	3	4	5	6
$P(X=x)$	0.01	0.25	0.4	0.3	0.04

Find mean and variance of X.

2. Suppose X has a probability mass function given by the table below.

x	11	12	13	14	15
$P(X=x)$	0.4	0.2	0.2	0.1	0.1

Find the mean and variance of X.

3. A random variable W has the probability distribution shown below.

w	0	1	2	3
$P(W=w)$	$2d$	0.3	d	0.1

Find the values of the constant d, hence find the mean and variance of W. Also find the mean and variance of $Y = 10X + 25$

4. A random variable X has the probability distribution shown below.

x	1	2	3	4	5
$P(X=x)$	$7c$	$5c$	$4c$	$3c$	c

Find the values of the constant c hence determine the mean and variance of X.

5. The random variable Z has the probability distribution shown below.

z	2	3	5	7	11
$P(Z=z)$	$1/6$	$1/3$	$1/4$	x	y

If $E(Z) = 4\frac{2}{3}$, find the values of x and y hence determine the variance of Z.

6. A discrete random variable M has the probability distribution

$$f(m) = \begin{cases} \frac{m}{36}, & m = 1, 2, 3, \ldots, 8 \\ 0, & \text{elsewhere.} \end{cases}$$

find the mean and variance of M.

7. For a discrete random variable Y the probability distribution is

$$f(y) = \begin{cases} \frac{5-y}{10}, & y = 1, 2, 3, 4 \\ 0, & \text{elsewhere.} \end{cases}$$

Calculate $E(Y)$ and $Var(Y)$.

8. Suppose X has a pmf given by

$$f(x) = \begin{cases} kx & \text{for } x = 1, 2, 3, 4 \\ 0, & \text{elsewhere} \end{cases}$$

find the value of the constant k, hence obtain the mean and variance of X.

Answers

Q1. $E(x) = \mu = 4.11$, $Var(x) = 0.7379$

Q2. $E(x) = 12.3$, $Var(x) = 1.81$

Q3. $d = 0.2$, $E(W) = \mu = 1$, $Var(x) = 1$

$E(Y) = 35$, $Var(Y) = 100$.

Q4. $c = 1/20$, $E(x) = 2.3$, $Var(x) = 4.05$

Q5. $x = \frac{1}{6}$, $y = 1/12$, $Var(z) = 6\frac{7}{18}$

Q6. $E(m) = 1$, $var(m) = 4\frac{2}{3}$

Q8. $K = 1/10$, $E(x) = 3$, $Var(x) = 1$

Q7. $E(Y) = 2$, $Var(Y) = 1$

- The function $p(x)$ is known as the probability mass function (pmf) for the RV of X.
- The function $\sum_{\text{all } x} P(X=x) = \sum_{\text{all}} p(x)$ is known as the cummulative Distribution Function (CDF) for RV X.
- Consider a pmf below.

x	1	2	3	4
$P(x)$	0.3	0.5	0.1	0.1

- The $E(X) = \mu = \sum_{x=1}^{4} x\, P(X=x)$

$$= (1\times 0.3) + (2\times 0.5) + (3\times 0.1) + (4\times 0.1)$$

$$= 0.3 + 1.0 + 0.3 + 0.4$$

$$= 2.0$$

- For any discrete random variable X with pmf there is associated Probability Generating Function (PGF).
- The PGF, denoted by $G(t)$ and defined by

$$G(t) = E(t^x) = \sum_{\text{all } x} t^x p(x)$$

Example 1:

For a discrete random variable X given below. Use PGF to find the mean of X.

x	1	2	3	4
$P(X=x)$	0.3	0.5	0.1	0.1

Solution

$$G(t) = \sum_{x=1}^{4} t^x p(x)$$

$$= (t^1 \times 0.3) + (t^2 \times 0.5) + (t^3 \times 0.1) + (t^4 \times 0.1)$$

$$G(t) = 0.3t + 0.5t^2 + 0.1t^3 + 0.1t^4$$

On differentiating

$$G'(t) = 0.3 + t + 0.3t^2 + 0.4t^3$$

$$G'(1) = 0.3 + 1 + 0.3 + 0.4$$

$$= 2.0$$

NB:

1. $G(t) = \sum_{\text{all } x} t^x P(x)$

2. $G(0) = \sum_{\text{all } x} 0^x P(x) = 0$

3. $G(1) = \sum_{\text{all } x} (1)^x p(x) = \sum_{\text{all}} P(x) = 1$

In general if

$$G(t) = \sum_{\text{all } x} t^{x} p(x)$$

$$G'(t) = \sum_{\text{all } x} x t^{x-1} P(x)$$

when $t = 1$ or putting $t = 1$

$$G'(1) = \sum_{\text{all } x} x (1)^{x-1} P(x)$$

$$= \sum_{\text{all } x} x(1) P(x)$$

$$= \sum_{\text{all } x} x P(x)$$

$$= E(x) \text{ or } \mu$$

<u>Computing Variance of a discrete random variable using PGF.</u>

$$G(t) = \sum_{\text{all } x} t^{x} p(x)$$

$$G'(t) = \sum_{\text{all } x} x t^{x-1} P(x)$$

$$G''(t) = \sum_{\text{all } x} x(x-1)\,t^{x-2}P(x)$$

$$= \sum_{\text{all } x} x^2 t^{x-2} P(x) - \sum_{\text{all } x} x t^{x-2} P(x)$$

Putting $t = 1$

$$G''(1) = \sum_{\text{all } x} x^2 (1)^{x-2} P(x) - \sum_{\text{all } x} x\, 1^{x-2} P(x)$$

$$= \sum_{\text{all } x} x^2 p(x) - \sum_{\text{all } x} x\, P(x)$$

$$G''(1) = E(x^2) - E(x)$$

Making $E(x^2)$ the subject

$$E(x^2) = G''(1) + E(x)$$

$$= G''(1) + G'(1) \quad \text{since } E(x) = G'(x)$$

But $Var(x) = E(x^2) - [E(x)]^2$

$$\therefore Var(x) = G''(1) + G'(1) - [G'(1)]^2$$

Example 2.

Use PGF to find the standard deviation for the following discrete random variable X given below:

x	1	2	3	4
$P(X=x)$	0.3	0.5	0.1	0.1

Solution

$$G(t) = E(t^x) = \sum_{x=1}^{4} t^x P(X=x)$$

$$= (t^1 \times 0.3) + (t^2 \times 0.5) + (t^3 \times 0.1) + (t^4 \times 0.1)$$

$$= 0.3t + 0.5t^2 + 0.1t^3 + 0.1t^4$$

$$G'(t) = 0.3 + t + 0.3t^2 + 0.4t^3$$

$$G''(t) = 0 + 1 + 0.6t + 1.2t^2$$

Putting $t = 1$

$$G'(1) = 0.3 + 1 + 0.3(1)^2 + 0.4(1)^3$$

$$= 0.3 + 1 + 0.3 + 0.4$$

$$= 2.0$$

$$G''(1) = 1 + 0.6(1) + 1.2(1)^2$$

$$= 1 + 0.6 + 1.2$$

$$= 2.8$$

$$Var(x) = G''(1) + G'(1) - [G'(1)]^2$$
$$= 2.8 + 2.0 - (2.0)^2$$
$$= 4.8 - 4.0$$
$$= 0.8$$

Standard deviation $= \sqrt{var(x)}$
$$= \sqrt{0.8}$$
$$= 0.894427191$$
$$\simeq \underline{\underline{0.8944}}$$

Example 3.

Use PGF to find the mean and standard deviation of a r.v X shown below

x	0	1	2
$p(X=x)$	$1/6$	$1/2$	$1/3$

Solution

$$G(t) = E(t^x) = \sum_{x=0}^{2} t^x p(X=x)$$
$$= \left(t^0 \times \frac{1}{6}\right) + \left(t^1 \times \frac{1}{2}\right) + \left(t^2 \times \frac{1}{3}\right)$$
$$G(t) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$$
$$G'(t) = 0 + \frac{1}{2} + \frac{2}{3}t$$
$$G''(t) = \frac{2}{3}$$

Putting $t = 1$

$$G'(1) = E(X) = \mu = \frac{1}{2} + \frac{2}{3} = \frac{7}{6}$$

$$Var(X) = G''(1) + G'(1) - [G'(1)]^2$$

$$= \frac{2}{3} + \frac{7}{6} - \left(\frac{7}{6}\right)^2$$

$$= \frac{17}{36}$$

$$\sigma = \sqrt{Var(X)} = \sqrt{\frac{17}{36}} = 0.68718427$$

Example 4.

Suppose X has a pmf given by

$$f(x) = \begin{cases} kx, & \text{for } x = 1, 2, 3, 4 \\ 0, & \text{elsewhere} \end{cases}$$

find the value of the constant k, hence using PGF obtain the mean and variance.

Solution

$$\sum_{x=1}^{4} p(x) = 1$$

x	1	2	3	4
$f(x)$	k	2k	3k	4k

$$k + 2k + 3k + 4k = 1$$

$$10k = 1$$

$$k = \frac{1}{10}$$

Next

x	1	2	3	4
$f(x)$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

$$G(t) = E(t^x) = \sum_{x=1}^{4} t^x f(x)$$

$$= \left(t^1 \times \frac{1}{10}\right) + \left(t^2 \times \frac{2}{10}\right) + \left(t^3 \times \frac{3}{10}\right)$$

$$+\left(t^4 \times \frac{4}{10}\right)$$

$$G(t) = \frac{t}{10} + \frac{t^2}{5} + \frac{3}{10}t^3 + \frac{2t^4}{5}$$

$$G'(t) = \frac{1}{10} + \frac{2}{5}t + \frac{9t^2}{10} + \frac{8t^3}{5}$$

$$G''(t) = \frac{2}{5} + \frac{9}{5}t + \frac{24}{5}t^2$$

Putting $t = 1$

$$G'(1) = E(x) = \mu = \frac{1}{10} + \frac{2}{5}(1) + \frac{9}{10}(1^2) + \frac{8}{5}(1)^3$$

$$= \frac{1}{10} + \frac{2}{5} + \frac{9}{10} + \frac{8}{5}$$

$$\therefore E(x) = 3$$

$$G''(1) = \frac{2}{5} + \frac{9}{5}(1) + \frac{24}{5}(1)^2$$

$$= 7$$

$$Var(x) = G''(1) + G'(1) - [G'(1)]^2$$

$$= 7 + 3 - 3^2$$

$$= 7 + 3 - 9$$

$$= 1$$

Exercises

Do questions 1, 2, 4, 6, 7 using a PGF from the previous exercises.