DISCRETE PROBABILITY DISTRIBUTIONS.

Introduction:

In application of probability, we are often interested in a number associated with the outcome of a random experiment. Such a quantity whose value is determined by the outcome of a random experiment is called a random variable.

A discrete random variable is a function whose range is finite and / or countable i.e it can only assume values in a finite or countably infinite set of values.

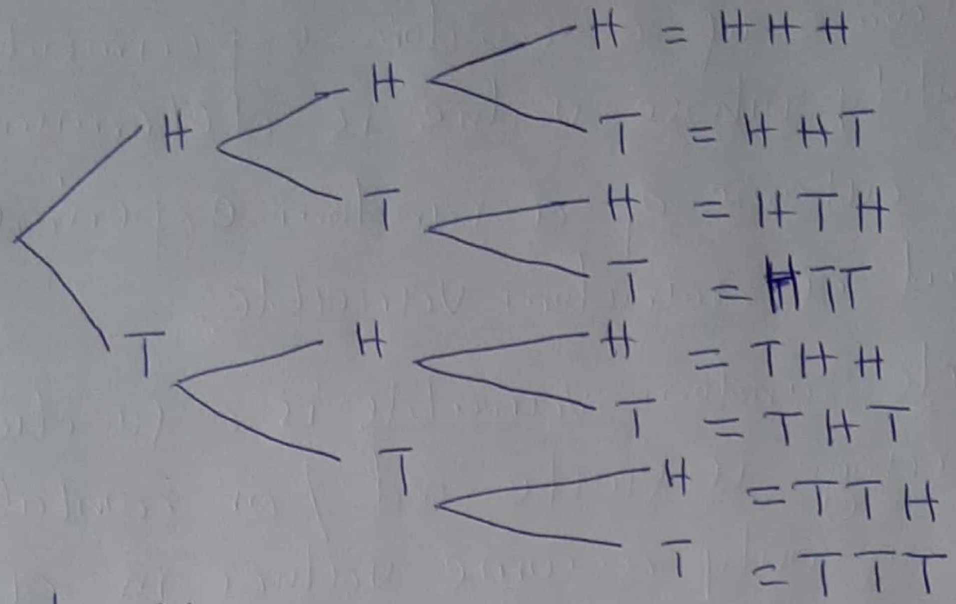
A continuous random variable is one that can take any value in an interval of real numbers. (There are unaccountably many real numbers in an interval of positive length.)

Discrete Random Variables

A random variable X is said to be discrete if it can take on only a finite or countable number of possible values x.

Consider the experiment of flipping a fair coin three times. The number of tails that appear is noted as a discrete random variable. X = "number of tails

that appear in 3 flips of a fair coin".
There are 8 possible outcomes of the experiment.



Namely the sample space consists of

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$X = \{0, 1, 1, 2, 1, 2, 2, 3\}$

are the corresponding values taken by the random variable X.

Now, what are the possible values that X takes on and what are the probabilities of X taking a particular value?
From the above we see that the possible values of X are the 4 values

$X = \{0, 1, 2, 3\}$

i.e the sample space is a disjoint union of the 4 events $\{X = j\}$ for $j = 0, 1, 2, 3$.

Specifically in our example:

$\{X = 0\} = \{HHH\}$,

$\{X = 1\} = \{HHT, HTH, THH\}$

$\{X = 2\} = \{TTH, HTT, THT\}$

$\{X = 3\} = \{TTT\}$

Since for a fair coin we assume that each element of the sample space is equally likely (with probability $\frac{1}{8}$), we find that the probabilities for the various values of X, called the probability distribution of X or the probability mass function (pmf). These can be summarized in the following table listing the possible values beside the probability of that value

x	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$3/8$	$3/8$	$\frac{1}{8}$

<u>Note</u>: The probability that X takes on the value x, i.e $P(X=x)$, is defined as the sum of the probabilities of all points in S that are assigned the value x.

- We can say that this pmf places mass $3/8$ on the value $X=2$.
- The "masses" (or probabilities) for a pmf should be between 0 and 1.
- The total mass (i.e total probability) must add up to 1.

<u>Definition</u>:

The <u>probability mass function</u> of a discrete variable is a table, formula or graph that specifies the proportion (or probabilities) associated with each possible value the random variable can take.

The mass function $P(X=x)$ (or simply $p(x)$) has the following properties:

(i) $0 \leq p(x) \leq 1$

(ii) $\sum_{\text{all } x} p(x) = 1$

More generally, let X have the following properties.

(i) It is a discrete variable that can only assume values $x_1, x_2, \ldots, x_n$.

(ii) The probabilities associated with these values $P(X=x_1)=P_1$, $P(X=x_2)=P_2, \ldots$ $P(X=x_n)=P_n$.

Then X is a discrete random variable if $0 \leq P_i \leq 1$ and $\sum_{i=1}^{n} P_i = 1$.

Remark:

We denote random variables with capital letters while realized or particular values are denoted by lower case letters.

Example 1.

Two tetrahedral dice are rolled together once and the sum of the scores facing down was noted. Find the pmf of the random variable "the sum of the scores facing down.

Solution

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

$X = \{1, 2, 3, 4, 5, 6, 7, 8\}$

Therefore the pmf is given by the table below

x	2	3	4	5	6	7	8
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

This can also be written as a function

$$P(X=x) = \begin{cases} \frac{x-1}{16} & \text{for } x = 2, 3, 4, 5 \\ \frac{9-x}{16}, & \text{for } x = 6, 7, 8 \end{cases}$$

Example 2.

The pmf of a discrete random variable W is given by the table below

w	-3	-2	-1	0	1
$P(W=w)$	0.1	0.25	0.3	0.15	d.

(i) Find the value of the constant d.

(ii) ~~Fi~~ Find $P(-3 \leq W < 0)$

(iii) ~~F~~ Find $P(W > -1)$

(iv) Find $P(-1 < W < 1)$.

Solution

(i) $\sum_{\text{all } x} P(W=w) = 1$

$0.1 + 0.25 + 0.3 + 0.15 + d = 1$

$0.8 + d = 1$

$d = 0.2$

(ii) $P(-3 \le W < 0) = P(W=-3) + P(W=-2) + P(W=-1)$

$= 0.1 + 0.25 + 0.3$

$= 0.65$

(iii) $P(W > -1) = P(W=0) + P(W=1)$

$= 0.15 + 0.2$

$= 0.35$

(iv) $P(-1 < W < 1) = P(W=0) = 0.15$

Example3:

A discrete random variable Y has a pmf given by the table below.

y	0	1	2	3	4
$P(Y=y)$	c	$2c$	$5c$	$10c$	$17c$

(i) Find the value of the constant c.

(ii) Hence compute $P(1 \le Y < 3)$.

Solution

(i) $\sum_{\text{all } y} P(Y=y) = 1$

$$c + 2c + 5c + 10c + 17c = 1$$

$$35c = 1$$

$$c = \frac{1}{35}$$

(ii) $P(1 \leq Y < 3) = P(Y=1) + P(Y=2)$

$$= \frac{2}{35} + \frac{5}{35}$$

$$= \frac{7}{35}$$

$$= 1/5$$

Exercises

1. A die is loaded such that the probability of a face showing up is proportional to the face number. Determine the probability of each sample point.
2. Roll a fair die and let X be the square of the score that show up. Write down the probability distribution of X. Hence compute $P(X < 15)$ and $P(3 \leq X < 30)$.
3. Let X be the random variable the number of fours observed when two dice are rolled together once. Show that X is a discrete random variable.

4. The pmf of a discrete random variable X is given by $P(X=x) = kx$ for $x = 1,2,3,4,5,6$.

(i) Find the value of the constant k.

(ii) Find $P(X<4)$.

(iii) Find $P(3 \leq X < 6)$.

5. A fair coin is flip until a head appears. Let N represent the number of tosses required to realize a head.

(i) Find the pmf of N.

(ii) Find $P(N<2)$

(iii) Find $P(N \geq 2)$.

6. A discrete random variable Y has a pmf given by $P(Y=y) = c\left(\frac{3}{4}\right)^y$ for $y = 0,1,2,\ldots$.

(i) Find the value of constant c.

(ii) Find $P(X<3)$.

(iii) Find $P(X \geq 3)$.

7. Verify that $f(y) = \dfrac{2y}{k(k+1)}$ for $y = 0,1,2,\ldots,k$

can serve as a pmf of a random variable X.

8. For each of the following determine c So that the function can serve as a

(i) $f(x) = c$ for $x = 1, 2, 3, 4, 5$

(ii) $f(x) = cx$ for $x = 1, 2, 3, 4, 5$

(iii) $f(x) = cx^2$ for $x = 0, 1, 2, \ldots, k$

(iv) $f(x) = \frac{c}{2}$ for $x = -1, 0, 1, 2$

(v) $f(x) = \frac{x-2}{c}$ for $x = 1, 2, 3, 4, 5$

(vi) $f(x) = \frac{x^2 - x + 1}{c}$ for $x = 1, 2, 3, 4, 5$

(vii) $f(x) = c(x^2 + 1)$ for $x = 0, 1, 2, 3$

(viii) $f(x) = cx({}^3C_x)$ for $x = 1, 2, 3$

(ix) $f(x) = c\left(\frac{1}{6}\right)^x$ for $x = 0, 1, 2, \ldots$

(x) $f(x) = c2^{-x}$ for $x = 0, 1, 2, \ldots$

9. A coin is loaded so that heads is three times likely as the tails.

(i) For 3 independent tosses of the coin find the pmf of the total number of heads realized and the probability of realizing at most 2 heads.

(ii) A game is played such that you earn 2 points for a head and loss 5 points for a tail. Write down the probability distribution of the total scores after 4 independent tosses of the coin.

10. For an online electronics retailer,
X = "the number of Zony digital cameras returned per day"
follows the distribution given by

x	0	1	2	3	4	5
$P(X=x)$	0.05	0.1	t	0.2	0.25	0.1

Find the value of t and $P(X>3)$.

Expectation and Variances of a Random Variable

One of the most important things we would like to know about a random variable is: What value does it take on average? What is the average price of a computer? What is the average value of a number that rolls on a die? The value found as the average of all possible values, weighted by how often they occur i.e probability

Definition:
Let X be a discrete r.v with probability $p(x)$. Then the expected value of X, denoted $E(X)$ or μ, is given by $E(X)=\mu=\sum_{x=-\infty}^{\infty} x\,p(X=x)$

Theorem: Let X be a discrete r.v with probability $p(X=x)$ and let $g(x)$ be a real valued function of X ie $g: X \to x$, then the expected value of $g(x)$ is given by

$$E[g(x)] = \sum_{x=-\infty}^{\infty} g(x)\, p(X=x)$$

Theorem: Let X be a discrete r.v with probability function $p(x)$. Then

(i) $E(c) = c$, where c is any real constant;

(ii) $E[ax+b] = a\mu + b$ where a and b are constants.

(iii) $E[k\, g(x)] = k\, E[g(x)]$ where $g(x)$ is a real valued function of X.

(iv) $E[a g_1(x) \pm b g_2(x)] = aE[g_1(x)] \pm bE[g_2(x)]$

and in general

$$E\left[\sum_{i=1}^{n} c_i g_i(x)\right] = \sum_{i=1}^{n} c_i E[g_i(x)]$$

where $g_{i's}(x)$ are real-valued functions of X. This property of expectation is called Linearity property.

Proof

(i) $E(c) = \sum_{\text{all } x} c\, p(X=x) = c \sum_{\text{all } x} p(X=x) = c(1) = c$

(ii) $E[ax+b] = \sum_{\text{all } x} (ax+b)\, p(x)$

$= \sum_{\text{all } x} ax\, p(x) + \sum_{\text{all } x} b\, p(x)$

$= a \sum_{\text{all } x} x\, p(x) + b \sum_{\text{all } x} p(x)$

$= a\mu + b(1)$

$= a\mu + b$

(iii) $E[k g(x)] = \sum_{\text{all } x} k\, g(x)\, P(X=x)$

$= k \sum_{\text{all } x} g(x)\, P(X=x)$

$= k\, E[g(x)]$

(iv) $E[a g_1(x) \pm b g_2(x)] = \sum_{\text{all } x} a g_1(x)\, P(X=x)$

$\pm \sum_{\text{all } x} b g_2(x)\, P(X=x)$

$= a \sum_{\text{all } x} g_1(x)\, P(X=x) \pm b \sum_{\text{all } x} g_2(x)\, P(X=x)$

$= a\, E[g_1(x)] \pm b\, E[g_2(x)]$

Variance and standard Deviation

Definition: Let X be a r.v with mean $E(X) = \mu$, the variance of X, denoted σ^2 or $Var(X)$, is given by

$$Var(x) = \sigma^2 = E[X-\mu]^2.$$

The units for variance are square units. The quantity that has the correct units is standard deviation, denoted σ. It is actually the positive square root of $Var(X)$.

$$\sigma = \sqrt{var(x)} = \sqrt{E(x-\mu)^2}$$

Theorem: $Var(X) = E(X-\mu)^2 = E(x^2) - [E(x)]^2$
$$= E(x^2) - \mu^2$$

Proof:

$$Var(X) = E(X-\mu)^2 = E(x^2 - 2X\mu + \mu^2)$$
$$= E(x^2) - 2\mu E(X) + \mu^2$$
$$= E(x^2) - 2\mu\mu + \mu^2$$
$$= E(x^2) - 2\mu^2 + \mu^2$$
$$= E(x^2) - \mu^2$$

Theorem: $Var(aX+b) = a^2 var(x)$

Proof:

Recall that $E[ax+b] = a\mu + b$

Therefore

$$\begin{aligned}Var(aX+b) &= E[(aX+b)-(a\mu+b)]^2 \\ &= E[aX+b-a\mu-b]^2 \\ &= E[aX-a\mu]^2 \\ &= E[a(X-\mu)]^2 \\ &= E[a^2(X-\mu)^2] \\ &= a^2E(X-\mu)^2 \\ &= a^2 Var(X).\end{aligned}$$

Remarks:

(i) The expected value of X always lies between the smallest and largest values of X.

(ii) In computations, bear in mind that variance cannot be negative.