Credit Manager - Credit Value At Risk (C-VaR) Model Validation

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Abstract

Credit Manager provides a comprehensive framework to calculate Credit Value at risk which is the value at risk on a portfolio of fixed income instruments due to credit migrations from issuers. Their model is based on credit migration analysis which is the probability of transitioning from a credit quality to another in a specified time horizon, say 1 year. Interest rates are assumed to be determinists, this means no interest rate simulation is carried out and the interest rates observed today are the one used to reprice the portfolio at the risk horizon. Credit Manager provides users with available market data ranging from transition matrices from rating agencies to government spreads and Libor spreads. The idea behind their methodology is quite simple from a birds' eye perspective and this paper explains the methodology employed to replicate and validate the model. We find the replication to be very straightforward and the model produces consistent results. The methodology also is mathematically correct.

Brief Methodology Summary

The Methodology used by Credit Manager for the Monte Carlo credit value at risk simulation is pretty straightforward. They use equity returns as a proxy for the firm's assets returns. Second, they ask that the user defines regression statistics as input to their model. Therefore, the user has to run a regression (OLS more often than not) of equity returns against a set of MSCI indices that will offer both economic sense and be quantitatively optimal as per time series analysis criterions. Once regressions are run, the user collects the R^2 of the regressions as well as all the weights on the MSCI indices as input to the model. Credit Manager takes the reins from there and computes adjusted weights as well as implied idiosyncratic weights – the weights on firm specific noises – in order to ease the simulation process to only simulate standard normal variables. The simulation consists of simulating normally distributed variables for all MSCI indices as well as for idiosyncratic returns and to draw uniform numbers for recovery rates. At this point, we can calculate every issuer's return at the risk horizon based on the weights and simulated MSCI returns. This return is then mapped to a rating ranging from "Aaa" to "Default" – depending on the rating system chosen - for each issuer under each scenario. The portfolio can then be priced at the risk horizon for all scenarios using the spread and base curves provided by Credit Manager for all ratings which have been validated in a previous study[1]. In case of default, the user has to specify mean and variance for recovery rates. Those will be used as input to fit a Beta Distribution and simulate a recovery value when an issuer defaults. The user can also decide to

[1] See Credit Manager Spread Validation Report, V:\Risk\Credit Risk\Analysts\Frederick Ete 2016\SpreadValidation\Spread Validation Report.docx

use mean a variance of recovery rates provided by Credit Manager based on studies[2]. The VaR at risk horizon is then the mean value of the portfolio at the risk horizon minus the nth percentile portfolio price at risk horizon.

Replicating the model: a step by step guide

The thinking behind the next manipulations is simply meant to express the weights the user obtained from regressions against MSCI indices as a linear combination of weights on standard normal correlated MSCI indices and weights on standard normal uncorrelated firm specific factors i.e.: idiosyncratic returns.

The first step consists of applying transformation to the weights on MSCI indices coming from regressions conducted by the user. Let W be the initial m by n weight matrix where m is the number of obligors/issuers and n is the total number of MSCI indices considered. Thus, W should look something like this, along with the computed R^2 statistics:

		Relevant Market Index					
		MSCI CA Consumer Discr Sector	MSCI DE Germany	MSCI Europe Industrials Sector	MSCI US Automobiles & Components	MSCI US Energy	MSCI US USA
Obligor Name	R^2	Relative Weights on MSCI Index					
Obligor_01	0.30	0	0	100	0	0	0
Obligor_02	0.35	0	100	0	0	0	0
Obligor_03	0.25	0	0	0	70	0	30
Obligor_04	0.40	100	0	0	0	0	0
Obligor_05	0.36	0	0	0	0	100	0
Obligor_06	0.42	0	0	0	0	0	100

Since R^2 is defined as the portion of obligor's variance explained by the MSCI market factors, we can compute the implied idiosyncratic weights for each obligor. We have[3]:

$$R_i^2 = \frac{\tilde{W_i}^T \cdot C \cdot \tilde{W}_i}{\tilde{W_i}^T \cdot C \cdot \tilde{W}_i + \tilde{w}_{i,0}^2} = \frac{\tilde{W_i}^T \cdot C \cdot \tilde{W}_i}{\sigma_{Z_i}^2}$$

Where W_i is a column/row vector from the weight matrix W, and C is the covariance matrix of the MSCI market indices. It follows[4] that:

$$\widetilde{w}_{i,0} = \sqrt{\left(\frac{1}{R^2} - 1\right) B}$$

[2] **Schuermann, Till** - What Do We Know About Loss Given Default?, 2004, p. 16-17

[3] See proof 1.2 in appendix

[4] See *proof 1.3* in appendix.

where:

$$B = \widetilde{W}_i^T \cdot C \cdot \widetilde{W}_i$$

It becomes clear that we can extract implied idiosyncratic weights on synthetic factors for each obligor. What we are ultimately looking to do here is transform the weight matrix provided by the user in order to be able to express these weights as weights on synthetic independent factors with a N(0,1) distribution for simulation purposes. The benefits of this approach are significant. In order to achieve our goal, we still need to go through some more steps. We need to normalize obligor's equity returns; this can be achieved by using central limit theorem i.e.: taking the original returns, subtracting the mean and then dividing by the standard deviation, this yields a series that follows and standard normal distribution.

More rigorously, we need to subtract the mean first:

$$\tilde{Z}_i - \bar{Z}_i = \tilde{w}_{i,1} \cdot (\tilde{S}_1 - \bar{r}_i) + \cdots + \tilde{w}_{i,M} \cdot (\tilde{S}_M - \bar{r}_M) + \tilde{w}_{i,0} \cdot \varepsilon_i$$

Then, utilizing the equation we had for R^2, we have that:

$$\sigma_{\bar{Z}_i} = \sqrt{\frac{\tilde{W}_i^T \cdot C \cdot \tilde{W}_i}{R_i^2}}$$

Putting it all together we therefore obtain:

$$Z_i = \frac{\tilde{Z}_i - \bar{Z}_i}{\sigma_{\bar{Z}_i}} = \frac{\tilde{w}_{i,1}}{\sigma_{\bar{Z}_i}} \cdot \sigma_{\bar{r}_1} \frac{(\tilde{S}_1 - \bar{r}_1)}{\sigma_{\bar{r}_1}} + \cdots + \frac{\tilde{w}_{i,M}}{\sigma_{\bar{Z}_i}} \cdot \sigma_{\bar{r}_M} \frac{(\tilde{S}_M - \bar{r}_M)}{\sigma_{\bar{r}_M}} + \frac{\tilde{w}_{i,0}}{\sigma_{\bar{Z}_i}} \cdot \varepsilon_i$$

If we define the new weights as:

$$\hat{w}_{i,j} = \frac{\tilde{w}_{i,j}}{\sigma_{\bar{Z}_i}} \sigma_{\bar{r}_j}$$

and the weight on idiosyncratic factor as:

$$\hat{w}_{i,0} = \frac{\tilde{w}_{i,0}}{\sigma_{\bar{Z}_i}},$$

we can rewrite the previous equation for obligor's return where the market factors are normalized but still correlated as:

$$Z_i = \hat{w}_{i,1} \cdot \hat{S}_1 + \cdots + \hat{w}_{i,M} \cdot \hat{S}_M + \hat{w}_{i,0} \cdot \varepsilon_i$$

At this stage we simply need to simulate correlated normal variables with mean 0 and variance 1. This implies that we must not use the usual Cholesky decomposition of the covariance matrix (Σ) as this yields a random matrix ~N (0, Σ)[5]. Instead, we must use the Cholesky decomposition of the correlation (C) matrix which will yield a random matrix S ~ N (0, C), implying variance of 1 since for standard normal variables, Σ = C.[6] More rigorously here are the steps to generate a correlated normal matrix with mean 0 and variance 1 and correlation matrix C.

1. Obtain L matrix of Cholesky factorization such that L is lower triangular and $C = LL'$
2. Simulate an m by n matrix Z of uncorrelated standard normal variables, where m is the number of synthetic factors and n is the number of simulation scenarios.
3. The correlated scenarios matrix S is obtained by: $S = LZ \sim N(0,C)$
4. We therefore obtain the simulated returns R for all obligors by multiplying the weight matrix W by the simulated synthetic factors returns: $R = WS$

Another alternative to the Cholesky factorization is the singular value decomposition of the correlation matrix. Singular value decomposition of a symmetric matrix finds matrices U, S *and* V such that $C = USV'$ where S is a diagonal matrix containing eigenvalues of C and $U = V$ contain eigenvectors of C on their columns. Let's detail the similarities between this approach and the Cholesky factorization approach in order to help determine what the transformation to a matrix $Z \sim N(0,I)$ must be.

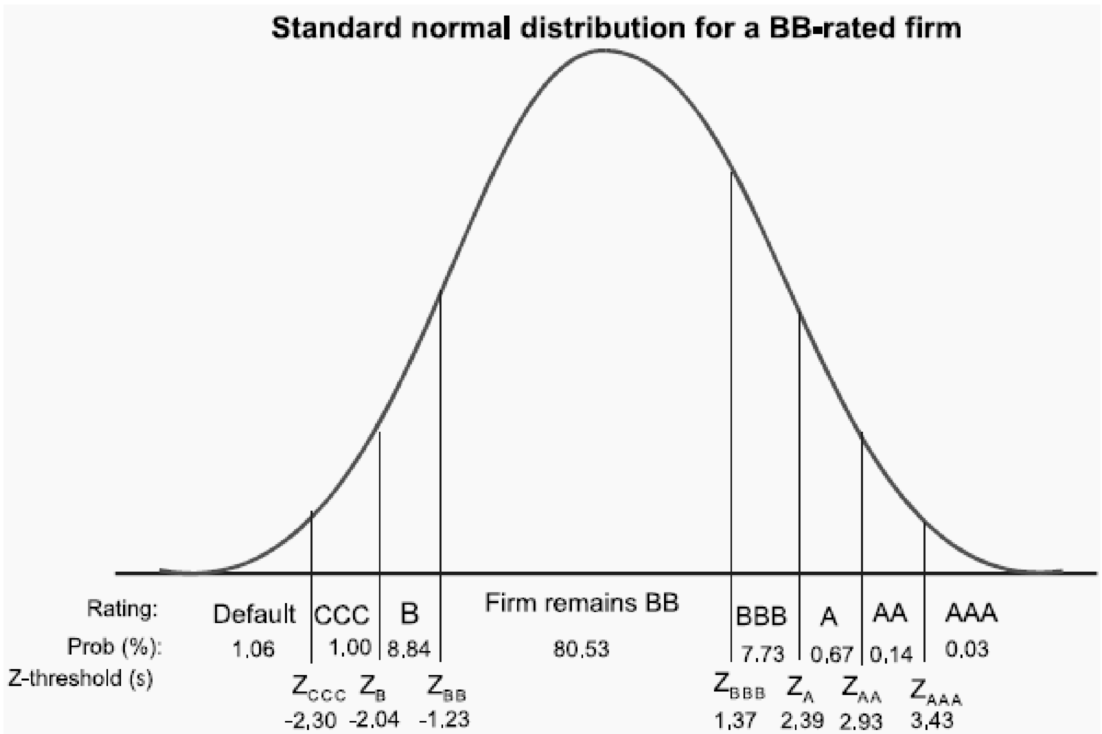
1. In the Cholesky factorization approach, we found L such that $C= LL'$. Here we found U and S such that $C = USU'$. Let's observe that since S is diagonal, it follows that $S = S'$.
2. Define $B=US^{1/2}$. We therefore have $C = BB'$, implying that the transformation to Z is simply $S = BZ \sim N(0,C)$[7]
3. Similar to the Cholesky approach, we therefore obtain the simulated returns R for all obligors by multiplying the weight matrix W by the simulated synthetic factors returns: $R = WS$

Once the simulation for the synthetic market factors is conducted and the returns for obligors have been calculated, we can simply simulate a m by n matrix of uncorrelated standard normal variables that will represent the return on idiosyncratic factors for each obligor, where m is the number of obligors in the portfolio and n is the number of simulations. We then multiply the idiosyncratic weight $\boldsymbol{w_{i,0}}$ for each obligor obtained earlier in this note by the simulated returns and add that to the already calculated returns implied by synthetic factors. Finally, we end the simulation process by drawing a n by m matrix of random uniform: U(0,1) numbers that simulate recovery rates in case of default for each obligor on every scenario.

Finally the rest of the procedure is pretty straightforward. Based on the transition matrix, we are able to compute a threshold matrix. In fact, the threshold matrix is obtained by slicing a normal distribution. The idea behind this method is that by obtaining z-scores for each rating based on the transition matrix, we can directly map an obligor's simulated return to a rating at the risk horizon by computing the z-score implied by this return and comparing it with the thresholds or z-scores for different ratings at horizon. In practice, the implementation of the threshold matrix is simple. If we take a specific initial rating and retrieve the row associated with it in the transition matrix, we obtain transition probabilities $p_{horizonRating}$ for this particular initial rating. At this point, for each horizon ratings, we want to find a z-score, such that a return beyond that z-score will yield a lower rating at horizon.

[5] **Glasserman, Paul** – Monte Carlo Methods for Financial Engineering, 2003, p.82
[6] See *proof* 1.1 in appendix
[7] For a more rigorous explanation of this result, see **Glasserman, Paul** – Monte Carlo Methods for Financial Engineering, 2003, p.75

If we take for example B as our initial rating and BB as our horizon rating, we have p_{BB}, the probability of moving from B to BB at horizon. Z_{BB} is such that $P(X < Z_{BB}) = p_{AAA} + p_{AA} + p_A + p_{BBB} + p_{BB}$ and, similarly, Z_{BBB} is such that $P(X < Z_{BBB}) = p_{AAA} + p_{AA} + p_A + p_{BBB}$ therefore implying that $P(Z_{BB} < X < Z_{BBB}) = p_{BB}$. The figure below represents the normal slicing we described here in an easy to understand fashion[8]. Of course, one needs to start its calculation at either end to obtain Z_{AAA} and/or $Z_{Default}$ first.



We are therefore able to translate all simulated returns into actual ratings at the risk horizon; this allows us to price the portfolio at the risk horizon for every scenario using the spreads associated with each credit rating. At this point we are almost done; we simply need to determine what our valuation of defaulted obligors' bond will be. The approach used by Credit Manager might be a little bit simplistic. In fact, they seem to ask the user for input as for the mean recovery rate and standard deviation of recovery rates. Even though Credit Manager offers parameters from research papers for all class of debts, the validity of these two parameters is crucial and any user of the model can obtain flawed results if incorrect or biased parameters are provided. Credit Manager then uses these two parameters - mean and variance of recovery rates – and fits a beta distribution[9] and maps the values in the simulated U(0,1) number for a specific defaulted obligor and computes a recovery value for the bond at the risk horizon. This is the heart of the C-VaR calculation process and we feel that even though it is backed by research papers, the parameters might not always accurately reflect the reality of the market. Therefore, the estimation of the mean and variance of recovery rates might need further validation. Nonetheless, we obtain a price for all defaulted securities and are able tu compute the value of the portfolio at risk horizon for every simulated scenario.

[8] **CROUHY, Michel** – A comparative analysis of current credit risk models, Journal of Banking and Finance, 2000, p.75

[9] See appendix for information on beat distributions

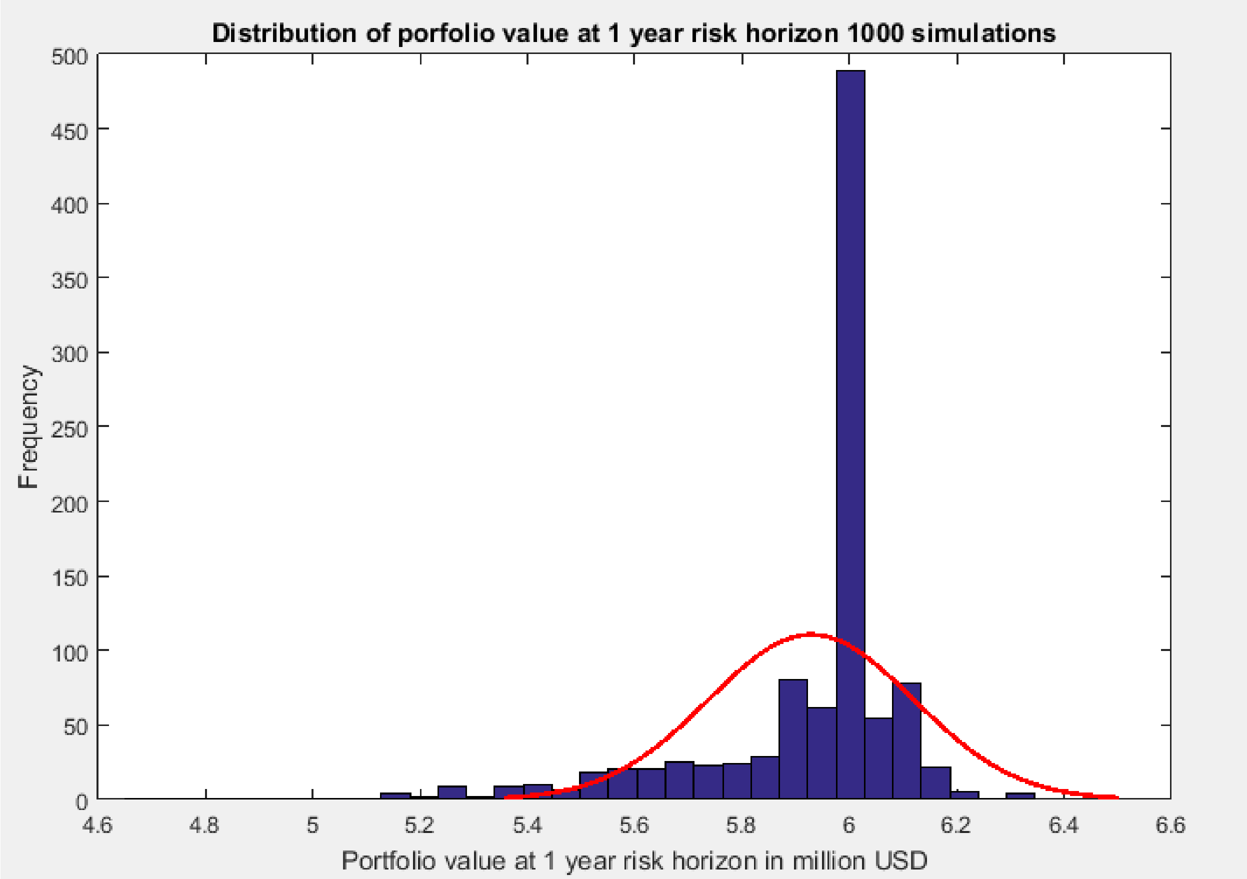
Once the portfolio is priced, we compute the mean horizon value (*mhv*) by taking the mean of all portfolio values at the horizon. We then rank all portfolio values in order and retrieve the n^{th} percentile value (*nV*). The Value At Risk due to Credit is then simply :

$$\$CVaR = mhv - nV$$

Results

We tested and replicated the algorithm for CVaR calculation using MATLAB and expose our results here. We will happily provide our dataset and sample portfolio as well as our code to anyone who asks. Simply send an email to **Credit_Risk_Group@investpsp.ca**. The slight difference of 1.5% or less we obtained is most likely due to recovery rate simulation as Credit Manager does not let the user set the seed for this particular draw from U(0,1) – we used Matlab's "default" seed.

	CVaR Simulation Results 1 000 000 simulations same seeds				
Confidence interval	Credit Manager VaR	PSP - Simulated VaR (SVD)	PSP - Simulated VaR (Chol)	Difference SVD	Difference Chol
95.00%	$ 486,962	$ 484,536	$ 483,783	0.50%	0.65%
97.00%	$ 580,016	$ 578,241	$ 577,941	0.31%	0.36%
99.00%	$ 749,933	$ 746,490	$ 745,649	0.46%	0.57%
99.75%	$ 1,053,028	$ 1,042,262	$ 1,040,161	1.02%	1.22%
99.99%	$ 1,605,342	$ 1,584,234	$ 1,583,069	1.31%	1.39%



Post replication review

In this brief validation, we were able to replicate the results of the Credit-VaR for a dummy portfolio of 6 straight bonds. In true institutional environments, asset managers will never own only straight bonds and this is why this validation has to be taken with a grain of salt and cannot confirm that the model is 100% reliable for all fixed income products. Even though the idea remains the same, the difficulty comes in accurately pricing securities at the risk horizon, which for some products, might require more advanced simulation techniques. Also, Credit Manager offers a choice between importance sampling and straight monte carlo simulation in order to generate more scenarios in the tail ends of the distributions giving accurate simulation results with less scenarios. Unfortunately, their importance sampling methodology has not been validated here. We feel like the results we obtained are strong evidence that the model is robust and mathematically correct, although there might be a degree of uncertainty regarding the estimation of mean and variance of recovery rates. Other models offered by different providers differ quite a bit in their methodology.

Appendix

Proof 1.1:

From the definition of correlation for a single variable we have:

$$\rho_{i,j} = \frac{Cov[i,j]}{\sqrt{Var[i]} \cdot \sqrt{Var[j]}}$$

In the case of multivariate standard normal variables we have:

$$Var[i] = Var[j] = 1$$

Therefore:

$$\rho_{i,j} = Cov[i,j] \ \blacksquare$$

Proof 1.2:

This definition of R^2 is correct since we express - following the regressions - each obligor's returns as:

$$r_i = w_{1,i}S_{1,i} + w_{2,i}S_{2,i} + \cdots + w_{n,i}S_{n,i} + w_0\epsilon_i$$

Where r_i is the return on obligor's equity at time i, $w_{j,i}$ is the weight on the j^{th} index at time i, w_o is the implied weight on ε_i; a normally distributed idiosyncratic factor for that particular obligor. Therefore, the variance of r_i can be expressed as:

$$Var[r_i] = w_{1,i}^2 Var\left[S_{1,i}\right] + \cdots + w_{n,i}^2 Var\left[S_{n,i}\right] + 2 \cdot w_{(1,i)} w_{2,i} Cov\left[S_{(1,i)S(2,i)}\right] + \ \ldots + w_0^2 Var[\epsilon_i]$$

Since we normalize ϵ_i to be normal with mean 0 and variance 1 we find:

$$Var[r_i] = w_{1,i}^2 Var\left[S_{1,i}\right] + \cdots + w_{n,i}^2 Var\left[S_{n,i}\right] + 2 \cdot w_{1,i} w_{2,i} \cdot Cov\left[S_{1,i} \cdot S_{2,i}\right] + \ \ldots + w_0^2$$

Which, in matrix form, is equivalent to:

$$Var[r] = \widetilde{W^T} \cdot C \ \cdot \widetilde{W} + \widetilde{w_0^2} \ \blacksquare$$

Proof 1.3:

We have:

$$R_i^2 = \frac{\tilde{W_i}^T \cdot C \cdot \tilde{W_i}}{\tilde{W_i}^T \cdot C \cdot \tilde{W_i} + \tilde{w}_{i,0}^2} = \frac{\tilde{W_i}^T \cdot C \cdot \tilde{W_i}}{\sigma_{\tilde{Z}_i}^2}$$

Where W_i is the i^{th} column of the weight matrix, C is the covariance matrix and $w_{i,0}^2$ is the variance on a standard normal idiosyncratic factor for the i^{th} obligor.

Define:

$$B = \widetilde{W_\iota}^T \cdot C \cdot \widetilde{W_\iota}$$

Thus, we have:

$$R^2 = \frac{B}{B + \widetilde{w_{\iota,0}^2}}$$

We find:

$$B + \widetilde{w_{\iota,0}^2} = \frac{B}{R^2}$$

We therefore have:

$$\widetilde{w_{\iota,0}} = \sqrt{\left(\frac{1}{R^2} - 1\right) \cdot B} \ \blacksquare$$

<u>The Beta Distribution:</u>

The Beta distribution is a very flexible distribution entirely determined by two (2) parameters: α and β where:

$$\mu = \frac{1}{1+\frac{\beta}{\alpha}}$$

and

$$\sigma^2 = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

It is therefore completely determined with the mean and variance provided by the user as we can isolate α and β with basic manipulations, we leave this up to the reader.