Monotonicity for the Frog Model with Drift on Trees

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Polymath Jr. (arXiv:2309.14443)

Abstract

The frog model $\mathrm{FM}(d,p)$ with drift p starts with an active particle at the root of the infinite d-ary tree and dormant particles at each non-root site. In discrete time, active particles move towards the root with probability p, and otherwise move to a uniformly sampled child vertex. When an active particle moves to a site containing dormant particles, all the particles at the site become active. The critical drift p_d is the infimum over all p for which infinitely many particles visit the root almost surely (that is, the system is *recurrent*). We have improved bounds on $\sup_{d\geq m} p_d$ and proved monotonicity of critical values associated to a self-similar variant.

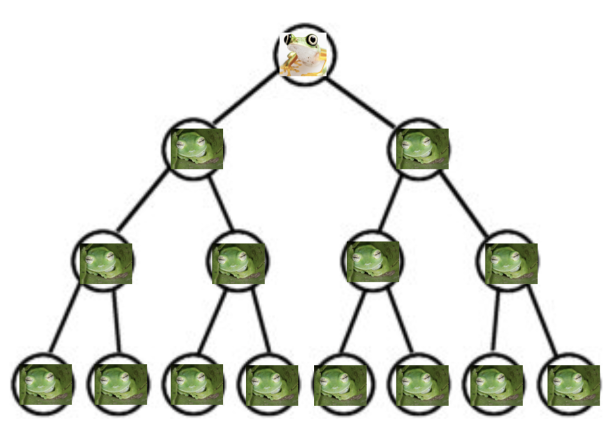

Fig. 1: frog model on the 2-ary tree

Background

The frog model can be viewed as a model of combustion, rumor spread, infection, etc. The behavior of frogs on the lattice $\mathbb{Z}^d$ has been well-studied, but their behavior on trees remains a difficult question.

Things we knew previously:

- $p_d \geq \frac{1}{d+1}$ for all d
- $p_1 = \frac{1}{2} = 0.5$
- $p_2 = \frac{1}{3} \approx 0.33$ [HJJ17]
- $p_3 \in [\frac{1}{4}, \frac{5}{17}] \approx [0.25, 0.294]$ [BJL23]
- $p_4 \in [\frac{1}{5}, \frac{27}{100}] = [0.2, 0.27]$ [BJL23]

From the above one might be tempted to conjecture that $p_d = \frac{1}{d+1}$ for all d. So this next result may be surprising:

- $p_d \geq \frac{2-\sqrt{2}}{4} \approx 0.1464$ for all d. [BFJ+19]

Things we do not know (conjectures):

1. Is p_d strictly decreasing in d?
2. Does $\lim_{d\to\infty} p_d$ equal $\frac{2-\sqrt{2}}{4}$?
3. Does a larger d or p always correspond to more root visits V_{FM}?

Improvement on the upper bounds

Define $S_m = \sup_{d\geq m} p_d$, then it satisfies the following:

m	2	3	4	5	6	7	8	9	10	11	12	13
$S_m \leq$	$\frac{55}{159}$	$\frac{42}{145}$	$\frac{40}{153}$	$\frac{23}{94}$	$\frac{46}{197}$	$\frac{23}{102}$	$\frac{38}{173}$	$\frac{20}{93}$	$\frac{15}{71}$	$\frac{5}{24}$	$\frac{7}{34}$	$\frac{11}{54}$.
$\approx$	.346	.290	.261	.245	.234	.225	.220	.215	.211	.208	.206	.204

Establishing these bounds relies heavily on a modified *self-similar frog model* $\mathrm{SFM}(d,p)$. It has several key properties:

- Every vertex now has independent Poi(1) dormant frogs initially
- No backtracking is allowed
- Only one active frog is ever allowed per subtree
- Frogs die upon hitting the root.

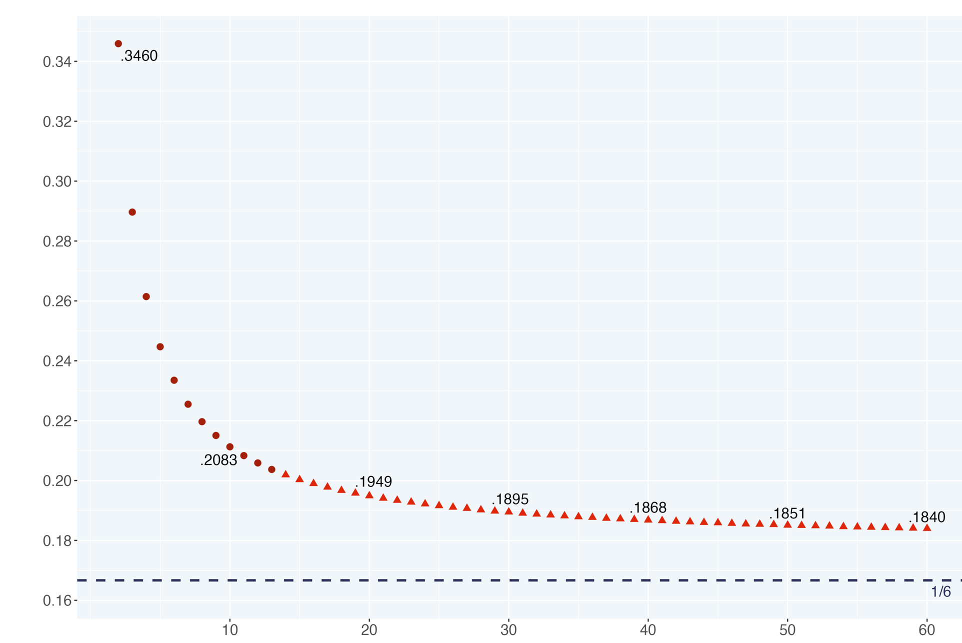

Fig. 2: bounds on S_m for $2 \leq m \leq 60$

The self-similar structure and the Poi(1)-per-vertex setup simplify the analysis significantly. In particular, if p makes $\mathrm{SFM}(d,p)$ recurrent, then it makes $\mathrm{FM}(d,p)$ recurrent.

We have shown that the number of root visits in SFM stochastically increases as d, the degree of the tree, increases.

Methods for establishing bounds

The jump probabilities in $\mathrm{SFM}(d,p)$ now depend on two quantities:

$$p_d^* = \frac{p(d-1)}{d-(d+1)p} \quad \text{and} \quad \hat{p} = \frac{p}{1-p}.$$

The *frog star* introduced in [BJL23] can be viewed as a single self-similar unit of SFM. Let U denote the number of vertices among $v_2, \ldots, v_d$ that are visited in the end.

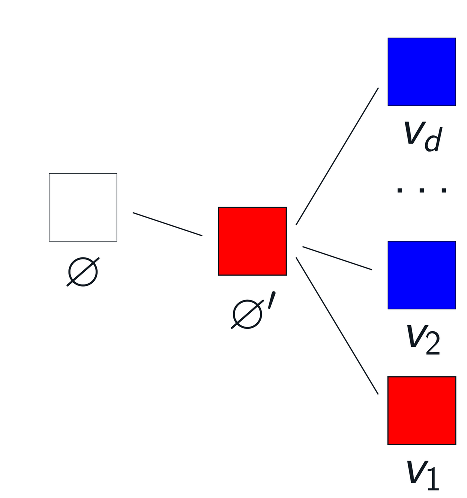

Fig. 3: frog-star model

For a given pair (d,p), define

$$f^{d,p}(\lambda) = e^{-p_d^*} \sum_{u=0}^{d-1} e^{(1-\hat{p}(1+u))\lambda} \, \mathrm{P}(U=u).$$

It can be shown that a sufficient condition for SFM to be recurrent is if

$$\sup_{\lambda\geq 0} f^{d,p}(\lambda) < 1. \tag{1}$$

Hence if we find a probability p that satisfies the above, then $p \geq p_d$.

What is the distribution of U? In [BJL23] the authors were able to compute its distribution by hand for $d = 2,3,4$. In our research we found an inductive way to compute the distribution of $U(d,p,\lambda)$ for higher d's (summarized using Figure 4).

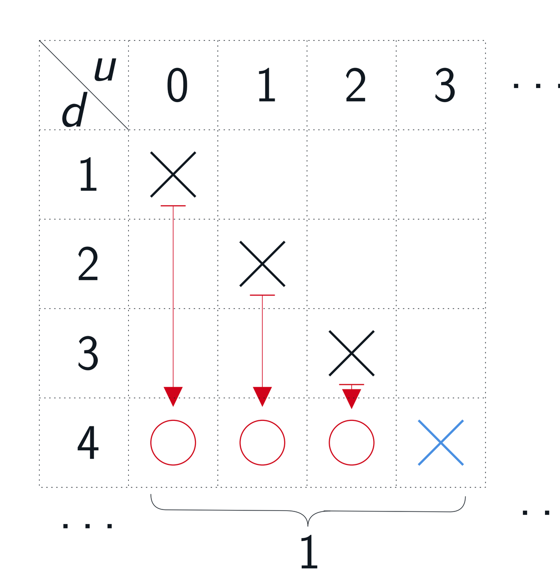

Fig. 4: induction scheme

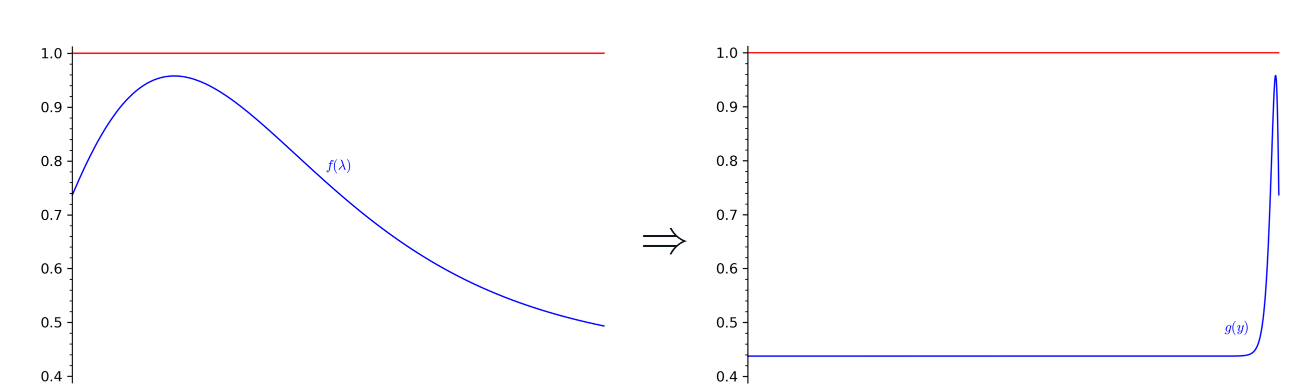

Fig. 5: change of variables when $d = 4$ and $p = 0.27$

Under a simple change of variables (see Figure 5), our $f(\lambda)$ becomes a polynomial $g(y)$ on $(0,1]$. This reduces our question to finding the unique maximum of the polynomial $g(y)$ and showing that it is less than 1, which is not hard to implement by computers. We get rigorous bounds on p_m for $2 \leq m \leq 13$.

The coupling argument

A coupling argument tells us that $\mathrm{SFM}(d,p)$ has stochastically more root visits as d increases. Therefore the bound on p_m we computed makes $\mathrm{SFM}(d,p)$, and hence $\mathrm{FM}(d,p)$, recurrent for all $d \geq m$.

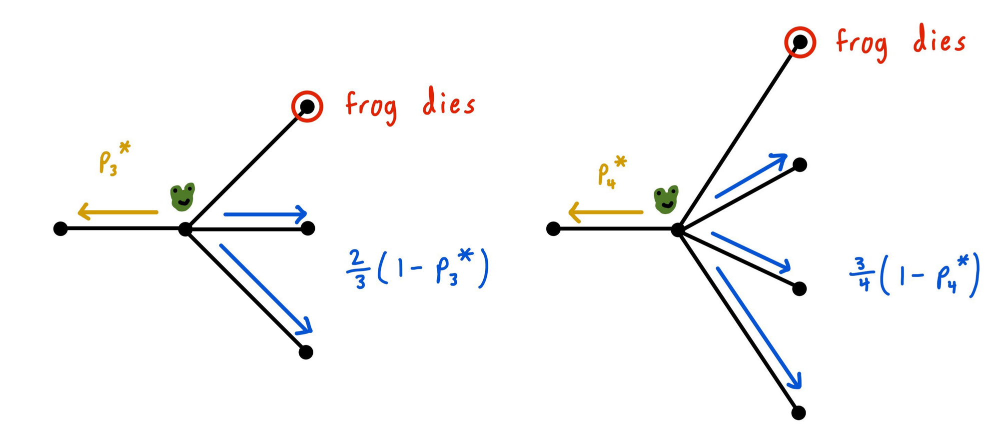

Fig. 6: the coupling argument

We remark that $V_{\mathrm{SFM}(d,p)} \preceq V_{\mathrm{SFM}(d+1,p)}$ supports conjecture 3.

A follow-up result

We define

$$q_d = \inf\{p : \sup_{\lambda\geq 0} f^{d,p}(\lambda) < 1\},$$

the minimal probability that satisfies equation (1). Using a basic analysis argument, we have shown that this q_d is strictly decreasing in d. This provides evidence towards conjecture 1.

Acknowledgements

This research was done under the guidance of Professor Matthew Junge (CUNY Baruch) and Professor Si Tang (Lehigh) during the summer 2023 Polymath Jr. Research Program.

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