Def. $f: (a,b) \to \mathbb{R}$.

f is differentiable at $c \in (a,b)$. if

$\exists \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \Big/ \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$

$f'(c)$

Since $f'(x)$ is a limit, it is a local property.

* c has to be an interior pt. of the domain:

$\because$ $f(c)$ and every $f(x)$ on $0<|x-c|<\delta$ should be defined to characterize derivative.

yet for limit, it only requires "c" to be a limit point s.t. $0<|x-c|<\delta$ cannot be trivally true for some $\delta>0$.

* $|x|' = x/|x|$. w/ $x \neq 0$.

For cont., no requirement as long as $f(c)$ is defined (recall cont. at an isolated pt.)

Differentiable $\Rightarrow$ continuous.

$\lim_{x\to c} f(x) = f(c)$

$\lim_{x\to c}\left(\frac{f(x)-f(c)}{x-c}\right) \lim_{x\to c}(x-c) + f(c) = f'(c)\cdot 0 + f(c)$

Thm. $f: (a,b) \to \mathbb{R}$ then f is differentiable at $c \in (a,b)$.

iff $\exists$ a constant $A \in \mathbb{R}$ and a function $r: (a-c, b-c) \to \mathbb{R}$

s.t. $f(c+h) = f(c) + Ah + r(h)$.

$\lim_{h\to 0} \frac{r(h)}{h} = 0$.

$(f\cdot g)'(c) = f'(c)g(c) + f(c)g'(c)$

$\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c}$

$= \lim_{x\to c} \frac{f(x)g(x) - f(c)g(x)}{x-c} + \lim_{x\to c} \frac{f(c)g(x) - f(c)g(c)}{x-c}$.

$(f/g)' = \frac{f'g - fg'}{g^2}$ (given $g(c) \neq 0$)

$$\lim_{x\to c} \frac{(f/g)(x) - (f/g)(c)}{x-c}$$

$$= \lim_{x\to c} \frac{f(x)g(c) - f(c)g(x)}{x-c} \cdot \frac{1}{g(x)g(c)}$$

$$= \lim_{x\to c} \frac{1}{g(x)g(c)} \left[\frac{f(x)-f(c)}{x-c} g(x) + f(c) \frac{g(c)-g(x)}{x-c} \right]$$

$$= \frac{1}{g^2(c)} [f'g - fg'].$$

Carathéodory Thm.

A function f is differentiable at $c \in \mathrm{dom}(f)$.

$\Leftrightarrow$ $\exists$ h on $\mathrm{dom}(f)$ that is cont. @ c and satisfies.

$$f(x) - f(c) = h(x)(x-c).$$

in this case, $h(c) = f'(c)$.

Prf. $\Leftarrow$: $h(x) = \dfrac{f(x)-f(c)}{x-c}$ when $x \neq c$. ($x=c$ is trivially true).

$$\lim_{x\to c} h(x) = h(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

$$\exists\ f'(c) = h(c)$$

$\Rightarrow$: Define $h(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & x \neq c. \\ f'(c) & x = c \end{cases}$

$\Rightarrow$ h is cont. @ c.

Chain Rule: Let $f: A \to \mathbb{R}$ and $g: B \to \mathbb{R}$ where $A \subseteq \mathbb{R}$ and $f(A) \subseteq B$.

Suppose c is an interior point of A, and $f(c)$ is an interior point of B.

If f is diff. @ c, and g is diff. @ $f(c)$.

then $(g \circ f)$ is diff. @ c w/

$(g \circ f)'(c) = g'(f(c)) f'(c)$.

$$(g \circ f)'(c) = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x - c}$$

By C Thm. $g(y) - g(f(c)) = h(y)(y - f(c))$.

$h(f(c)) = g'(f(c))$.

$$\lim_{x \to c} \frac{h(f(x))(f(x) - f(c))}{x - c}$$

$$= g'(f(c)) \cdot f'(c).$$

Fermat's Theorem. If $f: A \to B$ has a local extreme value at $c \in A$ (max/min of an open interval).

f is diff. @ $c \Rightarrow f'(c) = 0$.

$$\left.\begin{array}{ll} \leq 0 & \text{if } 0 < h < \delta \\ \geq 0 & \text{if } -\delta < h < 0 \end{array}\right\} \Rightarrow \; = 0.$$

Check for endpoints (for intervals) / boundary pts (for closed sets).

$$\text{Critical pts} \begin{cases} \text{interior pts} & \text{where } f \text{ is not diff.} \\ \quad \cdots & \text{where } f'(c) = 0 \text{ (stationary pts)} \end{cases}$$

Rolle's Thm. $f: [a,b] \to \mathbb{R}$

cont. on $[a,b]$
diff. on (a,b) w/ $f(a) = f(b)$.

$\exists c \in (a,b) \quad f'(c) = 0$

By EVT, and then consider the case at endpoints and not at endpoints
$\downarrow$ const. $\downarrow$ Fermat.

Lagrange MVT cont. diff.

$\exists c \in (a,b)$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

or $f(b) - f(a) = f'(c)(b-a)$.

Consider $g(x) = f(x) - \underbrace{L(x)}$.

$$= f(a) + \frac{f(b) - f(a)}{b-a}(x-a).$$

We are finding the pt. c s.t. $f'(c) = L'(c)$

the slope of AB.

Cauchy's MVT, cont. diff.

$\exists c \in (a,b)$ s.t. $f'(c)(g(b) - g(a))$
$= g'(c)(f(b) - f(a))$.

Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$.

monotonicity f is cont. on $[a,b]$ and diff. on (a,b).

(1) $f'(x) = 0 \quad \forall x \in (a,b) \Leftrightarrow f$ is const. $\quad \forall c \in (a,b)$

(2) $f'(x) \geq 0 \quad \forall x \in (a,b) \Leftrightarrow f$ is $\nearrow$. $\quad \Leftarrow$: $f(x) - f(c) \geq 0$ when $x - c > 0$
≤ 0 < 0.

(3) ≤ 0 $\Leftrightarrow f$ is $\searrow$.

(4) >0 $\Rightarrow$ strictly $\nearrow$.

(5) <0 $\Rightarrow$ strictly $\searrow$

$\Leftarrow$:

if $x>c \Rightarrow f(x)>f(c)$.

$$\lim \frac{f(x)-f(c)}{x-c} \geq 0 \text{ by the OLT.}$$

$\Rightarrow$: $\underbrace{f(x_1)-f(x_2)} = f'(c)\underbrace{(x_1-x_2)}$

$=0$

>0 $\quad$ >0

≥ 0. $\quad$ ≥ 0

From (1) we know $f'(x)=0$, f is const.

then Cor. f, g be diff. on (a,b) w/ $f'(x)=g'(x)$ $\forall x \in (a,b)$.

$$f(x)=g(x)+C \quad \forall x \in (a,b).$$

$$h(x)=f(x)-g(x).$$

then $h'(x)=0 \Rightarrow h$ is const.

Darboux's Thm (IVP for deri).

if f is diff on $[a,b]$ w/ α between $f'(a)$ and $f'(b)$

then $\exists c \in (a,b)$ w/ $f'(c)=\alpha$.

$\Updownarrow$

$f(c)=(\alpha x)'|_c$

Consider $f(x)-\alpha x = g(x)$. $\Rightarrow f'(x)-\alpha = g'(x)$

WLOG, $g'(a)<0$ $g'(b)>0$, Since g is cont. on a compact set $[a,b]$,

it has its global min / max at $c \in (a,b)$

because it cannot appear at endpoints by the def of deri.

Taylor's Thm w/ Lagrange Remainder.

$f: (a,b) \longrightarrow \mathbb{R}$ has $(n+1)$ derivatives on (a,b), and $c \in (a,b)$

$\forall x \in (a,b) \quad \exists \xi$ between c & x

s.t. $f(x) = P_n(x) + R_n(x)$

where $P_n(x) = f(c) + f'(c)(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$

and $R_n(x) = f^{(n+1)}(\xi) / (n+1)! \, (x-c)^{n+1}$

Proof: Note that $f(c) = P_n(c)$

$f'(c) = P_n'(c)$

$\vdots$

$f^{(n)}(c) = P_n^{(n)}(c)$

Fix $c, x \in (a,b)$

Consider $g(t) = P_n(t) + M/(n+1)! \, (t-c)^{n+1} - f(t)$.

We are trying to use Rolle's Theorem on $[c, x]$.

$g(x) = 0$

$g^{(i)}(c) = 0$ w/ $i = 0, 1, \cdots, n$.

$g^{(n+1)}(t) = M - f^{(n+1)}(t)$

We can apply Rolle's Thm to $g(x)$ and $g(c)$

$g'(x_1) = 0$

$\cdots$

$g^{(n+1)}(x_{n+1}) = 0$.

$M - f^{(n+1)}(x_{n+1})$

ξ

Riemann Integral.

$P = \{ \underset{a}{\overset{\shortparallel}{x_0}}, x_1, \cdots, \underset{b}{\overset{\shortparallel}{x_n}} \}$

$\|P\| = \max_{1 \le i \le n} (x_i - x_{i-1})$

Q is a refinement of P if $P \subseteq Q$. $\|Q\| \le \|P\|$.

Upper Sum: $U(f, P) = \sum_i M_i(f) \, \Delta x_i$.

Lower Sum: $L(f, P) = \sum_i m_i(f) \, \Delta x_i$.

$*$ If $P \subseteq Q$, then $L(f, P) \le L(f, Q)$ and $U(f, P) \ge U(f, Q)$.

$$m_k(x_k - x_{k-1}) \le \underset{\inf_{[z, x_k]} f(x)}{m_k'} (x_k - z) + \underset{\inf_{[x_{k-1}, z]} f(x)}{m_k''} (z - x_{k-1})$$

and then induction.

$\forall$ refinement P, Q.

$$\underbrace{\underset{\le L(f, P \cup Q)}{L(f, P)} \le \underset{\ge U(f, P \cup Q)}{U(f, Q)}}_{\le}$$

$\Rightarrow \quad \sup_P L(f, P) \le U(f, P)$

$\Rightarrow \quad \underset{\underline{\int}_a^b f \, dx}{\sup_P L(f, P)} \le \underset{\overline{\int}_a^b f \, dx}{\inf_P U(f, P)} \quad (=: \text{R. I.})$

* Dirichlet function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{Q}^c \end{cases}$$

not. R.I.

important counterexample.

Cauchy Criterion: bounded $f_1 : [a,b] \to \mathbb{R}$ is R.I.

iff. $\forall \varepsilon > 0$, $\exists$ a partition P_ε of $[a,b]$

s.t. $U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \varepsilon$

($<$).

Proof. $\Leftarrow$: $0 \leq \inf_P U(f,P) - \sup_P L(f,P)$

$\leq U(f, P_\varepsilon) - L(f, P_\varepsilon) \leq \varepsilon$

By Thm 1.2.6 (Abbott)

$\underbrace{\overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx}$. $= \underbrace{\int_a^b f\,dx}$

$\forall \varepsilon > 0$, $\cdots < \varepsilon$

$\Rightarrow$: $\forall \varepsilon > 0$, because of the property of inf & sup

$\exists P_1, P_2$ s.t. $U(f, P_1) < \inf_Q U(f, Q) + \frac{1}{2}\varepsilon$.

$L(f, P_2) > \sup_Q L(f, Q) - \frac{1}{2}\varepsilon$.

Let $P_\varepsilon = P_1 \cup P_2$, $U(f, P_\varepsilon) - L(f, P_\varepsilon)$

$\leq U(f, P_1) - L(f, P_2)$

$< \underbrace{U(f) - L(f)}_{0} + \varepsilon$.

2 ways to show. $U(f,P) - L(f,P) \le \varepsilon$

$$\Leftrightarrow \sum_i \underbrace{(M_i(f) - m_i(f))}_{a_i} \underbrace{\Delta x_i}_{b_i} \le \varepsilon.$$

1° bound $\sum_i a_i \le C$, let $b_i \le \frac{\varepsilon}{C}$. $\Rightarrow (\sum_i a_i)\, b_i \le \varepsilon$.

2° bound $\sum_i b_i \le C$, let $a_i \le \frac{\varepsilon}{C} \Rightarrow (\sum_i b_i)\, a_i \le \varepsilon$.

Thm. every cont. func. $f: [a,b] \to \mathbb{R}$.

is R. I.

since f is cont. on a closed set, f is uniformly cont. on $[a,b]$.

then $\forall \varepsilon > 0$, $\exists \delta > 0$. s.t. $|x-y| \le \delta$.

$$\Downarrow$$

$$|f(x) - f(y)| \le \frac{\varepsilon}{b-a}.$$

Now let $P = \{x_i \mid i = 0, 1, \cdots, n\}$ be a partition of $[a,b]$ w/

$$\|P\| = \max_i (x_i - x_{i-1}) \le \delta.$$

then by EVT, $\forall\ i = 1, 2, \cdots, n$

$\exists\ c_i, d_i \in [x_{i-1}, x_i]$ in every interval partition.

s.t. $M_i(f) = \max\limits_{[x_{i-1}, x_i]} f(x) = f(c_i)$

($\because$ max = sup

min = inf

by the EVT)

$m_i(f) = \min\limits_{[x_{i-1}, x_i]} f(x) = f(d_i)$

$$|c_i - d_i| \le |x_i - x_{i-1}| \le \delta$$

and by uniform continuity, $|M_i(f) - m_i(f)|$

$$= |f(c_i) - f(d_i)| \le \varepsilon/(b-a)$$

thus, $U(f,P) - L(f,P) = \sum_i \Delta x_i \cdot (M_i(f) - m_i(f))$

$$\leq \sum_i \Delta x_i \cdot \varepsilon/(b-a)$$

$$= \varepsilon.$$

By Cauchy's Criterion.

f is R.I. on $[a,b]$.

Rmk. $\sum \underbrace{(M_i(f) - m_i(f))} \Delta x_i$

try to bound this $\underbrace{f(c_i) - f(d_i)}$

$|c_i - d_i|$ let this be small

by uniform cont.

Thm. monotonic bounded function on $[a,b] \to$ R.I.

(bounded closed interval)

WLOG, $f \nearrow$ $\quad f(a) \leq f(b)$.

$\forall \varepsilon > 0$, let $P = \{x_0, x_1, \dots x_n\}$ be a partition of $[a,b]$.

w/ $\|P\| = \max_i (x_i - x_{i-1}) \leq \varepsilon / [f(b) - f(a)]$

$\forall i = 1, \dots, n.$

$M_i = \sup_{[x_{i-1}, x_i]} f(x) = f(x_i)$, $\quad m_i = \inf_{[x_{i-1}, x_i]} f(x) = f(x_{i-1})$

Thus, $U(f,P) - L(f,P)$ $\quad$ (you bound Δx_i by $\|P\|$.)

$\leq [f(b) - f(a)] \, \|P\| \leq \varepsilon.$ $\quad$ R.I.

Darboux $\Leftrightarrow$ Riemann Integrability. $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = T = \int_a^b f(x)\,dx$

$\boxed{\forall \varepsilon>0,\ \exists \delta>0, \text{ for all } P \text{ w/ } \|P\|<\delta, \quad \left|\sum_{i=1}^{n} f(t_i)\Delta x_i - T\right| < \varepsilon. \quad \text{for any } t_i \in [x_{i-1}, x_i]}$

for some T.

$\overset{\Longleftrightarrow}{\text{easy}}$ $\forall \varepsilon>0\ \exists \delta>0$ s.t. for all P w/ $\|P\|<\delta$

$U(f,P)-L(f,P)<\varepsilon$

$\overset{\Longleftrightarrow}{\text{hard}}$ $\forall \varepsilon>0,\ \exists P_\varepsilon$ s.t.

$U(f,P_\varepsilon)-L(f,P_\varepsilon)<\varepsilon$

Let $R[a,b]$ be the set of all R.I. functions

$R[a,b] \to \mathbb{R}$ is a linear order preserving map.

Thm. $\forall f,g \in R[a,b]$ and $\alpha \in \mathbb{R}$.

i) $f+g \in R[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

ii) $\alpha f \in R[a,b]$ and $\int_a^b \alpha f\,dx = \alpha \int_a^b f\,dx$.

iii) $f(x) \le g(x)\ \forall x \in [a,b] \Rightarrow \int_a^b f \le \int_a^b g$.

Proof. $\forall \varepsilon>0,\ \exists \delta>0$, if $\|P\|\le\delta$

then $\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx\right| \le \frac{\varepsilon}{2}$.

$\left|\sum_{i=1}^{n} g(t_i)\Delta x_i - \int_a^b g(x)\,dx\right| \le \frac{\varepsilon}{2}$

i) Use triangular inequality.

ii) insert $|\alpha|$.

$\underbrace{\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)\,dx\right| \le \frac{\varepsilon}{2}.}_{\lim\limits_{\|P\|\to 0} \sum_{i=1}^{n} f(t_i)\Delta x_i = \int_a^b f(x)\,dx}$

$$|\alpha| \left| \sum_{i=1}^{n} f(t_i) \Delta x_i - \int_a^b f(x)\,dx \right| \leq |\alpha| \frac{\varepsilon}{2}$$

iii) $\sum_{i=1}^{n} f(t_i) \Delta x_i \leq \sum_{i=1}^{n} g(t_i) \Delta x_i.$ for any $t_i \in [x_{i-1}, x_i]$.

then by the order theorem, $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(t_i) \Delta x_i \leq \lim_{\|P\| \to 0} \sum_{i=1}^{n} g(t_i) \Delta x_i.$

N.B.

$f \in R[a,b] \Rightarrow |f| \in R[a,b]$ ($\because$ $\sup|f| - \inf|f|$

$\leq \sup f - \inf f$

in each subinterval.

$-|f| \leq f \leq |f|$

$\Rightarrow \left| \int_a^b f\,dx \right| \leq \int_a^b |f|\,dx.$

$f \in R[a,b] \Rightarrow f^2 \in R[a,b].$

first prove the case when $f \geq 0$.

$\forall \varepsilon > 0$, $\exists$ a partition P s.t. $U(f,P) - L(f,P) \leq \varepsilon.$

Note $M_i(f^2) = (M_i(f))^2$; $m_i(f^2) = (m_i(f))^2$

$\because f \geq 0.$

thus, $U(f^2, P) - L(f^2, P)$

$= \sum_i (M_i(f^2) - m_i(f^2)) \Delta x_i$

$= \sum_i \underbrace{(M_i(f) + m_i(f))}_{\text{bound by } 2M} \underbrace{(M_i(f) - m_i(f)) \Delta x_i}_{\varepsilon}$

$\leq 2M \underbrace{\sum_i (M_i(f) - m_i(f))\,\Delta x_i}_{\leq \varepsilon}$ By C.C. $f^2 \in R[a,b]$.

Note that

$$f \in R[a,b] \Rightarrow |f| \in R[a,b] \Rightarrow f^2 \in R[a,b].$$

$f, g \in R[a,b]$, then $f \cdot g$ is R.I.

Also, if $g \neq 0$, $\frac{1}{g}$ is bounded, $f/g \in R[a,b]$.

Proof. (i) $f \cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2] \in R[a,b]$.

(ii) $f \cdot \frac{1}{g}$

$\forall \varepsilon > 0$, $\exists$ a partition P

s.t. $U(g,P) - L(g,P) \leq \varepsilon$.

$$= \underbrace{\sum_i [M_i(g) - m_i(g)]\,\Delta x_i}$$

$M_i(\frac{1}{g}) - m_i(\frac{1}{g})$?

$\forall x, y \in [x_{i-1}, x_i]$

$$\frac{1}{g(x)} - \frac{1}{g(y)}$$

$$\leq \frac{|g(y) - g(x)|}{|g(x)\,g(y)|}$$ $\leftarrow$ bound on $\frac{1}{g} \Rightarrow \frac{1}{|g|}$.

$$\leq M^2 \underbrace{|g(y) - g(x)|}_{\leq |M_i(g) - m_i(g)|}$$

$$U(\tfrac{1}{g}, P) - L(\tfrac{1}{g}, P) \le M^2 \varepsilon. \quad \Rightarrow \quad f/g \in R[a,b]$$

1st MVT, $f, g \in R[a,b]$ w/ $g(x) \ge 0$. $\forall x \in [a,b]$.

$$\int_a^b f(x) g(x)\, dx = c \int_a^b g(x)\, dx.$$

geometrically think of c as f weighed over g.

where $c \in [m, M]$

$$m = \inf_{[a,b]} f(x)$$

$$M = \sup_{[a,b]} f(x).$$

1° If $g(x) = 1$, then $\int_a^b f(x)\, dx = c\,(b-a)$

$$c = \frac{1}{b-a} \int_a^b f(x)\, dx.$$

2° If $f(x)$ is cont. on $[a,b]$, by IVT.

$\exists\, x_0 \in [a,b]$ s.t. $f(x_0) = c$

$$\int_a^b f(x) g(x)\, dx = f(x_0) \int_a^b g(x)\, dx.$$

Proof. $m g(x) \le f(x) g(x) \le M g(x)$

by the order property of integral, $m \int_a^b g \le \int_a^b fg \le M \int_a^b g$.

If $\int_a^b g \ne 0$, let $c = \int_a^b fg \,/\, \int_a^b g$.

$= 0$, any $c \in [m, M]$. works.

$$\int_a^b fg = 0 = \left(\int_a^b g\right) \cdot c = 0 \cdot c.$$

FTC Part I.

$f \in R[a,b]$, and $F: [a,b] \to \mathbb{R}$ is cont. on $[a,b]$

and diff. in (a,b). If $F'(x) = f(x)$ (F is the antiderivative of f;

f is R.I.) on (a,b);

then $\int_a^b f(x)\,dx = F(b) - F(a)$

Hint:

$\underbrace{F(b) - F(a) = \sum_i F(x_i) - F(x_{i-1})}$

$= \sum_i \underbrace{F'(t_i)}_{f(t_i)} \Delta x_i$

by MVT

$t_i \in (x_{i-1}, x_i) \subseteq [x_{i-1}, x_i]$

$= \sum_i f(t_i)\Delta x_i \xrightarrow{\|P\| \to 0} \int_a^b f(x)\,dx$ by the def of R.I.

alternatively, $\forall P$,

$L(f,P) \le \cdots \le u(f,P)$

N.B. We just need f is R.I.

at the endpoints

F does not require to be

diff. @ a, b.

$\star$ Thm. $f: [a,b] \longrightarrow \mathbb{R}$ and $c \in (a,b)$.

then f is R.I. on $[a,b] \Leftrightarrow f$ is R.I. on $[a,c]$ and $[c,b]$.

w/ $\int_a^b f = \int_a^c f + \int_c^b f$. (Hint: common refinement)

$\Rightarrow$: $\forall \varepsilon > 0$, $\exists$ a partition P of $[a,b]$ w/ $u(f,P) - L(f,P) < \varepsilon$.

$P' := P \cup \{c\}$. also a refinement of P.

$P' \cap [a,c]$ // $P' \cap [c,b]$. two partitions.

$\forall P$, We have $L(f,P) \leq L(f,P') = L(f,P_1) + L(f,P_2)$

$$\leq \underline{\int_a^c} f + \underline{\int_c^b} f$$

Similarly, $U(f,P) \geq U(f,P') = U(f,P_1) + U(f,P_2)$

$$\geq \overline{\int_a^c} f + \overline{\int_c^b} f$$

$$\therefore \underbrace{\overbrace{\left(\overline{\int_a^c} f - \underline{\int_a^c} f\right)}^{\geq 0} + \overbrace{\left(\overline{\int_c^b} - \underline{\int_c^b} f\right)}^{\geq 0.}}_{0} < \varepsilon. \quad \forall \varepsilon > 0.$$

both are R.I.

$$\underline{\int_a^c} f = \int_a^c f, \quad \underline{\int_c^b} f = \int_c^b f$$

$L(f,P) \leq \int_a^c f + \int_c^b f \leq U(f,P)$ for all partitions P of $[a,b]$.

$$\therefore \underbrace{\underline{\int_a^b} f \leq \qquad \leq \overline{\int_a^b}}_{=} \qquad \therefore \int_a^b f = \int_a^c + \int_c^b.$$

$\Leftarrow$: partition Q of $[a,c] \subsetneq [a,b]$

R of $[c,b] \subsetneq [a,b]$ $\qquad P = Q \cup R.$

$$U(f,P) - L(f,P) \leq (U(f,Q) + U(f,R)) - (L(f,Q) + L(f,R))$$
$$= (U(f,Q) - L(f,Q)) + (U(f,R) - L(f,R))$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\therefore f \in R[a,b]$ $\quad \forall x \in [a,b]$, $\quad F(x) = \int_a^x f(t)\,dt$ is always well-defined.

We want to prove FTC Part II:

Suppose f is R.I. on $[a,b]$ and $F : [a,b] \to \mathbb{R}$

is $F(x) = \int_a^x f(t)\,dt$.

If f is cont. at $c \in [a,b]$ $\Rightarrow$ F is diff. at c w/

$F'(c) = f(c)$.

A common idea might be

$$F'(c) = \lim_{h\to 0} \frac{F(c+h)-F(c)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(t)\,dt = f(c)$$

By MVT, We need $[a,b] \supseteq [c, c+h]$ cont.

s.t. $\int_c^{c+h} f(t)\,dt = f(c_h)\cdot\int_c^{c+h} 1\,dt$

$= h f(c_h)$

$c_h \in (c, c+h)$

when $h\to 0$, $c_h \to c$.

$$\lim_{c_h\to c} f(c_h) = f(c)$$

$\because f$ is cont at the limit point c.

* with $f \in R[a,b]$ only,

Lemma: f cont. @ a

then $\lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} f(x)\,dx$

$= f(a)$

* $\frac{1}{h}\int_a^{a+h} f(x)\,dx - f(a)$

$= \frac{1}{h}\left[\int_a^{a+h} f(x)\,dx - \int_a^{a+h} f(a)\,dx\right]$

$= \frac{1}{h}\left[\int_a^{a+h} (f(x)-f(a))\,dx\right] < \varepsilon$

Aim: $< \varepsilon$

By the def.: $\forall \varepsilon > 0$, since f is cont. at a $0 < h < \delta$.

$\exists\, \delta > 0$ s.t. $x \in [a, a+h) \subseteq [a, a+\delta) \Rightarrow |f(x) - f(a)| < \varepsilon$

$$\Rightarrow \left| \frac{1}{h}\int_a^{a+h} f(x)\,dx - f(a) \right| = \frac{1}{h}\left| \int_a^{a+h} (f(x)-f(a))\,dx \right|$$

$$\le \frac{1}{h}\int_a^{a+h} |f(x)-f(a)|\,dx$$

$$< \frac{1}{h}\int_a^{a+h} \varepsilon\,dx$$

$$= \varepsilon.$$

N.B. $\underbrace{\frac{1}{h}\int_a^{a+h} f(a)\,dx}$ is the integral of const. func. which we already know.

$$\therefore \lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} f(x)\,dx = f(a).$$

Similarly, cont. @ b, $\displaystyle\lim_{h\to 0^+} \frac{1}{h}\int_{b-h}^{b} f(x)\,dx = f(b)$

cont. @ $c \in (a,b)$, $\displaystyle\lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(x)\,dx = f(c).$

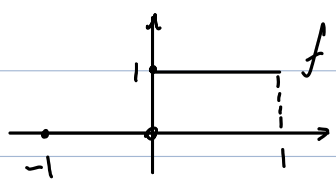
Thus, to show $F'(c) = f(c)$

$$F'(c) = \lim_{h\to 0} \frac{F(c+h)-F(c)}{h}$$

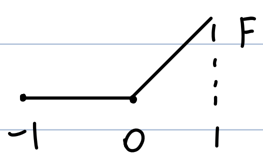
$$= \lim_{h\to 0} \frac{1}{h}\int_c^{c+h} f(x)\,dx = f(c).$$

same @ endpoints

Rmk. to ensure F is diff @ c, f has to be cont. @ c.

e.g. 

f dis cont. @ $c=0$.



we can have

F not diff. @ c

Cor. f is cont. on $[a,b]$

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = \frac{d}{dx}F(x) = f(x) \quad \because f \text{ is cont.}$$

and $$\frac{d}{dx}\left(\int_x^b f(t)\,dt\right) = -f(x)$$

if $\varphi(x)$ is diff. on $[a,b]$.

$$\frac{d}{dx}\left(\int_a^{\varphi(x)} f(t)\,dt\right) = \frac{d}{dx}F(\varphi(x)) = F'(\varphi(x))\cdot\varphi'(x).$$

Thm. If $f \in R[a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is Lipschitz cont.

in the sense that $\exists\, M \geq 0$ s.t. $\forall\, x, y \in [a,b]$,

$$|F(x) - F(y)| \leq M\,|x-y|$$

First, by additivity, $\forall\, x \in [a,b]\ \exists\, F(x)$

$\forall x, y \in [a,b]$ w/ $x \leq y$.

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| \leq \int_x^y |f(t)|\,dt \\ &\leq \int_x^y M\,dt \\ &= M\cdot|y-x|\end{aligned}$$

(M is an upper bound of $|f|$ on $[a,b]$).

$\because$ f must be bounded to be R.I.

Integration by Parts f, g cont. on $[a,b]$

diff. in (a,b). $f', g' \in R[a,b]$.

then $\int_a^b fg' = -\int_a^b f'g + f(b)g(b) - f(a)g(a)$.

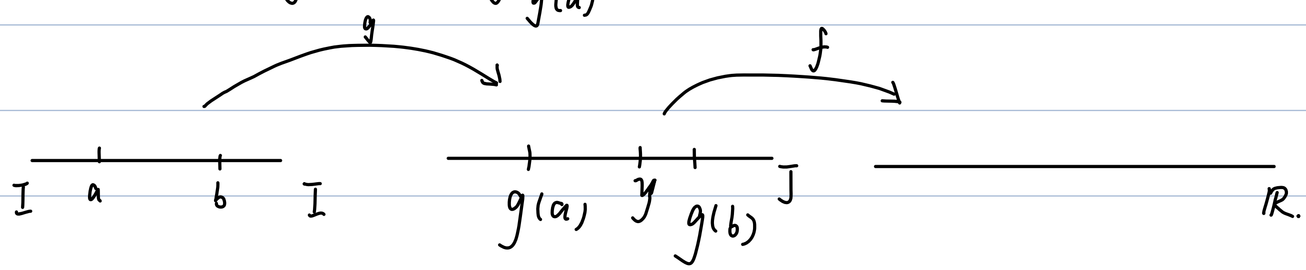
$\because (fg)' = \underbrace{f'g + fg'}_{R.I.} \qquad \int_a^b (fg)'\,dx = f(b)g(b) - f(a)g(a)$

Change of Variables. (see further remarks on MAT 127C Change of var. notes)

Suppose that I & J are two open intervals.

Let $g: I \to J$ be a differentiable function with g' integrable on any closed interval inside I. If $f: J \to \mathbb{R}$ is cont. then $\forall\, a, b \in I$.

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$



Let $F(y) = \int_{g(a)}^{y} f(u)\,du \quad \forall y \in J$.

f is cont. in J $\underset{\text{by FTC Part II}}{\Longrightarrow}$ F is diff. in J. w/ $F'(y) = f(y)$.

By Chain Rule, $\because g$ is also diff, $\quad F \circ g : I \to \mathbb{R}$ is diff.

$$(F\circ g)'(x) = \underbrace{f(g(x))\cdot g'(x)}_{\text{integrable}}.$$

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_a^b (F\circ g)'(x)\, dx = F(g(b)) - F(g(a)).$$

$$= \int_{g(a)}^{g(b)} f(u)\, du.$$

Lebesgue Criterion.

A bounded function $f: [a,b] \to \mathbb{R}$ is R.I.

iff $\exists$ a null set Z s.t. f is cont. at every $x \in [a,b] \setminus Z$.

Def Null set: $Z \subseteq \mathbb{R}$ has measure 0 if $\forall \varepsilon > 0$, $\exists$ a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ s.t. $Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k)$ and $\sum_{k=1}^{\infty} (b_k - a_k) < \varepsilon$.

N.B. any countable set is a null set.

Reverse is not true. e.g. Cantor Set

Proof. $A = \{a_i\}_1^{\infty}$ $\forall \varepsilon > 0$, let

$$I_1 = (a_1 - \frac{\varepsilon}{2^2}, a_1 + \frac{\varepsilon}{2^2}) \quad |I_1| = \frac{\varepsilon}{2}.$$

$$I_2 = (a_2 - \frac{\varepsilon}{2^3}, a_2 + \frac{\varepsilon}{2^3}) \quad |I_2| = \frac{\varepsilon}{4}$$

$$\cdots$$

$$I_k = (a_k - \frac{\varepsilon}{2^{k+1}}, a_k + \frac{\varepsilon}{2^{k+1}}) \quad |I_k| = \frac{\varepsilon}{2^k}$$

$$\cdots$$

$$A \subseteq \bigcup_{k=1}^{\infty} I_k \quad \text{and} \quad \sum_{k=1}^{\infty} |I_k| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Important Examples: cont. func. $Z = \phi$.

mono. Z is countable subset of $[a,b]$.

$$f(x) = \begin{cases} 1 & x \in E = \{0, 1, \frac{1}{2}, \cdots\} \\ 0 & x \notin E \end{cases} \qquad Z = E.$$

**. Thm. If $f: [a,b] \to J$ is R.I.

$g: J \to \mathbb{R}$ is cont.

then So is $g \circ f: [a,b] \to \mathbb{R}$ is also R.I.

$\exists$ null set Z s.t. $\underbrace{f \text{ cont. on } [a,b]\setminus Z}$

$(g \circ f)$ cont. on $[a,b]\setminus Z$

Improper Integrals.

R. I. bounded function on a closed interval. $[a,b]$.

unbounded / infinite interval.

Def. Let I be an interval.

A function $f: I \to \mathbb{R}$ is said to be locally integrable on I

if f is R.I. on each closed interval $[c,d] \subseteq I$.

$I = (a,b]$

Def. $\underbrace{\int_a^b f\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^b f\,dx}$

$\underbrace{\int_a^b f\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f\,dx}$

$I = [a,b)$

$I = (a,b)$

$\underbrace{\int_a^b f\,dx = \lim_{\varepsilon_1 \to 0} \int_{a+\varepsilon_1}^c f\,dx + \lim_{\varepsilon_2 \to 0^+} \int_c^{b-\varepsilon_2} f\,dx}$ $\forall c \in (a,b)$

$I = [a,\infty)$ def. $\int_a^\infty f\,dx = \lim_{r\to\infty} \int_a^r f\,dx$

$I = (-\infty, b]$ def. $\int_{-\infty}^b f\,dx = \lim_{r\to\infty} \int_{-r}^b f\,dx.$

$I = (-\infty,\infty)$, consider $\int_{-\infty}^c + \int_c^\infty$ and apply (iii).

if $\exists$, $\underbrace{\text{calculate via } c=0}$.

Improper Integral converges. $\iff$ corresponding limit exists

* important results: $\int_1^\infty \frac{1}{x^p}\,dx = \begin{cases} \infty & \text{if } 0<p\le 1 \\ \frac{1}{p-1} & \text{if } p>1 \end{cases}$ $x\in[1,\infty)$

$\int_0^1 \quad \begin{cases} \infty & \text{if } p\geq 1 \\ \frac{1}{1-p} & \text{if } 0<p<1. \end{cases}$ $x\in(0,1]$

When $I=(0,\infty)$, $\int_0^\infty \frac{1}{x^p}\,dx=\infty \quad \forall p>0.$

Linearity, order, $\left|\int_I f\,dx\right| \le \int_I |f|\,dx.$ are all preserved by taking limits

***.

Improper integrals of non-negative functions

f is locally integrable on I, $f\geq 0$.

By def. on $I=[a,b)$ $\int_a^b f\,dx = \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f\,dx.$

$\varepsilon \downarrow 0^+$, $\int_a^{b-\varepsilon} f\,dx \nearrow \because f\geq 0.$

build a sequence $\{\varepsilon_n\}$, by sequential criterion of limit & MCT.

$\int_a^b f\,dx = \lim_{\varepsilon\to 0^+} \int_a^{b-\varepsilon} f\,dx$ is ∞ / finite (if bdd)

* f i.i. $\iff$ $\int_I f\,dx$ DN diverge to ∞.

Comparison Theorem for i.i.

f, g locally integrable on I w/ $0\le f(x)\le g(x)$ $\forall x\in I$.

if g i.i. on I, so is f, by $\int_I f(x)\,dx \le \int_I g(x)\,dx < \infty$.

$\int_I f$ DN diverge to ∞ $\Rightarrow$ f is improperly integrable.

Ex. $f(x) = \dfrac{|\sin x|}{x^p}$ Obviously i.i. on $[1,\infty)$ w/ $p > 1$.

$\Rightarrow g(x) = \dfrac{\sin x}{x^p}$

for i.i. $|f|$ i.i. $\Rightarrow f$ is i.i.

Suppose f is locally integrable on I.

(i) f abs. int. on I if $|f|$ is i.i.

(ii) f cond. int. on I if f is i.i. yet $|f|$ is not.

$\underbrace{|f| \Rightarrow f}$ $\quad 0 \le |f| + f \le 2|f|$ Similar to the DCT and abs. conv. in Series.

Ex. $f(x) = \dfrac{\sin x}{x}$ is cond. int. on $[1,\infty)$

$$\int_1^r \frac{\sin x}{x}\,dx = \underbrace{-\frac{\cos x}{x}\Big|_1^r}_{C} - \underbrace{\int_1^r \frac{\cos x}{x^2}\,dx}_{\text{abs int.}}$$

$$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{r\to\infty}\int_1^r \frac{\sin x}{x}\,dx = \cos 1 - \int_1^\infty \frac{\cos x}{x^2}\,dx$$

Now show $\displaystyle\int_1^\infty \frac{|\sin x|}{x}\,dx = \infty$

$\xrightarrow{\hspace{2em}}$

common trick

$$\int_1^{n\pi} \frac{|\sin x|}{x}\,dx \ge \int_\pi^{n\pi} \frac{|\sin x|}{x}\,dx = \sum_{k=2}^{n} \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x}\,dx$$

$$\left(\because x \le k\pi \Rightarrow \frac{1}{x} \ge \frac{1}{k\pi},\right) \qquad \ge \sum_{k=2}^{n} \frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi} |\sin x|\,dx$$

$$= \sum_{k=2}^{n} \frac{1}{k\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi} \sum_{k=2}^{n} \frac{1}{k} \to \infty$$

$\therefore \int_1^\infty \frac{|\sin x|}{x} = \infty$ by the order theorem.

f is bounded & locally integrable on I.

g is abs. int. on I.

$\Rightarrow f \cdot g$ abs. int. on I

$\because \quad 0 \leq |f \cdot g| \leq M|g|$.

f, g i.i. on I $\not\Rightarrow$ $f \cdot g$ i.i. on I. when $x \in (0,1]$.

$f = 1/x^p \quad 0<p<1 \quad g = \frac{1}{x^{p'}} \quad 0<p'<1 \quad f \cdot g = 1/(x^{p+p'}) \quad p+p' \geq 1.$

$p = p' = 1/2$

$I = (2, \infty)$ $\quad f(x) = g(x) = \begin{cases} n^2 & \text{if } x \in [n, n + 1/n^4] \text{ for } n \geq 2 \ n \in \mathbb{Z}. \\ 0 & \text{otherwise} \end{cases}$

$$\int_2^\infty f(x)\,dx = \int_2^\infty g(x)\,dx = \sum_{n=2}^{\infty} n^2 \cdot 1/n^4 = \sum_{n=2}^{\infty} \frac{1}{n^2} < \infty.$$

$$\int_2^\infty f \cdot g\,dx = \sum_{n=2}^{\infty} n^4 \cdot \frac{1}{n^4} = \infty$$

f, g are i.i. while $f \cdot g$ is not.

Sequence and Series of Functions.

Let (f_n) be a seq of functions defined on $S \subseteq \mathbb{R}$. $n \in \mathbb{N}$

$f_n \to f$ ptws on S if $\forall x \in S$

$$\lim_{n\to\infty} f_n(x) = f(x).$$

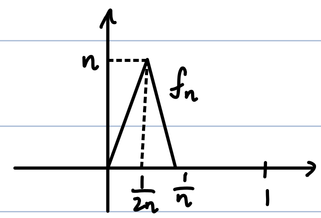
Ex.

$\left(\frac{x}{n}\right) \to 0 \Big/ \left(\frac{n}{nx+1}\right) \to \frac{1}{x}$; does not preserve boundedness.

$\left(1+\frac{x}{n}\right)^n \to e^x$

$(x^n) \to \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ on $[0,1]$ $\times$ continuity.

(*) $f_n(x) = \begin{cases} 2nx^2 & x \in [0, \frac{1}{2n}] \\ 2n^2\left(\frac{1}{n} - x\right) & x \in \left(\frac{1}{2n}, \frac{1}{n}\right] \\ 0 & x \in \left(\frac{1}{n}, 1\right] \end{cases}$



$\forall x$, when $\frac{1}{n} \le x \Leftrightarrow n \geq \frac{1}{x}$ $f_n(x) = 0$

$\therefore f_n(x) \to 0$ as $n \to \infty$

If $x = 0$, $f_n(x) = 0$ for all n,

$\therefore \forall x \in [0,1]$ $f_n \to 0$ ptws.

yet $\int_0^1 f_n(x)\,dx = \frac{1}{2} \neq \int_0^1 0\,dx$, $\times$ integral

We also do not have $\left.\begin{matrix} f_n \to f \\ f_n \text{ diff.} \end{matrix}\right\} \not\Rightarrow$ f' diff. or $f_n' \to f'$

(also not for uniform conv.)

$f_n(x) = \frac{\sin(nx)}{n}$ $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} \frac{\sin(nx)}{nx} \cdot x = 0.$

$f_n(x) \longrightarrow f(x) \equiv 0$ ptws on $\mathbb{R}$.

$f_n'(x) = \cos(nx)$, which is not ptws conv. on $\mathbb{R}$.

Or. $f_n(x) = \dfrac{x^2}{\sqrt{x^2 + \frac{1}{n}}}$

$$\left.\begin{aligned} &x \neq 0, \quad \lim_{n\to\infty} f_n(x) = |x|. \\ &x = 0, \quad f_n(0) = \frac{0}{\sqrt{0 + 1/n}} = 0. \end{aligned}\right\} \text{ ptws on } \mathbb{R}.$$

and $f_n'(x) = \dfrac{x^3 + \frac{2x}{n}}{(x^2 + \frac{1}{n})^{3/2}} \longrightarrow \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

* $\therefore \left.\begin{aligned} &f_n \longrightarrow f \\ &f_n \text{ diff.} \end{aligned}\right\} \not\Rightarrow f$ diff. even $f_n' \to g$ ptws.

$$\left(\begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \quad f' = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \right.$$

$\not\Rightarrow f' = g$.

Ptws on S:

$\forall x \in S \quad \forall \varepsilon > 0, \quad \exists N(\varepsilon, x)$ s.t. $|f_n(x) - f(x)| < \varepsilon$ whenever $n \geq N$.

Uniform Convergence on S:

Let $S \subseteq \mathbb{R}$ and (f_n) be a seq of functions defined on S

$f_n \longrightarrow f$ uniformly on S:

$\forall \varepsilon > 0, \quad \exists N(\varepsilon)$ s.t. for $n \geq N$, $\forall x \in S$, $|f_n(x) - f(x)| < \varepsilon$.

Geometrically, it means that $\forall \varepsilon > 0$, f_n is completely contained in the

ε-strip when $n \geq N$ (being large enough).

e.g.

$(*)$ is not uniformly conv. $\left(\lim_{n\to\infty} \left[\sup_{x \in S} | f_n(x) - f(x) | \right] = \lim_{n\to\infty} n \neq 0. \right)$

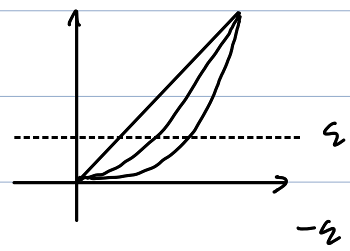
Thm. $(f_n) \longrightarrow f$ uniformly on S

$$\iff \lim_{n\to\infty} \left[\sup_{x\in S} | f_n(x) - f(x) | \right] = 0.$$

Proof is easy.

EX.

$f_n(x) = x^n$ on $[0,1]$ $\quad f_n(1) - f(1) = 1 \neq 0$



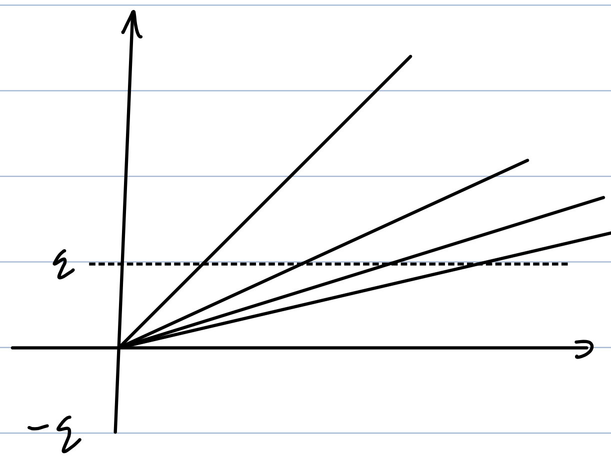
$f(x) = \begin{cases} 0 & [0,1) \\ 1 & 1 \end{cases}$

not contained in ε-strip.

— — — — —

$f_n(x) = x/n$ on $\mathbb{R}$ is not uniform

on $[a,b]$ is.



$$\sup_{x\in\mathbb{R}} | f_n(x) - f(x) | = \sup_{x\in\mathbb{R}} |x|/n \xrightarrow[n\to\infty]{} \infty$$

$f = 0$

$$\sup_{x\in[a,b]} |x|/n = \frac{\max\{|a|, |b|\}}{n} \xrightarrow{n\to\infty} 0.$$

(recall cont. on a compact set implies uni. cont.)

— — — —

$$f_n(x) = \frac{\sin(nx)}{n} \text{ on } \mathbb{R} \quad f(x) = 0.$$

$$0 \leq \sup_{x\in\mathbb{R}} \frac{|\sin(nx)|}{n} \leq \frac{1}{n}$$ Squeeze $\rightarrow$ uni.

$f_n(x) = x/(1+nx^2) \quad x \in \mathbb{R} \quad ; \quad f(x) = 0.$

$$0 \leq \frac{|x|}{1+nx^2} \leq \left\{ \begin{array}{ll} |x|/2\sqrt{n}\,|x| = \dfrac{1}{2\sqrt{n}} & x \neq 0 \\ 0 & x = 0 \end{array} \right\} \leq 1/2\sqrt{n}$$

$0 \leq \sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \leq \frac{1}{2\sqrt{n}}$ Squeeze $f_n \xrightarrow{\text{uni}} f$ on $\mathbb{R}$.

* N.B.

conceptually, ptws: $\forall x \in \mathbb{R}$, $\lim_{n\to\infty} f_n(x) = f(x)$

uni: $\lim_{n\to\infty} f_n(x) = f(x)$ independently of x.

thus f_n in the ε-strip for all x.

$\lim_{n\to\infty} f_n(x) = f(x)$ for all x at the same time.

(think of the use of squeeze theorem

in the last two examples

that make $\underbrace{\lim |f_n(x) - f(x)|}_{=0}$ ind. of x)

Preservation of Properties in Uniform Conv.

Boundedness: $f_n \xrightarrow{\text{uni}} f$

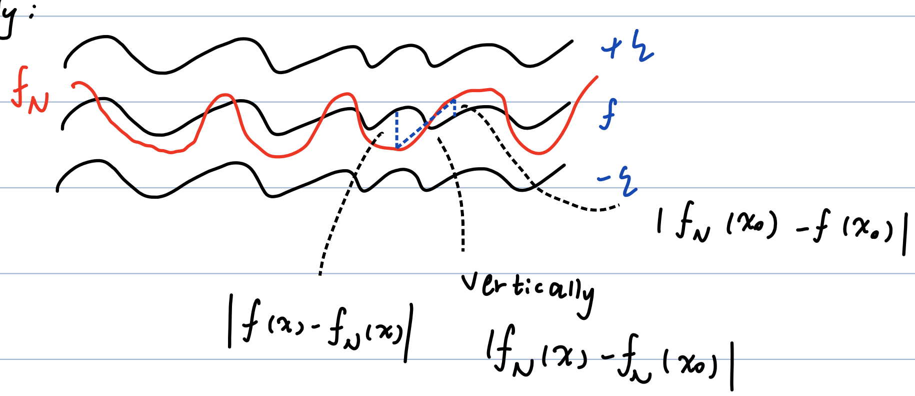
bdd $\Rightarrow$ bdd.

take $\varepsilon = 1$, $n = N$, $\left. \begin{array}{l} |f_N(x) - f(x)| \leq 1 \\ \underbrace{|f_N(x)| \leq M} \end{array} \right\} |f(x)| \leq 1 + M \quad \forall x \in S.$

Preservation of Cont. (the $\varepsilon/3$ method)

Aim: $|f(x) - f(x_0)| < \varepsilon.$

Conceptually:



$\forall \varepsilon > 0$, by uni. conv. $\exists N$ s.t. $n \geq N \Rightarrow |f_n(x) - f(x)| < \frac{1}{3}\varepsilon.\ \forall x \in S$

take $n = N$

$|f_N(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall x \in S$

$\therefore$ same for $x = x_0$

By cont. of f_N @ x_0.

$\exists \delta > 0 \quad |x - x_0| < \delta \Rightarrow |f_N(x) - f_N(x_0)| < \frac{1}{3}\varepsilon.\quad \forall x \in S.$

$\therefore$ under $|x - x_0| < \delta \quad x \in S.$

by Δ-ineq. $|f(x) - f(x_0)| < \varepsilon.$

Preservation of Integral.

$f_n : [a,b] \to \mathbb{R}$ R.I. for each n.

$f_n \to f$ uni. on $[a,b]$

$\Longrightarrow f$ is R.I. w/ $\int_a^b f\,dx = \lim_n \int_a^b f_n\,dx$

Namely, you can change the order of lim & $\int$.

$$\int_a^b \lim_n f_n\,dx = \lim_n \int_a^b f_n\,dx.$$ (this can be very useful for "unintegrable" f_n)

Proof. $\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t. $n \geq N$, $|f_n(x) - f(x)| \leq \varepsilon/3(b-a)$ $\forall x \in [a,b]$.

(why we use $\frac{\varepsilon}{3(b-a)}$ $\because$ we are considering Upper & Lower Sum over $\sum_i \Delta x_i = b-a$ here)

in particular $\forall x \in [a,b]$, $|f_N(x) - f(x)| \leq \frac{\varepsilon}{3(b-a)}$

Since f_N is R.I. $\forall \varepsilon > 0$, $\exists$ partition P s.t.

$$U(f_N, P) - L(f_N, P) \leq \frac{\varepsilon}{3}$$

Consider $U(f, P) - L(f, P) = \sum_i (M_i(f) - m_i(f))\Delta x_i$

$\because$ $f_N(x) - \frac{\varepsilon}{3(b-a)} \leq f(x) \leq f_N(x) + \frac{\varepsilon}{3(b-a)}$ $\quad x \in [a,b]$

$\therefore$ on $[x_{i-1}, x_i]$, $\sup f(x) \leq \sup f_N(x) + \frac{\varepsilon}{3(b-a)}$

$$\inf f_N(x) - \frac{\varepsilon}{3(b-a)} \leq \inf f(x)$$

$\therefore$ $M_i(f) - m_i(f) \leq M_i(f_N) - m_i(f_N) + \frac{2\varepsilon}{3(b-a)}$

$\therefore$ $U(f, P) - L(f, P) \leq U(f_N, P) - L(f_N, P) + \frac{2\varepsilon}{3(b-a)} \cdot (b-a) = \varepsilon.$

$\therefore$ $f \in R[a,b]$.

And $\left| \int_a^b f\,dx - \int_a^b f_n\,dx \right| \leq \int_a^b |f - f_n|\,dx$

(when $n \geq N$,) $\quad \leq \int_a^b \frac{\varepsilon}{3(b-a)}\,dx$ $\quad (\forall x \in [a,b])$

$$\leq \varepsilon/3.$$

$$\lim_n \int_a^b f_n\,dx = \int_a^b f\,dx.$$

N.B. Differentiability is neither preserved in Uni Conv.

$$\left.\begin{array}{l} f_n \to f \text{ uni on } [a,b] \\ f_n \text{ diff.} \end{array}\right\} \not\Rightarrow f \text{ diff. w/ } f_n' \to f'$$

e.g. $f_n(x) = \frac{1}{n}\sin(nx)$ on $[0,\pi]$.

* But we have:

Theorem: $f_n : (a,b) \longrightarrow \mathbb{R}$ be a seq of diff. func.

$f_n' : (a,b) \longrightarrow \mathbb{R}$ are locally integrable. on (a,b).

If $f_n \to f$ ptws on (a,b)

$f_n' \to g$ uni on (a,b) w/ g being cont.

then $f : (a,b) \to \mathbb{R}$ is cont. diff. (C') w/ $f' = g$.

Proof. Fix $c \in (a,b)$ $\forall x \in (a,b)$ by FTC part I. ($x < c$ included)

$f_n(x) = f_n(c) + \int_c^x f_n'(t)\,dt$ on the closed interval between c & x.

take $n \to \infty$, $\underbrace{f(x) = f(c)}_{\substack{\text{ptws at} \\ c,\,x \in (a,b)}} + \int_c^x \underbrace{g(t)}_{\substack{f'(t) \to g(t) \text{ uni.} \\ \text{and } f' \text{ locally integrable}}}\,dt$

$\Rightarrow g$ is integrable on $[c,x]$ / $[x,c]$ w/ the same integral.

$\therefore$ by FTC part II, $\because g$ is cont. at x

$$f'(x) = \frac{d}{dx}\int_c^x g(t)\,dt = g(x)$$

$\therefore f$ is cont. diff. on (a,b) ($\because \forall x \in (a,b)$).

P.S. See Hunter: intro analysis (Theorem 9.18) for:

$(f_n):(a,b) \to \mathbb{R}$ diff. w/ $\begin{cases} f_n \to f & \text{ptws} \\ f_n' \to g & \text{uni.} \end{cases}$ for some $f, g: (a,b) \to \mathbb{R}$.

$\Rightarrow f$ diff. on (a,b)

w/ $f' = g$

And in particular, if each f_n is C^1 (i.e. f_n' is cont. $\Rightarrow \begin{cases} \text{L.I. on } (a,b) \\ g \text{ cont.} \end{cases}$)

then $\left.\begin{array}{l} f_n \to f \text{ ptws.} \\ f_n' \to g \text{ uni.} \end{array}\right\} \Rightarrow f$ is C^1 w/ $f' = \lim_n f_n'$

Cauchy Criterion: We use when we do not know the limit.

Recall

Theorem: (x_n) conv. $\Leftrightarrow$ (x_n) Cauchy on $\mathbb{R}$ (remember Cauchy $\Rightarrow$ Conv. requires the use of the completeness of $\mathbb{R}$ (least upper bound property).

Def. (f_n) on S is uni Cauchy if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall x \in S$, $|f_n(x) - f_m(x)| < \varepsilon$ for $m, n \geq N$.

Uni Conv $\Leftrightarrow$ Uni Cauchy:

$\Rightarrow$: $\triangle$-ineq. which we also did for Cauchy Seq.

"$\Leftarrow$": in seq. we used B-W Theorem / limsup - liminf to prove the reverse direction. (based on the completeness of $\mathbb{R}$)

Here: Fix x, $(f_n(x))$ is a Cauchy Seq of Real Numbers

$\Rightarrow f_n(x)$ is conv. $\forall x \in S$.

Define $f: S \to R$ s.t. $f(x) = \lim_{n\to\infty} f_n(x)$ at all $x \in S$.

$f_n(x) \to f(x)$ ptws on S.

Now we need uniformity:

take $m \to \infty$. $f_m(x) \to f(x)$

$\Rightarrow \underbrace{|f_n(x) - f(x)| \le \varepsilon.}_{\text{order theorem}}$ $\forall n \geq N$, $x \in S$.

$f_n \to f$ uni on S.

Rmk. in both cases, if we conclude $(f_n(x))$ as Cauchy / Convergent at every fixed x, it only entails ptws conv. ; we dropped the common N independent of x.

We have to further show its uniformity

"$\Rightarrow$" : better to avoid completely & use $\triangle$-ineq for all x.

One additional: if $x_n \to x_0$ & $f_n \rightrightarrows f$, then $f_n(x_n) \to f(x_0)$ (proved in HW 9)

Series of Functions: $\sum_{k=1}^{\infty} f_k(x)$ limit of (S_n)

$S_n: A \to R$ $S_n(x) = \sum_{k=1}^{n} f_k(x)$ $\mathrm{Dom}(f_k) = A$.

$S(x) = \sum_{k=1}^{\infty} f_k(x) = \lim_{n\to\infty} \sum_{k=1}^{n} f_k(x)$ $\quad S_n \to S$ ptws.

uni.

Cauchy Criterion: (Recall How Cauchy Criterion is useful in Series of Numbers)

$\sum_{k=1}^{\infty} f_k$ conv. uni. on A iff

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t. $\left| \sum_{k=m+1}^{n} f_k(x) \right| < \varepsilon \quad \forall x \in A \quad n > m \geq N$

$\Updownarrow$

$\underbrace{\left| \sum_{1}^{n} f_k - \sum_{1}^{m} f_k \right| < \varepsilon}$ $\quad | \quad \ldots$

$\Updownarrow$

(S_n) convergent uniformly.

* take $n = m+1$

$\Rightarrow (f_k)$ converges uniformly.

*** Weierstrass M-Test:

$(f_k): A$; if $\forall k \in \mathbb{N} \ \exists M_k \geq 0$ s.t. $(x \in A, \ |f_k(x)| \leq M_k)$

Namely the uniform bound for all f_k

w/ $\sum_{k}^{\infty} M_k < +\infty \implies \sum_{k=1}^{\infty} f_k(x)$ conv. uni. on A

$\Downarrow$

(we only need tail boundedness)

and at every $x \in A$, $\underbrace{\sum_{k} f_k(x) \text{ is abs conv.}}$

Proof: the ptws abs conv. on A. is obvious by the Direct Comparison Test.

$0 \leq |f_k(x)| \leq M_k$ w/ $\sum M_k$ being tail conv.

$\therefore \sum_{k=1}^{\infty} f_k \to S$ on A ptws.

Uni: $0 \leq \sup_{x \in A} |S(x) - S_n(x)| = \sup_{x \in A} \left| \sum_{k=n+1}^{\infty} f_k(x) \right| \leq \sup_{x\in A} \sum_{k=n+1}^{\infty} |f_k(x)|$

(Note here it uses infinite Δ-ineq.) $\leq \sum_{k=n+1}^{\infty} M_k$ (becomes independent of x)

$\because \quad \sum_{k=1}^{\infty} M_k = \underbrace{\lim_{n\to\infty} \sum_{k=n+1}^{\infty} M_k + \lim_{n\to\infty} \sum_{k=1}^{n} M_k}_{=L<+\infty}$

$\lim_{n\to\infty} \sum_{k=n+1}^{\infty} M_k = 0$

By Squeeze Theorem: $\lim_{n\to\infty} [\sup_{x\in A} |S_n(x) - S(x)|] = 0$ $\square$.

Ex 1. $\sum_{k=1}^{\infty} \underbrace{\frac{1}{2^k + x^{2k}} x^k \sin(kx^2)}_{B(k,x)}$ conv. uni on $[-1, 1]$.

$\because \quad |x| \leq 1 \quad |\sin(kx^2)| \leq 1$

$|B(k,x)| \leq \frac{1}{2^k + x^{2k}} \leq \frac{1}{2^k}$, whose $\sum_k \frac{1}{2^k} < +\infty$; by W. MT.

$\Rightarrow$ uni. conv.

Ex 2. $\sum_{k=0}^{\infty} x^k$ $\quad \lim_n S_n(x)$ ptws $= \begin{cases} 1/(1-x) & |x|<1 \\ \text{DNE} & |x| \geq 1 \end{cases}$

Yet it is only ptws on $(-1, 1)$

given that $\sup_{x\in(-1,1)} |S_n(x) - S(x)| = \sup_{x\in(-1,1)} \left|\frac{x^{n+1}}{1-x}\right| \geq \lim_{x\to 1^-} \left|\frac{x^{n+1}}{1-x}\right| = \infty$

/ $\frac{1}{1-x}$ being unbounded, yet every $S_n(x) = 1 + x + x^2 + \dots + x^n$

$< n+1$. and is bounded.

$*$

$S_n(x) \xrightarrow{uni} S(x) = \frac{1}{1-x}$ on $[-\rho, \rho]$ for all $0 \leq \rho < 1$.

$\forall x \in [-\rho, \rho] \quad |x^k| \leq \rho^k$ and $\sum_{k=0}^{\infty} \rho^k = \frac{1}{1-\rho} < +\infty$

By W.MT $\sum_k x^k$ conv. uni.

(closed interval makes it possible to be bounded by a series

that actually conv.)

Ex 3. $W_n(x) = \sum_{k=1}^{n} \frac{1}{2^k} \cos(3^k x)$ by W.MT converges uni. on $\mathbb{R}$

to $W(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cos(3^k x)$.

which is cont. $\because$ every partial sum

(Σ of individual cont. func.)

is cont.

But $W(x)$ is not diff at any $x \in \mathbb{R}$ "fractal graph"

Power Series: $\sum_{k=0}^{\infty} a_k (x-c)^k$ centered at c. w/ coefficients a_k.

$r = \overline{\lim_{n \to \infty}} |a_k|^{\frac{1}{k}}$ Root Test for Real Series.

abs. conv. $0 \leq r < 1$.

div. $1 < r \leq \infty$.

inconclusive $r = 1$.

For the power series $\sum_k a_k (x-c)^k$, consider $\overline{\lim_{k \to \infty}} \sqrt[k]{|a_k (x-c)^k|}$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k (x-c)^k|} = \left(\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|}\right) \cdot |x-c|$$

$$= \beta\ |x-c|.$$

when $0 \le |x-c| < \frac{1}{\beta}$ abs. conv.

$\frac{1}{\beta} < |x-c| \le \infty$ div.

$|x-c| = \frac{1}{\beta}$ inconclusive.

We let the radius of conv.

$\underbrace{R = \frac{1}{\beta} \text{ if } \beta = 0;}\ \underbrace{R = 0 \text{ if } \beta = \infty}$

based on the convention of the extended real number system.

Check Convergence : $\beta = \underbrace{\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|}$

N. B.

Root Test is stronger than Ratio Test:

$\underbrace{\text{if } \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \beta, \text{ then } \overline{\lim_{k\to\infty}} |a_k|^{\frac{1}{k}} = \beta.}$

When Ratio Test shows conv., Root Test shows conv.

For $\{c_n\}$ w/ $c_n > 0$,

(See Rudin 3.37)

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \le \liminf_{n\to\infty} \sqrt[n]{c_n} \le \limsup_{n\to\infty} \sqrt[n]{c_n} \le \underbrace{\limsup_{n\to\infty} \frac{c_{n+1}}{c_n}}$$

if $= +\infty$. ✓

if $= \alpha < +\infty$ choose $\beta < \alpha$.

$\lim_{n\to\infty}\left(\sup_{k\ge n} \frac{c_{k+1}}{c_k}\right) = \alpha < \beta$

$\exists N$ s.t. $n \ge N$, $\sup_{k\ge n} \frac{c_{k+1}}{c_k} \le \beta$

$\Longleftrightarrow \frac{c_{n+1}}{c_n} \le \beta$

$c_n \le c_N \cdot \beta^{n-N}$ for any $n \ge N$.

$\Rightarrow \sqrt[n]{c_n} \le \sqrt[n]{c_N \beta^{-N}} \cdot \beta$

$$\limsup_{n\to\infty}\left(\sqrt[n]{c_N \beta^{-N}} \cdot \beta\right)$$

$$= \limsup_{n\to\infty}\left(\sqrt[n]{c_N \beta^{-N}}\right) \cdot \beta = \beta.$$

$\therefore \limsup\limits_{n\to\infty} \sqrt[n]{c_n} \le \beta.$ since this is true $\forall \beta > \alpha.$

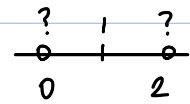
$\limsup\limits_{n\to\infty} \sqrt[n]{c_n} \le \alpha.$

$$\left(\begin{array}{l} \because \sup\limits_{k\ge n} \sqrt[n]{c_n} \le \sup\limits_{k\ge n} (\cdots) \\ \lim\limits_{n\to\infty} \cdots \le \lim\limits_{n\to\infty} \cdots \end{array} \right)$$

Radius $= 1/\beta.$ Check $x = c \pm R.$

Ex. $\sum\limits_{k=1}^{\infty} \underbrace{\frac{(-1)^{k+1}}{k}}_{a_k} (x-1)^k$ Ratio: $\beta = 1$ = Root : 1

$R = 1/\beta$



div. A.S.T. conv.

$\ln x = \sum\limits_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$ r.c. $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$

$\frac{1}{1+x} = 1 - x + x^2 - \cdots$

Ex. $\sum\limits_{k=0}^{\infty} 2^k x^{2k}$

$a_n = \begin{cases} 2^k & n \text{ even} = 2k \\ 0 & n \text{ odd.} \end{cases}$

$|a_n|^{1/n} = \begin{cases} (2^k)^{1/2k} = \sqrt{2} \\ 0^{1/n} = 0 \end{cases}$ $\beta = \sqrt{2} \Rightarrow R = \sqrt{2}/2.$

at $x = \pm \frac{\sqrt{2}}{2}$ $\Sigma 1$ diverges. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

Now, given the conv. of $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ on $(c-R, c+R).$

Def. $\underbrace{S(x) := \sum\limits_{k=0}^{\infty} a_k (x-c)^k}$ on $(c-R, c+R)$

$S_n(x) = \sum_{k=0}^{n} a_k (x-c)^k$ is a polynomial. $\Rightarrow$ cont.

Aim:
$S_n \rightrightarrows S$ on $[c-\rho, c+\rho]$ w/ $0 < \rho < R$.

Proof. $\forall x \in (c-R, c+R)$ $\sum_k a_k (x-c)^k$ is abs conv. (in the radius of conv. by the root test)

At $x = c+\rho$, $\sum_k |a_k (x-c)^k| = \sum_k |a_k| \rho^k < +\infty$. being abs conv.

$\forall x \in [c-\rho, c+\rho]$, $|a_k (x-c)^k| = |a_k| \, |x-c|^k \le |a_k| \rho^k$

whose $\sum_k$ is conv.

By W. M-Test, $\sum_k a_k (x-c)^k$ converges uniformly on $[c-\rho, c+\rho]$

$\therefore$ We can conclude cont. of $S(x)$ on $[c-\rho, c+\rho]$

But we do not stop here, we want to show cont. on $(c-R, c+R)$.

"Proof." $|c-x_0| < \rho < R$ find the ρ between two $|c-x_0|$ and R

$\Rightarrow$ cont. of x_0 in $[c-\rho, c+\rho]$.

$\because \forall x_0$, $S(x)$ cont. on $(c-R, c+R)$.

* be careful about the endpoints for every $[c-\rho, c+\rho]$	that does not extend onto $(c-R, c+R)$
cont. at the endpoint is the 1-side cont.	

Endpoint Continuity : 1st Abel's Lemma: $\{b_n\} \geq 0$, $\downarrow$

$\sum_{n=1}^{\infty} a_n$ has its partial sums bounded.

i.e. $\exists A > 0$ $\forall n \in \mathbb{N}$

$|a_1 + a_2 + \cdots + a_n| \leq A$.

By Abel's summation formula, $\forall n \in \mathbb{N}$ $\left| \sum_{i=1}^{n} a_i b_i \right| \leq A b_1$

(See Abbott 6.5.3 for detail.)

2nd. Key : Abel's Theorem $S(x) = \sum_{k=0}^{\infty} a_k x^k$ converging at $x = R > 0$

$S_n(x) \rightrightarrows S(x)$ on $[0, R]$.

(most importantly, it determines the cont. of $S(x)$ @ R from the left since $S_n(x)$ is originally cont. at $x = R$ on $[0, R]$.)

Centered at c / at $-R$, we have similar results.

Aim : construct a $\downarrow$ $\{b_n\} \geq 0$.

$$S(x) = \sum_{k=0}^{\infty} \underbrace{(a_n R^n)}_{\text{partial sum bdd.}} \underbrace{\left(\frac{x}{R}\right)^n}_{\downarrow}$$

use Cauchy's criterion @ R.

By the Conv. @ R, $\exists N \in \mathbb{N}$ s.t. when $n > m \geq N$,

$$|a_{m+1} R^{m+1} + \cdots + a_n R^n| \leq \varepsilon.$$

fix m, based on Abel's Lemma: $\sum_{j=1}^{\infty} a_{m+j} R^{m+j}$

has its partial sums bdd, and $\left\{ \left(\frac{x}{R} \right)^{m+j} \right\}_{j=1}^{n} \downarrow$

$$\Rightarrow \left| \left(a_{m+1} R^{m+1}\right) \left(\frac{x}{R}\right)^{m+1} + \cdots + \left(a_n R^n\right)\left(x/R\right)^n \right| \leq \varepsilon \cdot \underbrace{\left(\frac{x}{R}\right)^{m+1}}_{\leq 1} \leq \varepsilon$$

Term-by-Term $\int$.

$R>0$ for any $[a,b] \subseteq (c-R, c+R)$

$$\int_a^b \sum_{k=0}^{\infty} a_k (t-c)^k \, dt = \sum_{k=0}^{\infty} \int_a^b a_k (t-c)^k \, dt$$

(* interchange $\int$ and finite sum is obviously correct by the linearity of $\int$)

for infinite sum we need uniform conv.

$$\int_c^x \sum_{k=0}^{\infty} a_k (t-c)^k \, dt = \sum_{k=0}^{\infty} \int_c^x a_k (t-c)^k \, dt$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1}$$

Proof. On $[c-\rho, c+\rho]$ $(0<\rho<R)$ that contains $[a,b]$.

$S_n \rightrightarrows S$

$$\int_a^b \underbrace{\lim_n S_n}_{S} \, dt = \lim_n \int_a^b S_n \, dt$$

$$= \int_a^b \sum_{k=0}^{\infty} a_k (t-c)^k \, dt$$

$$= \lim_n \int_a^b \sum_{k=0}^{n} a_k (x-c)^k \, dt$$

$$= \lim_n \sum_{k=0}^{n} \int_a^b a_k (x-c)^k \, dt$$

$$= \sum_{k=0}^{\infty} \int_a^b a_k (x-c)^k \, dt$$

Ex. $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ $|x|<1$. centered at $x=0$.

$\forall x \in (-1,1)$ $\int_0^x \frac{1}{1-t}\,dt = \int_0^x \sum_{k=0}^{\infty} t^k\,dt = \sum_{k=0}^{\infty} \int_0^x t^k\,dt$

$$\underbrace{-\ln(1-x) \qquad = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}}$$

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

$$\Rightarrow \ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \qquad |x|<1.$$

Extend via Abel's Theorem to $x=1$.

when $x=1$, by the A.S.T. $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ conv. at $x=1$

$\Rightarrow$ by uni. conv. on $[0,1]$

and cont. of individual partial sums.

$\Rightarrow$ cont. of the power series at $x=1$.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k \Big|_{x=1} = \lim_{x\to 1^-} \underbrace{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k} = \lim_{x\to 1^-} \ln(1+x) = \ln 2$$

cont. of "ln" at 1^-.

$x \in (-1,1)$

in the left-side neighborhood of 1. : $\sum_{k=1}^{\infty} = \ln(1+x)$

— — — — — — — — —

if we are given an explicit formula $f(x) = \sum_{k=0}^{\infty} a_k x^k$ on $(-R,R)$

and the series $S(x)$ converges $\Rightarrow$ cont. at R.

then $S(R) = \lim_{x\to R^-} S(x) = \underbrace{\lim_{x\to R^-} f(x)}$

the limit of $f(x)$ must exist s.t. $S(R)$ exists.

Now we show $\sum_{k=0}^{\infty} a_k (x-c)^k$ has radius or conv. R, then the power series $\sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$ and $\sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1}$. have the same R.

Proof. $R = \dfrac{1}{\overline{\lim}_{k\to\infty} \sqrt[k]{a_k}} = \dfrac{1}{\beta}$

$$\Rightarrow \overline{\lim_{k\to\infty}} \sqrt[k]{k|a_k|} = \overline{\lim_{k\to\infty}} \sqrt[k]{k}\sqrt[k]{|a_k|}$$

$$= \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} = \beta.$$

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|/(k+1)} = \overline{\lim_{k\to\infty}} \sqrt[k]{|a_k|} / \sqrt[k]{k+1} = \beta/1 = \beta.$$

Term-by-term diff. $S(x) = \sum_{k=0}^{\infty} a_k (x-c)^k$ has radius $R>0$.

$\Rightarrow$ S diff. on $(c-R, c+R)$ w/ $S'(x) = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$

Idea: $\left.\begin{array}{l} S_n \to S \text{ ptws} \\ S_n' \rightrightarrows g \\ S_n \text{ is } C' \end{array}\right\} \Rightarrow \begin{array}{l} S \text{ is } C' \\ S' = g = \lim_n S_n' \end{array}$

Same trick: $\forall x \in (c-R, c+R)$ $|x-c| < \rho < R$.

then $S_n(t) = \sum_{k=0}^{n} a_k (t-c)^k \longrightarrow S(t)$ ptws on $[c-\rho, c+\rho] \subseteq (c-R, c+R)$

and for the series of functions $S_n'(t) = \sum_{k=1}^{n} k a_k (t-c)^{k-1} \rightrightarrows \sum_{k=1}^{\infty} k a_k (t-c)^{k-1}$ on the same $[c-\rho, c+\rho]$.

since the new $S_n'(x)$ has the same R.

Also, S_n is C' (S_n' is a poly.)

$\therefore$ $S(t)$ diff. on $[c-\rho, c+\rho]$ w/ $S'(t) = \sum_{k=1}^{\infty} k a_k (t-c)^{k-1}$

esp. @ x.

$\forall x \in (c-R, c+R)$, $S'(x) = \sum_{k=1}^{\infty} k\, a_k (x-c)^{k-1}$

N.B. this implies $S(x)$ is infinitely diff. in $(c-R, c+R)$

$$S^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\cdots(k-n+1)\, a_k (x-c)^{k-n}$$

At $x = c$, $S^{(n)}(c) = n \cdot (n-1) \cdot \cdots \cdot (n-n+1) a_n = n!\, a_n$

$$\Rightarrow a_n = \frac{S^{(n)}(c)}{n!} \quad n = 0, 1, 2, \ldots$$

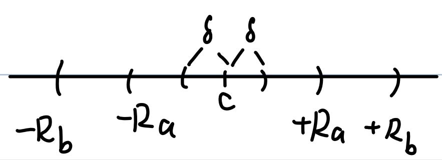
even just around the neighborhood of c

$\therefore$ if we have an explicit $S(x)$ for $\sum_{n=0}^{\infty} a_n (x-c)^n$ w/ $R > 0$.

then S is inf. diff. $a_n = \dfrac{S^{(n)}(c)}{n!}$ for $n = 0, 1, 2 \ldots$

(the opp. of course is false)

Cor. If $\sum_{n=0}^{\infty} a_n (x-c)^n$ and $\sum_{n=0}^{\infty} b_n (x-c)^n$ have R_a and $R_b > 0$.



if. $\forall x \in (c-\delta, c+\delta)$

$$\boxed{S = \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} b_n (x-c)^n}$$

given $R_a, R_b > 0$.

we have $a_n = \dfrac{S^{(n)}(c)}{n!}$; $b_n = \dfrac{S^{(n)}(c)}{n!}$. $\forall n \in \mathbb{Z}$.

as the n-th derivative exists in the δ-neighborhood of c.

$\therefore$ they are the same power Series w/ $R_a = R_b$.

Def. f real-valued function on (a,b). (in general open $D \subseteq \mathbb{R}$)

as D is Unions (possibly ∞) of open neighborhoods.

1) $f \in C^n(a,b)$ if it has cont. deri. $f^{(j)}$ of orders $1 \le j \le n$ on (a,b).

2) $f \in C^\infty(a,b)$ (f is smooth & infinitely diff.)

if it has cont. deri. of all orders on (a,b).

3) f is (real) analytic if $\forall c \in (a,b)$.

$f(x)$ can be written as a power series centered at c.

$\sum_{n=0}^{\infty} a_n (x-c)^n$ w/ $R_c > 0$.

(Namely in some neighborhood of c.)

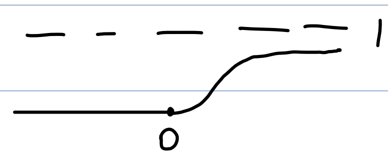
Analytic $\Rightarrow$ Smooth, infinitely diff. w/ coefficients given by Taylor.

Yet, the opposite is not true.

We do not have: $\forall c$, $f(x)$ can be Written as

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k \quad \text{in some } (c-R, c+R)$$

Counterexample: $f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$



Smooth: has deri. of all orders on $\mathbb{R}$.

$$f^{(n)}(0) = 0 \text{ for all } n \in \mathbb{N}.$$

This means: $\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 0 \neq f(x)$ when $x > 0$. Not-analytic.

When $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$

$\Longleftrightarrow$ $P_n(x) \to f(x)$ as $n \to \infty$.

$\Longleftrightarrow$ $R_n(x) \longrightarrow 0$

Taylor's Theorem. $f: (a,b) \longrightarrow \mathbb{R}$ has $(n+1)$ deri. on (a,b)

$c \in (a,b)$ $\quad f(x) = P_n(x) + R_n(x)$.

Lagrange Remainder: $\exists\, \xi$ between x and c.

s.t. $R_n(x) = \dfrac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$

Integral Remainder: If $f^{(n+1)}$ is locally R.I. then

$$R_n(x) = \frac{1}{n!} \int_c^x f^{(n+1)}(t)\,(x-t)^n \, dt$$

Cor. If f has deri. of all orders on (a,b) (can be written as Taylor Series around all c)
we have $\forall x \in (a,b)$, $|f^{(n)}(x)| \le M$ for all n.

then $R_n(x) \xrightarrow{n \to \infty} 0$ at all $x \in (a,b)$.

$\therefore f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!}(x-c)^k$ around all $c \in (a,b)$. (f is analytic)

(in particular, can check conv. of the Taylor Series around c via $|f^{(n+1)}(t)| \le M$.

for all t between c and x ($t \in (c-R, c+R)$)).

Proof. $R_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)(x-c)^{n+1}$

$$0 \le |R_n(x)| \le \frac{1}{(n+1)!} \underbrace{|f^{(n+1)}(\xi)|}_{\le M} |(x-c)^{n+1}|$$

Squeeze

$$\boxed{\lim_{n\to\infty} |R_n(x)| = \lim_{n\to\infty} R_n(x) = 0.}$$

$$\frac{|x|^n}{n!} \longrightarrow 0. \quad \text{as } n \to \infty$$

(to show the abs conv, consider $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\quad \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{1}{n+1} = 0.$

$R = \frac{1}{0} = \infty$

$\therefore$ for all $x \in \mathbb{R}$; $\to 0$

EX. For $\sin(x)$, $f^{(n)}(x) = \begin{cases} 0 & n = 2k \\ (-1)^k & n = 2k+1 \end{cases}$

$\forall x \in \mathbb{R}$, $|f^{(n)}(x)| \le 1$ for all n.

$\sin(x)$ is real analytic on $\mathbb{R}$, in particular around $c=0$,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in \mathbb{R}.$$

diff.

$$\Rightarrow \quad \cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \quad \forall x \in \mathbb{R}.$$

A common trick:

$f(x) = e^x$ we have $|f^{(n)}(x)| = |e^x|$ unbounded on $\mathbb{R}$.

but $\forall x_0 \in \mathbb{R}$, let $M = x_0 + 1$, then $|f^{(n)}(x)|$

$= |e^x| \leq e^M$

by the Cor. for all $x \in (-M, M)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$ for all $x \in (-M, M)$, in particular at $x_0 \in \mathbb{R}$.

$\therefore \forall x \in \mathbb{R}$, $e^x = \sum_{k=0}^{\infty} x^k / k!$