Notes on the Exponential and Logarithmic Functions

based on Priestley's *Introduction to Complex Analysis*, second edition

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We know when a power series converges within its radius of convergence, then it is holomorphic within its radius of convergence, and the derivative can be taken term-by-term. Now we define the complex *exponential function*

$$\exp(z) = e^z = \sum_{i=0}^{\infty} \frac{z^n}{n!}, \text{ where } z \in \mathbf{C}.$$

By the theorem mentioned, we immediately know that $\frac{d}{dz}e^z = e^z$ for all $z \in \mathbf{C}$. In addition, we know by definition that $e^0 = 1$.

We can show that $e^{z+w} = e^z e^w$. To show this, fix $c \in \mathbf{C}$, and consider $f(z) = e^z e^{c-z}$. It has derivative $e^z e^{c-z} - e^z e^{c-z} = 0$. This means that $f(z)$ is constant on the complex plane. Let $z = 0$ and we see that $e^c = f(z) = e^z e^{c-z}$. Replace c by $z + w$.

The special case $e^z e^{-z} = e^0 = 1$, which shows e^z is nonzero for all $z \in \mathbf{C}$.

We now show Euler's identity using our definition: e^{iy} $(y \in \mathbf{R})$

$$\begin{aligned} &= \sum_{i=0}^{\infty} \frac{i^n y^n}{n!} = \sum_{\text{even } i} \frac{i^n}{n!} y^n + i \sum_{\text{odd } i} \frac{i^{n-1}}{n!} y^n \\ &= \cos y + i \sin y \end{aligned}$$

by the Taylor expansion of cos and sin over $\mathbf{R}$. (Note that technically we need to write out the partial sums and retake limits.) It follows first that $|e^{iy}| = 1$. Second, for complex $z = x + iy$, where $x, y \in \mathbf{R}$

$$\begin{aligned} e^z = e^x e^{iy} &= e^x \cos y + i e^x \sin y \\ &= re^{i\theta}, \text{ where } r = e^x \text{ and } \theta = y. \end{aligned} \tag{1}$$

N.B. $r = e^x = |e^z|/|e^{iy}| = |e^z|$, and $e^{i\theta} = e^z/|e^z|$.

From this we know that $\exp : \mathbf{C} \to \mathbf{C}\backslash\{0\}$ is *onto*, because $r = e^x$ can be any positive real number by choosing the appropriate $x \in \mathbf{R}$ (i.e., the real exponential function is onto $\mathbf{R}^+$), and $\theta = y$ can be in any direction.

Since cos and sin have fundamental period 2π, the exponential function exp has fundamental period $2\pi i$. This directly implies that unlike in the real case, where we can define $\log(\cdot)$ simply as the inverse of $\exp(\cdot)$ because $\exp : \mathbf{R} \to \mathbf{R}^+$ is one-to-one and onto, the complex logarithm must be defined as a multifunction.

Before introducting logarithm, to make things clearer we define the *argument* of a complex number. For nonzero z, $[\![\arg z]\!] := \{\theta \in \mathbf{R} : z = |z|e^{i\theta}\}$, which is a *set*. It contains $2k\pi + \theta$, where θ

is any chosen real satisfying $e^{i\theta} = e^z/|e^z|$, as discussed in (1).

Now given w, we want to solve the z in $w = e^z = e^x e^{iy}$. From our discussion above, $e^x = |w|$, which implies $x = \ln(w)$. y clearly can any number from $[\![\arg z]\!]$. Thus we may define the complex *logarithm* for $w \neq 0$,

$$[\![\log(w)]\!] = \{\ln|w| + i\theta : \theta \in [\![\arg z]\!]\}.$$

Note that we can take the logarithm of any nonzero complex number. In $\mathbf{C}$ there is no > 0 or < 0. The direction of w is determined by the e^{iy} part of the expansion.

With the definiton of complex logarithm, we are now allowed to define the general complex power of a complex number, just as what we did in the real case. However, here the general power has to be defined as a multifunction. For $\alpha \in \mathbf{C}$, we define, for $z \neq 0$,

$$[\![z^\alpha]\!] = \{e^{\alpha(\ln|z|+i\theta)} : \theta \in [\![\arg z]\!]\}. \tag{2}$$

This matches are definition of $\exp : \mathbf{C} \to \mathbf{C}\backslash\{0\}$ at the beginning. In addition, *only when α is an integer does $[\![z^\alpha]\!]$ gives a singleton set containing z^α*, because we cannot break down the 2π period of θ in (2). However, if we choose $\alpha = \frac{1}{n}$, where $n \in \mathbf{Z}^+\backslash\{1\}$, then

$$[\![z^{1/n}]\!] = \{|z|^{1/n} \cdot e^{2k\pi i/n} : k \in \mathbf{Z}_n\},$$

the n-th root of the length (modulus) times the n-th roots of unities, as expected.

We do not give any particular properties of the general complex power. Working with multifunctions can be really tricky.

P.S. For the definition and discussion of complex trignometric functions and holomorphic branches of complex logarithm and general powers, see Chapter 7 of the book.