Lineage of the Theory of Invariant Integrals on Groups Hurwitz, Schur, Weyl, Haar, Neumann, Kakutani, Weil, Kakutani-Kodaira

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*

Abstract. This is mainly a translation of Proc. of the 29th Symp. on History of Mathematics, Tsuda University, held Oct. 2018, of my talk.¹⁾ The first occasion when I studied the history of the theory of invariant integrals (or measures) was an unintended opportunity where I was asked to write "Kaisetsu" (explanatory and commentary article) to the new book²⁾ of Prof. M. Saito, a first translation into Japanese of the famous Weil's book 'L'intégrations dans les groupes topologiques et ······'. Personally, for my professional work, it was needed only to read this Weil's original book and several text books on measure theory. For writing the Kaisetsu mentioned above, other than several mathematical papers and also Weil's non-mathematical works, it was sufficient for me to read Haar's original paper roughly and similarly for other historical classics. Thus I felt the necessity of further study and now I make intendedly a historical study. Reading original papers due to Hurwitz, Schur, Weyl and so on in detail, I explain their contents from my point of view, and give the relationships among them.³⁾

¹⁾ Cf. [Hir18] in **References**.

²⁾ This book [Sai15] was published as a volume of the series of pocket-size book of reproduction of natural science classics, called *Chikuma Gakugei Bunko*. Each book has an additional part called *Kaisetsu* of about 20 pages, written by an expert of the subject of the book and attracts readers in addition to those of the main body. In this case I wrote an article of 30 pages and Prof. Saito thanked me writing 'I think it's a very interesting commentary with rich of contents' in his afterword. To prepare this article, I read several mathematical papers, in particular around Weil representations, and also Weil's *Souvenirs d'apprentissage*, *Commentaire* in his *Collected Papers* Vols I–III and so on. Moreover, Mr. I. Ebihara, the editor of the book, wrote in the back-page-campaign of the bookcover as 'The study of integrals on the group-space has begun with Hurwitz at the end of 19th Century. · · · ', asking me as 'Is that correct?' I answered 'Yes', not knowing well about his work. And so naturally I was forced later to study his paper [Hur1897] to ensure my answer 'Yes'.

³⁾ Mathematical Subject Classification: Primary 22-02, 22-06; Secondary 20-03, 01-02. Keywords and Phrases. invariant integral on groups and homogeneous spaces, quasi-invariant measure on homogeneous spaces, uniqueness of invariant measures, Frobenius character theory, Hurwitz-Schur coordinates on rotation groups

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1 Hurwitz's pioneering work on integrals on Lie groups

The title of his paper [Hur1897] is 'On the generation of invariants through integration' and its main objective is to generate invariant polynomials under natural actions of Lie groups G, especially rotation groups SO(n) and also unitary groups SU(n).

For this end, he proposed to use invariant integrals on Lie groups G.

This 20 pages paper has an 'Introduction' of about 1 page and is not separated by sections. But it has several separations indicated by a horizontal segment in the middle of blank lines. Take appropriate 4 separations as sections and put appropriate titles to each sections, then the structure of the paper is approximately as follows (the technical terms are modernized as those of today):

| | Introduction | p.70 |
|--------|----------------------------------------------------------------|------|
| § 1 | Generation of invariants on continuous groups using invariant | |
| | integration, in particular, case of $SO(n)$ | p.71 |
| $\S 2$ | Global coordinates on $SO(n)$ and formula of invariant measure | p.75 |
| $\S 3$ | Global coordinates on $SU(n)$ and formula of invariant measure | p.80 |
| ξ4 | General theory for Lie groups | p.86 |

1.0 On 'Introduction'

Hurwitz explained first as

Assume that a finite group G acts on variables x_1, x_2, \ldots, x_n by permuting them. Then G acts on a function $f(x_1, x_2, \ldots, x_n)$, and we get an invariant function φ by averaging gf over $g \in G$. This process can be applicable in some cases of infinite number of variables.

Then continued, in German original, as (use machine translation if necessary)

····· Ich habe nun Gedanke verfolgt, dieses sich so zu sagen von selbst darbietende Verfahlen zur Erzeugung der Invarianten auf die continuierlichen Gruppe
zu übertragen, wo dann naturgemäß bestimmte Integrale an die Stelle der Summen treten. Dabei richtete ich mein Augenmerk zunächst auf die ganzen rationalen Invarianten der algebraischen Formen, also auf diejenigen ganzen rationalen Functionen der Coefficienten einer Form, die sich nicht ändern, wenn
man auf die Variabeln der Form eine beliebige lineare, homogene, unimodulare
Substitution ausübt. Die Untersuchung führte mich in dessen bald dazu, ·····

then continued, in short, as

More extended cases we study are in particular rotation groups SO(n), and also special unitary groups SU(n).

1.1 On '§1 Generation of invariants on continuous groups using invariant integration...'

Please admit me to use modern notation G = SO(n). Take an element $g = (r_{\alpha,\beta})_{\alpha,\beta\in I_n}$ of the rotation group G, with $I_n := \{1, 2, ..., n\}$, and consider its natural action on the variables $x := {}^t(x_1, x_2, ..., x_n)$ under

(1.1)
$$x_{\alpha} = \sum_{\beta} r_{\alpha\beta} x_{\beta}' \quad \text{or} \quad x = gx',$$

(1.2)
$$\sum_{\gamma} r_{\alpha\gamma} r_{\beta\gamma} = \delta_{\alpha\beta} , \quad |r_{\alpha\beta}| = 1,$$

where $|r_{\alpha\beta}|$ denotes determinant. The object ("Gebilde") in \mathbb{R}^{n^2} of $(r_{\alpha\beta})_{\alpha,\beta\in\mathbb{I}_n}$ defined by the fundamental equation (1.2) is denoted by R, which is a submanifold corresponding

to G. A measure on R is denoted by dR in [Hur1897], but permit me to denote it also as $dg (g \in G)$. When the substitution ("Substitution") (1.1) is put together with

(1.3)
$$x'_{\alpha} = \sum_{\beta} s_{\alpha\beta} x''_{\beta}, \qquad h = (s_{\alpha\beta})$$

we have $gh = (r'_{\alpha\beta})$ with

(1.4)
$$x_{\alpha} = \sum_{\beta} r'_{\alpha\beta} x''_{\beta}, \qquad r'_{\alpha\beta} = \sum_{\gamma} r_{\alpha\gamma} s_{\gamma\beta}.$$

Now consider a form

(1.5)
$$\Phi(a;x) = \sum_{1 \le k \le m} a_j \, p_j(x)$$

of polynomials in $x = {}^t(x_1, x_2, \ldots, x_n)$ of homogeneous degree $p \geq 0$ with coefficients $a = {}^{t}(a_1, a_2, \dots, a_m), \text{ where }$

$$m = \frac{(n+p-1)!}{p!(n-1)!},$$
 $p_j(x)$ different monomials of degree p ,

When x is replaced by x' under (1.1), we have

$$p_{j}(x) \rightarrow p\left(\sum_{\gamma} r_{1\beta} x'_{\beta}, \sum_{\gamma} r_{2\beta} x'_{\beta}, \dots, \sum_{\gamma} r_{n\beta} x'_{\beta}\right)$$

$$= \sum_{k} P_{g}^{jk} p_{k}(x'), \qquad P_{g} := (P_{g}^{jk})_{j,k \in \mathbf{I}_{m}},$$

$$(1.6) \qquad \qquad \therefore \qquad \Phi(a; x) = \Phi(a'; x'), \qquad a' = {}^{t}P_{g}^{-1}a.$$

Here we have $P_g P_h = P_{gh}$, and every element of P_g is a polynomial in $r_{\alpha\beta}$. Now let F(a) be an arbitrary polynomial in $a = {}^t(a_1, a_2, \ldots, a_m)$ of coefficients in $\Phi(a;x)$. Then, under "Substitution": $x \to x'$ in (1.1), F(a) is transformed to F(a'). The first assertion of Hurwitz is written in p.74, with an invariant measure dR on R, as

Das über das Gebilde Rausgedehte Integral

(1.7)
$$J(a) = \int F(a') dR$$

stellt nun eine ortogonale Invariante der Form $\Phi(a;x)$ dar.

Every "orthogonale Invariante der Form $\Phi(a;x)$ " is given by this integration.

But "das Abzählungsproblem der Anzahl der orthogonalen Invarianten der Form $\Phi(a;x)$ " cannot be solved at this stage, whereas the finiteness of this number is secured by the first fundamental theorem of Hilbert.

Explanation 1.1. Let π be a natural matrix representation $\pi(g) := g$ $(g \in G)$ on the representation space $V(\pi) := \mathbb{C}^n$. Then the space of polynomials of degree r of variables x_1, x_2, \ldots, x_n is nothing but the space of symmetric tensor product $S^rV(\pi)$ in $\otimes^rV(\pi)$. The form $\Phi(a;x)$ gives the dual pairing of $S^rV(\pi)$ and its dual $(S^rV(\pi))^*$, and both spaces receive the natural action of G = SO(n), and the actions of both sides are denoted by T(g) and $T^{\vee}(g) := {}^tT(g)^{-1}$ respectively. The replacement F(a) by F(a') can be expressed as $(g^{-1}F)(a) := F(a') = F(T^{\vee}(g)a)$, and the integration (1.7) is expressed as

(1.8)
$$J(a) = \int_{G} (g^{-1}F)(a) dg \quad \text{or} \quad J = \int_{G} g^{-1}F dg \ \Big(= \int_{G} gF dg \Big).$$

Das Abzählungsproblem für "orthogonale Invariante der Form $\Phi(a;x)$ " is proposed.

1.2 On '§2 Global coordinates on SO(n) and formula of invariant measure'

In general, let $\xi = (\xi_1, \xi_2, \dots, \xi_{\tau})$ be orthonormal coordinates in τ -dimensional \mathbf{R}^{τ} and consider a submanifold \mathfrak{R} of dimension σ defined by a formula

(1.9)
$$\xi_i = \varphi_i(p_1, p_2, \dots, p_\sigma) \qquad (i \in \mathbf{I}_\tau).$$

Then the line element on \mathfrak{R} defined by

(1.10)
$$ds^{2} := d\xi_{1}^{2} + d\xi_{2}^{2} + \dots + d\xi_{\tau}^{2} = \sum_{\lambda,\mu \in \mathbf{I}_{\sigma}} B_{\lambda,\mu} dp_{\lambda} dp_{\mu}$$

is invariant under an orthogonal transformation $R \in O(\tau)$ leaving stable the submanifold \mathfrak{R} . Further define a volume element on \mathfrak{R} using square-root of the discriminant of the quadratic form (1.10) as

(1.11)
$$\sqrt{|B_{\lambda,\mu}|} \, dp_1 \, dp_2 \cdots dp_\sigma \, .$$

Then it is invariant under R too.

Hurwitz applied this generality to the case of $G = SO(n) \subset \mathbb{R}^{n^2}$ with the defining equation (1.2), where $\tau = n^2$, $\sigma = n(n-1)/2$. For $1 \le \alpha < n$, denote by $E_{\alpha}(\varphi)$ the 2-dimensional rotation of angle φ in the space of $(x_{\alpha}, x_{\alpha+1})$ given as

(1.12)
$$\begin{cases} x_{\alpha} = \cos \varphi \ x'_{\alpha} + \sin \varphi \ x'_{\alpha+1}, \\ x_{\alpha+1} = -\sin \varphi \ x'_{\alpha} + \cos \varphi \ x'_{\alpha+1}, \\ x_{\beta} = x'_{\beta} \qquad (\beta \neq \alpha, \alpha + 1). \end{cases}$$

Introduce n(n-1)/2 angles as

$$(1.13) \varphi_{0,1}; \varphi_{0,2}, \varphi_{1,2}; \varphi_{0,3}, \varphi_{1,3}, \varphi_{2,3}; \dots; \varphi_{0,n-1}, \varphi_{1,n-1}, \dots, \varphi_{n-2,n-1}$$

and take orthogonal transformations as

(1.14)
$$\begin{cases} E_1 &= E_{n-1}(\varphi_{01}), \\ E_2 &= E_{n-2}(\varphi_{12})E_{n-1}(\varphi_{02}), \\ E_3 &= E_{n-3}(\varphi_{23})E_{n-2}(\varphi_{13})E_{n-1}(\varphi_{03}), \\ \cdots &= \cdots \\ E_{n-1} &= E_1(\varphi_{n-2,n-1})E_2(\varphi_{n-3,n-1})\cdots E_{n-1}(\varphi_{0,n-1}), \end{cases}$$

and put

$$(1.15) R = E_1 E_2 \cdots E_{n-1}.$$

When the parameter φ_{rs} $(0 \le r < s < n)$ runs over

(1.16)
$$0 \le \varphi_{0s} < 2\pi, \quad 0 \le \varphi_{rs} < \pi \ (1 \le r < s < n),$$

the element $R = (r_{\alpha\beta})_{\alpha,\beta\in I_n}$ runs over G once. Put $ds^2 = \sum_{\alpha} (dr_{\alpha\beta})^2$ as the line element on G.

The main results in this case are the following:

$$(1.17) dR = 2^{\frac{n(n-1)}{4}} \prod_{0 \le r < s \le n} (\sin \varphi_{rs})^r d\varphi_{rs},$$

(1.17)
$$dR = 2^{\frac{n(n-1)}{4}} \prod_{0 \le r < s < n} (\sin \varphi_{rs})^r d\varphi_{rs},$$

$$M = \int_G dR = \frac{2^{\frac{(n-1)(n+4)}{4}} \cdot \pi^{\frac{n(n+1)}{4}}}{\Gamma(1/2)\Gamma(2/2)\Gamma(3/2)\cdots\Gamma(n/2)}.$$

These formulas are very interesting, but they were not cited, except by Schur, in modern papers which I read until now.

Another form of global coordinates on SO(n)1.3

The above explicit form of global coordinates is not so familiar with me, and taking into account the later result by Schur, I would like to give another form of it well visible from a geometric point of view. For $i, j \in I_n$, $i \neq j$, take two-dimensional rotation $r_{ij}(\phi)$, in the space spanned by (x_i, x_i) , given as

$$r_{ij}(\phi)$$
: $\begin{pmatrix} x'_i \\ x'_j \end{pmatrix} \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x_i \\ x_j \end{pmatrix}$.

Put $G_n = SO(n)$ and consider a descending series of subgroups $G_n \subset G_{n-1} \subset \cdots G_3 \subset G_2$ and the corresponding series of quotient spaces $S^{n-1} = G_n/G_{n-1}$, $S^{n-2} = G_{n-1}/G_{n-2}$, \cdots , $S^2 = G_3/G_2$. Take a rotation $r_{j-1} \in G_j$ $(2 \le j \le n)$ given below as

$$r_{n-1} = r_{12}(\phi_{1,n-1})r_{23}(\phi_{2,n-1})\cdots r_{n-1,n}(\phi_{n-1,n-1})r_{n-2} = r_{12}(\phi_{1,n-2})r_{23}(\phi_{2,n-2})\cdots r_{n-2,n-1}(\phi_{n-2,n-2})r_{n-2,n-2}, \cdots = \cdots r_{n-2,n-1}(\phi_{n-2,n-2})r_{n-2,n-2}, \cdots = r_{n-2,n-2}(\phi_{n-2,n-2})r_{n-2,n-2}, \cdots = r$$

Then we have the following

Theorem 1.2. Put a general element u in $G_n := SO(n)$ as

$$u = r_{n-1}r_{n-2}\cdots r_2r_1.$$

For a fixed j $(1 \le j < n)$, when the parameter $\phi_{1,j}, \phi_{2,j}, \cdots, \phi_{j,j}$ of r_j runs over

$$0 \le \phi_{i,j} \le \pi \ (1 \le i < j), \quad 0 \le \phi_{i,j} < 2\pi,$$

it gives a spherical coordinates on j-dimensional sphere $S^j \cong G^{j+1}/G^j$. From the total point of view over $1 \leq j < n$, the radius of sphere decreases smaller and smaller depending on j, and when $(\phi_{ij})_{1 \leq i \leq j < n}$ moves over the above region, the total image of u above covers the whole of $G_n = SO(n)$.

Proof. Let (e_1, e_2, \cdots, e_n) be the standard orthonormal frame in a Euclidean space E^n around the origin and put $(e'_1, e'_2, \cdots, e'_n)$, $e'_i := ue_i$. Then u is determined by this frame. As is known rather well, the unit vector e'_n is obtained as $r_{n-1}^{-1}e'_n = e_n$ with an appropriate parameter $(\phi_{i,n-1})_{1 \leq i \leq n-1}$. Take a new frame $(e''_1, e''_2, \dots, e''_n)$, $e''_i = r_{n-1}^{-1}e'_i$, then we have $e''_n = e_n$. Here, replacing $S^{n-1} \subset E^n$ by $S^{n-2} \subset E^{n-1}$, we repeat the same discussion as above, that is, choose r_{n-2} so that $(r_{n-2})^{-1}e''_{n-1} = e_{n-1}$. Again we choose a new frame $e_i^{(3)} := (r_{n-2})^{-1}e''_i$ $(i \in I_n)$ in such a way that $e_i^{(3)} = e_i$ (i = n - 1, n). This time, we discuss on $S^{n-3} \subset E^{n-2}$. Thus, repeating this process, we arrive at the last stage such that $(r_1^{-1}r_2^{-1}\cdots r_{n-1}^{-1})e'_i = e_i$ $(i \in I_n)$. This gives $u = r_{n-1}\cdots r_2r_1$.

1.4 On '§3 Global coordinates on SU(n) and formula of invariant measure'

Now return to the original [Hur1897]. Denote by a^0 the conjugate of a complex number a. Then an element $T = (c_{\alpha\beta})$ of the group G = SU(n) is defined as

(1.19)
$$\sum_{\gamma} c_{\gamma\alpha} c_{\gamma\beta}^0 = \delta_{\alpha\beta} \; ; \quad |c_{\alpha\beta}| = 1.$$

Introduce a two-dimensional unitary transformation in $(x_{\alpha}, x_{\alpha+1})$ -space as

(1.20)
$$\begin{cases} x_{\alpha} = ax'_{\alpha} + bx'_{\alpha+1}, \\ x_{\alpha+1} = -b^{0}x'_{\alpha} + a^{0}x'_{\alpha+1}, \\ x_{\beta} = x'_{\beta}, \end{cases}$$

(1.21)
$$a = \cos \varphi e^{\psi i}, \quad b = \sin \varphi e^{\chi i}, \quad a^0 = \cos \varphi e^{-\psi i}, \quad b^0 = \sin \varphi e^{-\chi i},$$

which gives all transformation in SU(2). Denote it by $E_{\alpha}(\varphi, \psi, \chi)$, and put

(1.22)
$$\begin{cases} E_{1} &= E_{n-1}(\varphi_{01}, \psi_{01}, \chi_{1}), \\ E_{2} &= E_{n-2}(\varphi_{12}, \psi_{12}, 0)E_{n-1}(\varphi_{02}, \psi_{02}, \chi_{2}), \\ E_{3} &= E_{n-3}(\varphi_{23}, \psi_{23}, 0)E_{n-2}(\varphi_{13}, \psi_{13}, 0)E_{n-1}(\varphi_{03}, \psi_{03}, \chi_{3}), \\ \cdots &= \cdots, \\ E_{n-1} &= E_{1}(\varphi_{n-2,n-1}, \psi_{n-2,n-1}, 0)E_{2}(\varphi_{n-3,n-1}, \psi_{n-3,n-1}, 0) \cdots \\ &\cdots & E_{n-2}(\varphi_{1,n-1}, \psi_{1,n-1}, 0)E_{n-1}(\varphi_{0,n-1}, \psi_{0,n-1}, \chi_{n-1}), \end{cases}$$

$$(1.23)$$

$$T = E_{1}E_{2}E_{3} \cdots E_{n-1},$$

where the parameters run over the region

(1.24)
$$0 \le \varphi_{rs} < \frac{\pi}{2}, \quad 0 \le \psi_{rs} < 2\pi, \quad 0 \le \chi_s < 2\pi.$$

Then, by simple calculation, we have

$$\begin{array}{lll} E_{1}e'_{n} & = & -e^{i\varphi_{0,1}}\sin\psi_{0,1}\,e^{-i\chi_{1}}e'_{n-1} + e^{-i\varphi_{0,1}}\cos\psi_{0,1}\,e^{-i\chi_{1}}e'_{n}\,, \\ E_{2}e'_{n} & = & e^{i(\varphi_{1,2}+\varphi_{0,2})}\sin\psi_{1,2}\sin\psi_{0,2}\,e^{-i\chi_{2}}e'_{n-2} \\ & & -e^{i(-\varphi_{1,2}+\varphi_{0,2})}\cos\psi_{1,2}\sin\psi_{0,2}\,e^{-i\chi_{2}}e'_{n-1} + e^{-i\varphi_{0,2}}\cos\psi_{0,2}\,e^{-i\chi_{2}}e'_{n}\,, \\ \cdots & = & \cdots \cdots \\ E_{n-1}e'_{n} & = & (-1)^{n-1}e^{i(\varphi_{n-2,n-1}+\cdots+\varphi_{2,n-1}+\varphi_{1,n-1}+\varphi_{0,n-1})}\sin\psi_{n-2,n-1}\sin\psi_{n-3,n-1}\cdots \\ & & \times\sin\psi_{2,n-1}\sin\psi_{1,n-1}\sin\psi_{0,n-1}\,e^{-i\chi_{n-1}}e'_{1} + \\ & & (-1)^{n-2}e^{i(-\varphi_{n-2,n-1}+\cdots+\varphi_{2,n-1}+\varphi_{1,n-1}+\varphi_{0,n-1})}\cos\psi_{n-2,n-1}\sin\psi_{n-3,n-1}\cdots \\ & & \times\sin\psi_{2,n-1}\sin\psi_{1,n-1}\sin\psi_{0,n-1}\,e^{-i\chi_{n-1}}e'_{2} \\ & \pm & \cdots \cdots \\ & & + e^{i(-\varphi_{2,n-1}+\varphi_{1,n-1}+\varphi_{0,n-1})}\cos\psi_{2,n-1}\sin\psi_{1,n-1}\sin\psi_{0,n-1}\,e^{-i\chi_{n-1}}e'_{n-2} \\ & & - e^{i(-\varphi_{1,n-1}+\varphi_{0,n-1})}\cos\psi_{1,n-1}\sin\psi_{0,n-1}\,e^{-i\chi_{n-1}}e'_{n-1} \\ & & + e^{-i\varphi_{0,n-1}}\cos\psi_{0,n-1}\,e^{-i\chi_{n-1}}e'_{n}\,, \end{array}$$

and see that $E_{n-1}e'_n$ covers once $SU(n)/SU(n-1) \cong B^{n-1} := \{x \in \mathbb{C}^n ; ||x|| = 1\}$. Here for unitary group SU(n), the line element ds^2 chosen for $T = (c_{\alpha\beta})$ is

$$ds^{2} = \sum_{\alpha,\beta} dc_{\alpha\beta} dc_{\alpha\beta}^{0} \qquad (c_{\alpha\beta}^{0} := \overline{c_{\alpha\beta}})$$

and the corresponding invariant measure dT in the global coordinates $(\varphi_{rs}, \psi_{rs}, \chi_s)$ is

(1.25)
$$dT = \sqrt{n!} \ 2^{\frac{n(n-1)}{2}} \cdot \prod_{r,s} \cos \varphi_{rs} \left(\sin \varphi_{rs} \right)^{2r+1} d\varphi_{rs} \ d\psi_{rs} \cdot \prod_{s} d\chi_{s} .$$

1.5 On '§4 General theory for Lie groups'

In this last part pp.86–90 of [Hur1897], I feel that Hurwitz discussed generally for Lie groups, maybe in the direction of something like Maurer-Cartan formula. But I could not follow it up in detail.

2 Schur's simplification of Representation Theory for finite groups

In the paper [Sch05], Schur simplified in a very reasonable way the theory of characters due to Frobenius. As will be explained a little later, Schur's method for finite groups can be applied directly to a compact group G, if the existence of invariant integrals on G is admitted.

2.1 Frobenius' first three papers

The world's first foundation of Theory of Representations of Groups begun with three papers [Fro1896a], [Fro1896b] and [Fro1897] of Frobenius. We survey shortly them as preceding results of Schur.

2.1.1. Survey of the first paper [Fro1896a].

Let \mathfrak{H} be a finite group with the identity element E, $h := |\mathfrak{H}|$ its order, α a conjugacy class, $h_{\alpha} := |\alpha|$ the order of α , and $\alpha' := \{A^{-1}; A \in \alpha\}$. Put, for three conjugacy classes α, β, γ ,

$$(2.1) h_{\alpha\beta\gamma} := \sharp \{ (A, B, C) ; A \in \alpha, B \in \beta, C \in \gamma, ABC = E \}$$

Let k be the number of conjugacy classes, then the set (χ_{α}) of k numbers is called a Gruppencharakter if it satisfies the following equation:

(2.2)
$$h_{\beta}h_{\gamma}\chi_{\beta}\chi_{\gamma} = f\sum_{\alpha}h_{\alpha'\beta\gamma}\chi_{\alpha}, \qquad \text{(character equation)}$$

where f is called as *Proportionalitätsfactor*. Using results in [F51],⁴⁾ he proved, as the main result in §2, that there exist exactly k different solutions as

(2.3)
$$\chi_{\alpha} = \chi_{\alpha}^{(\kappa)}, \quad f = f^{(\kappa)} \qquad (\kappa = 0, 1, \dots, k - 1)$$

and $k \times k$ type determinant $|\chi_{\alpha}^{(\kappa)}| \neq 0$.

Explanation 2.1. In modern languages. Taking into account the results in §5, I can explain as follows.

Let $Conj(\mathfrak{H}) := \{\alpha\}$ be the set of all conjugacy classes of \mathfrak{H} and put functions on \mathfrak{H} as

$$F^{(\kappa)}(A) := (f^{(\kappa)})^{-1} \chi_{\alpha}^{(\kappa)} \quad \text{for} \quad A \in \alpha \subset \mathfrak{H}$$

starting from $(\chi_{\alpha}^{(\kappa)})$. On the other hand, consider the group algebra K[G] of a group G

⁴⁾ [F51] Über vertauschbare Matrizen, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin 1896, 601–614. [F51] denotes the number of paper in Gesammelte Abhandlungen.

with coefficient in $K = \mathbf{Z}, \mathbf{R}$ or \mathbf{C} . Then we know that the center $K[\mathfrak{H}]^o$ is spanned by elements $X_{\alpha} := \sum_{A \in \alpha} A \ (\alpha \in \operatorname{Conj}(\mathfrak{H}))$. The defining relations are calculated as

(2.4)
$$X_{\beta} \cdot X_{\gamma} = \sum_{\alpha \in \operatorname{Conj}(\mathfrak{H})} \frac{h_{\alpha'\beta\gamma}}{h_{\alpha}} X_{\alpha}.$$

Put $e_{\alpha} := h_{\alpha}^{-1} X_{\alpha}$, then we have $h_{\beta} h_{\gamma} e_{\beta} e_{\gamma} = \sum_{\alpha} h_{\alpha'\beta\gamma} e_{\alpha}$. So the function $F^{(\kappa)}(A)$ for $A \in \mathfrak{H}$ gives a representation of the commutative algebra $\mathbf{Z}[\mathfrak{H}]^{o}$, because

$$h_{\beta}h_{\gamma}F^{(\kappa)}(B)F^{(\kappa)}(C) = \sum_{\alpha} h_{\alpha'\beta\gamma}F^{(\kappa)}(A).$$

This explains the birthplace of the character equation, and $F^{(\kappa)}$ is normalized as $F^{(\kappa)}(E) = 1$ at the identity element E.

For direct relation with characters of irreducible linear representations of the group \mathfrak{H} , see the survey below of the third paper [Fro1897].

In §§8–10, all *Gruppencharakters* are calculated explicitly for groups, \mathfrak{A}_4 , $\mathfrak{S}_4/\mathbb{Z}_2^2$, \mathfrak{S}_4 , \mathfrak{A}_5 , \mathfrak{S}_5 (in §8), and for groups $PSL(2,\mathbb{Z}_p)$ with p odd prime (in §§9–10).

2.1.2. Survey of the second paper [Fro1896b].

This paper consists of Introduction and §§1–12. Let \mathfrak{H} be a group and h its order. Consider a set of variables $x = (x_P)_{P \in \mathfrak{H}}$ with index $P \in \mathfrak{H}$, and $h \times h$ type determinant

$$\Theta(x) := \det(x_{PQ^{-1}}),$$

where P, Q run over \mathfrak{H} . This is called as *Gruppendeterminante*, and is the main object to be studied here. First Θ is divisible by a linear form $\xi := \sum_{R \in \mathfrak{H}} x_R$, as is easily seen. In Introduction, main results of the paper are summarized, and I explain them below.

(1) The number of prime factors of $\Theta(x)$ is equal to the number k of conjugacy classes of \mathfrak{H} . Let them be $\Phi^{(\lambda)}(x)$ $(1 \le \lambda \le k)$ with $f^{(\lambda)} := \dim \Phi^{(\lambda)}$. Then Θ is decomposed as

$$\Theta = \prod_{1 \le \lambda \le k} \left(\Phi^{(\lambda)} \right)^{e^{(\lambda)}}.$$

- (2) The exponent $e^{(\lambda)}$ is equal to the dimension $f^{(\lambda)}$.
- (3) The dimension $f^{(\lambda)}$ divides the order h.
- (4) By a certain linear transformation, $\Phi^{(\lambda)}$ becomes a function of $(f^{(\lambda)})^2$ number of variables.
- (5) Collecting all such variables over $1 \le \lambda \le k$, we obtain just $h = \sum_{\lambda} (f^{(\lambda)})^2$ number of independent variables.
- (6) Put, for every conjugacy class α , $x_A = x_B$ for $A, B \in \alpha$. Then we have just k number of variables. For a prime factor $\Phi^{(\lambda)}(x)$, there exists a Charakter $(\chi_{\alpha}^{(\lambda)})_{\alpha \in \text{Conj}(\mathfrak{H})}$

having the following property: define a function $\chi^{(\lambda)}$ on \mathfrak{H} as $\chi^{(\lambda)}(A) := \chi_{\alpha}^{(\lambda)}$ for $A \in \alpha$, and a linear function $\xi^{(\lambda)}$ as

$$\xi^{(\lambda)} := \frac{1}{f^{(\lambda)}} \sum_{R} \chi^{(\lambda)}(R) x_R,$$

then $\Phi^{(\lambda)} = \xi^{(\lambda)} f^{(\lambda)}$.

- (7) The set of functions $\xi^{(\lambda)}$, $1 \leq \lambda \leq k$, are mutually linearly independent.
- (8) Charakter $(\chi_{\alpha}^{(\lambda)})$ determines $\Phi^{(\lambda)}(x)$ for general $x = (x_R)_{R \in \mathfrak{H}}$ completely. Studies on $\Theta(x)$ can be reduced to those of $\xi^{(\lambda)}$'s, in particular, when $x_{AB} = x_{BA}$ $(A, B \in \mathfrak{H})$,

$$\Theta = \prod_{\lambda} \left(\xi^{(\lambda)} \right)^{f^{(\lambda)^2}}.$$

(9) The calculation of the above decomposition of degree h can be reduced to that of size k determinant for $(\alpha, \beta) \in \text{Conj}(\mathfrak{H}) \times \text{Conj}(\mathfrak{H})$ given as

$$\det\left(\sum_{\gamma} \frac{1}{h_{\alpha}} h_{\alpha\beta'\gamma} x_{\gamma}\right)_{\alpha,\beta} = \prod_{\lambda} \xi^{(\lambda)}.$$
In fact, since
$$h_{\beta} h_{\gamma} \chi_{\beta}^{(\lambda)} \chi_{\gamma}^{(\lambda)} = f^{(\lambda)} \sum_{\alpha} h_{\alpha'\beta\gamma} \chi_{\alpha}^{(\lambda)},$$
and
$$\xi^{(\lambda)} = \frac{1}{f^{(\lambda)}} \sum_{\gamma} h_{\gamma} \chi_{\gamma}^{(\lambda)} x_{\gamma},$$
we have
$$h_{\alpha} \chi_{\alpha}^{(\lambda)} \xi^{(\lambda)} = \frac{1}{f^{(\lambda)}} \sum_{\gamma} h_{\alpha} \chi_{\alpha}^{(\lambda)} h_{\gamma} \chi_{\gamma}^{(\lambda)} x_{\gamma}$$

$$= \sum_{\gamma} \sum_{\beta} h_{\alpha\gamma\beta'} \chi_{\beta}^{(\lambda)} x_{\gamma} = \sum_{\beta} \left(\sum_{\gamma} h_{\alpha\beta'\gamma} x_{\gamma}\right) \chi_{\beta}^{(\lambda)},$$

$$\therefore \sum_{\beta} \left(\sum_{\gamma} h_{\alpha\beta'\gamma} x_{\gamma} - h_{\alpha\beta'} \xi^{(\lambda)}\right) \chi_{\beta}^{(\lambda)} = 0,$$

$$\therefore \det\left(\frac{1}{h_{\alpha}} \sum_{\gamma} h_{\alpha\beta'\gamma} x_{\gamma} - \delta_{\alpha\beta'} r\right)_{\alpha,\beta'} = 0 \quad \text{for } r = \xi^{(\lambda)} \quad (1 \le \lambda \le k) \text{ (eigenvalues)}.$$

Explanation 2.2. Personal impression on [Fro1896a] and [Fro1896b].

If we recognize that the matrix determinant $\Theta(x)$ expresses the left regular representation of a finite group \mathfrak{H} on $\ell^2(\mathfrak{H})$, then the results from (1) to (9) listed above are familiar with us, except the assertion (3), that is, f divides h. However, their proofs are all purely algebraic, and very new and refreshing for me, and also very difficult to read.

So, comparing to my personal self-education process in my student age, this kind of composition of the theory of group representations is rather astonishing and I never knows that it is possible, I mean, the order of introducing: firstly character equation and secondly matrix determinant, and linear representation of \mathfrak{H} is hidden in the backyard.

2.1.3. Survey of the third paper [Fro1897].

In §1, Charakter of a Group \mathfrak{H} is redefined as a function $\chi(R)$ on \mathfrak{H} by a system of equations as follows: with $h = |\mathfrak{H}|$,

$$\chi(E) = f,$$

(2.6)
$$\chi(AB) = \chi(BA) \qquad (A, B \in \mathfrak{H}),$$

(2.7)
$$h\chi(A)\chi(B) = f\sum_{R} \chi(AR^{-1}BR),$$

(2.8)
$$h = \sum_{R} \chi(R) \chi(R^{-1}).$$

In §2, Darstellung durch lineare Substitutionen of a group \mathfrak{H} is defined as a correspondence of $A \in \mathfrak{H}$ to a regular matrix (A) satisfying (A)(B) = (AB) for $A, B \in \mathfrak{H}$. Introducing variables x_R $(R \in \mathfrak{H})$, Frobenius associates a determinant $F(x) := |\sum (R)x_R|$ and studies its factorization into prime factors, and assert that

"every prime factor appears in Gruppendeterminante $\Theta(x)$."

A Darstellung (R) is called *prime* if the polynomial F(x) is prime.

In §4, from a representation (R), its (trace) character $\chi(R)$ is defined as $\chi(R) := \operatorname{tr}(R)$.

In §6, the relation of characters χ with linear representations (R) is clarified. For any conjugacy class α of \mathfrak{H} , the operator $J_{\alpha} := \sum_{A \in \alpha} (A)$ commutes with any (R), $R \in \mathfrak{H}$.

So, if the representation is prime (or irreducible), J_{α} must be a scalar multiple of the identity operator (E), and so a number χ_{α} can be defined as

On the other hand, the relation (2.4) gives us

(2.10)
$$J_{\beta} \cdot J_{\gamma} = \sum_{\alpha} \frac{h_{\alpha'\beta\gamma}}{h_{\alpha}} J_{\gamma} \quad \text{and so} \quad h_{\beta}\chi_{\beta} \cdot h_{\gamma}\chi_{\gamma} = f \sum_{\alpha} h_{\alpha'\beta\gamma} h_{\alpha}\chi_{\alpha}.$$

This last equation is just equal to the character equation in (2.2).

My personal impression on the third paper [Fro1897].

In this paper, the fundamental notions of linear representations and their (trace) characters are introduced. The relations with group algebra $K[\mathfrak{H}]$ and its center $K[\mathfrak{H}]^o$ and also with intertwining operators are studied, always in algebraic way. Important parts are closely related to the results in the previous two papers. So, in total, I feel, the side of treatments from the stand points of functional analytic ways is weak, and is still basically under way of developing.

2.2 Schur's simplification of Character Theory for finite groups

As explained above, the above three papers of Frobenius are difficult to read and not easy to approach. Schur, a pupil of Frobenius, wrote the paper [Sch05], wishing to dissolve

them, apart from efforts of Burnside. Schur changed the order of introducing notions, first linear representations, not necessary irreducible, and then their characters and irreducible decompositions and so on, as is in modern text books. Thus he aimed to facilitate and transparentize the theory. This method is better, not only for finite groups but also for general groups.

Schur himself wrote at the beginning of the paper as

Die vorligende Arbeit enthält eine durchaus elementare Einhürung in die Hrn. FROBENIUS begründete Theorie der Gruppencharaktere, die auch als die Lehre von Darstellung der endlichen Gruppen durch lineare homogene Substitutionen bezeichnet werden kann.

The paper consists of Preface and Sections 1 to 6 and well-organized. It contains 17 Assertions written in *italic* and numbered from I to XVII, and Important Equalities numbered as $(I.) \sim (III.)$, (III'.), $(VI.) \sim (VI)$, (VII.), $(VII.) \sim (XIV.)$

We notice three important results of general character.

First one is the so-called Schur's lemma on intertwining operators between two irreducible representations. This corresponds to Assertions I and II.

Second one is the so-called *orthogonality relation* for matrix elements of irreducible representations. This corresponds Equalities (I.) and (II.).

Third one is also *orthogonality relation* among irreducible characters. ⁵⁾ This corresponds to Equalities (IX.), (X.) and (XII.).

Let us explain a little more in detail. Schur used repeatedly the averaging of a function f on a finite group G. Despite of losing the good flavor of the classics, I use intentionally modern notation of integration in such a way that

(2.11)
$$\int_{G} f(g) dg := \frac{1}{|G|} \sum_{g \in G} f(g).$$

Here dg denotes the normalized invariant measure on G. In the space of functions C(G) on G, introduce inner product as

(2.12)
$$\langle f_1, f_2 \rangle := \int_G f_1(g) \, \overline{f_2(g)} \, dg \,,$$

then we get a Hilbert space $L^2(G)$. In the main part of the paper, Schur proved, in principle, the following assertions (which I write in modern languages in a form of theorem, for the sake of brevity):

Theorem 2.3. (i) Let π be a linear representation of G on a vector space $V(\pi)$. Then an inner product can be introduced in $V(\pi)$ in such a manner that $\pi(g)$, $g \in G$, are all unitary, that is, π becomes a unitary representation.

(ii) In $V(\pi)$, take a complete orthonormal system and, with respect to it, express $\pi(g)$ by a unitary matrix $(t^{\pi}_{\alpha\beta})_{1 \leq \alpha, \beta \leq d_{\pi}}$, with $d_{\pi} = \dim \pi$ $(t^{\pi}_{\alpha\beta})_{1 \leq \alpha, \beta \leq d_{\pi}}$. Suppose

⁵⁾ trace characters of irreducible linear representations.

 π is irreducible, and take another irreducible unitary representation ρ , not equivalent to π . Then

(2.13)
$$\langle t_{\alpha\beta}^{\pi}, t_{\gamma\delta}^{\pi} \rangle = \frac{1}{d_{\pi}} \, \delta_{\alpha,\beta} \, \delta_{\gamma,\delta} \qquad (1 \le \alpha, \beta, \gamma, \delta \le d_{\pi}),$$

(2.14)
$$\langle t_{\alpha\beta}^{\pi}, t_{\gamma\delta}^{\rho} \rangle = 0 \qquad (1 \le \alpha, \beta \le d_{\pi}, \ 1 \le \gamma, \beta \le d_{\rho}).$$

(iii) The character of linear representation π is defined as $\chi_{\pi}(g) := \operatorname{tr}(\pi(g))$ $(g \in G)$, which is invariant under inner automorphisms. Suppose π is irreducible and ρ is another irreducible representation not equivalent to π . Then

(2.15)
$$\|\chi_{\pi}\|^2 = \langle \chi_{\pi}, \chi_{\pi} \rangle = 1, \qquad \langle \chi_{\pi}, \chi_{\rho} \rangle = 0 \quad \text{or} \quad \chi_{\pi} \perp \chi_{\rho},$$

where ||f|| denotes the norm of f in $L^2(G)$.

I found that Schur's methods themselves for proving these assertions, can be directly applied to any group G if it has a *finite invariant measure* on it. For instance, take the equalities (2.13) and (2.14) above, which correspond respectively Equalities (I.) and (II.) in [Sch05]. I will rewrite Schur's proof on two pages of [Sch05] by using integral symbol $\int_G f(g)dg$ in place of $\sum_{g\in G} f(g)$. Then it looks like as follows:

Let π be an irreducible matrix representation of dimension $f (= d_{\pi})$. Take an arbitrary matrix $U = (u_{\alpha\beta})$ of degree f, and put

(3.)
$$V = \int_{G} \pi(g^{-1}) U \pi(g) \, dg \,.$$

Then, for any $g_0 \in G$, $\pi(g_0^{-1})V\pi(g_0) = \int_G \pi(gg_0)^{-1}U\pi(gg_0) dg$. Since the measure dg is right-invariant, this integral is equal to $\int_G \pi(g^{-1})U\pi(g) dg = V$, whence $\pi(g_0^{-1})V\pi(g_0) = V$ for any $g_0 \in G$. Therefore, by Schur's lemma,

$$V = vE_f$$

with scalar v and f-dimensional identity matrix E_f . Using the arbitrariness of $(u_{\alpha\beta})$, we can obtain the equality (2.13) easily.

I emphasize that this method of proof can be applied directly to any compact group G, if once the existence of invariant measure on it with finite volume is known. Similar comment is valid for the orthogonality (2.14) and (II.).

When I recognized this fact, the generality of Schur's method, first time, I was very much impressed by his originality despite of his comment "durchaus elementare Einfürung" in the top line of Introduction.

3 Schur's work and communications with Weyl

3.1 Schur's classification of irreducible representations of n-th rotation groups as application of invariant integral

After 27 years from Hurwitz's paper [Hur1897], there appeared as its application Schur's papers [Sch24a] and [Sch24b].

3.1.1 On the first paper [Sch24a].

The contents are, firstly a general theory of the method of applying invariant integrals on groups to the theory of invariant functions or forms, and secondly a general explanation for the method of classification of irreducible representations of n-th orthogonal groups \mathfrak{D} (:= SO(n)). Schur quotes Hurwitz's result and ameliorates it for a global coordinates on \mathfrak{D} and for explicit expression of invariant measures.

The second part is carried out by proving orthogonality relations of matrix elements, an extension of Theorem 2.3 (ii) for finite groups to the rotation group \mathfrak{D} . Then Schur determines irreducible characters explicitly by using the orthogonality relations among them, which is also an extension to \mathfrak{D} of Theorem 2.3 (iii) for finite groups.

To explain a little more on the contents, I first made a table of contents as below:

Erster Teil. Projektive Invarianten.

- §1. Ein Hilfssatz über unitäre Substitutionen.
- §2. Der Integrationsprozeß zur Erzeugung projektiver Invarianten.
- §3. Beziehungen zum Ω -Prozeß.

Zweiter Teil. Die Homomorphismen⁶⁾ der Drehungsgruppe und das Abzärungsproblem für Orthogonalinvarianten.

- §4. Hurwitzsche Integralkalkül.
- §5. Einige Eigenschaften der Homomorphismen der Gruppe D.
- §6. Die Grundrelationen für die enfachen Charakteristiken.⁷⁾
- §7. Das Abzählungsproblem für Orthogonalinvarianten.
- §8. Die Fälle n=2 und n=3.
- §9. Beliebige orthogonale Transformtionen.

In §9, there appear discussions for the full orthogonal groups \mathfrak{D}' (:= O(n)) too. In addition to this, I will quote the last paragraph of his Introduction:

Im Falle der Orthogonalinvarianten sheint der Hurwitzsche Integrationsprozeß keine ähnliche Umgestaltung zuzulassen. Ich zeige aber, daß gerade im diesen Falle der von Hurwitz entwickelte Kalkül noch andere wichtige Anwendungen gestattet. Insbesondere liefelt er eine elegante Lösung des "Abzählungsproblems", nämlich der Aufgabe, die genau Anzahl der zu f(a,x) gehörenden linear unhabhängigen Orthogonalinvarianten von vorgegebenem Grade r zu

⁶⁾ Here "Homomorphism" means linear representation.

⁷⁾ Here "Charakteristik" means (trace) character.

bestimmen. Man gelangt zu dieser Lösung auf dem im Falle einer endlichen Gruppe & vom Molien eingeschlagen Wege, indem man das an und für sich wichtige Studium der mit der "Drehungsgruppe" D homomorphen Gruppen linearer homogener Substitutionen weiterverfolgt und eine Theorie entwickelt, die weitgehende Analogien mit der schönen Frobeniusschen Theorie der Gruppencharaktere aufweist.

I understand, in short, that the way of calculation of Hurwitz plays an important role in 'Counting-up Problem,' for f(a, x) for the orthogonal group \mathfrak{D} , and also in 'Classification Problem' of irreducible linear representations for \mathfrak{D} through an extension of Frobenius' character theory from the case of finite groups to that of compact Lie groups \mathfrak{D} .

3.1.2 On the second paper [Sch24b].

In this paper Schur actually succeeded to classify irreducible linear representations of the rotation group $\mathfrak{D} = SO(n)$ and the full orthogonal group $\mathfrak{D}' = O(n)$. To arrive to these results, the theory of characters plays a decisive role. The paper contains Satz I to Satz X, printed in *italic*. To explain the contents, I made table of contents shown below:

After two and a half pages of Introduction,

- §1. Allgemeine Vorbemerkungen.
- §2. Die Fälle n=2 und n=3.
- §3. Eine Hilfsbetrachtung.
- §4. Die einfachen Charakteristiken der Gruppe \mathfrak{D}' .
- §5. Fortsetzung und Schuluß des Beweises.
- §6. Folgerungen aus dem Satze IV.

Quoting the third paragraph of Introduction, we continue to explain the contents of the paper:

Mein Hauptergebnis lautet: Alle stetigen Homomorphismen⁸⁾ der Gruppen $\mathfrak D$ und $\mathfrak D'$ lassen sich allein unter Benutzung ganzer rationaler Funktionen herstellen. Um eine Übersicht über die Gesamtheit der irreduziblen Gruppen linearer homogener Substitutionen zu gewinnen, die der Gruppe $\mathfrak D$ bzw. $\mathfrak D'$ homomorph sind, verfahre man folgendermaßen. Bedeutet ν die Zahl $\left[\frac{n}{2}\right]$, so bilde man für jedes System von ν nicht negativen ganzen Zahlen

(1)
$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\nu}$$

· · · · · · (I continue the explanation in my words) · · · · · · ·

⁸⁾ In these ages, Schur put an equal footing for a linear representation (say π) of G and its image $\pi(G)$, and the word "Homomorphism der Gruppe \mathfrak{D} " means linear representation of \mathfrak{D} . He does not use any symbol (such as π) to denote an irreducible linear representations, but, in place of such symbols, he uses images of it, for instance for n odd, to denote irreducible representations associated with α , he uses the symbol "eine mit \mathfrak{D} homomorphe irreducible Substitutionsgruppe $\mathfrak{G}_{\alpha_1,\alpha_2,\ldots,\alpha_{\nu}}$ in $N_{\alpha_1,\alpha_2,\ldots,\alpha_{\nu}}^{(n)}$ Variabeln".

Schur gave explicitly dimension formulas for irreducible linear representations of \mathfrak{D} and \mathfrak{D}' corresponding to $\boldsymbol{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_{\nu})$, clearly depending on the parity of n, with commentary explanations in detail. He continued as

In the next paragraph, Schur commented two papers of É. Cartan⁹⁾ and thanked H. Weyl for a notice on these Cartan's works.

A note from my point of view. The group $\mathfrak{D} = SO(n), n \geq 3$, has universal covering group $\mathrm{Spin}(n)$, and so \mathfrak{D} has two-valued representations, called spin and their highest weights $\alpha = (\alpha_i)_{i \in I_{\nu}}$ consist of half-integers (halfs of odd integers) α_i 's. In the works of Cartan, linear representations are treated on the level of Lie algebras and there does not encounter with strict differences between spin and $\mathrm{non\text{-}spin}$ cases. In these days, study of covering groups has not yet begun.

3.2 Weyl's idea inspired by Schur (an advanced announcement)

3.2.1. On the paper [Wey24].

This paper of Weyl has subtitle "From a letter to Mr. I. Schur". Actually, before this letter, Schur kindly sent to Weyl the drafts of two papers [Sch24b] and [Sch24c] before their publication, and Weyl's letter is nothing but its reply, from which the contents of this paper are taken. Weyl got to the heart of Schur's method, which I can explain as follows:

The essential of Schur's method is not global explicit expression of invariant measure on the group $\mathfrak D$ and its application, but is the integration of invariant functions (such as characters), the role of Cartan subgroups, and Weyl groups and the so-called Weyl's integration formula, not on the whole of the group $\mathfrak D$ but essentially on Cartan subgroup (cf. **Explanation 4.1** in the next section).

Weyl takes classical groups $\mathfrak{G} = SL(n, \mathbb{C})$, $\mathfrak{C} = Sp(2n, \mathbb{C})$, and $\mathfrak{D} = SO(n, \mathbb{C})$, and recognized that it is sufficient to work on their compact real forms $\mathfrak{G}_u = SU(n)$, $\mathfrak{C}_u = SpU(2n) := Sp(2n, \mathbb{C}) \cap U(2n)$, and $\mathfrak{D}_u = SO(n)$, and on their Cartan subgroups. He explains in detail how to classify irreducible unitary representations of these compact classical groups by determining irreducible characters (the so-called unitarian trick or "die unitäre Beschränkung").

3.2.2. On the third paper [Sch24c].

⁹⁾ É. Cartan, Les groupes projectifs qui ne laissent invariants aucune multiplicité plane, Bull. Soc. Math. France, **41**(1913), 53–96, and —, Les groupes projectifs continues réels qui ne laissent invariants aucune multiplicité plane, J. de Math., Sér. 6, **10**(1914), 149–186.

In Introduction, there are comments on Theory of Study on rotation invariants and Weyl's integration formula on invariant functions in comparison to Hurwitz integral. I made a list of contents as follows:

- §1. Einige Hilfsformeln.
- §2. Der vereinfachte Integralkalkül.
- §3. Der Abzählungskalkül für Orthogonalinvarianten.
- §4. Über die reellen Darstellungen der Gruppe D.

In addition, with reference to the niceness of the exchanges with Schur and Weyl, one can see about the details of these papers in my article¹⁰⁾ written in Japanese.

3.3 Schur's addresses to introduce these articles to Sitzungsberichte

Academician Schur gave addresses to introduce these articles to Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin 1924, and I feel that these addresses give testimony about interrelations among Hurwitz, Schur and Weyl. So, I would like to copy them from mimutes of the academy.¹¹⁾

Introductive address of [Sch24a] at the meeting on the 1st of January:

Der von A. Hurwitz angegebene Integralkalkül zur Erzeugung von Invarianten läßt sich im Falle der projektiven Invarianten durch einen einfacheren Kalkül ersetzen. Für den Fall der Orthogonalinvarianten liefelt die Hurwitzsche Methode auch eine Lösung des Abzählungsproblems. Dies gelingt, indem man für die Homomorphismen der Gruppe der reellen orthogonalen Substitutionen eine Theorie entwickelt, die weitgehende Analogien mit der Frobeniusschen Theorie der Gruppencharaktere aufweist.

Introductive address of [Sch24b] on the 13th of November:

Mit Hilfe der vom Verfasser früher entwickelten analytischen Methode werden alle irreduziblen Darstellungen der reellen Drehungsgruppe durch lineare homogene Substitutionen näher bestimmt und die zugehörigen Variabelnanzahlen genau berechnet. Für die Gruppe aller reellen orthogonalen Substitutionen läßt sich auch die Gesamtheit aller einfachen Charakteristiken angeben.

Introductive addresses of [Wey24] and [Sch24c] on the 11th of December:

For [Wey24].

Durch eine Modifikation der von Hrn. Schur im Falle der reellen Drehungsgruppe entwickelten Methode gelingt es dem Verfasser, für alle einfachen und

¹⁰⁾ T. Hirai, On three papers [S51], [S52], [S53] of Schur and a paper [W61] of Weyl (in Japanese), in Proc. of the 27th Symp. on History of Mathematics, held October 2016, Report of Tsuda University IMCS, 38(2017), 50–67. https://www2.tsuda.ac.jp>math>suugakushi>sympo27

¹¹⁾ F. Frobenius (1849/10/26 - 1917/08/03), A. Hurwitz (1853/03/26 - 1919/11/18),

I. Schur (1875/01/10 - 1941/01/10), H. Weyl (1885/11/09 - 1955/12/08).

halbeinfachen kontinuierlichen Gruppen das schon von Hrn. E. Cartan behandelte Darstellungsproblem in abgeschlossenerer Form zu lösen.

For [Sch24c].

Für die reellen Drehungsgruppe wird auf analytischem Wege eine Umgestaltung des Hurwitzschen Integralkalküls gewonnen, die, wie Hr. Weyl durch eine geometrische Betrachtung gezeigt hat, auch bei allen anderen im Betracht kommenden Gruppen durchgeführt werden kann.

4 Weyl's classification of irreducible representations of complex simple Lie groups

By the method announced in [Wey24], Weyl succeeded to classify (finite-dimensional) irreducible representations of all classical simple groups $\mathfrak{G} = SL(n, \mathbb{C})$, $\mathfrak{C} = Sp(2n, \mathbb{C})$, $\mathfrak{D} = SO(n, \mathbb{C})$. Then he applied the same method to complex simple groups of exceptional type. The results were published in four parts in [Wey25-6].

The most important new invention of Weyl for the step-up from Schur's method is the so-called integration formula of Weyl for compact simple Lie groups, which is applicable to determine explicitly irreducible characters. Taking its importance into account, I'd like to continue my explanation.

Explanation 4.1. Let G be a complex simple Lie group and $\mathfrak{g} = \text{Lie}(G)$ its Lie algebra. Take a compact real form \mathfrak{g}_u of \mathfrak{g} and let $G_u \subset G$ be the closed subgroup of G corresponding to \mathfrak{g}_u . For instance, for $G = \mathfrak{G}$, \mathfrak{C} and \mathfrak{D} above, we have $G_u = SU(n)$, SpU(2n) and SO(n) respectively. To simplify the situation, we take here $G = \mathfrak{D} = SO(n, \mathbb{C})$ and $G_u = SO(n)$.

For $1 \le i < j \le n$, let $r_{ij}(\varphi)$ be two-dimensional rotation in (x_i, x_j) -space and A_{ij} its basis matrix in \mathfrak{g}_u given as

(4.1)
$$r_{ij}(\varphi) = \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}, \qquad A_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let \mathfrak{h}_u be the Cartan subalgebra of \mathfrak{g}_u consisting of elements

$$X(\varphi) := \varphi_1 A_{12} + \varphi_2 A_{34} + \dots + \varphi_{\nu} A_{2\nu-1,2\nu}, \qquad \varphi := (\varphi_i)_{i \in I_{\nu}},$$

and let H_u be the centralizer $Z_{G_u}(\mathfrak{H}_u)$ of \mathfrak{h}_u in G_u , then H_u is connected and is a Cartan subgroup of G_u consisting of $h(\varphi) := \exp X(\varphi) = r_{12}(\varphi_1)r_{34}(\varphi_2) \cdots r_{2\nu-1,2\nu}(\varphi_{\nu})$.

Let \mathfrak{h} be the complexification of \mathfrak{h}_u , and consider root system for the pair $(\mathfrak{g}, \mathfrak{h})$ and introduce an appropriate order, and let e_j be an element of the dual space \mathfrak{h}^* given as $\langle e_j, X(\varphi) \rangle := i\varphi_j$, and for $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\nu})$, we put $e(\alpha) := \sum_{j \in I_{\nu}} \alpha_j e_j$ and

(4.2)
$$\xi_{\alpha}(h(\varphi)) := \exp\langle e(\alpha), X(\varphi) \rangle = \exp(\alpha_1 \varphi_1 + \alpha_2 \varphi_2 + \dots + \alpha_{\nu} \varphi_{\nu}).$$

The set Σ^+ of positive roots and a half of the sum of positive roots $e(\rho)$ are

(4.3)
$$\Sigma^{+} = \{ \boldsymbol{e}_{j} \pm \boldsymbol{e}_{k} \ (j < k, \ j, k \in \boldsymbol{I}_{\nu}), \ \boldsymbol{e}_{j} \ (j \in \boldsymbol{I}_{\nu}) \}, \quad \rho = \left(\frac{2\nu - 1}{2}, \frac{2\nu - 3}{2}, \cdots, \frac{1}{2}\right),$$

in case $n = 2\nu + 1 \ (\nu \ge 2)$, or of type B_{ν} ,

(4.4)
$$\Sigma^{+} = \{ \boldsymbol{e}_{j} \pm \boldsymbol{e}_{k} \ (j < k, j, k \in \boldsymbol{I}_{\nu}) \}, \quad \rho = (\nu - 1, \nu - 2, \cdots, 1, 0),$$

in case $n = 2\nu \ (\nu \ge 4)$, or of type D_{ν} .

Introduce the so-called Weyl denominator D(h), $h \in H_u$, as

(4.5)
$$D(h) := \xi_{\rho}(h) \prod_{\alpha \in \Sigma^{+}} \left(1 - \xi_{-\alpha}(h) \right) = \prod_{\alpha \in \Sigma^{+}} \left(\xi_{\alpha/2}(h) - \xi_{-\alpha/2}(h) \right),$$

where, the most right hand side is a symbolic expression, and in case $n = 2\nu + 1$, $\xi_{\rho}(h)$ is doubly-valued but becomes single-valued on the universal covering group. Moreover, denote by $N_{G_u}(H_u)$ (resp. $Z_{G_u}(H_u)$) the normalizer (resp. centralizer) in G_u of H_u . Then $Z_{G_u}(H_u) = H_u$ and the quotient $W_{G_u}(H_u) := N_{G_u}(H_u)/Z_{G_u}(H_u) = N_{G_u}(H_u)/H_u$ is a finite group called Weyl group, which acts on H_u .

Now take an invariant measure dg on G_u , normalized as $\int_{G_u} dg = 1$. For a continuous function f on G_u , since dg is two-sided invariant, we can define an invariant function f^o by

(4.6)
$$f^{o}(g) := \int_{G} f(vgv^{-1}) dv \qquad (g \in G_{u}).$$

On the other hand, any element $g \in G_u$ is conjugate to an $h \in H_u$, and so the invariant function f^o is uniquely determined by its restriction on H_u . Further, if g is regular, $f^{(2)}$ then $f^{(2)}$ is regular too. For a regular element $f^{(2)}$ is $f^{(2)}$ then $f^{(2)}$ is regular too. For a regular element $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ and $f^{(2)}$ is $f^{(2)}$ is surjective and (except lower dimensional sets) $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ in $f^{(2)}$ is $f^{(2)}$ is $f^{(2)}$ in $f^{(2)}$ is $f^{(2)}$ in $f^{(2)}$ in $f^{(2)}$ in $f^{(2)}$ in $f^{(2)}$ is $f^{(2)}$ in $f^{(2)$

The celebrated integration formula of Weyl for the group G_u is given as

Theorem 4.2. For a continuous function f on a compact semisimple Lie group G_u , there holds the following integration formula:

$$\int_{G_u} f(g) dg = \frac{1}{|W_{G_u}(H_u)|} \int_{H_u} \int_{G_u} f(vhv^{-1}) dv |D(h)|^2 dh$$
$$= \frac{1}{|W_{G_u}(H_u)|} \int_{H_u} f^o(h) |D(h)|^2 dh,$$

where dg, dv and dh are the normalized invariant measures on G_u , G_u and H_u .

Personal souvenir. Very long ago in my student age, I was studying Weyl's book *Classical Groups* and found that his short proof of Integration formula (corresponding to

¹²⁾ By definition, g is regular if $sh \neq h$ ($\forall s \in W_{G_n}(H_u), \neq e$).

the above Theorem 4.2) was difficult to understand for me, and felt that it is written by a great Math genius, and struggled myself to get my own proof. Very later, some years ago, when I read Chern-Chevalley paper on É. Cartan and his work, ¹³⁾ I found a sentence (p.226), very touched me, as

····· Whereas Weyl's line of attack was, if we may say so, brutally global, depending essentially on the method of integration on the whole group, the work of Cartan puts the emphasis on the connection between the local and the global. ······

At that moment, I couldn't help smiling since I was feeling the original proof of Weyl in the book as quite 'brutally geometric' at least for me at that age. Anyhow it is a geometric proof as Schur commented "wie Hr. Weyl durch eine geometrische Betrachtung gezeigt hat," in the last sentence of the previous section.

For more details of [Wey25-6], see pp.178–182 of the paper by Chevalley-Weil. 14)

5 Theorem of Peter and Weyl

At the stage when the existence of invariant measures on Lie groups was known, Peter and Weyl [PW27] proved the completeness of irreducible unitary representations for compact Lie groups \mathfrak{G} , or irreducible decomposition of the regular representations on $L^2(\mathfrak{G})$. I made Table of Contents as

| §1. | Grundlagen. Die Orthogonalitätsrelationen | pp.737-741 |
|---------|---------------------------------------------------|--------------|
| $\S 2.$ | Besselsche Ungleichung. Ansatz des Problems | pp.741-744 |
| §3. | Konstruktion der höchsten zu einer Gruppenzahl | |
| | gehörigen Darstellung | pp.744-745 |
| $\S 4.$ | Zerfällung der gewonnenen Darstellung | pp.746-749 |
| §5. | Iteration. Beweis der Vollständigkeitsrelation | pp.749-752 |
| §6. | Entwicklungssatz. Approximationssatz. Anwendungen | pp.753 - 755 |

The contents of §1 are separated into four Parts as

- 1. Gruppe. Volummessung in der Gruppenmannigfaltigkeit.
- 2. Darstellung. 3. Charakteristik.
- 4. Die Orthogonalitätsrelationen. Jede Darstellung ist einer solchen äquivalent, deren Matrizen E(s) unitär sind.

In Part 4, there quoted Schur's results as follows: Take two irreducible matrix representations, not mutually equivalent, $E(s) = (e_{ik})_{i,k \in I_n}$, $E'(s) = (e'_{i\kappa})_{i,\kappa \in I_{-l}}$ $(s \in \mathfrak{G})$, then

(5.1)
$$\int_{\mathfrak{G}} e_{ik}(s)e'_{\kappa\iota}(s^{-1}) ds = 0,$$

 $^{^{13)}}$ S.-S. Chern and C. Chevalley, \acute{E} . Cartan and his mathematical work, Bull. Amer. Math. Soc., 58 (1952), 217–250.

 $^{^{14)}}$ C. Chevalley et A. Weil, Hermann Weyl (1885 – 1955), L'Enseignement Mathématique, III-3 (1957), 157–187.

(5.2)
$$\int_{\mathfrak{G}} e_{ik}(s)e_{\kappa\iota}(s^{-1}) ds = \frac{1}{n} \delta_{i\iota}\delta_{k\kappa},$$

where ds denotes the normalized invariant measure on \mathfrak{G} , and $n = \dim E$.

Note that ds is both left- and right-invariant and also inversion-invariant $d(s^{-1}) = ds$. The character ('Charakteristik') is defined as $\chi(s) := \operatorname{tr}(E(s))$, $\chi'(s) := \operatorname{tr}(E'(s))$. When the representations E and E' are assumed to be unitary, then in the Hilbert space $\mathcal{H} := L^2(\mathfrak{G}, ds)$, matrix elements e_{ik} and $e'_{i\kappa}$ are mutually orthogonal, and characters χ and χ' are unit vectors mutually orthogonal.

The contents of §2 is an explanation of the method pursuing analogy with Fourier transformation on compact abelian groups, e.g., $\mathfrak{G} = \mathbf{T}^k$.

For a compact Lie group \mathfrak{G} , denote by $C(\mathfrak{G})$ the space of complex-valued continuous functions on \mathfrak{G} and introduce *Multiplikation* (convolution product): for $x, y \in C(\mathfrak{G})$,

(5.3)
$$xy(s) := \int x(sr^{-1})y(r) dr$$
 or
$$xy(st^{-1}) := \int x(sr^{-1})y(rt^{-1}) dr .$$

An element $x \in C(G)$ is called *Gruppenzahl* (nowadays, $C(\mathfrak{G})$ is called group algebra).

Explanation 5.1. In principle, Peter and Weyl consider the right regular representation R on the Hilbert space \mathcal{H} , in which $C(\mathfrak{G})$ is densely contained. Define for $s \in \mathfrak{G}$ the right translation operator as (R(s)f)(u) := f(us) for $f \in \mathcal{H}$, $u \in \mathfrak{G}$. For a continuous kernel function K(u,v) on $\mathfrak{G} \times \mathfrak{G}$, define an integral operator I_K on \mathcal{H} as follows:

(5.4)
$$(I_K f)(u) := \int_{\mathfrak{S}} K(u, v) f(v) \, dv \quad \text{for } f \in \mathcal{H}.$$

(In this case we can define the *trace* of I_K by $\operatorname{tr}(I_K) := \int_{\mathfrak{G}} K(u, u) \, du$.)

Suppose I_K commutes with all R(s) (or I_K is an intertwining operator for R with itself), that is, $R(s)I_K = I_KR(s)$. Then $K(us,v) = K(u,vs^{-1})$, whence $K(u,v) = K(uv^{-1},o)$. Here o denotes the identity element of \mathfrak{G} . Put $x(u) := K(u,o) \in C(G)$, $I(x) := I_K$, then

(5.5)
$$(I(x)f)(u) := (I_K f)(u) = \int_{\mathfrak{G}} x(uv^{-1})f(v) dv =: (x * f)(u),$$

where x * f is the *convolution product*, and

$$I(xy) = I(x)I(y),$$
 $\operatorname{tr}(I(x)) = x(o) =: S(x)$ (put).

Let L be the left regular representation on \mathcal{H} defined by $(L(s)f)(u) := f(s^{-1}u)$, and assume that I_K commutes with L(s) for all $s \in \mathfrak{G}$. Then we see similarly as for the above case that $K(u,v) = K(v^{-1}u,o)$, and so $(I_K f)(u) = (f*x)(u)$.

If I_K commutes with both R and L, then the kernel function x satisfies $x(uv^{-1}) = x(v^{-1}u)$ or $x(vuv^{-1}) = x(u)$ $(u, v \in \mathfrak{G})$, that is, x is an invariant function or a class function. Of course, every character is a class function.

On the other hand, for any matrix representation E(s) of \mathfrak{G} , we can define a representation of the group algebra $C(\mathfrak{G})$, with *-action $x^*(s) := \overline{x(s^{-1})}$, and its trace as

(5.6)
$$E(x) := \int_{\mathfrak{G}} E(s)x(s) \, ds$$
 and
$$\operatorname{tr}(E(x)) = \int_{\mathfrak{G}} \chi(s)x(s) \, ds.$$

Moreover define a bounded operator on \mathcal{H} from the character χ as

(5.7)
$$P_{\chi} := \chi(o) \int_{\mathfrak{G}} R(s) \chi(s^{-1}) \, ds = \dim E \int_{\mathfrak{G}} R(s) \overline{\chi}(s) \, ds,$$
 then
$$(P_{\chi} f)(u) = \dim E \int_{\mathfrak{G}} \chi(us^{-1}) f(s) \, ds \quad (f \in \mathcal{H}, \dim E = n),$$

and $(P_{\chi})^2 = P_{\chi}$, $(P_{\chi})^* = P_{\chi}$. Hence P_{χ} is an orthogonal projection, and its image $P_{\chi}\mathcal{H} \subset \mathcal{H}$ is spanned by matrix elements $e_{jk}(u)$, $j,k \in \mathbf{I}_n$, and carries exactly n-multiple of the representation E. However, the stand point of [PW27] is different from this point of view, as is explained below.

For any unitary matrix representation E(s), 'Matrix' $\mathbf{A}(x)$ and 'Zahl' $\chi(x)$ are defined as an analogy of Fourier transform: with $\chi(s) := \operatorname{tr}(E(s))$,

(5.8)
$$\mathbf{A}(x) := \int x(s)E^{*}(s^{-1}) ds = \int x(s)\overline{E}(s) ds \ (= \overline{E}(x)),$$

$$\mathbf{A}(x) = (\alpha_{ik}(x)), \quad \alpha_{ik}(x) := \int x(s)\overline{e}_{ik}(s) ds \quad (\text{Fourier-coefficients}),$$
(5.9)
$$\alpha(x) := \text{tr}(\mathbf{A}(x)) = \int_{\mathfrak{S}} x(s)\overline{\chi}(s) ds.$$

In original notation in [PW27], $\widetilde{x}(s) := \overline{x(s^{-1})}$ and so, $A(\widetilde{x}) = A(x)^*$. For $z = x\widetilde{x}$, $A(z) = A(x)A(x)^*$ is positive Hermitian, and $\operatorname{tr}(A(z)) = \sum_{i,k} |\alpha_{ik}(x)|^2$. Fourier transform of x is, in a sense, a decomposition of of the delta functional S(x) := x(o) at the origin o, into a positive linear sum of Fourier coefficients of x. Applying it to Hermitian element $z = x\widetilde{x}$, we will get an L^2 -type equality.

To establish this equality (called here as 'Die Vollständichkeits der primitiven Darstellung') is the main purpose of the paper.

One step before the equality, we will get an inequality called 'Besselsche Ungleichung' given as follows: with $n = \dim E$, and E runs over representatives of all equivalence classes of irreducible unitary representations,

(5.10)
$$\sum_{E} n \operatorname{tr}(\mathbf{A}(x\widetilde{x})) = \sum_{E} \dim E \sum_{i,k} |\alpha_{ik}(x)|^{2} \le \int |x(s)|^{2} ds.$$

From this inequality, to arrive at the equality, the following two facts are important in my opinion.

- (1°) The integral operator I_K with continuous kernel K is compact, that is, for any bounded sequence f_1, f_2, \ldots in \mathcal{H} , we can find a subsequence f_{i_1}, f_{i_2}, \ldots such that the series $I_K f_{i_k}$ converges as $k \to \infty$, or equivalently, the image of any bounded subset of \mathcal{H} is relatively compact.
- (2°) For any $x \in C(\mathfrak{G})$, the integral operator $I(x\widetilde{x}) = I(x)I(x)^*$ is Hermitian positive definite, and only has eigenvalues and eigenspaces.

Actually Peter and Weyl noted as

Die Vollständigkeitsrelation gewinnen wir aus der Theorie der Eigenwerte und Eigenfunktionen von Kernen der besonderen Gestart $x(st^{-1})$.

6 Long-awaited general theory of invariant measures on groups

Haar's result [Haa33] is a long-awaited general theory of invariant measures on groups. Its main result is for locally compact groups, not necessarily Lie groups.

Theorem 6.1. If a locally compact group is metrisable and separable, then there exists on it an outer measure which is right-invariant (or left-invariant).

In its Introduction he referred to the above works of Hurwitz, Schur and Weyl, as we quote below from the top of Introduction:

1. Der Ausgangspunkt der Lieschen Theorie der kontinuierlichen Gruppen, die sog. Infinitesimaltransformation, wird bekantlich mittels eines Differentiationsprozesses gewonnen; deshalb ist die Liesche Theorie in ihrer ursprünglichen Form nur auf solche Gruppen anwendbar, welche durch solche Gleichungen dargestellt sind, die die fraglichen Differentiationsbedingungen erfüllen. Dieser Theorie steht eine andere gegenüber, die von Hurwitz in einer berümten Arbeit angebahnt wurde, welche man treffend als eine Integrationstheorie der kontinuierlichen Gruppen bezeichnet hat; diese Theorie wurde insbesondere im letzten Jahrzehnt durch eine Reihe von wichtigen Arbeiten gefördert, von denen wir hier nur die schönen Arbeiten von Schur und Weyl erwähnen.

It continues as

Es liegt daher der Gedanke nahe, die Frage zu unterzuchen, ob man

Diese Frage ist offenbar damit gleichwertig, ob man in der Gruppenmannigfaltigkeit einen Inhalts- bez. Maß begriff einfüren kann, der invariant gegenüber
den Transformationen der Gruppe ist, d. h. der (2 lines omitted)
Unsere Untersuchungen gelten sogar für noch allgemeinere kontinuierliche Gruppen; wir werden im wesentlichen nur annehmen daß die Gruppenmannigfartigkeit metrisch, separabel und im Kleinen kompakt ist.

The paper [Haa33] is separated into 15 Parts, numbered as $\mathbf{1}-\mathbf{15}$, and to introduce its contents, we make up Table of Contents as

| §1 | Der Inhalts. | Part 2–6 | pp.148–155, |
|--------|---------------------------------------|------------|-------------|
| $\S 2$ | Eigenshaften der Inhaltes. | Part 7–8 | pp.155-160, |
| $\S 3$ | Das Analogon des Lebesgueschen Maßes. | Part 9–12 | pp.160–166, |
| 84 | Anwendungen. | Part 13–15 | pp.166–169. |

In §4, there discussed three kinds of applications, and the third one is an extension of Theorem of Peter-Weyl on compact Lie groups to the case of a locally compact groups metrizable and separable. The top of Part 15 is

15. Ist die Gruppenmannigfaltigkeit \mathfrak{G} kompakt, so kann mann ohne Schwierigkeiten die shöne Theorie von F. Peter und H. Weyl¹⁵) über die Darstellung der geschlossenen Lieschen Gruppen auf den vorliegenden Fall übertragen, da in diesen Untersuchungen lediglich nur der invariante Integrationsprozeß benutzt wird. Um dies kurz anzudeuten, \cdots .

As remarked before, the essential points of the proof are (1^o) and (2^o) at the end of §5.

7 Two different proofs of uniqueness of Haar measure

Neumann gave two different kind of proofs of the uniqueness of Haar measure in the papers [Neu35] and [Neu36].

7.1. The first paper [Neu35].

In its Part 2, Neumann introduced, in place of Haar-Lebesgue type measure theory, another approach as follows. Let $C_{\mathbf{R}}(G)$ be the space of all real continuous functions on a compact group G. Consider real functional M (called "Mittel" = mean) on $C_{\mathbf{R}}(G)$ which satisfies the conditions 1) to 7) below:

- 1) $M(\alpha f(x)) = \alpha M(f(x)) \quad (\alpha \in \mathbf{R}).$
- 2) M(f(x) + g(x)) = M(f(x)) + M(g(x)).
- 3) If $f(x) \ge 0$ $(x \in G)$, then $M(f(x)) \ge 0$.
- 4) If $f(x) = 1 \ (x \in G)$, then M(f(x)) = 1.
- 5) If M(f(xa)) = M(f(x)) $(a \in G)$.
- 6) If M(f(ax)) = M(f(x)) $(a \in G)$.
- 7) If $M(f(x^{-1})) = M(f(x))$.

Neumann proved that such an M gives a Haar-Lebesgue type measure on G, saying as

Mit der Hilfe eines solchen Mittels kann nämlich ein HAAR-LEBESGUESches Maß eingeführt werden, wie die folgenden Überlegungen zeigen:

¹⁵⁾ Mathematische Annnalen, Bd. 97, S.737–755.

In Part 3, he constructed such an M by managing 'averages' of right translations f(xa) (= R(a)f(x)), with very delicate technique, and get the so-called "recht-Mittel". Then in Part 4, he proved that the constructed M actually satisfies the conditions 1) to 7). Thus he proved the existence and the uniqueness of Haar measure, at the same time. The obtained M is called "Mittel stetiger Funktion".

Theorem 7.1. On a compact group G, without any set theoretical or topological restriction, there exists a right-invariant Haar measure, unique up to a constant multiple. Moreover it is left-invariant and also invariant under the inversion $x \to x^{-1}$.

7.2. The second paper [Neu36].

The paper is divided into 11 Parts, and organized as follows:

| Introduction | Parts 1–2, |
|------------------------------------|-------------|
| Proof of the Theorem of uniqueness | Parts 3–8, |
| Consequences | Parts 9–10, |
| Appendix | Part 11. |

In Part 2 of Introduction, Neumann remarked the following.

The extension of the above process in case of compact groups to get "Mittel stetiger Funktion" can be extended to non-compact case. However the result is not an exterior measure at all, but the generalization of the integral mean for *almost periodic functions* of any group which is both left- and right-invariant.

Then, as for the case of locally compact group, I quote from his text as

Thus, in the case of a general locally compact group an independent treatment of the problem of uniqueness is needed. This will be given in this paper: in fact, we shall prove the

Theorem of uniqueness. The left- as well as the right-invariant exterior measure in G is unique.

8 Simple proof for extended existence-uniqueness theorem

8.1. The first paper [Kak36].

In this paper, in connection to Haar's existence theorem of invariant measures on locally compact metrisable and separable groups, S. Kakutani treated the metrisability on topological groups. His results announced here is

Satz. Wenn die topologische Gruppe G dem ersten Abzählbarkeitsaxiom genükt, dann kann man in G eine metrik $\rho(x,y)$ einführen, welche ausser den drei Distanzaxiomen noch der isomerischen Relation

(8.1)
$$\rho(zx, zy) = \rho(x, y)$$

genükt.

Thus, for a topological group, the metrisability is equivalent to the first countability axiom. To me, his proof here is very interesting and simple.

8.2. The second paper [Kak38].

Let's quote from the top of Part 1 the following:

1. For a topological group G, which is locally compact and separable, the uniqueness of Haar's left-invariant measure is proved by J. v. Neumann. Although the method used by him is very interesting and powerful, his proof is somewhat long. The notion of right-zero-invariance is not necessary for the proof. In this paper we shall give a simplified proof. \cdots

Kakutani proposes moreover to treat the generalized case of $G \times S$, where S is a topological space, and G acts transitively on S as a group of homeomorphisms:

 \cdots Since the separability plays no essential rôle in our proof, it can also be, by slight modifications applied to the case of a non-separable group (the case of a locally bicompact topological group, which is treated by A. Weil), and moreover we can prove, in the same manner, the theorem of the uniqueness of Haar's measure even for the case, when the field G is no longer a topological group, that is, G is simply a topological space S, \cdots

A Borel set E of S is called μ -invariant if $\mu(E\Delta\sigma E)=0^{16}$ ($\forall \sigma\in G$), and G is called ergodic on S if for any invariant totally additive non-negative set function¹⁷⁾ μ on S and for any μ -invariant Borel set E of S, either $\mu(E)=0$ or $\mu(S\setminus E)=0$. In the special case where S=G, and G acts on S=G through left (or right) translation, μ -invariant set function on S=G is called left (or right)- μ -invariant and G is called left (or right)- μ -ergodic if G is ergodic on itself by respective translations.

Let assume G be locally compact and *separable* (despite the previous statement).¹⁸⁾ In the case of S = G, he asserts as follows:

- I. If G is left-ergodic, then the left-invariant measure of G is unique (up to a constant multiple).
- II. G is left-ergodic.

In Part 3, the assertion I is proved, and here the conditions of separability and of local compactness are used. In Part 4, the assertion II is proved. In Part 5, the general case of non-separable locally compact groups S = G is treated. Unfortunately, maybe because of the limitation of pages for Proc. Imp. Acad., detailed comments of the general case of S is missing.

¹⁶⁾ $E_1 \Delta E_2 := (E_1 \setminus E_2) \bigsqcup (E_2 \setminus E_1).$

Not assumed that $\mu(\overline{U}) > 0$ for open set U.

¹⁸⁾ Since the relation between uniqueness and ergodicity is more prominent in this case (by Kakutani).

9 The definitive work on invariant measure theory on groups

The second chapter (§§6–9) of this book treats invariant measures on topological groups and relative-invariant measures on quotient spaces. Table des Matières for Chapter 2 is

CHAPITRE II La mesure de Haar § 6. Mesures et Intégrales, p. 30. § 7. Mesure de Haar, p. 33. § 8. Propriétés de la mesure de Haar, p. 38. § 9. Mesures dans les éspaces homogènes, p. 42.

In §6, Rodon measure is explained. For a locally compact topological space X, let $\mathfrak{M}_c(X)$ be the σ -algebra of subsets of X generated by the set of all compact subsets, and a measure defined on $\mathfrak{M}_c(X)$ is called **Radon measure**. Radon measure can be defined by a real linear functional on the space $C_c(X)$ of real-valued continuous functions with compact supports on X. This way of treatment was used in [Neu35] and later it is the way of Bourbaki to treat integrals.¹⁹⁾ A real linear functional φ on $C_c(X)$ is called *positive* if $\varphi(f) \geq 0$ for $f \geq 0$, $f \in C_c(X)$. The basic proposition here is

Proposition 9.1. For a positive real linear functional φ , there exists uniquely a Radon measure μ on $\mathfrak{M}_c(X)$ such that

(9.1)
$$\varphi(f) = \int_X f(x) \, d\mu(x) \qquad (f \in C_c(X)).$$

Conversely, for a Radon measure μ on X, consider the integral in the right hand side of (9.1), then it gives a positive real linear functional on $C_c(X)$.

Let X be a locally compact group G. Then a Radon measure μ is left-invariant if and only if so is the corresponding positive functional φ , that is, $\varphi(L_s f) = \varphi(f)$ for $s \in G$. Based on this fact, Weil proved

Theorem 9.2. Let G be a locally compact group (not assumed any countability axiom). Then there exists a left-invariant positive functional φ , unique up to a constant multiple.

The proof in §7 (pp.33–38) uses Zermelo's Axiom of Choice, according to Weil, in the form of Tychonoff's therem: "A direct product of compact spaces is also compact."

Furthermore, in Chapter 5 of the book, the so-called Peter-Weyl Theorem is proved in a complete form for general compact groups.

Note. The book [Wei40], known for its beautiful text, is finally translated into Japanese [Sai15] by M. Saito, waited from long ago.

¹⁹⁾ A. Weil himself was an important member of the founders of Bourbaki group.

10 Equivalency between Borel measure and Baire measure

The paper [KK44] consists of short Introduction and four sections. Let's quote the top part of Introduction to see the problem studied here:

Zur Definition des Haarschen Masses m in einer lokal bikompakten, nicht separabeln Gruppe gibt es zwei Möglichkeiten. Nach der ersten gewönlichen Definition wird m zunächst für alle Borelschen Mengen erklärt und dann zum vollständigen Mass vervollständigt; nach der zweiten wird dagegen m zunächst nur für die Mängen mit Baireschen charakteristische Funktionen 20 — wir wollen solche Mänge Bairesche nennen — definiert und dann vervollständigt. Sind nun diese zwei Definitionen äquivalen? In der vorliegenden Note soll diese Frage bejahend beantworted werden. \cdots

The principal result of this paper is the following one for left-invariant measures on locally compact groups:

Theorem 10.1. The first method and the second method are mutually equivalent, that is, they give the same completed measure.

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²¹⁾ bold-style number [S7] denotes the number of paper in Schur's Gesammelte Abhandlungen.

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