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Hahn's Proof of the Hahn Decomposition Theorem, and Related Matters

Neil Falkner

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Abstract. The easiest proof of the Hahn decomposition theorem for signed measure spaces is Hahn's own proof, yet it has been forgotten. This note aims to redress that and also aims to draw attention to some facts about the Radon–Nikodym theorem and the Lebesgue decomposition theorem that should be better known than they are.

1. INTRODUCTION. The Hahn decomposition theorem is the key to the most enlightening proof of the Radon–Nikodym theorem and the quickest proof of the Lebesgue decomposition theorem, so it is desirable to have a proof of the Hahn decomposition theorem that is as easy as possible. The proofs of the Hahn decomposition theorem that are found in commonly used textbooks, such as [**4**, **9**, **10**], involve a tricky inductive construction.[1] Hahn's proof involved no inductive construction. Instead, Hahn cleverly exploited the fact that the result is obvious in the special case where the σ-algebra, or more generally the σ-ring,[2] is a finite set.

2. HAHN'S PROOF. We begin by recalling some definitions and basic facts. Let $\mathscr{R}$ be a ring of subsets of a set X and let $\mu\colon \mathscr{R} \to [-\infty, \infty]$ be additive. Then μ assumes at most one of the values $-\infty$ and ∞. If E is a subset of X, then to say that E *is μ-positive* (respectively, *μ-negative*, *μ-null*) means that for each $R \in \mathscr{R}$, we have $E \cap R \in \mathscr{R}$ and $\mu(E \cap R) \geq 0$ (respectively, $\mu(E \cap R) \leq 0$, $\mu(E \cap R) = 0$). To say that (P, N) *is a Hahn decomposition for* μ means that P and N are disjoint subsets of X, P is μ-positive, N is μ-negative, and $P \cup N = X$. If (P, N) and (P', N') are two Hahn decompositions for μ, then the symmetric differences $P \mathbin{\Delta} P'$ and $N \mathbin{\Delta} N'$ (which are actually the same set) are μ-positive and also μ-negative, so they are μ-null. In this sense, a Hahn decomposition for μ is μ-essentially unique, if it exists.

Lemma. *Let $\mathscr{R}$ be a ring of subsets of a set X, let $\mu\colon \mathscr{R} \to (-\infty, \infty]$ be additive, let $N \in \mathscr{R}$, and let $P = X \setminus N$. Then the following are equivalent.*

(a) *The ordered pair (P, N) is a Hahn decomposition for μ.*

(b) *For each $R \in \mathscr{R}$, we have $\mu(R) \geq \mu(N)$.*

Proof. Suppose (a) holds. Let $R \in \mathscr{R}$. Then $\mu(P \cap R) \geq 0 \geq \mu(N \setminus R)$, so $\mu(R) = \mu(N \cap R) + \mu(P \cap R) \geq \mu(N \cap R) + \mu(N \setminus R) = \mu(N)$. Thus (b) holds if (a) does.

Conversely, suppose (b) holds. Then $-\infty < \mu(N) \leq \mu(\emptyset) = 0$. Now for each $R \in \mathscr{R}$, we have $N \cap R \in \mathscr{R}$ and $\mu(N) + \mu(N \cap R) \leq \mu(N \setminus R) + \mu(N \cap R) = \mu(N)$, so $\mu(N \cap R) \leq 0$. Thus N is μ-negative. Also, for each $R \in \mathscr{R}$, we have

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[1] In [**11**], the Hahn decomposition theorem is presented as a consequence of the Radon–Nikodym theorem but the inductive construction is still present, though it occurs in a different place, to show that a complex measure on a σ-algebra is automatically bounded.

[2] When it adds no complication, we shall usually work with rings of sets and with σ-rings, but the reader who prefers to consider just algebras of sets and σ-algebras should feel free to do so.

$P \cap R = R \setminus N \in \mathscr{R}$ and $\mu(N) \leq \mu(N \cup R) = \mu(N) + \mu(R \setminus N)$, so $0 \leq \mu(P \cap R)$. Thus P is μ-positive. Therefore (P, N) is a Hahn decomposition for μ. Thus (a) holds if (b) does. ■

The Hahn Decomposition Theorem. *Let $\mu\colon \mathscr{R} \to (-\infty, \infty]$ be σ-additive, where $\mathscr{R}$ is a σ-ring on a set X. Then there is a Hahn decomposition (P, N) for μ such that $N \in \mathscr{R}$.*

Proof. Let $a = \inf\{\mu(R) : R \in \mathscr{R}\}$. By the lemma, it suffices to show that there exists $N \in \mathscr{R}$ such that $\mu(N) = a$. Since $\mathscr{R} \neq \emptyset$, we can choose a sequence (D_n) in $\mathscr{R}$ such that $\mu(D_n) \to a$. Hahn's clever idea was to replace the sequence (D_n) by a better sequence (C_n), as follows. Let $\mathscr{R}_n$ be the ring of sets generated by $D_1, \ldots, D_n$ and let μ_n be the restriction of μ to $\mathscr{R}_n$. Now $\mathscr{R}_n$ is nonempty and finite,[3] so μ achieves a minimum on $\mathscr{R}_n$, say at $C_n \in \mathscr{R}_n$. Then $\mu(C_n) \leq \mu(\emptyset) = 0$. By the lemma, C_n is μ_n-negative and $X \setminus C_n$ is μ_n-positive. Observe that $C_1, \ldots, C_n \in \mathscr{R}_n$, because $\mathscr{R}_1 \subseteq \cdots \subseteq \mathscr{R}_n$. Let $A_m = \bigcap_{k=m}^{\infty} C_k$. For $n \geq m$, let $A_m^n = \bigcap_{k=m}^{n} C_k$. Now $A_m^n \downarrow A_m$ as $n \to \infty$ and $-\infty < \mu(A_m^m) = \mu(C_m) < \infty$, so $\mu(A_m^n) \to \mu(A_m)$ as $n \to \infty$. For $n > m$, we have

$$\mu(A_m^{n-1}) = \mu(A_m^n) + \mu((X \setminus C_n) \cap A_m^{n-1}) \geq \mu(A_m^n),$$

because $X \setminus C_n$ is μ_n-positive and because $A_m^{n-1} \in \mathscr{R}_n$. Hence

$$\mu(D_m) \geq \mu(C_m) = \mu(A_m^m) \geq \mu(A_m^{m+1}) \geq \mu(A_m^{m+2}) \geq \cdots,$$

so $\mu(D_m) \geq \mu(A_m) \geq a$. Thus $\mu(A_m) \to a$. Now the sequence (A_m) is increasing. Let $N = \bigcup_m A_m$. Then $\mu(A_m) \to \mu(N)$. Thus $\mu(N) = a$, as desired. ■

Remarks.

(a) Most proofs of the Hahn decomposition theorem use the principle of dependent choice. The preceding proof, which does not involve an inductive construction, just uses the axiom of countable choice.

(b) The preceding proof is adapted from [6, pp. 401–403]. It shows that $\liminf_n C_n$ is a suitable choice for N. By a similar method, one can show that $\limsup_n C_n$ is a suitable choice for N. See [7, p. 17]. Since N is uniquely determined to within a μ-null set, it follows that $1_{C_n} \to 1_N$ pointwise except on a μ-null set.

(c) One can take C_n to be the union of the atoms of $\mathscr{R}_n$ that have negative μ-measure. Both [6] and [7] essentially do this but they do not mention the rings $\mathscr{R}_n$. Instead, they give a complicated-looking construction of C_n in terms of $D_1, \ldots, D_n$. Perhaps this is why their approach has been forgotten. I think it deserves to be remembered, since in my opinion, it is by far the easiest proof of the Hahn decomposition theorem. How remarkable it is that the original proof should be the easiest!

(d) It follows immediately from the Hahn decomposition theorem that μ is bounded below. However, this is by no means obvious a priori, which accounts for much of the difficulty in proving the theorem. Indeed, if a as defined above is known to be finite then it is well known and easy to check that if we choose (D_n) so that $\mu(D_n) \leq a + 2^{-n}$ for each n, then[4] we may take N be $\liminf_n D_n$ or, equally well, $\limsup_n D_n$.

[3] $\mathscr{R}_n$ is equal to the algebra of subsets of $\bigcup_{k=1}^n D_k$ generated by $D_1, \ldots, D_n$.

[4] Let $A \in \mathscr{R}$. If $A \subseteq D_n$, then $\mu(A) \leq 2^{-n}$, for otherwise $\mu(D_n \setminus A) < a$. Let $F_m = \bigcup_{n \geq m} D_n$. If $A \subseteq F_m$, then $\mu(A) = \sum_{n \geq m} \mu(A \cap (D_n \setminus \bigcup_{m \leq \ell < n} D_\ell)) \leq \sum_{n \geq m} 2^{-n} = 2^{1-m}$. Let $F = \limsup_n D_n$.

3. ABOUT THE RADON–NIKODYM THEOREM. The most enlightening way to prove the Radon–Nikodym theorem is to deduce it from the Hahn decomposition theorem, as follows. Let μ and ν be measures on a σ-algebra $\mathscr{A}$ on a set X. Suppose μ is σ-finite[5] and ν is absolutely continuous with respect to μ. We seek a measurable function $f\colon X \to [0, \infty]$ such that for each $A \in \mathscr{A}$, we have $\nu(A) = \int_A f\, d\mu$. It is easy to reduce to the case where μ is finite, so let us just consider that case. If f is as desired, then it is obvious that

$$\begin{aligned}&\text{for each } s \in [0, \infty)\text{, the ordered pair } \big(\{f > s\}, \{f \le s\}\big)\\ &\text{is a Hahn decomposition for the signed measure } \nu - s\mu.\end{aligned} \tag{1}$$

It is important to note that the converse also holds. This is straightforward to prove by suitably partitioning the range of f. Now for each rational number $r > 0$, let (P_r, N_r) be a Hahn decomposition of the signed measure $\nu - r\mu$. Let $f = \sup_r r 1_{P_r}$. Then it is easy to check that f satisfies (1), so since (1) is equivalent to what we want, we are done. For more details, see [9], [12], or [3, p. 661].

4. *s*-FINITE MEASURES. To say that a measure is *s-finite* means that it is a sum of countably many finite measures. The s-finite measures constitute a natural class of measures that contains the class of σ-finite measures but that is not mentioned in any real analysis textbook of which I am aware. Attention was drawn to this class of measures as early as 1957, when G. A. Hunt, on the first page of the first paper [8] in his famous trilogy on Markoff processes and potentials, comments that each measure considered on the state space of a Markoff process will be "understood to be…the sum of countably many bounded positive measures."

An image of a σ-finite measure need not be σ-finite[6] but such an image is s-finite. Indeed, an image of an s-finite measure is obviously s-finite. Furthermore, any s-finite measure is an image of a σ-finite measure.[7] Thus the s-finite measures constitute the smallest class of measures that contains all σ-finite measures and is closed under passage to images.

It is elementary that an s-finite measure has the same sets of measure zero as a suitable finite measure. The converse also holds, since it follows from the Radon–Nikodym theorem[8] that a measure that is absolutely continuous with respect to a finite measure is s-finite.

Fubini's theorem is usually stated for σ-finite measures but once it has been established for finite measures, it is obvious that it can be extended to s-finite measures,[9] provided one uses the product measure that is defined by iterated integration, as is done in [11], for instance. Then we have $\big(\sum_{j\in\mathbb{N}} \lambda_j\big) \otimes \big(\sum_{k\in\mathbb{N}} \mu_k\big) = \sum_{j,k\in\mathbb{N}} \lambda_j \otimes \mu_k$

Then $F = \bigcap_m F_m$. If $A \subseteq F$, then $\mu(A) \le 2^{1-m}$ for each m, so $\mu(A) \le 0$. Thus F is μ-negative. Similarly, letting $E = \liminf_n D_n$, it is easy to check that $X \setminus E$ is μ-positive. Thus $F \setminus E$ is μ-null, both of $(X \setminus E, E)$ and $(X \setminus F, F)$ are Hahn decompositions for μ, and $1_{D_n} \to 1_E = 1_F$ μ-a.e.

[5] In most commonly used textbooks, such as [4, 10, 11], both μ and ν are assumed to be σ-finite in the Radon–Nikodym theorem but, as is recognized in [9], no σ-finiteness assumption on ν is needed and indeed the method in [9], which is the method sketched above, handles this greater generality for free.

[6] For example, consider the projection of Lebesgue measure in the plane onto the x-axis. This may seem artificial but in abstract settings, things like this can happen without its being obvious a priori.

[7] Suppose $\mu = \sum_n \mu_n$ where (μ_n) is a sequence of finite measures. For each n, let δ_n be the unit point mass at n. Then μ is the image of the σ-finite measure $\sum_n \mu_n \otimes \delta_n$ under the projection $(x, n) \mapsto x$.

[8] Note that for this application of the Radon–Nikodym theorem, it is important that we do not need to assume that ν is σ-finite

[9] The key to seeing this is the easy fact that for any family (μ_i) of measures on a measurable space and for any nonnegative measurable function f on the same space, if $\mu = \sum_i \mu_i$, then μ is a measure and $\int f\, d\mu = \sum_i \int f\, d\mu_i$. To prove Fubini's theorem for s-finite measures, we just need this fact for the case where the family (μ_i) is countable but it actually holds even if (μ_i) is uncountable.

whenever the measures λ_j and μ_k are s-finite. Also, if λ' and μ' are images of s-finite measures λ and μ, respectively, then $\lambda' \otimes \mu'$ is the corresponding image of $\lambda \otimes \mu$. By the way, neither of these desirable relations would hold in such pleasant generality if we defined product measure by the outer measure construction.[10]

See [3, pp. 662–663] for some simple applications of s-finite measures. For a modern presentation of aspects of Markoff processes and potentials, which includes important applications of s-finite measures, see [5].

5. ABOUT THE LEBESGUE DECOMPOSITION THEOREM. Some commonly used textbooks, such as [4] and [11], present a combined proof of the Radon–Nikodym theorem and the Lebesgue decomposition theorem. Others, such as [9] and [10], deduce the Lebesgue decomposition theorem from the Radon–Nikodym theorem. In fact, though, the Lebesgue decomposition theorem can be obtained as an immediate corollary of the Hahn decomposition theorem. Let μ and ν be measures on a σ-algebra $\mathscr{A}$ on a set X. For the Radon–Nikodym theorem, we supposed that μ was σ-finite but we needed no finiteness condition on ν. For the Lebesgue decomposition theorem, we need no finiteness condition on μ but we shall suppose that ν is s-finite. We wish to show that ν can be expressed as the sum of two measures ν_a and ν_s such that ν_a is absolutely continuous with respect to μ and ν_s is singular with respect to μ. Since ν is s-finite, it has the same sets of measure zero as a suitable finite measure $\tilde{\nu}$ on $\mathscr{A}$. Let (P, N) be a Hahn decomposition for the signed measure $\lambda = \infty \cdot \mu - \tilde{\nu}$. Since $\lambda(N) \leq 0$, we have $\mu(N) = 0$. For each $E \in \mathscr{A}$, if $\mu(E) = 0$, then $0 \leq \tilde{\nu}(P \cap E) = -(\infty \cdot \mu(P \cap E) - \tilde{\nu}(P \cap E)) = -\lambda(P \cap E) \leq 0$, so $\tilde{\nu}(P \cap E) = 0$, so $\nu(P \cap E) = 0$. It follows that we may take ν_a and ν_s to be the measures on $\mathscr{A}$ defined by $\nu_a(E) = \nu(P \cap E)$ and $\nu_s(E) = \nu(N \cap E)$.

For another simple proof of the Lebesgue decomposition theorem, which does not even require the Hahn decomposition theorem and which extends to the case where the measure ν takes values in a normed linear space, see [1] and [2].

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[10] Some commonly used textbooks in which product measure is defined by the outer measure construction are [4, 9, 10]. This yields the largest product measure but it may be too large. For instance, let λ be Lebesgue measure on $[0, 1]$, let $\mu = \infty \cdot \lambda$, let ν be the product measure that is defined from λ and μ by iterated integration, and let $\tilde{\nu}$ be the product measure that is defined from λ and μ by the outer measure construction. Then Fubini's theorem holds for ν, because λ and μ are s-finite, but it does not hold for $\tilde{\nu}$. Indeed, $\nu(\Delta) = 0$ but $\tilde{\nu}(\Delta) = \infty$, where $\Delta = \{(x, x) : x \in [0, 1]\}$ is the diagonal. I learned this example from P. J. Fitzsimmons in 1989. (It is a variation on a standard example. See [9, p. 311, problem 25], for instance.) Note that $\mu = \sum_{k \in \mathbb{N}} \mu_k$, where $\mu_k = \lambda$ for each k, and μ is also the image of the σ-finite measure $\lambda \otimes \gamma$ under the map $(y, k) \mapsto y$, where γ is counting measure on $\mathbb{N}$.

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The Ohio State University, Department of Mathematics, 231 West 18th Avenue, Columbus, OH 43210
falkner@math.ohio-state.edu