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Stirling's Approximation for $n!$: the Ultimate Short Proof?

Dan Romik

For two real sequences x_n and y_n, we write $x_n \sim y_n$ if $\lim_{n\to\infty} x_n/y_n = 1$. Stirling's approximation is $n! \sim \sqrt{2\pi n}\,(n/e)^n$. We first prove that $n! \sim C\sqrt{n}\,(n/e)^n$ for some nonzero constant C, then give a short proof that $C = \sqrt{2\pi}$.

Lemma 1. *The limit $C = \lim_{n\to\infty} e^n n!/n^{n+1/2}$ exists.*

Proof: Denote as usual $\lfloor x \rfloor = \max\{n \in \mathbf{Z} : n \le x\}$ and $\{x\} = x - \lfloor x \rfloor$. Then

$$\log n! = \sum_{k=1}^{n} \log k = \sum_{k=1}^{n} \int_1^k \frac{dx}{x} = \int_1^n \frac{n - \lfloor x \rfloor}{x}\, dx$$

$$= \int_1^n \frac{n + \frac{1}{2} + (\{x\} - \frac{1}{2}) - x}{x}\, dx = \left(n + \tfrac{1}{2}\right)\log n - n + 1 + \int_1^n \frac{\{x\} - \frac{1}{2}}{x}\, dx$$

$$= \left(n + \tfrac{1}{2}\right)\log n - n + 1 + \int_1^\infty \frac{\{x\} - \frac{1}{2}}{x}\, dx + o(1),$$

since $\int_1^t (\{x\} - \frac{1}{2})\, dx$ is bounded and $1/x$ goes to 0 as $x \to \infty$. Thus

$$C = \exp\left(1 + \int_1^\infty \frac{\{x\} - \frac{1}{2}}{x}\, dx\right) > 0. \qquad \blacksquare$$

This is one of the standard proofs; it is a simple example of Euler-Maclaurin summation. It follows that $\binom{2n}{n} \sim \sqrt{2} \cdot 2^{2n}/C\sqrt{n}$.

Lemma 2. *Let f be an $(n + 1)$-times continuously-differentiable function on $\mathbf{R}$. Then for all $x \in \mathbf{R}$, $f(x) = f(0) + f'(0)x + f''(0)x^2/2! + \cdots + f^{(n)}(0)x^n/n! + R_n(x)$, where*

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)(x - t)^n\, dt.$$

Proof: Induction on n, using integration by parts. $\blacksquare$

We now apply Lemma 2 with $f(x) = (1 + x)^{2n+1}$ to calculate $R_n(1)/2^{2n+1}$:

$$\frac{1}{2^{2n+1}} R_n(1) = \frac{1}{2^{2n+1}} \cdot \frac{1}{n!}\int_0^1 (2n + 1)(2n)\cdots(n + 1)(1 + t)^n(1 - t)^n\, dt$$

$$= \frac{2\binom{2n}{n}}{2^{2n+1}}\left(n + \tfrac{1}{2}\right)\int_0^1 (1 - t^2)^n\, dt$$

$$= \frac{\binom{2n}{n}\sqrt{n}}{2^{2n}}\left(1 + \frac{1}{2n}\right)\int_0^{\sqrt{n}}\left(1 - \frac{u^2}{n}\right)^n du$$

$$\xrightarrow[n\to\infty]{} \frac{\sqrt{2}}{C}\int_0^\infty e^{-u^2}\, du = \frac{\sqrt{2}}{C} \cdot \frac{\sqrt{\pi}}{2}$$

The convergence of the integrals is justified by the fact that $0 \leq (1 - u^2/n)^n \leq e^{-u^2}$ in the domain of integration, and $(1 - u^2/n)^n \to e^{-u^2}$ uniformly on compacta. On the other hand, $R_n(1)/2^{2n+1} = \sum_{n<k\leq 2n+1}\binom{2n+1}{k}/2^{2n+1} = \frac{1}{2}$. Therefore $C = \sqrt{2\pi}$, as claimed.

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The Hankel Determinant of Exponential Polynomials

Richard Ehrenborg

The Hankel matrix of order $n + 1$ of a sequence $a_0, a_1, \ldots$ is the $n + 1$ by $n + 1$ matrix whose (i, j) entry is a_{i+j}, where the indices range between 0 and n. The *Hankel determinant* of order $n + 1$ is the determinant of the corresponding Hankel matrix, that is,

$$\det(a_{i+j})_{0\leq i,j\leq n} = \det\begin{pmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{pmatrix}.$$

The purpose of this note is two-fold. First we present evaluations of Hankel determinants of sequences of combinatorial interest related to partitions and permutations. Many such computations have been carried out by Radoux in his sequence of papers [**2**]–[**5**]. His proof methods include using a functional identity due to Sylvester and factoring the Hankel matrix. Second, unlike Radoux, we instead give bijective proofs to reveal the underlying structure of these identities.

A partition $\pi = \{B_1, \ldots, B_k\}$ of a finite set S is a collection of non-empty subsets $B_1, \ldots, B_k$, called *blocks*, such that the blocks are disjoint and their union is the set S. Let $|\pi|$ denote the number of blocks in the partition π. The *exponential polynomials* $e_n(x)$ are defined by

$$e_n(x) = \sum_{\pi} x^{|\pi|},$$

where π ranges over all partitions of an n-element set. A few properties of the exponential polynomials are (i) $e_n(1)$ is equal to the nth Bell number, (ii) $e_n(x) = \sum_{k=0}^{n} S(n,k)x^k$ where $S(n,k)$ is the Stirling number of the second kind, (iii) $e_n(x) = e^{-x}(x \cdot d/dx)^n e^x$, (iv) $\sum_{n\geq 0} e_n(x)t^n/n! = \exp(x(e^t - 1))$. For more on the properties of exponential polynomials, see [**7**, Section 13], which is [**6**, pp. 7–82].

Theorem 1 (Radoux [2]). *The Hankel determinant of order $n + 1$ of the exponential polynomials $e_n(x)$ is given by*

$$\det\big(e_{i+j}(x)\big)_{0\leq i,j\leq n} = x^{(n+1)n/2} \cdot \prod_{i=0}^{n} i!.$$