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Disasters in topology without the axiom of choice

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Abstract. We show that some well known theorems in topology may not be true without the axiom of choice.

1. Introduction and terminology

The *countable axiom of choice* CAC (Form 8 in [4]) is the assertion:

"*For every set* $\mathscr{A} = \{A_i : i \in \omega\}$ *of nonempty pairwise disjoint sets there exists a set* C *consisting of one and only one element from each element of* $\mathscr{A}$."

The *countable multiple choice axiom*, CMC, is the proposition:

"*For every set* $\mathscr{A} = \{A_i : i \in \omega\}$ *of nonempty sets there exists a family* $\mathscr{F} = \{F_i : i \in \omega\}$ *of finite nonempty sets such that for every* $i \in \omega$ $F_i \subseteq A_i$."

CAC_ω is CAC with the additional requirement that the members of $\mathscr{A}$ are countable sets and CMC_ω is CMC with the same requirement. ω-CMC is the statement:

"*For every countable family* $\mathscr{A}$ *of nonempty disjoint sets there exists a set* C *such that for every* $A \in \mathscr{A}$ $0 < |C \cap A| \leq \omega$."

Lemma 1.1. *(i) CMC iff* ω*-CMC+*CMC_ω*.*
(ii) CAC iff ω*-CMC+*CAC_ω*.*

Remark 1. If in ω-CMC we do not require that $\mathscr{A}$ be a family of disjoint sets then Lemma 1.1 (i) is false. Indeed, CMC is true in the Second Fraenkel Model,

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model $\mathcal{N}2$ from [4] (the set of atoms A is countable, i.e. $A = \{a_i : i \in \omega\}$, the group of permutations $\mathcal{G}$ is the group of permutations on A which leave the set $\{\{a_{2i}, a_{2i+1}\} : i \in \omega\}$ pointwise fixed, and supports are finite), but the modified ω-CMC is false. To see this let $\mathcal{A} = \{A_i : i \in \omega\}$ where $A_0 = A$,

$$A_1 = A - \{a_0, a_1\}, \dots , A_i = A - \bigcup_{j=0}^{i-1}\{\{a_{2j}, a_{2j+1}\}\}, \dots .$$

Let C be a set given by ω-CMC. Since C meets nontrivially each member of $\mathcal{A}$ and $0 < |C \cap A_0| \leq \omega$, $C \cap A$ must be an infinite well ordered subset of A but it is known that A does not have such subsets.

Lemma 1.2. *CMC iff for every countable family $\mathcal{A}$ of disjoint nonempty sets there exists an infinite set $C \subset \cup\mathcal{A}$ such that for every $A \in \mathcal{A}$ $0 \leq |C \cap A| < \omega$.*

Proof. It suffices to show ($\leftarrow$). Fix $\mathcal{A} = \{A_n : n \in \omega\}$ a family of disjoint nonempty sets. Put

$$\mathcal{B} = \{B_n = \prod_{m \leq n} A_m : n \in \omega\}.$$

By the hypothesis there exists an infinite set $C \subset \cup\mathcal{B}$ such that for every $n \in \omega$ $0 \leq |C \cap B_n| < \omega$. Based on C and taking projections we can easily construct a set $\mathcal{F} = \{F_n : n \in \omega\}$ satisfying CMC for $\mathcal{A}$. □

For the undefined notation we refer the reader to [13], [15], [12], [9] and [11]. All product spaces in this paper are given as products of **countably** many factors and carry the **Tychonoff** topology.

Our main aim here is to show that well known theorems of ZF+AC (see [15]) such as:

Theorem 1.3. *Countable products of metrizable spaces are metrizable,*

Theorem 1.4. *Countable products of second countable spaces are second countable,*

Theorem 1.5. *Countable products of first countable spaces are first countable,*

Theorem 1.6. *Countable products of separable T_2 spaces are separable,*

Theorem 1.7. *A metric space is countably compact iff each of its sequences has a cluster point,*

are not provable in ZF^- (= ZF minus foundation) without AC.

Van Douwen in [1] showed that it is consistent with ZF without AC that:

Theorem 1.8. *There exists a family $\{X_i : i \in \omega\}$ of countable compact metrizable spaces whose disjoint topological union is not metrizable.*

We show in Theorem 2.2 that their product $\prod\{X_i : i \in \omega\}$ may be neither metrizable nor second countable nor first countable nor separable, demonstrating the horrors of topology without AC. For an example of a family $\{X_i : i \in \omega\}$ of countable metrizable spaces whose disjoint topological union, as well as product, are metrizable but not separable or second countable, the reader is referred to a Cohen model given in [3] Theorem 4. For additional information about the permutational version of this model we refer the reader to [5], Example 2 and [6], Theorem 10.

Another well known instance where there is trouble in topology without AC occurs when taking Tychonoff products of families (countable or uncountable) of compact spaces. For more information about this matter and related bibliography we refer the reader to [14], [4] and [7]. In this paper we shall be concerned, as pointed out before, with products of countably many factors and the following two notions of compactness:

Definition 1.9 . *(i) A topological space (X, T) is countably compact iff each countable open cover has a finite subcover.*
(ii) A topological space (X, T) is sequentially limit compact (s.l.c. for abbreviation) iff each sequence of X has a cluster point.

Under AC the notions of countable compactness and s.l.c. are equivalent and s.l.c. is sometimes (see [15], p. 125) used to define countable compactness.

Clearly, in ZF^-, we have:

Theorem 1.10. *A topological space (X, T) is countably compact iff each countable family of closed sets with the finite intersection property has a nonempty intersection.*

As a corollary to theorem 1.10 we get that:

Corollary 1.11. *If the space (X, T) is countably compact then X is also s.l.c..*

Proof. Indeed, if $(x_n)_{n\in\omega}$ is a sequence in X, then

$$G = \{G_n = \overline{\{x_m : m \geq n\}} : n \in \omega\}$$

is a family of closed sets with the finite intersection property. Hence, $\cap G \neq \emptyset$ and any point in $\cap G$ is a cluster point of $(x_n)_{n\in\omega}$. □

For countable spaces it is easily seen that the notions of countable compactness and s.l.c. coincide. But in general they do not. We demonstrate this fact in Theorem 2.5.

In Theorem 2.4 we show that CAC implies the proposition:

A metric space (X, d) is compact iff it is countably compact.

In [2] the authors ask whether CAC implies the statement

PCC = *the countable product of compact spaces is compact*.

In Theorem 2.7 we find a partial answer to this question. We show that CAC implies

PCMC = *countable products of compact metrizable spaces are compact metrizable*.

We do not know the full answer to this question. We only know, see [7], that in permutational models CAC $\Leftrightarrow$ PCC.

2. Products of metric spaces and the axioms cac and cmc

In this section we shall be concerned with the following statements:

(1) : *Countable products of metrizable spaces are metrizable*.
(2) : *Countable products of second countable spaces are second countable*.
(3) : *Countable products of first countable spaces are first countable*.
(4) : *Countable products of separable T_2 spaces are separable*.

Theorem 2.1. *(i) CAC implies each of (2) and (4).*
(ii) CMC implies (1) and (3).

Proof. (i) To see (2) follow any standard text such as [13] and replace any occurrence of AC with CAC.

To see (4), fix $\mathscr{A} = \{(X_i, T_i) : i \in \omega\}$ a family of disjoint separable nonempty topological spaces and let $X = \prod_{i \in \omega} X_i$ be the Tychonoff product of $\mathscr{A}$. Fix, by CAC, for each $i \in \omega$ a countable dense subset Q_i of X_i and let $x = \{x_i : i \in \omega\}$ be a choice set for $\{X_i : i \in \omega\}$. For every $n \in \omega$ put

$$G_n = \{y \in X : \forall i \leq n, y(i) \in Q_i \text{ and } \forall i > n, y(i) = x_i\}.$$

Since $\prod_{i \leq n} Q_i$ is countable it follows that G_n is countable and, by CAC,

$$G = \cup\{G_n : n \in \omega\}$$

is also countable. It is straightforward to verify that $\overline{G} = X$. Thus X is separable as required.

(ii) (1). Fix $\{(X_n, T_n) : n \in \omega\}$ a family of metrizable spaces. Put $\mathscr{A} = \{A_n : n \in \omega\}$, where A_n is the set of all metrics on X_n producing T_n. Let $\mathscr{F} = \{F_n : n \in \omega\}$ be a multiple choice set for $\mathscr{A}$. For every $n \in \omega$ define a metric d_n on X_n by requiring

$$d_n((x, y)) = (\sum_{\rho \in F_n} \rho((x, y)))/(1 + \sum_{\rho \in F_n} \rho((x, y))).$$

Clearly, d_n is a metric on X_n producing T_n and we may finish the proof of (1) as in [15].

To see (3), fix $\{(X_i, T_i) : i \in \omega\}$ a family of disjoint first countable nonempty topological spaces and let $X = \prod_{i \in \omega} X_i$ be their Tychonoff product. Fix $x \in X$. We show that x has a countable neighborhood base in X.

Since, for each $i \in \omega$, X_i is first countable, it follows that each point $y \in X_i$ has a countable neighborhood base $\mathscr{V} = \{V_n : n \in \omega\}$. Without loss of generality we may assume that $\mathscr{V}$ is strictly descending. That is we may assume that $\mathscr{V}$ is well-ordered by $\supseteq$. Thus, $A_i = \{\mathscr{V} : \mathscr{V} = \{V_n : n \in \omega\}$ is a neighborhood base for $x(i)$ and the enumeration is compatible with $\supseteq\}$ is nonvoid for all $i \in \omega$. Put $\mathscr{A} = \{A_i : i \in \omega\}$ and let $\mathscr{F} = \{F_i : i \in \omega\}$ be a multiple choice set for $\mathscr{A}$. For every $i \in \omega$ define a neighborhood base $V_i = \{V_{in} : n \in \omega\}$ for $x(i)$ by letting

$$V_{in} = \cap\{V_n : V_n \in \mathscr{V} \in F_i\}.$$

It can be readily verified that

$$\mathscr{U} = \{U_n : n \in \omega\}, U_n = \cap\{\pi_i^{-1}(V_{in}) : i \leq n\}$$

is a countable neighborhood base for x in X finishing the proof of the theorem. □

Theorem 2.2. *CMC_ω is implied by each one of (1) and (3), and CAC_ω is implied by each one of (2) and (4).*

Proof. Fix $\mathscr{A} = \{A_i : i \in \omega\}$ a family of countable nonempty disjoint sets. Let X_i be the one point compactification of the discrete space A_i by adjoining the point A_i, and let X be the product of the spaces X_i.
(1) → CMC_ω. Clearly X_i is metrizable. (If $\{a_{in} : n > 0\}$ enumerates A_i then

$\rho_i : X_i \times X_i \to \mathbb{R}$, $\rho_i(a_{in}, a_{im}) = |1/n - 1/m|$, $\rho_i(a_{in}, A_i) = \rho_i(A_i, a_{in}) = 1/n$ and $\rho_i(A_i, A_i) = 0$,

is easily seen to be a metric on X_i producing its topology.) By (1) X is also metrizable. Let d be a metric on X generating its topology and $a \in X$ the element satisfying: $a(i) = A_i$, for all $i \in \omega$. Consider the family $\{D_{1/n}(a) : n > 0\}$, where $D_{1/n}(a)$ denotes the open disk of d radius $1/n$, centered at a. Clearly, $D_{1/n}(a)$ is an open set of the product space X. As

$$\{n \in \omega : \pi_i(D_{1/n}(a)) \neq X_i\} \neq \emptyset,$$

(if $\pi_i(D_{1/n}(a)) = X_i$ for all $n > 0$, then $\{D_{1/n}(a) : n > 0\}$ fails to be a neighborhood base),

$$n_i = min\{n \in \omega : \pi_i(D_{1/n}(a)) \neq X_i\}$$

exists for every $i \in \omega$. It is straightforward to verify that

$$\mathscr{F} = \{F_i = X_i \backslash \pi_i(D_{1/n_i}(a)) : i \in \omega\}$$

satisfies CMC_ω for $\mathscr{A}$.

(3) → CMC_ω. Work as in (1) → CMC_ω.

(2)$\rightarrow$CAC$_\omega$. By (2) X is second countable. Let $B = \{b_n : n \in \omega\}$ be a basis for X. For every $i \in \omega$, let

$$n_i = min\{n \in \omega : \pi_i(b_n) \neq \{A_i\} \text{ is a singleton}\}.$$

Clearly

$$c = \{x_{n_i}(i) : i \in \omega, x_{n_i} \in b_{n_i}\}$$

is a choice set for $\mathscr{A}$.

(4)$\rightarrow$CAC$_\omega$. By (4) X is separable. Fix $D = \{d_n : n \in \omega\}$ a countable dense set. For every $i \in \omega$, let

$$n_i = min\{n \in \omega : d_n(i) \neq A_i\}.$$

Clearly, $c = \{c_i = d_{n_i}(i) : i \in \omega\}$ is a choice set for $\mathscr{A}$. □

Corollary 2.3. *(A) CMC iff (1) + ω-CMC iff (3) + ω-CMC.*
(B) CAC iff (2) + ω-CMC iff (4) + ω-CMC.
(C) The statement "the product of countably many metrizable spaces is first-countable" implies CMC$_\omega$.

The Second Fraenkel Model $\mathcal{N}2$ satisfies CMC but not CAC$_\omega$, see [4], p. 178. Thus in $\mathcal{N}2$ (1) and (3) hold but (2) and (4) are false.

Of course, if a metric space is compact then it is also countably compact. All known (to us) proofs of the converse,

CCMC = a countably compact metric space is compact,

use the axiom of dependent choices DC. Furthermore,

- *CCMC* implies "*every infinite set A can be written as a countably infinite union of disjoint nonempty sets*".

Indeed, A with the discrete topology is not compact. Hence, by CCMC, A has a countable cover $\mathscr{U} = \{U_n : n \in \omega\}$ without a finite subcover. Using $\mathscr{U}$ we can easily find a countable partition $V = \{V_n : n \in \omega\}$ of A.

Since the statement "*every infinite set is the union of two disjoint infinite sets*", form 64 in [4], is false in the Basic Fraenkel Model, model $\mathcal{N}1$ in [4] (the set of atoms A is countably infinite, the group of permutations $\mathscr{G}$ is the group of all permutations on A and the ideal of supports $\mathscr{S}$ is the set of all finite subsets of A), it follows from the above that if we endow A with the discrete metric d then CCMC fails in $\mathcal{N}1$. If not then A, in view of the above, has a partition $V = \{V_n : n \in \omega\}$. Clearly $Q_1 = \cup\{V_{2n} : n \in \omega\}$ and $Q_2 = \cup\{V_{2n+1} : n \in \omega\}$ form a partition of A into two infinite sets which is impossible, see [4], p. 176. Hence, CCMC is not valid in $\mathcal{N}1$ and CCMC is not provable in ZF$^-$.

Next we show that a form of choice weaker than DC suffices for the proof of CCMC.

Theorem 2.4. *CAC implies CCMC.*

Proof. Fix (X, d) a countably compact metric space. For every $\varepsilon > 0$ $\mathscr{U}_\varepsilon$ will denote the set of all open ε-discs of X. First we show that:

Claim 1. A separable metric space (X, d) is compact iff it is countably compact.

Proof of Claim 1. Fix (X, d) a separable metric space. It suffices to show that if X is countably compact then X is compact. It is known (see Theorem 16.11, p. 112 in [15]) that a separable metric space (X, d) has a base $\mathscr{B} = \{B_n : n \in \omega\}$. Let $\mathscr{U}$ be an open cover of X. Express each member U of $\mathscr{U}$ as $U = \cup\{B \in \mathscr{B} : B \subseteq U\}$ and use the fact that $\mathscr{B}$ is countable and X is countably compact to get finitely many members of $\mathscr{B}$, say $B_1, B_2...B_n$ all subsets of members of $\mathscr{U}$, covering X. For each $B_1, B_2...B_n$ pick $U_1, U_2...U_n \in \mathscr{U}$ with $B_j \subseteq U_j$. It follows that $U_1, U_2...U_n$ is a finite subcover of $\mathscr{U}$ and X is compact as required.

Claim 2. CAC implies the statement: "A countably compact metric space (X, d) is *precompact* (for every $\varepsilon > 0$ $\mathscr{U}_\varepsilon$ has a finite subcover)".

Proof of Claim 2. Assume the contrary that (X, d) is countably compact but not precompact i.e., there exists an $\varepsilon > 0$ such that $\mathscr{U}_\varepsilon$ has no finite subcover. This clearly implies that for all $n > 0$, the set A_n of all n-tuples $(x_0, x_1...x_{n-1})$ of elements of X satisfying $d(x_i, x_j) \geq \varepsilon$ for all $i, j \in n, i \neq j$ is nonempty. Put $\mathscr{A} = \{A_n : n \in \omega\backslash 1\}$ and let by CAC, $\mathscr{Q} = \{Q_n \in A_n : n \in \omega\backslash 1\}$ be a choice set of $\mathscr{A}$. Clearly, the set $\mathscr{G}$ of all elements of X appearing in some member of $\mathscr{Q}$ is countable. $Y = \overline{\mathscr{G}}$, being closed is countably compact, and since it is separable, by Claim 1, Y is also compact, hence precompact. Thus, there exists a finite number of elements, say $D(x_1, \varepsilon/3), D(x_2, \varepsilon/3)...D(x_n, \varepsilon/3)$, of $\mathscr{U}_{\varepsilon/3}$ covering Y. Hence, there are $x, y \in Q_{n+1}$ and $m \leq n$ with $x, y \in D(x_m, \varepsilon/3)$. We have

$$\varepsilon \leq d(x, y) \leq d(x_m, y) + d(x, x_m) < \varepsilon/3 + \varepsilon/3 = 2\varepsilon/3.$$

This contradiction terminates the proof of Claim 2.

We prove now that (X, d) is compact. Assume the contrary and let $\mathscr{U}$ be an open cover of X without a finite subcover. Put $\mathscr{F} = \{\mathscr{F}_n : n \in \omega\backslash 1\}$ where $\mathscr{F}_n$ is the set of all finite subcovers of $\mathscr{U}_{1/2^n}$. By Claim 2, it follows that each $\mathscr{F}_n \neq \emptyset$. Let $\{\mathscr{V}_n \in \mathscr{F}_n : n \in \omega\backslash 1\}$ be a choice set for $\mathscr{F}$. In view of CAC we may assume that there is a function assigning to each n a well-ordering of $\mathscr{V}_n$.

We shall choose, for every $n = 1, 2, 3...$ a set $D(x_n, 1/2^n) \in \mathscr{V}_n$ which cannot be covered by finitely many elements of $\mathscr{U}$.

For $n = 1$ we let $D(x_1, 1/2) \in \mathscr{V}_1$ be the first disc of $\mathscr{V}_1$ which cannot be covered by finitely many members of $\mathscr{U}$.

Assume that

$$D(x_1, 1/2) \in \mathscr{V}_1, D(x_2, 1/2^2) \in \mathscr{V}_2...D(x_n, 1/2^n) \in \mathscr{V}_n$$

have been chosen in such a way that for all $1 < m \leq n$,

$$D(x_{m-1}, 1/2^{m-1}) \cap D(x_m, 1/2^m) \neq \emptyset$$

and $D(x_m, 1/2^m)$ cannot be covered by finitely many members of $\mathscr{U}$. Since $\mathscr{V}_{n+1}$ covers X, $\mathscr{V}_{n+1}$ also covers $D(x_n, 1/2^n)$. As $D(x_n, 1/2^n)$ is not covered by finitely many members of $\mathscr{U}$, we see that some member of $\mathscr{V}_{n+1}$ which meets $D(x_n, 1/2^n)$ is not covered by finitely many members of $\mathscr{U}$. Let $D(x_{n+1}, 1/2^{n+1})$ be the first element of $\mathscr{V}_{n+1}$ which has this property.

As X is countably compact it follows that $(x_n)_{n\in\omega}$ has a convergent subsequence, say $(x_n)_{n\in\omega}$ again, to some point $x \in X$. As $\mathscr{U}$ covers X, $x \in U$ for some $U \in \mathscr{U}$. Pick m large enough so that

$$D(x, 1/2^m) \subseteq U.$$

Fix $v > m + 3$ with $x_v \in D(x, 1/2^{m+3})$. Then,

$$D(x_v, 1/2^v) \subseteq D(x, 1/2^m) \subseteq U.$$

This contradicts the fact that $D(x_v, 1/2^v)$ is not covered by finitely many elements of $\mathscr{U}$ and finishes the proof of the theorem. □

Tychonoff's theorem for countable products of finite discrete spaces, hence compact metrizable, is equivalent to the *axiom of choice for countable families of finite sets* (see [10]). Hence the statement PCMC (defined at the end of Section 1), in view of Theorem 2.2, implies CAC_ω. Next we show that CAC implies PCMC. For this reason we shall need the following characterization of CAC.

Theorem 2.5. *CAC iff every s.l.c. space is countably compact.*

Proof. ($\rightarrow$) Fix a space X such that each of its sequences has a cluster point. Let $U = \{O_n : n \in \omega\}$ be an open cover of X. If U has no finite subcover then

$$Q = \{Q_n = \cup\{O_m : m \leq n\} : n \in \omega\}$$

is an ascending open cover of X such that no Q_n covers X. Use CAC to pick a sequence $(x_n)_{n\in\omega}$, $x_n \in X\backslash Q_n$. Let x be a cluster point of $(x_n)_{n\in\omega}$ and let O_n contain x. Then O_n must contain infinitely many terms of $(x_n)_{n\in\omega}$ contradicting the choice of x_n's. Hence U has a finite subcover as required.

($\leftarrow$) If CAC fails then (see Form 8 in [4]) there exists a family $\mathscr{A} = \{A_i : i \in \omega\}$ of nonempty sets having no infinite subfamily with a choice function. Topologize $X_i = A_i \cup \{A_i\}$, $A_i \notin A_i$ by declaring neighborhoods of $x \in A_i$ to be all cofinite subsets of A_i and, $\{A_i\}$ to be a neighborhood of A_i. Let X be the product of the spaces X_i.

Claim. In X every sequence $(x_n)_{n\in\omega}$ has a cluster point.

Proof of Claim. As no infinite subfamily of $\mathscr{A}$ has a choice function, it follows that, for every x in X, for some $n \in \omega$ $x(i) = A_i$ for all $i \geq n$. Furthermore, we can readily verify, using the well ordering of $(x_n)_{n\in\omega}$, that there exists $n \in \omega$ such that $x_m(i) = A_i$ for all $m \in \omega$ and $i \geq n$. Since $X_1 \times X_2 \times \ldots \times X_{n-1}$ is a compact space homeomorphic to the subspace

$$G = \{x \in X : x(i) = A_i \text{ for all } i \geq n\}$$

and $(x_n)_{n\in\omega} \subseteq G$, it follows by the compactness of G, that $(x_n)_{n\in\omega}$ has a cluster point $x \in G$. It is easy to see that x is a cluster point of $(x_n)_{n\in\omega}$ in X.

By the claim, X satisfies "*every sequence has a cluster point*". Hence X satisfies "*each countable open cover of X has a finite subcover*". Thus $F = \{\pi_i^{-1}[A_i] : i \in \omega\}$ being a countable family of closed sets with the finite intersection property, satisfies $\cap F \neq \emptyset$ meaning that $\mathscr{A}$ has a choice function which is a contradiction. Hence CAC holds finishing the proof of the theorem. □

Remark 2. Clearly, in view of Theorem 2.5, if CAC fails then there is an s.l.c. space X which is not countably compact. We would like to point out here that if Form 13 in [4] i.e. the statement "every infinite subset of the real line has a countably infinite subset" fails there exists an s.l.c. metric space (X, d) which is not countably compact. Indeed, if A is an infinite subset of the real line $\mathbb{R}$ having no countably infinite subset then clearly A with the standard metric d is s.l.c. but not countably compact.

Corollary 2.6. *CAC is equivalent to the conjunction: "Every sequentially compact metric space X is countably compact" + CMC.*

Proof. ($\rightarrow$) This can be proved as in Theorem 2.5.

($\leftarrow$) Assume that CAC fails but CMC holds. Then there exists a family $\mathscr{A} = \{A_i : i \in \omega\}$ of nonempty finite sets having no infinite subfamily with a choice function. Let X_i and X be as in Theorem 2.2. Working as in Theorem 2.5 we show that $\mathscr{A}$ has a choice set reaching a contradiction. □

Theorem 2.7. *CAC implies PCMC.*

Proof. Fix $\mathscr{X} = \{(X_i, T_i) : i \in \omega\}$ a family of disjoint compact metrizable spaces. By Theorem 2.1, X the product of $\mathscr{X}$, is a metric space. To complete the proof of the theorem it suffices, in view of Theorem 2.4, to show that X is countably compact. In view of Theorem 2.5, it suffices to show that X is s.l.c. To this end, fix $G = \{x_n : n \in \omega\} \subseteq X$ and let $Y = \prod_{i\in\omega} Y_i$, $Y_i = \overline{\pi_i(G)}$. For each $n \in \omega$ fix, by CAC, a countable base $\mathscr{B}_n$ for Y_n. (Let, by CAC, d_i be a metric producing T_i. $\pi_i(G)$ is a countable dense subset of Y_i and consequently $\mathscr{B}_n$ the set of all open d_i-disks of radius $1/m$, $m \in \omega\backslash 1$ centered at points of $\pi_i(G)$ is the required countable base $\mathscr{B}_n$.)

Let $\mathscr{B}$ be the set of all open sets b in Y such that

$$b = \prod_{i\in\omega}^{<n} O_i,$$

meaning $O_i = Y_i$ for all $i \geq n$ and $O_i \in \mathscr{B}_i$, $O_i \neq Y_i$, $Diameter(O_i) < 1/n$ for all $i < n$. Clearly, $\mathscr{B}$ is a countable base for Y. Using the fact that $\mathscr{B}$ is countable we can pick, via an easy induction, a set

$$B = \{b_n = \prod_{i \in \omega}^{<n} O_i \in \mathscr{B} : n \in \omega \backslash 1\}$$

such that:

$b_n \subseteq b_m$ for all $n, m \in \omega$ with $m \leq n$ and each b_n includes a subsequence of $(x_n)_{n \in \omega}$.

Set

$$C_j = \{\overline{O_{jn}} \subseteq Y_j : j < n \text{ and } O_{jn} \text{ appears in the expression of } b_n \in B\}.$$

By the compactness of Y_j it follows that $\cap C_j \neq \emptyset$ for all $j \in \omega$. Furthermore, since the diameter of the O_{jn}'s tends to 0 as n tends to ∞, it follows that $\cap C_j = \{c_j\}$ is a singleton. As B is a neighborhood base for the element

$$c \in Y, c(j) = c_j,$$

it follows that c is a cluster point of $(x_n)_{n \in \omega}$ finishing the proof of the theorem. □

Corollary 2.8. *CAC iff PCMC + ω-CMC.*

Question. Does PCMC imply ω-CMC?

We recall that a topological space (X, T) is *limit point compact*, l.p.c. for abbreviation, iff every infinite subset of X has a limit point. It is not hard to verify (see exercise 17F p. 125 in [15]) that CAC implies:

A T_1 topological space is l.p.c iff it is countably compact.

If (X, T) is T_1 and l.p.c., then it is easily seen to be s.l.c. and this without any choice. For the converse, i.e., the statement

a T_1 s.l.c. topological space is l.p.c.

some choice is needed because it does not hold in the Basic Fraenkel Model $\mathcal{N}1$. Indeed, the set of atoms A with the discrete topology, as seen before, is countably compact and it has no limit point.

In complete analogy with Theorem 2.5 we have:

Theorem 2.9. *CMC holds iff for every T_1 topological space, l.p.c. is equivalent to countable compactness.*

Proof. ($\rightarrow$). Working as in the first part of Theorem 2.5, one can readily verify that CMC implies:

l.p.c. T_1 topological spaces are countably compact.

For the converse we need the following easy result.

Claim. CMC implies for every infinite set A there exists a denumerable (countably infinite) subset B of $[A]^{<\omega}$.

Proof of the claim. Put $\mathscr{G} = \{G_n = [A]^n : n \in \omega\backslash 1\}$ and let $\mathscr{F} = \{F_n : n \in \omega\backslash 1\}$ be a multiple choice for $\mathscr{G}$. For every $n \in \omega\backslash 1$ set $K_n = \cup F_n$. Without loss of generality we may assume that each $B_n = K_{n+1}\backslash K_n$ is a nonempty set. Then $B = \{B_n : n \in \omega\}$ is the required denumerable set.

Now we can complete the proof of ($\rightarrow$) as in Corollary 1.11 with $\cup B$ in place of the sequence $(x_n)_{n\in\omega}$.

($\leftarrow$). If CMC fails then, by Lemma 1.2, there exists a family $\mathscr{A} = \{A_i : i \in \omega\}$ having no infinite subfamily with a multiple choice. Let X_i and X be as in Theorem 2.5.

Claim. In X every infinite set Q has a limit point.

Proof of Claim. As $\mathscr{A}$ has no partial multiple choice, it follows that for every $q \in Q$ there exists $n \in \omega$ such that $q(i) = A_i$ for all $i \geq n$. Thus,

$Q = \cup\{Q_n : n \in \omega\}$, $Q_n = \{q \in Q : q(i) = A_i$ for all $i \geq n\}$.

Furthermore, as $\mathscr{A}$ has no partial multiple choice, it follows that not all Q_n's are finite. Thus, there exists $n \in \omega$ with Q_n infinite. Since $X_1 \times X_2 \times ... \times X_{n-1}$ is a compact space homeomorphic to the subspace

$$G = \{x \in X : x(i) = A_i \text{ for all } i \geq n\}$$

and $Q_n \subseteq G$, it follows by the compactness of G, that Q_n has a limit point $x \in G$. (If not we can find, by the compactness of G, a finite open cover $\mathscr{U}$ of G each member of whose includes finitely many members of Q_n). It is easy to see that x is a limit point of Q_n in X.

By the claim, X is l.p.c. and consequently X is countably compact. It follows that $\cap F \neq \emptyset$, where F is as in Theorem 2.5 and consequently $\mathscr{A}$ has a choice function which is a contradiction. Hence CMC holds as required. □

Corollary 2.10. *If CMC fails then there is an l.p.c. space X which is not countably compact.*

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