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# Countable sums and products of metrizable spaces in ZF

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We study the role that the axiom of choice plays in Tychonoff's product theorem restricted to countable families of compact, as well as, Lindelöf metric spaces, and in disjoint topological unions of countably many such spaces.



## 1 Notation and terminology

In this paper we continue with the study of compact and Lindelöf metric spaces which we started in [12] and [13]. Besides the notation and terminology used in the above-mentioned papers we shall need to establish some more here. Let $(X,T)$ be a topological space. $X$ is said to be *metrizable* iff there is a metric $d$ on $X$ such that the topology $T_d$ on $X$ induced by $d$ coincides with $T$.

Form 9: *Every infinite set has a countably infinite subset.*

Form 418: *The disjoint union of countably many metrizable spaces is metrizable.*

$\mathrm{CPM}(P,Q)$: *If* $\{(X_i,d_i) : i\in\mathbb{N}\}$ *is a family of metric spaces each having the property* $P$, *then the* (*Tychonoff*) *product* $X=\prod_{i\in\mathbb{N}}X_i$ *has the property* $Q$.

$\mathrm{CPM_{le}}(P,Q)$: *If* $\{(X_i,T_i) : i\in\mathbb{N}\}$ *is a family of metrizable topological spaces each having the property* $P$, *then the* (*Tychonoff*) *product* $X=\prod_{i\in\mathbb{N}}X_i$ *is metrizable and has the property* $Q$.

$\mathrm{CSM}(P,Q)$: *If* $\{(X_i,d_i) : i\in\mathbb{N}\}$ *is a disjoint family of metric spaces each having the property* $P$, *then the sum* (*disjoint topological union, see* [19]) $X=\sum_{i\in\mathbb{N}}X_i$ *has the property* $Q$.

$\mathrm{CSM_{le}}(P,Q)$: *If* $\{(X_i,T_i) : i\in\mathbb{N}\}$ *is a disjoint family of metrizable topological spaces each having the property* $P$, *then the sum* $X=\sum_{i\in\mathbb{N}}X_i$ *is metrizable and has the property* $Q$.

$\mathrm{CAC(ML)}$ (resp. $\mathrm{CAC(MC)}$): *If* $\{(X_i,d_i) : i\in\mathbb{N}\}$ *is a family of Lindelöf* (*resp. compact*) *metric spaces, then* $\{X_i : i\in\mathbb{N}\}$ *has a choice function.*

Notice that CAC(ML) (resp. CAC(MC)) is equivalent to the proposition: *Products of countably many Lindelöf* (*resp. compact*) *metric spaces are non-empty.*

CAC($\mathrm{M_{le}}$L) (resp. CAC($\mathrm{M_{le}}$C)) : *If* $\{(X_i,T_i) : i\in\mathbb{N}\}$ *is a family of Lindelöf* (*resp. compact*) *metrizable topological spaces, then* $\{X_i : i\in\mathbb{N}\}$ *has a choice function.*

## 2 Introduction and some known results

It is well known that the countable version of Tychonoff's compactness theorem, i. e., countable products of compact topological spaces are compact, implies the axiom of countable choice CAC, see [11]. The status of the reverse implication is still an open problem, see [1, 5, 8]. However, it is known, see [12], that if we restrict to the class of compact metric spaces, then Tychonoff's theorem for countable products of compact metric

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spaces, CPM(C,C), is a theorem of $\mathsf{ZF} + \mathrm{CAC}$. In this paper we elucidate more on the set-theoretic strength of CPM(C,C) and prove that CPM(C,C) iff CAC(MC) iff M(C,S) iff CPM(C,S) iff CSM(C,S) (Theorem 8), and that $\mathrm{CPM_{le}}$(C,C) is strictly weaker than CAC in $\mathsf{ZF}^0$ (= $\mathsf{ZF}$ without the axiom of foundation) (Theorem 13). That CAC implies $\mathrm{CPM_{le}}$(C,C) and that $\mathrm{CPM_{le}}$(C,C) implies $\mathrm{CAC}_\omega$ has been established in [12]. We prove in Theorem 13 that $\mathrm{CPM_{le}}$(C,C) does not imply CAC answering a relative question from [12].

In Theorem 7 we prove that CAC implies $\mathrm{CPM_{le}}$(L,L), that $\mathrm{CPM_{le}}$(L,L) implies $\big(\mathrm{CAC(ML)} + \mathrm{CAC}_\omega\big)$, and that $\big(\mathrm{CAC(ML)}+\mathrm{CAC}(\mathbb{R})\big)$ implies CUC. In Theorem 14 we show that in $\mathsf{ZF}^0$, CUC does not imply CAC($\mathrm{M_{le}}$L) and that $\big(\mathrm{CUC} + \mathrm{Form 9}\big)$ does not imply Form 418.

The following table summarizes the positive and independence results obtained in this paper, as well as open problems. A "0" at position $(P, Q)$ means that it is unknown whether the row statement $P$ implies the column statement $Q$. A "1" at position $(P, Q)$ means that $P$ implies $Q$ in $\mathsf{ZF}$, and a "2" at $(P, Q)$ means that $P$ does not imply $Q$ in $\mathsf{ZF}$. Because of the equivalences $\big($CPM(L,L) iff CPM(L,S) iff CSM(L,S)$\big)$, and $\big($CAC(MC) iff CPM(C,C) iff CPM(C,S) iff CSM(C,S) iff M(C,S)$\big)$, in the table we will only use CPM(L,L) as a representative of the first set of equivalences, and CAC(MC) as a representative of the second set.

| | CPM(L,L) | CSM(L,L) | M(L,S) | M(L,hL) | CAC(ML) | CAC(MC) | CUC |
|---|---|---|---|---|---|---|---|
| CPM(L,L) | 1 | 2 | 1 | 2 | 1 | 1 | 0 |
| CSM(L,L) | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| M(L,S) | 0 | 2 | 1 | 2 | 0 | 1 | 0 |
| M(L,hL) | 0 | 0 | 1 | 1 | 0 | 1 | 0 |
| CAC(ML) | 0 | 2 | 0 | 2 | 1 | 1 | 0 |
| CAC(MC) | 0 | 2 | 0 | 2 | 0 | 1 | 0 |
| CUC | 0 | 2 | 0 | 0 | 0 | 0 | 1 |

The positive or independence results which appear in the above table but are not proved here, have been established in the following theorems to which we shall be referring frequently in the sequel.

**Theorem 1** ([7]) *The following statements are equivalent:*

(i) $\mathrm{CAC}(\mathbb{R})$.

(ii) $\mathbb{N}$*, the discrete space of natural numbers, is Lindelöf.*

(iii) *Every topological space with a countable base is Lindelöf.*

(iv) $\mathbb{R}$ *is Lindelöf.*

(v) *Every subspace of* $\mathbb{R}$ *is Lindelöf.*

(vi) *Every subspace of* $\mathbb{R}$ *is separable.*

(vii) $\mathrm{M(S, hS)}$ (see [9, 10]).

**Theorem 2** ([6]) $\neg\mathrm{CAC}(\mathbb{R})$ *iff* $\mathrm{M(L, C)}$.

**Theorem 3** ([13]) $\mathrm{CAC}(\mathbb{R})$ *implies* $\mathrm{M(S, hL)}$.

**Theorem 4** ([14])

(i) $\mathrm{M(L, hL)}$ *implies* $\mathrm{M(C, hL)}$*, and* $\mathrm{M(C, hL)}$ *implies* $\mathrm{CAC}(\mathbb{R})$.

(ii) $\mathrm{M(L, hL)}$ *iff* $\mathrm{M(L, hS)}$ *iff* $\big(\mathrm{M(L, S)} + \mathrm{CAC}(\mathbb{R})\big)$.

(iii) $\mathrm{M(C, hL)}$ *iff* $\mathrm{M(C, hS)}$.

**Theorem 5** ([16] and [14])

(i) $\mathrm{M(C, S)}$ *iff* $\mathrm{M(C, 2)}$.

(ii) $\mathrm{M(L, S)}$ *iff* $\mathrm{M(L, 2)}$.

Let $\mathcal{F} = \{(X_i, d_i) : i \in \mathbb{N}\}$ be a family of disjoint metric spaces and $X = \prod_{i\in\mathbb{N}} X_i$, $Y = \sum_{i\in\mathbb{N}} X_i$ be respectively the product and the sum of the family $\mathcal{F}$. Clearly, we can replace any unbounded metric $d_i$ with an equivalent metric $\varrho_i$ producing the same topology on $X_i$ and being bounded by 1. It is known, see any standard text of general topology such as [19], that $d : X \times X \longrightarrow \mathbb{R}$,

$$(1) \qquad d(x,y) = \textstyle\sum_{i\in\mathbb{N}} \dfrac{\varrho_i(x(i), y(i))}{2^i}$$

is a metric on $X$ inducing the product topology on $X$.

Likewise, one can easily verify that

$$(2) \qquad \sigma(x,y) = \begin{cases} \varrho_i(x,y)/i & \text{if } x,\, y \in X_i, \\ \max\{1/i, 1/j\} & \text{if } x \in X_i,\, y \in X_j \text{ and } i \neq j \end{cases}$$

is a metric on $Z = \bigcup\{X_i : i \in \mathbb{N}\}$ inducing the disjoint union topology on $Z$. In the sequel we shall always assume that in CPM(S,T) $X$ carries the metric $d$ given by (1), and in CSM(S,T) $Y$ carries the metric $\sigma$ given by (2).

## 3 Main results

We begin this section by observing the following straightforward characterization of Form 418:

$(*)$ Form 418 iff *countable products of metrizable spaces are metrizable.*

Proof. Let $\{(X_i, T_i) : i \in \omega\}$ be a pairwise disjoint family of metrizable topological spaces. For each $i \in \omega$, let $Y_i = X_i \cup \{\infty_i\}$, $\infty_i \notin X_i$, and expand $T_i$ by requiring $\infty_i$ to be isolated. Clearly, $Y_i$ is metrizable; if $d_i$ is a metric on $X_i$ bounded by 1, then $d(x,y) = d_i(x,y)$ if $x,\, y \in X_i$, and otherwise $d(x,y) = 1$ if $x \neq y$ and $d(x,x) = 0$, is a metric on $Y_i$ inducing its topology. Let $Y$ be the Tychonoff product of the $Y_i$'s. If $Y$ is metrizable, let $d$ be a metric on $Y$ inducing its topology. Then since $X_i$ is homeomorphic to the subspace $Z_i = X_i \times \prod_{j\in\omega\setminus\{i\}}\{\infty_j\}$ of $Y$, it is clear that $X_i$ has a definable metric inducing $T_i$ for all $i \in \omega$. Hence, the sum of the $X_i$'s is metrizable (see (2)). Conversely, if the sum $X$ of the $X_i$'s has a metric $d$ inducing the disjoint union topology, then $d|_{X_i}$ is a metric on $X_i$ inducing $T_i$, hence the product of the $X_i$'s is metrizable (see (1)). □

The following implications are also evident: $\mathrm{CPM_{le}(L,L)} \to \mathrm{CPM(L,L)}$, $\mathrm{CSM_{le}(L,L)} \to \mathrm{CSM(L,L)}$, and $\mathrm{CAC(M_{le}L)} \to \big(\mathrm{CAC(M_{le}C), CAC(ML)}\big) \to \mathrm{CAC(MC)}$.

**Theorem 6** $\mathrm{CAC} \to \mathrm{CSM_{le}(L,L)} \to \big(\mathrm{CAC}(\mathbb{R}) + \mathrm{CUC}\big)$.

Proof.

($\mathrm{CAC}\to\mathrm{CSM_{le}}$(L,L)). Fix a disjoint family $\{(X_i, T_i) : i \in \omega\}$ of Lindelöf metrizable spaces and let $X = \bigcup\{X_i : i \in \omega\}$. By CAC fix, for each $i \in \omega$, a metric $\varrho_i$ on $X_i$ which is bounded by 1 and induces the topology $T_i$. Then $d : X \times X \longrightarrow \mathbb{R}$ defined by $d(x,y) = \varrho_i(x,y)$ if $x, y \in X_i$, and $d(x,y) = 1$ for $x \neq y$ otherwise, is a metric on $X$ inducing the disjoint union topology on $X$. Now, let $\mathcal{U}$ be an open cover of $X$ and for every $i \in \omega$, put $\mathcal{U}_i = \{U \cap X_i : U \in \mathcal{U}\}$. Since each $X_i$ is Lindelöf, it follows that each $\mathcal{U}_i$ has a countable subcover. Fix, by CAC, for every $i \in \omega$ a countable subcover $\mathcal{V}_i = \{V_{i,n} : n \in \omega\}$ of $\mathcal{U}_i$. Clearly, $V = \{V_{i,n} : i,n \in \omega\}$ is countable. For every $i,n \in \omega$, let $A_{i,n} = \{U \in \mathcal{U} : V_{i,n} \subset U\}$. It can be readily verified that the range of any choice function on the family $\mathcal{A} = \{A_{i,n} : i,n \in \omega\}$ is a countable subcover of $\mathcal{U}$ and $X$ is Lindelöf as required.

(CSM(L,L)$\to$CAC($\mathbb{R}$)) follows from Theorem 1 and the observation that if $X_i = \{i\}$ for every $i \in \mathbb{N}$, then the sum $X = \sum_{i\in\mathbb{N}} X_i$ is homeomorphic to $\mathbb{N}$ taken with the discrete metric (i. e., $d(n,m) = 1$ for all $n,\, m \in \mathbb{N}$ with $n \neq m$, and $d(n,n) = 0$ for all $n \in \mathbb{N}$).

(CSM(L,L)$\to$CUC). Fix a disjoint family $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ of countable sets. Let $d$ denote the discrete metric on $X = \bigcup\mathcal{A}$. From the previous paragraph, it follows that each $A_i$ is Lindelöf. Thus $X$ is Lindelöf and the open cover $\mathcal{U} = \{\{x\} : x \in X\}$ of $X$ must be countable. It readily follows that $\bigcup\mathcal{A}$ is countable as required. □

**Theorem 7**

(i) $\mathrm{CAC} \to \mathrm{CPM_{le}(L,L)} \to \mathrm{CAC(ML)} \to \mathrm{CAC_{fin}}$.

(ii) $\mathrm{CPM_{le}(L,L)} \to \big(\mathrm{CAC(M_{le}L)} + \mathrm{CAC}_\omega\big)$.

(iii) $\big(\mathrm{CAC(ML)} + \mathrm{CAC}(\mathbb{R})\big) \to \mathrm{CUC}$.

Proof.

(i) (CAC→$\mathrm{CPM_{le}}$(L,L)). Fix a family $\{(X_i, T_i) : i \in \mathbb{N}\}$ of Lindelöf metrizable spaces, and by CAC pick, for each $i \in \mathbb{N}$, a metric $\varrho_i$ on $X_i$ which is bounded by 1 and induces $T_i$. Let $(X = \prod_{i\in\mathbb{N}} X_i, d)$ be the product metric space of the $X_i$'s, where $d$ is the metric on $X$ given by (1). If $X = \emptyset$, then $(X, d)$ is trivially Lindelöf, so let $x \in X$. Since CAC implies M(L,S) (see, e. g., [13]) it follows that for every $i \in \mathbb{N}$,

$$A_i = \{f \in X_i^{\mathbb{N}} : \overline{\mathrm{ran}(f)} = X_i\} \neq \emptyset,$$

where $X_i^{\mathbb{N}}$ is the set of all functions $f : \mathbb{N} \longrightarrow X_i$. By CAC let $g = \{(i, f_i) : i \in \mathbb{N}\}$ be a choice function of $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$. Then the set $D = \{f_i(n) : i, n \in \mathbb{N}\}$ is countable. It can be readily verified now that $\mathcal{D} = \{d \in X : (\exists n \in \mathbb{N})[(\forall i \leq n)\, d(i) \in \mathrm{ran}(f_i) \wedge (\forall i > n)\, d(i) = x(i))]\}$ is a countable, dense subset of $X$. Thus, by Theorem 1(iii), $X$ is Lindelöf as required.

(i) (CPM(L,L)→CAC(ML)). Let $\{(X_i, d_i) : i \in \mathbb{N}\}$ be a family of Lindelöf metric spaces, and $(Y, \varphi)$ be the product metric space of the family $\{(Y_i, \varphi_i) : i \in \mathbb{N}\}$, where $Y_i = X_i \cup \{*_i\}$, $*_i \notin X_i$, and $\varphi_i$ is the metric on $Y_i$ given by

$$\varphi_i(x, y) = \varphi_i(y, x) = \begin{cases} \varrho_i(x, y) & \text{if } x, y \in X_i, \\ 1 & \text{if } x \in X_i \text{ and } y = *_i, \\ 0 & \text{if } x = y = *_i, \end{cases}$$

where $\varrho_i$ is an equivalent metric to $d_i$ bounded by 1. By CPM(L,L), $(Y, \varphi)$ is Lindelöf. If $X \neq \emptyset$ we are done. Assume $X = \emptyset$. Then, $\mathcal{W} = \{\pi_i^{-1}(*_i) : i \in \mathbb{N}\}$ is an open cover of $Y$ having no finite subcover. For every $i \in \mathbb{N}$ and $x \in X_i$, let $x^i \in Y$ be given by $x^i(i) = x$ and for all $j \in \mathbb{N}\backslash\{i\}$, $x^i(j) = *_j$. Clearly, $\mathcal{U} = \{\pi_i^{-1}(*_i)\backslash\{x^{i+1}\} : i \in \mathbb{N}, x \in X_{i+1}\}$ is an open cover of $Y$. Let $\mathcal{V}$ be a countable subcover of $\mathcal{U}$. On the basis of $\mathcal{V}$ we can readily verify that there exists an infinite subfamily $\{X_{n_i} : i \in \mathbb{N}\}$ of $\{X_i : i \in \mathbb{N}\}$ having a choice function. This suffices for the family $\{X_i : i \in \mathbb{N}\}$ to have a choice function. Indeed, if for every $n \in \mathbb{N}$, $Z_n = \prod_{i\leq n} X_i$ and $\psi_n$ is the product metric on $Z_n$, then $\{(Z_n, \psi_n) : n \in \mathbb{N}\}$ is, by CPM(L,L), Lindelöf. Furthermore, if $f$ is a choice function of any infinite subfamily $\{Z_{n_i} : i \in \mathbb{N}\}$ of $\{Z_n : n \in \mathbb{N}\}$, then on the basis of $f$ one can easily define a choice function for the family $\{X_i : i \in \mathbb{N}\}$.

(i) (CAC(ML)→$\mathrm{CAC_{fin}}$). This readily follows from the observation that if $\mathcal{A} = \{A_i : i \in \mathbb{N}\}$ is a family of non-empty, finite sets, then for every $i \in \mathbb{N}$, $(A_i, d_i)$, where $d_i$ is the discrete metric on $A_i$, is a compact metric space.

(ii). The first assertion can be established similarly to the proof of CPM(L,L)→CAC(ML). For the second assertion, follow the proof of [12, Theorem 2.2] and immediately deduce that $\mathrm{CPM_{le}}$(L,L) implies the weak choice form: *for every family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty, countable sets there is a function $f$ such that $f(A_i)$ is a non-empty, finite subset of $A_i$*. The conclusion now follows from the implication $\mathrm{CPM_{le}}$(L,L)→$\mathrm{CAC_{fin}}$.

(iii). Fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of pairwise disjoint, countable sets. For each $i \in \omega$, let $\varrho_i$ be the discrete metric on $A_i$ and $d_i$ the product metric on $A_i^\omega$. For all $i \in \omega$, let $B_i = \{f \in A_i^\omega : f \text{ is a bijection}\}$. We assert that $(B_i, d_i|_{B_i})$ is a Lindelöf space for all $i \in \omega$. Fix an $i \in \omega$. Since $A_i$ is a countable set, it can be readily verified that $(A_i^\omega, d_i)$ is homeomorphic to the Baire metric space $\omega^\omega$. Furthermore, as $\omega^\omega$ is a separable metric space, it follows from Theorem 3 that $\omega^\omega$, hence $A_i^\omega$, is hereditarily Lindelöf. Thus, $(B_i, d_i|_{B_i})$ is a Lindelöf metric space as claimed. By CAC(ML) fix a choice function $g = \{(i, f_i) : i \in \omega\}$ on the family $\mathcal{B} = \{B_i : i \in \omega\}$. It is evident that $\bigcup \mathcal{A} = \{f_i(n) : i, n \in \omega\}$ is a countable set and the proof is complete. □

**Corollary 1**

(i) $\mathrm{CAC(MC)} \to \mathrm{CAC_{fin}}$.

(ii) *In all permutation models, each one of* $\mathrm{CAC(ML)}$ *and* $\mathrm{CPM(L,L)}$ (*hence* $\mathrm{CAC(M_{le}L)}$ *and* $\mathrm{CPM_{le}(L,L)}$) *implies* CUC.

P r o o f. The conclusion (ii) follows from parts (i) and (iii) of Theorem 7 and the fact that CAC($\mathbb{R}$) holds in every permutation model, see [9]. □

**Theorem 8** *The following statements are equivalent:* (i) $\mathrm{CAC(MC)}$. (ii) $\mathrm{M(C,S)}$. (iii) $\mathrm{CPM(C,C)}$. (iv) $\mathrm{CPM(C,S)}$. (v) $\mathrm{CSM(C,S)}$.

P r o o f.
(i)→(ii). Fix a compact metric space $(X, d)$. For each $n \in \mathbb{N}$, let

$$F_n = \{f \in X^m : m \in \mathbb{N} \wedge (\forall x \in X)\,(d(x, \mathrm{ran}(f)) \leq 1/n)\}.$$

Since $X$ is compact it is straightforward to verify that $F_n \neq \emptyset$ for all $n \in \mathbb{N}$. (By the compactness of $X$ there is a finite subset $F$ of $X$ such that $\{D(f, 1/n) : f \in F\}$ covers $X$, where $D(f, 1/n)$ is the open disc of radius $1/n$ centered at $f$. Then any bijection $g : |F| \longrightarrow F$ belongs to $F_n$.) For each $n \in \mathbb{N}$ let

$$k_n = \min\{m \in \mathbb{N} : \exists f\,(f \in (X^m \cap F_n)\}.$$

Put $Z_{k_n} = X^{k_n} \cap F_n$.

**Claim** *For each $n \in \mathbb{N}$, $Z_{k_n}$ is a compact metric space.*

P r o o f. Fix $n \in \mathbb{N}$. Since $Z_{k_n} \subset X^{k_n}$ and $X^{k_n}$ is (in $\mathsf{ZF}$) a compact metric space, it suffices to show that $Z_{k_n}$ is closed in $X^{k_n}$. To this end, let $f \in X^{k_n} \setminus Z_{k_n}$. Then there exists $x \in X$ such that $d(x, \mathrm{ran}(f)) = r > 1/n$. Let $\varepsilon = r - (1/n)$ and $V_f = \prod_{i<k_n} D(f(i), \varepsilon)$. Then $V_f$ is a neighborhood of $f$ which avoids $Z_{k_n}$. If not, let $g \in V_f \cap Z_{k_n}$. Since $d(x, \mathrm{ran}(g)) \leq 1/n$ and $\{d(x, g(i)) : i < k_n\}$ is a finite set, there exists $i < k_n$ such that $d(x, g(i)) \leq 1/n$. We have that $d(x, f(i)) \leq d(x, g(i)) + d(g(i), f(i)) < (1/n) + r - (1/n) = r$. This contradicts the fact that $d(x, \mathrm{ran}(f)) = r$ and completes the proof of the claim. □ Claim

By CAC(MC), let $f$ be a choice function for the family $\{Z_{k_n} : n \in \mathbb{N}\}$. Put $G = \{f(n)(i) : n \in \mathbb{N},\ i < k_n\}$. Clearly, $G$ is a countable set, and it is easily seen that $G$ is also dense in $X$. Thus, $X$ is separable as required.

(ii)→(iii). Let $\{(X_i, d_i) : i \in \mathbb{N}\}$ be a disjoint family of compact metric spaces and $(X, d)$ the product metric space of the $X_i$'s, where $d$ is the metric on $X$ given by (1). Let also $(Y, \sigma)$ be the sum of the $X_i$'s, where $\sigma$ is the metric on $Y$ given by (2). Set $Z = Y \cup \{*\}$, $* \notin Y$, and let $\varrho : Z \times Z \longrightarrow \mathbb{R}$ defined by

$$\varrho(x, y) = \varrho(y, x) = \begin{cases} \sigma(x, y) & \text{if } x, y \in Y, \\ 1/i & \text{if } x \in X_i \text{ and } y = *, \\ 0 & \text{if } x = y = *. \end{cases}$$

It can be readily verified that $(Z, \varrho)$ is a compact metric space, thus by M(C,S), $Z$ has a countable, dense subset. In [16, Theorem 2.1] we showed that in $\mathsf{ZF}$, every compact, separable metric space has a choice function for the family of its non-empty, closed subsets. Therefore, the family $\{F : (\exists i \in \mathbb{N})\,(F \text{ is a non-empty, closed subset of } X_i)\}$ has a choice function. Similarly now to the proof of [3, Theorem 15] we may conclude that $X$ is compact.

(iii)→(iv). In view of (i)→(ii), it suffices to show that CPM(C,C) implies CAC(MC). Let $\mathcal{X}$ be a family $\{(X_i, d_i) : i \in \mathbb{N}\}$ of non-empty, compact metric spaces. For each $i \in \mathbb{N}$, let $(Y_i, \varphi_i)$ be the metric space defined in the proof of CPM(L,L)→CAC(ML) of Theorem 7, and $(Y, \varphi)$ the product metric space of the $Y_i$'s. Clearly, $(Y_i, \varphi_i)$ is a compact metric space, hence by CPM(C,C), $Y$ is compact. If $X = \prod_{i\in\mathbb{N}} X_i = \emptyset$, then $U = \{\pi_i^{-1}(*_i) : i \in \mathbb{N}\}$ is an open cover of the compact space $Y$ having no finite subcover. This is a contradiction, thus $X \neq \emptyset$ and any element $f \in X$ is a choice function on the family $\mathcal{X}$.

(iv)→(v). Let $\{(X_i, d_i) : i \in \mathbb{N}\}$ be a disjoint family of compact metric spaces and $(X, d)$, $(Y, \sigma)$ be respectively the product and the sum of the $X_i$'s. By CPM(C,S), let $D = \{d_n : n \in \mathbb{N}\}$ be a dense subset of $X$. It is straightforward to verify that $G = \{d_n(i) : n, i \in \mathbb{N}\}$ is a countable, dense subset of $Y$.

(v)→(i). This is straightforward. □

**Theorem 9** $\mathrm{CPM(L,S)}$ *iff* $\mathrm{CSM(L,S)}$.

P r o o f. We only show (⇐) as the other direction can be proved as (iv)→(v) in the proof of Theorem 8. Let $\{(X_i, d_i) : i \in \mathbb{N}\}$ be a disjoint family of compact metric spaces and $(X, d)$, $(Y, \sigma)$ let respectively the product and the sum of the $X_i$'s. By CSM(L,S), let $D = \{d_i : i \in \mathbb{N}\}$ be a countable, dense subset of $Y$. Then $D_i = D \cap X_i$ is a countable, dense subset of $X_i$ for all $i \in \mathbb{N}$. We may define now a countable, dense subset of $X$ as in the proof of CAC→CPM$_{\text{le}}$(L,L)) of Theorem 7. □

**Theorem 10** *The following statements are equivalent:* (i) $\big(\mathrm{CPM(L,L)} + \mathrm{CAC}(\mathbb{R})\big)$. (ii) $\mathrm{CPM(L,hL)}$. (iii) $\mathrm{CSM(L,hL)}$. (iv) $\mathrm{CSM(L,L)}$.

Proof.
(i)→(ii). In [14, Corollary 14] it is shown that $\big((\mathrm{L}\times\mathrm{L}=\mathrm{L}) + \mathrm{CAC}(\mathbb{R})\big)$ implies M(L,S) (where $(\mathrm{L}\times\mathrm{L}=\mathrm{L})$ means "*the product of two Lindelöf metric spaces is Lindelöf*"). The conclusion now follows from Theorem 3.

(ii)→(iii). Clearly, CPM(L,hL) implies M(L,hL) (If $(X, d)$ is a Lindelöf metric space, then $X$, being homeomorphic to the – by CPM(L,hL) – hereditarily Lindelöf space $X \times \prod_{i\in\mathbb{N}}\{i\}$, is hereditarily Lindelöf.) Taking into account Theorem 4(ii) and the proof of (iv)→(v) of Theorem 8, we may conclude that CPM(L,hL) implies CSM(L,S). The conclusion now follows from Theorem 4(i) and Theorem 3.

(iii)→(iv). This is straightforward.

(iv)→(i). By Theorems 6 and 1 we conclude that CSM(L,L) implies the statement $(\mathbb{N}+\mathrm{L}=\mathrm{L})$ (i. e., *the sum of the discrete space* $\mathbb{N}$ *with a Lindelöf metric space is Lindelöf*). By Theorem 6 again we have that CSM(L,L) implies the principle CUC$|_{\mathrm{ML}}$: *A countable union of countable subsets of a Lindelöf metric space is countable.* In [14, Corollary 14] it is shown that the conjunction $\big((\mathbb{N}+\mathrm{L}) = \mathrm{L}) + \mathrm{CUC}|_{\mathrm{ML}}\big)$ is equivalent to M(L,hL). Thus, by Theorem 4(ii) we deduce that CSM(L,L) implies CSM(L,S), and by Theorem 9, also CPM(L,S). The conclusion now follows from Theorems 6 and 3. □

**Lemma 1** $\mathrm{M(L,hL)}$ *iff for every Lindelöf metric space* $(X, d)$*,* $X^\omega$ *is hereditarily Lindelöf.*

Proof.
(⇒). Let $(X, d)$ be a Lindelöf metric space. By Theorem 4(ii), it follows that $X$ is separable, hence $X^\omega$ is separable (as in the proof of CAC→ CPM$_{\text{le}}$(L,L) of Theorem 7). By Theorem 4(i), CAC($\mathbb{R}$) holds, and, by Theorem 3, it follows that $X^\omega$ is hereditarily Lindelöf as required.

(⇐). Fix a Lindelöf metric space $(X, d)$ and let $x \in X$. By our hypothesis, it follows that the subspace $Y = X \times \prod_{i\in\mathbb{N}}\{x\}$ of $X^\omega$ is hereditarily Lindelöf. Since $X$ is homeomorphic to $Y$, it follows that $X$ is hereditarily Lindelöf and the proof of the Lemma is complete. □

**Theorem 11**

(i) $\mathrm{CSM(L,L)}$ *implies* $\mathrm{CSM(L,S)}$.
(ii) $\mathrm{CPM(L,L)}$ *iff* $\mathrm{CPM(L,S)}$.
(iii) $\mathrm{CSM(L,L)}$ *iff* $\big(\mathrm{M(L,hL)} + \mathrm{CAC(ML)}\big)$.
(iv) $\mathrm{CPM(L,L)}$ *iff* $\big(\mathrm{M(L,S)} + \mathrm{CAC(ML)}\big)$.

Proof.
(i). This follows immediately from the proof of (iv)→(i) of Theorem 10.

(ii) (⇒). If CAC($\mathbb{R}$) fails, then by Theorem 2, Lindelöf=Compact, and the conclusion follows from Theorem 8. If CAC($\mathbb{R}$) holds, then the conclusion follows at once from Theorem 10, (i) and Theorem 9.

(ii) (⇐). If CAC($\mathbb{R}$) fails, then the conclusion follows from Theorems 2 and 8. If CAC($\mathbb{R}$) holds, then the conclusion follows from Theorem 1(iii).

(iii) (⇒). CSM(L,L)→CAC(ML) follows easily from (i), and CSM(L,L)→M(L,hL) follows from the proof of (iv)→(i) of Theorem 10.

(iii) (⇐). Fix a pairwise disjoint family $\{(X_i, d_i) : i \in \mathbb{N}\}$ of Lindelöf metric spaces. From Lemma 1 it follows that $X_i^{\mathbb{N}}$ is hereditarily Lindelöf for all $i \in \mathbb{N}$. For every $i,\, n \in \mathbb{N}$, put

$$Y_{i,n} = \{f \in X_i^{\mathbb{N}} : (\forall x, y \in \mathrm{ran}(f))\,(x \neq y \to d(x, y) \geq 1/n) \wedge (\forall x \in X_i)\,(d(x, \mathrm{ran}(f)) < 2/n)\}.$$

As in the proof of the Claim in [14, Theorem 13] we may show that $Y_{i,n} \neq \emptyset$ for all $i,\, n \in \mathbb{N}$. By M(L,hL), $Y_{i,n}$ is Lindelöf for all $i \in \mathbb{N}$. By CAC(ML), let $f = \{((i, n), g_{i,n}) : i,\, n \in \mathbb{N}\}$ be a choice function for the family $\{Y_{i,n} : i, n \in \mathbb{N}\}$. It is evident that $G = \{g_{i,n}(m) : i, n, m \in \mathbb{N}\}$ is countable and dense in the sum $X = \sum_{i\in\mathbb{N}} X_i$. Since CAC($\mathbb{R}$) holds, see Theorem 4(i), it follows by Theorem 1(iii) that $X$ is Lindelöf.

(iv) ($\Rightarrow$). From (ii) and Theorem 9, it follows that CPM(L,L) iff CSM(L,S). Thus, it suffices to show that CSM(L,S) implies M(L,S). Fix a Lindelöf metric space $(X, d)$. Clearly, $X$ can be considered as a subspace of the sum $X + \sum_{i\in\mathbb{N}}\{i\}$ which, in view of CSM(L,S), has a countable, dense subset $D$. Then $D \cap X$ is countable and dense in $X$. Therefore, M(L,S) holds. The second assertion follows from Theorem 7(i).

(iv) ($\Leftarrow$). If CAC($\mathbb{R}$) fails, the conclusion follows from Theorem 8. If CAC($\mathbb{R}$) holds, then by Theorem 4(ii) we have that M(L,hL) holds, thus, by (iii) CSM(L,L) is true, and by (i), (ii), CPM(L,L) is also true. □

**Corollary 2** $\big(\mathrm{M(L,hL)} + \mathrm{CAC(ML)}\big)$ *iff* $\big(\mathrm{CPM(L,L)} + \mathrm{CAC}(\mathbb{R})\big)$.

**Theorem 12** *The following statements are equivalent:* (i) $\mathrm{M(C,hL)}$. (ii) $\mathrm{CPM(C,hL)}$. (iii) $\mathrm{CSM(C,hL)}$.

Proof.

(i)→(ii). By Theorem 4(iii) we have that (i) implies M(C,S), hence by Theorem 8, CPM(C,C). Therefore, CPM(C,hL) holds.

(ii)→(iii). This can be proved as (ii)→(iii) of Theorem 10.

(iii)→(i). Let $(X, d)$ be a compact metric space. By CSM(C,hL), the sum $Y = X + \sum_{i\in\mathbb{N}}\{i\}$ is hereditarily Lindelöf, hence the subspace $X$ of $Y$ is hereditarily Lindelöf. □

## 4 Independence results

**Theorem 13**

(i) $\mathrm{CPM_{le}(C,C)}$, $\mathrm{CPM_{le}(L,L)}$ *and* $\mathrm{CSM_{le}(L,L)}$ *are strictly weaker than* CAC *in* $\mathsf{ZF}^0$. *Hence,* $\mathrm{CPM(C,C)}$, $\mathrm{CPM(L,L)}$ *and* $\mathrm{CSM(L,L)}$ *are strictly weaker than* CAC *in* $\mathsf{ZF}^0$.

(ii) *In* ZF, CUC *does not imply* $\mathrm{CSM(L,L)}$.

Proof.

(i) In [15] we showed that M(L,S), hence M(C,S), and M(L,hL) hold true in the basic Fraenkel permutation model, Model $\mathcal{N}1$ in [9]. Let us state three more facts which are valid in $\mathcal{N}1$, see [9]:

(a) A well ordered union of well orderable sets is well orderable, hence the axiom of choice for well ordered families of non-empty, well orderable sets (Form 165 in [9]) is valid in $\mathcal{N}1$;

(b) $\mathbb{R}$ is well orderable (this is true in every permutation model);

(c) the powerset of a well orderable set is well orderable (this is also true in every permutation model).

Let $\{(X_i, T_i) : i \in \mathbb{N}\}$ be a family of compact metrizable spaces in $\mathcal{N}1$. Since M(C,S) is true in $\mathcal{N}1$, we deduce that $|X_i| \le |\mathbb{R}|$ for all $i \in \mathbb{N}$. Thus, by (b) and (c) it follows that $X_i$ and $\wp(X_i)$ are well orderable for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ put

$$D_i = \{d : d : X_i \times X_i \longrightarrow [0, 1] \text{ is a metric on } X_i \text{ inducing } T_i\}.$$

By our hypothesis, each $D_i$ is non-empty. Furthermore, $D_i$ is well orderable ($D_i \subset \wp((X_i \times X_i) \times [0, 1])$ and since $X_i$ and $[0, 1]$ are well orderable, $X_i \times X_i \times [0, 1]$ is well orderable, e. g., lexicographically, and so by (c), $D_i$ is well orderable). By (a) it follows that there is a choice function $f$ for the family $\{D_i : i \in \mathbb{N}\}$. Since M(C,S) is true in $\mathcal{N}1$, it follows from Theorem 8 that the Tychonoff product of the family $\{(X_i, f(i)) : i \in \mathbb{N}\}$ is a compact metrizable space. Therefore, $\mathrm{CPM_{le}}$(C,C) is true in $\mathcal{N}1$.

Now, fix a family $\{(X_i, T_i) : i \in \mathbb{N}\}$ of pairwise disjoint, Lindelöf, metrizable spaces in $\mathcal{N}1$, and for any $i \in \mathbb{N}$ let $d_i \in \mathcal{N}1$ be a metric on $X_i$ which is bounded by 1 and induces $T_i$ (the choice of the metrics $d_i$ can be done as in the previous paragraph using now M(L,S) in $\mathcal{N}1$ and the statements (a), (b), and (c)). Let $(X, \sigma)$ be the sum of the $X_i$'s, where $\sigma(x, y) = d_i(x, y)$ if $x, y \in X_i$, and $\sigma(x, y) = 1$ for $x \ne y$ otherwise. For each $i \in \mathbb{N}$ let $A_i = \{f \in X_i^{\mathbb{N}} : \overline{\mathrm{ran}(f)} = X_i\}$. By M(L,S), we have that each $A_i$ is a non-empty set, and since $|X_i| \le |\mathbb{R}|$, and $|\mathbb{R}^\omega| = |\mathbb{R}|$ in ZF, it follows that for all $i \in \mathbb{N}$, $A_i$ is a well orderable set. By (a) let $f = \{(i, f_i) : i \in \mathbb{N}\}$ be a choice function for the family $\{A_i : i \in \mathbb{N}\}$. It can be readily verified that $D = \{f_i(n) : i, n \in \mathbb{N}\}$ is a countable and dense subset of $X$. By CAC($\mathbb{R}$) which holds in $\mathcal{N}1$ due to (b), $(X, \sigma)$ is Lindelöf, see Theorem 1. Therefore, $\mathrm{CSM_{le}}$(L,L) is true in $\mathcal{N}1$, and by Theorem 10, $\mathrm{CPM_{le}}$(L,L) is also true.

Finally, it is known, see [9], that CAC fails in $\mathcal{N}1$ and the independence results follow.

(ii) In the basic Cohen model $\mathcal{M}1$ in [9], CUC holds whereas CAC($\mathbb{R}$) fails, see [9]. By Theorem 6 it follows that CSM(L,L) also fails in $\mathcal{M}1$. □

Next we show that CUC does not imply CAC($\mathrm{M_{le}L}$) in $\mathsf{ZF}^0$ and also that none of CUC and Form 9 implies 418. In [4, Theorem 10] it was shown that in $\mathsf{ZF}^0$, CUC does not imply Form 418 but our proof here is easier. Furthermore, the status of the implication Form 9→Form 418 was indicated as unknown in the table of implications in [4].

**Theorem 14**

(i) *In* $\mathsf{ZF}^0$, CUC *does not imply* $\mathrm{CAC(M_{le}L)}$. *In particular,* $\mathrm{CAC}_\omega$ *does not imply* $\mathrm{CPM_{le}(L,L)}$.

(ii) *In* $\mathsf{ZF}^0$ *none of* CUC *and* Form 9 *implies* Form 418.

P r o o f. We give the description of a permutation model. The set of atoms $A = \bigcup\{A_n : n \in \omega\}$, where $A_n = \{a_{n,x} : x \in \mathbb{R}\}$ and $A_n$ is ordered like the reals by $\le_n$. $\mathcal{G}$ is the group of all permutations $\pi$ on $A$ such that $\pi|_{A_n} \in \mathrm{Aut}(A_n, \le_n)$ for all $n \in \omega$, where $\mathrm{Aut}(A_n, \le_n)$ is the group of all order automorphisms on $A_n$. The ideal of supports is the ideal generated by the set of all finite unions $\bigcup_{i\le n} A_i$, $n \in \omega$. Let $\mathcal{N}$ be the resulting permutation model. It follows that $A$ can be linearly ordered in $\mathcal{N}$ by requiring $a_{n,x} <^* a_{m,y}$ iff $n < m$, or $n = m$ and $x < y$. $<^*$ is in the model since it has empty support, i. e. every permutation fixes $<^*$.

(i) For each $n \in \omega$, let $T_n$ be the order topology on $A_n$ induced by $\le_n$. Clearly, $(A_n, T_n)$, being homeomorphic to $\mathbb{R}$ in $\mathcal{N}$ (the mapping $x \longmapsto a_{n,x}$, $x \in \mathbb{R}$, is in $\mathcal{N}$ since it has $A_n$ as its support), is a metrizable topological space in $\mathcal{N}$, and since CAC($\mathbb{R}$) holds in every permutation model, see [9], it follows from Theorem 1(v) that $A_n$ is a Lindelöf space. The family $T = \{T_n : n \in \omega\}$ is in the model $\mathcal{N}$ since for every permutation $\pi \in \mathcal{G}$, $\pi(T_n) = T_n$ for all $n \in \omega$ (every permutation maps basic open sets to basic open sets), hence $\pi(T) = T$. Therefore, $\{(A_n, T_n) : n \in \omega\}$ is a countable family in $\mathcal{N}$ (every permutation fixes the enumeration $\{(n, (A_n, T_n)) : n \in \omega\}$) consisting of Lindelöf, metrizable topological spaces. It is a routine work to verify that the family $\mathcal{A} = \{A_n : n \in \omega\}$ has no choice function in $\mathcal{N}$. Hence, CAC($\mathrm{M_{le}L}$) is false in $\mathcal{N}$.

We show next that CUC is true of $\mathcal{N}$. To this end, fix $X = \{X_i : i \in \omega\} \in \mathcal{N}$ a family of countable sets with $E = A_0 \cup \cdots \cup A_k$ as a support for $X$ and for $X_i$ for all $i \in \omega$. We will be done once we show that, for each $i \in \omega$, $E$ supports every element of $X_i$. Fix an $i \in \omega$. Suppose on the contrary that $\mathrm{fix}(E) \not\subseteq \mathrm{fix}(X_i)$. Then there exists a $y \in X_i$ and a $\varphi \in \mathrm{fix}(E)$ such that $\varphi(y) \ne y$. Let $E_y$ be a support for $y$. Then without loss of generality we may assume that $E_y = E \cup A_{k+1}$ and $\varphi$ fixes $A \setminus A_{k+1}$ pointwise. Denote the subgroup $\mathrm{fix}(A \setminus A_{k+1})$ ($\simeq \mathrm{Aut}(A_{k+1}, \le_{k+1})$) of $\mathcal{G}$ by $\boldsymbol{G}$. Put $D = \{\psi(y) : \psi \in \boldsymbol{G}\}$. $D$ is in the model, since it has $E_y$ as its support. Furthermore, $D \subseteq X_i$, since for all $\psi \in \boldsymbol{G}$, $\psi(X_i) = X_i$. Thus, $D$ is a countable set. Consider now the subgroup $H = \{\psi \in \boldsymbol{G} : \psi(y) = y\}$ of $\boldsymbol{G}$. $H$ is a proper subgroup of $\boldsymbol{G}$ since $\varphi \in \boldsymbol{G} \setminus H$. Since $D$ is countable, it can be readily verified that the index $|\boldsymbol{G} : H|$ of $H$ in $\boldsymbol{G}$ is countable. Now we use the following result of John Truss:

**Lemma 2** ([18]) *If* $H$ *is a subgroup of* $\mathrm{Aut}(\mathbb{R}, \le)$ (*the group of order-preserving permutations of* $\mathbb{R}$ ) *having index less than* $2^{\aleph_0}$, *then* $H = \mathrm{Aut}(\mathbb{R}, \le)$.

Thus we see that $H = \boldsymbol{G}$ which is a contradiction. Therefore, for each $i \in \omega$, $\mathrm{fix}(E) \subseteq \mathrm{fix}(X_i)$, hence $\bigcup X$ is supported by $E$ and $\bigcup X$ is well orderable in $\mathcal{N}$. Since $\bigcup X$ is countable in the ground model for $\mathsf{ZF}^0 + \mathrm{AC}$, it readily follows that $\bigcup X$ is countable in $\mathcal{N}$ as required.

(ii) For each $n \in \omega$, put $B_n = A_n \cup \{x_n, y_n\}$, where $x_n$, $y_n$ are distinct elements of $\mathcal{N}$ which do not belong to $A_n$, and extend the linear order $\le_n$ of $A_n$ by requiring $x_n$ and $y_n$ to be respectively the least and the greatest element of $B_n$. Clearly, each $B_n$ is a compact, metrizable space since it is homeomorphic to $[0, 1]$ in $\mathcal{N}$. If $B = \bigcup\{B_n : n \in \omega\}$ is metrizable, then it is normal (in $\mathsf{ZF}$, every metric space is normal; the proof of [19, Example (c) on page 100] goes through in $\mathsf{ZF}$ with some minor changes). Therefore, since the sets $X = \{x_n : n \in \omega\}$ and $Y = \{y_n : n \in \omega\}$ are closed and disjoint, there exist disjoint, open sets $U$ and $V$ in $B$ such that $X \subseteq U$ and $Y \subseteq V$. Then $f = \{(n, \sup(X \cap A_n)) : n \in \omega\}$ is a choice function for $\{A_n : n \in \omega\}$ which, in view of the proof of (i) is a contradiction. Since CUC is valid in $\mathcal{N}$, it follows that CUC does not imply Form 418 as required.

In order to complete the proof of the theorem it suffices to show that every infinite set $X \in \mathcal{N}$ has an infinite well orderable subset. Without loss of generality assume that $X$ is not well orderable and let $E = A_0 \cup \cdots \cup A_k$ be a support of $X$. Fix $y \in X$, $\varphi \in \mathrm{fix}(E)$ and let $E_y$, $\boldsymbol{G}$, $D$, and $H$ be as in the proof of (i). $D$ is a well orderable subset of $X$ in $\mathcal{N}$ since $\mathrm{fix}(E_y) \subseteq \mathrm{fix}(D)$. Indeed, let $\pi \in \mathrm{fix}(E_y)$. Then for all $\psi \in \boldsymbol{G}$ and all $a \in E_y$ we have that $(\pi \circ \psi)(a) = \pi(\psi(a)) = \psi(a)$ because $\psi(a) \in E_y$, and since $E_y$ is a support for $y$ and $\pi \circ \psi$, $\psi$ agree on $E_y$, it follows that $\pi(\psi(y)) = \psi(y)$, hence $\pi$ fixes $D$ pointwise. On the other hand, $D \subseteq X$ since any $\psi \in \boldsymbol{G}$ fixes $E$ pointwise, hence fixes $X$ setwise. We will be done once we show that $D$ is an infinite set. Assume the contrary. Then clearly $|\boldsymbol{G} : H|$ is finite, hence less than $2^{\aleph_0}$. By Lemma 2, $H = \boldsymbol{G}$ and this contradicts the fact that $\varphi \in \boldsymbol{G} \backslash H$. Thus, $D$ is infinite and the proof is complete. □

**Theorem 15**

(i) *In* ZF*,* CPM(L,L)*, hence* CAC(ML)*, does not imply any of* CSM(L,L)*,* M(L,hL)*, and* $\mathrm{CAC}(\mathbb{R})$*.*

(ii) *In* ZF*,* $\mathrm{CAC}(\mathbb{R})$ *does not imply any of* CAC(MC)*,* CSM(L,L)*,* CPM(L,L)*, and* M(L,hL)*.*

Proof.

(i) In [14] it is shown that M(C,S) holds true in the basic Cohen model $\mathcal{M}1$ of [9]. Thus, by Theorem 8 it follows that CPM(C,C) is true in $\mathcal{M}1$. Since $\mathrm{CAC}(\mathbb{R})$ fails in $\mathcal{M}1$, it follows by Theorem 4(i) that M(L,hL) fails in $\mathcal{M}1$, and by Theorem 2 that Lindelöf=Compact in $\mathcal{M}1$. Thus, CPM(L,L) is true in $\mathcal{M}1$ and by Theorem 7, CAC(ML) is also true in this model. On the other hand, by Theorem 6 we conclude that CSM(L,L) fails in $\mathcal{M}1$.

(ii) In [17, Theorem 2.1] a forcing model $\mathcal{M}$ was constructed such that $2^{\aleph_0} = \aleph_1$ in $\mathcal{M}$, hence $\mathrm{CAC}(\mathbb{R})$ holds in $\mathcal{M}$, whereas M(C,S) failed in $\mathcal{M}$. Thus, by Theorem 4(ii), M(L,hL) fails in $\mathcal{M}$, and by Theorem 8, CAC(MC), hence CAC(ML), fails in $\mathcal{M}$. Now by Theorems 7 and 10 we conclude that the statements CPM(L,L) and CSM(L,L) also fail in the model $\mathcal{M}$. □

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