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Continuing horrors of topology without choice

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Abstract

Various topological results are examined in models of Zermelo–Fraenkel set theory that do not satisfy the Axiom of Choice. In particular, it is shown that the proof of Urysohn's Metrization Theorem is entirely effective, whilst recalling that some choice is required for Urysohn's Lemma. $\mathbb{R}$ is paracompact and ω_1 may be paracompact but never metrizable. An example of a nonmetrizable paracompact manifold is given. Suslin lines, normality of LOTS and consequences of Countable Choice are also discussed.

Keywords: Axiom of Choice; Suslin line; Urysohn's Lemma; Urysohn's Metrization Theorem; ω_1

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1. Introduction

It is well known that there are theorems in both topology and functional analysis which are effectively equivalent to the Axiom of Choice (AC). For example, Kelly [11] proves that Tychonoff's Theorem is equivalent to AC. Bell and Fremlin [1] prove that AC is equivalent to the statement that the unit ball of the Banach dual of a normed vector space has an extreme point. Other results require only fragments of Choice. For instance, Tychonoff's Theorem for T_2 spaces is equivalent to the Boolean Prime Ideal Theorem which is equivalent to the existence of the Stone–Čech compactification, which in turn implies the Hahn–Banach Theorem (which is strictly weaker than BPI).

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We are interested here in the standing of various topological results in models of Zermelo–Fraenkel set theory (ZF) which do not satisfy (all of) AC. Some of these results are seen to be effective; others are shown to require Choice in some guise. We also obtain more examples of the sort van Douwen would describe as horrors.

In particular, we show that Urysohn's Metrization Theorem is a theorem of ZF (answering a question of Läuchli), but point out that Urysohn's Lemma is not. Furthermore, we answer a question of van Douwen, by proving in ZF that if every linearly ordered space is normal then every linearly ordered space is monotonically normal.

To give an illustration of the topological difficulties encountered without AC, consider the following result:

Theorem 1.1 [3] (The basic Cohen model). *It is consistent with ZF that there is a dense subset K of the reals $\mathbb{R}$ that has no countable subset.*

Corollary 1.2 (Con ZF). *A subspace of a separable metric space need not be separable.*

Corollary 1.3 [6] (Con ZF). *There is a metric space in which every Cauchy sequence converges but which is not Čech complete.*

2. Two pathological Suslin lines

Recall that a Suslin line is a linearly ordered space that is CCC but not separable. Suslin's hypothesis (SH) is the statement that there are no Suslin lines, and it is consistent with and independent of ZFC [12]. Assuming the existence of a Suslin line, one can prove in ZFC that its square cannot have the countable chain condition [12]. However, as a further consequence of Theorem 1.1, we have

Corollary 2.1 (Con ZF). *There is a Suslin line whose square is CCC.*

Proof. It is straightforward to prove from ZF that every second countable space is hereditarily CCC. Furthermore, we point out later on that in ZF every separable metric space is second countable. $\mathbb{R}$ is second countable (the rationals are a countable dense subset), and so is $\mathbb{R}^2$. Hence our set K is a Suslin line and $K \times K$ is CCC. Notice that this Suslin line is a subset of the reals! □

In [13], Läuchli gives an example (consistent with ZF) of a normal, locally compact space on which every continuous real-valued function is constant. Specifically, this highlights two facts:

Corollary 2.2 (Con ZF). *Normal spaces need not be completely regular, i.e., Urysohn's Lemma may fail.*

Corollary 2.3 (Con ZF). *Locally compact spaces need not be completely regular.*

(The proof in [4], for instance, appeals to Urysohn's Lemma.)

We reinforce these facts by presenting a compact Suslin line that demonstrates them both.

Example 2.4 (Con ZF). A compact Suslin line on which every continuous real-valued function is constant.

Proof. It is an exercise in [9, P96Ex15] to show that there is a model of ZF in which there is an infinite linearly ordered set whose only subsets are finite unions of intervals and points. Let $(X, <)$ be this linearly ordered set with the order topology. Without loss of generality, we may assume that X has a greatest and a least element, p and q respectively, so that $X = [p, q]$. We show that (a subset of) this space has the required properties.

Note 1. If $C \subseteq X$ is a nonempty closed set bounded below, then C has a (unique) least element; after all, C is the union of a finite number of closed intervals each bounded below, so certainly has a unique least element – just consider the endpoints.

Note 2. Consider all ordered pairs $\langle a, b\rangle$ such that $a < b$ and there is no c with $a < c < b$ (i.e., all "gaps"). The number of such gaps must be finite.

Proof. Suppose X had an infinite number of gaps, $A = \{\langle a_i, b_i\rangle : i \in I\}$.

Case 1: *A has an infinite subcollection of successive gaps, i.e., there exists an infinite $G \subseteq X$ such that* (a) *if $c_0 < c < c_1$ with $c_0, c_1 \in G$ then $c \in G$ and* (b) *if $c \in G$ then $c = a_i$ or $c = b_i$ for some $i \in I$.* Pick any $c_0 \in G$. As every point in G has a unique successor or a unique predecessor and G is infinite, we can construct a sequence $\{c_n \in G : n \in \omega\}$ with either $c_n < c_m$ whenever $n < m$ or $c_m < c_n$ whenever $n < m$. But then the set $\{c_{2n} : n \in \omega\}$ cannot be expressed as a finite union of intervals and points.

Case 2: *A has no infinite subcollection of sucessive gaps.* Then consider $B = \{b_i : i \in I\}$. B cannot be represented as a finite union of intervals or points because there are infinitely many $a_i \notin B$.

Therefore, there is some p', q' such that $[p', q'] \cap X$ is infinite and for all $a, b \in [p', q']$ whenever $a < b$ there is some $c \in [p', q']$ with $a < c < b$. Again, without loss of generality, $p' = p$ and $q' = q$, i.e., X is densely ordered.

X is connected. Suppose C and D were disjoint closed sets with $C \cup D = X$ and $p \in C$. Let b be the least element of D (so for all $c < b$, $c \in C$) and a be the greatest element of $C \cap [p, b]$. As $C \cup D = X$, we have $(a, b) = \emptyset$, a contradiction.

X is compact. Let $\mathscr{U}$ be an open cover of $[p, q]$. Define $\mathscr{U}_1$ to be the collection of all open intervals that refine $\mathscr{U}$ and A be the set of points that are contained in some finite chain of elements of $\mathscr{U}_1$ that contain the point p. Clearly A is open.

If $X-A$ were nonempty, by Note 1 it would have a least element a. Any interval in $\mathscr{U}_1$ containing a must meet A, as X is connected. Hence we can construct a finite chain from p to a, a contradiction.

Therefore $A=X$ and there is a finite chain of elements of $\mathscr{U}_1$ from p to q and hence we can find a finite subcover of $\mathscr{U}$.

X has no countably infinite subsets. Suppose $A=\{a_n: n\in\omega\}$ were a countable subset. As X is compact, A has a limit point, a. Either $[p, a)\cap A$ or $(a, q]\cap A$ is infinite. Without loss of generality, suppose the latter. Define $n_0\in\omega$ to be minimal such that $a_{n_0}\in(a, q]\cap A$ and $n_{m+1}\in\omega$ to be minimal such that $a_{n_{m+1}}\in(a, a_{n_m})\cap A$. Then a_{n_m} is a strictly decreasing sequence of elements of A. As above, $\{a_{n_{2m}}: m\in\omega\}$ cannot be expressed as a finite union of intervals or points.

If $f: X\to\mathbb{R}$ is continuous then f is constant. Suppose not. Then, without loss of generality, there are points $a<b$ with $f(a)=0$ and $f(b)=1$. Define $C_n=f^{-1}(1/n)$, which is nonempty (X is connected) and closed. So we may define x_n to be the least element of $C_n\cap[a, b]$. But the set of x_n's is now a countable subset.

X is CCC. The union of a collection of pairwise disjoint open sets in X can only be expressed as a finite union of intervals and points if the collection is finite, since X is connected.

X is not separable. It has no countably infinite subsets. □

Both Corollary 2.1 and Example 2.4 are worth comparing with Steprāns' result [18] that an uncountable tree is Suslin if and only if it has no uncountable continuous image in $\mathbb{R}$.

Also, the proof that X in Example 2.4 is compact is reminiscent of the proof of the Heine–Borel Theorem, that every closed bounded interval in $\mathbb{R}$ is compact. Indeed, the Heine–Borel Theorem is a theorem of ZF, as proved in [19].

3. On paracompactness

In this section, we present two contrasting results. Firstly, one can prove from the ZF axioms that $\mathbb{R}$ is paracompact. This is reassuring, if not unexpected. On the other hand, one cannot prove from ZF that ω_1 is not paracompact! As every set of ordinals has a prescribed well order, one might expect at first sight that ZF statements about ordinals could largely be settled without the explicit use of Choice. But evidently this is not the case.

Proposition 3.1 (ZF). *Every metric space X with a well-orderable dense subset is paracompact.*

Proof. Let $\mathscr{U}$ be an open cover of X and $E = \{x_\alpha: \alpha < \kappa\}$ be a well-ordered dense subset. For $\alpha < \kappa$, let $n_\alpha = \min\{m \in \omega: B_{1/m}(x_a) \subseteq U$ for some $U \in \mathscr{U}\}$. Then $\mathscr{V} = \{B_{1/n_\alpha}(x_\alpha): \alpha < \kappa\}$ is a well-ordered open refinement of $\mathscr{U}$. Now follow the standard proof of "metric implies paracompact" (e.g. [16]) to get a locally finite open refinement of $\mathscr{V}$ and hence of $\mathscr{U}$. □

Corollary 3.2 (ZF). *Every separable metric space is paracompact.*

Reassuringly, this implies that $\mathbb{R}$ is paracompact.

According to [10], Feferman and Lévy have constructed a model of ZF in which the cardinal $\aleph_1$ is singular: there is an increasing sequence of ordinals $\{\alpha_n: n \in \omega\}$ whose limit is ω_1. In fact, $\aleph_1$ is singular in any model in which the set of real numbers can be expressed as a countable union of countable sets. We show that in any model of ZF where ω_1 is singular, ω_1 with the order topology is paracompact. For the remainder of this section, we assume that we are in such a model.

Lemma 3.3 (ZF). *Let $\mathscr{V}$ be an open cover of ω_1 and $\beta < \gamma < \omega_1$. Then there is a (constructible) finite collection of disjoint clopen sets $\mathscr{U}$ that refines $\mathscr{V}$ and covers $(\beta, \gamma]$.*

Proof. Let $\gamma_0 = \gamma$. If $\gamma_n > 0$, let

$$\gamma_{n+1} = \min\{\delta < \gamma_n: \exists V \in \mathscr{V} \text{ and } (\delta, \gamma_n] \subseteq V\}.$$

This sequence of ordinals is well defined and $\gamma_{n+1} < \gamma$ for each n. There must be some n for which $\gamma_n \leqslant \beta$ (for otherwise we would have an infinite decreasing sequence of ordinals, contradicting the well order on ω_1) and hence $\{(\gamma_{i+1}, \gamma_i]: 0 \leqslant i < n\}$ is the required collection of sets. □

Corollary 3.4 (model). *ω_1 is paracompact.*

Proof. Let $\{\alpha_n: n \in \omega\}$ be a monotone increasing cofinal sequence in ω_1, with $\alpha_0 = 0$. Then $\{0\} \cup \{(\alpha_n, \alpha_{n+1}]: n \in \omega\}$ is an open cover of ω_1. Let $\mathscr{V}$ be any open cover of ω_1. By Lemma 3.3, there is a constructible disjoint open refinement $\mathscr{U}_n$ of $\mathscr{V}$ that covers $(\alpha_n, \alpha_{n+1}]$. Therefore $\{0\} \cup \bigcup_{n \in \omega} \mathscr{U}_n$ is certainly a locally finite open refinement. □

Notice that ω_1 can never be CCC or separable – consider the collection of successor ordinals smaller than ω_1. Furthermore, in the models with which we are concerned, ω_1 is not countably compact.

Lemma 3.5 (model). *ω_1 is $\aleph_1$-compact.*

Proof. Suppose that $A \subseteq \omega_1$ were an uncountable set. If $A \cap \alpha_n$ were finite for every $n \in \omega$, then A would be countable: $A_n = A \cap \alpha_n$ is increasing, so, using the well order of ω_1, a "countable" function can be constructed. Therefore, for some n, $A \cap \alpha_n$ is infinite. But $[0, \alpha_n]$ is compact, so $A \cap \alpha_n$ has a limit point in α_n and hence in ω_1. □

Corollary 3.6 (model). *ω_1 is DCCC.*

Proof. If $\mathscr{U}$ were an uncountable discrete collection of open sets, define $\beta_U = \min\{\beta \in \omega_1 : \beta \in U\}$ for each $U \in \mathscr{U}$. The collection $\{\beta_U : U \in \mathscr{U}\}$ is an uncountable closed discrete subset. □

Corollary 3.7 (model). *ω_1 is Lindelöf.*

Proof. If $\mathscr{V}$ were an open cover that had no countable subcover, then it would have an open refinement consisting of disjoint clopen sets, both uncountable and discrete. □

4. On metrizability

In contrast to Corollary 3.4 showing that ω_1 can be paracompact, ω_1 can never be metrizable. The following result is due to Robin Knight, which is included here with his kind permission.

Lemma 4.1 (ZF). *$\omega_1 + 1$ with the order topology is not metrizable.*

Proof. Suppose, for a contradiction, that $(\omega_1 + 1, <)$ were a metric space with associated metric d. Define $f : {}^{<\omega}\omega \to \omega_1 + 1$ as follows:

$$f(\emptyset) = \omega_1,$$

$$\text{for successor } f(s), \quad f(\widehat{sn}) = f(s) - 1 \text{ (or } \emptyset \text{ if } f(s) = \emptyset),$$

$$\text{for limit } f(s), \quad f(\widehat{sn}) = \min\{\alpha : d(\alpha, f(s)) < 1/2^n,\ \alpha < f(s)\}.$$

Notice that if $f(s)$ is a limit, $\{f(\widehat{sn}) : n \in \omega\}$ is cofinal in $f(s)$. We show that f is onto. For suppose that $\alpha \notin \operatorname{ran} f$. Then for all n, $\alpha + n \notin \operatorname{ran} f$, so $\alpha + \omega \notin \operatorname{ran} f$. At successor β, $\alpha + \beta \notin \operatorname{ran} f$ implies that $\alpha + \beta + 1 \notin \operatorname{ran} f$. For limit λ, $[\alpha, \lambda) \cap \operatorname{ran} f = \emptyset$ implies $\lambda \notin \operatorname{ran} f$. Hence $[\alpha, \omega_1] \cap \operatorname{ran} f = \emptyset$, contradiction.

So f must be onto. There is certainly a bijection $g : \omega \to {}^{<\omega}\omega$, e.g.

$$g(2^{n_1}3^{n_2} \dots p_m^{n_m}) = \langle n_1, n_2, \dots, n_m \rangle,$$

hence there is a surjection $h : \omega \to \omega_1$. □

Lemma 4.2 (ZF). *If $\mathrm{cf}(\omega_1) = \omega$, then ω_1 is not metrizable.*

Proof. Suppose $\{\alpha_n\colon n \in \omega\}$ were cofinal in ω. Define $f\colon {}^{<\omega}\omega \to \omega_1$ by $f(\emptyset) = \emptyset$, $f(\langle n \rangle) = \alpha_n$, and, for $|s| > 1$,

$$\text{for limit } f(s), \quad f(\widehat{sn}) = \min\{\alpha\colon d(\alpha, f(s)) < 1/2^n,\ \alpha < f(s)\},$$

$$\text{for successor } f(s), \quad f(\widehat{sn}) = f(s) - 1 \text{ (or } \emptyset \text{ if } f(s) = \emptyset).$$

Now follow the proof of Lemma 4.1 above. □

Lemma 4.3 (ZF). *Suppose* $\mathrm{cf}(\omega_1) = \omega_1$.

(1) *If A and B are closed unbounded subsets of ω_1, then $A \cap B \neq \emptyset$.*

(2) *If A is a closed unbounded subset and U is an open set containing A, then there is an $\alpha < \omega_1$ such that $(\alpha, \omega_1) \subseteq U$.*

Proof. (1) Define $\alpha_0 = \max\{\min A, \min B\}$, and inductively,

$$\alpha_{n+1} = \max\{\min\{a \in A\colon a > \alpha_0, \ldots, \alpha_n\}, \min\{b \in B : b > \alpha_0, \ldots, \alpha_n\}\}.$$

Then $\sup_{n \in \omega} \alpha_n \in A \cap B$.

(2) If not, then $B = X - U$ is closed and unbounded with $A \cap B = \emptyset$. □

Corollary 4.4 (ZF). *If* $\mathrm{cf}(\omega_1) = \omega_1$, *then ω_1 is not perfect, and hence cannot be metrizable.*

Proof. Let A be the set of limit ordinals smaller than ω_1. Then A is closed and unbounded. Suppose $\{U_n\colon n \in \omega\}$ is any collection of open sets, each containing A. Let α_n be the least ordinal such that $(\alpha_n, \omega_1) \subseteq U_n$ (by Lemma 4.3(2)). Then $(\sup_{n \in \omega} \alpha_n) + 1 \in \bigcap_{n \in \omega} U_n - A$. Thus A is not a G_δ-set in ω_1, and hence ω_1 is not perfect. □

Combining these results, we have shown that in ZF, ω_1 with the order topology is not a metric space. (Similar arguments show that it is also not a Moore space.)

Example 4.5. In any model where ω_1 is paracompact, the long line is a paracompact nonmetrizable manifold.

Proof. Just as in Lemma 3.3 and Corollary 3.4. □

We have already pointed out in Example 2.4 that Urysohn's Lemma cannot be proved from ZF alone, even for locally compact first countable spaces (the usual proof employs Countable Dependent Choice). In [13], Läuchli states that Urysohn's Metrization Theorem – that T_3 second countable spaces are metrizable – can be proved in ZF for locally compact spaces, but remarks that the general metrization theorem remains unsettled. We show that indeed the theorem holds true in ZF, by paying careful attention to details involving Choice in the standard proof.

Proposition 4.6 (ZF). *If X is T_3 and second countable, then X is normal.*

Proof. Let $\mathscr{B}$ be a countable base and H, K be disjoint closed sets. Let $\mathscr{B}^{(1)} = \{B \in \mathscr{B}\colon\ B \cap H \neq \emptyset$ and $\overline{B} \cap K = \emptyset\}$ and $\mathscr{B}^{(2)} = \{B \in \mathscr{B}\colon\ B \cap K \neq \emptyset$ and $\overline{B} \cap H = \emptyset\}$. As X is T_3, $\mathscr{B}^{(1)}$ covers H and $\mathscr{B}^{(2)}$ covers K. We can order $\mathscr{B}^{(1)} = \{B_n^{(1)}\colon\ n \in \omega\}$ and $\mathscr{B}^{(2)}$ using the order on $\mathscr{B}$.

Now define $U_n = \bigcup_{i=0}^{n} B_i^{(1)} - \overline{\bigcup_{j=0}^{n} B_j^{(2)}}$ and $V_n = \bigcup_{j=0}^{n} B_j^{(2)} - \overline{\bigcup_{i=0}^{n} B_i^{(1)}}$. Then $U = \bigcup_{n \in \omega} U_n$ and $V = \bigcup_{n \in \omega} V_n$ are disjoint open sets containing H and K respectively. □

In fact, we have shown that X is *monotonically* normal: if we let $D(H, K)$ be the set U described above, then D is a monotone normality operator for X (see [7]). Moreover, given the countable base $\mathscr{B}$, we have a constructive recipe for $D(H, K)$. We can now use this operator to give

Corollary 4.7 (ZF). *If X is second countable and regular, then it satisfies Urysohn's Lemma, i.e., that for any disjoint closed sets H, K there is a continuous $f: X \to \mathbb{R}$ with $f(H) \subseteq \{0\}$ and $f(K) \subseteq \{1\}$.*

Sketch of proof. Follow the traditional proof of Urysohn's Lemma (e.g. [4]). Use the operator D described above to define (rather than arbitrarily choose) the open sets in the construction of f. As D specifies which open sets to choose at each stage, f is constructed without Choice. □

This proof also shows that Urysohn's Lemma holds for monotonically normal spaces.

Corollary 4.8 (ZF) (Urysohn's Metrization Theorem). *If X is T_3 and second countable then X is metrizable.*

Proof. Follow the proof in [22], constructing the necessary functions using Corollary 4.7. □

5. More on monotone normality

Assuming AC, every linearly ordered set endowed with the order topology is normal. In [2], Birkhoff asked whether choice was required. It was later shown that some choice is required, but not its full strength [8,21].

Let LN be the statement "every linearly ordered space is normal" and LMN be the statement "every linearly ordered space is monotonically normal". In van Douwen's "horrors" paper [21], he asks whether LN and LMN are equivalent. Let

$<$ be a linear order on a set X. Then $*$ is said to be an *extreme choice function* of X if $*$ is a choice function on the collection of nonempty open intervals such that

$$a < b < c \text{ implies } a * c \in \{a * b, b, b * c\},$$
$$\text{where } a * c \text{ is the choice from } (a, c).$$

Let EC be the statement "every complete linear order has an extreme choice function". It is easy to see that AC implies EC: let $(X, <)$ be a complete linearly ordered set and take a well order, $\prec$, on X. Then set $a * b = \min_{\prec}(a, b)$. Morillon [14] has established in ZF that LN and EC are equivalent. From this we deduce the following:

Theorem 5.1 [5]. *In Zermelo–Fraenkel set theory, LN and LMN are equivalent.*

Proof. Clearly LMN implies LN in ZF, so we concentrate on the converse.

To this end, let $(X, <)$ be a linearly ordered space. Since monotone normality is hereditary in ZF and every linearly ordered space can be embedded in a complete linearly ordered space (again in ZF), we may suppose that $(X, <)$ is complete. Also we note that, in ZF, a space is monotonically normal if and only if there is an operator $V(.,.)$ which assigns to each $x \in X$ and basic open neighbourhood U of x a basic open neighbourhood $V(x, U)$ of x such that

$$V(x, U) \cap V(x', U') \neq \emptyset \text{ implies } x \in U' \text{ or } x' \in U.$$

So it is sufficient to define such a monotone normality operator for $(X, <)$.

For each $x \in X$ define

$$V(x, (x_0, x_1)) = (x_0 * x, x * x_1), \quad \text{where } x_0 < x < x_1,$$
$$V(x, (x_0, x]) = (x_0 * x, x], \quad \text{where } x_0 < x \text{ and } (x_0, x] \text{ is open},$$
$$V(x, [x, x_1)) = [x, x * x_1), \quad \text{where } x < x_1 \text{ and } [x, x_1) \text{ is open},$$
$$V(x, \{x\}) = \{x\}, \quad \text{where } \{x\} \text{ is open}.$$

Take $x < x'$ and basic neighbourhoods U, U' of x, x', respectively. Since the other cases are similar, we may suppose $U = (x_0, x_1)$, where $x_0 < x < x_1$ and $U' = (x'_0, x'_1)$, where $x'_0 < x' < x'_1$. Suppose for a contradiction that $V(x, U) \cap V(x', U') \neq \emptyset$, but $x \notin U'$ and $x' \notin U$. Then $x < x'_0 < x'_0 * x' < x * x_1 < x_1 < x'_1$.

From $x < x'_0 < x_1$, $x'_0 < x * x_1 < x_1$ and EC, it follows that $x * x_1 = x'_0 * x_1$.

From $x'_0 < x_1 < x'$, $x'_0 < x'_0 * x' < x_1$ and EC, it follows that $x'_0 * x' = x'_0 * x_1$.

Hence $x * x_1 = x'_0 * x_1 = x'_0 * x'$, which is impossible. □

6. On Countable Choice

It is well known that the Tychonoff Product Theorem is equivalent to the Axiom of Choice. Original proofs of this can be found in [11,20]. We present a proof which can easily be adapted to countable products.

Tychonoff's Theorem 6.1 (ZFC). *The product of compact spaces is compact.*

Proof. Suppose the product of every collection consisting of fewer than κ compact spaces is compact (which is provable in ZF for $\kappa = \omega$). Let $\{X_\alpha: \alpha < \kappa\}$ be a collection of compact spaces and $\mathscr{C}$ be a centred family of closed sets in $X = \prod_{\alpha<\kappa} X_\alpha$. Define

$$p_\beta : X \to \prod_{\alpha \leqslant \beta} X_\alpha \quad \text{by } p_\beta(x_0, x_1, \ldots, x_\alpha, \ldots) = (x_0, x_1, \ldots, x_\beta).$$

For $\alpha < \kappa$, pick $x_\alpha \in X_\alpha$ such that $(x_0, x_1, \ldots, x_\alpha) \in \bigcap\{\overline{p_\alpha(C)}: C \in \mathscr{C}\}$. This is always possible because $\Pi_{\alpha \leqslant \beta} X_\alpha$ is compact. We show that $\langle x_\alpha \rangle \in \bigcap \mathscr{C}$.

Suppose not. Then there is an open set $U = \Pi_{\alpha<\kappa} U_\alpha$ and $C \in \mathscr{C}$ such $\langle x_\alpha \rangle \in U$ and $U \cap C = \emptyset$. Pick γ so that $U_\alpha = X_\alpha$ whenever $\alpha > \gamma$, and let $V = \Pi_{\alpha \leqslant \gamma} U_\alpha$, open in $\Pi_{\alpha \leqslant \gamma} X_\alpha$. Notice that $(x_0, x_1, \ldots, x_\gamma) \in V$ so $V \cap p_\gamma(C) \neq \emptyset$, i.e., there is some $\langle y_\alpha \rangle \in C$ with $(y_0, y_1, \ldots, y_\gamma) \in V$. However, by the definition of V, $\langle y_\alpha \rangle \in U$, which contradicts $U \cap C = \emptyset$. $\square$

A modification of this proof shows that Countable Dependent Choice implies that countable products of compact spaces are compact. Moreover, by mimicking the usual proof that Tychonoff's Theorem implies AC, we see that if countable products of compact spaces are compact then Countable Choice follows.

Moreover, it is worth comparing Tychonoff's Theorem with product theorems for connected/Hausdorff spaces, results provable in ZF.

It is well known that for metric spaces, "being separable, Lindelöf, and second countable are equivalent". Countable Choice is enough to prove them equivalent. However, only one of the implications is provable just from ZF, namely, that separable metric spaces are second countable. The interested reader may like to complete the outlines of counterexamples and Proposition 6.5 below.

Example 6.2 (Con ZF). A separable space that is not Lindelöf: $\mathbb{R}$ in the basic Cohen model.

Example 6.3 (Con ZF). A second countable (hence CCC) metric space that is neither Lindelöf nor separable: the subset K of $\mathbb{R}$ in the basic Cohen model.

Example 6.4 (Con ZF). A compact (Lindelöf) metric space that is neither separable nor second countable. Let $\{A_n : n \in \omega\}$ be a countable collection of two-point sets whose union is uncountable [3] and $X = \bigcup A_n \cup \{p\}$. Isolate all points of $\bigcup A_n$ and define basic open sets containing p to consist of p and all but finitely many A_n.

Proposition 6.5 (ZF). *Each of the following statements imply those beneath it.*

(1) *The Countable Axiom of Choice.*

(2) *Every compact metric space is separable.*

(3) *The countable union of finite sets is countable.*

(4) *Every ω-tree either has an infinite chain or an infinite antichain.*
(5) *Every countable collection of finite sets has a choice function.*
(6) *The countable union of m-elements sets is countable* ($m \in \omega$).

Even statement (6) is unprovable in ZF: in [3], Cohen constructs a model where the Axiom of Choice fails for countable families of pairs.

(In a similar way, Countable Choice implies that every Lindelöf metric space is separable, which implies that countable unions of countable sets are countable.)

7. Questions

There are many topological problems that one could attempt to settle without Choice. Is there a ZF Dowker space, for instance? However, three questions of particular relevance to this discussion stand out:

Question 7.1. In ZF, metric spaces are monotonically normal which are, in turn, collectionwise normal and countably paracompact [17]. In ZF, are metric spaces paracompact?

The authors do not believe this is true [1].

Question 7.2. Let X be a linearly ordered topological space. Is X normal if and only if it is monotonically normal?

Question 7.3. Is Tychonoff's Theorem for countable products equivalent to Countable Choice or Countable Dependent Choice?

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References

[1] J.L. Bell and D.H. Fremlin, A geometric form of the axiom of choice, Fund. Math. 77 (1972) 167–170.
[2] G. Birkhoff, Lattice Theory (American Mathematical Society, Providence, RI, 1979).

[1] Added in proof: By modifying the example in [21], Steve Watson has settled this question by constructing a metric nonparacompact space consistent with ZF.

[3] P.J. Cohen, Set Theory and the Continuum Hypothesis (Benjamin, New York, 1966).
[4] R. Engelking, General Topology (Heldermann, Berlin, 1989).
[5] P.M. Gartside, Monotonicity in analytic topology, Ph.D. Thesis, Oxford (1993).
[6] R. Goldblatt, On the role of the Baire Category Theorem and dependent choice in the foundations of logic, J. Symbolic Logic 50 (1985) 412–422.
[7] G. Gruenhage, Generalized metric spaces, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 423–501.
[8] L. Haddad and M. Morillon, L'axiome de normalité pour les espaces totalement ordonnés, J. Symbolic Logic 55 (1990) 277–283.
[9] T. Jech, The Axiom of Choice (North-Holland, Amsterdam, 1973).
[10] T. Jech, About the Axiom of Choice, in: J. Barwise, ed., Handbook of Mathematical Logic (North-Holland, Amsterdam, 1977) 345–370.
[11] J.L. Kelly, The Tychonoff product theorem implies the Axiom of Choice, Fund. Math. 37 (1950) 75–76.
[12] K. Kunen, Set Theory, An Introduction to Independence Proofs (North-Holland, Amsterdam, 1980).
[13] H. Läuchli, Auswahlaxiom in der algebra, Comment. Math. Helv. 37 (1963) 1–18.
[14] M. Morillon, Extreme choices on complete lexicographic orders, Z. Math. Logik Grundlag. Math. 37 (1991) 353–355.
[15] M.D. Potter, Sets, An Introduction (Oxford University Press, New York, 1990).
[16] M.E. Rudin, A new proof that metric spaces are paracompact, Proc. Amer. Math. Soc. 20 (1969) 603.
[17] M.E. Rudin, Dowker spaces, in: K. Kunen and J.E. Vaughan, eds., Handbook of Set-Theoretic Topology (North-Holland, Amsterdam, 1984) 761–780.
[18] J. Steprāns, Trees and continuous mappings into the real line, Topology Appl. 12 (1981) 181–185.
[19] W.A. Sutherland, Introduction to Metric and Topological Spaces (Oxford University Press, Oxford, 1975).
[20] A. Tychonoff, Ein Fixpuntsatz, Math. Ann. 111 (1935) 767–776.
[21] E.K. van Douwen, Horrors of topology without AC: a nonnormal orderable space, Proc. Amer. Math. Soc. 95 (1985) 101–105.
[22] S. Willard, General Topology (Addison-Wesley, Reading, MA, 1970).