EXISTENCE OF HAAR MEASURE

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Abstract. In this presentation, I will prove that every compact topological group has a unique left-invariant measure with total measure 1. This presentation is for UT's real analysis prelim class.

Most of the measure theory we've done in this class has been within subsets of $\mathbb{R}^n$, even though measures are very general tools. Today, we will talk about a measure defined on a larger class of spaces, which generalizes the usual Lesbegue measure. I will define and prove its existence; Spencer will prove its uniqueness and provide some interesting examples; and Gill will talk about an application to harmonic analysis.

Definition. A *topological group* is a topological space G with a group structure: an identity, an associative multiplication map, and an inverse map, such that multiplication $G \times G \to G$ and inversion $G \to G$ are both continuous.

There are many examples, including $\mathbb{R}^n$ under addition; all Lie groups; and all finite groups (with the discrete topology).

Since a topological group G has a topology, we can talk about the σ-algebra of Borel sets on it, as usual generated by the open sets. But the group structure means we can also talk about *left- and right-invariance*: for every $g \in G$, we have continuous maps $\ell_g : G \to G$ and $r_g : G \to G$, sending $x \mapsto gx$ and $x \mapsto xg$, respectively. These should be read "left (resp. right) translation by g" (or "multiplication," or "action").

Often, we want something to be invariant under these maps. Specifically, we say that a measure is *left-invariant* if $\mu(S) = \mu(g \cdot S)$ for all $g \in G$, and define *right-invariance* similarly.

Theorem 1 (Haar). *Let G be a compact Hausdorff topological group. Then, there is a unique left-invariant Borel measure μ on G with total measure 1.*

This measure is called the *Haar measure*.
To prove this, we'll need the following theorem.

Theorem 2 (Riesz[1] representation theorem). *If X is a locally compact Hausdorff space, then for every positive linear functional $\Psi : C_c(X) \to \mathbb{R}$, there is a unique regular Borel measure μ such that for all $f \in C_c(X)$,*

$$\Psi(f) = \int_X f(x)\,\mathrm{d}\mu.$$

So really what we want is a left-invariant, positive linear functional $\Psi : C(G) \to \mathbb{R}$. We'll construct this in a few steps.

- Given a function $f \in C(G)$, let $L(f)$ denote the space of convex combinations of left-translates of f, and $R(f)$ denote that of right-translates of f. We'll show that $\overline{L(f)}$ is compact using the Arzelà-Ascoli theorem.
- Then, we will show that $\overline{L(f)}$ contains a constant.
- The first two steps work just as well for $\overline{R(f)}$; we'll show that there's a unique constant in $\overline{L(f)} \cap \overline{R(f)}$.
- Our functional Ψ will assign that unique constant to f. We'll show this is actually a linear functional, and that it has the needed properties.
- Then, we'll apply the Riesz representation theorem, and win.

Let's do this.

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[1]Sometimes also called the Riesz-Markov-Kakutani theorem.

Definition. If $f \in C(G)$, define

$$L(f) = \left\{ \sum_{i=1}^{n} a_i \ell_{g_i} f : n \geq 1, g_i \in G, a_i > 0, \sum_{i=1}^{n} a_i = 1 \right\}$$

$$R(f) = \left\{ \sum_{i=1}^{n} a_i r_{g_i} f : n \geq 1, g_i \in G, a_i > 0, \sum_{i=1}^{n} a_i = 1 \right\}.$$

That is, these really are the spaces of all convex combinations of left- or right-translations of f.

Lemma. $\overline{L(f)}$ *is compact.*

Proof. First, it's bounded:

$$\left\| \sum_{i=1}^{n} a_i \ell_{g_i} f \right\| \leq \sum_{i=1}^{n} a_i \| \ell_{g_i} f \| \leq \sum_{i=1}^{n} a_i \| f \|_\infty = \| f \|_\infty.$$

Then, we have to show that it's equicontinuous. Let $\varepsilon > 0$. Since multiplication is continuous, then the function $F(g, h) = f(g) - f(hg)$ is continuous, and for every $g \in G$, $F(g, e) = 0$, so there are neighborhoods M_g of g and N_g of e such that $|F(h, k)| < \varepsilon$ when $h \in M_g$ and $k \in M_g$. Since G is compact, it's covered by a finite subset of the M_g, say $M_{g_1}, \dots, M_{g_s}$, so let $N = \bigcap_{i=1}^{s} N_{g_i}$. This N is a neighborhood of the identity that has the property that if $g^{-1}h \in N$, then $|f(g) - f(h)| < \varepsilon$.

In this case, $(g_i g)^{-1} g_i^{-1} h = g^{-1} h \in N$ for each i, so $|f(g_i^{-1}g) - f(g_i^{-1}h)| < \varepsilon$. Thus, this is still true when we take a convex combination:

$$\left| \sum_{i=1}^{n} a_i \ell_{g_i} f(g) - \sum_{i=1}^{n} a_i \ell_{g_i} f(h) \right| \leq \sum_{i=1}^{n} a_i | \ell_{g_i} f(g) - \ell_{g_i} f(h)|$$

$$= \sum_{i=1}^{n} a_i | f(g_i^{-1} g) - f(g_i^{-1} h)| = \sum_{1}^{n} a_i \varepsilon = \varepsilon.$$

Thus, $\overline{L(f)}$ is equicontinuous, so by the Arzelà-Ascoli theorem, it's compact. $\boxtimes$

Lemma. $\overline{L(f)}$ *contains a constant function.*

Proof. Since G is compact, then for any $f \in C(G)$ we can define $M(f) = \max_{g \in G} f(g)$, $m(f) = \min_{g \in G} f(g)$, and $v(f) = M(f) - m(f)$. v is also continuous, so since $\overline{L(f)}$ is compact, v has a minimum f^* on it. It suffices to prove $f^* = 0$.

Suppose not; then, $M(f^*) > m(f^*)$, so the set $E = \{g \in G : f^*(g) > (M(f) + m(f))/2\}$ is a nonempty open set. Thus, $\{gE : g \in G\}$ is an open cover of G, meaning it has a finite subcover $g_1 E, \dots, g_n E$. Define $\phi : G \to \mathbb{R}$ by

$$\phi(g) = \sum_{i=1}^{n} \frac{1}{n} \ell_{g_i} f^*(g),$$

so $\phi \in C(G)$, and in fact $\phi \in \overline{L(f)}$, since $f^* \in \overline{L(f)}$. We also have that $M(\phi) \leq M(f^*)$, but $m(\phi) \geq m(f^*) + v(f^*)/2n$. This is not immediate; the reason is that for any $g \in G$, $g \in g_i E$ for some i, so $g_i^{-1} g \in E$ and $f^*(g_i^{-1}g) \geq (M(f^*) + m(f^*))/2$. Hence,

$$\phi(g) = \frac{f^*(g_i^{-1}g)}{n} + \sum_{j \neq i} f^*(g_j^{-1}g) > \frac{M(f^*) + m(f^*)}{2n} + \frac{(n-1)m(f^*)}{n} = m(f^*) + \frac{v(f^*)}{2n}.$$

Thus, $v(\phi) \leq M(f^*) - m(f^*) - v(f^*)/2n < v(f^*)$, which was the minimum.

This is a contradiction, so f^* must be constant. $\boxtimes$

Essentially the same proofs work for $\overline{R(f)}$ too, so we'll assume it's compact and contains some constant function too.

Lemma. $\overline{L(f)} \cap \overline{R(f)}$ *contains a unique constant function.*

Proof. Suppose α is a constant in $\overline{L(f)}$ and β is a constant in $\overline{R(f)}$; we want to show that $\alpha = \beta$.

Let $\varepsilon > 0$. We can choose $a_1, \ldots, a_n, b_1, \ldots, b_m > 0$ and $g_i, h_j \in G$ such that

$$\left\| \alpha - \sum_{i=1}^{n} a_i \ell_{g_i} f \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \beta - \sum_{j=1}^{m} b_j r_{h_j} f \right\| < \frac{\varepsilon}{2}.$$

In particular, this means for all $g \in G$ and for all j,

$$\left| \alpha - \sum_{i=1}^{n} a_i f(g_i^{-1} g h_j) \right| < \frac{\varepsilon}{2}.$$

Now, we bring in a convex combination:

$$\left| \alpha - \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j f(g_i^{-1} g h_j) \right| \le \sum_{j=1}^{m} b_j \left| \alpha - \sum_{i=1}^{n} a_i f(g_i^{-1} g h_j) \right| < \sum_{1}^{m} b_j \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

The same argument works for β, so by the triangle inequality, $|\alpha - \beta| < \varepsilon$. ⊠

Now, for a given $f \in C(G)$, there's a unique constant $c_f \in \overline{L(f)} \cap \overline{R(f)}$. Let $\Psi(f) = c_f$. The definitions of $L(f)$ and $R(f)$ imply that $\overline{L(\lambda f)} = \lambda \overline{L(f)}$ and similarly for $R(f)$, so $\Psi(\lambda f) = \lambda \Psi(f)$.

Then, we need to show that $\Psi(f_1 + f_2) = \Psi(f_1) + \Psi(f_2)$. This is actually by far the most involved calculation in the proof, and so I'm going to skip over it to avoid running out of time. The idea is to use a good approximation of $\Psi(f_1)$ in terms of f_1 to build an approximation for it in terms of f_2, and therefore also an approximation of $\Psi(f_2)$ in terms of f_1. This implies that $\Psi(f_1) + \Psi(f_2)$ can be approximated well by something in $\overline{L(f_1 + f_2)} \cap \overline{R(f_1 + f_2)}$, whose only constant function is $\Psi(f_1 + f_2)$.[2]

Since $\overline{L(f)}$ and $\overline{R(f)}$ are bounded by $\|f\|_\infty$, then $|\Psi(f)| \le \|f\|_\infty$, so Ψ is a bounded functional.

Since constant functions are preserved under left and right translation, then $\Psi(\ell_g f) = \Psi(f)$ and $\Psi(r_g f) = \Psi(f)$, and since $L(f)$ and $R(f)$ are defined using positive linear combinations, then if $f \ge 0$, then $\Psi(f) \ge 0$ as well. Thus, we have constructed a G-invariant, positive, bounded linear functional $C(G) \to \mathbb{R}$.

Now, we apply the Riesz representation theorem to get a measure μ on the Borel σ-algebra of G such that for any $f \in C(G)$, $\int_G f \, \mathrm{d}\mu = \Psi(f)$. Since Ψ is left- and right-invariant, then so is μ, and the total measure is

$$\mu(G) = \int_G \mathrm{d}\mu = \Psi(1),$$

but 1 is G-invariant, and all convex combinations of it are just 1 again, so the unique constant function in $\overline{L(1)} \cap \overline{R(1)}$ is 1, so $\mu(G) = 1$.

1. Jokes

(1) What do you call the Haar measure on a torus? The Les-bagel measure.
(2) What's a good name for measure theory on locally compact groups? Haarmonic analysis.
(3) What's the best response to a harmonic analysis joke? "Hardy Haar Haar."
(4) Since the Haar measure is unique, we conclude that Lesbeguers can't be choosers.
(5) The Haar measure is an integral tool in representation theory.
(6) The proof of existence of Haar measure for compact Lie groups is much cleaner: in that setting, we value form over function.

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[2]See http://www.math.ubc.ca/~feldman/m606/haar.pdf for the full story.