GENERAL COUNTABLE PRODUCT MEASURES

JUAN CARLOS SAMPEDRO

ABSTRACT. A new construction of product measures is given for an arbitrary sequence of measure spaces via outer measure techniques giving a coherent extension of the classical theory of finite product measures to countable many. Moreover, the Lebesgue spaces of this measures are simplified in terms of finite product measures. This decomposition simplifies all the considerations regarding infinite dimensional integration and gives to it a computational framework.

1. Introduction

The classical theory of product measures deals with two measure spaces (X, Σ_X, μ_X) and (Y, Σ_Y, μ_Y) in order to construct the product measure space $(X \times Y, \Sigma_X \otimes \Sigma_Y, \mu_X \otimes \mu_Y)$, where $\Sigma_X \otimes \Sigma_Y$ is the σ -algebra generated by $\mathcal{R}_{X \times Y} := \{A \times B : A \in \Sigma_X, B \in \Sigma_Y\}$ and $\mu_X \otimes \mu_Y$ is a measure on $\Sigma_X \otimes \Sigma_Y$ satisfying the identity

(1)
$$(\mu_X \otimes \mu_Y)(A \times B) = \mu_X(A) \cdot \mu_Y(B) \text{ for every } A \in \Sigma_X, B \in \Sigma_Y.$$

The most common method to prove the existence of this measure is through the celebrated Caratheodory extension theorem as follows. Denote by $\mathcal{U}(\mathcal{R}_{X\times Y})$ the family of finite unions of elements of $\mathcal{R}_{X\times Y}$, then it is easy to verify that $\mathcal{U}(\mathcal{R}_{X\times Y})$ is an algebra of subsets of $X\times Y$ and that every element of $\mathcal{U}(\mathcal{R}_{X\times Y})$ can be written as a finite union of pairwise disjoint members of $\mathcal{R}_{X\times Y}$. Define the set function

$$\mu_0: \quad \mathcal{U}(\mathcal{R}_{X\times Y}) \quad \longrightarrow \quad [0, +\infty] \\ \biguplus_{i=1}^N A_i \times B_i \quad \mapsto \quad \sum_{i=1}^N \mu_X(A_i) \cdot \mu_Y(B_i).$$

It is classical (see e.g. [3, 6]), that the set function μ_0 is well defined and σ -additive on $\mathcal{U}(\mathcal{R}_{X\times Y})$. Hence by Caratheodory extension theorem, there exists a measure μ on the σ -algebra $\Sigma_X \otimes \Sigma_Y$ that extends μ_0 . Therefore, μ satisfies identity (1) for each $A \in \Sigma_X$ and $B \in \Sigma_Y$.

Consider now a sequence $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ of measure spaces. Our aim is to generalize the classical product measure space to countable many, i.e., to construct the infinite product measure space $(\bigotimes_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i, \bigotimes_{i \in \mathbb{N}} \mu_i)$ where $\bigotimes_{i \in \mathbb{N}} \Sigma_i$ is the σ -algebra generated by

$$\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}} := \left\{ \sum_{i=1}^m \mathscr{C}_i \times \sum_{i=m+1}^\infty \Omega_i : \mathscr{C}_i \in \Sigma_i, \ \forall i \in \{1, 2, ..., m\} \text{ and } m \in \mathbb{N} \right\}$$

and $\bigotimes_{i\in\mathbb{N}}\mu_i$ is a measure satisfying an analogue of identity (1) for this general setting. For instance, if the finiteness condition $\prod_{i\in\mathbb{N}}\mu_i(\Omega_i)\in[0,+\infty)$ holds, the measure $\bigotimes_{i\in\mathbb{N}}\mu_i$ must satisfy the identity

²⁰¹⁰ Mathematics Subject Classification. 28A35 (primary), 28C20, 46G12, 46B25 (secondary). Key words and phrases. Infinite Dimensional Integration, Infinite Product Measures, L_p spaces.

(2)
$$\bigotimes_{i \in \mathbb{N}} \mu_i \left(\bigotimes_{i=1}^m \mathscr{C}_i \times \bigotimes_{i=m+1}^\infty \Omega_i \right) = \prod_{i=1}^m \mu_i (\mathscr{C}_i) \cdot \prod_{i=m+1}^\infty \mu_i (\Omega_i)$$

for each $\times_{i=1}^m \mathscr{C}_i \times \times_{i=m+1}^\infty \Omega_i \in \mathcal{R}(\Sigma_i)_{i \in \mathbb{N}}$.

The first first attempt to address this problem was for the particular case of probability spaces. In 1933, A. Kolmogoroff proved in [12] the existence of a probability measure $\bigotimes_{i\in\mathbb{N}} m_{[0,1]}$ on the measurable space $([0,1]^{\mathbb{N}},\bigotimes_{i\in\mathbb{N}}\mathcal{B}_{[0,1]})$, where $\mathcal{B}_{[0,1]}$ and $m_{[0,1]}$ are the Borel σ -algebra of [0,1] and the Lebesgue measure of [0,1] respectively, such that identity (2) holds for every

$$\underset{i=1}{\overset{m}{\times}} \mathscr{C}_i \times \underset{i=m+1}{\overset{\infty}{\times}} [0,1] \in \mathcal{R}(\mathcal{B}_{[0,1]})_{i \in \mathbb{N}}.$$

Kolmogoroff's proof was based on the compactness of the product space $[0,1]^{\mathbb{N}}$. More general cases were discussed by Z. Lomnicki and S. Ulam in 1934 on the reference [14]. In 1943, S. Kakutani generalized for general probability spaces the results of Kolmogoroff, Lomnicki and Ulam proving in [10] the next celebrated result.

Theorem 1.1. Given $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ a family of probability spaces, there exists an unique probability measure $\bigotimes_{i \in \mathbb{N}} \mu_i$ on the measurable space $(\bigotimes_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying for every $\mathscr{C} = \bigotimes_{i=1}^m \mathscr{C}_i \times \bigotimes_{i=m+1}^\infty \Omega_i \in \mathcal{R}(\Sigma_i)_{i \in \mathbb{N}}$ the following identity

$$\bigotimes_{i\in\mathbb{N}}\mu_i\left(\mathscr{C}\right) = \prod_{i=1}^m \mu_i(\mathscr{C}_i).$$

Kakutani's proof of this result has become standard in Probability and Measure Theory. The key tool of the proof relies on a result of E. Hopf (cf. [8], [19, Theorem 3.2]). In 1996, S. Saeki gives in [17] a new proof of Theorem 1.1, proving it in a more natural terms without the use of Hopf's result. If the measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ are not of probability but satisfy the finiteness condition $\prod_{i \in \mathbb{N}} \mu_i(\Omega_j) \in [0, +\infty)$, then normalizing each measure space, it can be also proven as a rather direct consequence of Theorem 1.1, the existence of a measure on $(\times_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying identity (2).

On other hand, if $\prod_{i\in\mathbb{N}} \mu_i(\Omega_i) = +\infty$, identity (2) is no longer useful for the construction of infinite product measures since its value is always infinite. Nevertheless, we can ask the measure to verify the new identity

(3)
$$\bigotimes_{i \in \mathbb{N}} \mu_i \left(\bigotimes_{i \in \mathbb{N}} \mathscr{C}_i \right) = \prod_{i \in \mathbb{N}} \mu_i (\mathscr{C}_i)$$

for each $\times_{i\in\mathbb{N}} \mathscr{C}_i \in \mathcal{F}(\Sigma_i, \mu_i)_{i\in\mathbb{N}}$, where

$$\mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}} := \left\{ \bigotimes_{i \in \mathbb{N}} \mathscr{C}_i : \mathscr{C}_i \in \Sigma_i, \ \forall i \in \mathbb{N} \ \text{and} \ \prod_{i \in \mathbb{N}} \mu_i(\mathscr{C}_i) \in [0, +\infty) \right\}.$$

Therefore, the natural extension of the classical theory for the nonfinite case is the product measure space $(X_{i\in\mathbb{N}} \Omega_i, \bigotimes_{i\in\mathbb{N}} \Sigma_i, \bigotimes_{i\in\mathbb{N}} \mu_i)$ where $\bigotimes_{i\in\mathbb{N}} \mu_i$ is a measure satisfying (3). The first purpose of this article is to provide a construction of this product space for an arbitrary family of measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i\in\mathbb{N}}$. This matter has not been ignored in the last century and several attempts have been made trying to formalize it.

In 1963, E.O. Elliott and A.P. Morse published a paper [5] constructing this kind of product spaces through a reformulation of the classical infinite product called *plus product*. Let $\mathfrak{a} = (a_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers and $\mathscr{A}(\mathfrak{a}) = \{n \in \mathbb{N} : a_n > 1\}$, then they defined the *plus product* of the sequence $(a_n)_{n \in \mathbb{N}}$ by

$$\prod_{n\in\mathbb{N}}^{+}a_{n}:=\prod_{n\in\mathscr{A}(\mathfrak{a})}a_{n}\cdot\prod_{n\notin\mathscr{A}(\mathfrak{a})}a_{n},$$

with the convention $0 \cdot \infty = \infty \cdot 0 = 0$ and setting the value of the empty product to 1. The main purpose of defining this concept lies in the fact that the plus product exists for every positive sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers, which does not happen with the classic product (take for instance the sequence defined by

$$a_n := \begin{cases} 2 & \text{if } n \equiv 0 \mod 2\\ \frac{1}{2} & \text{if } n \equiv 1 \mod 2. \end{cases}$$

for each $n \in \mathbb{N}$). The classical and the plus product do not, in general, coincide, but if the condition $\prod_{n \in \mathscr{A}(\mathfrak{a})} a_n < +\infty$ is satisfied, they coincide. We define the set of *finite plus rectangles* by

$$\mathcal{F}^+(\Sigma_i, \mu_i)_{i \in \mathbb{N}} := \left\{ \bigotimes_{i \in \mathbb{N}} \mathscr{C}_i : \mathscr{C}_i \in \Sigma_i, \forall i \in \mathbb{N} \text{ and } \prod_{i \in \mathbb{N}}^+ \mu_i(\mathscr{C}_i) \in [0, +\infty) \right\}.$$

Elliot and Morse proved that given a family of measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$, there exists a measure λ_{EM} on the measurable space $(\times_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying for every $\mathscr{C} = \times_{i \in \mathbb{N}} \mathscr{C}_i \in \mathcal{F}^+(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$, the following identity

(4)
$$\lambda_{EM}(\mathscr{C}) = \prod_{i \in \mathbb{N}}^{+} \mu_i(\mathscr{C}_i).$$

It must be observed that if a finite plus rectangle $\mathscr{C} = X_{i \in \mathbb{N}} \mathscr{C}_i \in \mathcal{F}^+(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ satisfies $\prod_i^+ \mu_i(\mathscr{C}_i) \neq 0$, then its plus product must coincide with the classical product $\prod_i \mu_i(\mathscr{C}_i)$. However, if $\prod_i^+ \mu_i(\mathscr{C}_i) = 0$, the products does not, in general, coincide. In consequence, this result does not stablish the existence of the required product measure since there are substantial sequence satisfying $\prod_i^+ \mu_i(\mathscr{C}_i) = 0$ with nonzero classical product (take for instance the sequence $a_n = \exp((-1)^{n+1}/n)$ for each $n \in \mathbb{N}$).

In 2004 R. Baker, proved in [2, Theorem I], the existence of the required product measure for the particular case in which the involved spaces $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ are locally compact metric and satisfy the property \mathcal{D} :

(\mathcal{D}) For every $i \in \mathbb{N}$ and $\delta > 0$, there exists a sequence $\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{B}_{X_i}$ such that $d_i(A_j) < \delta$ for each $j \in \mathbb{N}$ and

$$X_i = \bigcup_{j \in \mathbb{N}} A_j$$

where $d_i(A_j)$ stands for the diameter of A_j in X_i and \mathcal{B}_{X_i} for the borel σ -algebra of X_i .

He proved that given a sequence $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ satisfying the previous properties and a corresponding sequence of regular Borel measure spaces $\{(X_i, \mathcal{B}_{X_i}, \mu_{X_i})\}_{i \in \mathbb{N}}$, there exists a measure λ_B on the measurable space $(\bigotimes_{i \in \mathbb{N}} X_i, \bigotimes_{i \in \mathbb{N}} \mathcal{B}_{X_i})$ satisfying for every $\mathscr{C} =$

 $\times_{i\in\mathbb{N}}\mathscr{C}_i\in\mathcal{F}(\mathcal{B}_{X_i},\mu_{X_i})_{i\in\mathbb{N}}$, the following identity

$$\lambda_B(\mathscr{C}) = \prod_{i \in \mathbb{N}} \mu_{X_i}(\mathscr{C}_i).$$

In 2005, P. A. Loeb and D. A. Ross gave in [13, Theorem 1.1] another attempt of formalizing the product measure via Nonstandard Analysis techniques and Loeb Measures [1, §4]. They established that given a sequence of Hausdorff topological spaces $\{(X_i, \mathcal{T}_i)\}_{i \in \mathbb{N}}$ and a corresponding sequence of regular Borel measure spaces $\{(X_i, \mathcal{B}_{X_i}, \mu_{X_i})\}_{i \in \mathbb{N}}$, there exists a measure λ_{LR} on the measurable space $(X_{i \in \mathbb{N}}, X_i, \bigotimes_{i \in \mathbb{N}}, \mathcal{B}_{X_i})$ such that if $K_i \subset X_i$ is compact for all $i \in \mathbb{N}$ and $X_{i \in \mathbb{N}}, K_i \in \mathcal{F}(\mathcal{B}_{X_i}, \mu_{X_i})_{i \in \mathbb{N}}$, then the following identity holds

$$\lambda_{LR}\left(\bigotimes_{i\in\mathbb{N}}K_i\right)=\prod_{i=1}^{\infty}\mu_{X_i}(K_i).$$

Finally, in 2011 G.R. Pantsulaia presented in [15, Theorem 3.10] the best generalization of product measures to countable many till the date. He proved the following.

Theorem 1.2. Let $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be a family of σ -finite measure spaces satisfying $\mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}} \neq \emptyset$, then there exists a measure λ_P on $(\bigotimes_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying for each $\mathscr{C} = \bigotimes_{i \in \mathbb{N}} \mathscr{C}_i \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ the identity

$$\lambda_P(\mathscr{C}) = \prod_{i \in \mathbb{N}} \mu_i(\mathscr{C}_i).$$

This result provides a standard prove of the existence of the product measure without imposing any topological assumption over the involved spaces. However, the measure spaces $(\Omega_i, \Sigma_i, \mu_i)$ must be σ -finite for every $i \in \mathbb{N}$ and the measure λ_P is not constructed via outer measure techniques, that it is not a necessary condition but it always facilitates several computations like Fubini theorem's proof.

This paper is organized as follows. In section one, we give a construction of the product measure $\bigotimes_{i\in\mathbb{N}}\mu_i$ satisfying identity (3), via outer measure techniques without imposing any condition on the underlying measure spaces. This result will extend Theorem 1.2 establishing the best extension of the classical theory till the date. In section three and four we simplify the structure of Lebesgue spaces of this measure in terms of finite product measures via a theorem due to B. Jessen [9]. Roughly speaking, we decompose $L_p(\bigotimes_{i\in\mathbb{N}}\Omega_i,\bigotimes_{i\in\mathbb{N}}\Sigma_i,\bigotimes_{i\in\mathbb{N}}\mu_i)$ in terms of $L_p(\bigotimes_{i=1}^n\Omega_i,\bigotimes_{i=1}^n\Sigma_i,\bigotimes_{i=1}^n\mu_i)$, for each $1\leq p<\infty$. This decomposition allows to consider infinite dimensional functions as a sequence of finite dimensional ones and stabilises a computational method to compute the integral of this kind of maps. Finally, thanks to the previous considerations, in section five we construct a measure for each infinite dimensional separable Banach space X following the work of T. Gill, A. Kirtadze, G. Pantsulaia and A. Plichko [7] and we state an analogue of the cited decomposition for Lebesgue spaces of this measure.

2. Existence of the Product Measure

In this section, given an arbitrary sequence of measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$, we will prove the existence of a measure $\bigotimes_{i \in \mathbb{N}} \mu_i$ on the measurable space $(\bigotimes_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying the identity

(5)
$$\bigotimes_{i \in \mathbb{N}} \mu_i(\mathscr{C}) = \prod_{i \in \mathbb{N}} \mu_i(\mathscr{C}_i)$$

for each $\mathscr{C} = \underset{i \in \mathbb{N}}{\times} \mathscr{C}_i \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. A key fact we will use is the existence of this measure for the particular case in which the involved measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ satisfies the finiteness condition $\prod_i \mu_i(\Omega_i) \in [0, +\infty)$. More precisely, there exists a measure $\bigotimes_{i \in \mathbb{N}} \mu_i$ on the measurable space $(\underset{i \in \mathbb{N}}{\times} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying for each $\mathscr{C} = \underset{i=1}{\times} \mathscr{C}_i \times \underset{i=n+1}{\times} \Omega_i \in \mathcal{R}(\Sigma_i)_{i \in \mathbb{N}}$, the identity

$$\bigotimes_{i \in \mathbb{N}} \mu_i(\mathscr{C}) = \prod_{i=1}^n \mu_i(\mathscr{C}_i) \cdot \prod_{i=n+1}^\infty \mu_i(\Omega_i).$$

The existence of this measure is a rather direct consequence of Theorem 1.1. It is straightforward to verify that for this measure, identity (5) also holds.

Let $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be an arbitrary family of measure spaces. We define the *volume* map $\mathbf{vol} : \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}} \to [0, +\infty)$ by

$$\operatorname{\mathbf{vol}}\left(igtlephi_{i\in\mathbb{N}}\mathscr{C}_i
ight)=\prod_{i\in\mathbb{N}}\mu_i(\mathscr{C}_i).$$

The principal tool that is going to be used in the construction of the required product measure will be the next result, that establishes that the **vol** map is a good choice for the extension of the classical volume formula.

Theorem 2.1. Let $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be a sequence of measure spaces, $\mathscr{C} \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ and $\{\mathscr{C}_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ pairwise disjoint. Then

(1) If
$$\mathscr{C} = \biguplus_{n \in \mathbb{N}} \mathscr{C}_n$$
,

(6)
$$\operatorname{vol}(\mathscr{C}) = \sum_{n \in \mathbb{N}} \operatorname{vol}(\mathscr{C}_n).$$

(2) If $\biguplus_{n\in\mathbb{N}} \mathscr{C}_n \subset \mathscr{C}$,

(7)
$$\sum_{n \in \mathbb{N}} \mathbf{vol}(\mathscr{C}_n) \le \mathbf{vol}(\mathscr{C}).$$

(3) Suppose that $\{\mathscr{C}_n\}_{n\in\mathbb{N}}\subset\mathcal{F}(\Sigma_i,\mu_i)_{i\in\mathbb{N}}$ are not necessarily pairwise disjoint, then if $\mathscr{C}\subset\bigcup_{n\in\mathbb{N}}\mathscr{C}_n$,

$$\operatorname{vol}(\mathscr{C}) \leq \sum_{n \in \mathbb{N}} \operatorname{vol}(\mathscr{C}_n).$$

Before stating the proof of Theorem 2.1, we will give two lemmas.

Lemma 2.2. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two real positive sequences such that

- (1) $a_n \leq b_n$ for each $n \in \mathbb{N}$.
- (2) $\prod_{n\in\mathbb{N}} b_n \in [0,+\infty)$

then

(8)
$$\prod_{n \in \mathbb{N}} a_n \in [0, +\infty).$$

Moreover,

(9)
$$\prod_{n \in \mathbb{N}} a_n \le \prod_{n \in \mathbb{N}} b_n.$$

Proof. If $\prod_{n\in\mathbb{N}} b_n = 0$, the result is evident. Suppose $\prod_{n\in\mathbb{N}} b_n \in (0, +\infty)$. Then, since $(a_n/b_n)_{n\in\mathbb{N}} \subset [0, 1]$, the partial products are monotone and bounded by 1. In consequence $\prod_{n\in\mathbb{N}} \frac{a_n}{b_n} \in [0, +\infty)$. Using elementary properties of the limit

$$\lim_{m\to\infty} \prod_{n=1}^m a_n = \lim_{m\to\infty} \prod_{n=1}^m b_n \prod_{n=1}^m \frac{a_n}{b_n} = \prod_{n\in\mathbb{N}} b_n \prod_{n\in\mathbb{N}} \frac{a_n}{b_n} \in [0, +\infty).$$

This concludes the proof of (8). For (9) just note that the partial products satisfies $\prod_{i=1}^{n} a_n \leq \prod_{i=1}^{n} b_n$ for each $n \in \mathbb{N}$.

Lemma 2.3. Let $\mathscr{C}_1, \mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$, then $\mathscr{C}_1 \cap \mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. Moreover, for each $i \in \{1, 2\}$

(10)
$$\mathbf{vol}(\mathscr{C}_1 \cap \mathscr{C}_2) \leq \mathbf{vol}(\mathscr{C}_i).$$

Proof. Firstly, let us denote

$$\mathscr{C}_1 = \underset{i \in \mathbb{N}}{\times} \mathscr{C}_1^i \quad \text{and} \quad \mathscr{C}_2 = \underset{i \in \mathbb{N}}{\times} \mathscr{C}_2^i,$$

therefore, it is apparent that

$$\mathscr{C}_1 \cap \mathscr{C}_2 = \underset{i \in \mathbb{N}}{\bigvee} \mathscr{C}_1^i \cap \mathscr{C}_2^i.$$

Since Σ_i is a σ -algebra, $\mathscr{C}_1^i \cap \mathscr{C}_2^i \in \Sigma_i$ for each $i \in \mathbb{N}$. Moreover, since $\mu_i(\mathscr{C}_1^i \cap \mathscr{C}_2^i) \leq \mu_i(\mathscr{C}_1^i)$ for each $i \in \mathbb{N}$ and $\prod_{i \in \mathbb{N}} \mu_i(\mathscr{C}_1^i) = \mathbf{vol}(\mathscr{C}_1) \in [0, +\infty)$, it follows from Lemma 2.2, that

$$\prod_{i\in\mathbb{N}}\mu_i(\mathscr{C}_1^i\cap\mathscr{C}_2^i)\in[0,+\infty).$$

Thus $\mathscr{C}_1 \cap \mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. Inequality (10) follows from (9).

Proof of Theorem 2.1. Let $\mathscr{C} = \underset{i \in \mathbb{N}}{\times} \mathscr{C}^i \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. For each $i \in \mathbb{N}$, let us denote $(\mathscr{C}^i, \Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})$ the restriction of the measure space $(\Omega_i, \Sigma_i, \mu_i)$ to \mathscr{C}^i . Since the sequence of measure spaces $\{(\mathscr{C}^i, \Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})\}_{i \in \mathbb{N}}$ satisfies the finiteness condition

$$\prod_{i\in\mathbb{N}}\mu_{\mathscr{C}^i}(\mathscr{C}^i) = \prod_{i\in\mathbb{N}}\mu_i(\mathscr{C}^i) \in [0,+\infty),$$

there exists a measure $\bigotimes_{i\in\mathbb{N}}\mu_{\mathscr{C}^i}$ on the measurable space $(\mathscr{C},\bigotimes_{i\in\mathbb{N}}\Sigma_{\mathscr{C}^i})$ satisfying for each $\mathscr{D}=\bigotimes_{i\in\mathbb{N}}\mathscr{D}_i\in\mathcal{F}(\Sigma_{\mathscr{C}^i},\mu_{\mathscr{C}^i})_{i\in\mathbb{N}}$ the identity

(11)
$$\bigotimes_{i \in \mathbb{N}} \mu_{\mathscr{C}^i}(\mathscr{D}) = \prod_{i \in \mathbb{N}} \mu_{\mathscr{C}^i}(\mathscr{D}_i) = \prod_{i \in \mathbb{N}} \mu_i(\mathscr{D}_i) = \mathbf{vol}(\mathscr{D}).$$

It must be observed that the set map **vol** is well defined over $\mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})_{i \in \mathbb{N}}$ since $\mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})_{i \in \mathbb{N}} \subset \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. Moreover, by (11), the measure $\bigotimes_{i \in \mathbb{N}} \mu_{\mathscr{C}^i}$ coincides with **vol** over $\mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})_{i \in \mathbb{N}}$.

Let us begin proving item (1) and (2). In this cases $\mathscr{C}_n \subset \mathscr{C}$ for each $n \in \mathbb{N}$ and hence $\{\mathscr{C}_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})$ because if $\mathscr{D} \in \mathcal{F}(\Sigma_i, \mu_i)_{i\in\mathbb{N}}$ satisfies $\mathscr{D} \subset \mathscr{C}$, necessarily $\mathscr{D} \in \mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})_{i\in\mathbb{N}}$. Since $\bigotimes_{i\in\mathbb{N}} \mu_{\mathscr{C}^i}$ is a measure on $(\mathscr{C}, \bigotimes_{i\in\mathbb{N}} \Sigma_{\mathscr{C}^i})$ and it coincides with the map **vol** over $\mathcal{F}(\Sigma_{\mathscr{C}^i}, \mu_{\mathscr{C}^i})_{i\in\mathbb{N}}$, we deduce identities (6) and (7). Finally, suppose the hypothesis of case (3) holds. It is apparent that

(12)
$$\mathscr{C} = \bigcup_{n \in \mathbb{N}} \mathscr{C} \cap \mathscr{C}_n \subset \bigcup_{n \in \mathbb{N}} \mathscr{C}_n.$$

By Lemma 2.3, it becomes apparent that $\mathscr{C} \cap \mathscr{C}_n \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ and $\operatorname{vol}(\mathscr{C} \cap \mathscr{C}_n) \leq \operatorname{vol}(\mathscr{C}_n)$ for each $n \in \mathbb{N}$. Therefore, by (11) and (12), we deduce

$$\begin{aligned} \mathbf{vol}(\mathscr{C}) &= \bigotimes_{i \in \mathbb{N}} \mu_{\mathscr{C}^i}(\mathscr{C}) = \bigotimes_{i \in \mathbb{N}} \mu_{\mathscr{C}^i} \left(\bigcup_{n \in \mathbb{N}} \mathscr{C} \cap \mathscr{C}_n \right) \\ &\leq \sum_{n \in \mathbb{N}} \bigotimes_{i \in \mathbb{N}} \mu_{\mathscr{C}^i}(\mathscr{C} \cap \mathscr{C}_n) = \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathscr{C} \cap \mathscr{C}_n) \leq \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathscr{C}_n). \end{aligned}$$

This concludes the proof.

For the prove of the existence of a product measure for arbitrary family of measure spaces $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$, we consider the outer measure $\mu^* : \mathcal{P}(X_{i \in \mathbb{N}} \Omega_i) \to [0, +\infty]$, defined by

(13)
$$\mu^*(A) := \inf \left\{ \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathscr{C}_n) : \{\mathscr{C}_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}} \text{ and } A \subset \bigcup_{n \in \mathbb{N}} \mathscr{C}_n \right\}$$

for every $A \in \mathcal{P}(X_{i \in \mathbb{N}} \Omega_i)$. If no cover exists, we set $\mu^*(A) := +\infty$. We will prove that this outer measure is, in fact, a measure on the σ -algebra $\bigotimes_{i \in \mathbb{N}} \Sigma_i$ and that it satisfies identity (5) for each $\mathscr{C} \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$. We start stating some useful lemmas.

Lemma 2.4. Let $\mathscr{C}_i \subset \Omega_i$ for each $i \in \mathbb{N}$ and $\mathscr{C} = \bigotimes_{i \in \mathbb{N}} \mathscr{C}_i$, then

(14)
$$\mathscr{C}^{c} = \biguplus_{n \in \mathbb{N}} \left(\sum_{i=1}^{n-1} \mathscr{C}_{i} \times \mathscr{C}_{n}^{c} \times \sum_{i=n+1}^{\infty} \Omega_{i} \right).$$

Proof. It is clear that the union of the right hand of (14) is disjoint.

 \subseteq Let $(c_n)_{n\in\mathbb{N}}\in\mathscr{C}^c$, then $(c_n)_{n\in\mathbb{N}}\notin\mathscr{C}$ and there exists $n_0\in\mathbb{N}$ such that $c_{n_0}\notin\mathscr{C}_{n_0}$. Thus $c_{n_0}\in\mathscr{C}^c_{n_0}$. Define the subset

$$\mathcal{I}_{(c_n)_n} = \{ n \in \mathbb{N} : c_n \notin \mathscr{C}_n \} = \{ n \in \mathbb{N} : c_n \in \mathscr{C}_n^c \} \subset \mathbb{N}.$$

By the last argument, $\mathcal{I}_{(c_n)_n} \neq \emptyset$. Let $N_0 = \min \mathcal{I}_{(c_n)_n}$, then clearly

$$(c_n)_{n\in\mathbb{N}}\in \underset{i=1}{\overset{N_0-1}{\times}}\mathscr{C}_i\times\mathscr{C}_{N_0}^c\times\underset{i=N_0+1}{\overset{\infty}{\times}}\Omega_i.$$

 \supseteq) It is clear that for every $n \in \mathbb{N}$

$$\left(\sum_{i=1}^{n-1} \mathscr{C}_i \times \mathscr{C}_n^c \times \sum_{i=n+1}^{\infty} \Omega_i \right) \cap \mathscr{C} = \varnothing,$$

therefore

$$\biguplus_{n\in\mathbb{N}}\left(\bigotimes_{i=1}^{n-1}\mathscr{C}_i\times\mathscr{C}_n^c\times \bigotimes_{i=n+1}^{\infty}\Omega_i\right)\subset \mathscr{C}^c.$$

This concludes the proof.

Lemma 2.5. Let $\mathscr{C}_1 \in \mathcal{R}(\Sigma_i)_{i \in \mathbb{N}}$ and $\mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$, then $\mathscr{C}_1 \cap \mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$.

Proof. First, denote

$$\mathscr{C}_1 = \sum_{i=1}^m \mathscr{C}_{1,i} \times \sum_{i=m+1}^\infty \Omega_i \quad \text{and} \quad \mathscr{C}_2 = \sum_{i \in \mathbb{N}} \mathscr{C}_{2,i},$$

therefore it is apparent that

$$\mathscr{C}_1 \cap \mathscr{C}_2 = \underset{i=1}{\overset{m}{\times}} (\mathscr{C}_{1,i} \cap \mathscr{C}_{2,i}) \times \underset{i=m+1}{\overset{\infty}{\times}} \mathscr{C}_{2,i}.$$

Observe that $\mathscr{C}_{1,i} \cap \mathscr{C}_{2,i} \in \Sigma_i$ for each $i \in \{1,2,...,m\}$ and that $\mathscr{C}_{2,i} \in \Sigma_i$ for each $i \in \mathbb{N} \setminus \{1,2,...,m\}$. Finally, since $\mathscr{C}_2 \in \mathcal{F}(\Sigma_i,\mu_i)_{i\in\mathbb{N}}$,

$$\prod_{i=1}^{m} \mu_i(\mathscr{C}_{1,i} \cap \mathscr{C}_{2,i}) \cdot \prod_{i=m+1}^{\infty} \mu_i(\mathscr{C}_{2,i}) \in [0, +\infty).$$

Hence $\mathscr{C}_1 \cap \mathscr{C}_2 \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$.

Theorem 2.6 (Measurability). The family of subsets $\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}$ is μ^* -measurable.

Proof. Take $\mathscr{C} \in \mathcal{R}(\Sigma_i)_{i \in \mathbb{N}}$ and $B \in \mathcal{P}(\times_{i \in \mathbb{N}} \Omega_i)$. Consider $\varepsilon > 0$ and a family $\{\mathscr{B}_n\}_{n \in \mathbb{N}} \subset \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ such that

$$B \subset \bigcup_{n \in \mathbb{N}} \mathscr{B}_n$$

and

(15)
$$\sum_{n\in\mathbb{N}} \mathbf{vol}(\mathcal{B}_n) \le \mu^*(B) + \varepsilon.$$

If the cover $\{\mathscr{B}_n\}_{n\in\mathbb{N}}$ does not exist, then necessarily $\mu^*(B)=+\infty$ and the inequality

(16)
$$\mu^*(B) \ge \mu^*(B \cap \mathscr{C}) + \mu^*(B \cap \mathscr{C}^c)$$

follows. Suppose such cover exists, then by Lemma 2.4, there exists $\{\mathscr{C}_i\}_{i\in\mathbb{N}}\subset\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}$ pairwise disjoint of the form

$$\mathscr{C}_{i} = \sum_{j=1}^{n_{i}-1} \mathscr{C}_{i,j} \times \mathscr{C}_{i,n_{i}}^{c} \times \sum_{j=n_{i}+1}^{\infty} \Omega_{j}$$

such that

$$\begin{split} \mathcal{B}_n &= (\mathcal{B}_n \cap \mathcal{C}) \uplus (\mathcal{B}_n \cap \mathcal{C}^c) \\ &= (\mathcal{B}_n \cap \mathcal{C}) \uplus \left(\mathcal{B}_n \cap \biguplus_{i \in \mathbb{N}} \mathcal{C}_i \right) \\ &= (\mathcal{B}_n \cap \mathcal{C}) \uplus \left(\biguplus_{i \in \mathbb{N}} (\mathcal{B}_n \cap \mathcal{C}_i) \right). \end{split}$$

By Lemma 2.5, it follows that $\mathscr{B}_n \cap \mathscr{C}$, $\mathscr{B}_n \cap \mathscr{C}_i \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ and therefore, by Theorem 2.1 item (1), we arrive to the identity

$$\mathbf{vol}(\mathscr{B}_n) = \mathbf{vol}(\mathscr{B}_n \cap \mathscr{C}) + \sum_{i \in \mathbb{N}} \mathbf{vol}(\mathscr{B}_n \cap \mathscr{C}_i).$$

Hence, by equation (15) and the definition of the outer measure μ^* , (13), it becomes apparent that

$$\mu^*(B) + \varepsilon \ge \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathcal{B}_n) = \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathcal{B}_n \cap \mathcal{C}) + \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{N}} \mathbf{vol}(\mathcal{B}_n \cap \mathcal{C}_i)$$
$$= \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathcal{B}_n \cap \mathcal{C}) + \sum_{(n,i) \in \mathbb{N}^2} \mathbf{vol}(\mathcal{B}_n \cap \mathcal{C}_i)$$
$$\ge \mu^*(B \cap \mathcal{C}) + \mu^*(B \cap \mathcal{C}^c).$$

The last step follows from the following inclusion and the definition of infumum

$$B \cap \mathscr{C}^c \subset \left(\bigcup_{n \in \mathbb{N}} \mathscr{B}_n\right) \cap \mathscr{C}^c = \bigcup_{n \in \mathbb{N}} (\mathscr{B}_n \cap \mathscr{C}^c)$$
$$= \bigcup_{n \in \mathbb{N}} \biguplus (\mathscr{B}_n \cap \mathscr{C}_i) = \bigcup_{(n,i) \in \mathbb{N}^2} (\mathscr{B}_n \cap \mathscr{C}_i).$$

Taking $\varepsilon \to 0$, inequality (16) holds for every $B \in \mathcal{P}(X_{i \in \mathbb{N}} \Omega_i)$ and therefore \mathscr{C} is μ^* -measurable.

In Theorem 2.6 we have proved that $\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}$ is a subfamily of the Caratheodory σ algebra \mathfrak{C} and therefore by Caratheodory extension theorem, μ^* defines a measure on $\sigma(\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}) = \bigotimes_{i\in\mathbb{N}} \Sigma_i$. We will denote

$$\bigotimes_{i\in\mathbb{N}}\mu_i := \mu^*\big|_{\bigotimes_{i\in\mathbb{N}}\Sigma_i}.$$

Finally, we will prove that the outer measure μ^* satisfies identity (5) for each $\mathscr{C} \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$.

Proposition 2.7 (Volume). For each $\mathscr{C} \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$, the following identity holds

$$\mu^*(\mathscr{C}) = \mathbf{vol}(\mathscr{C}).$$

Proof. Let $\{\mathscr{C}_n\}_{n\in\mathbb{N}}\subset\mathcal{F}(\Sigma_i,\mu_i)_{i\in\mathbb{N}}$ be cover of \mathscr{C} , i.e.,

$$\mathscr{C} \subset \bigcup_{n \in \mathbb{N}} \mathscr{C}_n.$$

By Theorem 2.1 item (3), we deduce the following inequality

$$\mathbf{vol}(\mathscr{C}) \leq \sum_{n \in \mathbb{N}} \mathbf{vol}(\mathscr{C}_n).$$

Then, taking the infimum over all covers, we stablish that $\operatorname{vol}(\mathscr{C}) \leq \mu^*(\mathscr{C})$. Finally, considering the particular cover $\{\mathscr{C}_n\}_{n\in\mathbb{N}} \subset \mathcal{F}(\Sigma_i, \mu_i)_{i\in\mathbb{N}}$ defined by

$$\mathscr{C}_n := \left\{ egin{array}{ll} \mathscr{C} & ext{if } n = 1 \\ \varnothing & ext{if } n \neq 1 \end{array} \right.$$

it becomes apparent that

$$\mu^*(\mathscr{C}) \leq \mathbf{vol}(\mathscr{C}) \leq \mu^*(\mathscr{C})$$

which implies $\operatorname{vol}(\mathscr{C}) = \mu^*(\mathscr{C})$. This finished the proof.

In conclusion, we have proved the next existence theorem. The last statements of this result are proven analogously to those of [2, Theorem I].

Theorem 2.8. Let $\{(\Omega_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be a family of measure spaces, then there exists a measure $\bigotimes_{i \in \mathbb{N}} \mu_i$ on the measurable space $(\times_{i \in \mathbb{N}} \Omega_i, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$ satisfying for each $\mathscr{C} = \times_{i \in \mathbb{N}} \mathscr{C}_i \in \mathcal{F}(\Sigma_i, \mu_i)_{i \in \mathbb{N}}$ the identity

$$\bigotimes_{i\in\mathbb{N}}\mu_i(\mathscr{C})=\prod_{i\in\mathbb{N}}\mu_i(\mathscr{C}_i).$$

If each space Ω_i contains subsets A_i and B_i such that $\mu_i(A_i) = \mu_i(B_i) = 1$, then the measure $\bigotimes_{i \in \mathbb{N}} \mu_i$ is not σ -finite. Moreover, assuming that each $(\Omega_i, \Sigma_i, \circ_i)$ is a measurable group, if each μ_i is a left (respectively right)-invariant measure on $(\Omega_i, \Sigma_i, \circ_i)$, then $\bigotimes_{i \in \mathbb{N}} \mu_i$ is a left (respectively right)-invariant measure.

3. Decomposition Theorem for Finite Measures

In this section L_p of infinite product measures are studied and simplified in terms of L_p of finite product ones, for the particular case in which the finiteness condition is satisfied. Let $\{(\mathscr{C}_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be a family of measure spaces satisfying the finiteness condition $\prod_i \mu_i(\mathscr{C}_i) \in (0, \infty)$. We use the notation \mathscr{C}_i instead of Ω_i to emphasize that the involved measures satisfies the required condition. Consider the measure space $(\mathscr{C}, \bigotimes_{i \in \mathbb{N}} \Sigma_i, \bigotimes_{i \in \mathbb{N}} \mu_i)$, where $\mathscr{C} = \bigotimes_{i \in \mathbb{N}} \mathscr{C}_i$ and $\bigotimes_{i \in \mathbb{N}} \mu_i$ is the measure constructed in the last section in terms of the outer measure (13). In this section we will deal with the space

$$L_p\left(\mathscr{C}, \bigotimes_{i\in\mathbb{N}} \Sigma_i, \bigotimes_{i\in\mathbb{N}} \mu_i\right)$$

for $1 \leq p < \infty$. To simplify the notation, throughout this paper we will use unstintingly the notations $L_p(\mathscr{C})$ and $L_p(\mathscr{C}, \bigotimes_{i=1}^{\infty} \Sigma_i, \bigotimes_{i=1}^{\infty} \mu_i)$; and analogously $L_p(\bigotimes_{i=1}^n \mathscr{C}_i)$ and $L_p(\bigotimes_{i=1}^n \Sigma_i, \bigotimes_{i=1}^n \mu_i)$. Moreover, for each $n \in \mathbb{N}$, we will use the notation \mathscr{C}^n to denote $\bigotimes_{i=1}^n \mathscr{C}_i$. Consider for $1 \leq p < \infty$, the vector space

$$\bigoplus_{n\in\mathbb{N}} L_p\left(\mathscr{C}^n\right) := \left\{ (f_n)_{n\in\mathbb{N}} : f_n \in L_p\left(\mathscr{C}^n\right), \forall n \in \mathbb{N} \right\}$$

and the subspace

$$\lim_{n} L_{p}\left(\mathscr{C}^{n}\right) := \left\{ (f_{n})_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_{p}\left(\mathscr{C}^{n}\right) : \left\| \frac{f_{n}}{\prod_{i=n}^{m} \mu_{i}(\mathscr{C}_{i})} - f_{m} \right\|_{L_{n}\left(\mathscr{C}^{m}\right)} \xrightarrow{n, m \to \infty} 0 \right\} \middle/ \sim,$$

where given $(f_n)_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}}\in\bigoplus_{n\in\mathbb{N}}L_p(\mathscr{C}^n)$, we identify the sequences $(f_n)_{n\in\mathbb{N}}\sim(g_n)_{n\in\mathbb{N}}$ if

$$\lim_{n\to\infty} ||f_n - g_n||_{L_p(\mathscr{C}^n)} = 0.$$

Moreover, in the definition of $\lim_n L_p(\mathscr{C}^n)$, we are identifying for every n < m, $S_n^m(f) \equiv f$, where S_n^m is the linear embedding $S_n^m : L_p(\mathscr{C}^n) \hookrightarrow L_p(\mathscr{C}^m)$ defined by

$$S_n^m(f): \quad \mathscr{C}^m \longrightarrow \mathbb{R}$$

$$(\omega_1, \omega_2, \cdots, \omega_m) \mapsto f(\omega_1, \omega_2, \cdots, \omega_n)$$

for each $f \in L_p(\mathscr{C}^n)$. Hereinafter, we will use this identifications. If we consider the functional

$$\|\cdot\|_{\lim L_p}: \lim_n L_p(\mathscr{C}^n) \longrightarrow [0,+\infty)$$

$$(f_n)_{n\in\mathbb{N}} \mapsto \lim_{n\to\infty} \|f_n\|_{L_p(\mathscr{C}^n)}$$

then, for each $1 \leq p < \infty$, the pair $(\lim_n L_p(\mathscr{C}^n), \|\cdot\|_{\lim L_p})$ defines a normed space. The main result of this section states that the next spaces are isometrically isomorphic for $1 \leq p < \infty$

(17)
$$L_p(\mathscr{C}) \simeq \lim_n L_p(\mathscr{C}^n)$$

and consequently we would have the embedding

$$L_p\left(\mathscr{C}\right) \hookrightarrow \bigoplus_{n \in \mathbb{N}} L_p\left(\mathscr{C}^n\right),$$

that allows us to consider functions defined in spaces of infinite dimensions as a sequence of finite dimensional ones.

3.1. **Preliminary Lemmas.** We will give some lemmas concerning some dense subspaces of $L_p(\mathscr{C})$ and $\lim_n L_p(\mathscr{C}^n)$ that will be useful for the proof of (17). Throughout this section, we will identify for each $n \in \mathbb{N}$, the spaces

$$L_{p}\left(\mathscr{C}^{n}\right)\simeq\left\{f\cdot\chi_{\mathsf{X}_{i=n+1}^{\infty}\mathscr{C}_{i}}:f\in L_{p}\left(\mathscr{C}^{n}\right)\right\}\subset L_{p}(\mathscr{C}),$$

where $\chi_{\mathsf{X}_{i=n+1}^{\infty}\mathscr{C}_{i}}$ stands for the characteristic function of the set $\mathsf{X}_{i=n+1}^{\infty}\mathscr{C}_{i}$.

Lemma 3.1. The subspace $\bigcup_{n\in\mathbb{N}} L_p(\mathscr{C}^n)$ is dense in $L_p(\mathscr{C})$ for each $1\leq p<\infty$.

Proof. By definition, we have

$$\bigotimes_{i\in\mathbb{N}} \Sigma_i = \sigma(\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}).$$

Consequently since $\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}$ is an algebra of subsets, by Lemma 3.4.6 of [4], the space $\mathrm{Span}\{\chi_R: R\in\mathcal{R}(\Sigma_i)_{i\in\mathbb{N}}\}$ is dense in $L_p(\mathscr{C})$. Finally, due to the inclusion

$$\operatorname{Span}\left\{\chi_{R}: R \in \mathcal{R}(\Sigma_{i})_{i \in \mathbb{N}}\right\} \subset \bigcup_{n \in \mathbb{N}} L_{p}\left(\mathscr{C}^{n}\right),$$

the result follows.

Lemma 3.2. Let \mathcal{F}_{p}^{N} be the subspace of $\lim_{n} L_{p}\left(\mathscr{C}^{n}\right)$ defined by

$$\mathcal{F}_{p}^{N} := \left\{ (f_{n})_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} L_{p}\left(\mathscr{C}^{n}\right) : \text{ There exists } g \in L_{p}\left(\mathscr{C}^{N}\right) \right\}$$

such that
$$f_n = \frac{g}{\prod_{i=N+1}^n \mu_i(\mathscr{C}_i)}, \ \forall n \ge N$$
,

then for each $1 \leq p < \infty$, the subspace $\bigcup_{N \in \mathbb{N}} \mathcal{F}_p^N$ is dense in $\lim_n L_p(\mathscr{C}^n)$.

Proof. Let $(f_n)_{n\in\mathbb{N}} \in \lim_n L_p(\mathscr{C}^n)$ and consider the sequence $(\mathbf{F}^m)_{m\in\mathbb{N}} \subset \lim_n L_p(\mathscr{C}^n)$ defined by

$$\mathbf{F}_n^m = \begin{cases} 0 & \text{if } n < m \\ \frac{f_m}{\prod_{i=m+1}^n \mu_i(\mathscr{C}_i)} & \text{if } n \ge m \end{cases}$$

for each $n, m \in \mathbb{N}$. Then, by definition

$$\|\mathbf{F}^{m} - (f_{n})_{n \in \mathbb{N}}\|_{\lim L_{p}} = \lim_{n \to \infty} \|\mathbf{F}_{n}^{m} - f_{n}\|_{L_{p}(\mathscr{C}^{n})} = \lim_{n \to \infty} \left\| \frac{f_{m}}{\prod_{i=m+1}^{n} \mu_{i}(\mathscr{C}_{i})} - f_{n} \right\|_{L_{p}(\mathscr{C}^{n})}$$

and since $(f_n)_{n\in\mathbb{N}}\in\lim_n L_p(\mathscr{C}^n)$, it is apparent that

$$\lim_{m\to\infty} \|\mathbf{F}^m - (f_n)_{n\in\mathbb{N}}\|_{\lim L_p} = \lim_{m\to\infty} \lim_{n\to\infty} \left\| \frac{f_m}{\prod_{i=m+1}^n \mu_i(\mathscr{C}_i)} - f_n \right\|_{L_p(\mathscr{C}^n)} = 0.$$

Thus we have proved

$$(\mathbf{F}^m)_{m\in\mathbb{N}} \xrightarrow[m\to\infty]{} (f_n)_{n\in\mathbb{N}} \text{ in } \lim_n L_p\left(\mathscr{C}^n\right).$$

Consequently since $(\mathbf{F}^m)_{m\in\mathbb{N}}\subset\bigcup_{N\in\mathbb{N}}\mathcal{F}_p^N$, we conclude that the subspace $\bigcup_{N\in\mathbb{N}}\mathcal{F}_p^N$ is dense in $\lim_{n} L_{p}(\mathscr{C}^{n})$.

Another key result will be the Jessen's theorem. This theorem was proved by B. Jessen in 1934 in his PhD Thesis [9] and nowadays is proved by martingales techniques [18, Theorem 7.16. However, Jessen proved the result only for probability spaces. We need a version for the general case in which the involved spaces $(\mathscr{C}_i, \Sigma_i, \mu_i)$ are not necessarily of probability. This version is given in the next result which proof can be adapted mutatis mutandis from the proofs of Theorems II, III and IV of Baker's paper [2]. It must be observed that the extra conditions imposed by Baker on the measure spaces are not used in the proofs of Theorems II, III and IV.

Theorem 3.3. Let $\{(\mathscr{C}_i, \Sigma_i, \mu_i)\}_{i \in \mathbb{N}}$ be a family of measure spaces satisfying the finiteness condition $\prod_i \mu_i(\mathscr{C}_i) \in (0,+\infty)$, and consider $f \in L_p(\mathscr{C})$ with $1 \leq p < \infty$, then if we denote $\omega^n = (\omega_1, \omega_2, ..., \omega_n)$, $\overline{\omega}_n = (\omega_{n+1}, \omega_{n+2}, ...)$ and respectively for the variable x,

$$\int_{X_{i=1}^{n}\mathscr{C}_{i}} f(\omega^{n}, \overline{x}_{n}) \ d\omega \in L_{p}(\mathscr{C}) \quad and \quad \int_{X_{i=n}^{\infty}\mathscr{C}_{i}} f(x^{n-1}, \overline{\omega}_{n-1}) \ d\omega \in L_{p}(\mathscr{C})$$

for $\bigotimes_{i\in\mathbb{N}} \mu_i$ -a.e. $x \in \mathscr{C}$ and every $n \in \mathbb{N}$.

$$\lim_{n \to \infty} \int_{X_{i=1}^n \mathscr{C}_i} f(\omega^n, \overline{x}_n) \ d\omega = \int_{\mathscr{C}} f$$

for $\bigotimes_{i\in\mathbb{N}} \mu_i$ -a.e. $x\in\mathscr{C}$ and also with convergence in $L_p(\mathscr{C})$.

$$\lim_{n \to \infty} \int_{\mathbf{X}^{\infty}} \int_{\mathbf{X}^{n}} f(x^{n-1}, \overline{\omega}_{n-1}) \ d\omega = f$$

for $\bigotimes_{i\in\mathbb{N}} \mu_i$ -a.e. $x\in\mathscr{C}$ and also with convergence in $L_p(\mathscr{C})$.

3.2. **Decomposition Theorem.** Once the necessary machinery has been presented, we proceed to prove the identification (17).

Theorem 3.4. Let $1 \le p < \infty$ and consider the operators

$$\mathfrak{T}: \lim_{n} L_{p}(\mathscr{C}^{n}) \longrightarrow L_{p}(\mathscr{C})$$

$$(f_{n})_{n \in \mathbb{N}} \mapsto \lim_{L_{p}(\mathscr{C})} \frac{f_{n}}{\prod_{i=n+1}^{\infty} \mu_{i}(\mathscr{C}_{i})}$$

and

$$\mathfrak{S}: L_p(\mathscr{C}) \longrightarrow \lim_n L_p(\mathscr{C}^n)$$

$$f \longmapsto \left(\int_{\mathsf{X}_{i=n+1}^{\infty} \mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega \right)_{n \in \mathbb{N}}.$$

Then \mathfrak{T} is an isometric isomorphism with $\mathfrak{S} = \mathfrak{T}^{-1}$. In particular, $L_p(\mathscr{C})$ is isometrically isomorphic to $\lim_n L_p(\mathscr{C}^n)$,

$$L_p(\mathscr{C}) \simeq \lim_n L_p(\mathscr{C}^n)$$
.

Proof. Firstly, note that given $(f_n)_{n\in\mathbb{N}}\in\lim_n L_p(\mathscr{C}^n)$, the sequence

$$\left(\frac{f_n}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)}\right)_{n \in \mathbb{N}} \subset L_p(\mathscr{C})$$

is Cauchy in $L_p(\mathscr{C})$ since by Fubini's theorem

$$\left\| \frac{f_n}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} - \frac{f_m}{\prod_{i=m+1}^{\infty} \mu_i(\mathscr{C}_i)} \right\|_{L_p(\mathscr{C})} = \frac{1}{\prod_{i=m+1}^{\infty} \mu_i(\mathscr{C}_i)} \left\| \frac{f_n}{\prod_{i=n+1}^{m} \mu_i(\mathscr{C}_i)} - f_m \right\|_{L_p(\mathscr{C})}$$

$$= \frac{\prod_{i=m+1}^{\infty} \mu_i(\mathscr{C}_i)}{\prod_{i=m+1}^{\infty} \mu_i(\mathscr{C}_i)} \left\| \frac{f_n}{\prod_{i=n+1}^{m} \mu_i(\mathscr{C}_i)} - f_m \right\|_{L_p(\mathscr{C}^m)}$$

$$= \left\| \frac{f_n}{\prod_{i=n+1}^{m} \mu_i(\mathscr{C}_i)} - f_m \right\|_{L_p(\mathscr{C}^m)} \xrightarrow{n,m\to\infty} 0.$$

Thus \mathfrak{T} is well defined and it is linear. On the other hand, by Theorem 3.3 item (1), if $f \in L_p(\mathscr{C}, \bigotimes_{i \in \mathbb{N}} \Sigma_i)$, necessarily

$$\left(\int_{X_{i=n+1}^{\infty}\mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega\right)_{n\in\mathbb{N}} \in \bigoplus_{n\in\mathbb{N}} L_p\left(\mathscr{C}^n\right).$$

By Theorem 3.3, the next sequence converges

$$\lim_{L_p(\mathscr{C})} \frac{1}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} \int_{\underset{i=n+1}{\times} \mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega = \lim_{L_p(\mathscr{C})} \int_{\underset{i=n+1}{\times} \mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega = f$$

and consequently for n < m, we have

$$\left\| \frac{1}{\prod_{i=n+2}^{m+1} \mu_i(\mathscr{C}_i)} \int_{X_{i=n+1}^{\infty} \mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega - \int_{X_{i=m+1}^{\infty} \mathscr{C}_i} f(x^m, \overline{\omega}_m) \ d\omega \right\|_{L_p(\mathscr{C})}$$

$$= \prod_{i=m+2}^{\infty} \mu_i(\mathscr{C}_i) \cdot \left\| \frac{\int_{X_{i=n+1}^{\infty} \mathscr{C}_i} f(x^n, \overline{\omega}_n) \ d\omega}{\prod_{i=n+2}^{\infty} \mu_i(\mathscr{C}_i)} - \frac{\int_{X_{i=m+1}^{\infty} \mathscr{C}_i} f(x^m, \overline{\omega}_m) \ d\omega}{\prod_{i=m+2}^{\infty} \mu_i(\mathscr{C}_i)} \right\|_{L_p(\mathscr{C})} \xrightarrow{n, m \to \infty} 0.$$

We have proved that

$$\left(\int_{X_{i=n+1}^{\infty}\mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega\right)_{n \in \mathbb{N}} \in \lim_{n} L_{p}\left(\mathscr{C}^{n}\right)$$

or equivalently $\mathfrak{S}(f) \in \lim_n L_p(\mathscr{C}^n)$. This establishes that \mathfrak{S} is well defined and that it is linear. Once we have proved the consistency of the linear operators \mathfrak{T} and \mathfrak{S} , we will prove that they are isometries. Take $(f_n)_{n \in \mathbb{N}} \in \lim_n L_p(\mathscr{C}^n)$, then

$$\|\mathfrak{T}(f_n)_n\|_{L_p(\mathscr{C})} = \left\| \lim_{L_p(\mathscr{C})} \frac{f_n}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} \right\|_{L_p(\mathscr{C})} = \lim_{n \to \infty} \left\| \frac{f_n}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} \right\|_{L_p(\mathscr{C})}$$

$$= \lim_{n \to \infty} \prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i) \left\| \frac{f_n}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} \right\|_{L_p(\mathscr{C}^n)}$$

$$= \lim_{n \to \infty} \frac{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)}{\prod_{i=n+1}^{\infty} \mu_i(\mathscr{C}_i)} \|f_n\|_{L_p(\mathscr{C}^n)}$$

$$= \lim_{n \to \infty} \|f_n\|_{L_p(\mathscr{C}^n)} = \|(f_n)_{n \in \mathbb{N}}\|_{\lim L_p}.$$

Thus, \mathfrak{T} is an isometry. On the other hand, if $f \in L_p(\mathscr{C})$, we compute

$$\|\mathfrak{S}(f)\|_{\lim L_{p}} = \left\| \left(\int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right)_{n \in \mathbb{N}} \right\|_{\lim L_{p}} = \lim_{n \to \infty} \left\| \int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right\|_{L_{p}(\mathscr{C}^{n})}$$

$$= \left(\lim_{n \to \infty} \left\| \int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right\|_{L_{p}(\mathscr{C}^{n})} \right) \cdot \left(\lim_{n \to \infty} \prod_{i=n+1}^{\infty} \mu_{i}(\mathscr{C}_{i}) \right)$$

$$= \lim_{n \to \infty} \left\| \int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right\|_{L_{p}(\mathscr{C})} = \left\| \lim_{L_{p}(\mathscr{C})} \int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right\|_{L_{p}(\mathscr{C})}$$

$$= \|f\|_{L_{p}(\mathscr{C})}$$

where the last step follows from Theorem 3.3 item (3). Therefore, \mathfrak{S} is also an isometry. Finally we will see that $\mathfrak{S} = \mathfrak{T}^{-1}$. Consider $(f_n)_{n \in \mathbb{N}} \in \bigcup_{N \in \mathbb{N}} \mathcal{F}_p^N$ and let $g \in L_p(\mathscr{C}^M)$ such that $f_n = \frac{g}{\prod_{i=M+1}^n \mu_i(\mathscr{C}_i)}$ for $n \geq M$, then if we denote $g^n = g(x^{n-1}, \overline{\omega}_{n-1})$ for each $n \in \mathbb{N}$, we have

$$(\mathfrak{S} \circ \mathfrak{T})(f_n)_{n \in \mathbb{N}} = \mathfrak{S} \left(\lim_{L_p(\mathscr{C})} \frac{g}{\prod_{i=M+1}^n \mu_i(\mathscr{C}_i) \cdot \prod_{i=n+1}^\infty \mu_i(\mathscr{C}_i)} \right) = \frac{\mathfrak{S}(g)}{\prod_{i=M+1}^\infty \mu_i(\mathscr{C}_i)}$$

$$= \frac{1}{\prod_{i=M+1}^{\infty} \mu_{i}(\mathscr{C}_{i})} \left(\int_{\times_{i=2}^{\infty} \mathscr{C}_{i}} g^{2} d\omega, \dots, \int_{\times_{i=M}^{\infty} \mathscr{C}_{i}} g^{M} d\omega, g \prod_{i=M+1}^{\infty} \mu_{i}(\mathscr{C}_{i}), g \prod_{i=M+2}^{\infty} \mu_{i}(\mathscr{C}_{i}), \dots \right)$$

$$= \left(f_{1}, f_{2}, \dots, f_{M-1}, g, \frac{g}{\mu_{\mathscr{C}_{M+1}}(\mathscr{C}_{M+1})}, \frac{g}{\prod_{i=M+1}^{M+2} \mu_{i}(\mathscr{C}_{i})}, \dots \right) = (f_{n})_{n \in \mathbb{N}},$$

The last steps are justified by the equivalence relation defined on $\lim_n L_p(\mathscr{C}^n)$. Thus, we have proved that if $(f_n)_{n\in\mathbb{N}}\in\bigcup_{N\in\mathbb{N}}\mathcal{F}_p^N$

$$(\mathfrak{S} \circ \mathfrak{T})(f_n)_{n \in \mathbb{N}} = (f_n)_{n \in \mathbb{N}}.$$

Since $\bigcup_{N\in\mathbb{N}} \mathcal{F}_p^N$ is dense in $\lim_n L_p\left(\mathscr{C}^n\right)$ and $\mathfrak{S}\circ\mathfrak{T}$ is continuous (since it is an isometry), it is apparent that

$$\mathfrak{S} \circ \mathfrak{T} = I_{\lim_n L_p(\mathscr{C}^n)}.$$

On the other hand, consider $f \in \bigcup_{n \in \mathbb{N}} L_p(\mathscr{C}^n)$. Then $f \in L_p(\mathscr{C}^M)$ for some $M \in \mathbb{N}$ and if we denote $f^n = f(x^{n-1}, \overline{\omega}_{n-1})$ for each $n \in \mathbb{N}$, we have

$$(\mathfrak{T} \circ \mathfrak{S})(f) = \mathfrak{T} \left(\int_{X_{i=n+1}^{\infty} \mathscr{C}_{i}} f(x^{n}, \overline{\omega}_{n}) \ d\omega \right)_{n \in \mathbb{N}}$$

$$= \mathfrak{T} \left(\int_{X_{i=2}^{\infty} \mathscr{C}_{i}} f^{2} \ d\omega, \dots, \int_{X_{i=M}^{\infty} \mathscr{C}_{i}} f^{M} \ d\omega, \ f \prod_{i=M+1}^{\infty} \mu_{i}(\mathscr{C}_{i}), \ f \prod_{i=M+2}^{\infty} \mu_{i}(\mathscr{C}_{i}) \dots \right)$$

$$= \lim_{L_{p}(\mathscr{C})} \frac{f \cdot \prod_{i=n+1}^{\infty} \mu_{i}(\mathscr{C}_{i})}{\prod_{i=n+1}^{\infty} \mu_{i}(\mathscr{C}_{i})} = f.$$

Consequently, if $f \in \bigcup_{n \in \mathbb{N}} L_p(\mathscr{C}^n)$, it is apparent that

$$(\mathfrak{T} \circ \mathfrak{S})(f) = f.$$

Since $\bigcup_{n\in\mathbb{N}} L_p(\mathscr{C}^n)$ is dense in $L_p(\mathscr{C})$ and $\mathfrak{T}\circ\mathfrak{S}$ is continuous, we conclude that

$$\mathfrak{T} \circ \mathfrak{S} = I_{L_p(\mathscr{C})}.$$

This finishes the proof.

4. Decomposition Theorem for Infinite Measures

In this section, we will give the analogue of Theorem 3.4 for the case of the sequence of measure spaces $\{(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m_{\mathbb{R}})\}_{i\in\mathbb{N}}$ where $\mathcal{B}_{\mathbb{R}}$ is the borel σ -algebra of \mathbb{R} and $m_{\mathbb{R}}$ is the Lebesgue measure of \mathbb{R} . Observe that this sequence does not satisfy the finiteness condition since $\prod_i m_{\mathbb{R}}(\mathbb{R}) = +\infty$ and therefore Theorem 3.4 cannot be applied directly. Consider the corresponding product measure space

$$\left(\bigotimes_{i \in \mathbb{N}} \mathbb{R}, \bigotimes_{i \in \mathbb{N}} \mathcal{B}_{\mathbb{R}}, \bigotimes_{i \in \mathbb{N}} m_{\mathbb{R}} \right).$$

Let us denote $\mathbb{R}^{\mathbb{N}} := \times_{i \in \mathbb{N}} \mathbb{R}$. It should be observed that $\bigotimes_{i \in \mathbb{N}} \mathcal{B}_{\mathbb{R}} = \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$ where we are considering on $\mathbb{R}^{\mathbb{N}}$ the product topology. Moreover, it becomes apparent that the measure $\bigotimes_{i \in \mathbb{N}} m_{\mathbb{R}}$ is, in fact, the *Baker measure* λ_B , constructed in [2, Theorem I]. Thus, we will denote the measure $\bigotimes_{i \in \mathbb{N}} m_{\mathbb{R}}$ by λ_B . For the sake of notation, hereinafter, we will also denote for each $\mathfrak{a} = (a_n)_{n \in \mathbb{N}} \in \mathcal{N}$ and $m \in \mathbb{N}$

$$[0,1)^{\mathbb{N}}_{\mathfrak{a}} := \underset{n \in \mathbb{N}}{\times} [a_n, a_n + 1)$$
 and $[0,1)^m_{\mathfrak{a}} := \underset{n=1}{\overset{m}{\times}} [a_n, a_n + 1),$

where $\mathcal{N} := \mathbb{Z}^{\mathbb{N}}$. The next result simplifies considerably the integration issues on the measure space $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$.

Theorem 4.1. Let $f \in L_1(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ such that $f \geq 0$, then the set

$$\mathcal{O}_f = \left\{ \mathfrak{a} \in \mathcal{N} : \int_{[0,1)_{\mathfrak{a}}^{\mathbb{N}}} f \ d\lambda_B \neq 0 \right\}$$

is countable and

(18)
$$\int_{\mathbb{R}^{\mathbb{N}}} f \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} f \ d\lambda_B.$$

Proof. Suppose that \mathcal{O}_f is uncountable and let \mathfrak{F} be a finite subset of \mathcal{O}_f . Therefore, since

$$[0,1)^{\mathbb{N}}_{\mathfrak{a}} \cap [0,1)^{\mathbb{N}}_{\mathfrak{b}} = \emptyset$$
 for each $\mathfrak{a},\mathfrak{b} \in \mathcal{N}$, $\mathfrak{a} \neq \mathfrak{b}$,

it is apparent that

$$\int_{\biguplus\left\{[0,1)^{\mathbb{N}}_{\mathfrak{a}}:\mathfrak{a}\in\mathfrak{F}\right\}}f\ d\lambda_{B}=\sum_{\mathfrak{a}\in\mathfrak{F}}\int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}}f\ d\lambda_{B}\leq\int_{\mathbb{R}^{\mathbb{N}}}f\ d\lambda_{B}.$$

Thus, recalling the definition of uncountable sum, we arrive at

$$\sum_{\mathfrak{a}\in\mathcal{O}_f}\int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}}f\ d\lambda_B:=\sup\left\{\sum_{\mathfrak{a}\in\mathfrak{F}}\int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}}f\ d\lambda_B:\mathfrak{F}\subset\mathcal{O}_f\ \mathrm{finite}\right\}\leq\int_{\mathbb{R}^{\mathbb{N}}}f\ d\lambda_B.$$

Since \mathcal{O}_f is uncountable and

$$\int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} f \ d\lambda_B > 0,$$

for each $\mathfrak{a} \in \mathcal{O}_f$, by Proposition 0.20 of [6],

$$\sum_{\mathfrak{a}\in\mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} f \ d\lambda_B = +\infty$$

and this gets a contradiction since $f \in L_1(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$. Therefore \mathcal{O}_f is countable. Now, we prove identity (18). Let $f \in L_1(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ with $f \geq 0$, then since \mathcal{O}_f is countable, $f|_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} = 0$ λ_B -a.e. for each $\mathfrak{a} \in \mathcal{N} \setminus \mathcal{O}_f$. Hence, since

$$\mathbb{R}^{\mathbb{N}} = \biguplus \left\{ [0,1)_{\mathfrak{a}}^{\mathbb{N}} : \mathfrak{a} \in \mathcal{N} \right\}$$

it follows that

$$\int_{\mathbb{R}^{\mathbb{N}}} f \ d\lambda_B = \int_{\biguplus\{[0,1)_{\mathfrak{a}}^{\mathbb{N}}: \mathfrak{a} \in \mathcal{O}_f\}} f \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)_{\mathfrak{a}}^{\mathbb{N}}} f \ d\lambda_B.$$

This concludes the proof.

Define for each $\mathfrak{a}=(a_n)_{n\in\mathbb{N}}\in\mathcal{N}$, the translation

$$T_{\mathfrak{a}}: \quad [0,1)_{\mathfrak{a}}^{\mathbb{N}} \quad \longrightarrow \quad [0,1)^{\mathbb{N}} \\ (x_n)_{n \in \mathbb{N}} \quad \mapsto \quad (x_n - a_n)_{n \in \mathbb{N}}$$

Then, it is apparent that $T_{\mathfrak{a}}([0,1)^{\mathbb{N}}_{\mathfrak{a}}) = [0,1)^{\mathbb{N}}$. Moreover, since λ_B is translation invariant, we have that

(19)
$$\lambda_B(A) = \lambda_B(T_{\mathfrak{a}}(A)) = \lambda_B(T_{\mathfrak{a}}^{-1}(A)) \text{ for each } A \in \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}.$$

Define for each $1 \leq p < \infty$, the vector space

$$\bigoplus_{\mathbf{g}\in\mathcal{N}} L_p([0,1)^{\mathbb{N}},\lambda_B)$$

consisting in all sequence $\mathfrak{f}=(f_{\mathfrak{a}})_{\mathfrak{a}}$ satisfying

- (1) $f_{\mathfrak{a}} \in L_p([0,1)^{\mathbb{N}}, \lambda_B)$ for each $\mathfrak{a} \in \mathcal{N}$.
- (2) $f_{\mathfrak{a}} = 0$ λ_B -a.e. for each $\mathfrak{a} \in \mathcal{N} \setminus \mathcal{O}_{\mathfrak{f}}$ for some countable subset $\mathcal{O}_{\mathfrak{f}} \subset \mathcal{N}$.
- (3) $\|\mathfrak{f}\|_{p,\oplus} < +\infty$, where

(20)
$$\|\mathfrak{f}\|_{p,\oplus} := \left(\sum_{\mathfrak{a}\in\mathcal{O}_{\mathfrak{f}}} \int_{[0,1)^{\mathbb{N}}} |f_{\mathfrak{a}}|^p \ d\lambda_B\right)^{\frac{1}{p}}.$$

Under this definitions, it is clear that the pair

$$\left(\bigoplus_{\mathfrak{a}\in\mathcal{N}} L_p([0,1)^{\mathbb{N}},\lambda_B), \|\cdot\|_{p,\oplus}\right)$$

defines a normed space. It must be observed that the restriction of λ_B to $[0,1)^{\mathbb{N}}$ is the product measure $\bigotimes_{i\in\mathbb{N}} m_{[0,1)}$ constructed in section I. This measure satisfies the finiteness condition $\prod_i m_{[0,1)}([0,1)) = 1 < +\infty$ and therefore Theorem 3.4 can be applied to simplify the structure of $L_p([0,1)^{\mathbb{N}}, \lambda_B)$ for each $1 \leq p < \infty$ and to compute in a simple manner each term of (20). Hence, the structure of the space $\bigoplus_{\mathfrak{a}\in\mathcal{N}} L_p([0,1)^{\mathbb{N}}, \lambda_B)$ is simply determined. The next result establishes that the spaces $L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ and $\bigoplus_{\mathfrak{a}\in\mathcal{N}} L_p([0,1)^{\mathbb{N}}, \lambda_B)$ are isometrically isomorphic. This identification clarify the structure of $L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ establishing an analogue of the Theorem 3.4 for this case.

Theorem 4.2. The following spaces are isometrically isomorphic for $1 \le p < \infty$

$$L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B) \simeq \bigoplus_{\mathfrak{a} \in \mathcal{N}} L_p([0, 1)^{\mathbb{N}}, \lambda_B).$$

Moreover, given $f \in L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$, there exists a countable subset $\mathcal{O}_f \subset \mathcal{N}$ such that

(21)
$$\int_{\mathbb{R}^{\mathbb{N}}} |f|^p \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}} |f \circ T_{\mathfrak{a}}^{-1}|^p \ d\lambda_B.$$

Proof. Fix $1 \le p < \infty$ and define

$$\mathfrak{P}: L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B) \longrightarrow \bigoplus_{\mathfrak{a} \in \mathcal{N}} L_p([0, 1)^{\mathbb{N}}, \lambda_B) \\ f \longmapsto (f \circ T_{\mathfrak{a}}^{-1})_{\mathfrak{a}}.$$

Given $f \in L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$, necessarily $|f|^p \in L_1(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ and therefore an application of Theorem 4.1 yields the existence of a countable subset $\mathcal{O}_f \subset \mathcal{N}$ such that $f|_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} = 0 \ \lambda_B$ -a.e. for each $\mathfrak{a} \in \mathcal{N} \setminus \mathcal{O}_f$. Consequently, by identity (18)

$$||f||_{L_p(\lambda_B)}^p = \int_{\mathbb{R}^{\mathbb{N}}} |f|^p \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)_{\mathfrak{a}}^{\mathbb{N}}} |f|^p \ d\lambda_B.$$

On the other hand, for each $\mathfrak{a} \in \mathcal{N}$, by the *change of variable formula* for measures and identity (19)

$$\int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} |f|^p \ d\lambda_B = \int_{T_{\mathfrak{a}}([0,1)^{\mathbb{N}}_{\mathfrak{a}})} |(f \circ T_{\mathfrak{a}}^{-1})(\mathbf{x})|^p \ d\lambda_B(T_{\mathfrak{a}}^{-1}(\mathbf{x})) = \int_{[0,1)^{\mathbb{N}}} |f \circ T_{\mathfrak{a}}^{-1}|^p \ d\lambda_B.$$

Therefore, $f \circ T_{\mathfrak{a}}^{-1} \in L_p([0,1)^{\mathbb{N}}, \lambda_B)$ for each $\mathfrak{a} \in \mathcal{N}$ and $f \circ T_{\mathfrak{a}}^{-1} = 0$ for each $\mathfrak{a} \in \mathcal{N} \setminus \mathcal{O}_f$. Moreover

$$(22) ||f||_{L_p(\lambda_B)}^p = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)_{\mathfrak{a}}^{\mathbb{N}}} |f|^p d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}} |f \circ T_{\mathfrak{a}}^{-1}|^p d\lambda_B = ||(f \circ T_{\mathfrak{a}}^{-1})_{\mathfrak{a}}||_{p,\oplus}^p.$$

Consequently $(f \circ T_{\mathfrak{a}}^{-1})_{\mathfrak{a}} \in \bigoplus_{\mathfrak{a} \in \mathcal{N}} L_p([0,1)^{\mathbb{N}}, \lambda_B)$ and $||f||_{L_p(\lambda_B)} = ||(f \circ T_{\mathfrak{a}}^{-1})_{\mathfrak{a}}||_{p,\oplus}$ for each $f \in L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$. Thus \mathfrak{P} is well define and it is an isometry. Moreover, (22) proves identity (21). Finally, we will see that \mathfrak{P} is onto. Let $\mathfrak{f} = (f_{\mathfrak{a}})_{\mathfrak{a}} \in \bigoplus_{\mathfrak{a} \in \mathcal{N}} L_p([0,1)^{\mathbb{N}}, \lambda_B)$ and define

$$f := \sum_{\mathfrak{a} \in \mathcal{O}_{\mathfrak{f}}} (f_{\mathfrak{a}} \circ T_{\mathfrak{a}}) \cdot \chi_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}}.$$

By the monotone convergence theorem it follows that

$$\int_{\mathbb{R}^{\mathbb{N}}} |f|^p \ d\lambda_B = \int_{\mathbb{R}^{\mathbb{N}}} \sum_{\mathfrak{a} \in \mathcal{O}_{\mathfrak{a}}} |f_{\mathfrak{a}} \circ T_{\mathfrak{a}}|^p \cdot \chi_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_{\mathfrak{a}}} \int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} |f_{\mathfrak{a}} \circ T_{\mathfrak{a}}|^p \ d\lambda_B$$

and again by the change of variable formula

$$\int_{\mathbb{R}^{\mathbb{N}}} |f|^p \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_{\mathfrak{f}}} \int_{[0,1)^{\mathbb{N}}_{\mathfrak{a}}} |f_{\mathfrak{a}} \circ T_{\mathfrak{a}}|^p \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_{\mathfrak{f}}} \int_{[0,1)^{\mathbb{N}}} |f_{\mathfrak{a}}|^p \ d\lambda_B = \|\mathfrak{f}\|_{p,\oplus}^p < +\infty.$$

Therefore $f \in L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ and it is straightforward to verify that

$$\mathfrak{P}(f) = \left(f \circ T_{\mathfrak{a}}^{-1} \right)_{\mathfrak{a}} = \mathfrak{f}.$$

This concludes the proof.

5. Measures on Banach Spaces

Let $(X, \|\cdot\|)$ be an infinite dimensional separable Banach space and let $\mathfrak{X} = (x_n, x_n^*)_{n \in \mathbb{N}}$ be an absolutely convergent Markushevich basis (M-basis in short). Any infinite dimensional separable Banach space admits an absolutely convergent M-basis (see for instance [16]), therefore all the considerations given in this section are valid for any infinite dimensional separable Banach space. The aim of this section is to construct a normalized translation invariant borel measure for any Banach space clarifying the structure of its corresponding Lebesgue spaces. We start considering the linear map

$$\mathfrak{B}: (X, \mathcal{B}_X) \longrightarrow (\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}})$$

$$x \mapsto (x_n^*(x))_{n \in \mathbb{N}}.$$

Clearly \mathfrak{B} is continuous and injective and consequence, by Lusin-Souslin Theorem [11, Theorem 15.1], \mathfrak{B} maps borel subsets of X into borel subsets of $\mathbb{R}^{\mathbb{N}}$. We are considering in X the norm topology an in $\mathbb{R}^{\mathbb{N}}$ the product topology. Consider the set map $\mu_X : \mathcal{B}_X \to [0, +\infty]$ defined by

(23)
$$\mu_X(B) := \lambda_B(\mathfrak{B}(B))$$

for each $B \in \mathcal{B}_X$, where \mathcal{B}_X stands for the borel σ -algebra of X. It becomes apparent that μ_X is a measure on the measurable space (X, \mathcal{B}_X) . Moreover, by (23) and Theorem 2.8, the measure μ_X is translation invariant and if we define

$$Q := \{ x \in X : |x_n^*(x)| \le 1/2, \forall n \in \mathbb{N} \},$$

then, an straightforward computation yields

$$\mu_X(\mathcal{Q}) = \lambda_B(\mathfrak{B}(\mathcal{Q})) = \lambda_B\left(\left(\left(\sum_{n \in \mathbb{N}} \left[-\frac{1}{2}, \frac{1}{2} \right] \right) \right) = 1.$$

It is convenient to remark that the proof of the existence of a measure on (X, \mathcal{B}_X) satisfying this properties is not new since it was constructed by T. Gill, A. Kirtadze, G. Pantsulaia and A. Plichko in [7, Theorem 4.3]. Here, this type of measure has been constructed differently, via outer measure techniques developed in section I but we follow the philosophy of [7]. The next result establishes the structure of Lebesgue spaces of $(X, \mathcal{B}_X, \mu_X)$ and establishes an analogue of Theorem 4.2 for the category of Banach spaces. In the following result, given X, Y two Banach spaces, the notation $X \hookrightarrow Y$ means that there exists a linear isometry from X to Y.

Theorem 5.1. Let $(X, \mathcal{B}_X, \mu_X)$ be the constructed measure space, then for $1 \leq p < +\infty$

(24)
$$L_p(X, \mathcal{B}_X, \mu_X) \hookrightarrow \bigoplus_{\mathbf{a} \in \mathcal{N}} L_p([0, 1)^{\mathbb{N}}, \lambda_B).$$

Moreover, given $f \in L_p(X, \mathcal{B}_X, \mu_X)$, there exists a countable subset $\mathcal{O}_f \subset \mathcal{N}$ such that

(25)
$$\int_{X} |f|^{p} d\mu_{X} = \sum_{\mathfrak{a} \in \mathcal{O}_{f}} \int_{[0,1)^{\mathbb{N}}} |(f \circ \mathfrak{B}^{-1})\chi_{\mathfrak{B}(X)} \circ T_{\mathfrak{a}}^{-1}|^{p} d\lambda_{B}.$$

Proof. The operator \mathfrak{B} induces the map

(26)
$$\mathfrak{E}: L_p(X, \mathcal{B}_X, \mu_X) \hookrightarrow L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B) \\
f \mapsto (f \circ \mathfrak{B}^{-1}) \chi_{\mathfrak{B}(X)}.$$

This operator is an isometric embedding, since given $f \in L_p(X, \mathcal{B}_X, \mu_X)$, by the *change* of variable formula for measures

$$||f||_{L_p(\mu_X)}^p = \int_X |f|^p d\mu_X = \int_X |f(\mathbf{x})|^p d\lambda_B(\mathfrak{B}(\mathbf{x})) = \int_{\mathfrak{B}(X)} |f \circ \mathfrak{B}^{-1}|^p d\lambda_B$$
$$= \int_{\mathbb{R}^N} |f \circ \mathfrak{B}^{-1}|^p \cdot \chi_{\mathfrak{B}(X)} d\lambda_B = ||(f \circ \mathfrak{B}^{-1})\chi_{\mathfrak{B}(X)}||_{L_p(\lambda_B)}^p.$$

Therefore, \mathfrak{C} is well define and it defines an isometry. Consequently

$$L_p(X, \mathcal{B}_X, \mu_X) \hookrightarrow L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B).$$

Hence, an straightforward application of Theorem 4.2 yields embedding (24). Finally, we will prove identity (25). Let $f \in L_p(X, \mathcal{B}_X, \mu_X)$, then $(f \circ \mathfrak{B}^{-1})\chi_{\mathfrak{B}(X)} \in L_p(\mathbb{R}^{\mathbb{N}}, \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}, \lambda_B)$ and by (21), there exists a countable subset $\mathcal{O}_f \subset \mathcal{N}$ such that

$$\int_{\mathbb{R}^{\mathbb{N}}} |f \circ \mathfrak{B}^{-1}|^p \cdot \chi_{\mathfrak{B}(X)} \ d\lambda_B = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}} |(f \circ \mathfrak{B}^{-1})\chi_{\mathfrak{B}(X)} \circ T_{\mathfrak{a}}^{-1}|^p \ d\lambda_B.$$

This concludes the proof.

We finalize this section and the article summarizing the presented results in the following theorem.

Theorem 5.2. Let $(X, \|\cdot\|)$ be a infinite dimensional separable Banach space and let $\mathfrak{X} = (x_n, x_n^*)_{n \in \mathbb{N}}$ be an absolutely convergent M-basis. Then, there exists a translation invariant borel measure μ_X on the measurable space (X, \mathcal{B}_X) such that $\mu_X(\mathcal{Q}) = 1$, where

$$\mathcal{Q} := \{ x \in X : |x_n^*(x)| \le 1/2, \forall n \in \mathbb{N} \}.$$

Moreover, for $1 \le p < +\infty$

$$L_p(X, \mathcal{B}_X, \mu_X) \hookrightarrow \bigoplus_{\mathfrak{a} \in \mathcal{N}} L_p([0, 1)^{\mathbb{N}}, \lambda_B),$$

and given $f \in L_p(X, \mathcal{B}_X, \mu_X)$, there exists a countable subset $\mathcal{O}_f \subset \mathcal{N}$ such that

$$\int_X |f|^p d\mu_X = \sum_{\mathfrak{a} \in \mathcal{O}_f} \int_{[0,1)^{\mathbb{N}}} |(f \circ \mathfrak{B}^{-1}) \chi_{\mathfrak{B}(X)} \circ T_{\mathfrak{a}}^{-1}|^p d\lambda_B.$$

References

- [1] L. Arkeryd, N. Cutland, C.W. Henson. Nonstandard analysis. NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, vol. 493, Kluwer Academic Publishers Group, Dordrecht, (1997)
- [2] R. BAKER. Lebesgue Measure on \mathbb{R}^{∞} , II. Proceedings of the American Mathematical Society Volume 132 Number 9 (April 2004) pp. 2577-2591.
- [3] V. I. BOGACHEV. Measure Theory Vol I and II. Springer-Verlag Berlin Heidelberg (2007).
- [4] D. L. Cohn. Measure Theory. Second Edition, Springer Birkhäuser (2013).
- [5] E. O. Elliot, A. P. Morse. General Product Measures. Transactions of the American Mathematical Society Volume 110 (1963) pp. 245283.
- [6] G. B. Folland. Real Analysis: Modern Thechniques and their Applications. John Wiley & Sons (1999).
- [7] T. GILL, A. KIRTADZE, G. PANTSULAIA, A. PLICHKO. Existence and uniqueness of translation invariant measures in separable Banach spaces. Funct. Approx. Comment. Math. Volume 50, Number 2 (2014), pp. 401-419.
- [8] E. Hopf. Ergodentheorie. Berlin (1937) pp. 2.
- [9] B. Jessen. The Theory of Integration in a Space of Infinite Number of Dimensions. *Acta mathematica*, 63 (December, 1934), pp. 249-323.
- [10] S. KAKUTANI. Notes on Infinite Product Measure Spaces, I. Proc. Imp. Acad. Volume 19 Number 3 (March 1943) pp. 148-151.
- [11] A.S. Kechris. Classical descriptive set theory. Springer, New York, (1995).
- [12] A. Kolmogoroff. Grundbegriffe der Wahrscheinlichkeitsrechnung. Berlin (1933).
- [13] P. LOEB, P. ROSS. Infinite Products of Infinite Measures. *Illinois Journal of Mathematics 1 (Spring, 2005) pp. 153-158*.
- [14] Z. LOMNICKI, S. Ulam. Sur la theorie de la mesure dans les espaces combinatoires et son application au calcul des probabilites. I: Variables independantes. Fund. Math. vol. 23 (1934) pp. 237-278.
- [15] G. PANTSULAIA. On Ordinary and Standard Products of Infinite Family of σ-finite Measures and Some of Their Applications. Acta Mathematica Sinica, English Series Volume 27 Number 3 (February 2011) pp. 477496.
- [16] I. Singer. Bases in Banach spaces, II. Springer (1981).
- [17] S. SAEKI. A Proof of the Existence of Infinite Product Probability Measures. The American Mathematical Monthly Vol. 103, No. 8 (October, 1996), pp. 682-683.
- [18] K. R. Stromberg. Probability for Analysts. Chapman & Hall (1994).
- [19] Y. Yamasaki. Measures on Infinite Dimensional Spaces. World Scientific (1985).

DEPARTMENT OF MATHEMATICAL ANALYSIS AND APPLIED MATHEMATICS, FACULTY OF MATHEMATICAL SCIENCE, COMPLUTENSE UNIVERSITY OF MADRID, 28040-MADRID, SPAIN.

E-mail address: juancsam@ucm.es