A Proof of the Existence of Infinite Product Probability Measures

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In memory of my dear friend Karl Stromberg

Let $\{(\Omega_i, \mathscr{F}_i, P_i): i \in I\}$ be a nonempty collection of probability spaces, and let $\Omega := \Pi_i \Omega_i$ be the product space. A measurable cylinder in Ω is a subset A of Ω of the form $A = \Pi_i A_i$, where $A_i \in \mathscr{F}_i$ for each i and $A_i = \Omega_i$ for all but finitely many i's. For such a set A, define $P(A) := \Pi_i P_i(A_i)$. By definition, the product probability measure of the P_i's is the (necessarily unique) extension of P to a probability measure on $\mathscr{F}(\mathscr{M}c)$, where $\mathscr{M}c$ is the collection of all measurable cylinders in Ω and $\mathscr{F}(\mathscr{M}c)$ is the σ-field generated by $\mathscr{M}c$. The standard proof of the existence of the product probability measure is based upon Fubini's Theorem for finitely many factors; see [**HS**: pp. 429–435]. We give a simple proof that does not require Fubini's Theorem.

Lemma. *Let μ: $\mathscr{M}c \to [0,1]$ be a function such that $\sum_1^\infty \mu(A_n) = 1$ whenever (A_n) is a disjoint sequence in $\mathscr{M}c$ with union Ω. Then μ extends uniquely to a probability measure on $\mathscr{F}(\mathscr{M}c)$.*

Proof: Let $\mathscr{D}$ be the collection of all finite unions of measurable cylinders. It is easy to check that $\mathscr{D}$ is a field and each $A \in \mathscr{D}$ can be written as a finite disjoint union of members of $\mathscr{M}c$. In particular, A can be written as a countable disjoint union of members of $\mathscr{M}c$, say $A = \bigcup_1^\infty A_n$. Let $\mu'(A) := \sum_1^\infty \mu(A_n)$. To see that μ' is well-defined, write $\Omega \setminus A = \bigcup_1^m B_k$ with pairwise disjoint $B_k \in \mathscr{M}c$. Then

$$\sum_1^\infty \mu(A_n) = 1 - \sum_1^m \mu(B_k) \tag{1}$$

by our assumption on μ. Since the right-hand of (1) has nothing to do with the decomposition $\bigcup_1^\infty A_n$ of A, it follows that μ' is well-defined and therefore countably additive of $\mathscr{D}$. Hence the desired result is an immediate consequence of E. Hopf's extension theorem [**HS**: p. 142]. ∎

Theorem. *P extends uniquely to a probability measure on $\mathscr{F}(\mathscr{M}c)$.*

Proof: It suffices to prove that P satisfies the hypothesis of the lemma. Without loss of generality, assume that I is an infinite set. Let (A_n) be a disjoint sequence in $\mathscr{M}c$ with union Ω.

Case 1: I is countable. Then we may assume $I = \mathbb{N}$. Write $A_n = \prod_{i=1}^\infty A_{n,i}$, where $A_{n,i} \in \mathscr{F}_i$ for each i and $A_{n,i} = \Omega_i$ for all $i > i_n \in \mathbb{N}$. We claim that if $m \in \mathbb{N}$ and $x = (x_i)$ is an element of A_m and if $n \in \mathbb{N}$, then

$$\left\{\prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i)\right\} \prod_{i > i_m} P_i(A_{n,i}) = \delta_{m,n} \quad \text{(Kronecker's delta)}. \tag{2}$$

For $n = m$, this is trivial, so assume $n \neq m$. Then, since $\sum_1^\infty \chi_{A_k} = 1$ identically and $\chi_{A_m}(x_1, \ldots, x_{i_m}, y_{i_m+1}, \ldots) = 1$ for all $y_i \in \Omega_i$ with $i > i_m$, we have

$$\left\{ \prod_{i=1}^{i_m} \chi_{A_{n,i}}(x_i) \right\} \prod_{i>i_m} \chi_{A_{n,i}}(y_i) = 0 \tag{3}$$

for all such y_i's. Integrating each side of (3) finitely many times, we obtain (2) for $n \neq m$.

To get a contradiction, suppose $\sum_{n=1}^\infty P(A_n) \neq 1$. Then there must exist an $x_1 \in \Omega_1$ such that

$$\sum_{n=1}^{\infty} \chi_{A_{n,1}}(x_1) \prod_{i=2}^{\infty} P_i(A_{n,i}) \neq 1.$$

Hence an inductive argument yields an element $x = (x_i)$ of Ω such that

$$\sum_{n=1}^{\infty} \left\{ \prod_{i=1}^{k} \chi_{A_{n,i}}(x_i) \right\} \prod_{i=k+1}^{\infty} P_i(A_{n,i}) \neq 1 \tag{4}$$

for each $k \geq 1$. But $x \in A_m$ for some $m \in \mathbb{N}$. Hence (4) with $k = i_m$ contradicts (2).

Case 2: I is uncountable. Then we can choose a countable subset J of I such that $A_n = A'_n \times \Omega'$ for all $n \geq 1$, where each A'_n is a measurable cylinder in $\Pi_{i \in J} \Omega_i$ and $\Omega' = \Pi_{i \notin J} \Omega_i$. By Case 1 applied to (A'_n), we obtain $\sum_1^\infty P(A_n) = 1$. ■

Dedication. *Professor Karl Stromberg, my friend and colleague, died on July* 3, 1994. *He was an enthusiastic lover of the* Monthly. *When I presented the above proof in my seminar five to eight years ago, he liked it very much. Karl, I dedicate the present paper to you in the memory of our friendship. Have a peaceful sleep!*

REFERENCE

[HS] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, Berlin, 1965.

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A Problem

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TRIBUTE. I learned about this problem and its solution in 1935, when I was in the eighth grade, from my teacher of mathematics, Yakov Stepanovich Chaikovsky, a very young man at that time. Now, in retrospective of a few decades of my own