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REGULAR PROBABILITY MEASURES ON FUNCTION SPACE

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Introduction

Many arguments in the theory of stochastic processes involve conditions on a non-denumerable family of random variables. To carry out these arguments rigorously it is necessary to establish the measurability of the various sets and functions involved. Customarily, this is done by introducing a new σ-algebra or family of σ-algebras as each occasion arises, and proving measurability with respect to them. This leads to technical complications which obscure the real content of the theorems, and also has the disadvantage that the constructions are not canonical; i.e., not uniquely determined by the finite joint distributions. Thus there may be two versions x_t and x'_t of a stochastic process such that for each t, $x_t = x'_t$ a.e., but the conclusions of a given theorem may be true for x_t but not x'_t.

These complications usually arise in questions of a mixed topological and measure-theoretic nature, such as the question of the continuity of almost all sample functions. This suggests that the natural version of measure theory to use is that in which topology and measure are most intimately related: the theory of regular Borel measures. This approach was first suggested by Kakutani (see [2]). We shall attempt to show that the Kakutani approach yields a simple, canonical theory.

Section 1 contains a variant of the Kolmogoroff representation theorem [8] which establishes directly the existence of a regular probability measure defined on all Borel sets in function space. This is an immediate consequence of the Riesz-Markoff and the Stone-Weierstrass theorems. In theory, all properties of these measures are determined by the finite joint distributions. The next section shows how to make this reduction in practice. In Section 3 various subsets of function space, such as the set of all continuous functions or all functions continuous except for jumps, are shown to be Borel sets. The set of all (t, ω) at which a discontinuity occurs is a Borel set in $T \times \Omega$. In Section 4 we apply the Fubini theorem to this set to show that if μ is a measure on T such that the fixed points of discontinuity have μ measure 0, then almost every ω is continuous except on a set of measure 0. If there are no fixed points of discontinuity then almost every ω is continuous except on a set of the first category. Section 5 studies measurability questions

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connected with entering a set and first passage times.

The problem raised by Doob in [2] as to whether $\xi(t, \omega)$ is jointly measurable when, say, $\xi(s, \cdot)$ converges to $\xi(t, \cdot)$ in measure as $s \to t$, remains open. Thus it cannot be claimed that it is always appropriate to consider this topological version of a given stochastic process. However, this approach is quite useful in many situations where the topology of X plays an important role. We do not consider any special classes of stochastic processes here.

We deal throughout with stochastic processes taking values in a compact Hausdorff space. There is no loss of generality in not considering locally compact spaces, since these may always be compactified and then the behavior of the sample functions near infinity may be studied.

1. The canonical version of a stochastic process

Throughout this paper X is a compact Hausdorff space (the space in which the stochastic process takes its values), T is a set (the time parameter set), and Ω is the space of all functions from T to X, in the product topology. Thus Ω is the Cartesian product $\prod_{t \in T} X$, and is a compact Hausdorff space by the Tychonoff theorem. A base for the open sets of Ω is the class of all $\{\omega : \omega(t_1) \in G_1, \cdots, \omega(t_n) \in G_n\}$ where $G_1, \cdots, G_n$ are open sets in X. We shall sometimes assume additional structure on X or T.

If S is a subset of T, we shall denote by C_S the algebra of all real-valued continuous functions φ on Ω such that if $\omega(s) = \omega'(s)$ for all s in S then $\varphi(\omega) = \varphi(\omega')$, and by C_f the union of all C_S for finite subsets S of T. Then C_f is an algebra, and by the Stone-Weierstrass theorem [9] is dense in the algebra $C = C_T$ of all continuous functions on Ω, since it clearly separates points. The topology of C is that given by the supremum norm, $\| \varphi \| = \sup_{\omega \in \Omega} | \varphi(\omega) |$.

By the *Borel sets* $\mathscr{B}(Y)$ of a compact Hausdorff space Y is meant the smallest class of subsets of Y closed under the formation of differences and countable unions, and containing all compact sets. Since Y itself is compact, $\mathscr{B}(Y)$ is a σ-algebra and contains all open sets. The term *regular Borel measure*, or simply *regular measure*, means a positive finite measure μ defined on $\mathscr{B}(Y)$ such that for all Borel sets B, $\sup_F \mu(F) = \mu(B) = \inf_G \mu(G)$, where F ranges over all closed (i.e., compact) sets contained in B, and G ranges over all open sets containing B. The form of the Riesz-Markoff theorem established in [6] asserts that there is a one-to-one correspondence between positive linear functionals on the algebra $C(Y)$ of all real-valued continuous functions on Y and regular measures μ, the linear functional corresponding to μ being $f \to \int f(y)\,d\mu(y)$. We shall sometimes also denote the linear functional by the symbol μ,

$$\mu(f) = \int f(y)\,d\mu(y)\ .$$

THEOREM 1.1. *Let μ_S, for each finite subset S of T, be a positive linear functional on the algebra C_S such that $\mu_S(1) = 1$ and with the consistency property that if $S \subset S'$ then the restriction of $\mu_{S'}$ to C_S is μ_S. Then there exists a unique regular probability measure* Pr *on Ω such that for each S and φ in C_S, $\mu_S(\varphi) = \int \varphi(\omega)\,d\,\mathrm{Pr}(\omega)$.*

PROOF. For any φ in C_f there is by definition of C_f a finite set S such that φ is in C_S. Let $\mu(\varphi) = \mu_S(\varphi)$. This is well defined by the consistency property. If $\varphi \geqq 0$ then $\mu(\varphi) \geqq 0$, and $\mu(1) = 1$. Consequently μ is a continuous linear functional on C_f, and since C_f is dense in C, μ has a unique continuous extension to all of C. By continuity, μ on C is a positive linear functional with $\mu(1) = 1$, and so corresponds to a unique probability measure, which we shall denote Pr. The restriction of the linear functional $\varphi \to \int \varphi(\omega)\,d\,\mathrm{Pr}(\omega)$ to C_S is by definition μ_S, and any other regular measure having this property must give rise to μ on C_f, and hence on all of C, concluding the proof.

Theorem 1.1 implies the Kolmogoroff representation theorem [8], by taking μ_S to be the linear functional corresponding to the measure of the joint distribution of the random variables indexed by S. However, the measure Pr is defined on a larger σ-algebra if T is uncountable.

Following Doob [3], we shall define an X-valued *stochastic process* with index set T to be a family of functions x_t, t in T, defined on some probability space, such that for each Borel set B in X, $x_t^{-1}(B)$ is a measurable set in the probability space. We shall also call such an object a *version* of a stochastic process. Two stochastic processes x_t and x'_t will be called *versions of the same stochastic process* in case for each finite subset S of T the joint distribution of the x_s, s in S, is the same as that of the x'_s. This terminology is commonly used, and while it is not impeccable from the point of view of logical style, it has the advantage of not introducing the over-worked word "equivalent". By the *canonical version* of a stochastic process we shall mean that version in which the probability space is the space Ω of all functions from T to X, with σ-algebra the Borel sets $\mathscr{B}(\Omega)$, Pr is the regular Borel measure given by Theorem 1.1 where μ_S is the joint distribution of the random variables indexed by S, and the X-valued functions on Ω are the coordinate functions ξ_t given by $\xi_t(\omega) = \xi(t, \omega) = (t)\omega$. Thus Theorem 1.1 says that every stochastic process has a unique canonical version.

2. Reduction to countable parameter sets

Following Halmos [6], we use the term *Baire sets* of a compact Hausdorff space Y for the σ-algebra $\mathscr{B}_0(Y)$ generated by the compact G_δ sets. (A set is a G_δ if it is the intersection of a decreasing sequence of open sets, it is an F_σ if it is the union of an increasing sequence of closed (i.e., compact, since Y is compact) sets, an $F_{\sigma\delta}$ if it is the intersection of a decreasing sequence of F_σ sets, and so forth.)

THEOREM 2.1. *$\mathscr{B}_0(\Omega)$ is the σ-algebra generated by sets of the form $\{\omega : \omega(t) \in E\}$, t in T, E in $\mathscr{B}_0(X)$.*

Usually the space X is second countable, so that $B_0(X) = B(X)$. The important part of Theorem 2.1 is that Baire sets of Ω are in a σ-algebra generated by sets depending on only one coordinate t, so that each Baire set of Ω depends on only countably many coordinates. That is, Theorem 2.1 has the following immediate corollary :

COROLLARY 2.1. *If B is in $\mathscr{B}_0(\Omega)$ then there is a countable subset S of T such that if $\omega'(s) = \omega(s)$ for all s in S and ω is in B then ω' is in B.*

PROOF OF THEOREM 2.1. Clearly, each set of the form $\{\omega : \omega(t) \in E\}$, $t \in T$, $E \in \mathscr{B}_0(X)$, is a Baire set of Ω, so that $\mathscr{B}_0(\Omega)$ contains the σ-algebra they generate. Conversely, let Φ be a compact G_δ in Ω, Γ_n a decreasing sequence of open sets whose intersection is Φ. Let $\mathscr{G}_0(X)$ be the open Baire sets of X. Now an open set is an F_σ if and only if its complement is a compact G_δ (since X is compact), so that an open F_σ is a Baire set. But the open F_σ sets are a base for the open sets of X (see [6], p. 219, Ex. 6), and so $\mathscr{G}_0(X)$ is a base. (If X is second countable we may omit this argument.) Let $\mathscr{U}(\Omega)$ be the family of all sets of the form $\{\omega : \omega(t_1) \in G_1, \cdots \omega(t_n) \in G_n\}$ where the t_i are in T and the G_i are in $\mathscr{G}_0(X)$. By definition of the product topology and the fact that $\mathscr{G}_0(X)$ is a base for the open sets of X, $\mathscr{U}(\Omega)$ is a base for the open sets of Ω. Therefore each point ω in Φ has a neighborhood $\Psi_n(\omega)$ in $\mathscr{U}(\Omega)$ and contained in Γ_n. By compactness, Φ is contained in a finite union $\Psi_n = \Psi_n(\omega_1) \cup \cdots \cup \Psi_n(\omega_k)$ contained in Γ_n. Hence $\Phi = \bigcap_n \Psi_n$ and Φ is in the σ-algebra generated by sets of the required form, completing the proof.

If B is any subset of Ω and S is any subset of T, we shall define B_S to be the set of all ω in Ω such that there exists an ω' in B for which $\omega(s) = \omega'(s)$ for all s in S. If $\mathrm{B} = \mathrm{B}_S$ we shall say that B is *determined* by

S. Thus Corollary 2.1 says that a Baire set is determined by a countable subset of T. Notice that B is always contained in B_S, and that B_S is the smallest set determined by S which contains B.

If μ is a regular Borel measure, a set E is called *μ-measurable* in case there is a Borel set E' such that $(E - E') \cup (E' - E)$ is contained in a Borel set of μ measure 0; i.e., in case E is in the completed σ-ring of μ.

THEOREM 2.2. *Let* Pr *be a regular Borel measure on* Ω, Φ *a compact subset of* Ω. *Then there is a countable subset* S *of* T *such that* $\mathrm{Pr}(\Phi_S - \Phi) = 0$.

If B *is any* Pr*-measurable set then there is a countable subset* S *of* T *and a Baire set* B_0 *determined by* S *such that* $\mathrm{B} = \mathrm{B}_0$ a.e.

PROOF. For each n there is, by regularity, an open set Γ_n containing the compact set Φ, with $\mathrm{Pr}(\Gamma_n - \Phi) \leqq 1/n$. Since Φ is compact, there is an open set Ψ_1 containing Φ with $\overline{\Psi}_1 \subset \Gamma_1$, an open set Ψ_2 containing Φ with $\overline{\Psi}_2 \subset \Gamma_2 \cap \Psi_1$, and by induction an open set Ψ_n containing Φ with $\overline{\Psi}_n \subset \Gamma_n \cap \Psi_{n-1}$. Let $\Phi_0 = \bigcap_n \Psi_n$. Then Φ_0 contains Φ, Φ_0 is a G_δ, and Φ_0 is compact (since Φ_0 is also equal to $\bigcap_n \overline{\Psi}_n$), and $\mathrm{Pr}(\Phi_0 - \Phi) = 0$ (since Φ_0 is contained in each Γ_n). By Theorem 2.1 there is a countable subset S of T such that Φ_0 is determined by S. Since $\Phi \subset \Phi_0$ and Φ_0 is determined by S, $\Phi \subset \Phi_S \subset \Phi_0$. Therefore $\mathrm{Pr}(\Phi_S - \Phi) = 0$.

Now let $\mathcal{S}$ be the class of sets B for which there is a countable subset S (depending on B) of T and a Baire set B_0 determined by S such that $\mathrm{B} = \mathrm{B}_0$ a.e. It is clear that $\mathcal{S}$ is a σ-algebra. We have shown that $\mathcal{S}$ includes all compact sets, so that $\mathcal{S}$ contains all Borel sets. Finally, $\mathcal{S}$ includes any set equal almost everywhere to a set in $\mathcal{S}$, so that $\mathcal{S}$ includes all Pr-measurable sets, concluding the proof.

We shall need to use the fact that $\mathrm{Pr}(\Phi_S - \Phi) = 0$ for a class of sets Φ more general than the compact sets. A simple example shows that this is not true for all Borel sets. In fact, let T be uncountable, let $X = \{0, 1\}$, let Pr assign mass 1 to the function ω_0 which is identically 0, and let Γ be the open set $\Omega - \{\omega_0\}$. If S is any countable subset of T then $S \neq T$ and $\Gamma_S = \Omega$, so that $\mathrm{Pr}(\Gamma_S) = 1$. However, $\mathrm{Pr}(\Gamma) = 0$.

LEMMA 2.1. *If* S *is a fixed subset of* T, *the transformation* $\mathrm{B} \rightarrow \mathrm{B}_S$ *commutes with arbitrary unions and with intersections of decreasing sequences of compact sets.*

PROOF. Let $\pi : \Omega \rightarrow \prod_{s \in S} X$ be the projection mapping; $\pi(\omega)$ is the restriction of ω to S. Then π is a continuous mapping, and so $\mathrm{B} \rightarrow \pi(\mathrm{B})$ commutes with arbitrary unions and the intersection of decreasing chains of compact sets. Now $\mathrm{B}_S = \pi(B) \times \prod_{t \in T-S} X$, so that this transformation has the same two properties.

THEOREM 2.3. *Let* B *be of the form*

$$\mathrm{B} = \bigcup_J \bigcap_{i=1}^{\infty} \Phi_{j_1\cdots j_i} \tag{2.1}$$

where the union is over all infinite sequences $J = (j_1 j_2 \cdots)$ *of positive integers and the* $\Phi_{j_1\cdots j_i}$ *are compact sets indexed by the finite sequences* $j_1 \cdots, j_i$ *of positive integers, and such that* $\Phi_{k_1\cdots k_l} \subset \Phi_{j_1\cdots j_i}$ *whenever* $l \geqq i$ *and* $k_1 = j_1, \cdots, k_i = j_i$. *Let* Pr *be the completion of a regular Borel measure on* Ω. *Then there is a countable subset* S *of* T *such that* $\mathrm{B} = \mathrm{B}_S$ a. e.

PROOF. There are only countably many finite sequences $j_1 \cdots j_i$. Consequently there is a countable subset S of T such that $\Phi_{j_1\cdots j_i} = \Phi_{j_1\cdots j_i S}$ a.e. for all $j_1 \cdots j_i$, by Theorem 2.2. By Lemma 2.1, B_S is given by (2.1) with the $\Phi_{j_1\cdots j_i}$ replaced by $\Phi_{j_1\cdots j_i S}$. Now the set $\Theta = \bigcup (\Phi_{j_1\cdots j_i S} - \Phi_{j_1\cdots j_i})$, where the union is over all finite sequences, satisfies $\Pr(\Theta) = 0$. For each infinite sequence $J = (j_1 j_2 \cdots)$,

$$\bigcap_{i=1}^{\infty} \Phi_{j_1\cdots j_i S} - \bigcap_{i=1}^{\infty} \Phi_{j_1\cdots j_i} \subset \bigcup_{i=1}^{\infty} (\Phi_{j_1\cdots j_i S} - \Phi_{j_1\cdots j_i})$$

which is contained in Θ. Since this is true for each infinite sequence J, $\mathrm{B}_S - \mathrm{B} \subset \Theta$, concluding the proof.

COROLLARY 2.2. *Let* B *be an* $F_{\sigma\delta}$, Pr *the completion of a regular Borel measure on* Ω. *Then there is a countable subset* S *of* T *such that* $\mathrm{B} = \mathrm{B}_S$ a.e.

PROOF. Since B is an $F_{\sigma\delta}$, $\mathrm{B} = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \Phi_{ij}$, where the Φ_{ij} are compact. But we may also write this as $\mathrm{B} = \bigcup_J \bigcap_{i=1}^{\infty} \Phi_{ij_i}$. Now define $\Phi_{j_1\cdots j_i} = \Phi_{1j_1} \cap \Phi_{2j_2} \cap \cdots \cap \Phi_{ij_i}$. Then $\mathrm{B} = \bigcup_J \bigcap_{i=1}^{\infty} \Phi_{j_1\cdots j_i}$ is of the required form (2.1).

A set of the form (2.1) is called *analytic* over the compact sets, and the $\Phi_{j_1\cdots j_i}$ are called a *monotone Souslin scheme*. (See [4], V. The concluding portion of the proof of Theorem 2.3 follows Hahn's proof of 40.1.5 in [4].) It is worth emphasizing that the proof of Theorem 2.3 and its corollary use only the simplest properties of these objects. They occur again in Section 5, and it may be said that analytic sets in Ω are a natural and useful tool. We shall need Theorem 2.3 only for $F_{\sigma\delta}$ sets, as in the corollary, but the proof of this special case is no simpler.

The example given before Lemma 2.1 shows that not all Borel sets in Ω are analytic over the compact sets if T is uncountable (and X has at least two points). This is because such a space Ω is not second countable.

3. The Borel structure of various function classes

We shall assume that both X and T are second countable compact Hausdorff spaces. However, the results extend at once to the case that T is a countable union of such spaces T_n (for example, T may be an open or half-open interval). This is because all function classes studied in this section are defined by local properties, so that the given function class for T will be the intersection over n of the corresponding function classes for T_n.

The spaces X and T are metrizable, by Urysohn's theorem. The same symbol ρ will be used for the metric on X and T.

If s and t are in T and $\varepsilon > 0$, then $\Delta(s, t, \varepsilon)$ will denote the set

$$\{\omega : \rho(\omega(s), \omega(t)) \leqq \varepsilon\}.$$

If $\delta > 0$ then $D(\delta)$ will denote the set $\{(s, t) : \rho(s, t) \leqq \delta\}$.

THEOREM 3.1. *If T and X are second countable compact Hausdorff spaces, then the set Δ of all continuous functions ω from T to X is an $F_{\sigma\delta}$ and*

$$\Delta = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{(s,t)\in D(1/n)} \Delta(s, t, 1/m). \tag{3.1}$$

PROOF. The set $\Delta(s, t, 1/m)$ is closed, so that an arbitrary intersection of such sets is closed, and the right hand side of (3.1) is an $F_{\sigma\delta}$. By definition of uniform continuity (for all $\varepsilon = 1/m > 0$ there exists a $\delta = 1/n > 0$ such that for all s and t with $\rho(s, t) \leqq \delta$, i.e., (s, t) in $D(1/n)$, $\rho(\omega(s), \omega(t)) \leqq \varepsilon$, i.e., $\omega \in \Delta(s, t, 1/m)$) the right hand side of (3.1) is the set of uniformly continuous ω. Since T is compact, this is the set of all continuous ω, ending the proof.

By the *set of discontinuities* $O(\omega)$ of a function ω is meant the set of points t in T such that ω is not continuous at t. If $\varepsilon > 0$ then $O_\varepsilon(\omega)$ means the set of t in T such that for all $\delta > 0$ there are t_1, t_2 in the δ-neighborhood of t with $\rho(\omega(t_1), \omega(t_2)) \geqq \varepsilon$. Thus $O_\varepsilon(\omega)$ is a closed subset of T. Also, $O_\varepsilon(\omega) \subset O_{\varepsilon'}(\omega)$ if $\varepsilon \geqq \varepsilon'$, and $O(\omega) = \bigcup_{m=1}^{\infty} O_{1/m}(\omega)$ is an F_σ set in T.

THEOREM 3.2. *Let X and T be second countable compact Hausdorff spaces and let μ be a regular Borel measure on T. The set Δ_μ of all ω which are continuous except on a set of μ-measure* 0, i.e., $\mu(O(\omega)) = 0$, *is an $F_{\sigma\delta}$ in Ω. Let $\mathcal{U}$ be a countable base for the open sets of T, $T_{kl}(\mu)$ the k^{th} (in some enumeration) subset of T which is the union of a finite number of sets in $\mathcal{U}$ of total measure $\leqq 1/l$, and $D(k, l\,; 1/n)$ the set of all (s, t) such that $\rho(s, t) \leqq 1/n$ and $s \notin T_{kl}(\mu)$, $t \notin T_{kl}(\mu)$. Then*

$$\Delta_\mu = \bigcap_{m,l=1}^{\infty} \bigcup_{k,n=1}^{\infty} \bigcap_{(s,t)\in D(k,l;1/n)} \Delta(s, t, 1/m). \tag{3.2}$$

PROOF. The right hand side of (3.2) is an $F_{\sigma\delta}$ since each $\Delta(s, t, 1/m)$ is closed, and so the indicated intersection, which we will denote $\Phi(m, l, k, n)$, is closed. The set $\bigcup_{n=1}^{\infty} \Phi(m, l, k, n)$ is the set of all ω which are uniformly $1/m$-continuous on the complement of $T_{kl}(\mu)$. Since $O_{1/m}(\omega)$ is compact, it is of measure 0 if and only if it may be covered by a finite number of sets in $\mathcal{U}$ of arbitrarily small measure. That is $\mu(O_{1/m}(\omega)) = 0$ if and only if ω is in $\bigcap_{l=1}^{\infty} \bigcup_{k,n=1}^{\infty} \Phi(m, l, k, n)$. Hence $\mu(O(\omega)) = 0$ if and only if ω is in the right hand side of (3.2), concluding the proof.

COROLLARY 3.1. *If X and T are both compact intervals, the set of Riemann-integrable ω is an $F_{\sigma\delta}$.*

In fact, a function ω is Riemann-integrable if and only if $\mu(O(\omega)) = 0$, where μ is Lebesgue measure.

THEOREM 3.3. *Let X and T be second countable compact Hausdorff spaces. Then the set Δ_p of all ω with point discontinuities only, i.e., $O(\omega)$ of first category in T, is an $F_{\sigma\delta}$. Let $\mathcal{U}$ be a countable base for the open sets of T, T_0 a countable dense subset of T. Let T_{kl} be the k^{th} (in some enumeration) subset of T which is the union of a finite number of sets in $\mathcal{U}$ at positive distance from some finite $1/l$-dense set in T_0, and let $D'(k,l\,;1/n)$ be the set of all (s, t) such that $\rho(s, t) \leqq 1/n$ and $s \notin T_{kl}$, $t \notin T_{kl}$. Then*

$$(3.3) \qquad \Delta_p = \bigcap_{m,l=1}^{\infty} \bigcup_{k,n=1}^{\infty} \bigcap_{(s,t)\in D'(k,l;1/n)} \Delta(s, t, 1/m) .$$

PROOF. Again, the indicated intersection $\Phi'(m, l, k, n)$ of the $\Delta(s, t, 1/m)$ is closed, and the right hand side of (3.3) is an $F_{\sigma\delta}$. The set $O(\omega)$ is of first category if and only if each compact set $O_{1/m}(\omega)$ is nowhere dense. This is true if and only if for all l it is at positive distance from some $1/l$-dense set in T_0; i.e., if and only if for each m, ω is in

$$\bigcap_{l=1}^{\infty} \bigcup_{k,n=1}^{\infty} \Phi'(m, l, k, n) ,$$

which concludes the proof.

THEOREM 3.4. *Let $T = [a, b]$ be a compact interval and X a second countable compact Hausdorff space. Then the set Δ_j of all ω with jump discontinuities only, i.e., $\lim_{s\to t-}\omega(s)$ and $\lim_{s\to t+}\omega(s)$ existing for all t in $[a, b]$, is an $F_{\sigma\delta}$. Let $\Phi(m, l)$ be the intersection over all $a \leqq t_1 \leqq \cdots \leqq t_l \leqq b$ of $\{\omega : \min_{1\leqq i\leqq l-1} \rho(\omega(t_i), \omega(t_{i+1})) \leqq 1/m\}$. Then*

$$(3.4) \qquad \Delta_j = \bigcap_{m=1}^{\infty} \bigcup_{l=1}^{\infty} \Phi(m, l) .$$

PROOF. Each $\Phi(m, l)$ is closed, so that the right hand side of (3.4) is an $F_{\sigma\delta}$. Let $\Gamma(m, l)$ be the open set $\Omega - \Phi(m, l)$ and Ψ the $G_{\delta\sigma}$ set $\bigcup_{m=1}^{\infty}\bigcap_{l=1}^{\infty} \Gamma(m, l)$. A function ω is in Ψ if and only if for some $\varepsilon=1/m>0$ there are arbitrarily long increasing sequences of t_i in $[a, b]$ such that $\rho(\omega(t_i), \omega(t_{i+1})) > \varepsilon$; i.e., if and only if ω has a discontinuity of the second kind. Thus $\Delta_j = \Omega - \Psi$ and Δ_j is given by (3.4), concluding the proof.

We recall that the function ξ on $T \times \Omega$ is defined by $\xi(t, \omega) = \omega(t)$. We shall need some properties of ξ as a function of two variables.

LEMMA 3.1. *Let X be a compact Hausdorff space, T a set and S a subset of T. Let f be a continuous real-valued function on X. Then* $\inf_{s \in S} f(\xi(s, \omega))$ *and* $\sup_{s \in S} f(\xi(s, \omega))$ *are Borel measurable functions of ω.*

PROOF. This is because

$$\{\omega : \inf_{s \in S} f(\xi(s, \omega)) < c\} = \bigcup_{s \in S} \{\omega : \omega(s) \in f^{-1}((-\infty, c))\}$$

is open and $\{\omega : \sup_{s \in S} f(\xi(s, \omega)) \leqq c\} = \bigcap_{s \in S} \{\omega : \omega(s) \in f^{-1}((-\infty, c])\}$ is closed.

THEOREM 3.5. *Let X and T be second countable compact Hausdorff spaces. Then the set Θ of all (t, ω) such that ω is discontinuous at t is a Borel set in $T \times \Omega$. The function ξ is Borel measurable on $\Omega - \Theta$.*

PROOF. Let $G_{l1}, \cdots, G_{lk_l}$ be a covering of T by open sets of diameter $\leqq 1/l$, and let $g_{l1}, \cdots, g_{lk_l}$ be a partition of unity for this covering. Let f be a continuous real-valued function on X and let

$$f_l^+(t, \omega) = \sum_{j=1}^{k_l} g_{lj}(t) \sup_{s \in G_{lj}} f(\xi(s, \omega)) \tag{3.5}$$

with f_l^- defined similarly with sup replaced by inf. Let

$$f^+(t, \omega) = \limsup_{s \to t} f(\xi(s, \omega)) \tag{3.6}$$

and similarly for $f^-(t, \omega)$. Then

$$f^+(t, \omega) = \lim_{l \to \infty} f_l^+(t, \omega) \tag{3.7}$$

and similarly for $f^-(t, \omega)$. In fact, for each l

$$f^+(t, \omega) \leqq f_l^+(t, \omega) \leqq \sup_{s \in G} f(\xi(s, \omega)) \tag{3.8}$$

where G is the union of all G_{lj} containing t. Since the diameter of G is $\leqq 2/l$, the right hand term in (3.8) approaches the left hand term, proving (3.7). Consequently, $f^+(t, \omega) = f(\xi(t, \omega))$ if and only if $f(\xi(\cdot, \omega))$ is upper semicontinuous at t, and similarly $f^-(t, \omega) = f(\xi(t, \omega))$ if and only if $f(\xi(\cdot, \omega))$ is lower semicontinuous at t. Let

$$\Theta_f = \{(t, \omega) : f^+(t, \omega) > f^-(t, \omega)\}. \tag{3.9}$$

Then Θ_f is the set of all (t, ω) such that $f(\xi(\cdot, \omega))$ is discontinuous at t. Since X is second countable, there is a sequence of continuous functions f_n which separates points, so that $\Theta = \bigcup_{n=1}^{\infty} \Theta_{f_n}$. By Lemma 3.1, each f_l^+ and f_l^- is Borel measurable on $T \times \Omega$, so that f^+ and f^- are Borel measurable on $T \times \Omega$. Therefore each Θ_f is a Borel set, and so Θ is a Borel set, concluding the proof.

Using the proof of Lemma 3.1, it is easily seen that $\{(t, \omega) : f^+(t, \omega) < c\}$ is an F_σ, and dually for f^-. It follows that Θ_f, and hence Θ itself, is a $G_{\delta\sigma}$ set. Thus the set of all (t, ω) such that ω is continuous at t is an $F_{\sigma\delta}$ in $T \times \Omega$.

Each of the results of this section has a natural generalization to the case of an arbitrary compact Hausdorff space X. If f is a fixed continuous real-valued function defined on X then the set of ω such that $f(\omega(\cdot))$ is a continuous function except on a set which is respectively empty, of μ-measure 0, or of first category is an $F_{\sigma\delta}$. The proofs are identical, with the metric in X being replaced by the pseudo-metric

$$\rho'(x, y) = |f(x) - f(y)| .$$

Similarly, the set of ω such that $f(\omega(\cdot))$ has jump discontinuities only is an $F_{\sigma\delta}$ if T is an interval, and the set of (t, ω) such that $f(\omega(\cdot))$ is continuous at t is an $F_{\sigma\delta}$ in $T \times \Omega$.

4. Processes with no fixed discontinuities

Let $D(t, \delta)$ be the set of all (s_1, s_2) in $T \times T$ such that $\rho(s_1, t) \leqq \delta$, $\rho(s_2, t) \leqq \delta$, and let

$$\Delta_t^\varepsilon = \bigcup_{n=1}^\infty \bigcap_{(s_1, s_2) \in D(t, 1/n)} \Delta(s_1, s_2, \varepsilon) \tag{4.1}$$

where $\Delta(s_1, s_2, \varepsilon) = \{\omega : \rho(\omega(s_1), \omega(s_2)) \leqq \varepsilon\}$ as in Section 3, and let

$$\Delta_t = \bigcap_{m=1}^\infty \Delta_t^{1/m} . \tag{4.2}$$

Then Δ_t^ε is an F_σ, Δ_t is an $F_{\sigma\delta}$, and Δ_t is the set of all ω which are continuous at t.

Let Pr be a regular Borel probability measure on Ω. A point t in T will be called a *fixed point of discontinuity* in case $\Pr(\Delta_t) < 1$. The set of all fixed points of discontinuity will be denoted by $O(\Pr)$, and $O_\varepsilon(\Pr)$ will stand for the set of all t such that $\Pr(\Delta_t^\varepsilon) < 1$.

THEOREM 4.1. *Let X and T be second countable compact Hausdorff spaces, let* Pr *be a regular Borel probability measure on Ω, and let μ be a regular Borel measure on X. Then the set Δ_μ of all ω which are continuous except on a set of μ measure 0 satisfies $\Pr(\Delta_\mu) = 1$ if and only if $\mu(O(\Pr)) = 0$. If $\mu(O(\Pr)) = 0$ then ξ is $\mu \times$ Pr-measurable.*

PROOF. The set Θ of all (t, ω) such that ω is discontinuous at t is a Borel set by Theorem 3.5. Now $\{\omega: (t, \omega) \in \Theta\} = \Omega - \Delta_t$ and $\{t: (t, \omega) \in \Theta\} =$

$O(\omega)$. By Fubini's theorem, $\mu(O(\omega)) = 0$ for almost every ω if and only if $\Pr(\Omega - \Delta_t) = 0$ for almost all t; that is, if and only if $\mu(O(\Pr)) = 0$, proving the first statement. If indeed $\mu(O(\Pr)) = 0$, then $\mu \times \Pr(\Theta) = 0$. Consequently, $\xi(t, \omega)$ is $\mu \times \Pr$-measurable, since ξ is Borel measurable on $\Omega - \Theta$.

COROLLARY 4.1. *If X and T are compact intervals then the set of Riemann-integrable ω has probability one if and only if the fixed points of discontinuity have zero Lebesgue measure.*

The proof is as in the case of Corollary 3.1.

The theorem that ξ is $\mu \times \Pr$-measurable when $\mu(O(\Pr)) = 0$ is due to Doob ([3], II, §2). The proof of Theorem 3.5 is based upon the construction used by Doob.

If there are no fixed discontinuities then ξ is $\mu \times \Pr$-measurable for all μ.

THEOREM 4.2. *Let X and T be second countable compact Hausdorff spaces,* Pr *a regular Borel probability measure on Ω such that there are no fixed points of discontinuity. Then the set Δ_p of all ω with point discontinuities only satisfies* $\Pr(\Delta_p) = 1$.

PROOF. Using the notation of the proof of Theorem 3.5, for each t in T, $f^+(t, \omega) = f(\xi(t, \omega))$ for almost every ω, since almost every ω is continuous (and *a fortiori* upper semicontinuous) at t. By Theorem 3.3, Δ_p is an $F_{\sigma\delta}$. By Corollary 2.2, there is a countable subset S of T such that $\Delta_{pS} = \Delta_p$ a.e. Since S is countable, $f^+(s, \omega) = f(\xi(t, \omega))$ for all s in S, for almost every ω. For all ω in Ω, $f^+(\cdot, \omega)$ is, by (3.7), the pointwise limit of a sequence $f_i^+(\cdot, \omega)$ of continuous functions of t. By Baire's theorem, $f^+(\cdot, \omega)$ has point discontinuities only. But for almost every ω, $f(\xi(s, \omega)) = f^+(s, \omega)$ for all s in S. Since this is true for each f in a sequence of continuous functions which separate points on X, for almost every ω, $\xi(\cdot, \omega)$ is equal on S to a function with point discontinuities only. Therefore $\Pr(\Delta_{pS}) = 1$, and so $\Pr(\Delta_p) = 1$, concluding the proof.

If we knew the set Δ_1 of all functions of Baire class 0 or 1 to be analytic over the compact sets, then the above proof would show that $\Pr(\Delta_1) = 1$, using Theorem 2.3 instead of its corollary. This is an open question, but it is easy to show that the set Δ_2 of all functions of Baire class $\leqq 2$ is not analytic over the compact sets. Let $T = X = [0, 1]$, for simplicity. Then for any countable subset S of T, $\Delta_{2S} = \Omega$, since any function on S may be extended to a function of Baire class $\leqq 2$ on T by defining it to be 0 on $T - S$. If Δ_2 were analytic over the compact sets, then by Theorem 2.3 for any regular Borel probability measure Pr on Ω we would have

$\Pr(\Delta_2) = 1$. But we may take for Pr the mass 1 concentrated at a single element ω of Ω, and if we choose for ω a function which is not of Baire class $\leqq 2$ we have a contradiction.

5. Approaching a set

In some applications, notably the work of Hunt [7], it is important to establish the measurability of the set $\widetilde{E}$ of all ω in Ω such that $\omega(t)$ is in E for some t in T. Here we shall study the closely related set $\Sigma(E)$ of all ω in Ω such that some point in the closure of the range of ω is in E. For the Markoff processes studied by Hunt, $\Sigma(E) = \widetilde{E}$ a.e. The set $\Sigma(E)$ has the advantage that it is Pr-measurable for any regular Borel measure Pr, under the appropriate assumptions on E.

Let X be a compact Hausdorff space, T a set. If ω is in Ω, let $\overline{\omega}$ be the closure of the range of ω, so that $\overline{\omega}$ is a compact set in X. If $E \subset X$, let $\Sigma(E) = \{\omega : \overline{\omega} \cap E \neq 0\}$.

LEMMA 5.1. *The transformation* $E \to \Sigma(E)$ *commutes with arbitrary unions. If the* E_i *are a sequence of sets with* $\overline{E}_{i+1} \subset E_i$, *for* $i = 1, 2, \cdots$, *and if* $F = \bigcap_{i=1}^{\infty} E_i$, *then* $\Sigma(F) = \bigcap_{i=1}^{\infty} \Sigma(E_i)$.

In particular, Σ commutes with the intersection of decreasing sequences of compact sets, so that Σ has the two properties of the transformation $\mathrm{B} \to \mathrm{B}_S$ of Lemma 2.1.

PROOF. If $E = \bigcup_\alpha E_\alpha$, then $\overline{\omega} \cap E \neq 0$ if and only if $\overline{\omega} \cap E_\alpha \neq 0$ for some α, proving the first statement. To prove the second statement, note first that if $\overline{\omega} \cap F \neq 0$ then $\overline{\omega} \cap E_i \neq 0$ for each i, so that $\Sigma(F) \subset \bigcap_{i=1}^{\infty} \Sigma(E_i)$. Conversely, if $\overline{\omega} \cap E_i \neq 0$ for each i then *a fortiori* $\overline{\omega} \cap \overline{E}_i \neq 0$ for each i. But then $\overline{\omega} \cap \overline{E}_i$ is a sequence of non-empty compact sets decreasing to $\overline{\omega} \cap F$, so that $\overline{\omega} \cap F$ is non-empty. Thus $\bigcap_{i=1}^{\infty} \Sigma(E_i) \subset \Sigma(F)$, concluding the proof.

THEOREM 5.1. *Let* X *be a second countable compact Hausdorff space. If* G *is an open subset of* X *then* $\Sigma(G)$ *is an open set in* Ω. *If* F *is closed then* $\Sigma(F)$ *is a* G_δ. *If* E *is analytic over the compact sets then* $\Sigma(E)$ *is analytic over the* G_δ *sets.*

PROOF. If G is open and $\overline{\omega} \cap G \neq 0$ then $\omega(t)$ is in G for some t. That is, $\Sigma(G) = \bigcup_{t \in T} \{\omega \colon \omega(t) \in G\}$, which is an open set. If F is closed there is a sequence of open sets G_i with $\overline{G}_{i+1} \subset G_i$ and decreasing to F, since X is second countable. By the lemma and the fact that each $\Sigma(G_i)$ is open, $\Sigma(F)$ is a G_δ. A set E is analytic over the compact sets if it is given by a monotone Souslin scheme over the compact sets. By the lemma, Σ

commutes with such schemes, and so $\Sigma(E)$ is given by a Souslin scheme over the G_δ sets, since $\Sigma(F)$ is a G_δ if F is compact. This concludes the proof.

The corresponding theorem for a general compact Hausdorff space X is that $\Sigma(G)$ is open if G is open, $\Sigma(F)$ is a G_δ if F is a closed Baire set, and $\Sigma(E)$ is analytic over the G_δ sets if E is analytic over the compact Baire sets.

By known properties of Souslin schemes (see [4], [5]), it follows from Theorem 5.1 that $\Sigma(E)$ is Pr-measurable for any regular Borel measure Pr on Ω, provided that E is analytic over the compact sets (in particular, if E is a Borel set). However, it is also important to know how $\Pr(\Sigma(E))$ depends on E. The following theorem is easily derived from the Choquet extension theorem ([1], §30), just as in [7]. Notice that the class of sets analytic over the compact sets, as we have been using the term (following Hahn [4]), coincides with the class of K-analytic sets in Choquet's terminology. However, the sets $\Sigma(E)$, which are analytic over the G_δ sets, need not be K-analytic in the sense of Choquet.

THEOREM 5.2. *If X is a second countable compact Hausdorff space, E is analytic over the compact sets and* Pr *is a regular Borel measure on Ω, then*

$$\sup_F \Pr(\Sigma(F)) = \Pr(\Sigma(E)) = \inf_G \Pr(\Sigma(G)) \tag{5.1}$$

where F ranges over the compact subsets of E and G ranges over the open sets containing E.

PROOF. In case E is compact or open, (5.1) follows from Lemma 5.1 and the regularity of Pr. If $E, E_1, \cdots, E_n$ are analytic over the compact sets and $A(i \cdots l) = E \cup E_i \cup \cdots \cup E_l$, $1 \leqq i < \cdots < l \leqq n$, then

$$0 \leqq -\Pr(\Sigma(E)) - \sum (-1)^k \sum \Pr(\Sigma(A(i_1 \cdots i_k)) \leqq 1 \tag{5.2}$$

since it is equal to $\Pr((\Omega - \Sigma(E)) \cap \Sigma(E_1) \cap \cdots \cap \Sigma(E_n))$. In the terminology of Choquet, the set function $\Pr(\Sigma(\cdot))$ is a capacity which is alternating of order infinity. The Choquet extension theorem says that (5.1) holds for all sets E which are analytic over the compact sets, concluding the proof.

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