Putnam 2022 Problem A3:

Let $p > 5$ be a prime number. Let $f(p)$ denote the number of infinite sequences $\{a_n\}_{n=1}^{\infty}$ such that $a_n \in \{1, 2, \ldots, p-1\}$ and $a_n a_{n+2} \equiv 1 + a_{n+1} \pmod{p}$ for all $n \geq 1$. Show that $f(p) \equiv 0$ or $2 \pmod{5}$.

Proof. For a_n and $1 + a_{n+1}$ in $\mathbf{F}_p^*$, we know that the a_{n+2} is uniquely determined. Also note that if $1 + a_{n+1} = 0$, i.e., $a_{n+1} = p - 1$, then $p \mid a_n a_{n+2}$, so that the sequence must terminate. Therefore, the number of such infinite sequences is the number of pairs (a_1, a_2) such that $p - 1$ can never appear in the sequence.

Now suppose $p - 1$ appears in the sequence, and let $a_{n+2} = p - 1$. It follows that $a_{n+1} \equiv -1 - a_n \pmod{p}$. We move the index forward by 1, and then we have $a_{n-1}a_{n+1} \equiv 1 + a_n \pmod{p}$. This shows that $a_{n-1}(-1 - a_n) \equiv 1 + a_n \pmod{p}$, which shows that $a_{n-1} = p - 1$, which contradicts our assumption. Therefore, $p - 1$ must appear in either a_1, a_2, or a_3, if it can ever appear.

First consider a_1, which has $p - 2$ options (excluding $a_1 = p - 1$). For a_2, first $a_2 \neq p - 1$. Second, for every $a_1 \in \{1, 2, \ldots, p-2\}$, if $a_3 \equiv -1$, then $a_2 \equiv -1 - a_1$ for the corresponding a_1. Thus we cannot let $a_2 = p - 1 - a_1$. Meanwhile, $a_2 \neq p - 1$ either, and so only $(p - 2)(p - 3)$ pairs of (a_1, a_2) are valid.

Now it suffices to check for all $p \not\equiv 0 \pmod{5}$, $(p - 2)(p - 3) \equiv 0$ or $2 \pmod{5}$, which is indeed true. $\square$