Exponential Families and the Two Stirling Numbers

Mentee: Cheng, Feng & Xu, Chenlong
Mentor: Wu, Chutong

Definition Say we have a set P of "pictures."

1. A **card** $\mathcal{C}(S, p)$ consists of a finite set S (the label set) of positive integers and a picture $p \in P$.
   The weight of $\mathcal{C}$ is $|S|$.
   A *standard card* is one with the standard label set $[n] = \{1, 2, 3, ..., n\}$.

2. A **hand** H is a set of cards whose label sets form a partition of $[n]$, which means the label sets of the set of cards in H are pairwise disjoint and nonempty, and have a union of $[n]$.
   The weight of H is $\sum_{\mathcal{C} \in H} \text{weight}(\mathcal{C})$, which is equal to n here.

3. With a new label set S' such that $|S'| = |S|$, we can **relabel** $\mathcal{C}(S, p)$ into $\mathcal{C}(S', p)$.
   Standard relabeling means we relabel the original card with the standard label set, which mathematically has $S' = [|S|]$.

4. A **deck** $\mathcal{D}$ is a finite set of standard cards with the same weights and different pictures.
   The weight of $\mathcal{D}$ is the common weight of the standard cards.

5. An **exponetial family** $\mathcal{F}$ is a collection of decks $\mathcal{D}_n$ with $n = 1, 2, 3, ...$ and $\text{weight}(\mathcal{D}_n) = n$.

6. We write d_n as the number of standard cards in deck $\mathcal{D}_n$, and the **deck enumerator** of the family $\mathcal{D}(x) = d_1 x + \frac{d_2}{2!}x^2 + \frac{d_3}{3!}x^3 + ...$ is the exponential generating function (egf) of $\{d_n\}_1^\infty$.

7. We write $h(n, k)$ as the number of weight-n hands in k cards.
   We define the 2-variable **hand enumerator** of the family as

$$\mathcal{H}(x, y) = \sum_{n,k \geq 0} h(n, k) \frac{x^n}{n!} y^k.$$

The central question is to establish the relationship between $h(n, k)$ and the indivdidual numbers of decks $d_1, d_2, d_3, ...$. The central notion of an *exponential family* is that its hands and decks are both built from relabled indivdual cards, and thus combinatorially connected.

We have a very strong theorem that gives us the answer. The exponential formula states that

$$\mathcal{H}(x,\, y) = e^{y\mathcal{D}(x)}, \tag{1}$$

and then we have

$$h(n,\, k) = \left[\frac{x^n}{n!}\right]\left\{\frac{\mathcal{D}^k(x)}{k!}\right\} \tag{2}$$

following the definition.
From this we now have the number of hands of weight n

$$h(n) = \sum_k h(n,\, k) = \left[\frac{x^n}{n!}\right]\left\{e^{\mathcal{D}(x)}\right\}, \tag{3}$$

which gives us its succinct egf

$$\mathcal{H}(x) = e^{\mathcal{D}(x)}. \tag{4}$$

.

Also, we may sum $h(n,\, k)$ from (2) over restricted numbers of cards $k \in T$ to get

$$\{h_n(T)\}_0^\infty \overset{\text{egf}}{\longleftrightarrow} e_T(\mathcal{D}(x)), \tag{5}$$

which is the sum of $k \in T$ -th terms of $e^{\mathcal{D}(x)}$.

Application to the Two Stirling Numbers

The exponential formula can be directly applied to combinatorial counting problems: as long as we find the deck enumerator, we will get our hand enumerator, and therefore $h(n,\, k)$. Here we provide examples in regards to the Stirling numbers.

Stirling Number of the First Kind

The *unsigned Stirling number of the first kind* (denoted by $\left[{n \atop k}\right]$) can be defined as the number of permutations of n elements into k disjoint cycles. The exponential family here is all permutations of different weights since every permutation uniquely corresponds to a cycle decomposition. One can show how the definition provided here is equivalent to its alternative algebraic definition using rising factorial.

First note that every $\mathcal{D}(n)$ has $d_n = (n-1)!$ cards, and it follows that

$$\mathcal{D}(x) = \sum_{n\geq 1}(n-1)!\frac{x^n}{n!} = \ln\frac{1}{1-x},$$

which by (1) tells us $\mathcal{H}(x,\, y) = \exp\left(y\ln\frac{1}{1-x}\right) = (1-x)^{-y}$.

We can get $h(n,\, k) = \left[{n \atop k}\right]$ with two methods. Following (2) we simply have

$$\left[{n \atop k}\right] = \frac{1}{k!}\left[\frac{x^n}{n!}\right]\left\{\ln\frac{1}{1-x}\right\}^k,$$

which is a general result that is difficult to calculate (indeed Stirling number of the first kind has a fairly intricate recurrence relation that we will show later).

Alternatively, one can follow from the definition of hand enumerator to get

$$\sum_k \left[{n \atop k}\right] y^k = \left[\frac{x^n}{n!}\right](1-x)^{-y}$$

$$= n!\left[x^n\right](1-x)^{-y}, \text{ which by the binomial series implies}$$

$$= n!\binom{y+n-1}{n} = (y+n-1)\ldots(y+1)\,y.$$

Therefore, the unsigned Stirling number of the first kind $\left[{n \atop k}\right]$ is the coefficient of y^k in the expansion of the rising factorial $(y+n-1)\ldots(y+1)\,y$.

From this alterantive definition, one may work out the recurrence relation for $\left[{n \atop k}\right]$ as follows:

$$\left[{n+1 \atop k}\right] = \left[y^k\right]\{(y+n)\,(y+n-1)\ldots(y+1)\,y\}$$

$$= \left[y^k\right]\{y(y+n-1)\ldots\, y\} + \left[y^k\right]\{n\,(y+n-1)\ldots\, y\}$$

$$= \left[y^{k-1}\right]\{(y+n-1)\ldots\, y\} + n\left[y^k\right]\{(y+n-1)\ldots\, y\}$$

$$= \left[{n \atop k-1}\right] + n\left[{n \atop k}\right].$$

We may also verify this recurrence relation by a combinatorial interpretation.

Stirling Number of the Second Kind

We define the number of partitions of $[n]$ (the hand) into k disjoint sets (the cards) *Stirling Number of the Second Kind*. We will follow the steps similar to what we have done above.

Consider the exponential family with decks $\mathcal{D}_n$ all containing only 1 card of weight n with the standard label $[n]$. Now the hand is the partition of $[n]$ into k disjoint cards, which all have one-to-one correspondence with the relabeled standard card and are therefore unique.

These implies that $h(n,\,k) = \left\{ {n \atop k} \right\}$, the notation we use for Stirling number of the second kind.

Since all $d_n = 1$ for all $n \geq 1$, the egf of d_n is

$$\mathcal{D}(x) = \sum_n d_n \frac{x^n}{n!} = \sum_{n\geq 1} \frac{x^n}{n!} = e^x - 1,$$

which by (2) and (3) respectively gives us

$$\left\{ {n \atop k} \right\} = \left[\frac{x^n}{n!}\right] \left\{ \frac{(e^x - 1)^k}{k!} \right\},$$

the formula for $\left\{ {n \atop k} \right\}$, and

$$h(n) = \left[\frac{x^n}{n!}\right] \left\{ e^{e^x - 1} \right\},$$

the number of total partitions of $[n]$ (called the *Bell numbers* $b(n)$), which has a succinct egf e^{e^x-1}.

Admittedly, it is much more difficult to infer the recurrence relation for Stirling number of the second kind that $\left\{ {n+1 \atop k+1} \right\} = \left\{ {n \atop k} \right\} + (k+1) \left\{ {n \atop k+1} \right\}$ straight from the generating function formula.

However, note that through the binomial expansion, we have the following explicit formula:

$$\begin{aligned}
\left\{ {n \atop k} \right\} &= \left[\frac{x^n}{n!}\right] \left\{ \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} e^{ix} (-1)^{k-i} \right\} \\
&= \sum_{i=0}^{k} \left[\frac{x^n}{n!}\right] \left\{ \frac{1}{i!(k-i)!} e^{ix} (-1)^{k-i} \right\} \\
&= \sum_{i=0}^{k} \left\{ \frac{(-1)^{k-i}\, i^n}{i!(k-i)!} \right\}.
\end{aligned}$$

Apart from the two exponential families regarding Stirling numbers above, we want to focus on a subclass of permutation: the permutation of $[n]$ into even $(2m)$ cycles, all of which are of odd weight $(2r+1)$.

Similarly, for this exponential family, we first consider the deck enumerator $\mathcal{D}(x)$. In this case, $d_n = (n-1)!$ when n is odd and $d_n = 0$ when n is even. Thus,

$$\mathcal{D}(x) = \sum_{\text{odd } n} (n-1)!\frac{x^n}{n!} = \sum_{r\geq 0} \frac{x^{2r+1}}{2r+1}.$$

Given

$$\ln\frac{1}{1-x} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots \text{ and}$$

$$\ln\frac{1}{1+x} = -x + \frac{x^2}{2} - \frac{x^3}{3} + \ldots ,$$

we then have

$$\begin{aligned}\mathcal{D}(x) &= \frac{1}{2}\left(\ln\frac{1}{1-x} - \ln\frac{1}{1+x}\right)\\ &= \ln\sqrt{\frac{1+x}{1-x}}.\end{aligned}$$

By (5), we now restrict the number of cards to even, then the egf of $h_n(T)$ (where T is the set of even numbers) is the even terms of the Taylor expansion of $e^{\mathcal{D}(x)}$. Notice that

$$\cosh(x) = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots .$$

Hence, the desired egf is

$$\begin{aligned}\cosh\left(\ln\sqrt{\frac{1+x}{1-x}}\right) &= \frac{1}{2}\left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right)\\ &= \frac{1}{(1-x^2)^{\frac{1}{2}}}\end{aligned}$$

which by the binomial series would give us the explicit formula

$$e_T(\mathcal{D}(x)) = \sum_{m\geq 0} \binom{2m}{m}\frac{x^{2m}}{2^{2m}}.$$

Taking its exponential coefficient, we have our final answer to the number of such permutations $h_n(T)$:

$$\left[\frac{x^n}{n!}\right]\left\{\sum_{m\geq 0} \binom{2m}{m}\frac{x^{2m}}{2^{2m}}\right\} = \binom{n}{n/2}\frac{n!}{2^n}.$$

In addition, we have an interesting result that follows from this. If we divide $h_n(T)$ by total number of n-permutations $n!$, we have

$$P\Big(\text{even-cycled } n\text{-permutations with odd cycle lengths}\Big) = \binom{n}{n/2}\frac{1}{2^n},$$

which is exactly $P\Big(n \text{ fair coin tosses, } \frac{n}{2} \text{ tosses being head}\Big)$.

Conclusion

We can infer more properties about the Stirling Numbers using the method of generating function; for example, we may show their parity and concavity easily. In addition to the examples above regarding partition and permutation, the notion of an exponential family containing decks of cards that make up the hands may be applied to many other counting problems, especially the ones involving graphs.

Overall, the method of generating function often provides an interesting yet completely different way of looking at traditional combinatorial counting problems. Sometimes this method would offer us tremendous insight into certain questions.

Work Cited.

Wilf, Herbert S. *Generatingfunctionology.* A.K. Peters, 2006, pp. 79–90.