The Historical Development of Complex Numbers

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One has only to look at Fig. 6 to see the IMO problem as a special case, with XCB, CYA, BAZ right-angled isosceles triangles and P, Q, R similarly situated in them.

Generalisations

The generalisations of the fourth and fifth solutions are distinct. Put into simple language, solution 5 is (see Fig. 5):

> *If triangles ZRA, BPC are directly similar and triangles ZRB, AQC are directly similar, then triangles QRP, AZB are directly similar.*

The generalisation of solution 4 is (see Fig. 7):

> *If triangles ZRA, CQA are directly similar and triangles ZRB, CPB are directly similar, then*
>
> (i) $\angle QRP = \angle QAC + \angle CBP = \angle ARB - \angle AZB$,
> (ii) $QR/RP = (QA/AC) \times (CB/BP) = (AR/RB) \div (AZ/ZB)$.

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Early skirmishes

The Alexandrian Greek mathematician Heron—whom we associate with the formula $\sqrt{\{s(s-a)(s-b)(s-c)\}}$ for the area of a triangle—got involved in a calculation about a pyramid design which led to the evaluation of $\sqrt{(81-144)}$. This occurs in his book *Stereometria* (c. A.D. 75) and to 'solve' it the numbers are turned round thus: $\sqrt{(144-81)}$, to give $\sqrt{63}$ which is taken to be $7\frac{15}{16}$. (Is this a reasonable approximation for $\sqrt{63}$?) It is not known whether Heron made this transpositional error or whether a copier of his work was responsible. This seems to be the first occasion in which the square root of a negative number was stumbled across—a concept not properly understood for another 1750 years!

Another famous Greek—Diophantus—is known to have met the same difficulty which arose in his book *Arithmetica* (c. A.D. 275). His problem was: "A right-angled triangle has area 7 square units and perimeter 12 units. Find the lengths of its sides." Using modern notation:

$$\tfrac{1}{2}xy = 7, \quad \text{and} \quad x^2 + y^2 = (12 - x - y)^2.$$

Solving by substitution of y by $14/x$ yields

$$172x - 24x^2 - 336 = 0.$$

Thus the problem yields a quadratic equation whose solutions are:

$$x = \frac{43 \pm \sqrt{(-167)}}{12}.$$

Diophantus essentially saw this but although he said, in effect, that there could be solutions only if $(172/2)^2 \geqslant 24 \times 336$ he went no further and did not seem to be really aware of the basic difficulty—that of dealing with the square root of a negative number, which throughout this article will be designated by $\sqrt{-}$.

We have no further record of the difficulty arising for hundreds of years, until the Hindu mathematician Mahavira (c. A.D. 850) commented on it again:

"... as in the nature of things a negative (quantity) is not a square (quantity), it has therefore no square root." ([1], p. 261)

And so the matter was (temporarily) disposed of.

Some three hundred years later the position had advanced no further. The celebrated Indian mathematician and astronomer Bhascara (1114–c. 1185) wrote:

"The square of an affirmative or of a negative is affirmative; and the square root of an affirmative quantity is two-fold: positive and negative. There is no square root of a negative quantity; for it is not a square." ([1], p. 261)

At about this time (c. 1120) a Jew, Abraham bar Chiia, was solving the pair of equations $xy = 48$, $x + y = 14$. These give

$$x^2 - 14x + 48 = 0,$$

$$x = \frac{14 \pm \sqrt{(196 - 4 \times 48)}}{2} = \frac{14 \pm \sqrt{4}}{2},$$

$$x = 6 \text{ or } 8.$$

There is, of course, no difficulty here, but bar Chiia noted that trouble *could* arise if the square root had to be that of a negative quantity.

As mathematics began to be developed in Europe, it was inevitable that $\sqrt{-}$ would keep cropping up. For example, the Italian Luca Pacioli (c. 1445–1509+) stated in his book *Summa di arithmetica geometria*, which was published in 1494, that $x^2 + c = bx$ was soluble only if $\tfrac{1}{4}b^2 \geqslant c$, which indicated some awareness of the 'impossible' $\sqrt{-}$; and a French mathematician, Nicolas Chuquet (c. 1445–c. 1500) made similar remarks about 'impossible solutions' in an unpublished manuscript dated 1484. We see, then, that $\sqrt{-}$ kept appearing from time to time, much to the bewilderment or annoyance of the mathematicians.

Cardano and the Ars magna

Most credit for seriously considering the possibility of working with $\sqrt{-}$ usually goes to Girolamo Cardano (1501–1576)—also known as Jerome Cardan. In his famous *Ars magna* (1545) he became the first mathematician to use $\sqrt{-}$ in a computation. Cardano considered the problem: "Split 10 into two parts such that their product is 40" ([2], ch. 37). This gives

$$x + y = 10 \quad \text{and} \quad xy = 40,$$

with solutions $x = 5 + \sqrt{(-15)}$ and $y = 5 - \sqrt{(-15)}$ (or vice versa). Cardano, using ordinary rules of algebra, verified that these solutions did indeed satisfy $x + y = 10$ and $xy = 40$. He then tried (hopefully!) to use $5 + \sqrt{15}$ and $5 - \sqrt{15}$ instead, but discovered that the product was no longer 40. He concluded:

"So progresses arithmetic subtlety the end of which ... is as refined as it is useless." ([2], in Witmer's translation)

It must be emphasised that Cardano used primitive notation, to which we shall return later, and also felt obliged to offer geometrical demonstrations to support his algebraic arguments. These two factors made his task much more difficult than is apparent to us with our sophisticated and economical notation and wealth of experience.

Cardano's quadratic problem was of course somewhat artificial and in such cases the occurrence of complex roots indicates 'impossibility' in an obvious sense. However, a much more startling and insistent intrusion of complex numbers arose in solving certain cubic equations, a matter of great interest to the leading algebraists of the time, especially to Cardano himself. To solve equations of the type $x^3 + ax = b$ with a and b positive, Cardano's method worked as follows. Taking the example

$$x^3 + 9x = 24,$$

let $u - v = x$ and $uv = 9 \div 3 = 3$. Upon substitution for x we get

$$u^3 - v^3 = 24,$$

and on eliminating v,

$$u^6 - 24u^3 - 27 = 0,$$

so that $u^3 = 12 + \sqrt{171}$ and $v^3 = 12 - \sqrt{171}$. Hence

$$x = u - v = \sqrt[3]{(12 + \sqrt{171})} - \sqrt[3]{(12 - \sqrt{171})}.$$

(The reader may care to solve $x^3 + 6x = 20$ by this method, being the first example of this type which Cardano considered in his *Ars magna*, ch. 11.)

Cubics of the form $x^3 = ax + b$ (a and b positive) we today would see as being essentially the same as $x^3 + ax = b$, and we would simply transform the equation to give $x^3 + (-a)x = b$. However, Cardano and his contemporaries could not accept negative numbers and were none too sure about zero, so they always treated problems with positive coefficients only. Thus a 'different' method had to be used to solve, for example, $x^3 = 15x + 4$. This time the substitutions

$$u + v = x, \quad uv = 15 \div 3 = 5$$

are made. For the problem cited the appropriate method yields the solution

$$x = \sqrt[3]{\{2 + \sqrt{(-121)}\}} + \sqrt[3]{\{2 - \sqrt{(-121)}\}}.$$

Somewhat surprisingly, *complex* numbers arise using Cardano's formulae in cases where the cubic has three *real* distinct roots (described by Tartaglia as the 'irreducible' case).

Boyer ([3], p. 314) implies that Cardano tried the above problem, $x^3 = 15x + 4$, and, whilst he *knew* that a solution was $x = 4$, he could not see how the "sophistic" numbers could produce the real number solution. The writer can find no trace of this problem in the *Ars magna*, although Cardano was certainly aware of the limitations of his method of solution, and either some other less well known publication of Cardano's refers to this or it should be attributed to another Italian algebraist†.

It must be concluded that Cardano never really came to grips with the problem of handling the complex numbers which his formulae demanded in some instances.

Bombelli and his Algebra

The Italian Rafael Bombelli (c. 1526–1572) continued the research into cubic and biquadratic equations. Opinions differ as to Bombelli's contribution, but certainly he established the four rules for complex numbers, much as we use them today, in his efforts to summarise Cardano's work in a clear and comprehensive manner. Carl Boyer [3] tells us that Bombelli did make a significant discovery relating to 'irreducible' cubics such as the one already referred to, $x^3 = 15x + 4$, with 'solution'

$$\sqrt[3]{\{2 + \sqrt{(-121)}\}} + \sqrt[3]{\{2 - \sqrt{(-121)}\}}.$$

Bombelli noticed that the radicands, $2 + \sqrt{(-121)}$ and $2 - \sqrt{(-121)}$, were of the form $r + \sqrt{(-n)}$ and $r - \sqrt{(-n)}$. He surmised that the *cube roots*

† Professor Boyer has indicated in a private letter (29.11.75) to the writer that "I suspect that in referring to Cardan's knowledge that 4 is a root of the equation $x^3 = 15x + 4$, I was basing this statement on the fact that Cardan's knowledge of relations between roots and coefficients would have told him this ...".

themselves might also be of this general form. Moreover it was known that the sum of these two cube roots was 4, since $x = 4$ clearly satisfied the cubic equation. The above assumptions led to

$$r + \sqrt{(-n)} + r - \sqrt{(-n)} = 4,$$
$$r = 2.$$

Bombelli also could see that, since $2 + \sqrt{(-n)}$ was to equal to $\sqrt[3]{\{2 + \sqrt{(-121)}\}}$, simply cubing $2 + \sqrt{(-n)}$ would give an equation for n:

$$(2 + \sqrt{(-n)})^3 = 2 + \sqrt{(-121)},$$

so that

$$8 + 12\sqrt{(-n)} - 6n - n\sqrt{(-n)} = 2 + \sqrt{(-121)},$$
$$8 - 6n = 2, \quad \text{i.e. } n = 1.$$

(Also $(12 - n)\sqrt{(-n)} = \sqrt{(-121)}$, i.e. $n = 1$, as a further check.)

Bombelli had thus obtained the *conjugate pair* of complex numbers, $2 + \sqrt{(-1)}$ and $2 - \sqrt{(-1)}$, which the Cardano formula demanded. (However, the term "conjugate" itself was not introduced until 1821, by Cauchy.) Unfortunately Bombelli needed to know the root ($x = 4$) in order to find this conjugate pair, so the method was useless for actually solving a novel equation! Nevertheless it was now clear that *complex* numbers had to be reckoned with in *real* problems, and could no longer be swept under the carpet. Bombelli published his work in his book *Algebra* in 1572 [4], although he had written it in the period 1557–1560. (The reader may like to attempt to solve $x^3 = 7x + 6$, which is found in Bombelli's *Algebra*, by this method.)

It is indeed interesting to see that complex numbers were in effect rejected where quadratic equations were solved but forced themselves upon mathematics in the case of cubic equations. Had cubic equations not been studied so ardently by these Italian mathematicians, complex number work probably would not have been developed until much later. Nevertheless, D. J. Struik's observation—

"It is a curious fact that the first introduction of the imaginaries occurred in the theory of cubic equations, in the case where it was clear that real solutions existed though in an unrecognisable form, and not in the theory of quadratic equations, where our present textbooks introduce them." ([5], p. 114)

—is a little misleading in its implication that attempts to solve quadratic equations did not influence the early development of complex numbers.

Notation

It is always important to remember that notation was very different from, and inferior to, our own. The translator of Cardano's *Ars magna*, T. R. Witmer, points out that Cardano had nothing comparable to our parentheses, for example. (However Bombelli had in his *Algebra* of 1572.) Again, it was nearly a century after Cardano was working that Albert Girard (1595–1632), an important Dutch mathematician, introduced the symbol $\sqrt{-1}$, in 1629. Before that date some of the symbols used had been ℞m̄.1 (Cardano), di.m℞q.1 (Bombelli).

As an example of Cardano's notation (which was subject to several variations!), consider

℞.v.cub.℞.108p̄.10m̄.℞.v.cub.℞.108m̄.10.

The symbols had the following significance:

℞ means 'radix', i.e. $\surd$, or 'root'.

℞.v. means 'radix universalis', i.e. $\sqrt{}$, indicating that the root applied further than to just the first of the following items (the extent was by no means clear).

cub. means 'cube', so ℞.cub. would be $\sqrt[3]{}$, i.e. 'cube root'.

p̄ means 'plus'.

m̄ means 'minus'.

Can you, then, put into modern notation the above expression? (It in fact stood for $\sqrt[3]{\{\sqrt{108}+10\}} - \sqrt[3]{\{\sqrt{108}-10\}}$.)

Bombelli's similar notation was superior. For example, he could show the extent of the root. The reader should be able to 'decode' the following expression from Bombelli's *Algebra*:

$$℞^3\lfloor 2\mathrm{p}℞\lfloor 0\mathrm{m}121\rfloor\rfloor$$

(It is equivalent to $\sqrt[3]{\{2+\sqrt{(-121)}\}}$.)

Another difference between early mathematics and the way we work today is that for us x^2 and x^3 are quite abstract, and we can just as easily think about x^4 and x^5. However, early mathematicians thought of x^2 as a physical square of side x, and x^3 as a physical cube of side x, and Cardano for example used different words for each power. (However, Bombelli had a system closer to our own.) So although Cardano studied quadratic and cubic equations in depth, he gave higher degree equations very little attention—he could not see how they could exist meaningfully ("Nature does not permit it"). However, his curiosity as a mathematician led him to some limited investigations beyond what he saw as being 'sensible'.

Another great mathematician to contribute to complex number development was René Descartes (1596–1650), who was the first to use the terms "real" and "imaginary" and thus explicitly to separate out the two parts of what we call a complex number (a term introduced only in 1832, by Gauss).

Geometrical interpretation: Wallis

In the history of mathematics great strides forward have resulted from the realisation that geometry and arithmetic can be related—that real numbers can represent lines and vice versa. (Paradoxically, the insistence on geometric 'proofs' of algebraic results was also a hindrance.) An important stage in the development of complex number theory was begun by the great English mathematician, John Wallis (1616–1703), a contemporary of Newton. He recognised that $\sqrt{-}$ presented a great difficulty, but then so did -1, and he pointed out that perfectly good physical meaning (representation) was possible for negative numbers: to represent distances of lines in the opposite direction to the usual (positive) lines.

D. E. Smith records that, by analogy with negative lines, Wallis went on to say that we can have negative *area* which must have sides:

"Now what is admitted in lines must, on the same reason, be allowed in plains also ....

But now (supposing this negative plain, -1600 perches, to be in the form of a square;) must not this supposed square be supposed to have a side? And if so, what shall this side be?

We cannot say it is 40, nor that it is -40 ...

But thus rather that it is $\sqrt{(-1600)}$ or $10\sqrt{(-16)}$ or $20\sqrt{(-4)}$ or $40\sqrt{(-1)}$.

Where $\sqrt{}$ implies a mean proportional between a positive and a negative quantity. For like as $\sqrt{bc}$ signifies a mean proportional between $+b$ and $+c$; or between $-b$ and $-c$; ... so doth $\sqrt{(-bc)}$ signify a mean proportional between $+b$ and $-c$, or between $-b$ and $+c$." ([1], pp. 263–4)

Wallis first published his work in 1673, although the above quotation is from a work of 1693. He came close to drawing a line perpendicular to the $+$ and $-$ axes to represent $\sqrt{-}$, realising that $\sqrt{-}$ was off the real number line, but his rather complicated geometrical arguments did not really constitute much progress in themselves as he failed to connect the y-axis with the direction for imaginary numbers. However, a new avenue to explore had been indicated.

The research continues

Another great mathematician to contribute to complex number theory was Gottfried Wilhelm Leibniz (1646–1716). Although he showed in 1676 that $\sqrt{\{1+\sqrt{(-3)}\}}+\sqrt{\{1-\sqrt{(-3)}\}}=\sqrt{6}$ (a result which astonished many at that time) and also factorised x^4+a^4 as

$$(x+a\sqrt{\{-\sqrt{(-1)}\}}).(x+a\sqrt{\{\sqrt{(-1)}\}}).(x-a\sqrt{\{-\sqrt{(-1)}\}}).(x-a\sqrt{\{\sqrt{(-1)}\}}),$$

he remained unable to envisage the possibility of graphical representation. Leibniz's work led him to conclude that the sum of every pair of conjugate complex numbers must be real, and he established the validity of Cardano's formulae for the roots of cubic equations even in the 'irreducible case' when complex numbers are involved in producing a real answer.

A little known but important English scientist and mathematician Thomas Harriot (c. 1560–1621) was another who attempted a uniform treatment of all algebraic equations, and he arrived at the square root of a negative number as the solution to a biquadratic equation.

An important step forward in the theory of complex numbers is at least partly attributable to another little known but nevertheless great English mathematician, Roger Cotes (1682–1716). Newton recognised Cotes' ability and lamented his early death as robbing England of a brilliant mathematical mind. ("If Cotes had lived, we might have known something.") Cotes, in 1714, stated a theorem which in modern notation is:

$$\log_e(\cos\phi + i\sin\phi) = i\phi.$$

This, of course, implies the "Euler relation"

$$\cos\phi + i\sin\phi = e^{i\phi},$$

and leads to "de Moivre's theorem"

$$(\cos\phi + i\sin\phi)^n = \cos n\phi + i\sin n\phi,$$

which Abraham de Moivre (1667–1754) stated for various special cases and forms in his writings from 1707 onwards, particularly in 1722. It was left to Leonhard Euler (1707–1783), the most prolific mathematician of all time, to give an explicit statement and general proof of de Moivre's theorem, in 1748. Another important pair of relationships published by Euler in 1748 was

$$\sin\phi = \frac{e^{i\phi} - e^{-i\phi}}{2i}, \quad \cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}.$$

Euler was the first (1777) to use i for $\sqrt{(-1)}$, but it first appeared in print in 1794 and only gained acceptance after Gauss used it in his *Disquisitiones arithmeticae* of 1801.

Geometrical interpretation: Wessel and Argand

As has been mentioned, Wallis did not quite achieve an adequate graphical representation for complex numbers and it was a Norwegian surveyor, not a professional mathematician, Caspar Wessel (1745–1818) who in 1797 formulated the modern geometric theory in a paper presented to the Royal Academy of Denmark:

"Let us designate by $+1$ the positive rectilinear unit, by $+\varepsilon$ another perpendicular to the first and having the same origin; then the angle of direction of $+1$ will be equal to 0°, that of -1 to 180°, that of ε to 90°, and that of $-\varepsilon$ to $-90°$ or to 270°." ([1], p. 265)

Wessel went on to develop the isomorphism between angular rotation and the product of complex numbers, and he also showed how de Moivre's theorem could be used to prove that $\sqrt[3]{\{4\sqrt{3} + 4\sqrt{(-1)}\}}$ denotes a line of length 2 at an angle 10° above the positive x axis. (An interesting exercise for the reader!)

The Swiss, Jean-Robert Argand (1768–1822), after whom the geometrical diagrams are named, independently wrote about the method in 1806, further developing the relationship between rotation and complex multiplication. Argand is one of many who have in the past been credited with Wessel's great achievement, the obscurity of Wessel's publication undoubtedly being a contributory factor to this confusion. A biographer of Carl Friedrich Gauss (1777–1855) records that the graphical representation of complex numbers was perhaps derived by Gauss independently in about 1800 [6].

Pure mathematics is applied ...

Although complex numbers first occurred in the theoretical development of algebra, many practical applications arose and undoubtedly the demands of these applications provided a 'cultural stress', to use R. L. Wilder's phrase [7], for further study of their properties.

When Gauss published his first proof of the fundamental theorem of algebra, in 1799, complex numbers were given a more certain standing, but wide and confident acceptance was to take another half-century or so. Meanwhile mathematicians were busily applying these useful but little understood 'numbers'. For example, in 1752 the French mathematician Jean le Rond d'Alembert (1717–1783) used complex numbers in his study of hydrodynamics, which in turn led to the modern theory of aerodynamics; and in 1772 complex numbers were used by Johann Heinrich Lambert of Germany (1728–1777) in the construction of maps by a technique called "conformal conic projection". The important theory of functions of a complex variable was soon to be developed (1821 being an important date), by Augustin-Louis Cauchy (1789–1857) and others, and complex numbers today find many respectable applications throughout science and are readily handled by the average sixth former studying mathematics. It is to be hoped that such students are also introduced to some of the fascinating and significant history of the mathematics which they meet.

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