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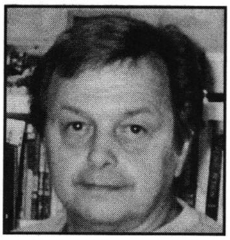

Al Cuoco (alcuoco@edc.org) directs the Center for Mathematics Education at Education Development Center. Before EDC, he taught high school mathematics for a long time. Both of these experiences have convinced him that young people are much smarter than many people think.

I have a conjecture: A great deal of classical mathematics was invented by teachers who wanted to make up problems that come out nice. Problems that come out nice allow students to concentrate on important ideas rather than messy calculations. They give students feedback that they are on the right track. They are easier to correct.

Let's call the problem of making up a nice problem a "meta-problem." Meta-problems in mathematics are not meta-mathematics problems. The idea is to apply mathematics to the design of pleasing mathematics problems. So, we are dealing with applied mathematics rather than with philosophy.

Talking with teachers over the years, I've found an amazing number of these meta-problems. Every now and then, someone publishes a method to solve some class of them (see **[1, 2, 3, 4]**, for example). One purpose of this paper is to show how classical algebra and number theory can be applied to meta-problems; another is to lobby for the inclusion of such applications in courses for prospective high school teachers.

The grand-daddy of them all. One of the earliest meta-problems must have been the generation of Pythagorean triples. Everyone has a favorite way of doing this. This one is based on the observation that if you square a Gaussian integer, the square has components that are (except for sign) legs of a Pythagorean triple. For example:

$$(2+i)^2 = 3+4i, \quad (3+i)^2 = 8+6i, \quad (5+2i)^2 = 21+20i.$$

Yes, you can see why this works by squaring $a+bi$, but, with a little more work, you can gain some insight and get a method that generalizes.

The connection between the Gaussian integers $\mathbb{Z}[i]$ and Pythagorean triples is via the norm map. If $z = a+bi \in \mathbb{Z}[i]$, then

$$N(z) = z\bar{z} = (a+bi)(a-bi) = a^2+b^2,$$

which is what we'd like to make a perfect square. But the norm is multiplicative:

$$N(zw) = N(z)N(w),$$

so

$$N(z^2) = (N(z))^2.$$

Hence, to make $N(z)$ a perfect square, make z a perfect square. It turns out that this gives you *all* primitive Pythagorean triples, and the proof involves the fact that $\mathbb{Z}[i]$ has a fundamental theorem of arithmetic.

If we apply our method to $v = x + yi$, then

$$v^2 = (x + yi)^2 = (x^2 - y^2) + 2xyi$$

so

$$N(v^2) = (x^2 - y^2)^2 + (2xy)^2.$$

But

$$N(v^2) = (N(v))^2 = (x^2 + y^2)^2,$$

and we get the celebrated identity

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$$

that shows up at least once every other month in the *Mathematics Teacher*.

Eisenstein triples. This meta-problem involves constructing nice law of cosine problems. For example, a triangle with side lengths 21, 16, and 19 has a 60° angle. Are there more triangles like this? If the sides of the triangle have lengths a, b, and c, then the law of cosines says that

$$c^2 = a^2 + b^2 - 2ab\cos 60°$$

or

$$c^2 = a^2 + b^2 - ab. \tag{1}$$

This looks like the first meta-problem, and, in fact, we can use the same method to solve it. Let's call triples of integers (a, b, c) that satisfy (1) *Eisenstein triples* in honor of George Eisenstein, a student of Gauss. Eisenstein was very fond of the ring of integers in the field of cube roots of unity: $\mathbb{Z}[\omega] = \{a + b\omega | a, b \in \mathbb{Z}\}$ where

$$\omega = \frac{-1 + i\sqrt{3}}{2}.$$

Notice that

$$\omega^2 = \overline{\omega} = -1 - \omega,\ \omega + \overline{\omega} = -1, \text{ and } \omega \cdot \overline{\omega} = 1.$$

If $z = a + b\omega$, the norm function acts like this:

$$\begin{aligned} N(z) &= (a + b\omega)(\overline{a + b\omega}) = (a + b\omega)(a + b\overline{\omega}) \\ &= a^2 + ab(\omega + \overline{\omega}) + b^2\omega \cdot \overline{\omega} = a^2 - ab + b^2. \end{aligned}$$

How nice! Norms from Eisenstein's integers are just what we want to make perfect squares. But, as before, if we want the norm to be a square in $\mathbb{Z}$, just make the thing a square in $\mathbb{Z}[\omega]$. In other words, to find an Eisenstein triple, take *any* Eisenstein integer v and square it. If $v^2 = a + b\omega$, then $(a, b, \sqrt{N(v^2)})$ is an Eisenstein triple. Some examples are

$$(5,8,7),\ (7,15,13),\ (9,24,21),\ (11,35,31), \text{ and } (13,48,43).$$

Does this method give us all primitive Eisenstein triples? Well, almost. $\mathbb{Z}[\omega]$ has a fundamental theorem, but it also has six units: $1, -1, \omega, -\omega, \omega^2, -\omega^2$. This allows us to build new triples from the ones generated from our algorithm. For example, if we multiply $8 + 5\omega$ by the six units, we discover another triple-producing integer: $3 + 8\omega = (8 + 5\omega)(-\omega^2)$.

There's nothing sacred about 60°. What's nice about 90° (Pythagoras) and 60° (Eisenstein) is that their cosines are rational (so there's a *chance* of an integer sided triangle). Suppose, for example, that we take $\theta = \cos^{-1}\left(\frac{3}{5}\right)$, and we let

$$\rho = -\cos\theta + i\sin\theta = \frac{-3+4i}{5}.$$

Then we can do arithmetic in the ring $\mathbb{Z}[\rho]$, and, in particular, if we take the norm of

$$z = (6+5\rho)^2 = 11 + 30\rho$$

we find that a triangle whose sides are 11, 30, and 25 has θ as one of its angles.

Clean cubics. The next meta-problem has crossed everyone's mind at one time or another. How do you generate cubic polynomial functions with integer coefficients, with three integral zeros, two integral extrema, and one integral inflection point? No cheating: the six integral points must all be distinct.

First some algebra. Let's suppose we are looking for

$$f(x) = ax^3 + bx^2 + cx + d.$$

Let $x = y - \frac{b}{3a}$. Then if $g(y) = 27a^3 f(x)$, we have that

$$g(y) = ry^3 + sy + t$$

for integers r, s, and t. So, we can assume that f is given by

$$f(x) = rx^3 + sx + t$$

for some r, s, and t. Next, by a scale change, we can assume our cubic is monic: if the cubic starts out as

$$f(x) = rx^3 + sx + t,$$

let $y = rx$, and let $g(y) = r^2 f(x)$. Then

$$g(y) = y^3 + rsy + r^2 t,$$

and we can work with g instead of f.

Suppose, then, that

$$f(x) = x^3 + cx + d.$$

This ensures that f'' has an integral zero (namely 0). If we put $c = -3q^2$ for some integer q, then $f'(x) = 3x^2 + c = 3(x^2 - q^2)$ will have integral zeros. So, now our function is

$$f(x) = x^3 - 3q^2 x + d.$$

If f has two integral zeros then it has three, so it's enough to make two zeros, say $-\alpha$ and β, integral. But if $f(-\alpha) = f(\beta) = 0$, we have

$$-\alpha^3 + 3q^2\alpha = \beta^3 - 3q^2\beta$$

or

$$\beta^3 + \alpha^3 = 3q^2(\alpha + \beta).$$

Since $\alpha + \beta \neq 0$ (we want our roots to be distinct), this is the same as

$$\alpha^2 - \alpha\beta + \beta^2 = 3q^2$$

or

$$N(\alpha + \beta\omega) = 3q^2.$$

Back to Eisenstein. This time, we're looking for an Eisenstein integer whose norm is three times a square. We see that

$$N(1+2\omega)=3$$

so we are looking for an Eisenstein integer z so that

$$N(z)=N(1+2\omega)q^2.$$

To find one, all we have to do is to take *any* Eisenstein integer v, and let $z=(1+2\omega)v^2$. Then

$$\begin{aligned} N(z) &= N((1+2\omega)v^2)=N(1+2\omega)(N(v))^2 \\ &= 3(\textit{something})^2. \end{aligned}$$

So, to build our cubic $F(x)=x^3+cx+d$,

- the coefficient of x is $-N(z)=-3q^2$.
- to find d, note that $z=\alpha+\beta\omega$ where $-\alpha$ and β are the roots of our cubic. Since

  $$-\alpha^3+3q^2\alpha+d=0,$$

  we see that

  $$d=\alpha^3-3q^2\alpha.$$

A computer or calculator can be used to generate values for $3q^2$ and d:

$$(147,286),(507,-506),(1323,-7722),(2883,-35282).$$

So, for example, $f(x)=x^3-147x+286$ is a nice cubic. And, if you'd like a cubic with an x^2 term, simply translate and look at

$$g(x)=(x+1)^3-147(x+1)+286=x^3+3x^2-144x+140.$$

Another example: $f(x+3)$ produces

$$h(x)=x^3+9x^2-120x-128.$$

If you allow for *rational* instead of integer points, you can get polynomials with small coefficients. A little experimenting (and an old-fashioned theory of equations book) produces

$$\frac{1}{108}f(6x+5)=2x^3+5x^2-4x-3,$$

which is the lead example in **[2]**.

The box problem. Every calculus course has the box problem where we cut little squares out of the corners of a rectangle and fold up the sides. The box meta-problem is this:

> A rectangle measures $a\times b$. Little squares are cut out of the corners, and the sides are folded up to make a box. Find a method to generate a and b if we insist that the size of the cut-out that maximizes the volume of the box is a rational number.

Well, as we tell our students, let the size of the cut-out be x. Then the volume is a function of x:

$$V(x) = (a - 2x)(b - 2x)x = 4x^3 - 2(a + b)x^2 + abx,$$

so

$$V'(x) = 12x^2 - 4(a + b)x + ab.$$

We want this to have rational zeros, so we want the discriminant $16(a + b)^2 - 48ab$ to be a perfect square. But 16 is a perfect square, so we want to make

$$(a + b)^2 - 3ab = a^2 - ab + b^2$$

a perfect square. We can do this by taking a and b to be the legs of an Eisenstein triple.

Really mean values. When you teach about the mean value theorem, you often assign problems where students must find the x-coordinate of the point where the tangent is parallel to the secant. Here's a meta-version:

> Find a cubic function f with integer coefficients and integers p and q so that
>
> $$f(q) - f(p) = f'(c)(q - p)$$
>
> with c a rational number.

We could use the same method that we used to generate nice cubics (think of how the mean value theorem is *proved*), but a separate calculation gives us an added bonus.

As before, we can assume that f is of the form

$$f(x) = x^3 + rx + t.$$

If our p and q are $-a$ and b, the equation

$$f(q) - f(p) = f'(c)(q - p)$$

becomes

$$a^2 + b^2 - ab = 3c^2$$

and we know what to do with that. The bonus is that a and b are *independent* of f. That is, endpoints of 2 and 11 and a c of 7 work for *any* cubic of the form under discussion.

Some well placed points. Here's a meta-problem that comes up when designing problem sets for geometry courses:

> Find three lattice points z, w, and u in the plane so that the distance between any two of the three points is an integer.

Clearly, solutions are invariant under translation by a lattice point, so we can assume that one of the points, say u, is at the origin. Look at the plane as if it were the complex plane. Then z and w are Gaussian integers. So, we want Gaussian integers z and w so that $|z|$, $|w|$, and $|z - w|$ are integers. If $z = a + bi$, then

$$|z| = \sqrt{a^2 + b^2} = \sqrt{N(z)}.$$

Hence, to make the length an integer, make the norm a perfect square. To make the norm a perfect square, make the Gaussian integer a perfect square in $\mathbb{Z}[i]$. That is, we want Gaussian integers z and w so that z, w, and $z-w$ are perfect squares in $\mathbb{Z}[i]$. So, we want to choose z and w so that

$$z=\alpha^2,\ w=\beta^2, \text{ and } z-w=\gamma^2 \text{ for some } \alpha,\beta,\gamma\in\mathbb{Z}[i].$$

That is, we want *Gaussian* integers, α, β, and γ so that

$$\alpha^2-\beta^2=\gamma^2 \quad \text{or} \quad \alpha^2=\beta^2+\gamma^2.$$

The punchline is that our favorite identity

$$(x^2+y^2)^2=(x^2-y^2)^2+(2xy)^2$$

works just fine in *any* commutative ring, including $\mathbb{Z}[i]$. So, the trick is to pick any *Gaussian* integers x and y, and to let

$$\alpha=x^2+y^2, \quad \beta=x^2-y^2.$$

Then let

$$z=\alpha^2, \quad w=\beta^2.$$

For example, starting with $x=3+2i$ and $y=2+i$, the method shows that $(0,0)$, $(-192,256)$, and $(-60,32)$ are vertices of an integer-sided triangle. And we can dilate to produce the more manageable triangle with vertices $(0,0)$, $(-48,64)$, and $(-15,8)$.

Why meta-problems? There are several reasons to include meta-problems in your algebra or number theory course:

- Students can make up their own.
- They show how mathematics can be applied to things other than trajectories of baseballs.
- In addition to their use as applications, they can be used as springboards to important and hard topics (see [5] for an example of this).
- They are great fun.

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