Creating 'Nice' Problems in Elementary Mathematics – I*

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We consider here the 'meta-problem' of creating 'nice' problems in elementary mathematics, 'nice' being defined as a problem in which the input data, as well as the answers, are rational numbers. The classical example of this is that of generating Pythagorean triples. Many examples of this kind arise when we study Euclidean geometry and the theory of equations. We consider a few problems of this genre.

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Introduction

Anyone who has taught high school mathematics will have encountered the task of composing 'nice' problems, where the input data and the answer are 'nice' numbers. Here we use the following operational definition of niceness: *Problems in which the input data and answer are rational numbers are nice. Problems in which either the input data or the answer involve irrational numbers are not nice.* In [1], Cuoco writes that nice problems "allow students to concentrate on important ideas rather than messy calculations" and makes a tongue-in-cheek conjecture: "A great deal of classical mathematics was invented by teachers who wanted to make up [nice] problems." The author calls the task of composing a nice problem a *meta-problem*. We study a few meta-problems in this three-part article. The first three are from Cuoco's paper but treated differently here.

The determination of Pythagorean triples is surely the best-known problem of this genre; it is encountered most naturally when we seek instances of right-angled triangles which yield rational val-

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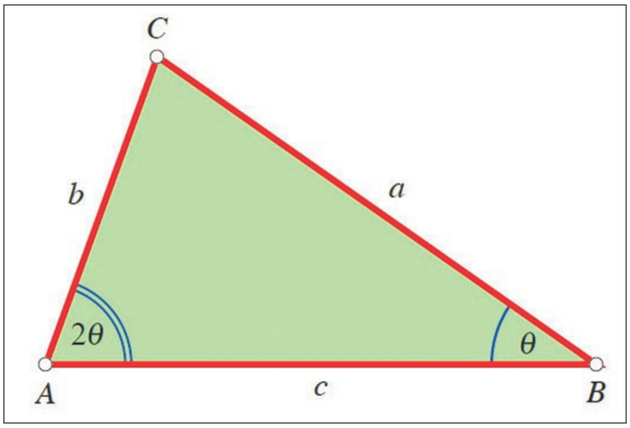

Figure 1. Triangle ABC with $\measuredangle A = 3\measuredangle B$.

Problems in which the input data and answer are rational numbers are nice.

ues for the sine and cosine—a task routinely faced by anyone who teaches trigonometry.

The French mathematician Jean d'Alembert (1717–83) once said, "Algebra is generous; she gives more than is asked of her." We will see the truth of this many times over in this article.

Terminology

Problems in which either the input data or the answer involve irrational numbers are not nice.

Many of the problems we shall study deal with integer-sided triangles with the property that the gcd of their side lengths equals 1. We call such a triangle *primitive*. (Sometimes, we apply the same word to a polynomial with integer coefficients; the meaning is that the gcd of its coefficients equals 1.)

1. Triangles With One Angle a Multiple of Another

Generating integer-sided triangles in which one angle is a 'nice' multiple of another angle is a task one encounters frequently, while teaching the multiple angle formulas of trigonometry and the solution of triangles. We study the cases when the ratio of the two angles is (a) 2 : 1 (b) 3 : 1.

Triangles With A = 2B

Let a, b, c be the sides of a primitive $\triangle ABC$ in which $\measuredangle A = 2\measuredangle B$. We shall first show that this leads to the relation $a^2 = b(b + c)$. If $\measuredangle B = \theta$ then the sine rule leads to

$$\frac{a}{\sin 2\theta} = \frac{b}{\sin\theta} = \frac{c}{\sin 3\theta}, \qquad \therefore \ \frac{a}{2\cos\theta} = b = \frac{c}{4\cos^2\theta - 1}. \quad (1)$$

Eliminating $\cos\theta$ from the last pair of equalities we readily get $a^2 = b(b + c)$. It is a nice exercise to show that if this relation holds, then $\measuredangle A = 2\measuredangle B$. Hence the relation completely characterizes such triangles. (Observe that the relation is homogeneous. This could have been expected, because a scaled up copy of the triangle will have the same geometric properties as the original one.) So our task reduces to finding primitive triples (a, b, c) of positive integers which satisfy the equation $a^2 = b(b + c)$.

"A great deal of classical mathematics was invented by teachers who wanted to make up [nice] problems."

The simplest way of doing so is to solve the equation over the rationals and then to scale up appropriately; this works because of homogeneity of the equation. Putting $a = 1$ and $b = t$ we get $t(t + c) = 1$, which yields $c = 1/t - t$. We thus get $(a,\ b,\ c) = (1,\ t,\ 1/t - t)$. To get a valid triangle from this triple, we must have:

The determination of Pythagorean triples is surely the best-known problem of this genre.

$$t > 0, \quad \frac{1}{t} - t > 0, \quad 1 + t > \frac{1}{t} - t, \quad 1 + \frac{1}{t} - t > t, \quad t + \frac{1}{t} - t > 1, \tag{2}$$

and these collectively yield $1/2 < t < 1$.

Let $t = m/n$ where m, n are coprime positive integers. After clearing fractions we get:

$$(a, b, c) = \left(mn,\ m^2,\ n^2 - m^2\right). \tag{3}$$

This provides a parametrization of the triples (a, b, c) of coprime integers for which $a^2 = b(b + c)$. Here a few of these triples:

t	m	n	a	b	c
2/3	2	3	6	4	5
3/4	3	4	12	9	7
3/5	3	5	15	9	16
5/6	5	6	30	25	11

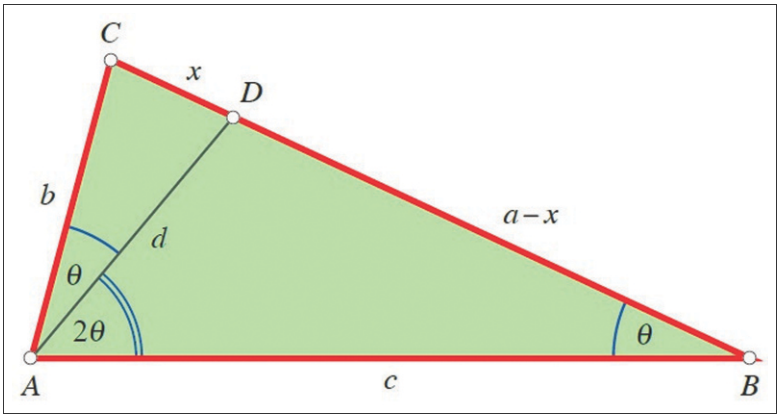

Figure 2. Triangle ABC with $\measuredangle A = 3\measuredangle B$.

"Algebra is generous; she gives more than is asked of her."

From the parametric form of (a, b, c) we deduce a result which we may not have anticipated: ***If the integers* a, b, c *are the sides of a primitive triangle* ABC *in which* $\measuredangle$A = 2$\measuredangle$B, *then* b = gcd(a, b)^2. *In particular,* b *is a perfect square.***

Triangles With A = 3B

Let a, b, c be the sides of a primitive triangle ABC in which $\measuredangle A = 3\measuredangle B$. We shall show that this leads to the homogeneous third-degree relation $bc^2 = (a-b)^2(a+b)$. Let $\measuredangle B = \theta$; then $\measuredangle A = 3\theta$ and $\measuredangle C = 180^{\circ} - 4\theta$. Locate a point D on side BC (*Figure* 2) such that $\measuredangle CAD = \theta$, $\measuredangle DAB = 2\theta$. Let $AD = d$ and $CD = x$.

Since $\triangle CAD \sim \triangle CBA$ (both triangles have angles $\theta, 3\theta, 180^{\circ} - 4\theta$), we get:

$$x : b : d = b : a : c, \qquad \therefore\ x = \frac{b^2}{a}, \quad d = \frac{bc}{a}, \quad \therefore\ a - x = \frac{a^2 - b^2}{a}. \tag{4}$$

In $\triangle DAB$ we have $\measuredangle DAB = 2\measuredangle DBA$. Hence the relation found earlier applies:

$$\left(\frac{a^2 - b^2}{a}\right)^2 = \frac{bc}{a} \cdot \left(\frac{bc}{a} + c\right). \tag{5}$$

Simplifying, we get $bc^2 = (a-b)^2 \cdot (a+b)$, as claimed.

We must solve this equation over the positive integers. As earlier, we first solve it over the rationals, exploiting its homogeneity.

This time we put $b = 1$; we get:

$$c^2 = (a-1)^2 \cdot (a+1). \tag{6}$$

From this we see that $a+1$ is the square of a rational number, say $a + 1 = t^2$ where $t \in \mathbb{Q}$. This yields $a = t^2 - 1$ and $c = t\left(t^2 - 2\right)$. Hence:

$$(a,\ b,\ c) = \left(t^2 - 1,\ 1,\ t^3 - 2t\right); \tag{7}$$

we have obtained the desired parametrization. To ensure that the triple yields a valid triangle, we must have:

$$t^2-1 > 0, \quad t^3-2t > 0, \quad t^2 > t^3-2t, \quad 1+t^3-2t > t^2-1, \quad t^3-2t+t^2-1 > 1. \tag{8}$$

Collectively these yield a union of two intervals: $\sqrt{2} < t < 2$, $-\sqrt{2} < t < -1$.

Finally we scale up; if $t = m/n$ where m, n are coprime integers, then

$$(a,\ b,\ c) = \left(n(m^2 - n^2),\ n^3,\ m(m^2 - 2n^2)\right). \tag{9}$$

From this formula we arrive at another unexpected result, much like the one we got earlier: ***If the integers* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are the sides of a primitive triangle* $\mathbf{ABC}$ *in which* $\measuredangle\mathbf{A} = 3\measuredangle\mathbf{B}$*, then* $\mathbf{b} = \mathbf{gcd}(\mathbf{a}, \mathbf{b})^3$*. In particular,* $\mathbf{b}$ *is a perfect cube.***

Listed below are some triples (a, b, c) obtained from this parametrization.

t	m	n	a	b	c
3/2	3	2	10	8	3
5/3	5	3	48	27	35
8/5	8	5	195	125	112

Extensions

We see the possibility of a cluster of related results, the shared setting being that $\measuredangle A = r\measuredangle B$ for some rational number r. The reader is invited to continue the exploration. The following question is surely worth a closer look: *Suppose a, b, c are the sides of a primitive $\triangle ABC$ in which $\measuredangle A = 4\measuredangle B$. Is it necessarily true that b is the fourth power of an integer?*

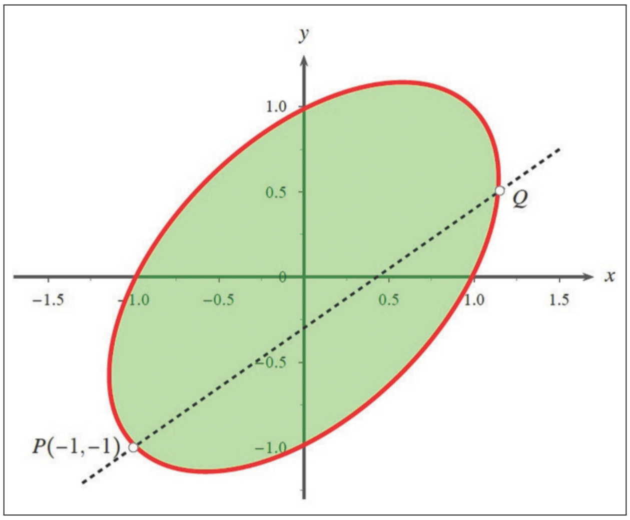

Figure 3. The ellipse E : $x^2 - xy + y^2 = 1$, and a line through $P(-1, -1)$.

2. Triangles With a 60 Degree Angle

How would one generate a triple of positive integers which are the sides of a primitive triangle with a 60^o *angle?* This is much like the problem of determining Pythagorean triples, but with 60^o in place of 90^o. The problem comes up naturally when one is teaching the cosine rule in plane trigonometry.

Let a, b, c be the sides of a triangle ABC for which $\measuredangle C = 60^o$. The cosine rule yields the following homogeneous relation:

$$c^2 = a^2 + b^2 - ab. \tag{10}$$

We must find triples (a, b, c) of coprime positive integers satisfying this relation. Let $x = a/c$, $y = b/c$; then

$$x^2 - xy + y^2 = 1. \tag{11}$$

In the coordinate plane this equation represents an ellipse E with eccentricity $\sqrt{2/3}$ passing through the points $(\pm 1, 0)$, $(0, \pm 1)$, $(1, 1)$ and $(-1, -1)$. It is sketched in *Figure* 3.

Our interest is in the rational points on E that are located in the first quadrant. Let ℓ be a line with rational slope t passing through the rational point $P(-1, -1)$ which lies on E. Since the equation of E is of the second degree and has rational coefficients, a line with rational slope passing through a rational point on E must intersect the curve again at a rational point Q (which could be P itself; this would be the case if ℓ is tangent to E). Conversely, a line through any two rational points on E must have a rational slope. Hence by solving a pair of simultaneous equations we can in principle enumerate all the rational points on E. (Actually we do miss one rational point—the point $Q = (-1, 0)$, for which the slope of PQ is undefined.)

The equation of line ℓ is $y+1 = t(x+1)$. Substituting $y = t(x+1)-1$ into $x^2 - xy + y^2 = 1$ we get the quadratic equation

$$x^2(t^2 - t + 1) + x(t - 1)(2t - 1) + t^2 - 2t = 0, \qquad (12)$$

one of whose roots is $x = -1$. Using the formula for the product of the roots of a quadratic equation, we get the other root:

$$x = \frac{t(2 - t)}{t^2 - t + 1}. \qquad (13)$$

Substituting this into the equation of ℓ we get $Q = (t(2 - t), 2t - 1)/(t^2 - t + 1)$. Let $t = m/n$ where m, n are positive integers. We then get:

$$Q = \left(\frac{m(2n - m)}{m^2 - mn + n^2}, \frac{n(2m - n)}{m^2 - mn + n^2}\right). \qquad (14)$$

We thus have a parametrization of the primitive integer triples (a, b, c) which can serve as the sides of a $\triangle ABC$ for which $\angle C = 60^{\circ}$:

$$a = m(2n - m), \quad b = n(2m - n), \quad c = m^2 - mn + n^2, \quad m, n \in \mathbb{N}. \qquad (15)$$

To ensure that we get positive integers we must have $1/2 < t < 2$, i.e., $n < 2m < 4n$. Here are some triples we get from this parametrization:

t	m	n	a	b	c
2/3	2	3	8	3	7
3/4	3	4	15	8	13
3/5	3	5	21	5	19
5/6	5	6	35	24	31

To get $(-1, 0)$ we must let $t \to \infty$.

One reason for choosing a 60° angle in this problem is that its cosine is a rational number. Using the same approach we can generate integer-sided triangles with one angle whose cosine is any given rational number between -1 and 1.

In solving the above problem we receive a bonus, for it happens that triangles in which one angle equals 60° have a large number of non-trivial properties; so the same problem comes up from many independent directions. (Here are some examples of such properties. Given any $\triangle ABC$ with circumcentre O and incentre I, let B_1, C_1 be the points where the extensions of BI, CI meet sides AC, AB respectively, and let B_2, C_2 be the reflections of B, C in lines CI, BI respectively. Then the following are true: B_1C_1 is parallel to OI if and only if $\angle A = 60^{\circ}$; B_2, C_2, I lie in a straight line if and only if $\angle A = 60^{\circ}$; if $\angle A = 60^{\circ}$, then O is equidistant from lines BI and CI. These are only a few of many such properties.)

In Closing . . .

We shall continue in this vein and study more problems of this genre in Part II and Part III of this article. As the reader will see now, there is much more to elementary mathematics than meets the eye!

Suggested Reading

[1] **Al Cuoco, Meta-problems in mathematics, *College Mathematics Journal*, Vol.31, No.5, Nov 2000.**

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