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90.45 on Integer-Sided Triangles Containing Angles of  $120^\circ$  or  $60^\circ$

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*Proof:* Let  $X, Y, Z$  be the reflections of  $P$  in  $BC, CA, AB$  respectively. Then, the triangles  $XCY, YAZ$  and  $ZBX$  are each isosceles with an angle of  $120^\circ$ . Also, the hexagon  $AZBXC Y$  has area double that of  $ABC$  (because  $\triangle AZB = \triangle APB$ , etc.), which is  $x^2\sqrt{3}/4$ . The triangle  $XYZ$  has sides  $a\sqrt{3}, b\sqrt{3}, c\sqrt{3}$  and therefore using Hero's formula, has area  $3\sqrt{s(s-a)(s-b)(s-c)}$ . Finally, by adding the areas of  $XCY, YAZ, ZBX$  and  $XYZ$  to obtain the hexagon, we get

$$\frac{\sqrt{3}}{4}(a^2 + b^2 + c^2) + 3\sqrt{s(s-a)(s-b)(s-c)} = \frac{x^2\sqrt{3}}{2}$$

and hence the stated result.

*Acknowledgement:* I thank the Editor for suggesting this very compact (and non-calculus) version of the initial proof.

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## 90.45 On integer-sided triangles containing angles of $120^\circ$ or $60^\circ$

### *Introduction*

Most secondary school pupils will have, at some stage of their mathematics course, come across a 3, 4, 5 or a 5, 12, 13 triangle. Such triangles are said to correspond to Pythagorean triples, in that they contain an angle of  $90^\circ$ , and the lengths of all their sides are integers. The classification of all such triangles is a classical problem, the solution to which is well documented. It is, however, a less well-known fact that a 3, 5, 7 triangle contains an angle of  $120^\circ$  or that a 3, 8, 7 triangle contains an angle of  $60^\circ$ . In this note, we consider triangles which contain angles of  $120^\circ$  or  $60^\circ$  and whose sides are positive integers, and, in so doing, provide an alternative approach to the work already done on this topic by [1, 2, 3, 4]. The methods and definitions which we shall use are similar to those used in the case of Pythagorean triples, and it is recommended that the reader familiarise themselves with the arguments and concepts involved, by referring to e.g. Jones and Jones [5, pp. 219-224].

Using obvious notation, we shall start off by considering  $120^\circ$  triangles.

### *$120^\circ$ triangles*

We define  $(a, b, c)$  to be a 120-triple if  $a, b, c$  are positive integers such that the triangle having sides of length  $a, b, c$  contains an angle of  $120^\circ$ . As a convention, we shall always assume that it is the third component of such a triple which represents the side opposite the  $120^\circ$  angle.

Alternatively, applying the cosine rule to a  $120^\circ$  triangle with sides of length  $a, b, c$ , we have

$$a^2 + b^2 + ab = c^2, \quad (1)$$

and thus we may think of a 120-triple as being positive integers satisfying equation (1).

A 120-triple  $(a, b, c)$  will be called primitive if  $\gcd(a, b, c) = 1$ . Now suppose that  $(a, b, c)$  is a 120-triple with  $\gcd(a, b, c) = h$ . If we write  $a = ha_1$ ,  $b = hb_1$ ,  $c = hc_1$  and substitute in equation (1) we see that  $a_1^2 + b_1^2 + a_1b_1 = c_1^2$  which means that  $(a_1, b_1, c_1)$  must be a primitive 120-triple. Thus any 120-triple will be a multiple of a primitive 120-triple and the corresponding triangles will be similar. We need therefore only consider primitive 120-triples.

Let us now assume that  $(a, b, c)$  is a primitive 120-triple. Equation (1) may be rewritten as:

$$3a^2 + (a + 2b)^2 = (2c)^2,$$

and thus

$$3a^2 = (2c + a + 2b)(2c - a - 2b). \quad (2)$$

We also have

$$3b^2 + (2a + b)^2 = (2c)^2,$$

and thus

$$3b^2 = (2c + 2a + b)(2c - 2a - b). \quad (3)$$

We also have

$$(a + b)^2 - ab = c^2,$$

and thus

$$ab = (a + b - c)(a + b + c). \quad (4)$$

Let us now consider equation (4). Putting  $u = a + b - c$  we see that  $(a + b + c) = 2(a + b) - u$ . Thus

$$ab = u[2(a + b) - u] = 2u(a + b) - u^2,$$

which may be re-written in the form

$$u^2 - 2u(a + b) + ab = 0. \quad (5)$$

From (5), we see that

$$(a - 2u)(b - 2u) = ab - 2u(a + b) + 4u^2 = 3u^2,$$

or, in other words,

$$3(a + b - c)^2 = (2c - a - 2b)(2c - 2a - b). \quad (6)$$

Note that equations (2) and (3) imply that both the quantities appearing on the right hand side of equation (6) are positive.

**Lemma 1:** Let  $(a, b, c)$  be a primitive 120-triple. Then

$$\gcd(2c - a - 2b, 2c - 2a - b) = 1.$$

*Proof:*

Let  $p$  be a prime which divides both  $2c - a - 2b$  and  $2c - 2a - b$ . Then, by equation (6),  $p^2 \mid 3(a + b - c)^2$ , so (even if  $p = 3$ )  $p \mid (a + b - c)^2$ , so  $p \mid (a + b - c)$ . Now

$$a = (2c - a - 2b) + 2(a + b - c),$$

$$b = (2c - 2a - b) + 2(a + b - c),$$

$$c = (2c - a - 2b) + (2c - 2a - b) + 3(a + b - c),$$

and hence,  $p$  must divide each of  $a, b, c$ . This, however, contradicts the fact that  $(a, b, c)$  is primitive.

Thus no prime  $p$  can exist which divides both  $2c - a - 2b$  and  $2c - 2a - b$  and hence  $\gcd(2c - a - 2b, 2c - 2a - b) = 1$ .

**Theorem 1:** Let  $(a, b, c)$  be a primitive 120-triple. Then there exist positive integers  $y, z$ , with  $\gcd(y, z) = 1$  and  $z$  not divisible by 3, such that, possibly with  $a$  and  $b$  interchanged,

$$z^2 = 2c - a - 2b \tag{7}$$

$$3y^2 = 2c - 2a - b \tag{8}$$

$$zy = a + b - c. \tag{9}$$

*Proof:* Equations (2), (3) and (4) imply that the RHS of (7), (8) and (9) respectively are positive. Equation (6) and Lemma 1 imply that one of the factors appearing on the RHS of equation (6) must be of the form  $z^2$  and the other of the form  $3y^2$ , where  $z$  and  $y$  are positive integers satisfying  $zy = a + b - c$ . Equations (7), (8) and (9) represent one such outcome, the other is given by interchanging  $a$  and  $b$  in these equations. Since Lemma 1 also implies that  $\gcd(3y^2, z^2) = 1$ , we must have  $\gcd(y, z) = 1$  and  $z$  not divisible by 3.

**Theorem 2:**

- (i) Let  $(a, b, c)$  be a primitive 120-triple. Then there exist positive integers  $y, z$ , with  $\gcd(y, z) = 1$  and  $z$  not divisible by 3, such that, possibly with  $a$  and  $b$  interchanged,

$$a = z^2 + 2yz,$$

$$b = 3y^2 + 2yz,$$

$$c = 3y^2 + z^2 + 3yz.$$

- (ii) Conversely, let  $y, z$  be positive integers with  $\gcd(y, z) = 1$  and  $z$  not divisible by 3. Define

$$\begin{aligned}a &= z^2 + 2yz, \\b &= 3y^2 + 2yz, \\c &= 3y^2 + z^2 + 3yz.\end{aligned}$$

Then  $(a, b, c)$  is a primitive 120-triple.

*Proof:*

- (i) This part follows from Theorem 1 by solving equations (7), (8) and (9) for  $a, b$  and  $c$ .
- (ii) In order to verify that  $(a, b, c)$  is a 120-triple, we must show that the given expressions for  $a, b, c$  satisfy equation (1). Checking for primitivity involves expressing  $z^2$  and  $3y^2$  in terms of  $a, b$  and  $c$  as in equations (7) and (8) and then using the given conditions on  $y$  and  $z$  to argue as in the proof of Lemma 1. Both exercises are left to the reader.

### 60° triangles

We shall now consider triangles which contain an angle of 60° and whose sides are positive integers. Using similar definitions to those used above, and the same argument as that used in equation (1), we note that  $(m, n, k)$  will be a 60-triple if

$$m^2 + n^2 - mn = k^2. \quad (10)$$

In this case, the third component of such a triple represents the side opposite the 60° angle.

A similar approach to that used above for 120-triples could also be used in the case of 60-triples. However, the following result, which exhibits a relationship between 120-triples and 60-triples, enables us to use the result of Theorem 2 to find all primitive 60-triples.

*Lemma 2:*

- (i) Let  $(a, b, c)$  be a primitive 120-triple. Then both  $(a, a + b, c)$  and  $(a + b, b, c)$  are primitive 60-triples.
- (ii) Every primitive 60-triple, other than  $(1, 1, 1)$  which corresponds to an equilateral triangle, is associated with a primitive 120-triple in this way.

*Proof:*

- (i) If  $(a, b, c)$  is a primitive 120-triple, then

$$a^2 + (a + b)^2 - a(a + b) = a^2 + b^2 + ab = c^2,$$

and, by equation (10),  $(a, a + b, c)$  is a 60-triple.

In the same way,  $(a + b, b, c)$  is also a 60-triple.

The fact that both are primitive is a simple exercise and is left to the reader.

- (ii) Let  $(m, n, k)$  be a primitive 60-triple which does not correspond to an equilateral triangle. Then either  $m > n$  or  $n > m$ .

If  $m > n$ , then

$$(m - n)^2 + n^2 + n(m - n) = m^2 + n^2 - mn = k^2,$$

and, by equation (1),  $(m - n, n, k)$  is a 120-triple.

In the same way, if  $n > m$ , then  $(m, n - m, k)$  is a 120-triple. The reader is again left to check for primitivity.

A geometrical interpretation of the results of part (i) of the above lemma is given in the diagram below.

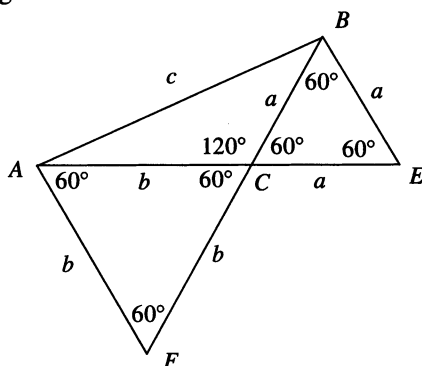


FIGURE 1

It is left to the reader to find similar geometrical interpretations for the results appearing in part (ii) of the above lemma.

We are now able to extend Theorem 2 to include the case of 60-triples.

*Theorem 3:*

- (i) Let  $y, z$  be positive integers with  $\gcd(y, z) = 1$  and  $z$  not divisible by 3. Define

$$a = z^2 + 2yz,$$

$$b = 3y^2 + 2yz,$$

$$c = 3y^2 + z^2 + 3yz,$$

$$d = 3y^2 + z^2 + 4yz.$$

Then  $(a, b, c)$  is a primitive 120-triple,  $(a, d, c)$  is a primitive 60-triple and  $(d, b, c)$  is also a primitive 60-triple.

- (ii) All primitive 120-triples and all primitive 60-triples other than  $(1, 1, 1)$  are of this form.

*Proof:*

Note that  $d = a + b$ . The result now follows from Theorem 2 and Lemma 2.

The following table illustrates the outcomes of applying Theorem 3 for small values of  $y$  and  $z$ .

$y$	$z$	$(a, b, c)$	$(a, d, c)$	$(d, b, c)$
1	1	(3, 5, 7)	(3, 8, 7)	(8, 5, 7)
1	2	(8, 7, 13)	(8, 15, 13)	(15, 7, 13)
2	2	(5, 16, 19)	(5, 21, 19)	(21, 16, 19)
3	1	(7, 33, 37)	(7, 40, 37)	(40, 33, 37)
3	2	(16, 39, 49)	(16, 55, 49)	(55, 39, 49)

The results of Theorem 2 and Theorem 3 are also proved in [1, Theorem 2] and [2, Theorem 2]. It is left to the reader to verify that the variables  $q$  and  $r$  which appear in both these theorems are related to the variables  $y$  and  $z$  used in this note by the transformations  $r = y$  and  $q = y + z$ .

*Further geometrical considerations*

[1], [2] and [3] have exhibited a simple geometrical relationship which exists between the two  $60^\circ$  triangles generated in Theorem 3. An alternative characterisation may be derived by considering the following approach.

Let  $(a, b, c)$  be a primitive 120-triple and  $d = a + b$  as above. Is it possible to construct a triangle in which two of the sides are of length  $c$  and  $d$  and the angle opposite the side of length  $c$  equal to  $60^\circ$ ?

This question may be answered by substituting  $d$  and  $c$  for  $m$  and  $k$  respectively in equation (10). Lemma 2 tells us that this equation will then be satisfied if  $n$  takes one of two values, namely  $n = a$  or  $n = b$ . Thus, there are in fact two triangles which satisfy the given criteria – the so called ‘ambiguous’ case which can arise in the solution of triangles. A geometrical representation is given in the sketch below for the case  $b > a$ .

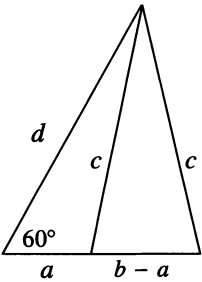


FIGURE 2

Finally, we consider the following question.

The diagram below shows a primitive 120-triple  $(a, b, c)$  and the corresponding triangle  $ABC$ . The line  $CD$  is the internal bisector of the  $120^\circ$  angle. Both the triangles  $ACD$  and  $BCD$  formed in this way contains an angle of  $60^\circ$ . What more can we say about these two triangles?

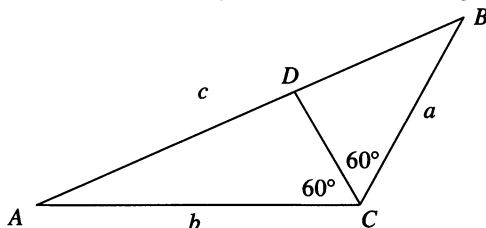


FIGURE 3

To Figure 1 let us add the line  $CD$ . It is clear that  $ACD$  and  $AEB$  are similar triangles and thus the primitive 60-triple corresponding to triangle  $ACD$  is  $(a, a + b, c)$ . In the same way,  $BCD$  and  $BFA$  are similar triangles and thus the primitive 60-triple corresponding to triangle  $BCD$  is  $(a + b, b, c)$ .

Note that these are precisely the two 60-triples associated with the original 120-triple  $(a, b, c)$  in the sense of Lemma 2.

#### Acknowledgement

I am indebted to the referee for providing me with a simplified proof of the above result.

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