

Creating ‘Nice’ Problems in Elementary Mathematics – III*

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We consider here the ‘meta-problem’ of creating ‘nice’ problems in elementary mathematics, ‘nice’ being defined as a problem in which the input data, as well as the answers, are rational numbers. The classical example of this is that of generating Pythagorean triples. Many examples of this kind arise when we study Euclidean geometry and the theory of equations. We consider a few problems of this genre.

Introduction

In Parts I and II of this article we defined a ‘nice’ problem as one *for which the input data and answer are rational numbers*. (Conversely, problems in which either the input data or the answer involve irrational numbers are considered ‘not nice’.) Anyone who has taught high school mathematics will routinely have had to compose nice problems; for, as Cuoco says [1], nice problems “allow students to concentrate on important ideas rather than messy calculations”. He calls the task of composing such problems a *meta-problem*. We continue our study of meta-problems in Part III of this article.

1. Pairs of Integer-sided Triangles with Equal Area

To exhibit pairs of non-congruent integer-sided triangles with equal perimeter is trivial; not so the task of exhibiting *pairs of non-congruent integer-sided triangles with equal area*. This problem may be encountered while teaching Heron’s formula for the area of a triangle in terms of its sides.



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Keywords

Heron’s formula, Brahmagupta–Fibonacci–Diophantus identity, two-squares theorem, infinite descent, Leonard Euler, Brahmagupta–Pell equation, cubic curve, chord-tangent construction, Vieté’s identities, law of composition, commutative group, elliptic curve, point at infinity, projective plane.

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However, the problem when stated in this form has too many degrees of freedom, because there are so many variables to play with. It becomes more interesting if we impose some additional constraints. So we pose the following problem: *How would one generate an integer-sided $\triangle OAB$ such that $OA \neq OB$, the length of AB is even, and the median OD from vertex O has integral length?* If we generate such a triangle, then $\triangle OAD$ and $\triangle OBD$ will be non-congruent and will have integer sides and equal area, and will thus answer the challenge posed above.

Let $OA = a$, $OB = b$, $AB = 2c$, $OD = d$, where a, b, c, d are positive integers; then by the theorem of Apollonius, $a^2 + b^2 = 2(c^2 + d^2)$. Expressing this in the following form,

$$a^2 + b^2 = (c + d)^2 + (c - d)^2,$$

We can proceed by looking for an integer that is a sum of two squares in two essentially different ways.

we see that we can proceed by *looking for an integer that is a sum of two squares in two essentially different ways*. The first few integers of this form are 25, 50, 65 and 85 (these are the only numbers which have the stated property and are less than 100).

However, we must impose some additional conditions on a, b, c, d ; namely, the following triples must all obey the triangle inequality:

$$(a, b, 2c), \quad (a, c, d), \quad (b, c, d).$$

Let's see what we get from $25 = 4^2 + 3^2 = 5^2 + 0^2$. The presence of 0 might bother us, but we shall see shortly what it leads to. The relations yield $(a, b, c, d) = (4, 3, 5/2, 5/2)$ which we scale up to $(8, 6, 5, 5)$. This corresponds to triangle OAB with $OA = 8$, $OB = 6$, $AB = 10$, $OD = BD = AD = 5$ (see Figure 1; the 0 has resulted in a triangle with a right angle at vertex O). So we get triangles with sides $(8, 5, 5)$ and $(6, 5, 5)$ which have equal area.

From $50 = 7^2 + 1^2 = 5^2 + 5^2$, we do not get anything of interest. If $(a, b, c, d) = (5, 5, 4, 3)$, we get an isosceles triangle (which means that triangles OAD and $OB D$ are congruent to each other), and if $(a, b, c, d) = (7, 1, 5, 0)$, we do not get a triangle at all.

More interesting is the case $65 = 7^2 + 4^2 = 8^2 + 1^2$; it yields $(a, b, c, d) = (7, 4, 9/2, 7/2)$. Scaling up, we get the 4-tuple $(14, 8, 9, 7)$.



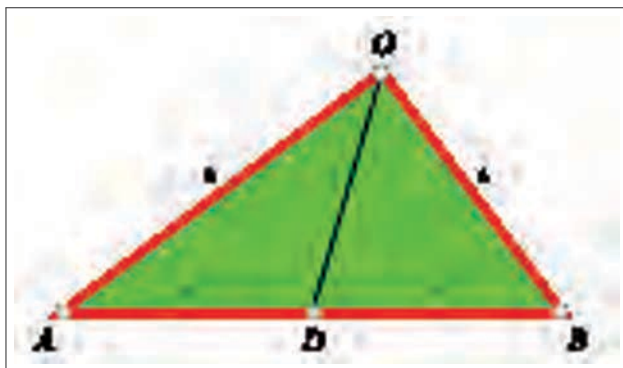
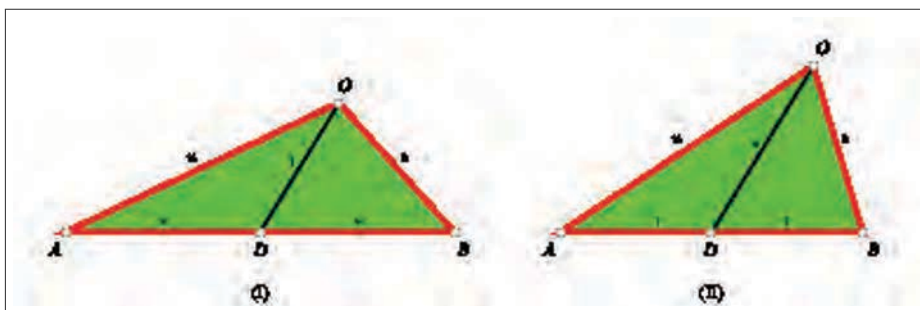


Figure 1. Generating a pair of integer-sided isosceles triangles with equal area via an integer-sided right triangle.



The resulting triangles with sides $(14, 9, 7)$ and $(8, 9, 7)$ have equal area and yield two different configurations as shown in *Figure 2*.

We can find such a solution using any two unequal primes of the form $1 \pmod{4}$; for, each such prime may be written as a sum of two squares (this is the famous ‘two-squares’ theorem of Fermat; for more on this result, see *Box 1* at the end of the article), and the product of any two such primes may be so written in two different ways, using the Brahmagupta-Fibonacci-Diophantus identity (see *Box 2* for details),

$$(p^2 + q^2) \cdot (r^2 + s^2) = (pr + qs)^2 + (ps - qr)^2 = (pr - qs)^2 + (ps + qr)^2. \quad (1)$$

For example, from $5 = 2^2 + 1^2$ and $17 = 4^2 + 1^2$ we get

$$85 = 7^2 + 6^2 = 9^2 + 2^2.$$

This yields $2(a, b, c, d) = (14, 12, 11, 7)$, and we get triangles with

Figure 2. Generating pairs of integer-sided triangles with equal area via a number that is the sum of two squares in two different ways.

Each such prime may be written as a sum of two squares (this is the famous ‘two-squares’ theorem of Fermat).



sides (14, 11, 7) and (12, 11, 7) which have equal area. (As earlier, two different configurations are possible.)

Box 1. Pierre de Fermat, the two-squares theorem, and infinite descent.

It is easy to see that a sum of two squares must be of one of the forms $0, 1, 2 \pmod{4}$. This implies that no prime of the form $3 \pmod{4}$ can be expressed as a sum of two squares. The special prime 2 can obviously be written in this form: $2 = 1^2 + 1^2$. This immediately invites an examination of the remaining primes, which are all of the form $1 \pmod{4}$.

Fermat's celebrated two-squares theorem states that every prime of the form $1 \pmod{4}$ can be written as a sum of two squares, and in just one way. For example:

$$5 = 2^2 + 1^2, \quad 13 = 3^2 + 2^2, \quad 17 = 4^2 + 1^2, \quad \dots \quad 61 = 6^2 + 5^2, \quad \dots,$$

and in each case the expression is unique.

The result was first stated by the French mathematician Albert Girard, in 1625. Fermat wrote about it to the French mathematician-priest Marin Mersenne in 1640. It is not known whether he possessed a proof of the claim. It is likely, however, that he had an idea of how the result can be proved — using the principle of infinite descent. This is a proof technique that Fermat had himself pioneered. We describe this approach here briefly.

The principle of descent can be thought of as a 'mirror image' of the principle of induction. Say we wish to prove a proposition \mathcal{P} about a subset S of the positive integers. To set up a proof by contradiction, we suppose \mathcal{P} to be false. Then there exists an element $a \in S$ for which \mathcal{P} is false. Suppose we can now prove that there must exist an element $b \in S$, with $b < a$, for which \mathcal{P} is false. If we are able to do this, then we can apply the same argument to b and deduce that there must exist an element $c \in S$, with $c < b$, for which \mathcal{P} is false. And we can continue in this manner indefinitely. But this would imply that we have a strictly decreasing infinitely long sequence of positive integers, which is clearly not possible. This contradiction would mean that our initial supposition is false, thereby proving \mathcal{P} .

Fermat had used this approach to solve some difficult problems in number theory. However, we do not know whether he was successful in proving the two-squares theorem. More than a century later, the theorem was proved by Leonard Euler using the principle of descent.

For more details, the reader could refer to [6] and [7].



Box 2. The Brahmagupta-Diophantus identity.

The identity

$$(p^2 + q^2) \cdot (r^2 + s^2) = (pr + qs)^2 + (ps - qr)^2 = (pr - qs)^2 + (ps + qr)^2 \quad (2)$$

was known to Diophantus of Alexandria. It is known as Diophantus's identity and also as the Brahmagupta-Fibonacci identity. Here is an obvious implication of the identity:

The product of two positive integers that can each be written as the sum of two squares can itself be written as the sum of two squares, and in two different ways.

For example, from $5 = 2^2 + 1^2$ and $17 = 4^2 + 1^2$ we get $85 = 7^2 + 6^2 = 9^2 + 2^2$.

Brahmagupta used a more general identity:

$$(p^2 + nq^2) \cdot (r^2 + ns^2) = (pr + nqs)^2 + n(ps - qr)^2 = (pr - nqs)^2 + n(ps + qr)^2. \quad (3)$$

This identity turns out to be indispensable in the study of the Brahmagupta-Pell equation, which was studied closely by Brahmagupta (and later by Bhaskara II; and still later, by Euler and by Lagrange).

Both the identities may be verified by direct simplification. However, how Diophantus and Brahmagupta found and proved these identities is not clear, especially if we keep in mind that symbolic algebra as we know it today did not exist in their time.

A simple way of reconstructing the Diophantus-Brahmagupta-Fibonacci identity is by using the *norm* of a complex number. Recall that the norm of the product of two complex numbers is equal to the product of the norms of the two numbers. In other words:

$$|(p + qi)| \cdot |(r + si)| = |(p + qi)(r + si)|. \quad (4)$$

On simplifying both sides of equation (4), we immediately obtain (2).

For more details, the reader could refer to [8] and [9].

2. Pairs of Isosceles Triangles with Equal Area and Equal Perimeter

The next meta-problem we study is the following, which follows from the one just studied: *How would one exhibit pairs of non-congruent, isosceles, integer-sided triangles which share equal area as well as equal perimeter?* Here's how we do this. We shall show that it is possible to exhibit any number of pairs of such triangles that share equal area and equal perimeter.



We start by scaling down so that both the triangles have semi-perimeter 1. Let the first triangle have sides $x, x, 2(1-x)$, and let the second triangle have sides $y, y, 2(1-y)$ where x and y are distinct positive rational numbers. The areas are then, respectively,

$$(1-x) \cdot \sqrt{x^2 - (1-x)^2} \quad \text{and} \quad (1-y) \cdot \sqrt{y^2 - (1-y)^2},$$

i.e.,

$$(1-x) \cdot \sqrt{2x-1} \quad \text{and} \quad (1-y) \cdot \sqrt{2y-1}.$$

Since the areas are equal, $(1-x)^2 \cdot (2x-1) - (1-y)^2 \cdot (2y-1) = 0$. Expanding the expression and factorizing we get

$$(x-y)(2x^2 + 2y^2 + 2xy - 5x - 5y + 4) = 0,$$

and since $x \neq y$,

$$2x^2 + 2y^2 + 2xy - 5x - 5y + 4 = 0.$$

We must find pairs (x, y) of unequal rational numbers satisfying this relation. Hence we must find rational points on the conic section K with equation $2x^2 + 2y^2 + 2xy - 5x - 5y + 4 = 0$, and they must not lie on the line $y = x$ (see *Figure 3*).

An easily-found rational point on K is $P = (1, 1)$. Let ℓ be a line through P with slope t where t is a rational number. The second point of intersection of ℓ with K will then be a rational point. Solving the equations, we find that this point is $Q(t)$ where

$$Q(t) = \left(\frac{2t^2 + t + 1}{2(t^2 + t + 1)}, \frac{t^2 + t + 2}{2(t^2 + t + 1)} \right). \quad (5)$$

This parametrization gives a rational point on K for every rational value of t (with $t \neq 1$).

Examples

1. Let $t = 1/2$. We get $Q = (4/7, 11/14)$. Hence the triangles with sides $4/7, 4/7, 6/7$ and $11/14, 11/14, 6/14$ have equal perimeter and equal area. (To see where the length of the base comes from, recall that the semi-perimeter is 1.) Scaling up, we deduce that the isosceles triangles with sides $8, 8, 12$ and $11, 11, 6$ have equal perimeter and equal area.



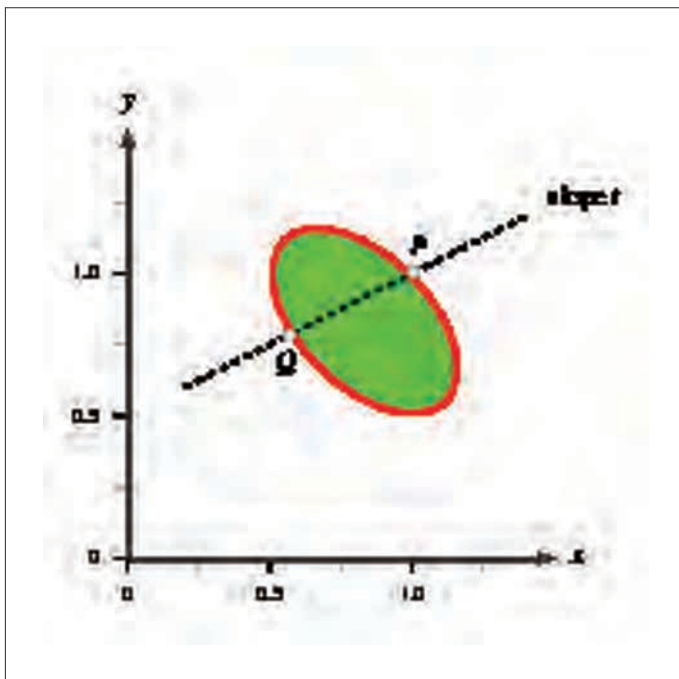


Figure 3. The ellipse $K : 2x^2 + 2y^2 + 2xy - 5x - 5y + 4 = 0$, and a line with rational slope t through the point P with coordinates $(1, 1)$.

2. Let $t = 1/3$. We get $Q = (7/13, 11/13)$. So the triangles with sides $7/13, 7/13, 12/13$ and $11/13, 11/13, 4/13$ have equal perimeter and equal area. Scaling up, we deduce that the isosceles triangles with sides $7, 7, 12$ and $11, 11, 4$ have equal perimeter and equal area.
3. Let $t = 1/4$. We get $Q = (11/21, 37/42)$. Scaling up, we deduce that the isosceles triangles with sides $22, 22, 40$ and $37, 37, 10$ have equal perimeter and equal area.

3. Pairs of Non-congruent, Non-isosceles, Integer-sided Triangles with Equal Area and Equal Perimeter

The last meta-problem we study is the following: *How would one exhibit pairs of non-congruent, non-isosceles, integer-sided triangles which share equal area as well as equal perimeter?* This may seem impossible, there being too many constraints—but it can be done. Here's how we approach the problem. As earlier, we do not attempt to solve the problem in full generality but only



offer an approach that generates such pairs of triangles. For an earlier account of this approach, see [3].

There is no loss in replacing the word ‘integer’ by ‘rational number’; for, if we have such a pair of triangles whose sides are rational numbers, then we can scale everything up suitably to obtain a pair of integer-sided triangles with the required property.

If two triangles with sides a, b, c and a', b', c' respectively have equal perimeter ($2s$) and equal area, then $a+b+c = 2s = a'+b'+c'$ and $\sqrt{s(s-a)(s-b)(s-c)} = \sqrt{s(s-a')(s-b')(s-c')}$, which yields

$$(s-a)(s-b)(s-c) = (s-a')(s-b')(s-c'). \quad (6)$$

Now note that

$$(s-a) + (s-b) + (s-c) = s = (s-a') + (s-b') + (s-c').$$

From the above equalities, we see that the triples $(s-a, s-b, s-c)$ and $(s-a', s-b', s-c')$ have the same sum (equal to the semi-perimeter of the triangle) and the same product (equal to the square of the area divided by the semi-perimeter of the triangle). Also, all six quantities in these triples are positive numbers (this follows from the triangle inequality).

So our task reduces to the following: *Find two unequal triples of distinct, positive rational numbers which share the same sum and the same product.*

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The shared sum of the two triples is s ; let the shared product be p . Both s and p are positive rational numbers (fixed to start with). We want to find two unequal triples (x, y, z) of distinct positive rational numbers such that

$$\begin{cases} x + y + z = s, \\ xyz = p. \end{cases} \quad (7)$$

If we treat z as a parameter, then the equations $x + y = s - z$, $xy = p/z$ yield a quadratic equation whose solutions are x, y . The discriminant of this quadratic is

$$(s-z)^2 - \frac{4p}{z}. \quad (8)$$



For given rational values of s and p , we require a rational value for z such that the above expression is the square of a rational number. Denoting this quantity by u^2 , we have the relation

$$u^2 z = z(s - z)^2 - 4p. \quad (9)$$

In (z, u) space, (9) defines a *cubic curve*. We require rational points on this curve.

This kind of number-theoretic problem was first studied by Fermat, and he came up with an elegant and insightful idea to generate such points: the *chord-tangent construction*. Using this approach, we may generate desired solutions.

Example

Consider the triangle with sides $(a, b, c) = (5, 6, 7)$. Note that it is not isosceles. Its semi-perimeter is $s = 9$, so $(s - a, s - b, s - c) = (4, 3, 2)$. We now look for a second triple of distinct positive rational numbers with the same sum and the same product as the triple $(4, 3, 2)$. Here the shared sum is $s = 9$, and the shared product is $p = 2 \cdot 3 \cdot 4 = 24$. Therefore the equation of the cubic curve in (u, z) space is

$$u^2 z = z(9 - z)^2 - 96. \quad (10)$$

Figure 4 displays a sketch of the curve. It is symmetric in the z -axis by virtue of the u^2 term. (We have only shown the portion of the curve lying in the first and fourth quadrants. There are branches in the second and third quadrants which we have not shown as they are not of relevance to us.)

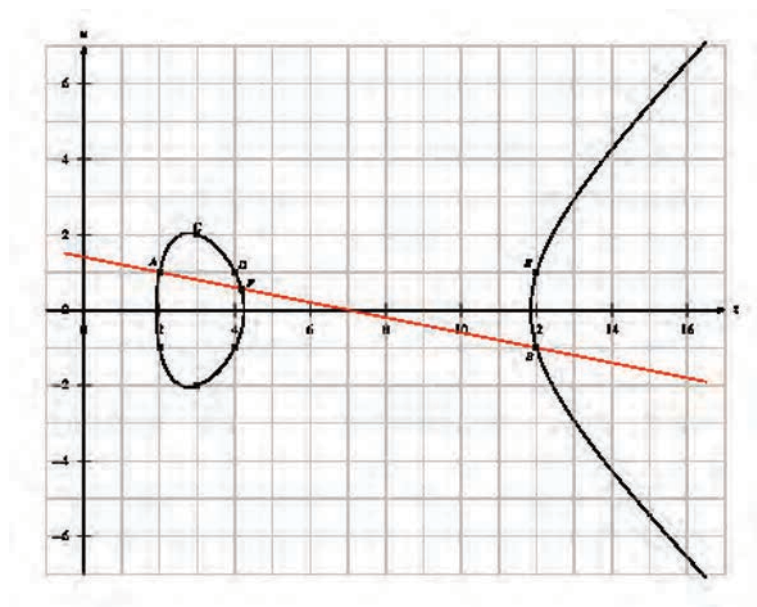
Our interest is in finding rational points on this curve. There are some visible candidates which are of no use to us; namely, the points corresponding to $z = 2, 3, 4$, as these correspond to the original triangle itself. These are the points A, C, D with coordinates $(2, 1)$, $(3, 2)$, $(4, 1)$ respectively (and their images under reflection in the z -axis).

The graph shows another rational point: the point E with coordinates $(12, 1)$. It too is of no use to us—for a different reason,

This kind of number-theoretic problem was first studied by Fermat, and he came up with an elegant and insightful idea to generate such points: the chord-tangent construction.



Figure 4. Graph of $u^2z = z(9 - z)^2 - 96$. Rational points $A = (2, 1)$, $B = (12, -1)$, $C = (3, 2)$, $D = (4, 1)$, $E = (12, 1)$, and $F = (25/6, 17/30)$ have been shown.



however: it corresponds to a triangle with negative sides! Indeed, the sides of the triangle turn out to be 10, 11, -3 , so it surely lives in an alternate universe! The reader may check that this ghostly triangle has the same semi-perimeter ($= 9$) and the same area ($= 6\sqrt{6}$) as the triangle with sides 5, 6, 7.

The Chord-tangent Construction

We now draw upon a powerful principle pioneered by Fermat. The underlying principle is that *for a cubic equation with rational coefficients, if any two roots are rational numbers, then the third root too is a rational number*. This follows from Viète's identities for the sum and product of the roots ("for the cubic equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, the sum of the roots is $-b/a$ and the product of the roots is d/a "). From this follows: *If A and B are two rational points on a cubic curve with rational coefficients, then the line AB intersects the curve in a rational point C*. (The possibility that these points are not distinct is permitted. If A and B coincide, then AB is the tangent to the curve at that point, and if that point also happens to be a point of inflection of

For a cubic equation with rational coefficients, if any two roots are rational numbers, then the third root too is a rational number.



the curve, then C coincides with A and B .)

We put this principle to use with the above curve. A study of the graph suggests that a promising choice of points might be $A = (2, 1)$ and $B = (12, -1)$, because AB intersects the curve again on the oval where A is located. (Another promising choice: A and D .) Line AB has equation $u = (7 - z)/5$. By solving the appropriate pair of equations, we find that the line intersects the curve again at the rational point $F = (25/6, 17/30)$. We therefore put $z = 25/6$ and find that $x = 32/15$, $y = 27/10$. Therefore the following sets of numbers,

$$\{2, 3, 4\}, \quad \left\{ \frac{32}{15}, \frac{27}{10}, \frac{25}{6} \right\},$$

share the same sum ($= 9$) and the same product ($= 24$).

As the lcm of the denominators 15, 10, 6 is 30, we scale up all the numbers by 30, and deduce that $\{60, 90, 120\}$ and $\{64, 81, 125\}$ share the same sum and the same product. The common sum is 270, so it follows that the triangles with sides $\{210, 180, 150\}$ and $\{206, 189, 145\}$ share the same perimeter ($= 540$) and the same area ($= 5400\sqrt{6}$). Note that these two triangles are non-isosceles and not similar to each other.

We get greedy now, and ask for a set of *three* non-congruent, non-isosceles integer-sided triangles with equal area and equal perimeter. Examining the graph again, we see that another rational point can be located on the curve by using E and the newly generated point F . The line EF has equation $u - 1 = 13(z - 12)/235$. By solving an appropriate pair of equations, we find that this line intersects the curve again at the rational point $G = (2209/1147, 23866/53909)$. We therefore put $z = 2209/1147$ and find that $x = 5476/1457$, $y = 5766/1739$. Therefore the following three sets of numbers,

$$\{2, 3, 4\}, \quad \left\{ \frac{32}{15}, \frac{27}{10}, \frac{25}{6} \right\}, \quad \left\{ \frac{5476}{1457}, \frac{5766}{1739}, \frac{2209}{1147} \right\},$$

share the same sum ($= 9$) and the same product ($= 24$). As the lcm of the denominators in the above fractions is 1617270,

If A and B are two rational points on a cubic curve with rational coefficients, then the line AB intersects the curve in a rational point C .



we scale up by this factor, and deduce that the following sets of numbers,

$$\{3234540, 4851810, 6469080\}, \{3450176, 4366629, 6738625\}, \\ \{6078360, 5362380, 3114690\},$$

share the same sum and the same product. The common sum is 14555430, so it follows that the triangles with the following sides,

$$\left\{ \begin{array}{l} \{11320890, 9703620, 8086350\}, \\ \{11105254, 10188801, 7816805\}, \\ \{8477070, 9193050, 11440740\}, \end{array} \right.$$

share the same perimeter ($= 29110860$) and the same area ($= 15693373517400\sqrt{6}$). We may check that these triangles are not similar to each other. And, of course, they are not isosceles.

Continuing, it is plausible to conclude that for any positive integer n , however large, there exist n non-congruent, non-isosceles, integer-sided triangles with equal area and equal perimeter.

Continuing, it is plausible to conclude that for any positive integer n , however large, there exist n non-congruent, non-isosceles, integer-sided triangles with equal area and equal perimeter.

Other Treatments of the Problem

For other treatments of this problem, see [4] and [5]. For another treatment of the isosceles triangles case, see [2]; the authors show that it is not possible to find more than two non-congruent, isosceles, integer-sided triangles that share the same area and the same perimeter.

4. Closing Remarks

In this three-part series of articles, we have looked at a number of problems that originate in Euclidean geometry but give rise in a natural way to problems in number theory; specifically, to problems where positive integer (or rational) solutions are sought to algebraic equations. In many cases, these equations give rise to second-degree curves (i.e., conic sections), and the problem thus



devolves to finding rational points on some associated second-degree curves. Thus, problems originating in Euclidean geometry have given rise in a natural way to problems in arithmetic geometry.

It is possible to define a law of composition on the set of points of a second-degree curve, thereby giving rise to a commutative group (this theme has been explored in the literature; e.g., see [10]). This enables us to define a law of composition on the set of solutions of the problem. Algebraic considerations thus enter the picture.

In the problem studied in Section III (above), we encountered a third-degree curve (i.e., a cubic curve). Such curves belong to an important category known as *elliptic curves*. It turns out that a law of composition can be defined on the set of points of such a curve, and if we include the ‘point at infinity’ in the set of solutions (which means that we are considering the curve over the projective plane and not the usual Cartesian plane), then we obtain a commutative group. Though we did not refer to this in the solution, it is made use of implicitly in the chord-tangent construction we described. These curves provide a rich field of study. For example, the proof of Fermat’s Last Theorem by Andrew Wiles draws deeply on the study of elliptic curves. For more on this subject, the reader could refer to [11].

These interconnections between Euclidean geometry and arithmetic algebraic geometry illustrate beautifully the deep unity within mathematics, and the richness of cross-disciplinary connections.

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Suggested Reading

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