Creating ‘Nice’ Problems in Elementary Mathematics – II*

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We consider here the ‘meta-problem’ of creating ‘nice’ problems in elementary mathematics, ‘nice’ being defined as a problem in which the input data, as well as the answers, are rational numbers. The classical example of this is that of generating Pythagorean triples. Many examples of this kind arise when we study Euclidean geometry and the theory of equations. We consider a few problems of this genre.

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Introduction

In Part I we defined a ‘nice’ problem as one *for which the input data and answer are rational numbers*. (Conversely, problems in which either the input data or the answer involve irrational numbers are considered ‘not nice’.) Anyone who has taught high school mathematics will routinely have had to compose nice problems; for, as Cuoco says [1], nice problems “allow students to concentrate on important ideas rather than messy calculations”. He calls the task of composing such problems a *meta-problem*. We continue our study of meta-problems in Part II of this article.

1. Lattice Triangles

In the coordinate plane, how would one generate a lattice $\triangle OAB$ whose sides OA, OB and AB have integral lengths? (A ‘lattice triangle’ or more generally a ‘lattice polygon’ is one whose vertices have integral coordinates.) This problem arises naturally when one is teaching the distance formula in coordinate geometry.

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Nice problems "allow students to concentrate on important ideas rather than messy calculations".

As the stated property is invariant under translation through a vector with integer coordinates, we may assume with no loss of generality that O lies at the origin. We now embed the triangle in the complex plane, $\mathbb{C}$. Let a and b be the complex numbers corresponding to points A and B, respectively. The problem now is:

Find pairs $a, b \in \mathbb{Z}[i]$ such that $|a|$, $|b|$ and $|a-b|$ are integers.

(Here $\mathbb{Z}[i]$ is the ring of Gaussian integers.) Rather than solve the problem in the most general way possible, we consider an approach that enables us to generate infinitely many solutions.

Here is how we proceed. *Let us choose a and b so that they are squares of Gaussian integers, say $a = \alpha^2$ and $b = \beta^2$, and in such a way that $\alpha^2 - \beta^2$ is itself the square of a Gaussian integer, say $\alpha^2 - \beta^2 = \gamma^2$ with $\alpha, \beta, \gamma \in \mathbb{Z}[i]$.* Then it will naturally happen that the norms of a, b and $a-b$ are integers. So let us look for a way to generate $\alpha, \beta, \gamma \in \mathbb{Z}[i]$ such that $\alpha^2 - \beta^2 = \gamma^2$.

Now the corresponding problem over the integers is very well-known—it is the familiar problem of generating Pythagorean triples. The method generally followed to find integer triples (α, β, γ) satisfying the equation $\alpha^2 = \beta^2 + \gamma^2$ is to choose arbitrary integers c, d and then let

$$\beta = c^2 - d^2, \qquad \gamma = 2cd, \qquad \alpha = c^2 + d^2.$$

(Common factors can be factored out if so desired, giving us a primitive solution. Or we can insist at the start that c and d are coprime and have opposite parity.) Very conveniently for us, the same prescription works for solutions in $\mathbb{Z}[i]$. (While this may seem a great piece of luck, it is actually not such a great mystery; it happens because the domain $\mathbb{Z}[i]$ has unique factorization.)

In the coordinate plane, how would one generate a lattice $\triangle OAB$ whose sides OA, OB and AB have integral lengths?

So for the lattice point problem, the prescription reduces to this: *Pick any two Gaussian integers c and d, and let*

$$\alpha = c^2 + d^2, \qquad \beta = c^2 - d^2.$$

Then the triangle with vertices 0, α^2 and β^2 has the desired property. Common factors can be factored out if needed, giving a primitive solution.

Examples

The stated property is invariant under translation through a vector with integer coordinates.

1. Let $c = 2 + i$, $d = 1 - i$; then we get

$$\alpha = 3 + 2i, \quad \beta = 3 + 6i, \quad \alpha^2 = 5 + 12i, \quad \beta^2 = -27 + 36i,$$

and we get our first such triangle, with vertices $(0, 0)$, $(5, 12)$ and $(-27, 36)$. Its sides have lengths 13, 40, 45.

2. Let $c = 4 + i$, $d = 1 + i$; then we get

$$\alpha = 15+10i, \quad \beta = 15+6i, \quad \alpha^2 = 125+300i, \quad \beta^2 = 189+180i,$$

and we get another such triangle, with vertices $(0, 0)$, $(125, 300)$ and $(189, 180)$. Its sides have lengths 136, 261, 325.

2. Lattice Triangles and the Theorem of Apollonius

How would one generate a lattice $\triangle OAB$ for which the sides OA and OB and the median OD from vertex O have integral lengths? This problem arises naturally when teaching the theorem of Apollonius. (The theorem states that for any triangle OAB, if D is the midpoint of side AB, i.e., if OD is a median through O, then $OA^2 + OB^2 = 2(OD^2 + AD^2)$.)

Taking vertex O to be the origin (as above), we seek lattice points A, B such that OA, OB and OD are integers, where $D = (A+B)/2$. Embedding the problem in the complex plane, this amounts to seeking $a, b \in \mathbb{Z}[i]$ such that $|a|$, $|b|$ and $|\frac{1}{2}(a + b)|$ are integers. These conditions will be met if we put $a = \alpha^2$ and $b = \beta^2$ where $\alpha, \beta \in \mathbb{Z}[i]$ are chosen so that $\alpha^2 + \beta^2 = 2\gamma^2$ with $\gamma \in \mathbb{Z}[i]$. If we write $x = \alpha/\gamma$ and $y = \beta/\gamma$, then $x, y \in \mathbb{Q}(i)$, and $x^2 + y^2 = 2$. This is the equation that we must solve.

Rather than solve the problem in the most general way possible, we consider an approach that enables us to generate infinitely many solutions.

Let us now recall how we solve this equation over $\mathbb{Q}$ alone. The equation $x^2 + y^2 = 2$ yields a circle C with radius $\sqrt{2}$, centred at $(0, 0)$, and our task is to find the rational points on C. One such point is $P(-1, -1)$. Let ℓ be a line with rational slope t passing through P; it intersects C at another rational point Q. By allowing t to take all rational values we get all possible rational points $Q \in$

The corresponding problem over the integers is very well-known.

C. The equation of ℓ is $y = tx + t - 1$. Substituting for y in $x^2 + y^2 = 2$, we get the quadratic equation

$$x^2(t^2 + 1) + 2tx(t - 1) + t^2 - 2t - 1 = 0, \tag{1}$$

one of whose roots is $x = -1$. Using the formula for the product of the roots, we find the other root and hence the coordinates of Q:

$$Q = \left(-\frac{t^2 - 2t - 1}{t^2 + 1}, \frac{t^2 + 2t - 1}{t^2 + 1}\right). \tag{2}$$

Now we can generate as many rational points $Q \in \mathsf{C}$ as we wish. (But note that we miss the point $(-1, -1)$ for which we need $t \to \infty$.)

As in the case of the Pythagorean equation, this approach works for generating solutions over $\mathbb{Q}[i]$ as well. The only difference is that we permit t to take values from $\mathbb{Q}(i)$ rather than $\mathbb{Q}$.

Examples

1. Let $t = 1 + i$; we get

   $$x = \frac{3 - 6i}{5}, \quad y = \frac{9 + 2i}{5}, \quad x^2 = \frac{-27 - 36i}{25}, \quad y^2 = \frac{77 + 36i}{25},$$

   giving us the solution $A = (-27, -36)$, $B = (77, 36)$. *Check.* $OA = 45$, $OB = 85$, $D = (25, 0)$, $OD = 25$.

2. Let $t = 3 + i$; we get

   $$x = \frac{-11 - 10i}{39}, \quad y = \frac{55 - 2i}{39}, \quad x^2 = \frac{21 + 220i}{1521}, \quad y^2 = \frac{3021 - 220i}{1521},$$

   giving us the solution $A = (21, 220)$, $B = (3021, -220)$. *Check.* $OA = 221$, $OB = 3029$, $D = (1521, 0)$, $OD = 1521$.

The domain $\mathbb{Z}[i]$ has unique factorization.

3. Nice Quadratics

Consider the quadratic functions $x^2 + 5x + 6$ and $x^2 + 5x - 6$; their four roots are distinct integers (namely: $\{-2, -3\}$ and $\{1, -6\}$ respectively). *How would one generate ordered pairs* (a, b) *of* *integers such that the four roots of the quadratics $x^2 + ax + b$ and $x^2 + ax - b$ are distinct integers?*

Let $a, b \in \mathbb{Z}$ be such that all the roots of $x^2 + ax + b = 0$ and $x^2 + ax - b = 0$ are distinct integers. For this to happen it is necessary and sufficient that $b \neq 0$ and $a^2 - 4b$ and $a^2 + 4b$ are non-zero perfect squares, say $a^2 - 4b = c^2$ and $a^2 + 4b = d^2$ where $c, d \in \mathbb{Z}$, $c \neq 0$, $d \neq 0$. But this means that the squares c^2, a^2, d^2 form a three-term arithmetic progression whose common difference is a multiple of 4. Hence if we find a way of generating triples of distinct squares in arithmetic progression, our problem will be solved. (*Remark.* The condition that the common difference must be $0 \pmod 4$ is superfluous; for c, a, d must clearly have the same parity.)

Let $x = c/a$ and $y = d/a$; then since $c^2 + d^2 = 2a^2$, we get $x^2 + y^2 = 2$. But this is just the equation that we solved in Section II over $\mathbb{Q}$. The parametrization obtained earlier works here just as well, and we have the solution

$$x = -\frac{t^2 - 2t - 1}{t^2 + 1}, \quad y = \frac{t^2 + 2t - 1}{t^2 + 1}, \quad \text{where } t \in \mathbb{Q}. \tag{3}$$

Examples

1. Let $t = 3/2$; then $Q = (7/13, 17/13)$, corresponding to the triple $7^2, 13^2, 17^2$. This yields $a = 13$ and $b = \left(13^2 - 7^2\right)/4 = 30$, and these yield the quadratics $x^2 + 13x \pm 30$.

2. Let $t = 5/3$; then $Q = (7/17, 23/17)$, corresponding to the triple $7^2, 17^2, 23^2$. This yields $a = 17$ and $b = \left(17^2 - 7^2\right)/4 = 60$, and these yield the quadratics $x^2 + 17x \pm 60$.

Remark

If we push this investigation further we find a nice number theoretic connection.

Let **A** be the set of pairs of positive integers (a, b) such that both $a^2 - 4b$ and $a^2 + 4b$ are perfect squares. For example, $(5, 6)$ and

$(13, 30)$ are elements of $\mathbf{A}$. Clearly, if $(a, b) \in \mathbf{A}$ then $(ka, k^2 b) \in \mathbf{A}$ for any $k \in \mathbb{N}$. Define a partial order $\prec$ on $\mathbf{A}$, thus: if (a, b) and (a', b') are distinct elements of $\mathbf{A}$, and $a' = ka$ and $b' = k^2 a$ for some $k \in \mathbb{N}$, then $(a, b) \prec (a', b')$. It is of interest to find the minimal elements of $\mathbf{A}$ defined by this partial order. Here is a list of some minimal elements of $\mathbf{A}$:

(5, 6),	(13, 30),	(17, 60),	(25, 84),	(29, 210),
(37, 210),	(41, 180),	(53, 630),	(61, 330),	(65, 504),
(65, 924),	(73, 1320),	(85, 546),	(89, 1560),	(97, 2340),
(101, 990),	(109, 2730),	(113, 840),	(125, 2574)	....

If we list just the first elements of these pairs (i.e., only the 'a' component), we get the set

$$\{5,\ 13,\ 17,\ 25,\ 29,\ 37,\ 41,\ 53,\ 61,\ 65,\ 73,\ 85,\ 89,\ 97,\ 101,\ 109, 113,\ 125,\ \ldots\},$$

and this is precisely the set of $n \in \mathbb{N}$ all of whose prime factors are the form $1 \pmod 4$. A pretty connection! It owes to the fact that if $t = m/n$, where m, n are coprime, then the corresponding pair is $(a, b) = \left(m^2 + n^2,\ mn(m^2 - n^2)\right)$, showing that a is a sum of two squares; and it is well-known that the odd factors of a sum of two coprime squares are all of the form $1 \pmod 4$.

4. Nice Cubics

How would one generate a cubic polynomial $f(x)$ with integer coefficients, having integral zeros, integral extrema and an integral inflection point, these integers all being distinct? This task is encountered by the teacher while teaching the topics of turning points and points of inflection. This problem has been treated earlier in the literature; see, e.g., [2] and [3]. However, the problem is approached differently here.

This task is encountered by the teacher while teaching the topics of turning points and points of inflection.

The distinctness condition has been put in only to avoid the trivial instance $f(x) = x^3$.

We need a cubic polynomial $f(x) \in \mathbb{Z}[x]$ such that the roots of $f(x)$, $f'(x)$ and $f''(x)$ are six distinct integers. Let the roots of

$f(x)$ be $a, b, c \in \mathbb{Z}$. We may then suppose that f is monic, with:

$$\begin{cases} f(x) &= (x-a)\cdot(x-b)\cdot(x-c), \\ f'(x) &= 3x^2 - 2(a+b+c)x + (ab+bc+ca), \\ f''(x) &= 6x - 2(a+b+c). \end{cases} \tag{4}$$

We can ensure that $f''(x)$ has an integral root by requiring that $a+b+c$ is a multiple of 3.

But if this requirement has been met, then we can consider in place of $f(x)$ the translated polynomial

$$g(x) = f\left(x + \frac{a+b+c}{3}\right), \tag{5}$$

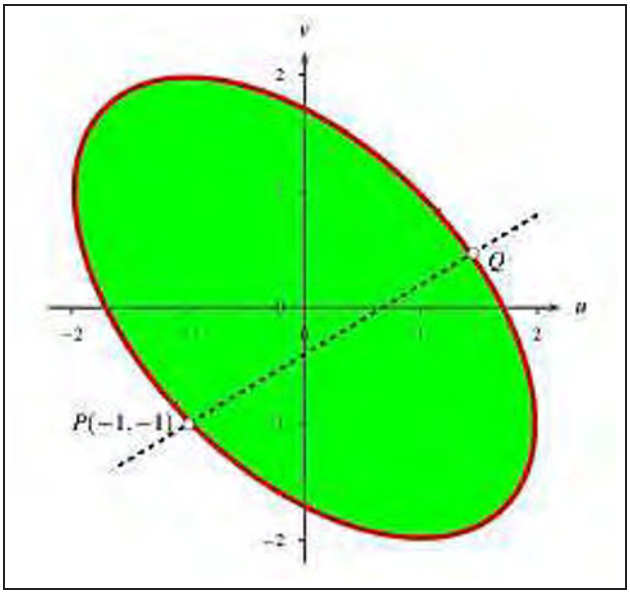
and this will have all the properties we require of f. Since the sum of the roots of g is 0, there is no loss of generality in assuming that this is true of f as well, i.e., in assuming that $a+b+c=0$, and we shall do so in the subsequent analysis. Since $c = -(a+b)$, we have:

$$\begin{cases} f(x) &= (x-a)\cdot(x-b)\cdot(x+a+b), \\ f'(x) &= 3x^2 - \left(a^2+ab+b^2\right), \\ f''(x) &= 6x. \end{cases} \tag{6}$$

To ensure that $f'(x)$ has integer roots, we must choose $a, b \in \mathbb{Z}$ so that $a^2+ab+b^2 = 3d^2$ with $d \in \mathbb{Z}$. To ensure distinctness, we must have $a \neq 0$, $b \neq 0$, $a^2 \neq b^2$, $d \neq 0$. (The last condition follows from the first two conditions, so we need not list it.)

We shall generate primitive solutions to $a^2+ab+b^2 = 3d^2$ using the approach followed earlier, in Section II. Let $u = a/d$ and $v = b/d$; then $u^2+uv+v^2 = 3$, which defines an ellipse E in (u, v)-space (see *Figure* 1). Our interest is in the rational points on E. One such point is $P(-1,-1)$. Let ℓ be any line through P, with rational slope t. As earlier, ℓ must intersect E again at a rational point Q (possibly P itself). Hence by solving a pair of simultaneous equations we obtain all the rational points on E in

Figure 1. The ellipse E : $u^2 + uv + v^2 = 3$, and a line through $P(-1, -1)$.



parametric form. (We miss only the point $(-1, 2)$, for which we must let $t \to \infty$.)

The equation of line ℓ is $v+1 = t(u+1)$. Substituting $v = t(u+1)-1$ into the equation of the curve ($u^2 + uv + v^2 = 3$), we get the quadratic equation

$$u^2(t^2 + t + 1) + u(2t^2 - t - 1) + t^2 - 2t - 2 = 0, \tag{7}$$

one of whose roots is $u = -1$. Solving for the other root we get the coordinates of Q:

$$Q = \left(-\frac{t^2 - 2t - 2}{t^2 + t + 1} \frac{2t^2 + 2t - 1}{t^2 + t + 1}\right). \tag{8}$$

We can use this formula to generate cubics having the desired property.

Examples

1. $t = 3/2$ yields $Q = (11/19, 26/19)$; we get the solution $(a, b, d) = (11, 26, 19)$ to the equation $a^2 + ab + d^2 = 3d^2$, and hence the following cubic:

$$f(x) = (x - 11) \cdot (x - 26) \cdot (x + 37) = x^3 - 1083x + 10582.$$

The sets of roots of $f(x)$, $f'(x)$, $f''(x)$ are, respectively: $\{11, 26, -37\}$, $\{19, -19\}$, $\{0\}$. We see that the roots are distinct, as desired.

2. $t = -5/4$ yields $Q = (-11/7, -2/7)$; we get the solution $(a, b, d) = (-11, -2, 7)$ to the equation $a^2 + ab + d^2 = 3d^2$, and hence the following cubic:

$$f(x) = (x + 11) \cdot (x + 2) \cdot (x - 13) = x^3 - 147x - 286.$$

The sets of roots of $f(x)$, $f'(x)$, $f''(x)$ are: $\{-11, -2, 13\}$, $\{7, -7\}$, $\{0\}$.

The conditions to be imposed on t to ensure distinctness of the numbers $\{a, b, -a - b, 0\}$ may be shown to be: $t \notin \{\pm 1, 0, 3/2\}$.

. . . *And What About Nice Quartics?*

Having explored 'nice quadratics' and 'nice cubics' one naturally wants to explore 'nice quartics' and so one asks: *How would one generate a quartic polynomial f with integer coefficients, such that the roots of f, f', f'' and f''' are distinct integers?* But this question appears to be difficult. (The difficulty arises from having to ensure that the roots of the cubic polynomial f' too are distinct integers.) If we drop the condition of distinctness, then an example of such a quartic is $f(x) = x^4 - 6x^2 + 8x - 3$. (Here are the roots of f are $-3, 1, 1, 1$; the roots of f' are $-2, 1, 1$; the roots of f'' are $-1, 1$; and the root of f''' is 0.) Perhaps some reader can carry forward this investigation.

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Suggested Reading

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