Analysis for Statistics ❦

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Abstract

This is an introduction to analytical statistics.

Contents

1 Background

1.1 Sets

Definition

Sets are an *unordered* collection of elements.

$$\{1,2,3\} = \{3,2,1\} = \{2,3,1\}$$
$$(1,2) \neq (2,1)$$

Definition

Let $A, B \subseteq \Omega$, then

$$\begin{aligned} A \setminus B &= A - B \\ &= A \cap B^c \end{aligned}$$

Example 1.1. Let $\Omega = \{1,2,3,4,5\}$, $A = \{1,2\}$, $B = \{2,3,4\}$, then:

$$\begin{aligned} B^c &= \{1,5\} \\ A - B &= \{1,2\} - \{2,3,4\} = \{1\} \\ A \cap B^c &= \{1,2\} \cap \{1,5\} = \{1\} \end{aligned}$$

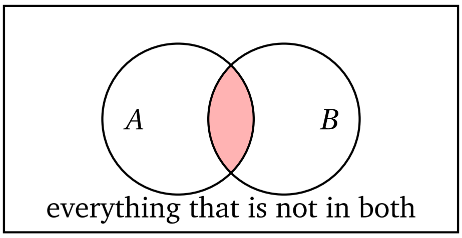
Definition

$A^c = \{x \in \Omega : x \notin A\}$

Definition

The symmetric difference between two sets A and B is:

$$A \Delta B = \{x : x \in A \setminus B \text{ or } x \in B \setminus A\}$$



Definition

The power set of a set A is the set of all subsets of A. This includes $\emptyset$ and A. It is denoted as 2^A.

Proposition

If A is a finite set, then:

$$|2^A| = 2^{|A|}$$

Example 1.2. What is the power set for $A = \{1, 2, 3\}$?

$$2^{\{1,2,3\}} = 2^{|A|} = 2^3 = 8$$

$$\mathscr{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$$

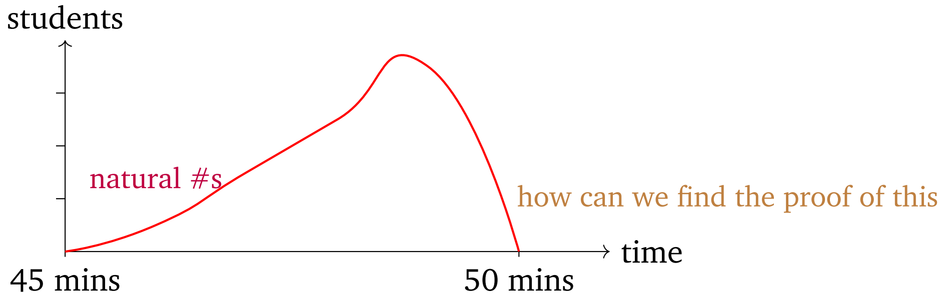
Note

The power set of the natural numbers $\mathbb{N}$ has cardinality:

$$|2^{\mathbb{N}}| = 2^{|\mathbb{N}|} = \mathfrak{c}, \quad \text{not countable}$$

Motivation

We may want to extend our powersets beyond natural numbers. For example, suppose we want the probability of students turning in exams. Most may turn in their exams within the last 5 minutes.



$$|2^{\mathbb{R}}| = 2^{|\mathbb{R}|} = \mathfrak{c}_3, \quad |\mathbb{N}| = \aleph_0$$

Proposition

$$S \setminus \bigcup_{\alpha \in \mathbb{Q}} A_\alpha = \bigcap_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$$

Proof. Let $x \in S \setminus \bigcup_{\alpha \in \mathbb{Q}} A_\alpha$. Then $x \notin A_\alpha$ for all α, so

$$x \in \bigcap_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$$

Conversely, let $y \in \bigcap_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$. Then $y \in S$ and $y \notin A_\alpha$ for all α, hence

$$y \in S \setminus \bigcup_{\alpha \in \mathbb{Q}} A_\alpha$$

$\square$

Proposition

$$S \setminus \bigcap_{\alpha \in \mathbb{Q}} A_\alpha = \bigcup_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$$

Proof. If $x \in S \setminus \bigcap_{\alpha \in \mathbb{Q}} A_\alpha$, then $x \notin A_\alpha$ for some α, so $x \in S \setminus A_\alpha$ and therefore

$$x \in \bigcup_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$$

Conversely, if $y \in \bigcup_{\alpha \in \mathbb{Q}} (S \setminus A_\alpha)$, then $y \in S \setminus A_\alpha$ for some α, so

$$y \notin \bigcap_{\alpha \in \mathbb{Q}} A_\alpha \Rightarrow y \in S \setminus \bigcap_{\alpha \in \mathbb{Q}} A_\alpha$$

□

Definition

The sets are **mutually disjoint** if

$$\bigcap_{\alpha \in A} A_\alpha = \emptyset$$

Example 1.3. Let $A = \{1,2\},\ B = \{2,3\},\ C = \{3,4\}$. Then

$$A \cap B \cap C = \emptyset$$

A, B, C are mutually disjoint, **not pairwise disjoint**.

Definition

The sets U_α for $\alpha \in A$ are **pairwise disjoint** if

$$\alpha \neq \beta \Rightarrow U_\alpha \cap U_\beta = \emptyset$$

Example 1.4. Let $A = \{1,2\},\ B = \{3,4\},\ C = \{5,6\}$. These are pairwise disjoint.

Example 1.5. Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3,4\}$. These are **not** pairwise disjoint since:

$$A \cap B = \{2\} \neq \emptyset, \quad A \cap C = \emptyset, \quad B \cap C = \{3\} \neq \emptyset$$

Proposition

If sets are pairwise disjoint, then they are mutually disjoint. However, the converse is false: mutually disjoint $\not\Rightarrow$ pairwise disjoint.

Proposition A.1

Let $A, B, C \subseteq \Omega$

2. Double Complement Identity

$$(A^c)^c = A \quad \text{where} \quad A^c = \{x \in \Omega \mid x \notin A\}$$

Proof. Let $x \in (A^c)^c$ then $x \notin A^c$ so $x \in A$ hence $(A^c)^c \subseteq A$. Now let $x \in A$ then $x \notin A^c$ so $x \in (A^c)^c$ hence $A \subseteq (A^c)^c$. Therefore $A = (A^c)^c$ □

3. Empty Set Subset Property

$$\emptyset \subseteq A$$

Proof. It holds vacuously that for all $x \in \emptyset$ we have $x \in A$. This is true since nothing is in $\emptyset$ so everything must be true. □

Proof. Assume by contradiction that $\emptyset \not\subseteq A$. Then there exists an element in $\emptyset$ that is not in A. This contradicts the definition of the empty set. Hence $\emptyset \subseteq A$ □

Corollary (De Morgan's Laws)

$$\left(\bigcup_{\alpha\in\mathbb{Q}} A_\alpha\right)^c = \bigcap_{\alpha\in\mathbb{Q}} A_\alpha^c$$
$$\left(\bigcap_{\alpha\in\mathbb{Q}} A_\alpha\right)^c = \bigcup_{\alpha\in\mathbb{Q}} A_\alpha^c$$

Proof. Let $x \in \left(\bigcup_{\alpha\in\mathbb{Q}} A_\alpha\right)^c$ then $x \notin \bigcup_{\alpha\in\mathbb{Q}} A_\alpha$. That is for all $\alpha \in \mathbb{Q}$ we have $x \notin A_\alpha$. So $x \in A_\alpha^c$ for all α hence $x \in \bigcap_{\alpha\in\mathbb{Q}} A_\alpha^c$. The reverse follows similarly □

Proof. Let $x \in \left(\bigcap_{\alpha\in\mathbb{Q}} A_\alpha\right)^c$. Then $x \notin \bigcap_{\alpha\in\mathbb{Q}} A_\alpha$. So there exists $\alpha_0 \in \mathbb{Q}$ such that $x \notin A_{\alpha_0}$. Then $x \in A_{\alpha_0}^c \subseteq \bigcup_{\alpha\in\mathbb{Q}} A_\alpha^c$. Conversely suppose $y \in \bigcup_{\alpha\in\mathbb{Q}} A_\alpha^c$. Then there exists $\alpha \in \mathbb{Q}$ such that $y \notin A_\alpha$. So $y \notin \bigcap_{\alpha\in\mathbb{Q}} A_\alpha$. Hence $y \in \left(\bigcap_{\alpha\in\mathbb{Q}} A_\alpha\right)^c$ □

Example 1.6.
$$\bigcup_{n\in\mathbb{N}} \left[0, \frac{n}{n+1}\right)$$

For $n = 1$ we get $\left[0, \frac{1}{2}\right)$. For $n = 2$ we get $\left[0, \frac{2}{3}\right) \approx [0, 0.667)$. Inductively we get,

$$\bigcup_{n\in\mathbb{N}} \left[0, \frac{n}{n+1}\right) = [0, 1)$$

Example 1.7.
$$\bigcap_{n\in\mathbb{N}} \left[0, \frac{n}{n+1}\right)$$

Notice that $n = 1$ gives $\left[0, \frac{1}{2}\right)$. If $n > 1$, then $\frac{n}{n+1} > \frac{1}{2}$. Hence the only number in all intervals is $[0, \frac{1}{2})$.

Definition

The sets U_α, where $\alpha \in A$, form a partition of U if

$$\bigsqcup_{\alpha\in A} U_\alpha = U.$$

Example 1.8. Let $U = \{1, 2, 3, 4, 5, 6\}$ with partitioning sets $U_1 = \{1, 2, 3, 5\}$ and $U_2 = \{2, 4, 6\}$.

Notice that $U_1 \cup U_2 = U$, so U_1 and U_2 form a partition of U. Also, $U_1 \cap U_2 = \emptyset$, so they are pairwise disjoint and have unique elements. That is, the union of the pairwise disjoint sets U_α is U. For instance, $\{1,2,3\}$ and $\{4,5,6\}$ are equivalence classes.

Definition

An equivalence relation $\sim$ on a set A induces a partition of A as follows:

$$a \sim b \text{ if and only if } a \text{ and } b \text{ are in the same equivalence class.}$$

Example 1.9. $2 \sim 4$ if and only if 2 and 4 are in the same equivalence class.

This means 2 and 4 are in the same equivalence class because they are in the even set.

This is an equivalence relation. Let $A = \{a, b, c\}$, and define $a \sim b$ if and only if $a = b$. Then:

$$U_a = \{x \in A : (a, x) \in R\}$$

are the subsets of A determined by equivalence. Similarly define $U_b, U_c, \ldots$

Let

$$R = \{(a, a), (b, b), (c, c)\}.$$

1.2 Cardinality

Proposition A.3.1

Every infinite subset E of a countably infinite set A is countably infinite.

Proof. If $A \sim \mathbb{N}$, then $E \sim \mathbb{N}$ as well. That is, any infinite subset $E \subseteq A$, where A is countably infinite, is itself countably infinite.

Let A be countably infinite. Then there exists a bijection $f : \mathbb{N} \to A$. We write,

$$f(1) = a_1, \quad f(2) = a_2, \quad f(3) = a_3, \quad \ldots, \quad f(i) = a_i.$$

So $A = \{a_1, a_2, a_3, \ldots\}$.

Let $E \subseteq A$ be infinite. We define a map $g : \mathbb{N} \to E$ as follows, Let $e_1 = a_{i_1}$, where

$$i_1 = \min\{i \in \mathbb{N} : a_i \in E\}.$$

Then define $e_2 = a_{i_2}$, where

$$i_2 = \min\{i \in \mathbb{N} : a_i \in E \setminus \{e_1\}\},$$

and more generally,

$$e_{r+1} = a_{i_{r+1}}, \quad i_{r+1} = \min\{i \in \mathbb{N} : a_i \in E \setminus \{e_1, e_2, \ldots, e_r\}\}.$$

Define $g(j) = e_j$. We show that g is a bijection.

To show g is injective, assume $g(j) = g(k)$. Then $e_j = e_k$, so $a_{i_j} = a_{i_k}$. Since f is a bijection from $\mathbb{N} \to A$, we have $i_j = i_k$. Thus $j = k$, and g is injective.

To show g is surjective, let $e \in E$. Since $E \subseteq A$ and $f : \mathbb{N} \to A$ is a bijection, there exists some $m \in \mathbb{N}$ such that $f(m) = e$. Because $e \in E$, the selection rule used to define g ensures that there is some $j \in \mathbb{N}$ such that $e_j = e$. That is, every $e \in E$ appears in the image of g.

Therefore, $g : \mathbb{N} \to E$ is a bijection, so $E \sim \mathbb{N}$ and E is countably infinite. $\square$

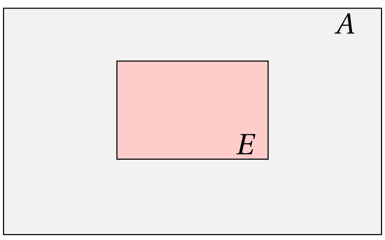
Definition

Let A be an infinite set. If $A \sim \mathbb{N}$, then A is a countably infinite set. That is, there exists a bijection to $\mathbb{N}$.

If $A \not\sim \mathbb{N}$, then A is an uncountable infinite set.

Proposition

If A is countably infinite and $E \subseteq A$, then E is also countably infinite.



A countably infinite
E countably infinite

Proposition

A countable union of countably infinite sets is countably infinite.

Let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of countably infinite sets. Then

$$\bigcup_{n\in\mathbb{N}} A_n$$

is countably infinite. Let $A_1, ..., A_i$ be countable sets then.

$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots \cup \underbrace{\bigcup A_i}_{\text{countable}}$$

Corollary

If A is countably infinite, then so is A^d for any $d \in \mathbb{N}$.

Proposition

Let $a < b$ and $d \in \mathbb{N}$. Then

$$\mathbb{N} \sim \mathbb{Z} \sim \mathbb{Q}$$

These are all bijected to $\mathbb{N}$.

However, the sets

$$\mathbb{N} \not\sim [a,b], \quad \mathbb{N} \not\sim (a,b), \quad \mathbb{N} \not\sim \mathbb{R}, \quad \mathbb{N} \not\sim \mathbb{R}^d$$

are uncountably infinite. These sets have the same cardinality as $\mathbb{R}$ and cannot be put into bijection with $\mathbb{N}$.

1.3 Functions

Definition

Let A and B be two sets. A function $f : A \to B$ assigns one element $b \in B$ for any element $a \in A$. That is,

$$f(a) = b.$$

Definition

For a function $f : A \to B$, we define the image (or range) as

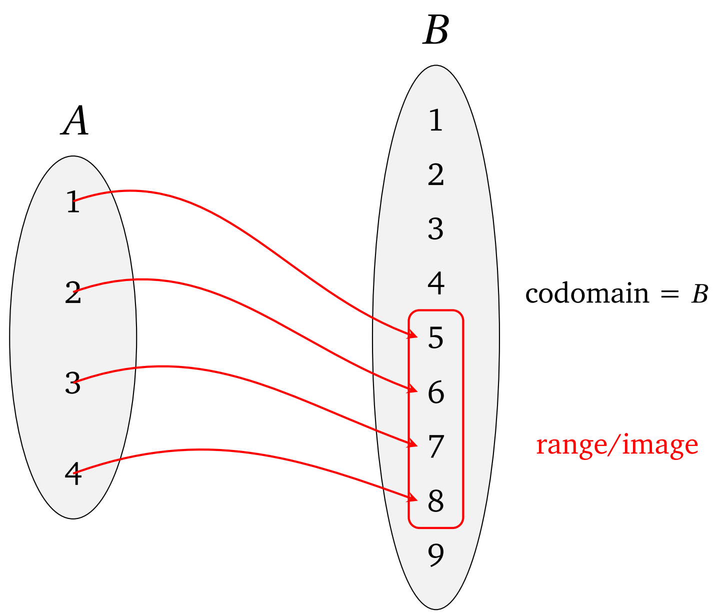
$$\text{range } f = \{f(a) : a \in A\}.$$

Example. Say our domain is $A = \{1, 2, 3, 4\}$.

Then,

$$\text{range} = \{5, 6, 7, 8\},$$

$$\text{codomain} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} = B.$$

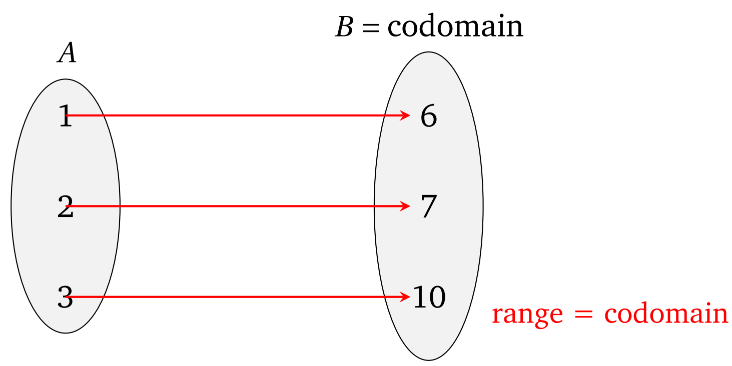


That is,

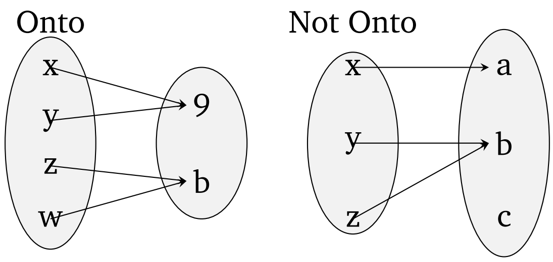
$$\text{range } f = \{f(a) : a \in A\} = \{f(1) = 5,\ f(2) = 6,\ f(3) = 7,\ f(4) = 8\}.$$

If range $f = B$, then f is **onto** (surjective).

Let $A = \{1, 2, 3\}$, $B = \{6, 7, 10\}$, and range $= B$. Then f is onto.

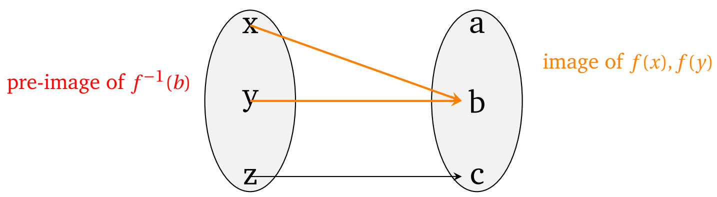


Examples of Onto and Not Onto Functions



Pre-image of a single element

$$f^{-1}(b) = \{a \in A : f(a) = b\}$$

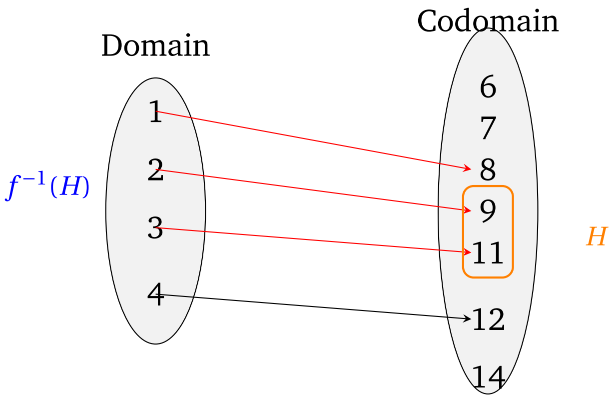


Then,

$$f^{-1}(b) = \{x, y\}$$

Pre-image of a set H

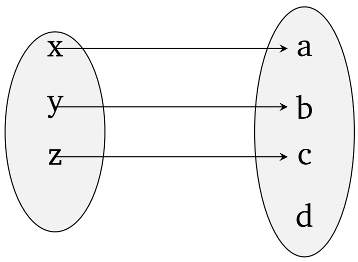
$$f^{-1}(H) = \{a \in A : f(a) \in H\}$$



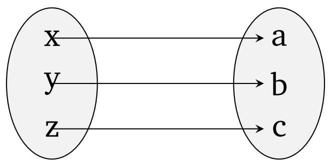
Here,

$$f^{-1}(H) = \{1, 3, 4\}$$

Injective Function: f is injective if each element in the domain maps to exactly one in the codomain and no input is shared.

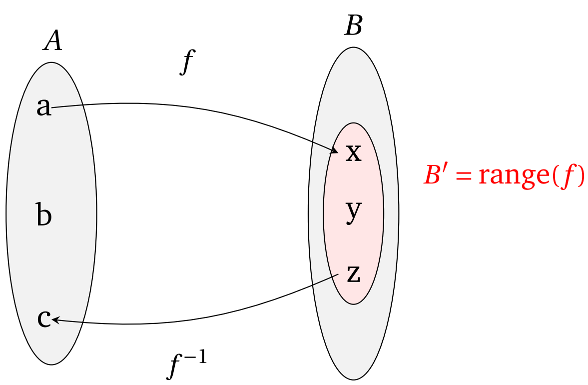


Bijective:



Definition

If $f : A \to B$ is injective, then f^{-1} is also a function $f^{-1} : B' \to A$, where $B' = \text{range}(f) \subseteq B$.



Definition

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

Proposition

For any function f and any indexed collection of sets U_α such that $\alpha \in A$:

(a) $f^{-1}\left(\bigcap_{\alpha \in A} U_\alpha\right) = \bigcap_{\alpha \in A} f^{-1}(U_\alpha)$

(b) $f^{-1}\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(U_\alpha)$

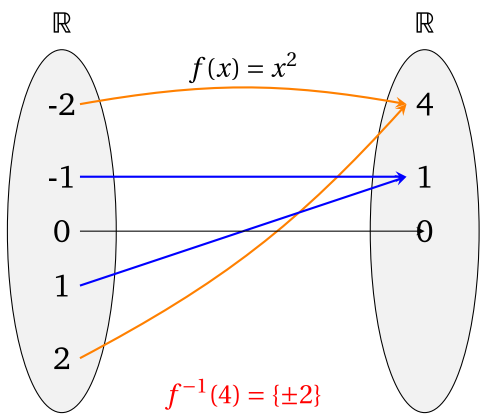
(c) $f\left(\bigcup_{\alpha \in A} U_\alpha\right) = \bigcup_{\alpha \in A} f(U_\alpha)$

Note that $f(U \cap V) = f(U) \cap f(V)$ is not generally true. But $f(U \cap V) \subseteq f(U) \cap f(V)$ is true.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Then

$$f^{-1}(x) = \{\sqrt{x}, -\sqrt{x}\}$$

so f is not injective.



The pre-image of 4 is $f^{-1}(4) = \{\pm 2\}$.

Definition

Let $A \subseteq \Omega$. The indicator function $I_A : \Omega \to \mathbb{R}$ is defined by

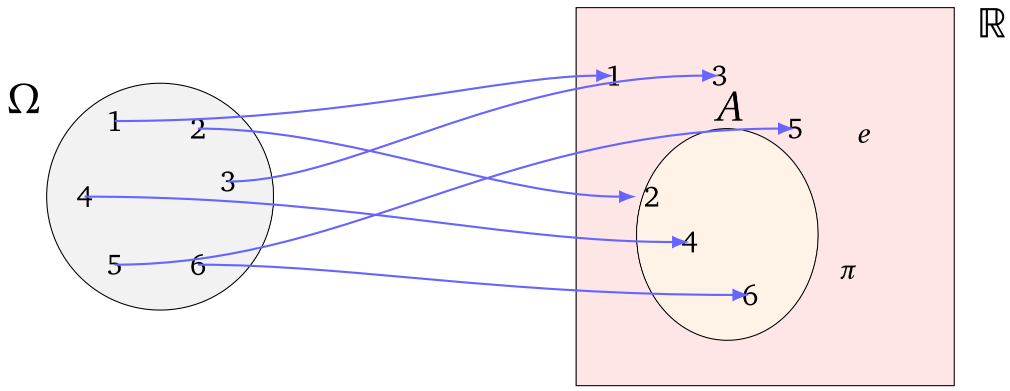
$$I_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

Example. The numbers from rolling a die form the sample space

$$\Omega = \{1, 2, 3, 4, 5, 6\}, \quad A = \{2, 4, 6\}$$

Then

$$I_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

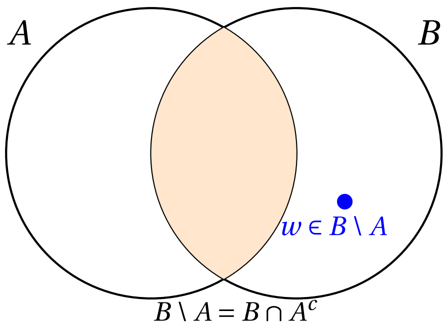


Property. For any set A,

$$I_A = 1 - I_{A^c}$$

$$(1 - I_{A^c})(w) = \begin{cases} 1 - 1 = 0 & \text{if } w \in A^c \\ 1 - 0 = 1 & \text{if } w \in A \end{cases} = I_A(w)$$

If $A \subseteq B$, then $I_A(w) \leq I_B(w)$ for all $w \in \Omega$. If $w \in A$, then $w \in B \Rightarrow I_A(w) = I_B(w) = 1$. If $w \in B \setminus A$, then $I_A(w) = 0$, $I_B(w) = 1$, so still $I_A(w) \leq I_B(w)$.



1.4 Metric Spaces

Definition

Let $X \neq \emptyset$. A **distance** or **metric** on X is a function

$$d : X \times X \to [0, \infty)$$

such that for all $x, y, z \in X$

1. **Non-negativity and Identity of Indiscernibles:**

$$d(x, y) = 0 \iff x = y$$

   The distance between any two points is not negative. It is zero if and only if the two points are the same.

2. **Symmetry:**

$$d(x, y) = d(y, x)$$

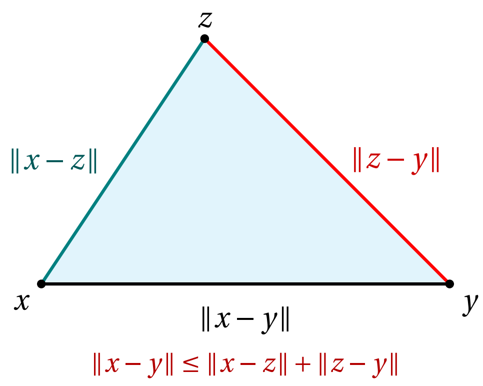
   The distance from x to y is the same as the distance from y to x. We do not take into account direction, energy, or effort.

3. **Triangle Inequality:**

$$d(x, y) \leq d(x, z) + d(z, y)$$

   The direct path from x to y is at most the sum of the paths from x to z and from z to y.

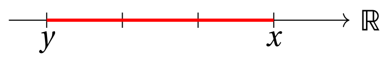
We say that (X, d) is a **metric space**.



Example 1.10. Let $X = \mathbb{R}$ and define

$$d : \mathbb{R} \times \mathbb{R} \to [0, \infty), \quad d(x, y) = |x - y|.$$

This is called the Euclidean metric on $\mathbb{R}$.



To show that this is a metric space, we check the definition.

1. $d(x, y) = 0 \iff x = y$

   Notice that the distance can never be negative, since

   $$|x - y| \geq 0.$$

   So,

   $$|x - y| = 0 \iff x = y.$$

2. $d(x, y) = d(y, x)$

   By the properties of absolute value,

   $$|x - y| = |y - x|.$$

3. $d(x, y) \leq d(x, z) + d(z, y)$

   The triangle inequality holds for all $x, y, z \in \mathbb{R}$,

   $$|x - y| \leq |x - z| + |z - y|.$$

Therefore, $d(x,y) = |x-y|$ defines a metric on $\mathbb{R}$.

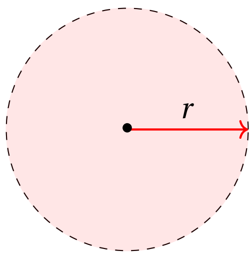
Example 1.11. Let $X \neq \emptyset$. Define $d : X \times X \to [0,\infty)$ where

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

Definition

Let (X,d) be a metric space and let $x_0 \in X$, $r > 0$. The open ball centered at x_0 with radius r is

$$B_r(x_0) = \{x \in X : d(x,x_0) < r\}$$



an open ball $B_r(x_0)$

Definition

A set $A \subseteq X$ is an open set if it is a union of open balls.

Definition

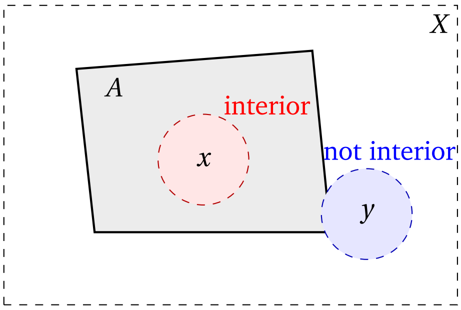
A set $A \subseteq X$ is a closed set if it is the complement of an open set.

Definition

A set A is bounded if $A \subseteq B_r(x)$ for some $r > 0$ and $x \in X$.

Definition

Let (X,d) be a metric space. Let $A \subseteq X$. We say that a point $x \in X$ is in the **interior** of A if there exists $r > 0$ such that $B_r(x) \subseteq A$.



Definition

The interior of A is the set of all points that are in the interior of A. It is denoted

$$\mathring{A} = \operatorname{int}(A) = \{x \in X : x \text{ is in the interior of } A\}.$$

Definition

A set A is called **open** if $\mathring{A} = A$.

Note that $\mathring{A} \subseteq A$.

Proof. Let $x \in \mathring{A}$, then $\exists r > 0$ such that $B_r(x) \subseteq A$. Since $x \in B_r(x)$, we have $x \in A$. □

As a consequence, to prove that a set is open, it is enough to show that each point in the set is actually in its interior. That is,

$$A \text{ is open} \iff A \subseteq \mathring{A} \iff \forall x \in A,\ x \in \mathring{A} \iff \forall x \in A,\ \exists r > 0 \text{ such that } B_r(x) \subseteq A.$$

Proposition

If G_1 and G_2 are open subsets of a metric space (X, d), then $G_1 \cap G_2$ is open.

Proof. Let $x \in G_1 \cap G_2$. Since G_1 and G_2 are open, there exist $r_1, r_2 > 0$ and points $x_1, x_2 \in X$ such that

$$x \in B_{r_1}(x_1) \subseteq G_1 \quad \text{and} \quad x \in B_{r_2}(x_2) \subseteq G_2.$$

Let

$$r = \min\{r_1 - d(x, x_1),\ r_2 - d(x, x_2)\}.$$

Then $r > 0$, and we claim $B_r(x) \subseteq G_1 \cap G_2$.

Let $y \in B_r(x)$. By the triangle inequality:

$$d(x_1, y) \le d(x_1, x) + d(x, y) < d(x_1, x) + r \le d(x_1, x) + r_1 - d(x_1, x) = r_1,$$

so $y \in B_{r_1}(x_1) \subseteq G_1$.

Similarly,

$$d(x_2, y) \le d(x_2, x) + d(x, y) < d(x_2, x) + r \le r_2,$$

so $y \in B_{r_2}(x_2) \subseteq G_2$. Hence, $y \in G_1 \cap G_2$.

Therefore, $B_r(x) \subseteq G_1 \cap G_2$, proving that x is an interior point of $G_1 \cap G_2$. Since x was arbitrary, $G_1 \cap G_2$ is open.

Let $x \in G_1 \cap G_2$. Since G_1 is open, there exists $r_1 > 0$ such that $B_{r_1}(x) \subseteq G_1$. Similarly, since G_2 is open, there exists $r_2 > 0$ such that $B_{r_2}(x) \subseteq G_2$.

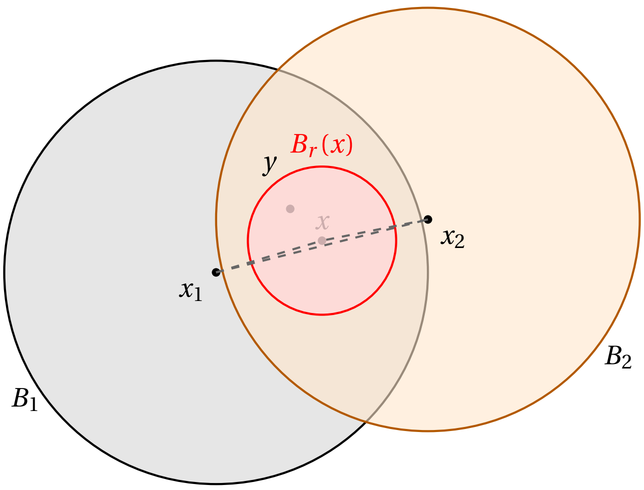
Let $r = \min(r_1, r_2)$, then $B_r(x) \subseteq G_1 \cap G_2$. Thus, for each $x \in G_1 \cap G_2$, there exists a ball $B_r(x) \subseteq G_1 \cap G_2$.

This implies that:

$$G_1 \cap G_2 = \bigcup_{x \in G_1 \cap G_2} B_r(x)$$

so $G_1 \cap G_2$ is open, since it's a union of open balls.

$\square$



Example 1.12. Let $(a, b) \subset \mathbb{R}$. Show that (a, b) is open.

We note that the interval can be written as an open ball,

$$(a, b) = B_{\frac{b-a}{2}}\left(\frac{a+b}{2}\right)$$

Hence, (a,b) is open.

Example 1.13. Show that $(a,\infty) = \bigcup_{n=1}^{\infty} B_1(a+n)$ is open.

Each $B_1(a+n) = (a+n-1, a+n+1)$ is open, and the union of open sets is open. So (a,∞) is open.

Example 1.14. Is $(-\infty, b)$ open?

Yes, same argument as above. It can be written as a union of open balls.

Example 1.15. Show that $[a,b]$ is closed.

We have:

$$[a,b]^c = (-\infty, a) \cup (b, \infty)$$

The complement is open, so $[a,b]$ is closed.

Example 1.16. Is $(a,b]$ open or closed?

It is neither. Suppose it is open. Then for any $x_0 \in (a,b]$, there exists $r > 0$ such that $B_r(x_0) \subseteq (a,b]$. Then $x_0 + r \leq b$, but this fails if $x_0 \to b$. So $(a,b]$ is not open. Its complement is also not open, so it's not closed either.

Example 1.17. Show that $\mathbb{R} = \bigcup_{n=1}^{\infty} B_n(0)$ is open.

Each $B_n(0)$ is open, and countable union of open sets is open. Therefore, $\mathbb{R}$ is open.

Example 1.18. Is the empty set $\varnothing$ closed?

Yes. Its complement is $\mathbb{R}$, which is open. So $\varnothing$ is closed.

Example 1.19. Show that $\varnothing = (1,2) \cap (3,4)$ is open.

The intersection of two open sets is open. So $\varnothing$ is open. Its complement $\mathbb{R}$ is open, so $\varnothing$ is also closed.

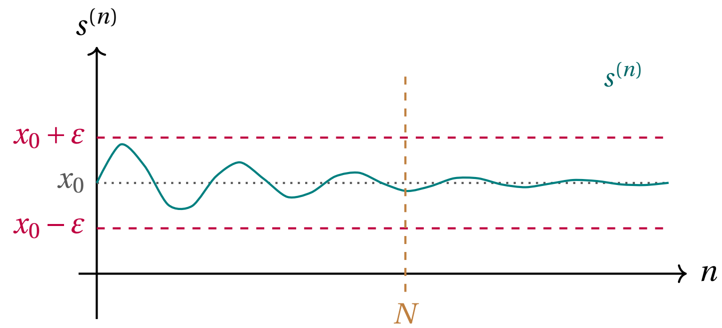
Definition

A sequence $s^{(n)} \to x_0$ means

$$\lim_{n\to\infty} s^{(n)} = x_0$$

if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

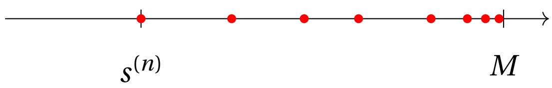
$$d(s^{(n)}, x_0) < \varepsilon.$$



Definition

Let $s^{(n)} \in \mathbb{R}$. We say $\lim s^{(n)} = \infty$ if

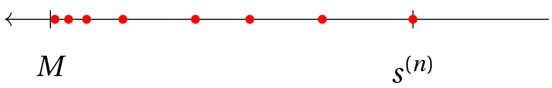
$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n > N \Rightarrow M \leq s^{(n)}.$$



Definition

Let $s^{(n)} \in \mathbb{R}$. We say $\lim s^{(n)} = -\infty$ if

$$\forall M \in \mathbb{R},\ \exists N \in \mathbb{N} \text{ such that } n > N \Rightarrow s^{(n)} < M.$$



Definition

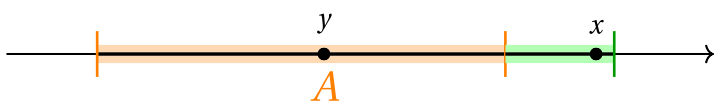
A sequence $s^{(n)}$ is:

- **Monotonic increasing** if $s^{(n)} \leq s^{(n+1)}$ for all n.
- **Monotonic decreasing** if $s^{(n+1)} \leq s^{(n)}$ for all n.

If either condition holds, we say the sequence is monotonic.

Definition

Let $A \subseteq \mathbb{R}$. An upper bound $x \in \mathbb{R}$ satisfies $x \geq y$ for all $y \in A$.

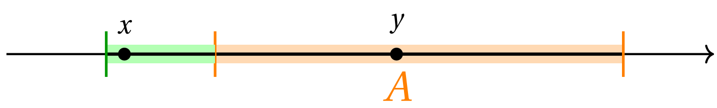


Definition

The smallest upper bound of A is α such that if $\beta < \alpha$, then β is not an upper bound.

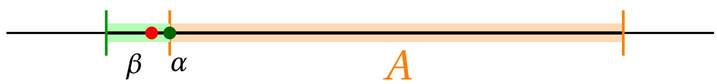
Definition

Let $A \subseteq \mathbb{R}$. A lower bound $x \in \mathbb{R}$ satisfies $x \leq y$ for all $y \in A$.



Definition

The greatest lower bound of A is α such that if $\beta > \alpha$, then β is not a lower bound. That is, $\inf A = \alpha$.



Definition

For a sequence $s^{(n)} \in \mathbb{R}$, $n \in \mathbb{N}$,

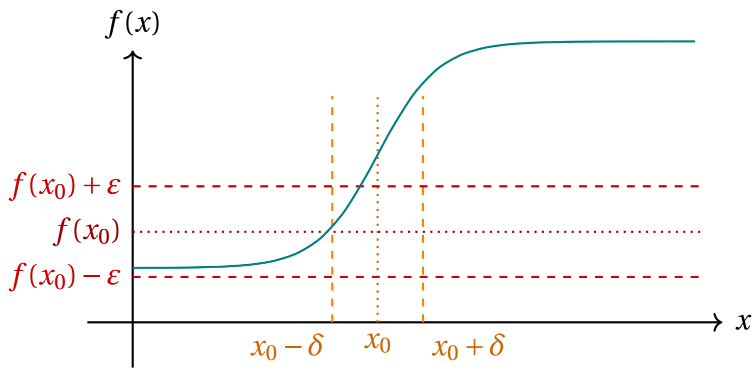
$$\liminf_{n\to\infty} s^{(n)} = \lim_{n\to\infty} \inf\left\{s^{(k)} : k \geq n\right\},$$
$$\limsup_{n\to\infty} s^{(n)} = \lim_{n\to\infty} \sup\left\{s^{(k)} : k \geq n\right\}.$$

Definition

A function $f : A \to \mathbb{R}$ is **continuous at a point** x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x \in A,\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon.$$

If f is continuous at every point in its domain, then f is continuous.



Definition

If $f : A \to B$ where A and B are metric spaces, then

$$\lim_{x\to p} f(x) = y_0 \quad \text{if } \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_1(x, p) < \delta \Rightarrow d_2(f(x), y_0) < \varepsilon.$$

Definition

Let A, B be metric spaces. A function $f : A \to B$ is *continuous on* A if

$$\forall a \in A, \forall \varepsilon > 0, \exists \delta > 0 \text{ such that } d_1(x, a) < \delta \Rightarrow d_2(f(x), f(a)) < \varepsilon.$$

Proposition

If $x^{(n)}$ is a sequence of elements in A that converges to $x_0 \in A$, and $f : A \to B$ is continuous, then

$$\lim_{n\to\infty} f(x^{(n)}) = f\left(\lim_{n\to\infty} x^{(n)}\right) = f(x_0).$$

Proof. Let $\varepsilon > 0$. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$d_1(x, x_0) < \delta \Rightarrow d_2(f(x), f(x_0)) < \varepsilon.$$

Since $x^{(n)} \to x_0$, for this $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$d_1(x^{(n)}, x_0) < \delta.$$

Thus, by continuity,

$$d_2(f(x^{(n)}), f(x_0)) < \varepsilon \quad \text{for all } n \geq N.$$

Therefore,

$$\lim_{n\to\infty} f(x^{(n)}) = f(x_0).$$

□

Proposition

Let $f : A \to B$ be continuous between metric spaces A and B. If $G \subseteq B$ is open, then $f^{-1}(G) \subseteq A$ is open.

Proof. Suppose $G \subseteq B$ is open. Then by definition, every point in G is an interior point:

$$G = \operatorname{int}(G).$$

Let $x \in f^{-1}(G)$. Then $f(x) \in G$. Since G is open, there exists $r > 0$ such that

$$B_r(f(x)) \subseteq G.$$

Because f is continuous, for this $r > 0$, there exists $\delta > 0$ such that

$$d_1(y,x) < \delta \Rightarrow d_2(f(y), f(x)) < r.$$

Hence, for such y,

$$f(y) \in B_r(f(x)) \subseteq G \Rightarrow y \in f^{-1}(G).$$

So $B_\delta(x) \subseteq f^{-1}(G)$, which means x is an interior point of $f^{-1}(G)$. Thus, $f^{-1}(G)$ is open. □

1.5 Archimedean Property

Proposition (Archimedean Property)

Let $a, b \in \mathbb{R}$ be positive real numbers. Then there exists $n \in \mathbb{N}$ such that

$$a < nb.$$

Equivalently, there exists $n \in \mathbb{N}$ such that

$$\frac{a}{b} < n.$$

Direct Proof. We can instead prove the equivalent statement: for all $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.

Let $x = \frac{a}{b}$, then if $n > \frac{a}{b}$, it follows that $nb > a$. Done. □

Proof by Contradiction. Suppose for contradiction that there exists $x \in \mathbb{R}$ such that $x \geq n$ for all $n \in \mathbb{N}$. That is, $\mathbb{N}$ is bounded above.

Then $\sup(\mathbb{N}) = S \in \mathbb{R}$. But then there exists $m \in \mathbb{N}$ such that

$$m > S - 2 \Rightarrow m + 2 > S,$$

which contradicts that S is an upper bound.

Hence, such an x cannot exist, and so for every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$. □

Example 1.20. Let $b = \varepsilon > 0$ and $a = 1$. Show that there exists $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

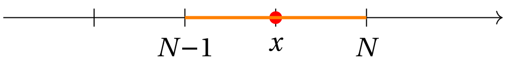
By the Archimedean property, since $1, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$1 < N\varepsilon \Rightarrow \frac{1}{N} < \varepsilon.$$

Corollary (Archimedean Property)

If $x > 0$ for $x \in \mathbb{R}$, then there exists $N \in \mathbb{N}$ such that

$$N - 1 \leq x < N.$$



Proof. Since $x > 0$, $1 > 0$, by the Archimedean property, there exists $N \in \mathbb{N}$ such that

$$x < N.$$

Let

$$S = \{n \in \mathbb{N} : x < n\}.$$

Then $S \subseteq \mathbb{N}$ is nonempty, and by the Well-Ordering Principle, it has a smallest element. Let $N_0 = \min S$. Then $x < N_0$, and by minimality,

$$N_0 - 1 \leq x.$$

If not, i.e. if $N_0 - 1 > x$, then $N_0 - 1 \in S$, which contradicts the minimality of N_0. So,

$$N_0 - 1 \leq x < N_0,$$

as desired. □

Example 1.21. If $0 < x \leq 1$, then there exists $N \in \mathbb{N}$, $N \geq 2$, such that

$$\frac{1}{N} < x < \frac{1}{N-1}.$$

Note that $0 < x \leq 1$.

Recall that if $0 < a < b$, then $\frac{1}{a} > \frac{1}{b}$. So,

$$\frac{1}{x} \geq 1.$$

By the corollary, there exists $N \in \mathbb{N}$ such that

$$N - 1 \leq \frac{1}{x} < N.$$

This only works when $N \geq 2$, because if $N = 1$, then

$$0 \leq \frac{1}{x} \leq 1,$$

which contradicts $\frac{1}{x} \geq 1$. So for $N \geq 2$, we take reciprocals to get:

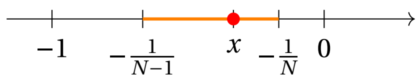
$$\frac{1}{N-1} \geq x \geq \frac{1}{N},$$

and thus,

$$\frac{1}{N} < x < \frac{1}{N-1}.$$

Example 1.22. If $-1 < x < 0$, then there exists $N \geq 2$ such that

$$-\frac{1}{N-1} \leq x < -\frac{1}{N}.$$



Let $-1 \leq x < 0$, then $1 \geq -x > 0$.

By the corollary, there exists $N \in \mathbb{N}$ such that

$$N - 1 \leq \frac{1}{-x} < N.$$

Taking reciprocals and switching the inequality (since $x<0$) gives:

$$-\frac{1}{N-1} \leq x < -\frac{1}{N}.$$

Example 1.23. Let $A_n = \left(-\frac{1}{n}, 1\right)$ for $n = 1, 2, 3, \ldots$. Show that

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, 1\right) = [0,1).$$

Proof. First, we show $x \in [0,1) \Rightarrow x \in \left(-\frac{1}{n}, 1\right)$ for all $n \in \mathbb{N}$. Since $x \geq 0$ and $-\frac{1}{n} < 0$ for all n, clearly $x \in \left(-\frac{1}{n}, 1\right)$ for every n. So $x \in \bigcap_n \left(-\frac{1}{n}, 1\right)$.

Now suppose $x \in \bigcap_n \left(-\frac{1}{n}, 1\right)$. We want to show $x \in [0,1)$. From the inclusion, we know,

$$-\frac{1}{n} < x < 1 \quad \text{for all } n \in \mathbb{N}.$$

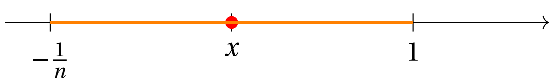
This implies $x < 1$. To show $x \geq 0$, suppose for contradiction that $x < 0$. Then, since $x \in \left(-\frac{1}{n}, 1\right)$ for all n, we would have

$$x > -\frac{1}{n} \quad \forall n \in \mathbb{N}.$$

But since $x < 0$, we can apply the Archimedean property to find $N \in \mathbb{N}$ such that

$$-\frac{1}{N-1} \leq x < -\frac{1}{N}.$$

So $x \notin \left(-\frac{1}{N}, 1\right)$, which contradicts the assumption that $x \in \left(-\frac{1}{n}, 1\right)$ for all n. Hence, $x \geq 0$. Therefore, $x \in [0,1)$. □



Summary of Results

We have shown the following:

⋆ If $x > 0$, then there exists $N \in \mathbb{N}$ such that

$$N - 1 \leq x < N.$$

⋆⋆ If $0 < x \leq 1$, then there exists $N \in \mathbb{N}$, $N \geq 2$, such that

$$\frac{1}{N} < x < \frac{1}{N-1}.$$

⋆⋆⋆ If $-1 \leq x < 0$, then there exists $N \in \mathbb{N}$, $N \geq 2$, such that

$$-\frac{1}{N-1} \leq x < -\frac{1}{N}.$$

1.6 More on Limits

Definition

For a sequence of sets A_n, where $n \in \mathbb{N}$, we have:

For $A_1, A_2, \ldots, A_k$:

1)

$$\inf_{k \geq n} A_k = \bigcap_{k=n}^{\infty} A_k$$

When $n = 1$, then

$$\inf_{k \geq 1} A_k = \bigcap_{k=1}^{\infty} A_k$$

When $n = 2$, then

$$\inf_{k \geq 2} A_k = \bigcap_{k=2}^{\infty} A_k$$

$$\vdots$$

They may grow or they may not grow. If each set is the empty set, i.e.,

$$A_1 = \varnothing = \cdots = A_k,$$

then it does not grow, i.e., may stay the same. The set of non-decreasing.

2)

$$\sup_{k\geq n} A_k = \bigcup_{k=n}^{\infty} A_k$$

When $n = 1$, then

$$\sup_{k\geq 1} A_k = \bigcup_{k=1}^{\infty} A_k$$

When $n = 2$, then

$$\sup_{k\geq 2} A_k = \bigcup_{k=2}^{\infty} A_k$$

Same as above, may be decreasing or not. That is non-increasing.

3)

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \inf_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} A_k$$

4)

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \sup_{k\geq n} A_k = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k$$

Proposition

$$\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c$$

Proof.

$$\begin{aligned}
\left(\liminf_{n\to\infty} A_n\right)^c &= \left(\bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k\right)^c \\
&= \bigcap_{n\in\mathbb{N}} \left(\bigcap_{k\geq n} A_k\right)^c \\
&= \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k^c \\
&= \limsup_{n\to\infty} A_n^c
\end{aligned}$$

□

Definition

For a sequence of sets A_n where $n \in \mathbb{N}$, if

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n,$$

then we define

$$\lim_{n\to\infty} A_n := \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

Example. Consider $A_k = \left[0, \frac{k}{k+1}\right)$

Notice that

$$A_1 = \left[0, \frac{1}{2}\right)$$
$$A_2 = \left[0, \frac{2}{3}\right)$$
$$A_3 = \left[0, \frac{3}{4}\right)$$
$$A_4 = \left[0, \frac{4}{5}\right)$$

For general n, we have:

$$\inf_{k\geq n} A_k = \left[0, \frac{k}{k+1}\right) \cap \left[0, \frac{k+1}{k+2}\right) \cap \cdots = \left[0, \frac{k}{k+1}\right)$$

$$\sup_{k\geq n} A_k = \left[0, \frac{1}{2}\right) \cup \left[0, \frac{2}{3}\right) \cup \left[0, \frac{3}{4}\right) \cup \cdots = [0, 1)$$

If $n = 1$, then:

$$\inf_{k\geq 1} A_k = \left[0, \frac{1}{2}\right) \cap \left[0, \frac{2}{3}\right) \cap \left[0, \frac{3}{4}\right) \cap \cdots = \left[0, \frac{1}{2}\right)$$

If $n = 2$, then:

$$\inf_{k\geq 2} A_k = \left[0, \frac{2}{3}\right) \cap \left[0, \frac{3}{4}\right) \cap \cdots = \left[0, \frac{2}{3}\right)$$

Now compute,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \inf_{k\geq n} A_k = \bigcup_{n\in\mathbb{N}} \left[0, \frac{k}{k+1}\right) \quad \text{(note: } k \geq n\text{)}$$

$$= \left[0, \frac{1}{2}\right) \cup \left[0, \frac{2}{3}\right) \cup \cdots = [0,1)$$

That said,

$$\inf_{k\geq 1} \left[0, \frac{1}{2}\right) \cap \left[0, \frac{2}{3}\right) \cap \left[0, \frac{3}{4}\right) = \left[0, \frac{1}{2}\right)$$

Now compute,

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \sup_{k\geq n} A_k = \bigcap_{n\in\mathbb{N}} [0,1) = [0,1) \cap [0,1) \cap \cdots = [0,1)$$

Proposition

$$\limsup_n (A_n \cap B_n) \subseteq \left(\limsup_n A_n\right) \cap \left(\limsup_n B_n\right)$$

Proposition A.4.1

Let A_n, $n \in \mathbb{N}$, be a sequence of subsets of Ω. Then

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k$$

Proof. Let $\omega \in \bigcap_{n\in\mathbb{N}} \bigcup_{k=n}^{\infty} A_k$. Then $\omega \in \bigcup_{k=n}^{\infty} A_k$ for all $n \in \mathbb{N}$, because it's in the intersection.

Since $\omega \in \bigcup_{k=1}^{\infty} A_k$, there exists k_1 such that $\omega \in A_{k_1}$. Then $\omega \in \bigcup_{k=k_1+1}^{\infty} A_k$, so there exists k_2 such that $\omega \in A_{k_2}$. Similarly, $\omega \in \bigcup_{k=k_2+1}^{\infty} A_k$, so there exists k_3 such that $\omega \in A_{k_3}$, and so on.

Then

$$\mathbb{1}_{A_{k_1}}(\omega) = 1 \quad \text{because } \omega \in A_{k_1}$$

$$\mathbb{1}_{A_{k_2}}(\omega) = 1 \quad \text{because } \omega \in A_{k_2}$$

$$\vdots$$

$$\sum_{i=1}^{\infty} \mathbb{1}_{A_{k_i}}(\omega) = \infty$$

$$\limsup_{n\to\infty} A_n = \left\{ \omega \in \Omega : \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n}(\omega) = \infty \right\}$$

Now let $\omega \in \Omega$ and $\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n}(\omega) = \infty$. Then for all $n \in \mathbb{N}$, there exists $k_n \geq n$ such that $\omega \in A_{k_n}$ (or $\mathbb{1}_{A_{k_n}}(\omega) = 1$). That is, for every $n \in \mathbb{N}$, we can find a k_n such that $\omega \in A_{k_n}$.

Otherwise, if that is not true, then there exists $n_0 \in \mathbb{N}$ such that $\omega \notin A_n$ for all $n \geq n_0$, hence

$$\mathbb{1}_{A_n}(\omega) = 0 \quad \text{for } n \geq n_0.$$

Then,

$$\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n}(\omega) = \sum_{n=1}^{n_0-1} \mathbb{1}_{A_n}(\omega) + \sum_{n\geq n_0} \mathbb{1}_{A_n}(\omega) = \sum_{n=1}^{n_0-1} \mathbb{1}_{A_n}(\omega) \leq n_0 - 1.$$

$$A_1, A_2, A_3, \ldots, A_{n_0-1}, A_{n_0}, A_{n_0+1}, \ldots$$

The first $n_0 - 1$ terms might contribute 1's or 0's. If (magically) they are all 1's, then the sum is still at most $n_0 - 1$, which is finite — contradicting our assumption.

For $n = 1$, there exists $k_1 \geq 1$ such that $\omega \in A_{k_1}$, where

$$\mathbb{1}_{A_{k_1}}(\omega) = 1.$$

This automatically means $\omega \in \bigcup_{k=1}^{\infty} A_k$, since once we know $\omega \in A_k$ for some k, we know it belongs to the union.

If $n = 1$ then we might have something like:

$$\omega \notin A_1,\ \omega \notin A_2,\ \omega \notin A_3,\ \omega \in A_4,\ \ldots \quad \Rightarrow \quad k_1 = 4$$

If $n = 2$, then there exists $k \geq 2$ such that $\omega \in A_{k_2}$, so

$$\mathbb{1}_{A_{k_2}}(\omega) = 1 \quad \Rightarrow \quad \omega \in \bigcup_{k=2}^{\infty} A_k$$

We then have:

$$\omega \in \bigcup_{k=1}^{\infty} A_k$$

$$\omega \in \bigcup_{k=2}^{\infty} A_k$$

$$\vdots$$

$$\omega \in \bigcup_{k\geq n} A_k$$

Therefore:

$$\omega \in \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \limsup_{n\to\infty} A_n$$

□

Proposition

$$\liminf_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} \mathbb{1}_{A_n^c}(\omega) < \infty\right\}$$

Proof. Assume $\omega \in \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$. Then there exists some n_0 such that:

$$\omega \in \bigcap_{k\geq n_0} A_k = A_{n_0} \cap A_{n_0+1} \cap \cdots$$

Then $\omega \in A_k$ for all $k \geq n_0$, So $\omega \notin A_k^c$ for all $k \geq n_0$, Thus,

$$\mathbb{1}_{A_k^c}(\omega) = 0 \quad \text{for } k \geq n_0.$$

Now,

$$\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n^c}(\omega) = \sum_{n=1}^{n_0-1} \mathbb{1}_{A_n^c}(\omega) + \sum_{n\geq n_0} \mathbb{1}_{A_n^c}(\omega) = \sum_{n=1}^{n_0-1} \mathbb{1}_{A_n^c}(\omega) \leq n_0 - 1 < \infty$$

Hence, even if all of $\mathbb{1}_{A_n^c}(\omega) = 1$ for $n = 1, \ldots, n_0 - 1$, the sum is still bounded by $n_0 - 1$, which is finite. Suppose $\omega \in \Omega$ and

$$\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n^c}(\omega) < \infty$$

Then there exists n_0 such that

$$\mathbb{1}_{A_k^c}(\omega) = 0 \quad \text{for all } k \geq n_0.$$

Eventually you get a sequence of zeros since the sum $\sum_{n\in\mathbb{N}} \mathbb{1}_{A_n^c}(\omega)$ is finite.

Otherwise, for all $n \in \mathbb{N}$, there exist $k_n \geq n$ such that

$$\mathbb{I}_{A^c_{k_n}}(\omega) = 1.$$

For $n = 1$, there exists $k_1 \geq 1$ such that

$$\mathbb{I}_{A^c_{k_1}}(\omega) = 1.$$

For $n = 2$, there exists $k_2 > k_1$ such that

$$\mathbb{I}_{A^c_{k_2}}(\omega) = 1.$$

We need a way to take into account that we do not have doubles. That is, if $k_1 = k_2$, then

$$\mathbb{I}_{A^c_{k_1}}(\omega) = 1 = \mathbb{I}_{A^c_{k_2}}(\omega),$$

and the sum would be 2, but they are the same term. Doing this can falsely lead us to think the sum is infinite.

Thus,

$$\sum_{n \in \mathbb{N}} \mathbb{I}_{A^c_{k_n}}(\omega) = \infty$$

□

Then for $\omega \in A_k$, $k \geq n_0$, we also have

$$\omega \in \bigcap_{k \geq n_0} A_k.$$

Then

$$\omega \in \bigcap_{k \geq 1} A_k \cup \bigcap_{k \geq 2} A_k \cup \cdots \cup \bigcap_{k \geq n_0} A_k \cup \bigcap_{k \geq n_0 + 1} A_k \cup \cdots$$

This includes $\bigcap_{k \geq n_0} A_k$ and more.

$$\Rightarrow \quad \omega \in \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} A_k = \liminf_{n \to \infty} A_n$$

Now recall,

$$\liminf_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} \mathbb{I}_{A_n^c}(\omega) < \infty\right\} \quad \subseteq \quad \limsup_{n\to\infty} A_n = \left\{\omega \in \Omega : \sum_{n\in\mathbb{N}} \mathbb{I}_{A_n}(\omega) = \infty\right\}$$

Example 1.24. Let A_n be such that $A_n = \Omega$ for $n \geq 4$, and empty otherwise.

Then

$$\limsup_{n\to\infty} \mathbb{I}_{A_n}(\omega) = \sum_{n=4}^{\infty} \mathbb{I}_{A_n}(\omega) = 0+0+0+1+1+\cdots = \infty$$

But

$$\liminf_{n\to\infty} A_n^c = \sum_{n=4}^{\infty} \mathbb{I}_{A_n^c}(\omega) = \mathbb{I}_{A_1^c} + \mathbb{I}_{A_2^c} + \mathbb{I}_{A_3^c} + 0 + 0 + \cdots = 3$$

Hence,

$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

It may not always be the case that the liminf equals limsup. Consider:

Let $\mathbb{I}_{A_n}(\omega)$ be the indicator for the set of all even n. Then:

$$\mathbb{I}_{A_1} + \mathbb{I}_{A_2} + \mathbb{I}_{A_3} + \mathbb{I}_{A_4} + \cdots = 0+1+0+1+\cdots = \infty$$

But for $\mathbb{I}_{A_n^c}(\omega)$, we get:

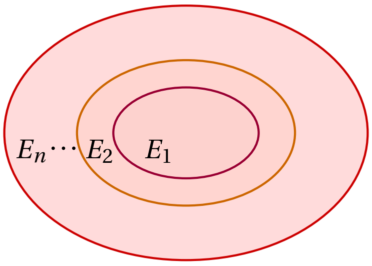
$$\mathbb{I}_{A_1^c} + \mathbb{I}_{A_2^c} + \mathbb{I}_{A_3^c} + \mathbb{I}_{A_4^c} + \cdots = 1+0+1+0+\cdots = \infty$$

Definition

A sequence of sets A_n, $n \in \mathbb{N}$, is said to be **monotonic non-decreasing** if

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

This is denoted $A_n \nearrow$. It means the sets may stay the same or get bigger.

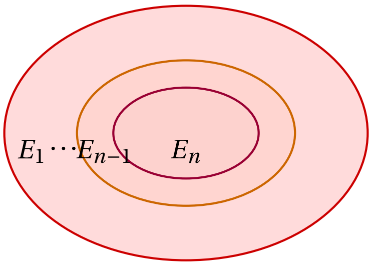


Definition

A sequence of sets A_n, $n \in \mathbb{N}$, is said to be **monotonic non-increasing** if

$$\cdots \subseteq A_3 \subseteq A_2 \subseteq A_1$$

This is denoted $A_n \searrow$. It means the sets may stay the same or get smaller.



If $\lim A_n = A$, then the sequence A_n is both monotonic increasing and decreasing (i.e., constant).

Proposition

If $A_n \nearrow$, then

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

Proof. We want to show,

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

Given that $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we proceed,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k$$

Note:

$$\bigcap_{k\geq 1} A_k = A_1 \cap A_2 \cap A_3 \cap \cdots = A_1$$

$$\bigcap_{k\geq 2} A_k = A_2 \cap A_3 \cap A_4 \cap \cdots = A_2$$

$$\vdots$$

$$\bigcap_{k\geq n} A_k = A_n \cap A_{n+1} \cap A_{n+2} \cdots = A_n$$

Therefore,

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

Now,

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

Note that

$$\bigcup_{k\geq 1} A_k = A_1 \cup A_2 \cup A_3 \cup \cdots = \bigcup_{n\in\mathbb{N}} A_n$$

$$\bigcup_{k\geq 2} A_k = A_2 \cup A_3 \cup \cdots = \bigcup_{n\geq 2} A_n$$

$$\vdots$$

Since all unions contain the same elements (due to monotonicity), their intersection is the same,

$$\limsup_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n$$

and in general,

$$\bigcup_{k\geq n} A_k = A_n.$$

Now observe,

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \bigcap_{n\in\mathbb{N}} A_n.$$

Since $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, we have,

$$\bigcap_{n\in\mathbb{N}} A_n = A_1.$$

As n increases, the intersections are just A_n due to inclusion, so,

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

Thus,

$$\limsup_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = \liminf_{n\to\infty} A_n,$$

and since $\limsup = \liminf$, they are equal to $\lim_{n\to\infty} A_n$. So,

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n.$$

□

Corollary

Let $A_1, A_2, \ldots$ be a sequence of sets. Define:

$$B_n = \bigcup_{k\geq n} A_k \quad \text{and} \quad C_n = \bigcap_{k\geq n} A_k$$

Then B_n and C_n form monotonic sequences.

Since $C_n = \bigcap_{k\geq n} A_k$, the sequence (C_n) is monotonic non-increasing. Likewise, $B_n = \bigcup_{k\geq n} A_k$ is monotonic non-decreasing.

Thus, we can write,

$$\liminf_{n\to\infty} A_n = \lim_{n\to\infty}\left(\inf_{k\geq n} A_k\right) \qquad \text{and} \qquad \limsup_{n\to\infty} A_n = \lim_{n\to\infty}\left(\sup_{k\geq n} A_k\right)$$

Let's now explore $B_n = \bigcup_{k\geq n} A_k$, which forms a monotonic non-decreasing (or non-increasing) sequence depending on the structure of A_k.

$$B_1 = \bigcup_{k\geq 1} A_k = A_1 \cup A_2 \cup A_3 \cup \cdots$$

Note,

$$B_2 = \bigcup_{k\geq 2} A_k = A_2 \cup A_3 \cup A_4 \cup \cdots$$

We also have:

$$B_n \subseteq B_{n-1} \subseteq \cdots \subseteq B_1$$

So (B_n) is a monotonic non-increasing sequence.

Then by the definition of limit for monotonic sequences, we know:

$$\lim_{n\to\infty} \inf B_n = \lim_{n\to\infty} \sup B_n = \lim_{n\to\infty} B_n = \bigcap_{n\in\mathbb{N}} B_n \quad \text{(by Proposition A.4.2)}$$

But since $B_n = \bigcup_{k\geq n} A_k$, we have,

$$\bigcap_{n\in\mathbb{N}} B_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k = \limsup_{n\to\infty} A_n$$

Also note that,

$$\liminf_{n\to\infty} B_n = \lim_{n\to\infty} \bigcup_{k\geq n} A_k = \limsup_{n\to\infty} A_k$$

Thus, from both facts ($\liminf B_n = \limsup A_n$ and $\lim B_n = \limsup A_n$), we conclude:

$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k$$

2 Measure Theory

2.1 Sigma Algebra

Proposition (Algebra of Sets)

Let Ω be a set. A collection $\mathscr{F}$ of subsets of Ω is called an *algebra over* Ω if:

i) $\Omega \in \mathscr{F}$

ii) If $A \in \mathscr{F}$, then $A^c \in \mathscr{F}$

iii) If $A, B \in \mathscr{F}$, then $A \cup B \in \mathscr{F}$

Example 2.1. Let $\Omega = \{1,2,3\}$ and define

$$\mathscr{F} = \{\emptyset, \{1,2,3\}, \{1\}, \{2,3\}\}.$$

Show that $\mathscr{F}$ is an algebra over Ω.

(i) $\Omega = \{1,2,3\} \in \mathscr{F}$

(ii) Check complements:

- $\emptyset \in \mathscr{F} \Rightarrow \emptyset^c = \{1,2,3\} \in \mathscr{F}$
- $\{1,2,3\} \in \mathscr{F} \Rightarrow \{1,2,3\}^c = \emptyset \in \mathscr{F}$
- $\{1\} \in \mathscr{F} \Rightarrow \{1\}^c = \{2,3\} \in \mathscr{F}$
- $\{2,3\} \in \mathscr{F} \Rightarrow \{2,3\}^c = \{1\} \in \mathscr{F}$

(iii) Check unions:

- $\{1\} \cup \{2,3\} = \{1,2,3\} \in \mathscr{F}$
- $\{1\} \cup \emptyset = \{1\} \in \mathscr{F}$
- $\{2,3\} \cup \emptyset = \{2,3\} \in \mathscr{F}$

Therefore, $\mathscr{F}$ satisfies all conditions and is an algebra over Ω.

From deMorgan's, if $A \in \mathscr{F}$ then $A^c \in \mathscr{F}$ from (ii). If $B \in \mathscr{F}$ then $B^c \in \mathscr{F}$ from (ii)

Then $A^c \cup B^c \in \mathscr{F}$ from (iii)

Then $(A^c \cup B^c)^c \in \mathscr{F}$ from (ii)

Then $A \cap B \in \mathscr{F}$

By induction, if $A_1, \ldots, A_n \in \mathscr{F}$ then $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathscr{F}$

Also, notice

If $A_1^c, A_2^c, \ldots, A_n^c \in \mathscr{F}$ then $A_1^c \cup A_2^c \cup \cdots \cup A_n^c \in \mathscr{F}$

Then $(A_1^c \cup A_2^c \cup \cdots \cup A_n^c)^c \in \mathscr{F}$

Then $A_1 \cap A_2 \cap \cdots \cap A_n \in \mathscr{F}$

Actually, (i) is redundant, if you need to show it's an algebra, show (ii), (iii).

Definition

A σ-algebra of subsets of Ω is a class of sets $\mathscr{F}$ such that

i) $\Omega \in \mathscr{F}$

ii) closed under countable unions, countable intersections

iii) closed under complements

Example 2.2. $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ is a sigma algebra over $\Omega = \{1,2\}$

i) $\Omega = \{1,2\} \in \mathscr{F}$

ii) any union of sets in $\mathscr{F}$ results in a set that is also in $\mathscr{F}$

$$\{1\} \cup \{2\} = \{1,2\} \in \mathscr{F}$$

$$\{1\} \cup \emptyset = \{1\} \in \mathscr{F}$$

$$\{2\} \cup \emptyset = \{2\} \in \mathscr{F}$$

iii) closure under complements

$$\{1\}^c = \{2\} \in \mathscr{F}$$

$$\{2\}^c = \{1\} \in \mathscr{F}$$

$$\emptyset^c = \{1,2\} \in \mathscr{F}$$

$$\{1,2\}^c = \emptyset \in \mathscr{F}$$

Fact

Let $\mathbb{R}$ be our whole space then $\mathscr{P}(\mathbb{R})$ is the biggest σ-algebra, $\{\emptyset, \mathbb{R}\}$ is the smallest σ-algebra, Let $\mathscr{F}$ be a σ-algebra over $\mathbb{R}$, then

$$\{\emptyset, \mathbb{R}\} \subseteq \mathscr{F} \subseteq \mathscr{P}(\mathbb{R})$$

Let $A_1, \ldots, A_n$ be a finite collection of sets in $\mathscr{F}$, since $\emptyset \in \mathscr{F}$

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots \in \mathscr{F} \quad \text{by ii}$$

But

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots = A_1 \cup \cdots \cup A_n$$

Then

$$\bigcup_{i=1}^{n} A_i \in \mathscr{F}$$

Likewise,

We know that $\mathscr{F}$ is closed under complements. Since $A_1, \ldots, A_n$ is a finite collection of sets in $\mathscr{F}$, the complements are also in $\mathscr{F}$. Hence

$$A_1^c \cup A_2^c \cup A_3^c \cup \cdots \cup A_n^c \in \mathscr{F}$$

Then

$$\left(A_1^c \cup A_2^c \cup \cdots \cup A_n^c\right)^c \in \mathscr{F}$$

Then

$$A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_n \in \mathscr{F}$$

Then

$$\bigcap_{i=1}^{n} A_i \in \mathscr{F}$$

$$\Omega \in \mathscr{F}$$

Since we know that it is closed under countable unions. We also know that it is closed under countable intersections. That is,

$$\emptyset = A \cap A^c \cap A^c \cap A^c \cap \ldots$$

Recap

- Since $\mathscr{F}$ is not empty, then there exists a set $A \subseteq \Omega$ such that $A \in \mathscr{F}$. We showed

  $$A_1 \cup A_2 \cup \cdots \cup A_n \cup \emptyset \cup \emptyset \cup \cdots \in \mathscr{F}$$

  That is, $\mathscr{F}$ is closed under countable unions. Then, by DeMorgan's, $\mathscr{F}$ is closed under countable intersections.

  $$A^c \in \mathscr{F} \quad \text{since} \quad \emptyset = A \cap A^c \cap A^c \cap A^c \cap \cdots \in \mathscr{F}$$

- Recall that for the set $\mathbb{N} = \{1, 2, 3, \ldots\}$, this is closed under finite addition since

  $$n_1 + n_2 + n_3 + \cdots + n_k = n \in \mathbb{N}$$

  But $\mathbb{N}$ is not closed under infinite addition as

  $$1 + 1 + 1 + \cdots + 1 = \infty \notin \mathbb{N}$$

  However,

  $$\mathbb{N} \cup \{\infty\} = \overline{\mathbb{N}} \quad \text{is closed under infinite addition.}$$

  Hence we include ∞ in the measure definitions.

Definition

A measurable space $(\Omega, \mathscr{F})$ consists of a set Ω and a σ-algebra of subsets of Ω.

Example 2.3. We showed that $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ is a sigma algebra over $\Omega = \{1, 2\}$.

Hence,

$$(\Omega, \mathscr{F}) = (\{1,2\}, \{\emptyset, \{1\}, \{2\}, \{1,2\}\})$$

is a measurable space.

Definition

A set of sets $\mathscr{A}$ is a σ-algebra iff

1. $\Omega \in \mathscr{A}$
2. If $A \in \mathscr{A}$, then $A^c \in \mathscr{A}$
3. If $A_n \in \mathscr{A}$, $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{A}$

Example 2.4. The set of sets $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ is a σ-algebra since the 3 conditions are satisfied.

Only thing missing is $\Omega \in \mathscr{A}$, which is the proof:

$$\emptyset = A \cap A^c \cap A^c \cap A^c \cap \cdots \in \mathscr{A}$$

Proposition E.1.2

An intersection of multiple σ-algebras is also a σ-algebra.

Let $\{\mathscr{F}_\alpha\}_{\alpha\in Q}$ be a set of σ-algebras on Ω. We want to show that $\bigcap_{\alpha\in Q} \mathscr{F}_\alpha$ is also a σ-algebra.

Example 2.5. We showed that

$$F' = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

is a sigma algebra over $\Omega = \{1,2\}$.

Likewise,

$$F^2 = \{\emptyset, \{1\}, \{2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$$

is a sigma algebra for $\Omega = \{1,2,3\}$. Then,

$$F' \cap F^2 = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$$

is a sigma algebra.

Note

F^2 is not a sigma algebra for $\{1,2\}$. Ω must be the biggest set, i.e. $\{1,2,3\}$.

Note

The intersections of multiple sigma algebras is also a sigma algebra when they are defined over the same Ω. In the example above, $\Omega \neq \Omega'$, but we were essentially looking at

$$F = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}.$$

Note:

$$1 \in \{1,2,3,4\} = S$$

$$\{1\} \subset \{1,2,3,4\}$$

$$\emptyset^c = \{1,2,3,4\}$$

$$\emptyset \notin \{1,2,3,4\}$$

Let $\mathscr{A}$ be a set of sets:

$$\mathscr{A} = \{\emptyset, \{1\}, \{2,5\}, \{1,2\}, \Omega\}$$

$$\emptyset \in \mathscr{A}, \quad \emptyset^c = \{1,2,5\} \in \mathscr{A}$$

$$\{1\}^c = \{2,5\}, \quad \{2,5\}^c = \{1\}, \quad \Omega^c = \emptyset$$

Example 2.6. Let $\mathscr{A}' = \{\{1\}, \{2,5\}, \{1,2\}\}$, then,

$$\emptyset \notin \mathscr{A}', \quad \emptyset^c = \Omega \notin \mathscr{A}'$$

Example 2.7. We know that powersets are sigma algebras. That is, let $\Omega = \{1,2,3,4\}$. Then the minimal (smallest) sigma algebra is

$$\{\emptyset, \Omega\} = \{\emptyset, \{1,2,3,4\}\}$$

The largest is 2^Ω.

Notice that

$$F = \{\emptyset, \Omega\} = \{\emptyset, \{1,2,3,4\}\}$$

is a sigma algebra over Ω because:

- $\Omega \in F$
- Closed under countable unions. There are only two elements in F, and thus

  $$\emptyset \cup \{1,2,3,4\} = \{1,2,3,4\} \in F$$

  Closed under countable intersections

  $$\emptyset \cap \{1,2,3,4\} = \emptyset \in F$$

- Closed under complements. Notice that

  $$\emptyset^c = \{1,2,3,4\} \in F, \quad \{1,2,3,4\}^c = \emptyset \in F$$

Hence, by definition, $F = \{\emptyset, \Omega\} = \{\emptyset, \{1,2,3,4\}\}$ is a sigma algebra over Ω.

Proposition 1.3

Given a class of sets $\mathscr{A}$ of Ω, then there exists a unique "minimal" sigma-algebra containing $\mathscr{A}$, this is the sigma-algebra generated by $\mathscr{A}$.

There is at least one sigma-algebra that contains $\mathscr{A}$: the powerset of Ω, 2^Ω.

We define the smallest sigma-algebra to be the intersection of all sigma-algebras containing $\mathscr{A}$. We call this $\sigma(\mathscr{A})$.

It is a sigma-algebra from Prop 1.2 and by construction is minimal in the sense that it is a subset of all sigma-algebras.

Important

In summation, given any set of sets, we can make a sigma-algebra out of it, and it will be the smallest sigma-algebra containing all sets of $\mathscr{A}$.

Example 2.8. Let $\mathscr{A} = \{\{B\},\{C\}\}$, then we can find the sigma-algebra generated by $\mathscr{A}$, denoted $\sigma(\mathscr{A})$:

$$\sigma(\mathscr{A}) = \{\emptyset, \{B\}, \{C\}, \{B,C\}\}$$

which is a sigma-algebra over $\Omega = \{B, C\}$.

Definition

Let $\mathscr{C}$ be a class of sets of Ω, and let U be a set in Ω. Then we define:

$$\mathscr{C} \cap U = \{C \cap U \mid C \in \mathscr{C}\}$$

Example 2.9. Let $\mathscr{C} = \{\{1,2\},\{3,4\},\{5\}\}$, $U = \{1,3,5,7\}$. Find $\mathscr{C} \cap U$.

$$\{1,2\} \cap \{1,3,5,7\} = \{1\}, \quad \{3,4\} \cap \{1,3,5,7\} = \{3\}, \quad \{5\} \cap \{1,3,5,7\} = \{5\}$$

$$\Rightarrow \mathscr{C} \cap U = \{\{1\},\{3\},\{5\},\emptyset\}$$

Example 2.10. Let $\mathscr{C} = \{\{1\},\{1,2\},\{1,3\},\{2,3\}\}$, $U = \{1,5\}$. Find $\mathscr{C} \cap U$.

$$\{1\} \cap \{1,5\} = \{1\}, \quad \{1,2\} \cap \{1,5\} = \{1\}, \quad \{1,3\} \cap \{1,5\} = \{1\}, \quad \{2,3\} \cap \{1,5\} = \emptyset$$

$$\Rightarrow \mathscr{C} \cap U = \{\{1\},\emptyset\}$$

Example 2.11. Show that $(A \setminus B)^c = A^c \cup (B \cap A)$

Let x be an arbitrary element.

$$x \in (A \setminus B)^c \iff x \notin A \setminus B \iff x \in A^c \cup B$$

$$x \in A^c \cup B \iff x \in A^c \cup (B \cap A)$$

Example 2.12. Show that $A^c \setminus B^c = B \setminus A$

Let $x \in A^c \setminus B^c$

$$\iff x \in A^c \cap (B^c)^c \iff x \in A^c \cap B \iff x \in B \cap A^c \iff x \in B \setminus A$$

Prop 1.4

Let $\mathscr{F}$ be a σ-algebra over Ω and $U \subseteq \Omega$, then $\mathscr{F} \cap U$ is a σ-algebra over U.

Proof. We want to show (i), (ii), and (iii).

(i) $U = \Omega \cap U \in \mathscr{F} \cap U$

(ii) Let $B \in \mathscr{F} \cap U$, then $\exists C \in \mathscr{F}$ such that $B = C \cap U$. We want to show $U \setminus B \in \mathscr{F} \cap U$.

$$\begin{aligned} U \setminus B &= U \cap B^c \\ &= U \cap (C \cap U)^c \\ &= U \cap (C^c \cup U^c) \\ &= (U \cap C^c) \cup (U \cap U^c) \\ &= U \cap C^c \\ &= (\Omega \setminus C) \cap U \end{aligned}$$

Since $\Omega \setminus C \in \mathscr{F}$, we conclude $U \setminus B \in \mathscr{F} \cap U$

(iii) Let $B_1, B_2, B_3, \ldots$ be a countable collection of elements in $\mathscr{F} \cap U$. Then

$$\bigcup_{n \in \mathbb{N}} B_n \in \mathscr{F} \cap U$$

Since $B_i \in \mathscr{F} \cap U$, there exist $C_i \in \mathscr{F}$ such that $B_i = C_i \cap U$. Then:

$$\begin{aligned}\bigcup_{n\in\mathbb{N}} B_n &= \bigcup_{n\in\mathbb{N}} (C_i \cap U) \\ &= \left(\bigcup_{n\in\mathbb{N}} C_i\right) \cap U\end{aligned}$$

Since $\bigcup_{n\in\mathbb{N}} C_i \in \mathscr{F}$, it follows that

$$\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{F} \cap U$$

Hence $\mathscr{F} \cap U$ is a σ-algebra.

□

Example 2.13. Recall that $\mathscr{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$ is a σ-algebra over $\Omega = \{1,2,3\}$. Claim that $\mathscr{F} \cap U$ is a σ-algebra over $U = \{2,3\}$.

$$\emptyset \cap \{2,3\} = \emptyset, \quad \{1\} \cap \{2,3\} = \emptyset, \quad \{3\} \cap \{2,3\} = \{3\}$$

$$\{1,2\} \cap \{2,3\} = \{2\}, \quad \{1,3\} \cap \{2,3\} = \{3\}, \quad \{2,3\} \cap \{2,3\} = \{2,3\}$$

i.e.

$$\mathscr{F} \cap U = \{\{2\}, \{3\}, \{2,3\}, \emptyset\}$$

Which is indeed a σ-algebra for $U = \{2,3\}$ since:

(i) $U = \{2,3\} \in \mathscr{F} \cap U$

(ii) $\{2\} \cup \{3\} = \{2,3\}$, all unions are closed and $\{2\} \cap \{3\} = \emptyset$, all intersections are closed.

(iii) All complements are here.

  (a) Notice that $u = \Omega \cap U \in \mathscr{F} \cap U$

  (b) Let $B \in \mathscr{F} \cap U \Rightarrow \exists C \in \mathscr{F}$ such that $B = C \cap U$

  Then:

$$B^c = U \setminus B = (\Omega \setminus C) \cap U \in \mathscr{F} \cap U$$

Proposition 1.5

Let $\mathscr{C}$ be a set and U be a set, then

$$\sigma(\mathscr{C} \cap U) = \sigma(\mathscr{C}) \cap U$$

We must find complements with respect to U. Recall, generated means the smallest σ-algebra that contains the set.

Example 2.14. Let $\Omega = \{1,2,3,4,5,6\}$ ("Die").

Then notice that

$$2^{\Omega} = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \dots, \{1,2,3,4,5,6\} = \Omega\}$$

If $\mathscr{C} = \{\varnothing, \{1\}\}$, then the smallest σ-algebra that has $\mathscr{C}$ is

$$\sigma(\mathscr{C}) = \{\varnothing, \{1\}, \{2,3,4,5,6\}, \Omega\}$$

Example 2.15. Again, let $\Omega = \{1,2,3,4,5,6\}$, $\mathscr{C} = \{\{1,3,5\}, \{2,4,6\}\}$, and $U = \{1,2\}$.

Then

$$\mathscr{C} \cap U = \{C \cap U : C \in \mathscr{C}\} = \{\{1\}, \{2\}\}$$

That is, the $\sigma(\mathscr{C} \cap U)$ on $U = \{1,2\}$ is

$$\sigma(\mathscr{C} \cap U) = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$$

Now,

$$\sigma(\mathscr{C}) \cap U = \{\varnothing, \{1,3,5\}, \{2,4,6\}, \Omega\} \cap \{1,2\}$$

$$\begin{aligned} \varnothing \cap \{1,2\} &= \varnothing \\ \{1,3,5\} \cap \{1,2\} &= \{1\} \\ \{2,4,6\} \cap \{1,2\} &= \{2\} \\ \Omega \cap \{1,2\} &= \{1,2\} \end{aligned}$$

Therefore,

$$\sigma(\mathscr{C}) \cap U = \{\varnothing, \{1\}, \{2\}, \{1,2\}\}$$

Proof. We want to prove that

$$\sigma(\mathscr{C} \cap U) = \sigma(\mathscr{C}) \cap U.$$

Since $\mathscr{C} \subseteq \sigma(\mathscr{C})$, and intersection with U preserves containment, we have

$$\mathscr{C} \cap U \subseteq \sigma(\mathscr{C}) \cap U.$$

So the σ-algebra generated by $\mathscr{C} \cap U$ is contained in any σ-algebra that contains $\mathscr{C} \cap U$, and in particular,

$$\sigma(\mathscr{C} \cap U) \subseteq \sigma(\mathscr{C}) \cap U.$$

Let us now define the set

$$\mathscr{G} = \{A \subseteq \Omega : A \cap U \in \sigma(\mathscr{C} \cap U)\}.$$

We will show that $\mathscr{G}$ is a σ-algebra.

(i) Since $\Omega \cap U = U \in \sigma(\mathscr{C} \cap U)$, we have $\Omega \in \mathscr{G}$.

(ii) Suppose $A \in \mathscr{G}$, so $A \cap U \in \sigma(\mathscr{C} \cap U)$. Then

$$\begin{aligned} A^c \cap U &= (U \cap A^c) \cup (U \cap U^c) \\ &= U \cap (A^c \cup U^c) \\ &= U \cap (A \cap U)^c \\ &= U - (A \cap U) \end{aligned}$$

since $\sigma(\mathscr{C} \cap U)$ is a σ-algebra. Hence, $A^c \in \mathscr{G}$.

(iii) Suppose $A_n \in \mathscr{G}$ for each $n \in \mathbb{N}$. Then $A_n \cap U \in \sigma(\mathscr{C} \cap U)$, and so

$$\left(\bigcup_{n \in \mathbb{N}} A_n\right) \cap U = \bigcup_{n \in \mathbb{N}} (A_n \cap U) \in \sigma(\mathscr{C} \cap U),$$

so $\bigcup_n A_n \in \mathscr{G}$.

Therefore, $\mathscr{G}$ is a σ-algebra. Also, since for every $C \in \mathscr{C}$, we have $C \cap U \in \sigma(\mathscr{C} \cap U)$, it follows that $C \in \mathscr{G}$, so $\mathscr{C} \subseteq \mathscr{G}$. Thus,

$$\sigma(\mathscr{C}) \subseteq \mathscr{G}.$$

Finally, since $A \in \sigma(\mathscr{C}) \Rightarrow A \in \mathscr{G} \Rightarrow A \cap U \in \sigma(\mathscr{C} \cap U)$, we conclude that

$$\sigma(\mathscr{C}) \cap U \subseteq \sigma(\mathscr{C} \cap U).$$

Combining both directions, we obtain the desired equality:

$$\sigma(\mathscr{C} \cap U) = \sigma(\mathscr{C}) \cap U.$$

□

2.2 Measures

Definition

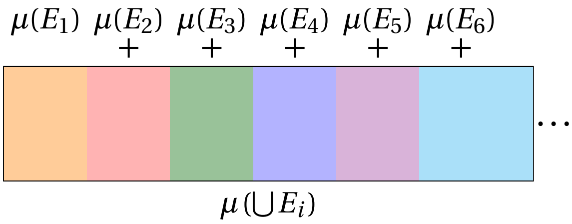
Let $\mathscr{F}$ be an algebra. A *measure* is a function

$$\mu : \mathscr{F} \to [0, \infty] \quad \text{i.e.} \quad \mu : \mathscr{F} \to [0, \infty)$$

such that:

1. **Empty set:** $\mu(\emptyset) = 0$
2. **Countable additivity:** If $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of disjoint events in $\mathscr{F}$ (i.e. $E_i \cap E_j = \emptyset$ whenever $i \neq j$), then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \left(\text{i.e. } \mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i) \Rightarrow \lim_{n \to \infty} \sum_{i=1}^{n} \mu(E_i)\right)$$



(Union of disjoint sets)

If $\mathscr{F}$ is a σ-algebra, then the triplet $(\Omega, \mathscr{F}, \mu)$ is called a *measure space*.

Example 2.16. Let $\Omega = \mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathscr{F} = \mathscr{P}(\mathbb{N}) = \{\emptyset, \{1\}, \{1, 2\}, \ldots, \{1, 2, \ldots, n\}, \ldots\}$.

Notice $\mathscr{P}(\mathbb{N})$ is a σ-algebra for $\mathbb{N}$.

Define the measure function μ as the *counting measure*. This assigns each subset of Ω its size. Then for $E \in \mathscr{F}$, define

$$\mu(E) = \sum_{x \in E} 1 = 0$$

Example 2.17. If $E_3 = \{a, b, c\}$ then $\mu(E_3) = 1 + 1 + 1 = 3$

Proof. Showing (i): "$\mu(\emptyset) = 0$". Notice that

$$\mu(\emptyset) = \sum_{x \in \emptyset} 1 = 0$$

since the empty set has no elements. Lets show countable additivity,

If $\{E_i\}_{i=1}^{\infty}$ is a countable collection of pairwise disjoint sets in $\mathscr{F}$, i.e. $E_i \cap E_j = \emptyset$ for $i \neq j$, then

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \sum_{x \in \bigcup_{i=1}^{\infty} E_i} 1 \\
&= \sum_{i=1}^{\infty} \sum_{x \in E_i} 1 \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}$$

For example, if

$$E_1 = \{1,2\}, \quad E_2 = \{3,4,5\}, \quad E_3 = E_4 = \cdots = \emptyset$$

then $E_1, E_2, E_3, \ldots$ are all disjoint.

$$\begin{aligned}
\mu\left(\bigcup_{i=1}^{\infty} E_i\right) &= \mu(E_1 \cup E_2 \cup E_3 \cup \cdots) \\
&= \mu(\{1,2\} \cup \{3,4,5\} \cup \emptyset \cup \cdots) \\
&= \mu(\{1,2,3,4,5\}) \\
&= 5 \\
&= 2 + 3 + 0 + \cdots \\
&= \mu(E_1) + \mu(E_2) + \mu(E_3) + \cdots \\
&= \sum_{i=1}^{\infty} \mu(E_i)
\end{aligned}$$

Since $\mathscr{P}(\mathbb{N})$ is a σ-algebra over $\Omega = \mathbb{N}$, then $(\mathbb{N}, \mathscr{P}(\mathbb{N}), \mu)$ is a measure space.

If $\mu(\Omega) = 1$, then μ is a probability measure.

$$\mu(\Omega) = \infty$$

$\square$

Example 2.18. Define the Dirac measure or point mass measure: Let $\Omega = X$, $x_0 \in X$, $\mathscr{F} = \mathscr{P}(X)$, and $E \in \mathscr{P}(X)$. Define

$$S_{x_0}(E) = \begin{cases} 1 & \text{if } x_0 \in E \\ 0 & \text{if } x_0 \notin E \end{cases}$$

Proof. (i) Showing the measure of the empty set is 0,

$$S_{x_0}(\emptyset) = \begin{cases} 1 & \text{if } x_0 \in \emptyset \\ 0 & \text{if } x_0 \notin \emptyset \end{cases} = 0$$

since $\emptyset$ has no elements.

(ii) Showing that $S_{x_0}\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} S_{x_0}(E_i)$,

Let $\{E_i\}_{i=1}^{\infty} \subset \mathscr{F}$ be disjoint sets.

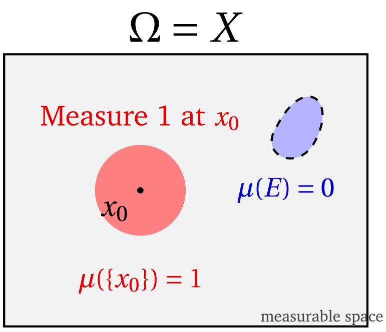
Assume $x_0 \in \bigcup_{i=1}^{\infty} E_i$. Then there exists a unique E_i such that $x_0 \in E_i$. Thus,

$$S_{x_0}\left(\bigcup_{i=1}^{\infty} E_i\right) = \begin{cases} 1 & \text{if } x_0 \in E_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases} = 1$$

Then $S_{x_0}(E_i) = 1$ for that index, and $S_{x_0}(E_j) = 0$ for $j \neq i$. Therefore,

$$\sum_{i=1}^{\infty} S_{x_0}(E_i) = 0 + 0 + 1 + 0 + \cdots = 1$$

If $x_0 \notin \bigcup_{i=1}^{\infty} E_i$, then both sides are 0. Hence, countable additivity holds for S_{x_0}. $\square$



Example 2.19. Let X be non-countable and let

$$\mathscr{F} = \{E \subseteq X : E \text{ is countable or } E^c \text{ is countable}\}.$$

Define

$$\mu(E) = \begin{cases} 1 & \text{if } E^c \text{ is countable} \\ 0 & \text{if } E \text{ is countable} \end{cases}$$

Show that μ is a measure.

Proof. (i) First, we show that $\mu(\emptyset) = 0$. Since $\emptyset$ has no elements, it is trivially countable, so:

$$\mu(\emptyset) = 0.$$

(ii) Let $\{E_i\}_{i=1}^{\infty} \subseteq \mathscr{F}$ be a sequence of disjoint sets. We consider two cases:

Case 1. If all of the E_i's are countable, then

$$\mu(E_i) = 0 \quad \forall i.$$

Then the countable union $\bigcup_{i=1}^{\infty} E_i$ is countable, hence

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = 0,$$

and

$$\sum_{i=1}^{\infty} \mu(E_i) = 0.$$

Case 2. Suppose not all E_i's are countable. Since the sets are disjoint, if $E_i \subseteq E_j^c$ and $E_j \subseteq E_i^c$, then there can be at most one i_0 such that $E_{i_0}^c$ is countable (i.e. E_{i_0} is uncountable).

So we write:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i \neq i_0} E_i \cup E_{i_0}\right).$$

The union of countable sets $\bigcup_{i \neq i_0} E_i$ is countable, and E_{i_0} is uncountable. Therefore:

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \mu\left(\bigcup_{i \neq i_0} E_i\right) + \mu(E_{i_0}) = 0 + 1 = 1,$$

and also

$$\sum_{i=1}^{\infty} \mu(E_i) = \sum_{i \neq i_0} \mu(E_i) + \mu(E_{i_0}) = 0 + 1 = 1.$$

In both cases, countable additivity holds. Hence, μ is a measure. □

Proposition 2.1 (The Continuity of Measure)

Any measure μ with $\mu(\Omega) < \infty$ satisfies the following properties:

- Finite additivity: For a finite sequence of disjoint $E_1, \ldots, E_k \in \mathscr{F}$,

$$\mu\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} \mu(E_i)$$

- Continuity from below: If $E_n \nearrow E$, $E \in \mathscr{F}$, and $E_n \in \mathscr{F}$ for all $n \in \mathbb{N}$, then

$$\mu(E_n) \nearrow \mu(E)$$

- Continuity from above: If $E_n \searrow E$, $E \in \mathscr{F}$, and $\mu(E_1) < \infty$, then

$$\mu(E_n) \searrow \mu(E)$$

To prove continuity from below, define the sets:

$$B_1 = E_1, \quad B_2 = E_2 \setminus E_1, \quad B_3 = E_3 \setminus E_2, \quad \ldots, \quad B_n = E_n \setminus E_{n-1}$$

Then the union is disjoint and

$$\begin{aligned}\mu(E) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) &= \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{k\to\infty} \sum_{n=1}^{k} \mu(B_n) = \lim_{k\to\infty} \mu\left(\bigcup_{n=1}^{k} B_n\right) = \lim_{k\to\infty} \mu(E_k)\end{aligned}$$

To prove continuity from above, suppose $E_n \searrow E$. Define

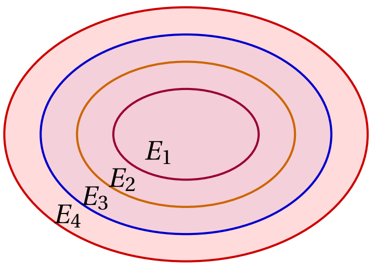
$$A_n = E_n \setminus E$$

Then $A_n \searrow \emptyset$, and since μ is countably additive and $\mu(E_1) < \infty$, we have

$$\mu(E_n) = \mu(E \cup A_n) = \mu(E) + \mu(A_n)$$

$$\lim_{n\to\infty} \mu(E_n) = \mu(E) + \lim_{n\to\infty} \mu(A_n) = \mu(E)$$

since $\mu(A_n) \to 0$.



From Prop A.42,

$$\begin{aligned}\lim_{n\to\infty}(E_1 \setminus E_n) &= \bigcup_{n\in\mathbb{N}}(E_1 \setminus E_n)\\ &= E_1 \setminus E\end{aligned}$$

Since

$$\begin{aligned}\bigcup_{n\in\mathbb{N}}(E_1 \setminus E_n) &= \bigcup_{n\in\mathbb{N}}\left(E_1 \cap E_n^c\right)\\ &= E_1 \cap \left(\bigcup_{n\in\mathbb{N}} E_n^c\right)\\ &= E_1 \cap \left(\bigcap_{n\in\mathbb{N}} E_n\right)^c\\ &= E_1 \cap E^c\\ &= E_1 \setminus E\end{aligned}$$

From 2 above, since

$$\begin{aligned}&E_1 \setminus E_n \uparrow E_1 \setminus E \text{ then}\\ &\qquad \lim_{n\to\infty}\mu(E_1 \setminus E_n) = \mu(E_1 \setminus E)\end{aligned}$$

But

$$\begin{aligned}\mu(E_1) &= \mu\left((E_1 \setminus E_n) \cup E_n\right)\\ &= \mu(E_1 \setminus E_n) + \mu(E_n)\end{aligned}$$

$$\text{Then} \quad \mu(E_1 \setminus E_n) = \mu(E_1) - \mu(E_n)$$

Then by limits

$$\lim_{n\to\infty} \left[\mu(E_1) - \mu(E_n)\right] = \mu(E_1) - \mu(E)$$
$$\Rightarrow \mu(E_1) - \lim_{n\to\infty} \mu(E_n) = \mu(E_1) - \mu(E)$$
$$\Rightarrow \lim_{n\to\infty} \mu(E_n) = \mu(E)$$

Proposition 2.1 (The Continuity of Measure)

Any measure with $\mu(\Omega) < \infty$ satisfies the following properties.

- **Finite Additivity.** For a finite sequence of disjoint sets $E_1, \ldots, E_k$,

$$\mu\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} \mu(E_i)$$

- **Continuity from Below.** If $E_n \nearrow E$, and each $E_n \in \mathscr{F}$, then

$$\mu(E_n) \nearrow \mu(E)$$

That is, if $E_n \subseteq E_{n+1}$ and $E = \bigcup_{n=1}^{\infty} E_n$, then

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$$

Let $B_1 = E_1$, $B_2 = E_2 \setminus E_1$, $B_3 = E_3 \setminus E_2$, $\ldots$, $B_n = E_n \setminus E_{n-1}$. Then

$$\mu(E_n) = \sum_{i=1}^{n} \mu(B_i)$$
$$\mu(E) = \sum_{i=1}^{\infty} \mu(B_i)$$

So the limit of the partial sums converges to the infinite sum.

- **Continuity from Above.** If $E_n \searrow E$, with $\mu(E_1) < \infty$, then

$$\mu(E_n) \searrow \mu(E)$$

That is, if $E_{n+1} \subseteq E_n$ and $E = \bigcap_{n=1}^{\infty} E_n$, then

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$$

To show this, note:

$$E_1 \setminus E_n = \bigcup_{k=n}^{\infty} (E_k \setminus E_{k+1})$$
$$\mu(E_1 \setminus E_n) \nearrow \mu(E_1 \setminus E)$$

So,

$$\mu(E_n) = \mu(E_1) - \mu(E_1 \setminus E_n)$$

and

$$\lim_{n\to\infty} \mu(E_n) = \mu(E_1) - \mu(E_1 \setminus E) = \mu(E)$$

- **Countable Subadditivity.** If $E_n \in \mathscr{F}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Define $B_1 = E_1$, $B_2 = E_2 \setminus E_1$, $B_3 = E_3 \setminus (E_1 \cup E_2)$, ..., so that

$$B_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$$

Then the B_n are disjoint and

$$\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} B_n$$

and since μ is countably additive on disjoint sets,

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Hence,

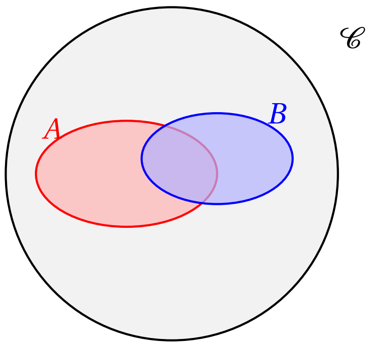
$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)$$

Next, we discus Carathéodory's Extension Theorem. It is hard to determine a measure even on a σ-algebra.It is even harder for a powerset. Then we will define a measure on a σ-algebra which will extend to a σ-algebra.

Def. E.3.1

A class of sets $\mathscr{C}$ is a π-system if

$$A, B \in \mathscr{C} \text{ then } A \cap B \in \mathscr{C}$$



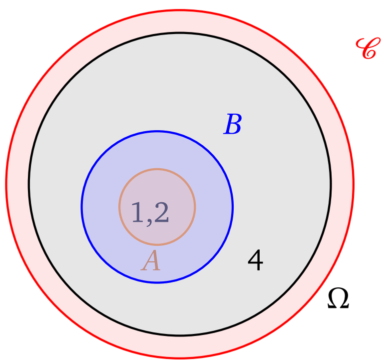
Example 2.20. Let $\Omega = \{1,2,3,4\}$ and let's construct $\mathscr{C}$ as a collection of subsets of Ω such that

$$\mathscr{C} = \{\{1,2\}, \{1,2,3\}, \Omega, \emptyset\}$$

Then indeed $\mathscr{C}$ is a π-system since the intersection of any of the sets are indeed in $\mathscr{C}$.

Notice $A = \{1,2\} \in \mathscr{C}$ but

$$\{\{1,2\}\} \subseteq \mathscr{C} \quad \text{does not imply} \quad \{\{1,2\},\{1,2,3\}\} \subseteq \mathscr{C}$$



Definition E.3.2

A class of sets $\mathscr{C}$ is a λ-system if

i) $\Omega \in \mathscr{C}$

ii) $A \in \mathscr{C} \Rightarrow A^c \in \mathscr{C}$

iii) If $A_n \in \mathscr{C}$, $n \in \mathbb{N}$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{C}$

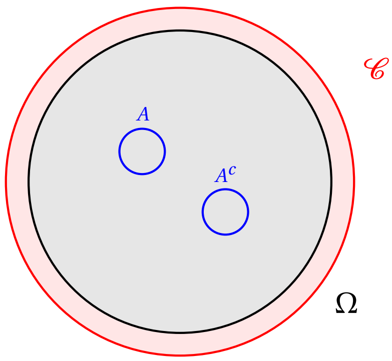
Example. $\mathscr{C} = \{\Omega, \{1,2\}, \{3,4,5,6\}, \emptyset\}$ where $\Omega = \{1,2,3,4,5,6\}$.

Proposition E.3.1

If $\mathscr{C}$ satisfies (i) and (iii) in the definition above, then $\mathscr{C}$ satisfies (ii) iff

$$A, B \in \mathscr{C}, \ A \subseteq B \Rightarrow B \setminus A \in \mathscr{C}$$

(ii') $A, B \in \mathscr{C}, A \subseteq B \Rightarrow B \setminus A \in \mathscr{C}$



the union of all disjoint sets are in here

Example 2.21. Let $\mathscr{C} = \{\Omega, \{1,2\}, \{3,4,5,6\}, \emptyset\}$ where $\Omega = \{1,2,3,4,5,6\}$. Show that $\mathscr{C}$ is a λ-system.

We check all three conditions:

i) $\Omega \in \mathscr{C}$ ✓

ii) $\{1,2\}^c = \{3,4,5,6\} \in \mathscr{C}$ and $\{3,4,5,6\}^c = \{1,2\} \in \mathscr{C}$ ✓

iii) $\{1,2\}$ and $\{3,4,5,6\}$ are disjoint, and $\{1,2\} \cup \{3,4,5,6\} = \Omega \in \mathscr{C}$ ✓

So $\mathscr{C}$ is a λ-system.

Proposition E.3.2

A set $\mathscr{C}$ that is both a π-system and a λ-system is a σ-algebra.

Proof. i) $\Omega \in \mathscr{C}$, since $\mathscr{C}$ is a λ-system.

ii) If $A \in \mathscr{C}$, then $A^c \in \mathscr{C}$, since $\mathscr{C}$ is a λ-system.

iii) Let $A_n \in \mathscr{C}$, $n \in \mathbb{N}$, and we want to show $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{C}$.

Define:

$$\begin{aligned} B_1 &= A_1 \\ B_2 &= A_2 \cap A_1^c \\ B_3 &= A_3 \cap A_2^c \cap A_1^c \\ &\vdots \\ B_n &= A_n \cap A_{n-1}^c \cap \cdots \cap A_1^c \end{aligned}$$

Each $B_n \in \mathscr{C}$ since $\mathscr{C}$ is a π-system (intersections) and closed under complements.

The B_n are disjoint, and:

$$\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n$$

Since $\mathscr{C}$ is a λ-system, the disjoint union $\bigcup_{n\in\mathbb{N}} B_n \in \mathscr{C}$. Hence, $\mathscr{C}$ is closed under countable unions.

Therefore, $\mathscr{C}$ is a σ-algebra. □

Proposition E.3.3

The intersection of several λ-systems is a λ-system.

(Analogous proof as Proposition E.1.2.)

Proposition E.3.4 (Dynkin's Theorem)

If $\mathscr{C}$ is a π-system and $\mathscr{D}$ is a λ-system, then if $\mathscr{C} \subseteq \mathscr{D}$, we have $\sigma(\mathscr{C}) \subseteq \mathscr{D}$.

Let $\mathscr{D}'$ be the smallest λ-system that contains $\mathscr{C}$. Then

$$\mathscr{C} \subseteq \mathscr{D}' \subseteq \mathscr{D}$$

We want to show that $\mathscr{D}'$ is a π-system. Then, from Proposition E.3.2, it will imply that $\mathscr{D}'$ is a σ-algebra containing $\mathscr{C}$. So then,

$$\mathscr{C} \subseteq \sigma(\mathscr{C}) \subseteq \mathscr{D}' \subseteq \mathscr{D}$$

That is, $\sigma(\mathscr{C}) \subseteq \mathscr{D}$, which would conclude the proof. That is, we will show that if $A, B \in \mathscr{D}'$ then $A \cap B \in \mathscr{D}'$.

Proof. We want to show that if $A \in \mathscr{D}'$, then the collection

$$D_A := \{B \subseteq \Omega : A \cap B \in \mathscr{D}'\}$$

is a λ-system.

(i) Since $A \cap \Omega = A \in \mathscr{D}'$, it follows that $\Omega \in D_A$.

(ii) Instead of proving that $B \in D_A \Rightarrow B^c \in D_A$, we prove a stronger closure property, if $B, B' \in D_A$ and $B \subseteq B'$, then $B' \setminus B \in D_A$.

Since $B, B' \in D_A$, we know,

$$A \cap B \in \mathscr{D}' \quad \text{and} \quad A \cap B' \in \mathscr{D}'.$$

Now compute,

$$\begin{aligned} A \cap (B' \setminus B) &= (A \cap B') \setminus (A \cap B) \\ &= (A \cap B') \cap (A \cap B)^c \\ &= (A \cap B') \cap (A^c \cup B^c) \\ &= \left[(A \cap B') \cap A^c\right] \cup \left[(A \cap B') \cap B^c\right] \\ &= \emptyset \cup A \cap (B' \setminus B) \\ &= A \cap (B' \setminus B). \end{aligned}$$

Since $A \cap B' \in \mathscr{D}'$ and $A \cap B \in \mathscr{D}'$, and $\mathscr{D}'$ is a λ-system, we know their difference is also in $\mathscr{D}'$, so,

$$A \cap (B' \setminus B) \in \mathscr{D}' \Rightarrow B' \setminus B \in D_A.$$

(iii) Let $B_n \in D_A$ be a sequence of disjoint sets. Then $A \cap B_n \in \mathscr{D}'$ for all n, and since

$\mathscr{D}'$ is a λ-system,

$$A \cap \left(\bigcup_{n \in \mathbb{N}} B_n \right) = \bigcup_{n \in \mathbb{N}} (A \cap B_n) \in \mathscr{D}'.$$

Thus, $\bigcup_n B_n \in D_A$. Therefore, D_A is a λ-system. □

Proof. Step 2A. Suppose $A \in \mathscr{C}$. Since $\mathscr{C} \subseteq \mathscr{D}'$, it follows that $A \in \mathscr{D}'$.

From Step 1, we showed that for any $A \in \mathscr{D}'$, the collection

$$D_A := \{B \subseteq \Omega : A \cap B \in \mathscr{D}'\}$$

is a λ-system. So D_A is a λ-system.

Furthermore, since $\mathscr{C} \subseteq \mathscr{D}'$, we claim $\mathscr{C} \subseteq D_A$. That is, for any $C \in \mathscr{C}$, we want to show $C \in D_A$, i.e. $A \cap C \in \mathscr{D}'$.

But since $\mathscr{C}$ is a π-system, and $A, C \in \mathscr{C}$, it follows that

$$A \cap C \in \mathscr{C} \subseteq \mathscr{D}'.$$

Hence, $C \in D_A$, and thus $\mathscr{C} \subseteq D_A$.

Now recall that $\mathscr{D}'$ is the smallest λ-system containing $\mathscr{C}$, and we have shown $\mathscr{C} \subseteq D_A$, where D_A is itself a λ-system. Therefore,

$$\mathscr{D}' \subseteq D_A.$$

That is, $A \in \mathscr{D}' \Rightarrow \mathscr{D}' \subseteq D_A$.

Step 2. Let $B \in \mathscr{D}'$. From Step 1, D_B is a λ-system. From Step 2A, we just showed that $\mathscr{D}' \subseteq D_B$.

That is, for any $A \in \mathscr{D}'$, we now have,

$$A \in D_B \quad \text{which means} \quad A \cap B \in \mathscr{D}'.$$

So we conclude that,

$$A, B \in \mathscr{D}' \Rightarrow A \cap B \in \mathscr{D}',$$

i.e., $\mathscr{D}'$ is closed under finite intersections.

Therefore, $\mathscr{D}'$ is a π-system.

□

Proof. Step 3. Let $A, B \in \mathscr{D}'$. We want to show that $A \cap B \in \mathscr{D}'$, i.e., $\mathscr{D}'$ is a π-system.

We already showed in Step 2 that for any $A \in \mathscr{D}'$, the collection

$$D_A := \{B \subseteq \Omega : A \cap B \in \mathscr{D}'\}$$

is a λ-system and that $\mathscr{D}' \subseteq D_A$. Hence, for any $B \in \mathscr{D}'$, we have $B \in D_A$, which means

$$A \cap B \in \mathscr{D}'.$$

Therefore, $\mathscr{D}'$ is closed under finite intersections:

$$A, B \in \mathscr{D}' \Rightarrow A \cap B \in \mathscr{D}'.$$

This shows that $\mathscr{D}'$ is a π-system. Now since $\mathscr{D}'$ is both a π-system and a λ-system, we conclude from Proposition E.3.2 that $\mathscr{D}'$ is a σ-algebra. Finally, recall that $\mathscr{C} \subseteq \mathscr{D}' \subseteq D$, where D was an arbitrary λ-system containing $\mathscr{C}$, and that $\sigma(\mathscr{C}) \subseteq \mathscr{D}'$ because $\mathscr{D}'$ is a σ-algebra containing $\mathscr{C}$. Therefore,

$$\sigma(\mathscr{C}) \subseteq \mathscr{D}' \subseteq D.$$

Since D was arbitrary, it follows that $\sigma(\mathscr{C}) \subseteq \bigcap D$, i.e., the smallest λ-system containing $\mathscr{C}$ contains $\sigma(\mathscr{C})$, so

$$\sigma(\mathscr{C}) \subseteq D.$$

This completes the proof of Dynkin's theorem.

□

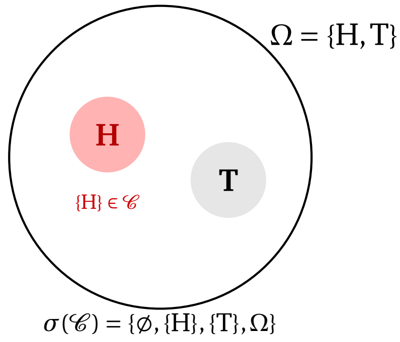
Example 2.22. Let our sample space be $\Omega = \{\mathrm{H}, \mathrm{T}\}$ and define the Dynkin system $D = \{\emptyset, \{\mathrm{H}\}, \{\mathrm{T}\}, \Omega\}$. Define the π-system $\mathscr{C} = \{\{\mathrm{H}\}\}$.

We want to verify that $\mathscr{C} \subseteq D$, and find $\sigma(\mathscr{C})$. Since the only set in $\mathscr{C}$ is $\{\mathrm{H}\}$, we note:

$$\sigma(\mathscr{C}) = \{\emptyset, \{\mathrm{H}\}, \{\mathrm{T}\}, \Omega\} \subseteq D.$$

This aligns with Dynkin's theorem, which ensures that:

$$\sigma(\mathscr{C}) \subseteq D.$$



This example shows that even with a minimal π-system, the generated σ-algebra can contain more subsets, but Dynkin's theorem ensures it still lies inside any Dynkin system D that contains the original π-system.

Corollary E.3.1

Let P_1, P_2 be two probability measures on $\sigma(\mathscr{C})$, where $\mathscr{C}$ is a π-system. If P_1 and P_2 agree on $\mathscr{C}$, then they also agree on $\sigma(\mathscr{C})$.

Example 2.23. Let $\mathscr{C} = \{\{1\}, \{2\}, \{3\}\}$ be the outcomes of rolling a die up to 3. Show $\mathscr{C}$ is a π-system and compute $\sigma(\mathscr{C})$.

Then $\sigma(\mathscr{C}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$.

Let P_1, P_2 be two probability measures such that:

$$P_1(\{1\}) = P_2(\{1\}) = \frac{1}{6}, \quad P_1(\{2\}) = P_2(\{2\}) = \frac{1}{6}.$$

Corollary E.3.1

Let P_1, P_2 be two probability measures on $\sigma(\mathscr{C})$, where $\mathscr{C}$ is a π-system. If P_1 and P_2 agree on $\mathscr{C}$, then they also agree on $\sigma(\mathscr{C})$.

Proof. Let $\mathscr{F}$ be a class where both P_1 and P_2 agree. Then $\mathscr{C} \subseteq \mathscr{F}$. We must show that $\mathscr{F}$ is a λ-system and π-system. Therefore, by Dynkin's theorem, $\mathscr{F}$ is a σ-algebra that has $\mathscr{C}$. Using Dynkin's theorem:

$$\mathscr{C} \subseteq \sigma(\mathscr{C}) \subseteq \mathscr{F} \quad \text{so} \quad P_1, P_2 \text{ agree on } \sigma(\mathscr{C}).$$

Step 1: Show $\Omega \in \mathscr{F}$. Since both P_1 and P_2 are probability measures,

$$P_1(\Omega) = P_2(\Omega) = 1 \Rightarrow \Omega \in \mathscr{F}.$$

Step 2: Show $A \in \mathscr{F} \Rightarrow A^c \in \mathscr{F}$. Let $A \in \mathscr{F}$, then

$$P_1(A) = P_2(A) \Rightarrow 1 - P_1(A) = 1 - P_2(A) \Rightarrow P_1(A^c) = P_2(A^c),$$

so $A^c \in \mathscr{F}$.

Step 3: Let $A_n \in \mathscr{F}$ **be pairwise disjoint. Show** $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$

$$P_1\left(\bigcup_{n=1}^{\infty} A_n\right) = P_2\left(\bigcup_{n=1}^{\infty} A_n\right)$$

because both P_1 and P_2 are countably additive,

$$P_1\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P_1(A_n) = \sum_{n=1}^{\infty} P_2(A_n) = P_2\left(\bigcup_{n=1}^{\infty} A_n\right).$$

Therefore, $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$, so $\mathscr{F}$ is a λ-system.

Now show that $\mathscr{F}$ is a π-system. Since we have already shown that $\mathscr{F}$ is closed under complements and unions, then $\mathscr{F}$ must be closed under finite intersections:

$$A, B \in \mathscr{F} \Rightarrow A^c, B^c \in \mathscr{F} \Rightarrow A^c \cup B^c \in \mathscr{F} \Rightarrow (A^c \cup B^c)^c = A \cap B \in \mathscr{F}.$$

So $\mathscr{F}$ is a π-system.

Since $\mathscr{F}$ is both a λ-system and π-system that contains $\mathscr{C}$, by Dynkin's theorem,

$$\mathscr{C} \subseteq \sigma(\mathscr{C}) \subseteq \mathscr{F}.$$

Therefore, P_1 and P_2 agree on $\sigma(\mathscr{C})$.

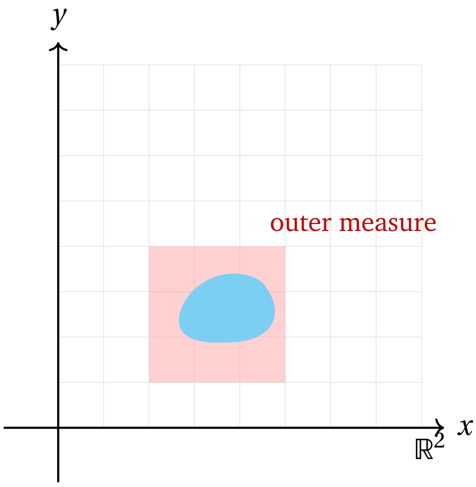
□

Outer Measure: *Approximate measure from outside* (Doesn't need to be measurable)

Note: If $A \not\subseteq B$, then $P(A)$ may not be defined. But if $A \subseteq \Omega$, then

$$P^*(A) = \inf P(\Omega) = 1$$

Idea: We are going to look at these overestimates and try to take the smallest infimum of these overestimates. It will tell us how to actually measure A.



In other words, it is the smallest way to cover the blue blob (or our sets).

Definition E.3.3

Let P be a probability measure on an algebra $\mathscr{C}$. For each set $A \subseteq \Omega$, the **outer measure** of A is

$$P^*(A) = \inf \sum P(A_n)$$

where the infimum is taken over all (finite or countable) covers $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathscr{C}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$.

Smallest way that we can cover A in these sets.

Let P be a probability measure on an algebra $\mathscr{C}$. Define another measure P^* by

$$P^*(A) = \inf \sum_n P(A_n)$$

where $A \subseteq \bigcup A_n$, and the cover $\{A_n\}$ is either finite or countable.

Definition: Measurable Set

Let P be a probability measure on an algebra $\mathscr{C}$. Then $A \subseteq \Omega$ is P^*-measurable if

$$P^*(A \cap E) + P^*(A^c \cap E) = P^*(E) \quad \text{for all } E \in \mathscr{C} \subseteq \Omega.$$

The set of all P^*-measurable sets is denoted $\mathscr{M}$.

Note that

$$P^*(A \cap E) + P^*(A^c \cap E) \geq P^*(E)$$

is always true by subadditivity. Hence, to verify measurability, it is enough to check

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E).$$

If $P^*(E) < \infty$, we must check this inequality. If $P^*(E) = \infty$, then the inequality trivially holds, so we only need to check the condition for E with finite outer measure.

Proposition E.3.5

The outer probability measure function P^* satisfies the following:

$$P^*(A) = \inf \sum P(A_n)$$

1. $P^*(\emptyset) = 0$
2. $P^*(A) \geq 0$
3. If $A \subseteq B$, then $P^*(A) \leq P^*(B)$ (monotonicity)
4. $P^*\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum_n P^*(A_n)$ (sub-countable additivity)

Proof. 1. $P^*(\emptyset) = \inf \sum_{n\in\mathbb{N}} P(\emptyset) = 0$

2. $P^*(A) \geq 0$

   Notice that $P^*(A) = \inf P(A)$, but P is a non-negative function, so,

$$P^*(A) = \inf P(A) \geq 0$$

3. If $A \subseteq B$, then $P^*(A) \leq P^*(B)$

   Let $A \subseteq B$. Then any cover of B must be a cover of A.

   That is,
$$P^*(A) \leq P^*(B) = \inf_{n\in\mathbb{N}} \sum P(B_n)$$

4. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets of Ω. For each A_n, let $\varepsilon > 0$, and for $\frac{\varepsilon}{2^n}$, obtain a cover,
$$\sum P(B_k^n) \leq P^*(A_n) + \frac{\varepsilon}{2^n}$$

   For example,
$$P^*(A_1) = \inf \sum P(B_n), \quad A_1 \subseteq B_n$$
$$P^*(A_2) = \inf \sum P(B_n), \quad A_2 \subseteq B_n$$

   This implies,
$$\bigcup_n A_n \subseteq \bigcup_n \left(\bigcup_k B_k^n \right)$$

   From the monotonicity property of P^*, $A \subseteq B \Rightarrow P^*(A) \leq P^*(B)$, so:
$$P^*\left(\bigcup_n A_n \right) \leq P^*\left(\bigcup_n \bigcup_k B_k^n \right)$$

   But $\bigcup_n \bigcup_k B_k^n$ is a particular cover of $\bigcup_n A_n$, hence,
$$\leq \sum_n \sum_k P(B_k^n)$$

   Recall that $A_n \subseteq \bigcup_k B_k^n$, so this is a particular cover of A_n. Hence,
$$\inf \sum_k P(B_k^n) = P^*(A_n) \leq \sum P(\bigcup_k B_k^n)$$

   Since $\bigcup_k B_k^n$ is a particular cover of A_n, we conclude,
$$P^*\left(\bigcup_n A_n \right) \leq \sum_n P^*(A_n) \quad (\checkmark)$$

□

Corollary E.3.2

A set A is in $\mathscr{M}$ if and only if for all $E \in \Omega$,

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E)$$

Proof. A set A is P^*-measurable if

$$P^*(A \cap E) + P^*(A^c \cap E) = P^*(E), \quad \forall E \in \Omega$$

where $\mathscr{M}$ is the set of all P^*-measurable sets.

$\Rightarrow$: Let $A \in \mathscr{M}$. Since $\mathscr{M}$ is the set of all P^*-measurable sets, then A is P^*-measurable. By definition,

$$P^*(A \cap E) + P^*(A^c \cap E) = P^*(E), \quad \forall E \in \Omega$$

$\Leftarrow$: If

$$P^*(A \cap E) + P^*(A^c \cap E) \leq P^*(E)$$

then we want to show that $A \in \mathscr{M}$. That is, we want to show that

$$\forall E \in \Omega, \quad P^*(A \cap E) + P^*(A^c \cap E) = P^*(E)$$

Indeed, the inequality always holds by subadditivity, but let's prove the reverse inequality.

Notice that

$$E = (A \cap E) \cup (A^c \cap E)$$

and the union is disjoint, so by subadditivity,

$$P^*(E) \leq P^*(A \cap E) + P^*(A^c \cap E)$$

Combining both inequalities, we get equality,

$$P^*(E) = P^*(A \cap E) + P^*(A^c \cap E)$$

So A is P^*-measurable and hence $A \in \mathscr{M}$. $\square$

Proposition E.3.4

$\mathscr{M}$ is an algebra, i.e. for all $E \in \Omega$,

i) $\Omega \in \mathscr{M}$

ii) $A \in \mathscr{M} \Rightarrow A^c \in \mathscr{M}$

iii) $A, B \in \mathscr{M} \Rightarrow A \cup B \in \mathscr{M}$

Proof. **(i)** $\Omega \in \mathscr{M}$

We want to show it is measurable,

$$P^*(\Omega \cap E) + P^*(\Omega^c \cap E) = P^*(E)$$

But $\Omega \cap E = E$ and $\Omega^c = \emptyset$, so:

$$P^*(E) + P^*(\emptyset) = P^*(E) + 0 = P^*(E)$$

So $\Omega \in \mathscr{M}$.

(ii) Suppose $A \in \mathscr{M}$, we want to show $A^c \in \mathscr{M}$

By definition of $\mathscr{M}$, for all $E \in \Omega$,

$$P^*(A \cap E) + P^*(A^c \cap E) = P^*(E)$$

Now plug in A^c in place of A, then,

$$P^*(A^c \cap E) + P^*((A^c)^c \cap E) = P^*(A^c \cap E) + P^*(A \cap E) = P^*(E)$$

Hence $A^c \in \mathscr{M}$.

(iii) Let $A, B \in \mathscr{M}$. We want to show $A \cup B \in \mathscr{M}$, i.e.,

$$P^*[(A \cup B) \cap E] + P^*[(A \cup B)^c \cap E] = P^*(E), \quad \forall E \in \Omega$$

Since $B \in \mathscr{M}$, we know,

$$P^*(B \cap E) + P^*(B^c \cap E) = P^*(E)$$

Now compute,

$$P^*(E) = P^*(B \cap E) + P^*(B^c \cap E)$$

Apply measurability of A to both $B \cap E$ and $B^c \cap E$, since $A \in \mathscr{M}$,

$$P^*(B \cap E) = P^*[A \cap (B \cap E)] + P^*[A^c \cap (B \cap E)]$$

$$P^*(B^c \cap E) = P^*[A \cap (B^c \cap E)] + P^*[A^c \cap (B^c \cap E)]$$

So:

$$P^*(E) = P^*(A \cap B \cap E) + P^*(A^c \cap B \cap E) + P^*(A \cap B^c \cap E) + P^*(A^c \cap B^c \cap E)$$

Group these by regions,

$$= P^*\big((A \cap B) \cap E\big) + P^*\big((A^c \cap B) \cap E\big) + P^*\big((A \cap B^c) \cap E\big) + P^*\big((A^c \cap B^c) \cap E\big)$$

Note:

$$A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$$

and all of these are disjoint, and,

$$(A \cup B)^c = A^c \cap B^c$$

So:

$$P^*(A \cup B \cap E) = P^*(A \cap B \cap E) + P^*(A \cap B^c \cap E) + P^*(A^c \cap B \cap E)$$

$$P^*((A \cup B)^c \cap E) = P^*(A^c \cap B^c \cap E)$$

Therefore,

$$P^*((A \cup B) \cap E) + P^*((A \cup B)^c \cap E) = P^*(E)$$

So $A \cup B \in \mathscr{M}$. □

Proposition E.3.7

For a finite or countable sequence A_n of pairwise disjoint sets in $\mathscr{M}$, we have

$$P^*\left(E \cap \bigcup_n A_n\right) = \sum_n P^*(E \cap A_n), \quad \forall E \subseteq \Omega$$

Each $A_n \in \mathscr{M}$ such that

A_1	A_2	A_3	$\cdots$	A_n

(disjoint)

a) Finite case: Use induction (requires only one base case). Show base case $n = 2$, then derive from there.

b) Countable case: To be addressed.

Proposition E.3.8

$\mathscr{M}$ is a σ-algebra and P^* restricted to $\mathscr{M}$ is countably additive.

Note: $\mathscr{M}$ is the set of all P^*-measurable sets. We will need to show it is a σ-algebra.

Proposition E.3.9

Let P be a probability measure on an algebra $\mathscr{C}$, where P^* is the outer measure on $\mathscr{M}$ (the set of P^*-measurable sets). Then:

i) For all $A \in \mathscr{C}$, $P^*(A) = P(A)$

ii) $\mathscr{C} \subset \mathscr{M}$

Proof. **i)** $P^*(A) = \inf \sum_n P(A_n)$ where A_n covers A and $A_n \in \mathscr{C}$.

If $A \in \mathscr{C}$, then the covering of A that attains this infimum is precisely A itself,

$$A \subset A, \quad A \subset A \cup A_1, \quad A \subset A \cup A_2 \cup A_3, \quad \text{etc.}$$

Then,

$$P^*(A) = P(A)$$

ii) $\mathscr{C} \subset \mathscr{M}$: Let $A \in \mathscr{C}$.

(WTS: $A \in \mathscr{M}$, i.e. $\forall E \in \Omega$, $P^*(E) = P^*(A \cap E) + P^*(A^c \cap E)$)

Let $E \subset \Omega$, select E_n, a cover of E, and $\varepsilon > 0$ such that,

$$\text{WTS: } P^*(E \cap A^c) \leq \sum P^*(E_n \cap A^c), \quad \text{and } P^*(E \cap A) \leq P^*\left(\bigcup E_n \cap A\right)$$

Since $\mathscr{C}$ is an algebra, then,

$$B_n = E_n \cap A \in \mathscr{C}, \quad C_n = E_n \cap A^c \in \mathscr{C}$$

Since $E \subset \bigcup E_n$ and E_n is a cover of E, then:

$$E \cap A \subset \left(\bigcup E_n\right) \cap A = \bigcup (E_n \cap A)$$

Then,

$$P^*(E \cap A) \leq P^*\left(\bigcup (E_n \cap A)\right) = P^*\left(\bigcup B_n\right) \leq \sum_n P^*(E_n \cap A)$$

Similarly,

$$P^*(E \cap A^c) \leq \sum_n P^*(E_n \cap A^c)$$

Therefore,

$$P^*(E) = P^*(E \cap A) + P^*(E \cap A^c) \leq \sum_n P^*(E_n \cap A) + \sum_n P^*(E_n \cap A^c)$$

$$= \sum_n \left[P^*(E_n \cap A) + P^*(E_n \cap A^c)\right]$$

Since $E_n \cap A \in \mathscr{C}$, $\quad E_n \cap A^c \in \mathscr{C}$, and $\mathscr{C}$ is an algebra

$$= \sum_n \left[P(E_n \cap A) + P(E_n \cap A^c)\right]$$

Since $E_n \cap A$ and $E_n \cap A^c$ are disjoint and P is a probability measure (finite additive),

$$= \sum_n P(E_n) \leq P^*(E) + \varepsilon$$

Since this holds for any $\varepsilon > 0$, we conclude,

$$P^*(E \cap A) + P^*(E \cap A^c) \le P^*(E)$$

And since E was arbitrary, $A \in \mathscr{M}$. Since A was arbitrary, $\mathscr{C} \subset \mathscr{M}$. Thus, $\mathscr{C} \subset \mathscr{M}$ and $P^*(A) = P(A)$ for all $A \in \mathscr{C}$.

Let $A \in \mathscr{M}$, since A was arbitrary. $\mathscr{C} \subset \mathscr{M}$ is an algebra, then

$$B_n = E_n \cap A \in \mathscr{C} \quad \text{and} \quad C_n = E_n \cap A^c \in \mathscr{C}$$

Notice that E_n is a cover of E,

$$\begin{aligned} E \cap A &\subset \left(\bigcup_n E_n \right) \cap A \\ &= \bigcup_n (E_n \cap A) \\ \Rightarrow P^*(E \cap A) &\le P^* \left(\bigcup_n (E_n \cap A) \right) \\ &\le \sum_n P^*(E_n \cap A) \end{aligned}$$

□

Proposition E.3.10 (Carathéodory's Extension Thm)

A prob measure P defined on an algebra $\mathscr{C}$ has a unique extension to a prob measure on $\sigma(\mathscr{C})$.

Proof. The existence of such a prob measure.

$\mathscr{M}$ is a σ-algebra that contains $\mathscr{C}$ (Proposition E.2.8 and E.3.9), then

$$\mathscr{C} \subseteq \sigma(\mathscr{C}) \subseteq \mathscr{M}$$

From Prop E.3.8, P^* (the outer measure associated with P) is countably additive on $\mathscr{M}$. Then P^* is also a prob measure on $\mathscr{M}$ and also P^* agrees with P on $\mathscr{C}$ since

$$P^*(C) = P(C) \quad \forall C \in \mathscr{C}$$

From Cor E.3.1: Since P^* and P agree on $\mathscr{C}$, an algebra, and an algebra is also a π-system, then P and P^* also agree on $\sigma(\mathscr{C})$ and P^* is the only extension of P on $\sigma(\mathscr{C})$. □

Discrete measures. Let Ω be finite or countably infinite and let 2^Ω be its power set. (Remember 2^Ω is a σ-algebra). We define the measure space $(\Omega, 2^\Omega)$ associated with a set of non-negative numbers $\{\rho_\omega : \omega \in \Omega\}$ as

$$\mu(A) = \sum_{\omega \in A} \rho_\omega$$

Example 2.24. $\Omega = \{1,2,3,4,5,6\}$

Then $2^\Omega = 2^6 = \{\Omega, \{1\}, \{1,2\}, \ldots, \emptyset\}$ where $\{\rho_\omega : \omega \in \Omega\} = \{\rho_1 = 1, \rho_2 = 2, \ldots, \rho_6 = 6\}$

$$\mu : 2^\Omega \to [0, \infty), \quad \mu(A) = \sum_{\omega \in A} \rho_\omega \quad \Rightarrow \quad \mu \text{ is a measure}$$

Lets find the discrete measure of Ω given below,

Example 2.25. Let $\Omega = \{1,2,3,4,5,6\}$

$$\mu(\Omega) = \sum_{\omega \in \Omega} \rho_\omega = \rho_1 + \rho_2 + \rho_3 + \rho_4 + \rho_5 + \rho_6 = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

Example 2.26. $\mu(\emptyset) = \sum_{\omega \in \emptyset} \rho_\omega = 0$ *(the sum of no non-negative numbers is zero)*

Example 2.27. Let $\Omega = \{0, 1, 2, \ldots, n, \ldots\}$

Let,

$$\rho_0 = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^0, \quad \rho_1 = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^1, \quad \rho_2 = \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^2, \quad \ldots$$

Then,

$$\mu(\Omega) = \sum_{\omega \in \Omega} \rho_\omega = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)\left(\frac{3}{4}\right)^n = \frac{\frac{1}{4}}{1 - \frac{3}{4}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1$$

That is,

$$\mu(\Omega) = \sum_{\omega \in \Omega} \rho_\omega = \rho_1 + \cdots + \rho_n = 1$$

It is hard to define a measure on $\mathbb{R}$ or on any interval in $\mathbb{R}$.

Definition

A metric space (Ω, d) is a function $d : \Omega \times \Omega \to \mathbb{R}$ such that,

1. $d(x,y) \geq 0$
2. $d(x,y) = 0$ iff $x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,y) \leq d(x,z) + d(z,y)$

Definition E.5.1

The Borel σ-algebra associated with a metric space (Ω, d) is $\sigma(\mathcal{O})$, where $\mathcal{O}$ is the set of all open sets in (Ω, d). This σ-algebra is denoted as $\mathcal{B}(\Omega)$ or simply $\mathcal{B}$.

That is, the Borel σ-algebra is the σ-algebra generated by open sets such as $\{x \in \mathbb{R} : a < x < b\}$.

For each distance we can define the following sets called *balls*:

$$B_r(x) = \{y \in \mathbb{R} : d(x,y) < r\}$$

(center $=$ x, radius $=$ r)

Definition

A set A is **open** if $\forall x \in A$, $\exists r > 0$ such that $B_r(x) \subset A$.

Example 2.28. $\Omega = \{1,2,3,4,5,6\}$ with the Euclidean metric $d : \Omega \times \Omega \to \mathbb{R}$.

This is the Euclidean distance, so:

$$d(1,2) = |1-2| = 1 d(1,1) = |1-1| = 0$$

Proposition E.5.1

Let $\mathscr{A}$ be the set of all open balls in $\mathbb{R}^n$. Then

$$\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R}^n)$$

$\sigma(\mathscr{A})$ is the sigma algebra generated by open balls.

σ-algebra generated by all open sets in $\mathbb{R}^n$

Definition

Let $\mathscr{B}(\mathbb{R})$ be the σ-algebra generated by all open sets in $\mathbb{R}$. That is,

$$\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{O}) = \text{refinement of all open sets in } \mathbb{R}$$

The σ-algebra generated by all open intervals in $\mathbb{R}^n$ is

$$\mathscr{B}(\mathbb{R}^n) = \sigma\left(\{(a,b) : a, b \in \mathbb{R}\}\right) = \sigma$$

Proposition E.5.2

$$\mathscr{B}(\mathbb{R}) = \sigma\left(\{(-\infty, x] : x \in \mathbb{R}\}\right)$$

For the proof, it is sufficient to show that

i) Let $(a,b) \in \mathscr{B}(\mathbb{R}) \Rightarrow (a,b) \in \sigma(\{(-\infty,x] : x \in \mathbb{R}\})$

ii) Let $(-\infty,x] \in \sigma(\{(-\infty,x] : x \in \mathbb{R}\}) \Rightarrow (-\infty,x] \in \mathscr{B}(\mathbb{R}) = \sigma(\{(a,b) : a < b \in \mathbb{R}\})$

Proof. Let $a < b \in \mathbb{R}$

i) Let $(a,b) \in \mathscr{B}(\mathbb{R})$ then notice that

$$\begin{aligned}(a,b) &= \bigcup_{n\in\mathbb{N}} \left((-\infty, b - \tfrac{1}{n}] \cap (-\infty, a]^c\right) \\ &= (-\infty, b) \cap (a, \infty)\end{aligned}$$

Therefore, since $(a,b) = (-\infty,b) \cap (a,\infty)$ where $(-\infty,b)$ is of the form $(-\infty,x]$, then

$$(a,b) \in \sigma(\{(-\infty,x] : x \in \mathbb{R}\})$$

That is,

$$\mathscr{B}(\mathbb{R}) = \sigma(\{(a,b) : a < b \in \mathbb{R}\}) \subset \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$$

ii) Now let $x \in \mathbb{R}$ such that $(-\infty, x] \in \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$

Notice that

$$(-\infty, x] = \bigcap_{n \in \mathbb{N}} (x, x+n)^c$$

If $n \in \mathbb{N}$, $x = 1$, then

$$(1,2)^c \cap (1,3)^c \cap (1,4)^c \cap \cdots$$

That is,

$$(-\infty, 1] \cup [2, \infty) \cap (-\infty, 1] \cup [3, \infty) \cap (-\infty, 1] \cup [4, \infty) \cap \cdots$$

Factor out a $(-\infty, 1]$,

$$(-\infty, 1] \cup \{[2, \infty) \cap [3, \infty) \cap \cdots\}$$

Otherwise, if $x \in \bigcap_{n \geq 2} [n, \infty)$, then $x \geq 2$. Then $a = x$, $b = 1$, then there exists $N \geq 2$ such that $a = x < b < N$

That is,

$$x < N \quad \forall x < N \Rightarrow x \notin \bigcap_{n \geq N} [n, \infty)$$

So

$$(-\infty, 1] \cup \{[2, \infty) \cap [3, \infty) \cap \cdots\} = (-\infty, 1]$$

Then $(-\infty, x] \in \mathscr{B}(\mathbb{R})$, i.e.

$$\sigma(\{(-\infty, x] : x \in \mathbb{R}\}) \subset \sigma(\{(a,b) : a < b \in \mathbb{R}\})$$

□

Proposition E.5.3

Let $\Omega = (\alpha, \beta]$, $\alpha, \beta \in \mathbb{R}$, and let $\mathscr{C}$ be the set of all unions of finite disjoint half intervals of the form $(a_i, b_i]$, where $\alpha \leq a_i \leq b_i \leq \beta$. That is,

$$\mathscr{C} = \left\{ \bigcup_{i=1}^{k} (a_i, b_i] : \text{the } (a_i, b_i] \text{ are pairwise disjoint and } \alpha \leq a_i \leq b_i \leq \beta \right\}.$$

Then $\mathscr{C}$ is an algebra.

Proof. **i)** Notice that $(\alpha,\beta]$ is a finite union of disjoint half intervals of the form $(a_i,b_i]$. That is, $\Omega\in\mathscr{C}$.

ii) *WTS: If $A\in\mathscr{C}$ then $A^c\in\mathscr{C}$.*

Let

$$A=\bigcup_{i=1}^{k}(a_i,b_i] \quad \text{s.t. } \alpha\le a_i\le b_i\le\beta.$$

Notice that:

$$A^c=(\alpha,a_1]\cup(b_1,a_2]\cup\cdots\cup(b_{k-1},a_k]\cup(b_k,\beta]$$

which is the same as

$$A^c=(\alpha,a_1]\cup\bigcup_{i=1}^{k-1}(b_i,a_{i+1}]\cup(b_k,\beta].$$

Then $A^c\in\mathscr{C}$.

iii) *Closed under finite unions.*

Let $A_1,A_2,\ldots,A_n\in\mathscr{C}$. Suppose

$$A_1=\bigcup_{i=1}^{k_1}(a_i^{(1)},b_i^{(1)}],\quad A_2=\bigcup_{i=1}^{k_2}(a_i^{(2)},b_i^{(2)}],\quad \ldots,\quad A_n=\bigcup_{i=1}^{k_n}(a_i^{(n)},b_i^{(n)}].$$

At least intuitively, we have

$$\bigcup_{i=1}^{n}A_i\in\mathscr{C}.$$

□

Example 2.29. Let $\Omega=(1,5]$, and

$$\mathscr{C}=\{(1,2]\cup(2,3]\cup(3,4]\cup(4,5]\}.$$

Let $\mu:\mathscr{C}\to[0,\infty]$ where $\mathscr{C}=\{(a_1,b_1]\cup(a_2,b_2]\cup\cdots\cup(a_k,b_k]\}$ as (half open intervals).

$$\mu\left(\bigcup_{i=1}^{k}(a_i,b_i]\right)=\sum_{i=1}^{k}|b_i-a_i|$$

Then, $\mathscr{M}$ is the Lebesgue measure on $\mathscr{C}$ and on the algebra $\mathscr{D}$. I.e.

* The Lebesgue measure of a single point is zero

i) $\mu \geq 0$ since $b_i > a_i$ and $|b_i - a_i| > 0$

ii) $\mu(\emptyset) = 0$ since the length of nothing is 0.

$$\mu(\emptyset)=\sum_{i=1}^{k}|0|=0$$

iii) Countably additive. Let A_n s.t. $n \in \mathbb{N}$ be a sequence of disjoint sets in $\mathscr{C}$ which is an algebra. Notice that $\bigcup_{n\in\mathbb{N}} A_n$ is not necessarily in $\mathscr{C}$.

$$\mu\left(\bigcup_{n\in\mathbb{N}}A_n\right)=\sum_{n=1}^{\infty}\mu(A_n)\quad\text{if }\bigcup_{n\in\mathbb{N}}A_n\in\mathscr{C}$$

*** The Lebesgue measure of a single point is zero**

$$\mu\left(\bigcup_{i=1}^{k}(a_i,a_i]\right)=\sum_{i=1}^{k}|a_i-a_i|$$

Then,

$$\mu\left(\bigcup_{i=1}^{k}(a_i,a_i]\right)=\sum_{i=1}^{k}|a_i-a_i|=0$$

Let $\Omega = (\alpha, \beta]$, $\alpha < \beta \in \mathbb{R}$. If $x \in (\alpha, \beta]$ then $\mu(\{x\}) = 0$.

Let $\varepsilon > 0$ s.t. $(x-\varepsilon, x] \subset (\alpha, \beta]$ then

$$\{x\}\subset(x-\varepsilon,x]\Rightarrow 0\leq\mu(\{x\})\leq\mu((x-\varepsilon,x])=x-(x-\varepsilon)=\varepsilon\Rightarrow 0\leq\mu(\{x\})\leq\varepsilon$$

Since ε can be made small,

$$\mu(\{x\})=0$$

Let $\mathscr{F} : (\alpha, \beta] \to [0, \infty) \cup \{\infty\}$ be a function. Define the Lebesgue–Stieltjes measure $\mu_{\mathscr{F}}$ on the algebra $\mathscr{C}$ by

$$\mu_{\mathscr{F}}\left(\bigcup_{i=1}^{k}(a_i,b_i]\right)=\sum_{i=1}^{k}[\mathscr{F}(b_i)-\mathscr{F}(a_i)]$$

for pairwise disjoint intervals $(a_i, b_i] \subset (\alpha, \beta]$.

Proposition E.5.5

There exists a unique measure $\mu_{\mathscr{F}}$ on $\sigma(\mathscr{C})$ that extends the above formula.

Definition E.6.1

Let $(\Omega_1, \mathscr{F}_1)$, $(\Omega_2, \mathscr{F}_2)$ be measurable spaces. A function $F : \Omega_1 \to \Omega_2$ is called $\mathscr{F}_1/\mathscr{F}_2$-measurable if for every $A \in \mathscr{F}_2$,

$$F^{-1}(A) \in \mathscr{F}_1.$$

Proposition E.6.1

Let $(\Omega_1, \mathscr{F})$, $(\Omega_2, \sigma(\mathscr{C}))$ be measurable spaces. We say $F : \Omega_1 \to \Omega_2$ is $\mathscr{F}/\sigma(\mathscr{C})$-measurable if

$$F^{-1}(C) \in \mathscr{F} \quad \text{for all } C \in \mathscr{C}.$$

Proposition E.6.2

The composition of two measurable functions f and g is a measurable function. Let $f : \Omega_1 \to \Omega_2$, where $\mathscr{F}_1, \mathscr{F}_2$ are σ-algebras on Ω_1 and Ω_2 respectively. Let $g : \Omega_2 \to \Omega_3$, with $\mathscr{F}_2, \mathscr{F}_3$ σ-algebras on Ω_2, Ω_3 respectively.

Then $g \circ f$ is measurable.

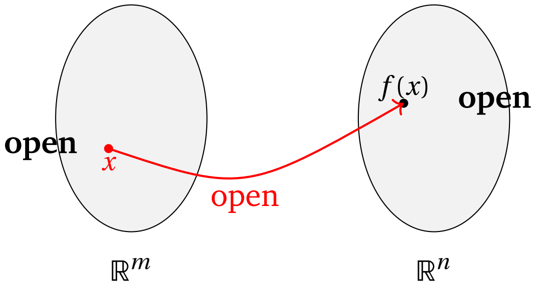
Proposition E.6.3

If a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, then it is measurable.

Proposition B.3.9

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is continuous, then if A is an open set in $\mathbb{R}^n$, the preimage $f^{-1}(A)$ is an open set in $\mathbb{R}^m$.

Since $\mathscr{F}_2 = \sigma(\text{all open sets in } \mathbb{R}^n)$, it suffices to show that if B is an open set in $\mathscr{F}_2$, then $f^{-1}(B) \in \mathscr{F}_1$. From Proposition B.3.9, if $f : A \to B$ is continuous, then for any open set $C \subseteq B$, the preimage $f^{-1}(C) \subseteq A$ is open in A. But since f is continuous, Proposition B.3.9 states this precisely: $f^{-1}(B) \in \mathscr{B}(\mathbb{R}^m)$, where $\mathscr{B}(\mathbb{R}^m)$ is the Borel σ-algebra on $\mathbb{R}^m$. Since $\mathscr{B}(\mathbb{R}^m)$ is the σ-algebra generated by all open sets in $\mathbb{R}^m$, the result follows.



Example 2.30. Let X be the time (in minutes) to complete an exam, with $X \in [0, 50]$. Then $\Omega = [0, 50]$.

Let $2^n = \chi_2$, and divide the interval $[0, 50]$ into five equal subintervals of width 10, each with equal probability mass.

We want the total area under the density to be 1. Since each subinterval has height h, and total width is 50, we get:

$$1 = h \cdot 50 \quad \Rightarrow \quad h = \frac{1}{50}.$$

The probability density function (pdf) is:

$$f(x) = \begin{cases} \frac{1}{50}, & 0 \leq x \leq 50, \\ 0, & \text{otherwise.} \end{cases}$$

Let $F(x) = P(X \leq x)$ be the cumulative distribution function (cdf).

- If $x < 0$, then
$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 0.$$

- If $0 \leq x \leq 50$, then

$$F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} \frac{1}{50}\,dt = 0 + \left[\frac{1}{50}t\right]_0^x = \frac{1}{50}x.$$

- If $x \geq 50$, then

$$F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{50} \frac{1}{50}\,dt + \int_{50}^{x} f(t)\,dt = 0 + 1 + 0 = 1.$$

Therefore, the cumulative distribution function is:

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{50}x, & 0 \leq x \leq 50, \\ 1, & x \geq 50. \end{cases}$$

2.3 Probability

Definition

$\binom{n}{r}$ is defined as:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Definition

A **sample space** is a pair $(\Omega, \mathscr{F})$ such that Ω is the set of all possible outcomes and $\mathscr{F}$ is a σ-algebra.

Example: Let us toss a coin. The set Ω is the set of symbols H and T, where H denotes head and T denotes tail. Then $\mathscr{F}$ is the class of all subsets of Ω, namely $\{\{H\}, \{T\}, \{H, T\}, \emptyset\}$.

If the coin is tossed two times, then:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

The event "at least one head" includes outcomes such as: $\{(H, H), (H, T), (T, H)\}$, which belong to $\mathscr{F}$ if $\mathscr{F}$ is the power set of Ω.

Definition: Random Variable

Let $(\Omega, \mathscr{F})$ be a measurable space. A single-valued function $X\colon \Omega \to \mathbb{R}$ is called a **random variable** if the inverse image under X of all Borel sets $B \subseteq \mathbb{R}$ is in $\mathscr{F}$. That is,

$$X^{-1}(B) \in \mathscr{F}, \quad \forall B \in \mathscr{B}$$

This means X is $\mathscr{F}/\mathscr{B}$-measurable.

Theorem 1

X is a random variable defined on $(\Omega, \mathscr{F})$ if and only if for all $x \in \mathbb{R}$,

$$\{\omega : X(\omega) \leq x\} = X^{-1}((-\infty, x]) \in \mathscr{F} = 2^{\Omega}$$

Proof. **(P ⇒ Q)**: If X is a random variable, then it is $\mathscr{F}/\mathscr{B}$-measurable. Since $(-\infty, x] \in \mathscr{B}$, we have:

$$X^{-1}((-\infty, x]) \in \mathscr{F}$$

(Q ⇒ P): From Proposition E.6.1, let $(\Omega_1, \mathscr{F}_1), (\Omega_2, \mathscr{F}_2)$ be measurable spaces. A function $f : \Omega_1 \to \Omega_2$ is $\mathscr{F}_1/\mathscr{F}_2$-measurable if and only if:

$$f^{-1}(C) \in \mathscr{F}_1 \quad \forall C \in \mathscr{F}_2$$

Since $\mathscr{B} = \sigma(\{(-\infty, x] : x \in \mathbb{R}\})$, and by hypothesis:

$$X^{-1}((-\infty, x]) \in \mathscr{F} \quad \Rightarrow \quad X^{-1}(B) \in \mathscr{F} \quad \forall B \in \mathscr{B}$$

□

Example 2.31. Toss a coin twice and observe the number of heads.

Let

$$\Omega = \{HH, HT, TH, TT\}, \quad \mathscr{F} = 2^{\Omega}, \quad X(\omega) = \text{\# of H's in } \omega$$

Then,

$$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$$

Preimages,

$$X^{-1}((-\infty,x]) = \begin{cases} \emptyset & \text{if } x<0 \\ \{TT\} & \text{if } 0\le x<1 \\ \{HT, TH, TT\} & \text{if } 1\le x<2 \\ \Omega & \text{if } x\ge 2 \end{cases}$$

CDF, $F(x) = \mathbb{P}(X \le x)$

$$F(x) = \begin{cases} 0 & x<0 \\ \frac{1}{4} & 0\le x<1 \\ \frac{3}{4} & 1\le x<2 \\ 1 & x\ge 2 \end{cases}$$

Is X discrete? Yes.

$$\mathbb{P}(X \in E) = \mathbb{P}(\{HH\}) + \mathbb{P}(\{HT\}) + \mathbb{P}(\{TH\}) + \mathbb{P}(\{TT\}) = 1$$

Probability Mass Table:

X	$\mathbb{P}(X = x_i)$
0	$\frac{1}{4}$
1	$\frac{3}{4}$
2	1

Example 2.32. Toss a coin and observe the side that lands up.

Let the sample space be:

$$\Omega = \{H, T\}, \quad \mathscr{F} = 2^{\Omega} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$$

Let $\mathscr{B}$ be the Borel σ-algebra on $\mathbb{R}$.

Define the function $X : \Omega \to \mathbb{R}$ by,

$$X(\omega) = \mathbf{1}_{\{T\}}(\omega) = \begin{cases} 1 & \text{if } \omega = T \\ 0 & \text{if } \omega \neq T \end{cases}$$

That is,

$$X(H) = 0, \quad X(T) = 1$$

Check measurability: Is X an $\mathscr{F}/\mathscr{B}$-measurable function?

We verify:

$$X^{-1}((-\infty, x]) \in \mathscr{F} \quad \forall x \in \mathbb{R}$$

- $x < 0$: $\quad X^{-1}((-\infty, x]) = \emptyset \in \mathscr{F}$
- $0 \le x < 1$: $\quad X^{-1}((-\infty, x]) = \{H\} \in \mathscr{F}$
- $x \ge 1$: $\quad X^{-1}((-\infty, x]) = \{H, T\} = \Omega \in \mathscr{F}$

Thus, for all $x \in \mathbb{R}$, $X^{-1}((-\infty, x]) \in \mathscr{F}$, so,

$$\{\omega \in \Omega : X(\omega) \le x\} \in \mathscr{F} \quad \Rightarrow \quad X \text{ is } \mathscr{F}/\mathscr{B}\text{-measurable}$$

More examples,

$$X^{-1}(\{1\}) = \{T\} \in \mathscr{F}, \quad (\text{since } \{1\} \in \mathscr{B})$$

$$X^{-1}((0, 1)) = \emptyset \in \mathscr{F}, \quad (\text{since no value of } X \text{ lies strictly between 0 and 1})$$

Hence X is measurable, and we can compute probabilities of events in $\mathbb{R}$ using preimages in Ω.

Example 2.33. Let X be the time to complete an exam in 50 minutes.

Let the sample space be $\Omega = [0,50]$, with $\mathscr{F} = 2^{\Omega}$, and codomain $\mathbb{R}$ with Borel σ-algebra $\mathscr{B}$.

Define the function,

$$X : \Omega \to \mathbb{R} \quad \text{such that} \quad X(\omega) = \omega$$

We examine the preimage,

$$X^{-1}((-\infty, x]) = \begin{cases} \emptyset & \text{if } x < 0 \\ [0, x] & \text{if } 0 \leq x \leq 50 \\ [0, 50] = \Omega & \text{if } x > 50 \end{cases}$$

Since all these sets belong to 2^{Ω}, we conclude that X is $\mathscr{F}/\mathscr{B}$-measurable.

Example 2.34. Toss a coin 10 times and observe the number of heads. Show that the function that counts the number of heads is $\mathscr{F}/\mathscr{B}$-measurable.

Let the sample space be,

$$\Omega = \{H, T\}^{10}, \quad \mathscr{F} = 2^{\Omega}, \quad \mathscr{B} = \text{Borel } \sigma\text{-algebra on } \mathbb{R}$$

Define $X : \Omega \to \mathbb{R}$ by,

$$X(\bar{\omega}) = \sum_{i=1}^{10} \mathbf{1}_{\{H\}}(\omega_i)$$

That is, $X(\bar{\omega})$ counts the number of heads in the sequence $\bar{\omega} = (\omega_1, \ldots, \omega_{10})$.

Example: If there are 4 heads, then:

$$X(\bar{\omega}) = 1 + 1 + 1 + 1 + 0 + 0 + \cdots + 0 = 4$$

We now compute the preimage,

$$X^{-1}((-\infty, x]) = \begin{cases} \emptyset & \text{if } x < 0 \\ \{\omega : \text{no heads}\} & \text{if } 0 \leq x < 1 \\ \{\omega : \text{at most 1 head}\} & \text{if } 1 \leq x < 2 \\ \vdots & \\ \{\omega : \text{at most } \lfloor x \rfloor \text{ heads}\} & \text{for general } x \end{cases}$$

Since each of these sets is a subset of Ω, and $\mathscr{F} = 2^{\Omega}$, it follows that,

$$X^{-1}((-\infty, x]) \in \mathscr{F}, \quad \forall x \in \mathbb{R}$$

Therefore,

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathscr{F} \quad \forall x \in \mathbb{R} \quad \Rightarrow \quad X \text{ is } \mathscr{F}/\mathscr{B}\text{-measurable}$$

Additional example:

$$\{10\} \in \mathscr{B}, \quad X^{-1}(\{10\}) = \{\omega \in \Omega : \omega \text{ is all heads}\} = \{H, H, \ldots, H\}$$

This also belongs to $\mathscr{F}$, so measurability is preserved.

Example 2.35. Toss a die and observe the number that comes up. Show that the function $X(\omega) = \omega$ is measurable and compute its distribution function.

Let $\Omega = \{1, 2, 3, 4, 5, 6\}$, and $\mathscr{F} = 2^{\Omega}$. Let $X : \Omega \to \mathbb{R}$ be defined by $X(\omega) = \omega$.

We compute the preimage of intervals $(-\infty, x]$,

$$X^{-1}((-\infty, x]) = \begin{cases} \emptyset & x < 1 \\ \{1\} & 1 \le x < 2 \\ \{1,2\} & 2 \le x < 3 \\ \{1,2,3\} & 3 \le x < 4 \\ \{1,2,3,4\} & 4 \le x < 5 \\ \{1,2,3,4,5\} & 5 \le x < 6 \\ \Omega = \{1,2,3,4,5,6\} & x \ge 6 \end{cases}$$

Since all these sets are in $\mathscr{F} = 2^{\Omega}$, we conclude that X is $\mathscr{F}/\mathscr{B}$-measurable.

The cumulative distribution function $F(x) = \mathbb{P}(X \le x)$ is given by,

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{6} & 1 \le x < 2 \\ \frac{2}{6} & 2 \le x < 3 \\ \frac{3}{6} & 3 \le x < 4 \\ \frac{4}{6} & 4 \le x < 5 \\ \frac{5}{6} & 5 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

Theorem

The random variable X defined on a measure space $(\Omega, \mathscr{F}, \mathbb{P})$ induces a **probability space** $(\mathbb{R}, \mathscr{B}, \mathbb{Q})$ by means of

$$\mathbb{Q}(B) = \mathbb{P}(X^{-1}(B)) = \mathbb{P}(\{\omega : X(\omega) \in B\}) \quad \forall B \in \mathscr{B}$$

We interpret this as a measurable mapping,

$$(\Omega, \mathscr{F}) \xrightarrow{X} (\mathbb{R}, \mathscr{B}) \quad \text{with } \mathbb{P} \text{ on } \Omega \text{ inducing } \mathbb{Q} \text{ on } \mathbb{R}$$

If you have a set $\Omega = [0,1]$, the probability of getting a specific point, say $\frac{1}{2}$, is nearly 0 because $[0,1]$ contains infinitely many points.

But we can find the probability of a set or interval.We claim that $\mathbb{Q}$ is a probability measure.

(1) $\mathbb{Q}(\mathbb{R}) = 1$, i.e.,

$$\mathbb{Q}(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1$$

(2) $\mathbb{Q} \geq 0$ since it is defined using $\mathbb{P}$

(3) If $B_n \subset \mathscr{B}$ are disjoint sets, then

$$\mathbb{Q}\left(\bigcup_n B_n\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_n B_n\right)\right) = \mathbb{P}\left(\bigcup_n X^{-1}(B_n)\right)$$

$$= \sum_n \mathbb{P}(X^{-1}(B_n)) = \sum_n \mathbb{Q}(B_n)$$

Definition

A real-valued function F defined on $(-\infty, \infty)$ that is non-decreasing, right-continuous, and satisfies

$$\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to \infty} F(x) = 1$$

is called a **distribution function (DF)**.

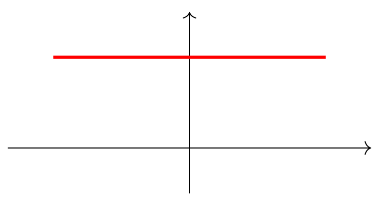
Distribution Function Properties

A distribution function (D.F.) can only have **jump discontinuities**.

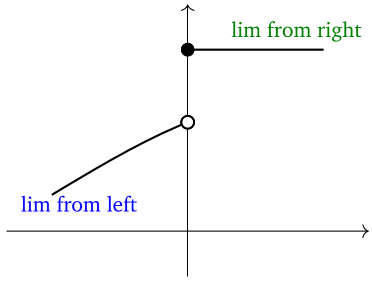
$$F(x^-) \leq F(x) \leq F(x^+) \quad \text{if } F \text{ is non-decreasing}$$

$$F(x^-) \leq F(x) = F(x^+) \quad \text{if } F \text{ is right-continuous}$$

(a) If $F(x^-) = F(x) = F(x^+)$, then F is **continuous** at x.



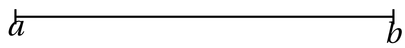
(b) If $F(x^-) < F(x) = F(x^+)$, then x is a **jump point**.



Definition

The set of discontinuity points of a D.F. is at most countable.

Let $[a, b]$ be a finite subset of the real numbers $\mathbb{R}$, and let S be the set of jump points in $[a, b]$.



$$\begin{aligned} n = 1 &\quad \text{how many points in } [a, b] \text{ with jump at most } 1 \\ n = 2 &\quad \text{how many points in } [a, b] \text{ with jump at most } 2 \\ &\quad \vdots \end{aligned}$$

For any n, the number of jump points with jump of at least $\frac{1}{n}$ is at most n.

That is, define

$$S_n := \left\{ x_i \in [a, b] : \frac{1}{n+1} < F(x_i) - F(x_i^-) \leq \frac{1}{n} \right\}$$

i.e,

$$S_1 := \left\{ x_i \in [a, b] : \frac{1}{2} < F(x_i) - F(x_i^-) \leq 1 \right\}$$

$$S_2 := \left\{ x_i \in [a, b] : \frac{1}{3} < F(x_i) - F(x_i^-) \leq \frac{1}{2} \right\}$$

Each S_n is empty or finite.

Let $S = \bigcup_{n=1}^{\infty} S_n$. Since $\bigcup$ contains all points at which the jump is positive on $[a,b]$, then S is the countable union of empty or finite sets. Hence, S is at most countable.

But

$$\mathbb{R} = \{[0,1) \cup [1,2) \cup [2,3) \cup \cdots \cup (-1,0] \cup \cdots\}$$

Therefore $\mathbb{R}$ can be expressed as a countable union of disjoint finite intervals of the form $(a,b]$.

The set of all jump points in the domain of the distribution function is at most countable.

Definition

Let X be a r.v. defined on $(\Omega, \mathscr{F}, \mathbb{P})$. Define the pointwise function

$$F(x) = Q((-\infty, x]) = \mathbb{P}(\{\omega : X(\omega) \le x\}) = \mathbb{P}(X^{-1}((-\infty, x]))$$

Theorem 2

$F(x)$ defined above is indeed a distribution function (DF).

Proof. **(a)** F is non-decreasing. **WTS:** If $x_1 < x_2 \Rightarrow F(x_1) \le F(x_2)$

Let

$$A = \{\omega : X(\omega) \le x_1\} \subset \{\omega : X(\omega) \le x_2\} = B$$

Since if $\omega \in A$, then $X(\omega) \le x_1 < x_2$, then $\omega \in B$. Likewise, the inverse images satisfy the same inclusion.

Equivalently,

$$X^{-1}((-\infty, x_1]) \subseteq X^{-1}((-\infty, x_2])$$

From monotonicity of $\mathbb{P}$,

$$F(x_1) = \mathbb{P}(X^{-1}((-\infty, x_1])) \le \mathbb{P}(X^{-1}((-\infty, x_2])) = F(x_2)$$

i.e.

$$\mathbb{P}(\text{“}X \le x_1\text{”}) \le \mathbb{P}(\text{“}X \le x_2\text{”})$$

(b) F is right continuous. **WTS:** $\lim_{x \to a^+} F(x) = F(a)$

It is sufficient to show that for any sequence $x_1 < x_2 < \cdots < a$ with $x_n \to a$, we have

$$F(x_n) \to F(a)$$

Let

$$A_k = \{\omega : X(\omega) \in (a, x_k]\}$$

Note that A_k is a non-increasing sequence of sets in $\mathscr{F}$. Then

$$\lim_{k\to\infty} A_k = \bigcap_{k\in\mathbb{N}} A_k = \varnothing$$

since if $\omega \in \bigcap_{k=1}^{\infty} \{\omega : X(\omega) \in (a, x_k]\}$, then $X(\omega) \in (a, x_k]$ for all k, which implies that $a < X(\omega) \le x_k$ for all $k \in \mathbb{N}$.

Take the limit as $k \to \infty$,

$$a < \lim_{k\to\infty} X(\omega) \le \lim_{k\to\infty} x_k = a$$

which is a contradiction. Hence, $a < X(\omega) \le a$ is impossible. Therefore, the intersection is empty and

$$F(x_k) \to F(a)$$

so F is right continuous.

From Proposition E.2.1, if $E_n \searrow E$, then $\mu(E_n) \searrow \mu(E)$. Also, if $A_k \searrow \emptyset$, then $\mathbb{P}(A_n) \searrow \mathbb{P}(\emptyset) = 0$.

But,

$$\mathbb{P}(A_k) = \mathbb{P}(\text{“}X \le x_k\text{”}) - \mathbb{P}(X \le a)$$

$$= F(x_k) - F(a)$$

where,

$$0 = \lim_{k\to\infty} \mathbb{P}(A_n) = \lim_{k\to\infty} (F(x_k) - F(a)) = 0$$

□

Theorem 4

Given a probability measure Q on $(\mathbb{R}, \mathscr{B})$, there exists a distribution function F

satisfying:

$$Q((-\infty, x]) = F(x) \quad \forall x \in \mathbb{R}.$$

Moreover, $Q(B) = \mathbb{P}(X^{-1}(B))$, so that

$$F(x) = Q((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])).$$

This defines $F(x) = Q((-\infty, x_k])$ properly. **However, this works in $\mathbb{R}$.**

Conversely, given a distribution function F, there exists a *unique* probability measure Q on $(\mathbb{R}, \mathscr{B})$ such that:

$$Q((-\infty, x]) = F(x).$$

In a less detailed manner, $Q \Rightarrow \exists F(x) = Q((-\infty, x])$. Conversely, $F(x) \Rightarrow Q$ uniquely. *(by Lebesgue–Stieltjes Theorem.*

Definition

A random variable X on $(\Omega, \mathscr{F}, \mathbb{P})$ is *discrete* if there exists a countable (finite or one-to-one with $\mathbb{N}$) set $E \subset \mathbb{R}$ such that

$$\mathbb{P}(X \in E) = 1.$$

The points of E are called *jump points*, and their probabilities are called the *jumps* of the distribution function (DF) of X.

Definition

The collection of numbers $\{p_i\}$ satisfying,

1. $\mathbb{P}(X = x_i) = p_i \geq 0$
2. $\sum_{i=1}^{\infty} p_i = 1$

is called the **probability mass function (pmf)** of the discrete random variable X.

Example 2.36. Toss a loaded coin once and record the side that comes up.

Let $\Omega = \{H, T\}$, and define $X : \Omega \to \mathbb{R}$ by

$$X(\omega) = \mathbf{1}_{\{T\}}(\omega) = \begin{cases} 1 & \text{if } \omega = T \\ 0 & \text{if } \omega = H \end{cases}$$

Assume,

$$\mathbb{P}(\{T\}) = \frac{3}{4}, \qquad \mathbb{P}(\{H\}) = \frac{1}{4}$$

We have,

$$\mathscr{F} = 2^{\Omega} = \{\varnothing, \{H\}, \{T\}, \{H, T\}\}$$

$$\mathscr{B} = \text{Borel } \sigma\text{-algebra on } \mathbb{R}$$

Then X is $2^{\Omega}/\mathscr{B}$-measurable, and the induced map,

$$X^{-1} : \mathscr{B} \to 2^{\Omega}$$

defines the measure,

$$Q(B) = \mathbb{P}(X^{-1}(B))$$

Thus, we can define the distribution function of X by,

$$F(x) = Q((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])) = \mathbb{P}(X \leq x)$$

Now let's find $X^{-1}((-\infty, x])$

$$X^{-1}((-\infty, x]) = \begin{cases} \varnothing & \text{if } x < 0 \\ \{H\} & \text{if } 0 \leq x < 1 \\ \{H, T\} & \text{if } 1 \leq x \end{cases}$$

Now, let's look at the probability:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{4} & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x \end{cases}$$

X is discrete if $\exists E \subset \mathbb{R}$ such that

$$\mathbb{P}(X \in E) = 1$$

Find E:

$$E = \{0, 1\} \quad \text{(i.e. our finite set for indicator function)}$$

Is $\mathbb{P}(X \in E) = 1$? **Notice:**

$$\mathbb{P}(X \in E) = \mathbb{P}(X^{-1}(\{0,1\})) = \mathbb{P}(\Omega) = 1$$

That is:

$$\mathbb{P}(X = 0) = \mathbb{P}(X^{-1}(\{0\})) = \mathbb{P}(H) = \frac{1}{4}$$

$$\mathbb{P}(X = 1) = \mathbb{P}(X^{-1}(\{1\})) = \mathbb{P}(T) = \frac{3}{4}$$

$$\mathbb{P}(X = x_i) = p_i \quad \text{(p.m.f.)}$$

$$\mathbb{P}(X = 0) = \frac{1}{4}, \quad \mathbb{P}(X = 1) = \frac{3}{4}$$

X	$\mathbb{P}(X = x_i)$
$x_1 = 0$	$\frac{1}{4}$
$x_2 = 1$	$\frac{3}{4}$

Definition

Let X be a random variable defined on $(\Omega, \mathscr{F}, \mathbb{P})$ with distribution function F. We say that F is **continuous** if F is **absolutely continuous**, i.e., there exists a

Lebesgue-integrable function $f(x)$ such that:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Example 2.37. Toss a fair coin once and define X by:

$$X(\omega) = \begin{cases} 0 & \text{if } \omega = T \\ 1 & \text{if } \omega = H \end{cases}$$

Then,

$$\mathbb{P}(X=0) = \mathbb{P}(T) = \frac{1}{2}, \quad \mathbb{P}(X=1) = \mathbb{P}(H) = \frac{1}{2}$$

X	$\mathbb{P}(X = x_i)$
$x_1 = 0$	$\frac{1}{2}$
$x_2 = 1$	$\frac{1}{2}$

$$X^{-1}((-\infty, x]) = \begin{cases} \varnothing & \text{if } x < 0 \\ \{T\} & \text{if } 0 \le x < 1 \\ \Omega & \text{if } 1 \le x \end{cases}$$

$$Q((-\infty, x]) = F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2} & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x \end{cases}$$

Is X discrete? That is, does there exist $E \subset \mathbb{R}$ such that

$$\mathbb{P}(X \in E) = 1\,?$$

Notice that,

$$\mathbb{P}(X \in E) = \mathbb{P}(X^{-1}(\{0,1\})) = \mathbb{P}(\Omega) = 1$$

References