Weaving through Incompleteness in Logical Thought and it's Aftermath ❦

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Abstract

Mathematics is a foundational discipline, essential for the advancement of other subjects. In fields like epidemiology, mathematical models and theorems play a crucial role. Thus, making the accuracy and reliability of mathematical principles indispensable. The extensive use of mathematics in such contexts relies on correctness. Mathematics must be correct and true for all discipline. Mathematicians are known for rigorously proving their claims. Despite this, a notable gap exists in mathematics. Not all true statements can be proven. This paradox highlights incomplete nature of mathematical knowledge.

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1 Historical Background

1.1 The Foolish Ignorabimus

In 1874, German mathematician Georg Cantor published a paper known as "*Ueber eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen*" *("On a Property of the Collection of All Real Algebraic Numbers")* [1]. This very paper propagated a new branch of mathematics known today as Set Theory.

Definition

A set is the mathematical model for a collection of different things; it contains elements or members, which can be mathematical objects of any kind.

For example, your body is a set which comprising of various elements that collectively define your physical existence. Objects such as your eyes, hair, fingers, heart, lungs, and plant life, as well as natural phenomena such as the ocean, rocks, volcanoes, and constructed entities like houses. Taking it further, our universe can be regarded as a set within which our solar system, consisting of our sun and Earth whom act as an element. The same concept of classification can be paralleled in the realm of mathematics. The form of categorization we speak of is seen in the grouping of numbers. This gave Cantor a brilliant way to categorize all the natural numbers, integers, real numbers, and so forth [1].

Cantor wondered if there were more natural numbers than real numbers between 0 and 1. Both sets contain an infinite number of numbers, so one might assume they are the same size. Right? However, Cantor discovered that the real numbers between 0 and 1 actually contain more elements than all the natural numbers. In other words, Cantor realized that there are indeed different sizes of infinity. Some infinities are larger than others. He classified these as countable infinities, like the natural numbers. Similarly uncountable infinities, like the reals between 0 and 1. Thus, Cantor demonstrated that the concept of infinity was more complex than mathematicians had previously defined it. This revelation caused a significant split among mathematicians at the end of the 1800s. A major debate ensued, with one side, known as the "intuitionists," dismissing

[1]This is a reference from my own book/ notes!

Cantor's work as absurd. Henri Poincaré famously stated that *"later generations will regard set theory as a disease from which one has recovered."* [4] Another mathematician, Leopold Kronecker, who opposed Cantor's work, labeled him a *"scientific charlatan* [2]*."* Kronecker also prevented Cantor's aspirations for a desired career.

On the other side of the debate were the "formalists," who were keen on Cantor's set theory. The leader of this group was none other than David Hilbert. At the time, Hilbert was a renowned mathematician who was known for his work not only in mathematics but also in physics. He competed with Einstein in publishing the theory of general relativity. Although Einstein ultimately beat him. Additionally, Hilbert discovered new mathematical concepts that significantly contributed to the study of quantum mechanics. Hilbert was convinced that Cantor's work was important mathematics. He famously stated, *"No one shall expel us from the paradise that Cantor has created"* [6]. The formalists believed that set theory would be a powerful mathematical tool.

Hilbert sought answers to three pivotal questions. The first question was: *Is mathematics complete? In other words, do all true statements have a proof?* The second question asked: *Is mathematics consistent? This means, is mathematics free of contradictions?* It's important to note that if you can prove a statement A and also its negation not A, then you could prove anything. No matter how nonsensical the statement might be. Lastly, the third question was: *Is mathematics decidable?* That is, does an algorithm exist that can determine whether a statement follows directly from the axioms? Hilbert was convinced that the answers to all these questions were true.

On September 8, 1930, in Königsberg, David Hilbert addressed the annual meeting of the Society of German Natural Scientists and Physicians. He concluded the conference with a powerful statement, *(we will not know): "In opposition to the foolish ignorabimus, our slogan shall be: We Must Know, We Will Know"* [7]. Ironically, just a day before, in the same conference but at a different meeting, a 24-year-old mathematician named Kurt Gödel claimed to have found the answer to Hilbert's first question. Surprisingly, his answer to that question was no.

[2]Essentially calling Cantor a fraud.

1.2 Kurt Gödel's Incompleteness Theorem

Kurt Gödel made the discovery that a complete system of mathematics was impossible. Initially, not many people paid attention to Gödel's findings. The only person who showed interest was John von Neumann who was a former student of Hilbert. However, a year later in 1931, Gödel published a proof of his incompleteness theorem titled *"On Formally Undecidable Propositions of Principia Mathematica and Related Systems I"* [3]. Unlike his recognition at the conference, this publication caught the attention of the entire mathematical community.

Gödel aimed to apply logic within mathematics to address questions about the systems of logic and mathematics themselves. To achieve this, he assigned numbers to the symbols used in mathematics. These were called Gödel numbers.

To see this consider the table [5].

Table 1: Mathematical symbols with their Gödel numbers.

Mathematical symbol	**Gödel number**
$\neg$	1
$\vee$	3
$\forall$	5
$=$	7
0	9
S	11
$+$	13
$\cdot$	15
E	17
$<$	19
(	21
)	23
v_i	$2i$

Using this we can decipher any natural number. For example, 1 is in our universe since $S(0) = 1$. The number 2 is written as $SS(0)$. Inductively we can form all the natural numbers as tedious as it might be just by using our successors and 0. Likewise, if we wish to to show the equation $0 = 0$ then notice that 0 corresponds to 9 and $=$ corresponds to 7. Raising each to consecutive prime powers and adding by 1 we have

$$2^{9+1}3^{7+1}5^{9+1}$$

Thus via the aid of the table we can concluded that the corresponding Gödel number of the equation $0 = 0$ is $2^{10}3^{8}5^{10}$.[3] Using this system one can write down Gödel numbers for any set of symbols. In other words, one can write Gödel Numbers for any equation. Notice by using the prime factorization of these numbers we can work backwards to receive the symbols.

In this system there will indeed be both true and false statements. Thus, in order to show that something is true then we must consider a set of axioms. Let's introduce one.

Peano Axiom 1

$$\neg(Sx = 0).$$

Which means that the successor of any number can not be 0.

This makes sense as there are no negative numbers in our system. Now lets prove that 1 does not equal 0.

Thereom

$$1 \neq 0.$$

Proof.

$$\neg(Sx = 0)$$

$$\neg(S0 = 0)$$

□

To obtain the Gödel Number of the theorem notice that $1 \neq 0$ simply just means $\neg = S00$. Thus,

[3] Absurdly brilliant.

$$
\begin{aligned}
\ulcorner 1 \neq 0 \urcorner = \ulcorner \neg = S00 \urcorner & \\
&=< 1, \ulcorner = S00 \urcorner > \\
&=< 1, < 7, \ulcorner S0 \urcorner, \ulcorner 0 \urcorner >> \\
&=< 1, < 7, < 11, \ulcorner 0 \urcorner >, < \ulcorner 0 \urcorner >>> \\
&=< 1, < 7, < 11, < 9 >>, < 9 >>> \\
&=< 1, < 7, < 11, 2^{10} >, 2^{10} >> \\
&=< 1, < 7, 2^{12} \cdot 3^{2^{10}+1}, 2^{10} >> \\
&=< 1, 2^8 \cdot 3^{2^{12} \cdot 3^{2^{10}+1} \cdot 5^{2^{10}+1}+1} > \\
&= 2^2 \cdot 3^{2^8 \cdot 3^{2^{12} \cdot 3^{2^{10}+1} \cdot 5^{2^{10}+1}+1}+1}
\end{aligned}
$$

This is a ridiculously large number and for our sanity, we will denote these numbers as letters. Let's assign the Gödel Number a to the proof $1 \neq 0$ above. Using this system, we can find a number x that corresponds to the proof that there is no proof for the statement with Gödel Letter x. However, this arises some complications. What this number x is implying is that the statement "*there is no proof for the statement with Gödel Number x*" is not provable [2]. Notice that if this statement is false and there is a proof, then what you have shown is that there is no proof. Thus, arriving at a contradiction due to x asserting its own provability. However, if x is true meaning that there is no proof for it, then our system contains unprovable statements and hence incomplete.[4] This is what gave birth to Gödel's incompleteness's theorem and showing that Hilbert was wrong.

In summation, any mathematical system that operates using basic arithmetic will always have true statements that are not provable. Surely, you may ask, what if we tack on more axioms to prove those unprovable statements? Adding axioms can help in proving previously unprovable statements. However, it leads to the creation of a new system with its own set of true but unprovable statements.

Gödel's Incompleteness Theorem demonstrates that truth and provability are not the same. This ultimately proved Hilbert's first question false. Hilbert had two other

[4] Yikes, do you see the paradox?

questions he was confident about. The second which stated the consistency of mathematics, meaning its freedom from contradictions. However, in 1931, Gödel published his Second Incompleteness Theorem. Yet again showing Hilbert wrong. The Second Incompleteness Theorem showed that any consistent system of mathematics cannot prove its own consistency. Therefore, the best one can hope for is a system that is consistent, though not complete. Nevertheless, we cannot know if the system we work within is inconsistent until we encounter a contradiction.

1.3 The Turnin Machine

This leads us to the final question that Hilbert and others were confident about. "*Is mathematics decidable?*" That is, does an algorithm exist that can always determine whether a statement follows directly from the axioms? It turns out that Alan Turing found a way to answer this question. To do so, well, he only had to conceptualize the modern computer[5]. Turing utilized a binary system in his design. The program consists of a set of instructions that tells the machine what to do based on the digit it reads and its current state. This machine could also transmit information to other machines. Thus, given sufficient memory and an appropriate program, a Turing machine could compute any algorithm. Of course if given enough time. When a Turing machine halts, the program is finished. The output then becomes a tape of binary numbers and your answer is solved. However, sometimes the Turing machine never halts, presumably getting stuck in an infinite loop.

Turing discovered that this problem closely paralleled the decidability problem in mathematics. If he could determine whether a Turing machine would halt, it might be possible to decide if a statement followed directly from the axioms. Surprisingly, he found out that no such machine existed. There was no way to predict whether a machine would halt given an input. He did this via a proof by contradiction. Turing assumed that there indeed existed a Turning machine that could decide if any Turing machine halted. He then constructed a logical contraction, showing that no such machine existed [6]. If mathematics was decidable then indeed the halting problem would

[5]Before the invention of modern computers, it was common for women to perform all computational work.

[6]This means that no matter how advance computers get, there will always be problems that they cannot solve.

have a solution. When combine with Gödel's incompleteness theorems, this led to the conclusion that mathematics is undecidable. Thus, shattering all three of Hilbert's dreams.

1.4 Proof of the Halting Problem

Thereom

No Turing machine H can exist that solves the halting problem for all Turing machines and inputs.

Proof. Let's consider a Turing machine H that decides the halting problem. This means for any Turing machine M and input I, $H(M,I)$ halts and correctly determines whether M halts or runs indefinitely on input I.

Now, construct a Turing machine D which takes M as an input and does the opposite of what H predicts.

$$D(M) = \begin{cases} \text{run indefinitely} & \text{if } H(M,M) \text{ halts,} \\ \text{halt} & \text{if } H(M,M) \text{ runs indefinitely.} \end{cases}$$

The paradox arises when D is given its own description as input, $D(D)$:

- If H determines if D halts with input D, $H(D,D)$ halts, then by the definition of D, $D(D)$ should run indefinitely.
- If H determines if D runs indefinitely with input D $H(D,D)$ runs indefinitely, then by the definition of D, $D(D)$ should halt.

This leads to a contradiction, as $D(D)$ cannot simultaneously halt and run indefinitely. Therefore, our initial assumption that such a Turing machine H exists is false. This implies that the halting problem is undecidable [8].

□

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