From Fundamentals to Functions: Intermediate Algebra

Jose Alfredo Alanis

To my students.

Contents

0.1 Editorial Board

A special thanks to my mother who read every word of this book and gave me motivation to keep moving forward. This book is inspired by a compilation of math notes from the Sacramento State department. Special thanks to Kyle Olson for providing these notes.

[1]Teaching Associate at *California State University, Sacramento*
[2]Student at *California State University, Sacramento*

0.2 Introduction

Mathematics is notorious for being challenging. To be quite honest with you, I have always struggled with the subject. However, I always keep going as a student. Even teaching this subject stumps me at times. It is not easy, just like anything in life. You only become better by practice. Regardless of what you want to do, you have to be consistent with it to get better results. Please, do not get discouraged. You will eventually understand the material with time, and do not let anyone tell you otherwise. It might take weeks, months, or even years, but it will come. Some of this material I fully learned as I taught it. Thus, do not sweat it! :)

If you catch any typos, have any questions, or comments, please contact me at josealanis2@csus.edu or josefreddyalanis@gmail.com.

1 Review

1.1 Prime Factors & Fractions

I begin this book by stating a fact. Mathematics holds the highest position among the subjects in STEM (Science, Technology, Engineering, and Mathematics). It is a statement that cannot be refuted, for as mathematicians, our realm is built upon proofs and logical reasoning. One simply needs to inquire: What would engineering, physics, chemistry, and other scientific disciplines be without the foundational framework of mathematics? I'll wait.

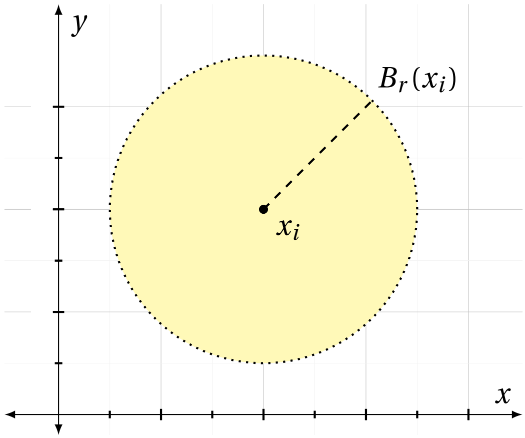
We are starting with the most important definition of the class.

Definition

The ball $B_r(x)$ of radius r centered at x_i is the set of all points y_i such that the absolute value of the difference between x_i and y_i is less than r. Formally,

$$B_r(x_i) = \{y_i \in \mathbb{R} \mid |x_i - y_i| < r\}$$

Let's look at a picture of this.



We will be using this later. I promise! Alright, shall we begin?

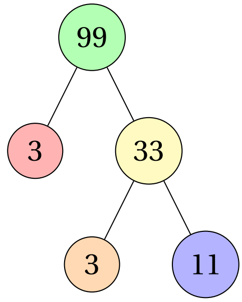
Prime factorization is an essential tool that you'll use frequently in this class, particularly when simplifying fractions. Before diving into fraction operations, it's essential to understand the concept of prime factorization. Remember that prime numbers are those that have only two divisors: 1 and themselves (excluding 1). For example $\{2,3,5,7,11,13,17,19,...\}$ are all prime numbers. Take for instance 19, the only numbers that divide 19 are 19 and 1. Developing proficiency in prime factorization will simplify the process of working with fractions, which is a crucial skill in mathematics. As the famous mathematician Khwarizmi [3] once said, "When I consider what people generally want in calculating, I found that it is always a number." However, depending on the natural phenomena being studied, these numbers may not be whole numbers. Therefore, mastering the operations involving fractions is fundamental to many areas of study beyond just mathematics. So, let's begin exploring prime factorization and fraction operations to enhance our understanding of mathematics! Let's begin with some examples.

Example 1.1. Write 99 as a product of prime factors.

$$\begin{aligned} 99 &= 3 \cdot 33 \\ &= 3 \cdot 3 \cdot 11 \\ &= 3^2 \cdot 11 \end{aligned}$$

Note that 3 and 11 are both primes. That is, the only numbers that divide them are 1 and themselves. Hence, they can not be simplified any further.

[3]Muhammad ibn Musa al-Khwarizmi, widely regarded as the father of algebra, was born in the year 780. He made significant contributions to the field of algebra and was also an accomplished astronomer. During his time, it was common for mathematicians to also study astronomy, a practice that continued until the era of Newton.

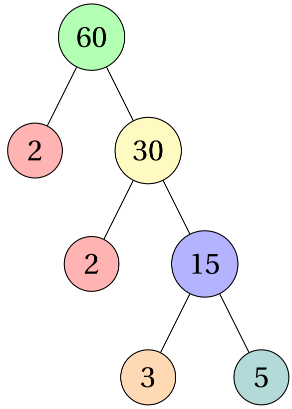


Here, we only care about the numbers that cannot be further broken down. Hence our answer is $3 \cdot 3 \cdot 11$.

Example 1.2. Write 60 as a product of prime factors.

$$\begin{aligned} 60 &= 2 \cdot 30 \\ &= 2 \cdot 2 \cdot 15 \\ &= 2^2 \cdot 3 \cdot 5 \end{aligned}$$

Here, 2, 3, and 5 are all prime numbers. They are only divisible by 1 and themselves, so they cannot be broken down into simpler factors.

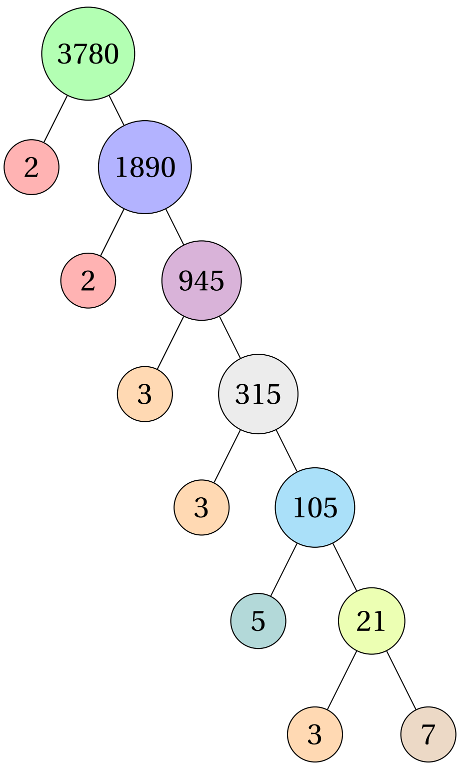


Again, we only get the numbers that were not factorable. That is, the two red 2s, orange 3 and teal 5.

Example 1.3. Write 3780 as a product of prime factors.

$$\begin{aligned} 3780 &= 2 \cdot 1890 \\ &= 2 \cdot 2 \cdot 945 \\ &= 2^2 \cdot 3 \cdot 315 \\ &= 2^2 \cdot 3 \cdot 3 \cdot 105 \\ &= 2^2 \cdot 3^2 \cdot 5 \cdot 21 \\ &= 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 3 \\ &= 2^2 \cdot 3^3 \cdot 5 \cdot 7 \end{aligned}$$

Note that 2, 3, 5, and 7 are all prime numbers. Let's now construct the factor tree. Remember that you want to break the number 3780 into products of primes. That is, numbers that are only divisible by 1 and themselves.



Prime factorization becomes relevant when you want to simplify mathematics. We can see this by the simplification of fractions. To simplify a fraction to its fullest, you must decompose the numerator (top part of the fraction) and the denominator (bottom part of the fraction) to their fullest and then simplify. Here is an example.

Example 1.4. Simplify completely: $\dfrac{12+7\cdot 6}{(8-5)^2}$

$$\frac{12+7\cdot 6}{(8-5)^2} = \frac{12+42}{(3)^2}$$

Remember *PEMDEAS* on top the 7 must first multiply the 6. The product then can be added to the 12. Likewise, in the denominator, the parenthesis goes first. Hence, we subtract the 8 by 5 then square it.

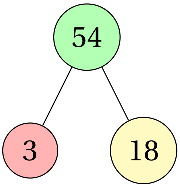
$$= \frac{54}{3\cdot 3}$$

Break into prime factors. Note that $54 = 18\cdot 3$. You can see this by the tree below.

$$= \frac{18\cdot \cancel{3}}{3\cdot \cancel{3}}$$

Cancel liked terms.

$$= \frac{18}{3}$$



That was not so bad after all. Let's attempt a more complicated example.

Example 1.5. Simplify completely: $\dfrac{15+8\cdot 4}{(9-6)^2}$

$$\frac{15+8\cdot 4}{(9-6)^2} = \frac{15+32}{3^2}$$

$$= \frac{47}{9}$$

Since 47 is a prime number and doesn't share any common factor with 9, it cannot be simplified further.

Example 1.6. Subtract and simplify completely: $\dfrac{3}{4} - \dfrac{5}{7}$.

To add or subtract fractions, first ensure a common denominator. We do this by multiplying each fraction by an equivalent of 1. Before we begin notice that $\frac{7}{7}$ and $\frac{4}{4}$ both equal 1.

$$\frac{3}{4} - \frac{5}{7} = \frac{7}{7}\cdot\frac{3}{4} - \frac{4}{4}\cdot\frac{5}{7}$$

Now, both fractions have the common denominator 28.

$$= \frac{21}{28} - \frac{20}{28}$$

Combine the numerators over the common denominator.

$$= \frac{21-20}{28}$$

Simplify to find the answer.

$$= \frac{1}{28}$$

Example 1.7. Divide and simplify completely: $\frac{7}{12} \div \frac{2}{9}$.

Note that this fraction is the same as

$$\frac{\frac{7}{12}}{\frac{2}{9}}$$

To that end, we can multiple this fraction by the multiplicative inverse of $\frac{2}{9}$ to terminate a fraction over a fraction.

$$\frac{7}{12} \div \frac{2}{9} = \frac{\frac{7}{12}}{\frac{2}{9}}$$

Flip and multiply. That is, switch the 2 with the 9

$$= \frac{7}{12} \cdot \frac{9}{2}$$

$$= \frac{7 \cdot 9}{12 \cdot 2}$$

Break down the terms into prime factors.

$$= \frac{7 \cdot \cancel{3} \cdot 3}{4 \cdot \cancel{3} \cdot 2}$$

Cancel the 3 terms.

$$= \frac{7 \cdot 3}{4 \cdot 2}$$

$$= \frac{21}{8}$$

1.2 Real Numbers, Prime numbers & Fractions

Humans have an innate tendency to classify objects according to their distinctive traits. In the field of biology, we humans [4] categorize ourselves as primates. In this set, we are connected with gorillas, chimpanzees, orangutans, bonobos, and gibbons [5]. In collaboration we form a group known as apes. Furthermore, our set extends to other creatures beyond the ape category, encompassing species like monkeys. This act of categorization establishes a framework for organizing living organisms. Through this framework, we gain the ability to delve into their collective behaviors and biology both as a group or individually. This same concept of classification can be paralleled in the realm of mathematics. The form of categorization we so speak of is seen in the form of grouping numbers.

In this section, I introduce different numerical spaces that facilitate efficient mathematical operations based on our specific needs. However, for most of this class, our focus will be on Integers. Towards the end of the book, we will transition to complex numbers and real numbers.

Definition

Natural numbers: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$
Integers: $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$
Whole numbers: $\mathbb{W} = \{0, 1, 2, 3, 4, 5, \ldots\}$
Rational numbers: $\mathbb{Q} = \left\{\frac{a}{b} \text{ where } a, b \text{ are integers.}\right\}$
Irrational numbers: $\mathbb{R} - \mathbb{Q} = \{$All the numbers that cannot be written as a rational number.$\}$

[4] The scientific name for us Humans is Homo Sapiens Sapiens.
[5] None of these species are MONKEYS! They are APES :)

Real numbers: $\mathbb{R} = \{$All rational and irrational numbers. $\}$

Let's proceed via definitions and examples.

Definition

A whole number is a ***factor*** of another when it divides that number exactly, with no remainders.

Example 1.8. Find all the factors of 36.

$$1, 2, 3, 4, 6, 9, 12, 18, 36$$

Notice that all these numbers divide 36 and produce a whole number (not a fraction.)

Definition

A ***prime number*** is a natural number with EXACTLY two factors. Those factors are 1 and itself. (Notice that 1 is not a prime number as its does not have two factors. Its only factor is 1.)

Definition

A ***prime factor*** is a factor that is prime.

Example 1.9. What is the prime factor of 16.

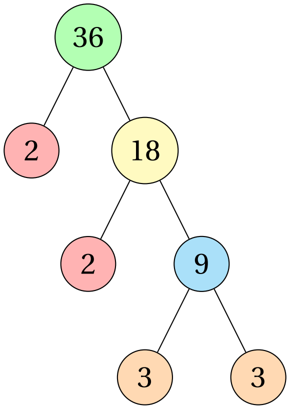
Notice that 16 factors into $2 \cdot 2 \cdot 2 \cdot 2$ and since 2 is a prime number then it indeed is a prime factor of 16.

Definition

To write the ***prime factorization*** of a number, we write the number as a product of its prime factors.

Example 1.10. What is the prime factorization of 36?

Notice that $36 = 2 \cdot 2 \cdot 3 \cdot 3$ or more simply $2^2 \cdot 3^2$.



Next let's define some important properties that you will use throughout your whole mathematical career. These tasks may appear insignificant, and you might not give them much thought as you perform them. However, their significance can vary depending on the context or environment in which you are working. Since we are mainly working with the integers, reals and imaginaries then we should not fret about my absurd claims.

Properties of Real Numbers

For any real numbers a, b, and c, the following are always true.

Commutative Property of Addition and Multiplication:

$a + b = b + a$ and $ab = ba$

(This says the order in which we add or multiply doesn't matter.)

Associative Property of Addition and Multiplication:

$$a+(b+c)=(a+b)+c \text{ and } a(bc)=(ab)c$$

(This says it doesn't matter how we group terms of the same operation.)

Distributive Property:

$$a(b+c)=ab+ac$$

(That is addition and multiplication work well together.)

Additive Property of Equality:

If $a=b$, then $a+c=b+c$.

(This says we can add the same value to both sides and still have a true statment.)

For any real numbers a, b, and c, the following are always true.

Multiplicative Property of Equality:

If $a=b$, then $a\cdot c=b\cdot c$.

(This says we can multiply the same value on both sides of an equation.)

Let's put these properties to use by doing some examples.

Example 1.11. Use the properties above to complete the following.

$$-3x(4\alpha+5+6y)-2x=(-3x)(4\alpha)+(-3x)(5)+(-3x)(6y)-2x$$

Distribute the $-3x$ into each term.

$$=-12x\alpha+-15x-18xy-2x$$

Group liked terms and then add.

$$= -12x\alpha + -15x - 2x - 18xy$$

$$= -12x\alpha + -17x - 18xy$$

We will now use algebra to manipulate structures. Generally, we aim to solve for a variable, which is always a number.

Example 1.12. Solve for m.

$$y = mx + b$$

Subtract b to both sides.

$$y - b = mx + b - b$$

$$y - b = mx + 0$$

We wish to get rid of x. That is, divide both sides by x.

$$\frac{y-b}{x} = \frac{m\cancel{x}}{\cancel{x}}$$

$$\frac{y-b}{x} = m$$

Example 1.13. Solve for y.

$$-4y + 6 = 2(y + 4)$$

Multiply the 2 to each term inside the parentheses to eliminate the parentheses.

$$-4y + 6 = 2y + 8$$

After distributing, move on to grouping like terms. To isolate y terms on one side, add $-2y$ to both sides. To get the constants on the other side, subtract 6 from both sides.

$$-4y + 6 + (-2y) = 2y + 8 + (-2y)$$

Adding $-2y$ to both sides to get all y terms on one side.

$$\begin{aligned} -4y - 2y + 6 &= 2y - 2y + 8 \\ -6y + 6 &= 0 + 8 \end{aligned}$$

Subtracting 6 from both sides to get all the constants on the other side.

$$\begin{aligned} -6y + 6 - 6 &= 8 - 6 \\ -6y + 0 &= 2 \end{aligned}$$

Simplify both sides of the equation.

$$y = \frac{2}{-6}$$

Isolate y by dividing both sides by -6.

$$y = -\frac{1}{3}$$

Simplify the fraction to get the final value for y.

Example 1.14. Solve for ϕ.

$$-\frac{7}{3}\phi - \frac{1}{2} = \frac{3}{4}$$

Add $\frac{1}{2}$ to both sides.

$$-\frac{7}{3}\phi = \frac{3}{4} + \frac{1}{2}$$

In order to add $\frac{3}{4}$ and $\frac{1}{2}$ we must get a common denominator.

$$-\frac{7}{3}\phi = \frac{3}{4} + \frac{2}{4}$$

$$-\frac{7}{3}\phi = \frac{5}{4}$$

Multiple both sides by 3.

$$-7\phi = \frac{5 \cdot 3}{4}$$

$$-7\phi = \frac{15}{4}$$

Divide both sides by $-\frac{1}{7}$.

$$\phi = \frac{15}{4} \cdot -\frac{1}{7}$$

Multiply straight across.

$$\phi = -\frac{15}{28}$$

Example 1.15. Solve for α.

$$x = y - \alpha z$$

Subtract y from both sides of the equation.

$$x - y = y - y - \alpha z$$

$$x - y = -\alpha z$$

Divide both sides by z to isolate $-\alpha$.

$$\frac{x-y}{z} = \frac{-\alpha \cancel{z}}{\cancel{z}}$$

Multiply both sides by -1 to change the sign of α.

$$\frac{x-y}{z} = -\alpha$$

$$-1\left(\frac{x-y}{z}\right) = -1(-\alpha)$$

Simplification gives the final equation.

$$-\frac{x-y}{z} = \alpha$$

1.3 Applications

In mathematics, studying logic becomes evident as a significant part of the subject. The way we express mathematical statements is crucial since slight variations can completely alter their meanings. We begin by giving all the commonly used words for addition, subtraction, multiplication and division. Allow me to further clarify via this table.

Definition

Addition	**Subtraction**
sum	difference
more	minus
plus	a less b
b more than a	a decreased by b
a increased by b	
the total of a and b	**Division**
	quotient
Multiplication	half of
product	a divided by b
twice/thrice	b divided into a
times	ration of a and b
double/triple	a over b
$\frac{a}{b}$ of c	a per b

Note: Some phrases like "less than" or "more than" can really depend on context. ***For example***, What's the difference between “Nine less than a number" and "A number is less than nine" when written as a math statement? ($x - 9$ vs. $x < 9$) We will disccus this notation in more detail in the next chapter.

When dealing with word problems, it is best to be as detailed as possible. When you are defining variables make sure that you give an accurate description. Answering questions with complete sentences is a valuable practice. For example, say "x = number of tickets sold" instead of "x = tickets".

Word problems can be challenging, but it is essential to identify key words, such as the ones provided in the table above. In most cases, you will typically only require three or so crucial words from a word problem to effectively solve it. Although word problems may not be enjoyable, it is necessary to tackle them head-on. It's worth noting that this won't be the last encounter with such problems.

The word "is" can sometimes cause confusion. A clear example can be seen when considering age. For instance, my cat, Mr. Pouch, is 4 years old. Hence we can claim Mr. Pouch [6] = 4 years old

Example 1.16. Two consecutive integers have a sum of 71. Find the integers.

$$\begin{aligned} \text{First integer } &= x \\ \text{Second integer } &= x+1 \end{aligned}$$

Now add them together and set the equal to 71

$$\begin{aligned} x+(x+1) &= 71 \\ 2x+1 &= 71 \\ 2x &= 70 \\ x &= 35 \end{aligned}$$

Since we were given that they are two consecutive numbers, then the second number is either going to be 34 or 36. However, $35+36=71$, hence our two numbers are 35 and 36.

Example 1.17. Three consecutive even integers have a sum of 48. Find the integers

[6] Fat boy! That is one of his several nicknames.

$$\begin{aligned}\text{First integer } &= x\\ \text{Second integer } &= x+2\end{aligned}$$

Notice that we chose 2 and 4 since we are dealing with even integers.

$$\text{Third integer } = x+4$$

Now add them together and set them equal to 48.

$$\begin{aligned}x+(x+2)+(x+4) &= 48\\ 3x+6 &= 48\\ 3x &= 42\\ x &= 14\end{aligned}$$

Notice that $14+16+18=48$.

Example 1.18. The sum of two numbers is 100.

1. One of the numbers is given to be 30. Find the other.

   $$100-30=70$$

   Hence the second number is 70.

2. One of the numbers is given to be -45. Find the other.

   $$\begin{aligned}100-(-45) &= 100+45\\ &= 145\end{aligned}$$

   Hence the other number is 145.

3. If one of the numbers is x, what expression represents the other number?

   $$100-x$$

Example 1.19. The sum of two numbers is A.

1. One of the numbers is given to be B. Find the other.

$$A - B = A - B$$

Hence the second number is $A - B$.

2. One of the numbers is given to be $-C$. Find the other.

$$\begin{aligned} A - (-C) &= A + C \\ &= A + C \end{aligned}$$

Hence the other number is $A + C$.

3. If one of the numbers is x, what expression represents the other number?

$$A - x$$

Example 1.20. The perimeter of a square is 288 yards. How long is each side?

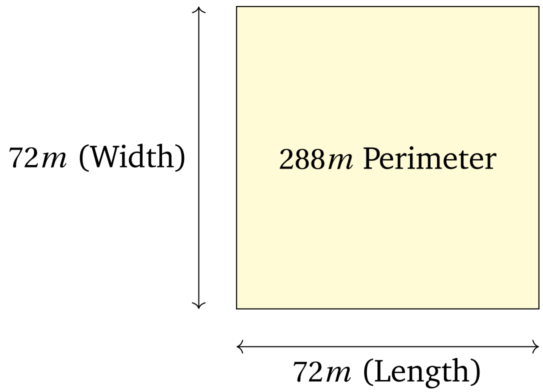
Notice that the formula for the perimeter is $P = 4s$ where $s =$ the side of the rectangle, hence

$$P = 4s$$

$$288 = 4s$$

$$\frac{288}{4} = s$$

$$72 = s$$



Before I give this next example, I will provide a list of relations.

Example 1.21. The perimeter of a rectangle is 340 meters. The length of the rectangle is 100 meters. What is the width of the rectangle?

Notice that the formula for the perimeter of a rectangle is $P = 2l + 2w$ where $l =$ the length of the rectangle and $w =$ the width of the rectangle. Hence,

$$P = 2l + 2w$$

Since the length is 100 $l = 100$

$$340 = 2(100) + 2w$$

Now we solve for the width w.

$$340 = 200 + 2w$$

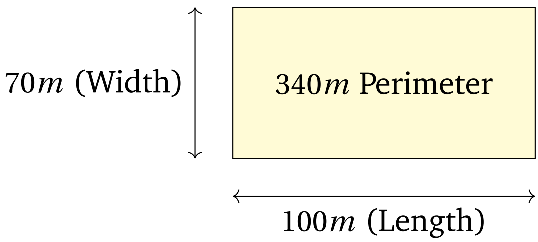
$$340 - 200 = 2w$$

$$140 = 2w$$

$$\frac{140}{2} = w$$

$$70 = w$$

So, the width of the rectangle is 70 meters.



Example 1.22. The area of a rectangle is 180 square meters, and the length of the rectangle is 15 meters. What is the width of the rectangle?

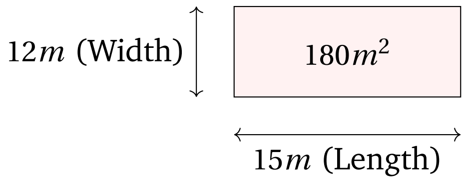
The formula for the area of a rectangle is given by $A = l \cdot w$ where A is the area, l is the length, and w is the width.

$$A = 180m^2$$
$$l = 15m$$

We need to find the width w. Rearranging the area formula gives us.

$$\begin{aligned} w &= \frac{A}{l} \\ &= \frac{180m^2}{15m} \\ &= 12m \end{aligned}$$

So, the width of the rectangle is 12 meters.



Example 1.23. Find the area of a rectangle whose length is 15 meters and width is 7 meters.

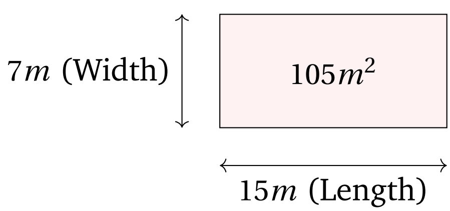
The formula for the area of a rectangle is given by $A = l \cdot w$ where A is the area, l is the length, and w is the width.

$$\begin{aligned} l &= 15m \ (m \text{ for meters}) \\ w &= 7m \end{aligned}$$

Thus, the area A can be calculated as follows.

$$\begin{aligned} A &= l \cdot w \\ &= 15 \cdot 7 \\ &= 105 \end{aligned}$$

Therefore, the area of the rectangle is 105 square meters.



It becomes evident that we must learn the process of algebra manipulation, as this universal tool is used for many purposes. For instance, the disaster of the Chernobyl Nuclear Power Plant located in northern Ukraine (part of the Soviet Union). On the 28th of April 1986, the 4th unit of the power plant was undergoing a safety test. This safety test mimicked a power outage to ensure that the reactor could safely shut down. However, during the test, a mixture of operator failures and reactor design flaws led to a failure of the system. Today, the city of Chernobyl remains uninhabitable for humans for the next $20,000$ years. When unit 4 exploded, a substance called corium was released from the reactor. Corium is made up of nuclear fuel, nuclear fission products, and damaged parts of the reactor mixed with air. 8 months later, in room 2172, we found a chunk of corium that we call the elephant's foot [7]. When it was discovered, the elephant's foot released around $10,000$ roentgen per hour. Roentgen is a measure of ionizing radiation. Moreover, it is used to quantify the exposure to X-rays and gamma rays. However, when found, it was sufficient to stand next to it for only 5 minutes to kill you. To show this, the dose received can be calculated using the formula.

$$D = \frac{D_{\text{rate}} \cdot t}{Q}$$

where,

- D = dose in sieverts (Sv) which measures the health effect of ionizing radiation on the human body. Anything above $4-5$ sieverts is a considered a lethal dose.

- D_{rate} = dose rate in sieverts per hour (Sv/h)

- t = time in hours (h)

- Q = quality factor (dimensionless, depends on radiation type; usually 1 for gamma rays. We will assume that $Q = 1$)

We will assume that 1 Roentgen $\approx$ 0.01 sieverts for gamma rays. Since we are only taking account for 5 minutes of ration exposure, $t = \frac{5}{60}$ hours $= 0.083$ hours. Above we have assumed that $Q = 1$. Therefore, all we need is to find D_{rate}. We define D_{rate} as,

$$D_{\text{rate}} \left(\text{in } \frac{Sv}{h}\right) = E_{\text{rate}} \left(\text{in } \frac{R}{h}\right) \cdot 0.01 \left(\text{in } \frac{Sv}{R}\right)$$

[7] It literally looks like the foot of an elephant.

Since the elephant's foot released around $10,000$ roentgen per hour and E_{rate} is the exposure rate in roentgens per hour (R/h).

$$\begin{aligned}
&= \left(\frac{10,000R}{h}\right) \cdot \left(\frac{0.01Sv}{R}\right) \\
&= \left(\frac{10,000\cancel{R}}{h}\right) \cdot \left(\frac{0.01Sv}{\cancel{R}}\right) \\
&= \left(\frac{10,000}{h}\right) \cdot \left(\frac{0.01Sv}{1}\right) \\
&= \frac{100Sv}{h}
\end{aligned}$$

Together,

$$\begin{aligned}
D &= \frac{D_{\text{rate}} \cdot t}{Q} \\
&= \frac{\frac{100Sv}{h} \cdot 0.083h}{1} \\
&= \frac{100Sv}{h} \cdot 0.083h \\
&= \frac{100Sv}{\cancel{h}} \cdot 0.083\cancel{h} \\
&= 100 \cdot 0.083Sv \\
&= 8.3Sv
\end{aligned}$$

Therefore, within those 5 minutes, your body would have been exposed to 8.3 sieverts. Since $8.3Sv > 4Sv$, exposure to radiation would have reached a threshold leading to death.

I once had a professor who told me, "In mathematics, there is no such thing as almost getting close to the correct answer." He justified giving zeros on exams for incorrect answers with this reasoning. Although I still don't agree with his grading rubric, I respect it. The questions we give you in class are not life-threatening. If you get it wrong, you only lose points. But if you get the maths wrong in the practical world, you put other people's lives on the line. Thus, there is no such thing as getting close to correct. In application, you must be precise. An aerospace engineer needs to know the maths, physics, chemistry, and other sciences that go into making a plane. If they didn't, who would ride planes if the probability of survival were low? Yeah, you may never use the rigorous math that we do in this book, but you will always depend on someone who does.

1.4 Additional Problems

1. Compute the following and clearly show each step.

   (a) $\frac{5}{7}-\frac{1}{5}$

   (b) $\frac{12}{7}\div\frac{4}{3}$

   (c) $\frac{1}{2}-\left(\frac{1}{4}\div\frac{3}{5}\right)$

   (d) $\frac{1}{3}-\left(\frac{1}{4}\div\frac{3}{7}\right)$

2. The equation to change from Fahrenheit (F) to Celsius (C) is written below: Solve for F in the equation above and clearly explain what the new equation represents.

   (a) $C=\frac{5(F-32)}{9}$

3. These two equations are the Kinetic and Potential Energy formulas studied in physics.

   (a) Solve for mass m: $K=\frac{1}{2}mv^2$

   (b) Solve for acceleration due to gravity g: $P=mgh$

4. Consider the ideal gas law, a fundamental equation in chemistry, given by $PV=nRT$ where P is the pressure, V is the volume, n is the number of moles, R is the universal gas constant, and T is the temperature.

(a) Solve for the number of moles n in terms of P, V, R, and T.

(b) Solve for the temperature T in terms of P, V, n, and R.

5. Consider the Hardy-Weinberg equation, which is used in population genetics. The equation is $p^2 + 2pq + q^2 = 1$, where p and q represent the frequency of the dominant and recessive alleles in a population, respectively.

   (a) Solve for the frequency of the dominant allele p in terms of q.

   (b) Solve for the frequency of the recessive allele q in terms of p.

6. Consider Kepler's Third Law of Planetary Motion, which relates the period of a planet's orbit around the Sun to its average distance from the Sun. The law is expressed as $T^2 = k \cdot a^3$, where T is the orbital period, a is the average distance from the Sun (semi-major axis), and k is a constant.

   (a) Solve for the average distance a from the Sun in terms of T and k.

7. Solve for acceleration a.

   (a) $V_f = V_0 + at$

8. Solve for x.

   (a) $\triangle = abcdefghijklmnopqrstuvwxyz - \nabla$

9. Solve for m.

   (a) $\dfrac{s-m}{9} = 2x$

   (b) $\dfrac{m-c}{1} = no + m - m$

   (c) $3mmuy = 4rm$

   (d) $math = sucks$

   (e) $d\left(\dfrac{\vec{\mathrm{V}}}{23}\right) = \dfrac{d}{t}$

10. The following is the dilution equation used in chemistry. Solve for M_2 (final molarity.)

    (a) $1 = \dfrac{M_2 V_2}{M_1 V_1}$

11. Solve for e.

    (a) $\dfrac{(18e - re)}{9t} = 0$

    (b) $e - mc^2 - 1 = -1$

    (c) $ey + 2e = 21h$

12. Are the following true? Explain your answer.

    (a) $(7+9)^2 = 7^2 + 9^2$

    (b) $(1+2)^2 = 1^2 + 2^2$

13. Solve for h

    (a) $A = \dfrac{1}{2}bh$

    (b) $A = \dfrac{(a+b)h}{2}$

14. The "Elephant's Foot" was measured at around 200 roentgens per hour (R/h) during a visit in 2018. This is still lethal if exposed for a long period of time. To get the same exposure as 5 minutes in 1986, you must be around the Elephant's Foot for 4.15 hours. Show the mathematical process. Use the derivation in discussion from page 27.

2 Inequalities, Absolute Values & Lines

2.1 Absolute Value Equations

Absolute values play a vital role in mathematics and science due to their exceptional properties. They possess the unique ability to transform negative values into positive values; thus, making them indispensable when studying functions. This property is crucial for ensuring that potential functions meet the necessary requirements to be considered valid functions. While discussing functions may seem premature, it is important to highlight the significance of absolute values, as they provide distinct characteristics that find broad application across various scientific disciplines.

Definition

Relations

$<$	$\leq$	$=$	$\geq$	$>$
less than	less or equal	equal	greater or equal	greater

Example 2.1. The sum of a number times 8 and 20 is less than or equal to 28.

A number times 8 will look like this.

$$8x$$

Less than or equal to 28 will look something like this

$$\leq 28$$

Notice that we are looking at the sum of these the two numbers 20 and $8x$ and this is less than or equal to 28. That is,

$$8x + 20 \leq 28$$

Now that we have the appropriate language [8], we are able to express mathematics using set builder notation. Set builder notation looks difficult at first, however when mastered, it is beneficial as it can be interchanged in English terms, graphically and mathematically.

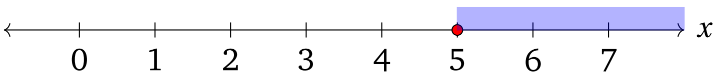
Definition

Set Builder Notation

$$\{x : x \geq 5\}$$

This reads, "All the numbers x such that x is greater than or equal to five." Or "All the numbers greater than or equal to five." The first is more appropriate for a mathematician. Note that ":" means such that.

Here is the $1D$ plot of the statment.



The x inside the number line may throw you off. However it simply just means that x can be any number from 5 to infinity. We denoted infinity as ∞.

Definition

Domain & Range Notation

$$], [\text{ and } \bullet = \text{The number is included.}$$
$$), (\text{ and } \circ = \text{The number is not included.}$$
$$[\ )\ \ (\ ] = \text{A mixture of both.}$$

[8]Languages are complicated. Today, approximately 7000 languages are still spoken. This includes our well-known languages such as English and Spanish. Just like everything else, languages face the risk of extinction. Anthropologists have observed throughout history that languages have disappeared due to diverse reasons, often driven by political, cultural, and popular factors. Pretty sad if you ask me. With every disappearance, history and cultures fade too.

That is consider the last example of all the numbers x such that x is greater than or equal to five.

Interval Notation

$$[5,+\infty)$$

Important: Note that ∞ and $-\infty$ will always have$),($ since ∞ is not a number we do not include it[9].

Some European countries such as France and Germany use $[a,b]$ for the inclusion of the numbers and $]a,b[$ for the exclusion of the number. For example, $[5,\pm\infty)$ is the same thing as $[5,\pm\infty[$. Here both ")" and "[" mean the same thing, the exclusion of a value.

Definition

Sets are a mathematical model used for collections of objects. In the following example A and B are each sets containing different elements.

Definition

Union and Intersection of Sets

The ***union*** of two sets contains every element that appears in

[9] Infinity is not a number it's a concept. Think about it, you can not really define infinity as one number. Consider the set of all odd numbers. This set contains an infinite amount of numbers because for every positive integer n, there exists an odd number $2n+1$. So, we can say that the set of odd numbers is infinite.

On the other hand, let's consider the set of all numbers, which includes both even and odd numbers. Similar to the set of odd numbers, the set of all numbers is also infinite. However, it is a different kind of infinity compared to the set of odd numbers. The set of all numbers contains an infinite amount of even numbers as well as an infinite amount of odd numbers.

Although both sets are infinite, they are different in terms of sets. The set of odd numbers only consists of odd numbers, while the set of all numbers includes both even and odd numbers. Thus, despite both sets being infinite, they are distinct due to the difference in the numbers they contain.

Whaaaaaat! HAI.

the first set ***OR*** the second set. The symbol $\bigcup$ stands for "union."
The ***intersection*** of two sets contains every element that appears in the first set ***AND*** the second set. The symbol $\bigcap$ stands for "intersection."
If two sets have no elements in common, then the intersection is the ***empty set***, denoted $\emptyset$.

Example 2.2. Consider the two sets $A = \{a, b, 1, 3, \text{cat}, 0, -1\}$ and $B = \{1, 2, 3, a, \text{cat}\}$.

$$A \cup B = \{a, b, -1, 0, 1, 2, 3, \text{cat}\}$$

Notice that this collected all of the numbers and put them each in a set. There are no replicas. That is 1 ***and*** 3 are in both A ***and*** B, but they only appear once in the set. [10]

Example 2.3. Consider the two sets $A = \{a, b, 1, 3, \text{cat}, 0, -1\}$ and $B = \{1, 2, 3, a, \text{cat}\}$.

$$A \cap B = \{a, 1, 3, \text{cat}\}$$

This contains exactly everything that is exactly in A ***and*** B. Lets now touch up on ***Compound Inequalities***. Sometimes the inequalities we encounter need to satisfy several conditions.

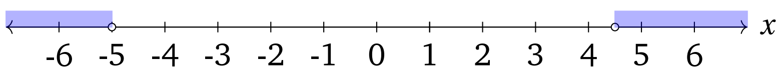
[10] Your body is a set, comprising various elements that collectively define your physical existence. This includes your eyes, hair, fingers, heart, lungs, and so on. Similarly, Earth can be seen as a set encompassing all living species. Humans and plant life, as well as natural phenomena such as the ocean, rocks, volcanoes, and constructed entities like houses. Taking it further, our universe can be regarded as a set, within which our solar system, consisting of our sun and Earth, acts as an element. A set within a set within a set, in this trivial example.

OR Statements

$$\begin{aligned} 4x-5>13 &\quad \text{or} \quad 2x-6<-16 \\ 4x>18 &\quad \text{or} \quad 2x<-10 \\ x>\frac{18}{4} &\quad \text{or} \quad x<-5 \\ x>\frac{9}{2} &\quad \text{or} \quad x<-5 \\ \left(\frac{9}{2},\infty\right) &\quad \text{or} \quad (-\infty,-5) \end{aligned}$$

Since this is an ***OR*** statement, values in either of these intervals will be in the solution set (i.e. the union of the solution sets).

Domain: $(-\infty,-5)\cup\left(\frac{9}{2},\infty\right)$



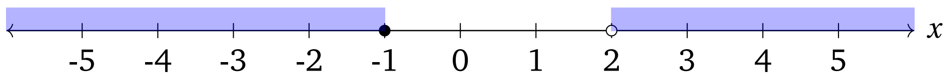
As seen in the number line above, you would plot both inequalities and since they don't intersect at any point, you would write the final union as two separate sets (the two inequalities in the statement) combined by a $\cup$. This is because a union includes all the values present in both sets.

OR Statements

$$\begin{aligned} 3x+4>10 &\quad \text{or} \quad 5x-3\leq -8 \\ 3x>6 &\quad \text{or} \quad 5x\leq -5 \\ x>\frac{6}{3} &\quad \text{or} \quad x\leq\frac{-5}{5} \\ (2,\infty) &\quad \text{or} \quad (-\infty,-1] \end{aligned}$$

Since this is an ***OR*** statement, values in either of these intervals will be in the solution set (i.e., the union of the solution sets).

Domain: $(-\infty,-1]\bigcup(2,\infty)$

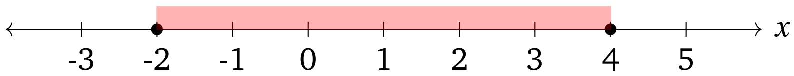


AND Statements

$$\begin{aligned} 2x-4 &\leq 4 & \text{and} \quad & 2x-1 \geq -5 \\ 2x &\leq 8 & \text{and} \quad & 2x \geq -4 \\ x &\leq 4 & \text{and} \quad & x \geq -2 \end{aligned}$$

Since this is an ***AND*** statement, we need to find the values that satisfy BOTH of these inequalities at the same time. This is the intersection $(-\infty,4)\cap(-2,+\infty)$.

Domain: $[-2,4]$

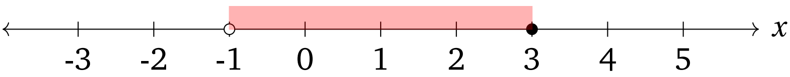


Example 2.4. Find the solution set and graph.

$$\begin{aligned} -3 < 2x-1 &\leq 5 \\ -2 < 2x &\leq 6 \\ -1 < x &\leq 3 \end{aligned}$$

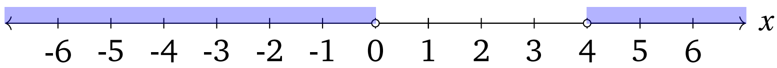
This says that x is between -1 and 3. However, due to the inequalities, x is not -1 but can be 3. We define this notation as the domain.

$$\text{Domain} = (-1,3]$$



Example 2.5. Find the solution set and graph.

$$\begin{aligned} 2x-5<-5 &\quad \text{or} \quad 3x+2>14 \\ 2x<-5+5 &\quad \text{or} \quad 3x>14-2 \\ 2x<0 &\quad \text{or} \quad 3x>12 \\ x<0 &\quad \text{or} \quad x>4 \\ \text{Domain} = (-\infty,0) &\quad \text{or} \quad (4,\infty) \end{aligned}$$



Formal Definition of absolute values

$$|x-0| = |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

This definition will be helpful when we are trying to algebraically solve equations and inequalities.

Example 2.6. (i) $|2| = 2$ (ii) $|-5| = -(-5) = 5$

Intuitive Definition of absolute values

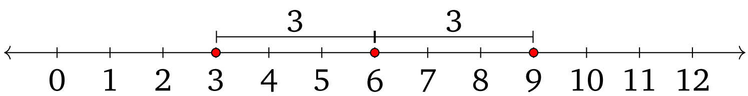
$|x|$ is the distance between x and zero.
$|x-k|$ is the distance between x and k.

Since distance is always positive, the absolute value is the perfect tool to describe distance.

Using the intuitive definition makes it easier to graph our solution set.

Example 2.7. Use the intuitive definition to solve for x, graph the solution, and write an English sentence. $|x-6| = 3$

Using the intuitive definition we can draw a graph to find the solutions, this says "The distance between a number and six is three."



The solution is $x = 3$ or $x = 9$. However, If we apply the formal definition we have the following.

$$|x-6| = 3$$

Case 1: $(x-6) \geq 0$

$$\begin{aligned} x-6 &= 3 \\ x &= 9 \end{aligned}$$

Case 2: $(x-6) < 0$

$$\begin{aligned} -(x-6) &= 3 \\ x &= 3 \end{aligned}$$

Notice that we arrived to the same answer! $x = 9$ or $x = 3$. **Since absolute values can either be negative or positive, we force case 1 is positive and case 2 is negative. This approach will be used in our other examples.**

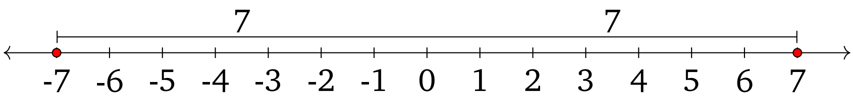
Example 2.8. Use the intuitive definition to solve for x, graph the solution, and write an English sentence. $|x| = 7$

$$\begin{aligned} |x| &= 7 \\ |x-0| &= 7 \end{aligned}$$

Using the definition, "the distance between a number and zero is seven". Moreover, using the formal definition we have.

$$|7| = \begin{cases} 7 & x \geq 0 \\ -7 & x < 0 \end{cases}$$

All the numbers whose distance from 0 is 7.



Example 2.9. Use the formal definition to solve for x, graph the solution, and write an English sentence. $|x-3| = 5$

$$|x-3| = 5$$

Using the definition.

$$|x-3| = \begin{cases} (x-3) & x-3 \geq 0 \\ -(x-3) & (x-3) < 0 \end{cases}$$

The distance between a number and three is five.

$$|x-3| = 5$$

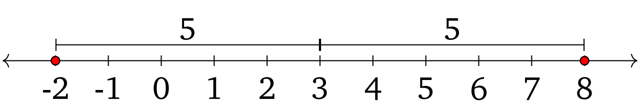
<u>Case 1: $(x-3) \geq 0$</u>

$$x - 3 = 5$$
$$x = 8$$

<u>Case 2: $(x-3) < 0$</u>

$$-(x-3) = 5$$
$$x = -2$$

That is! $x = 8$ or $x = 2$.



Example 2.10. Use the formal definition to solve for x, graph the solution, and write an English sentence. $|2x+4|=8$

$$|2x+4|=8$$

Using the definition.

$$|2x+4|=\begin{cases}(2x+4) & 2x+4\geq 0\\ -(2x+4) & 2x+4<0\end{cases}$$

The distance between two times a number plus four and zero is eight.

$$|2x+4|=8$$

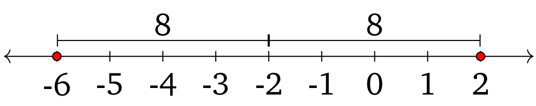
<u>Case 1: $2x+4\geq 0$</u>

$$2x+4=8$$
$$2x=4$$
$$x=2$$

<u>Case 2: $2x+4<0$</u>

$$-(2x+4)=8$$
$$-2x-4=8$$
$$x=-6$$

Thus, $x=2$ or $x=-6$.



Example 2.11. Use the formal definition to solve for y, graph the solution, and write an English sentence. $3|y-5|-4=11$

$$3|y-5|-4=11$$

First, we simplify and isolate the absolute value term.

$$3|y-5|-4=11$$

Add 4 to both sides.

$$3|y-5|=15$$

To get ride of the right hand 3, we divide 3 to both sides. Since $\frac{15}{3}=5$ then

$$|y-5|=5$$

Now, using the definition of absolute value.

$$|y-5|=\begin{cases}(y-5) & \text{if } y-5\geq 0\\ -(y-5) & \text{if } y-5<0\end{cases}$$

$$|y-5|=5$$

Case 1: $y-5\geq 0$

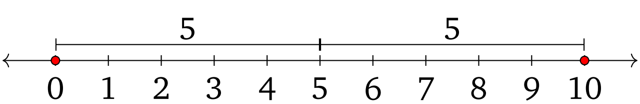
$$\begin{aligned} y-5&=5\\ y&=10\end{aligned}$$

Case 2: $y-5<0$

$$\begin{aligned} -(y-5)&=5\\ -y+5&=5\\ 5&=5+y\\ 5-5&=y\\ 0&=y\end{aligned}$$

Thus, $y=10$ or $y=0$.

The graph of this solution would show two points on the number line, at $y=0$ and $y=10$, representing the two solutions of the equation.



2.2 Absolute Value Inequalities

We follow the same ideas in this section, only now we are looking at inequalities.

Definition

$|x| < c$ says "the distance between x and zero is less than c."
$|x| > c$ says "the distance between x and zero is greater than c."

Example 2.12. $|x| < 4$ Says that the "distance between x and zero is less than 4. "

Now, let's establish the definitions of the distance between two objects.

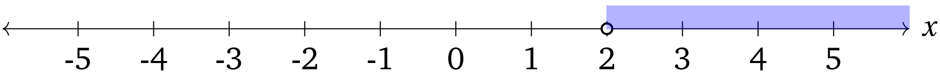
Definition

$|x-k| < c$ says "the distance between x and k is less than c."
$|x-k| > c$ says "the distance between x and k is greater than c."

Example 2.13. $|x-2| > 6$ Says "the distance between x and 2 is greater than 6."

Example 2.14. Multiplying and dividing by a negative number flips the inequality.

$$\begin{aligned} -6x+3 &< 15 \\ -6x &< 12 \\ x &> 2 \end{aligned}$$



Now, I introduce ***AND*** vs. ***OR*** statements.

Definition

AND Statement

$$|A| < c \quad \text{is equivalent to} \quad -c < A < c.$$

Note: This says $-c < A$ ***AND*** $c > A$.

Example 2.15. $|x-2| \leq 6$

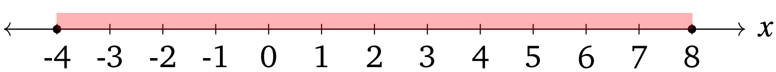
This is equivalent to

$$-6 \leq x-2 \leq 6.$$

Solving for x we get

$$-6+2 \leq x \leq 6+2$$
$$-4 \leq x \leq 8$$

Notice that the circles are shaded in because of the $\leq$ inequality. Thus, including the numbers -4 and 8.



Example 2.16. Solve for x and represent your set of solutions graphically and in interval notation. $\left|\frac{1}{2}x-3\right|-1<2$.

Notice that this is an and statement.

$$\left|\frac{1}{2}x-3\right|-1<2$$

$$\left|\frac{1}{2}x-3\right|<3$$

$$-3<\frac{1}{2}x-3<3$$

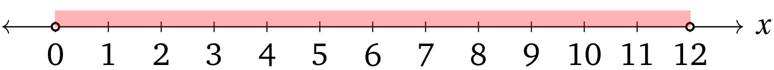
Add by 3 to all sides. Recall that we want to solve for x. Thus, we must get ride of that -3 in the middle. Therefore, it is suffice to add by 3.

$$0<\frac{1}{2}x<6$$

Again, we want to isolate x. Since $2\cdot\frac{1}{2}=1$ and $1\cdot x=x$ then it is enough to multiply all sides by 2.

$$0<x<12$$

$$(0,12)$$



Example 2.17. Solve for x and represent your solution set graphically and in interval notation. $-3<2-x\leq 5$.

First, we rearrange the inequality.

$$-3<2-x\leq 5$$

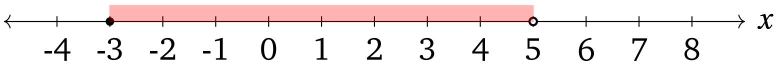
$$-3-2<-x\leq 5-2$$

$$-5<-x\leq 3$$

$$5>x\geq -3$$

x is in between -3 and 5 but does not include 5. Hence, the solution set is.

$$[-3,5)$$



Definition

OR Statement

$$|A| > c \quad \text{is equivalent to} \quad A < -c \;\; \textbf{\textit{OR}} \;\; A > c.$$

Example 2.18. $|x-2| \geq 6$

This is equivalent to

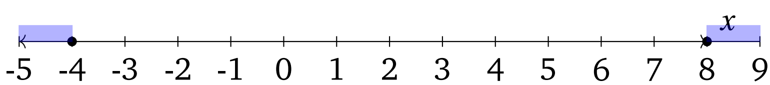
$$x-2 \leq -6 \;\; \textbf{\textit{OR}} \;\; x-2 \geq 6.$$

Solving for x we get.

$$x \leq -6+2 \;\; \textbf{\textit{OR}} \;\; x \geq 6+2.$$

$$x \leq -4 \;\; \textbf{\textit{OR}} \;\; x \geq 8.$$

Note: The solution set here is all the numbers that satisfy $x \leq -8$ ***OR*** $x \geq 4$, but not both at the same time.



Example 2.19. $|3x+1| > 9$

This is equivalent to

$$3x+1 > 9 \quad \textbf{\textit{OR}} \quad 3x+1 < -9.$$

Solving for x, we get two inequalities.
Case 1:

$$\begin{aligned} 3x+1 &> 9 \\ 3x &> 8 \end{aligned}$$

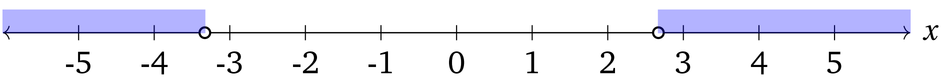
$$x > \frac{8}{3}$$

Or case 2:

$$\begin{aligned} 3x + 1 &< -9 \\ 3x &< -10 \\ x &< -\frac{10}{3}. \end{aligned}$$

Therefore, the solution set is.

$$x < -\frac{10}{3} \quad \textbf{\textit{OR}} \quad x > \frac{8}{3}.$$



Example 2.20. $2|4x - 3| + 5 < 19$

This is equivalent to

$$2|4x - 3| < 14.$$

Which is also equivalent to

$$|4x - 3| < 7.$$

Breaking down the absolute value, we get.

$$-7 < 4x - 3 < 7$$

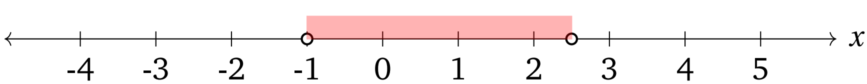
$$-7 + 3 < 4x < 7 + 3$$

$$-4 < 4x < 10$$

$$-1 < x < \frac{10}{4}$$

$$-1 < x < \frac{5}{2}$$

Notice, that although this was an **or** statement, the final answer was an and statement.



2.3 Additional Problems

1. Is the following true or false. Explain you answer.

   (a) $|x| = -2$

   (b) $|x| = 2$

   (c) $|5x+3| = -0$

2. Use the intuitive definition to solve for x.

   (a) $|5x| - 4 = 0$

   (b) $|9x+1| = 8$

   (c) $3|2x-7| + 9 = 36$

   (d) $4|5x-3| - 6 = 22$

   (e) $\frac{1}{3}|6x+5| + 2 = 4$

   (f) $\frac{1}{4}|8x-1| = 4$

   (g) $\frac{2}{1}|1+x| - 1 = 5$

3. Solve for x and represent your solution set graphically and in interval notation.

   (a) $|16+x| > 4$

   (b) $|9-3x| \leq 15$

   (c) $|2-3x| \geq 8$

   (d) $|x-4| < 5$

   (e) $7|x+3| - 5 \leq 9$

   (f) $6|-x+4| + 13 < 35$

   (g) $\frac{1}{2}|-5+x| > 6$

   (h) $\frac{1}{3}|-3x-1| - 3 \geq 5$

4. Translate each into a math equation (you do not need to solve).

   (a) Twice the difference of a number and 3 is 11.

   (b) The sum of 4 times a numbers and 5 is 7.

   (c) The quotient of x and 6 is at least 31.

   (d) Seven subtracted from the product of 8 and a number is a most -43.

3 Linear Functions

3.1 Slopes & Intercepts

Thus far, we have been working in a one-dimensional graph, which represents a number line and considers only the variable x. However, let's spice our analysis up by now working in a two-dimensional plane. In this plane, we introduce an additional variable y. The addition of a second variable allows us to explore the relationship between x and y simultaneously. By extending our graph into the $2D$ plane, we can visualize functions, equations, and inequalities in a richer and more comprehensive manner. This opens up new possibilities for understanding and analyzing mathematical concepts. Consider the following equation.

$$5x+2y=-1$$

Notice that this equation contains an x and a y term. Do you think there is a point on the Cartesian plane (x&y plane) that will give us a ***TRUE*** statement if we substitute its x and y coordinates into the equation? Is there more than one point that will make this equation a true statement? Let's try a few examples.

Example 3.1. Do these point make the statement true.

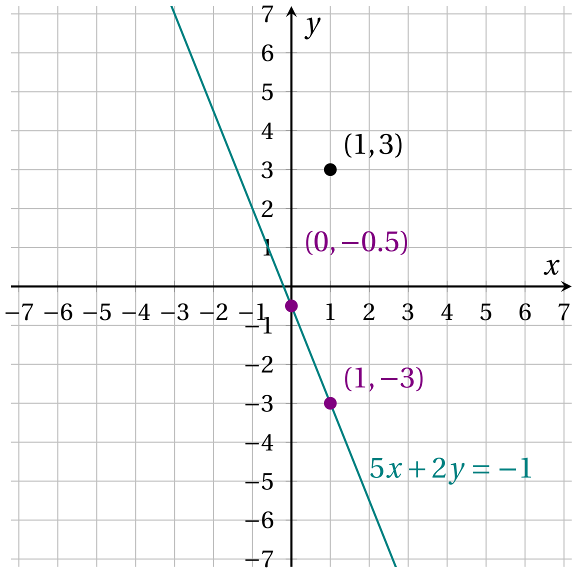
$(1,-3)$ **Yes**

$(1,3)$ **No**

$\left(0,-\frac{1}{2}\right)$ **Yes**

$(2,8)$ **No**

If we plot every single point that satisfies this equation we get a line. This is the solution set for the equation $5x+2y=-1$.



If you have an equation in two variables (like x and y) and neither is raised to a power, we can be sure the shape of the graph is a line.

Definition

The x ***-intercept*** is the point on the line whose y value is 0; i.e.
$(x,0)$ and the y ***-intercept*** is the point on the line whose x value is 0; i.e. $(0,y)$.

Example 3.2. Graph the solution set for the following linear equation by finding its x and y intercepts: $-5x+4y=-10$.

x-intercept:

$$-5x+4(0)=-10$$

$$-5x=-10$$

$$x=\frac{-10}{-5}$$

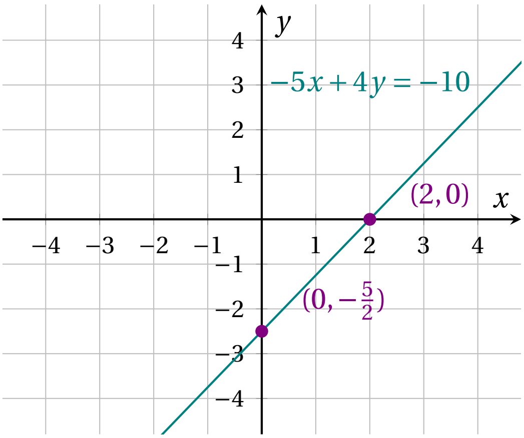
$$x=\frac{\cancel{-5}\cdot 2}{\cancel{-5}}$$

$$x = 2$$

Giving us the points $(2, 0)$.
***y*-intercept:**

$$-5(0) + 4y = -10$$
$$y = \frac{-10}{4}$$
$$y = \frac{-5 \cdot \cancel{2}}{2 \cdot \cancel{2}}$$
$$y = -\frac{5}{2}$$

Thus, our point is $\left(0, -\frac{5}{2}\right)$.



Definition

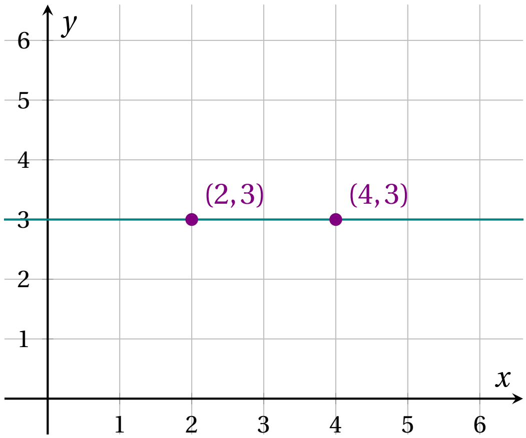
The ***slope*** of a line is a numerical value that gives the "steepness" of a line. For any two points (x_1, y_1) and (x_2, y_2) on the line, the slope m is found by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Rise}}{\text{Run}}$$

Example 3.3. To find the slope of a horizontal line, grab any two points from any horizontal line and use the slope formula. Here we will use $(2,3)$ and $(4,3)$ from the line above.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-3}{4-2} = \frac{0}{2} = 0$$

The slope of any horizontal line is $m = 0$.

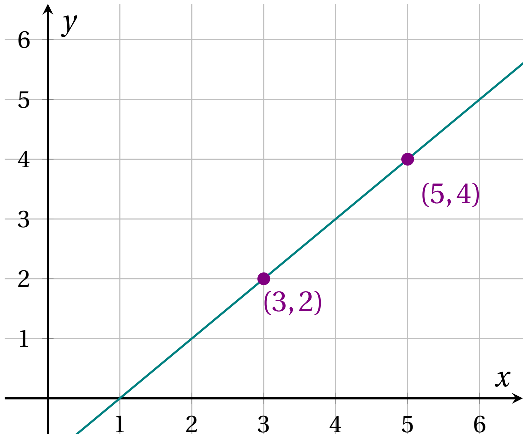


Example 3.4. Use the definitions to find the slope of the line passing through the points $(5,4)$ and $(3,2)$.

To find the slope of a line, use the slope formula with two points from the line. Here we will use $(5,4)$ and $(3,2)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{5-3} = \frac{2}{2} = 1$$

The slope of the line passing through the points $(5,4)$ and $(3,2)$ is $m = 1$.

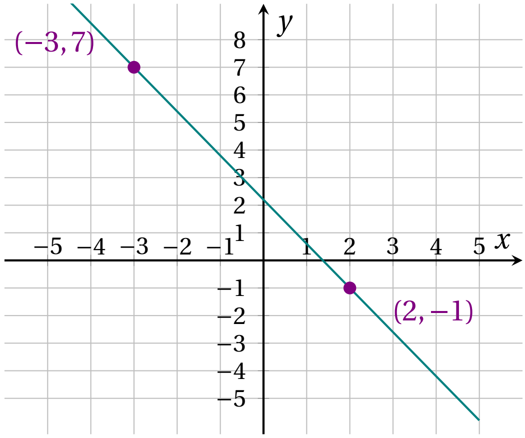


Example 3.5. Use the definitions to find the slope of the line passing through the points $(-3,7)$ and $(2,-1)$.

Again, to find the slope of a line, we must use the slope formula with two points from the line. Given that we have the two points $(-3,7)$ and $(2,-1)$.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-7}{2-(-3)} = \frac{-8}{5} = -\frac{8}{5}$$

The slope of the line passing through the points $(-3,7)$ and $(2,-1)$ is $m = -\frac{8}{5}$.

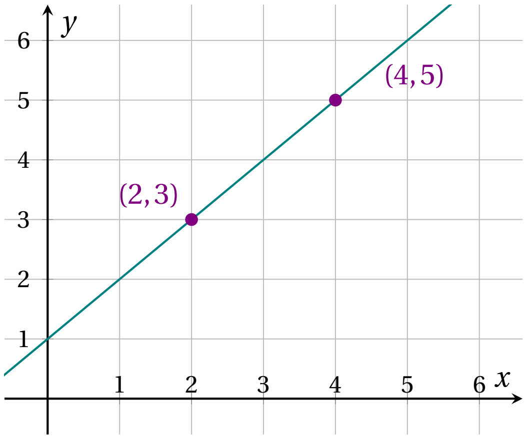


Example 3.6. Use the definitions to find the slope of the line passing through the points $(2,3)$ and $(4,5)$.

Using the definition of slopes on the points $(2,3)$ and $(4,5)$ we get.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{4-2} = \frac{2}{2} = 1$$

The slope of the line passing through the points $(2,3)$ and $(4,5)$ is $m = 1$.

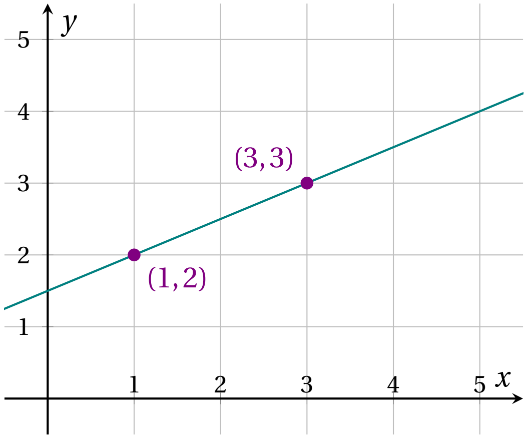


Example 3.7. Use the definitions to find the slope of the line passing through the points $(1,2)$ and $(3,3)$.

To find the slope of a line, use the slope formula with two points from the line. Here we will use $(1,2)$ and $(3,3)$.

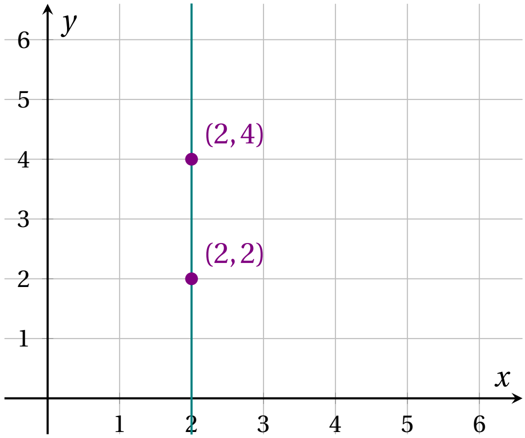
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-2}{3-1} = \frac{1}{2}$$

The slope of the line passing through the points $(1,2)$ and $(3,3)$ is $m = \frac{1}{2}$.



> **Example 3.8.** Use the definitions to find the $x-$ and $y-$intercepts of each line. We can see that vertical lines have an undefined slope using the same strategy. Consider the points $(2,4)$ and $(2,2)$.

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-2}{2-2} = \frac{2}{0} = DNE$. DNE is short for does not exist which is valid in our case. We cannot divide any number by zero. [11]



[11] You start dividing by zero and you can show non-sense. Did I offend any physicist?

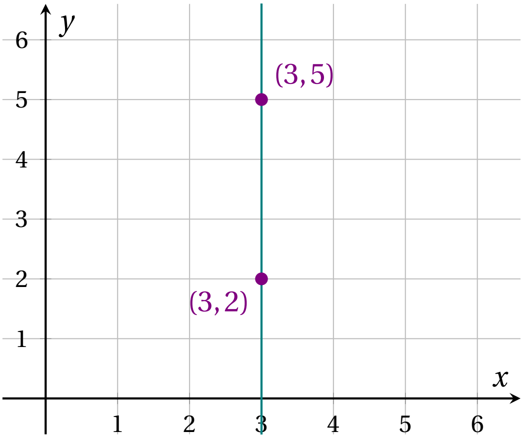
Here is an other example.

Example 3.9. Do the points $(3,2)$ and $(3,5)$ have a slope?

Remember, to find the slope of a vertical line, grab any two points from any vertical line and use the slope formula. Here we will use $(3,2)$ and $(3,5)$ from the line.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-2}{3-3} = \frac{3}{0}$$

The slope of any vertical line is undefined because we are dividing by zero.



3.2 Linear Equations & Applications

As previously mentioned, equations in two variables (such as x and y) without any exponents result in a straight line graph. Nevertheless, linear equations can be expressed in various forms. Let's look at some properties of a line.

Definition

Slope-Intercept form: $\boxed{y = mx + b}$

This is the most practical form to solve equations of lines and to quickly sketch a graph. It's got all the relevant information in the equation. Where m is the slope and b is the y-intercept.

Definition

Standard form: $\boxed{Ax + By = C}$ where A, B and C are real numbers.

This is a nice form for solving systems of equations.

Definition

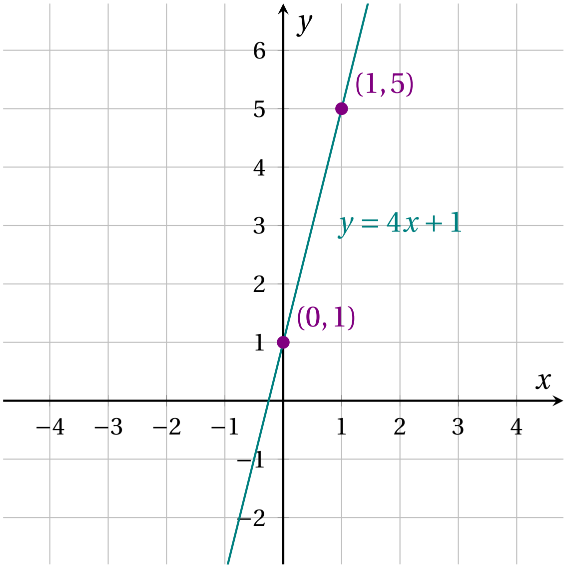
Point-slope form: $\boxed{y - y_0 = m(x - x_0)}$ where (x_0, y_0) is a point on the line.

Switching between different forms of linear equations can often be beneficial.

Example 3.10. Graph the solution set to the linear equation $-8x + 2y = 2$ by first putting it in ***Slope-Intercept form.***

$$-8x + 2y = 2$$
$$2y = 8x + 2$$
$$y = 4x + 1$$

When graphing a line, we can determine two points for plotting. By converting the equation into slope-intercept form, we can identify the y-intercept, which is located at $(0,1)$. The 1 in $y = 4x+1$ tells us to start at $(0,0)$, we always start at $(0,0)$, and go up by 1 to the point $(0,1)$. To find another point on the line, we can utilize the slope, which in this case is $m = 4$ (or $\frac{4}{1}$). We can then move up 4 units and to the right 1 unit from the initial point, thereby obtaining a second point $(5,1)$ that lies on the line. Connecting these two points by drawing a line allows us to represent all the points within the solution set. If the slope is positive you rise up. If the slope is negative you rise down.



Example 3.11. Find the equation of a line given the information $m = 2$ passing through $(3,5)$. We will express it in the form $y = mx + b$.

$$5 = 2(3) + b$$

$$5 = 6 + b$$

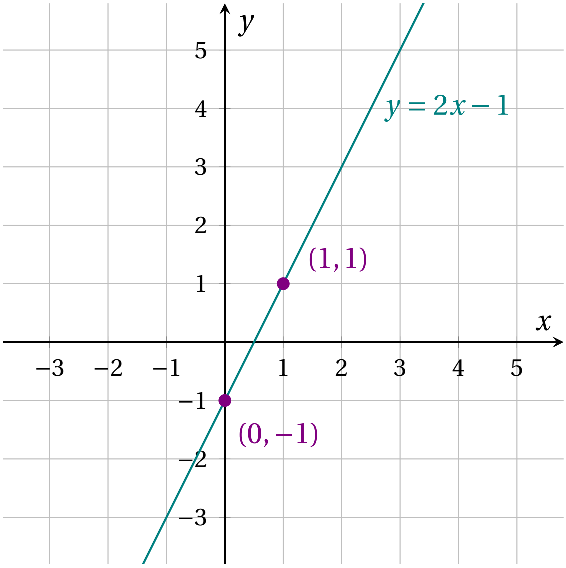
$$5 - 6 = b$$

$$-1 = b$$

Here, we substitute 3 for x and 5 for y in $y = mx + b$. Since $m = 2$ and $b = -1$, the equation becomes

$$y = 2x - 1.$$

The y-intercept is -1. The slope $(m = 2)$ indicates a rise over run of 2, meaning we go up by 2 and to the right by 1 from the point $(0, -1)$. This brings us to the point $(1, 1)$.



Example 3.12. Find the equation of a line given the points $(2,3)$ and $(4,7)$. First, calculate the slope (m), and then use it to find the equation in the form $y = mx + b$.

First, calculate the slope.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{7-3}{4-2}$$

$$m = \frac{4}{2}$$

$$m = 2$$

Therefore we can rewrite our equation as $y = 2x + b$. Now, use the slope and one of the points (let's use $(2,3)$) to find the equation.

$$y = 2x + b$$

$$3 = 2(2) + b$$

$$3 = 4 + b$$

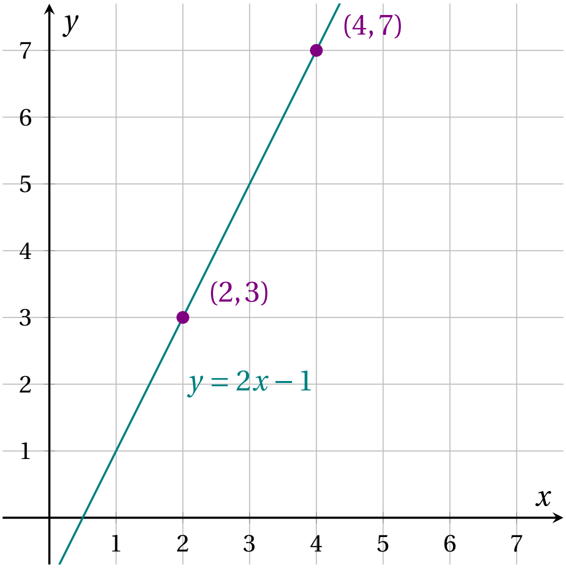
$$3 - 4 = b$$

$$-1 = b$$

Therefore, the equation of the line is:

$$y = 2x - 1.$$

Finally, the graph.



Example 3.13. Find the equation of line given the information $m = -\frac{1}{3}$ passing through $(-4, -2)$. Remember we want it in the form $y = mx + b$.

$$-2 = -\frac{1}{3}(-4) + b$$

$$-2 = \frac{4}{3} + b$$

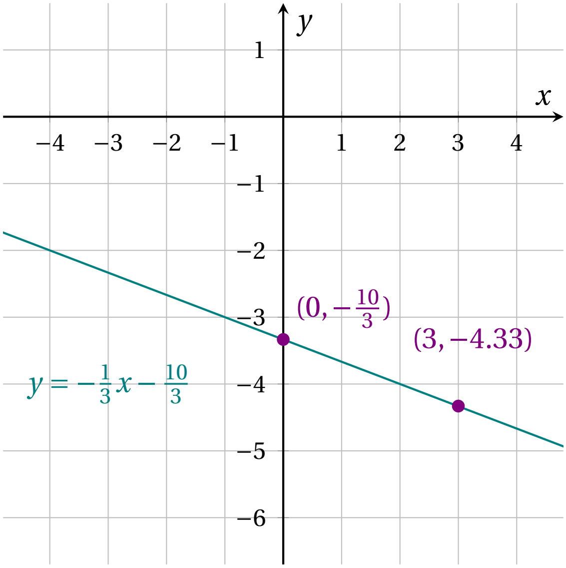
$$-2 - \frac{4}{3} = b$$

$$-\frac{10}{3} = b$$

Notice that we plug in -4 for x and -2 for y in $y = mx + b$. Since $m = -\frac{1}{3}$ and $b = -\frac{10}{3}$ then

$$y = -\frac{1}{3}x - \frac{10}{3}.$$

Note that our y- intercept is $\frac{10}{3} \approx 3.33$. Since our rise over run $(m = -\frac{1}{3})$, then we start at $(0, -3.33)$, go down by -1 and to the right by 3. Hence, arriving to the point $(3, -4.33)$.



Definition

Horizontal Lines: Since we know the slope of any horizontal line is 0 we can deduce that the equation of any horizontal line must be:

$$y = 0x + b \Longrightarrow \boxed{y = b}$$

where b is the y value of the y-intercept.

Vertical Lines: A little less intuitive than horizontal lines, a vertical line that intercepts the x-axis at $(c, 0)$ is represented by the equation:

$$x = c.$$

ex: The equation $x = 5$ represents the vertical line that passes through $(5, 0)$.

Definition

Two lines are ***parallel*** if they have the same slope and different y-intercepts.

Example 3.14. Given the line $y = 2x + 1$, find three lines parallel to it. To create parallel lines, we keep the slope the same (which is 2 in this case) and change the y-intercept (b) to any value other than 1.

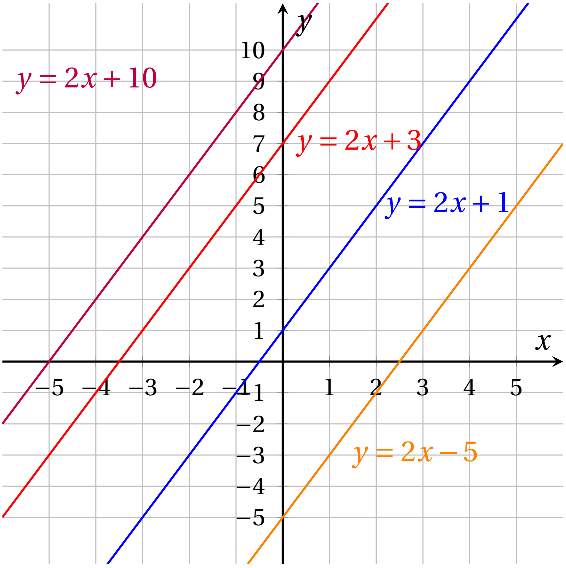
Recall that the original line is,

$$y = 2x + 1.$$

The three lines below are parallel to $y = 2x + 1$. To obtain a parallel line to $y = 2x + 1$, it suffices to change the b to any number other than 1. The word "any" is not used loosely; we may choose any number, no matter the size. For example, $y = 2x + 102873828$ is also parallel to $y = 2x + 1$. Although 102873828 is a stupendously huge b, it works.

- $y = 2x + 7$
- $y = 2x - 5$
- $y = 2x + 10$

Since these lines have the same slope as the original line but different y-intercepts, they are parallel to it. Graphically, we have the following.



Example 3.15. Consider the line $y = 2x + 1$. Find the equation of a line that is parallel and passes through $(2,3)$. Note: We need to find a new b for the point $(2,3)$.

Plug in 3 into y and 2 for x.

$$y = 2x + 1$$

$$3 = 2 \cdot 2 + b$$

$$3 = 4 + b$$

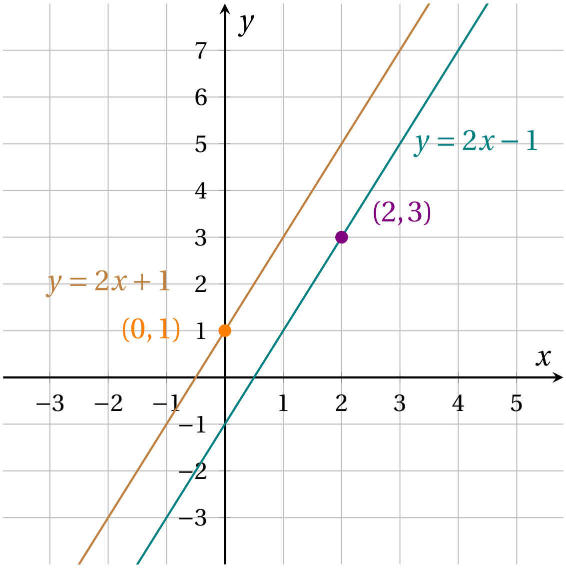
$$3 - 4 = b$$

$$b = -1$$

Plug in $b = -1$ into $y = 2x + b$.

$$y = 2x - 1$$

This is the line that is parallel to $y = 2x + 1$ and passes through $(2,3)$.



Example 3.16. Consider the line $y = 3x - 4$. Find the equation of a line that is parallel and passes through $(-7, -6)$. Note: All that we need to do is find a b with respect to the points $(-7, -6)$.

Plug in -6 into y and -7 for x

$$y = 3x - 4$$

$$-6 = 3(-7) + b$$

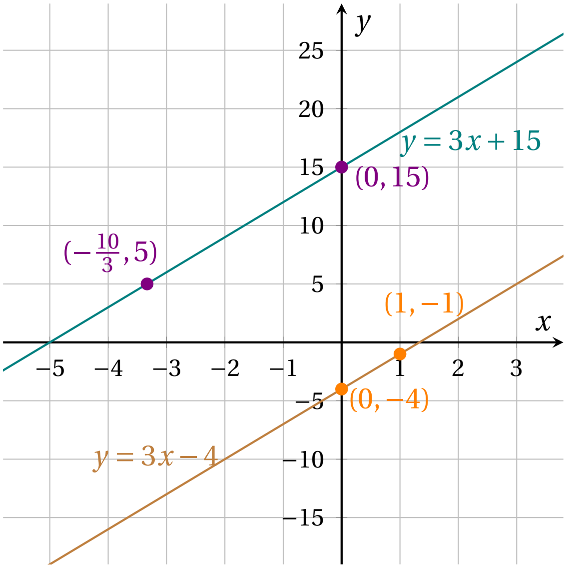
$$-6- = -21 + b$$

$$15 = b$$

Plug in $b = 15$ into $y = 3x + b$.

$$y = 3x + 15$$

This is the line that is parallel to $y = 3x - 4$ and passes through $(-7, -6)$.



Definition

Two lines are ***perpendicular*** if their slopes are opposite reciprocals. That is, if the slope of a line is

$$m = \frac{a}{b}$$

then the slope of any perpendicular line is

$$m_{\perp} = -\frac{b}{a}$$

Example 3.17. Given the line $y = 2x+1$, find three lines perpendicular to it. To create perpendicular lines, we change the slope to the negative reciprocal (which is $-\frac{1}{2}$ in this case) and change the y-intercept (b) to any value.

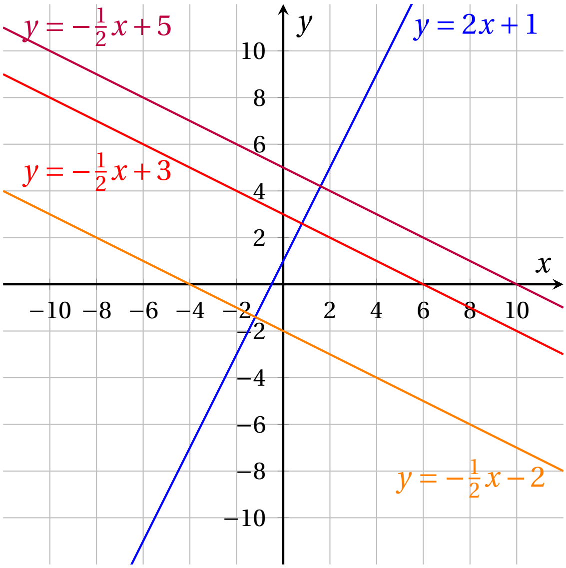
Recall that the original line is,

$$y = 2x + 1.$$

Again, just as example 3.14, we have an infinite amount of perpendicular lines to choose from. The three lines below are perpendicular to $y = 2x + 1$. To obtain a perpendicular line, it suffices to use the slope $-\frac{1}{2}$ and change the b to any number.

- $y = -\frac{1}{2}x + 3$
- $y = -\frac{1}{2}x - 2$
- $y = -\frac{1}{2}x + 5$

Since these lines have the perpendicular slope to the original line but different y-intercepts, they are perpendicular to it. Graphically, we have the following.



Example 3.18. Consider the line $y = x+2$. Find the equation of a line that is perpendicular and passes through $(1,-2)$. We are using the perpendicular definition. Since the slope m of the given line is 1, the slope of the perpendicular line will be $m_\perp = -\frac{1}{1} = -1$ (negative reciprocal).

$$y = m_\perp x + b$$

$$y = -1 \cdot x + b$$

Plug in -2 for y and 1 for x.

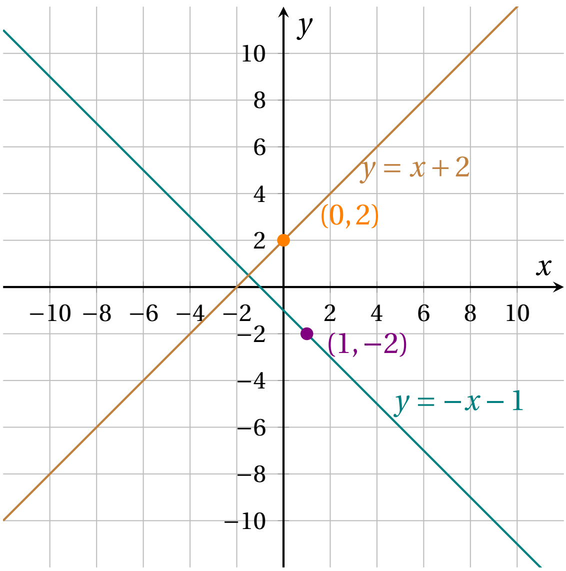
$$-2 = -1 \cdot 1 + b$$

$$-2 = -1 + b$$

$$b = -1$$

Hence the perpendicular line is,

$$y = -x - 1$$



Example 3.19. Consider the line $y = 3x - 4$. Find the equation of a line that is perpendicular and passes through $(-7, -6)$. We are using the perpendicular definition. Since $m = \frac{3}{1}$ then we will use $m_\perp = -\frac{1}{3}$.

Begin with $y = m_\perp x + b$ and plug in our $m_\perp = -\dfrac{1}{3}$.

$$y = -\frac{1}{3}x + b$$

Now we plug -6 for y and -7 for x.

$$-6 = -\frac{1}{3} \cdot (-7) + b$$

$$-6 = \frac{7}{3} + b$$

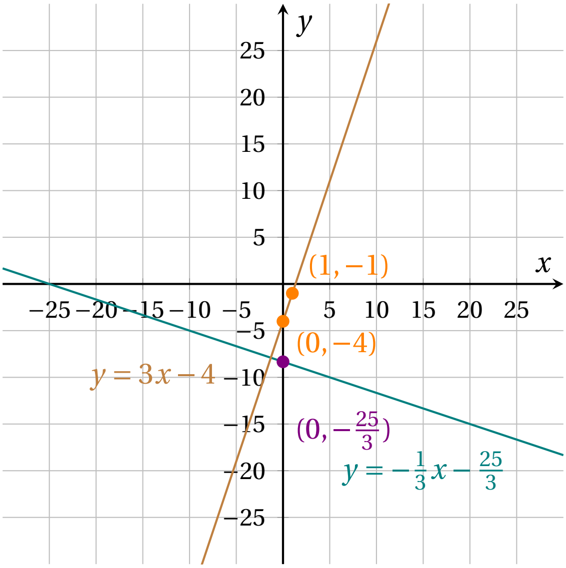
$$-6 - \frac{7}{3} = b$$

$$-\frac{18}{3} - \frac{7}{3} = b$$

$$-\frac{25}{3} = b$$

Hence the perpendicular line is,

$$y = -\frac{1}{3}x - \frac{25}{3}$$



Example 3.20. Alice is saving money for a vacation and needs to save \$15,000. She saves \$500 each month. The amount \$$V$ that she still needs to save after n months is given by the function

$$V(n) = 15000 - 500n$$

(a) How many months Alice has been saving if she still has \$10,000?

(b) How much money Alice still needs to save after 18 months

(a) To find out how many months Alice has been saving if she still has \$10,000 left to save, we set $V(n) = 10000$ and solve for n.

$$\begin{aligned} 10000 &= 15000 - 500n \\ 500n &= 15000 - 10000 \\ 500n &= 5000 \\ n &= \frac{5000}{500} \\ n &= 10 \end{aligned}$$

Thus, Alice has been saving for 10 months.

(b) To find out how much money Alice still needs to save after 18 months, substitute $n = 18$ into the equation.

$$\begin{aligned} V(18) &= 15000 - 500 \cdot 18 \\ V(18) &= 15000 - 9000 \\ V(18) &= 6000 \end{aligned}$$

That is, after 18 months, Alice still needs to save \$6,000.

Example 3.21. A new telescope is being used to observe a distant galaxy. The telescope captures images at a rate of 5 images per hour. The total number of images $I(n)$ captured after n hours is given by the function

$$I(n) = 5n$$

(a) How many hours have passed if the telescope has captured 75 images?

(b) How many images will the telescope have captured after 24 hours?

(a) To find out how many hours have passed if the telescope has captured 75 images, we set $I(n) = 75$ and solve for n.

$$\begin{aligned} 75 &= 5n \\ n &= \frac{75}{5} \\ n &= 15 \end{aligned}$$

Hence, 15 hours have passed.

(b) To find out how many images the telescope will have captured after 24 hours, substitute $n = 24$ into the equation.

$$\begin{aligned} I(24) &= 5 \cdot 24 \\ I(24) &= 120 \end{aligned}$$

Therefore, after 24 hours, the telescope will have captured 120 images.

Example 3.22. A certain species of cactus grows at a constant rate. The height of the cactus, $H(t)$ in centimeters, after t years is given by the function

$$H(t) = 10 + 2t$$

(a) How many years have passed if the cactus is currently 26 centimeters tall?

(b) What will be the height of the cactus after 5 years?

(a) To find out how many years have passed if the cactus is currently 26 centimeters tall, we set $H(t) = 26$ and solve for t.

$$\begin{aligned} 26 &= 10 + 2t \\ 2t &= 26 - 10 \\ 2t &= 16 \\ t &= \frac{16}{2} \\ t &= 8 \end{aligned}$$

Concluding that 8 years have passed.

(b) To find out the height of the cactus after 5 years, substitute $t = 5$ into the equation.

$$\begin{aligned} H(5) &= 10 + 2 \cdot 5 \\ H(5) &= 10 + 10 \\ H(5) &= 20 \end{aligned}$$

Therefore, after 5 years, the cactus will be 20 centimeters tall.

Example 3.23. The cost of feeding a significantly Fat cat increases due to the need for special diet food. Assume the basic monthly food cost for a regular cat is 40, but for every month an overweight cat is on a special diet, the cost increases by 5. The total food cost $F(m)$ for n months is given by the function

$$F(m) = 40m + 5m$$

(a) Calculate the total food cost for an overweight cat over a period of 6 months.

(b) If the total food cost spent is 345, how many months has the cat been on the special diet?

(a) To calculate the total food cost for an overweight cat over a period of 6 months, substitute $m = 6$ into the equation.

$$\begin{aligned} F(6) &= 40 \cdot 6 + 5 \cdot 6 \\ &= 240 + 30 \\ &= 270 \end{aligned}$$

Therefore, the total food cost for an overweight cat over a period of 6 months is 270 dollars.

(b) To find the number of months the cat has been on the special diet if the total food cost is 345, solve for n in the equation $F(n) = 345$.

$$\begin{aligned} 345 &= 40n + 5n \\ 345 &= 45n \\ \frac{345}{45} &= n \\ 7.67 &= n \end{aligned}$$

Thus, the cat has been on the special diet for approximately 7.67 months, which can be rounded to 8 months for practical purposes.

3.3 Functions

Functions are fundamental in the study of mathematics, and you have encountered them in various ways without explicitly recognizing them. As you continue your studies, you will realize that functions remain an integral part of mathematical concepts. In fact, you will come to rely on them extensively. Functions provide an excellent framework for describing the relationship between inputs and outputs. We will focus on linear functions in this section. However, it is worth noting that we will delve into quadratic functions, square root functions, exponential functions, and logarithmic functions in the future chapters. [12]

Before we can explore the applications and graphical representations of linear functions, it is essential to have a clear understanding of what a function actually is. This foundational knowledge will serve as a basis for further exploration and analysis.

Definition

A ***function*** is a special relation where each input has only a single output. If we have a function f and an input x, we call the output $f(x) = y$.

Example 3.24. Consider the function $y = 2x + 1$ or $f(x) = 2x + 1$.

Notice that we can plug in x values and get an output "y or $f(x)$" values. Together we can plot points and create a line. This is just as effective as the $y = mx + b$ method. Let's begin by plugging in the x values $-3, -2, -1, 0, 1, 2,$ and 3 into $f(x) = 2x + 1$.

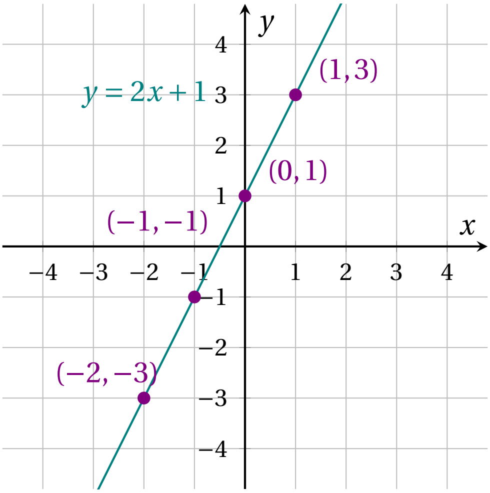
$$f(-3) = 2(-3) + 1 = -5$$

[12] Can you guess my favorite function?

$$f(-2) = 2(-2) + 1 = -3$$
$$f(-1) = 2(-1) + 1 = -1$$
$$f(-0) = 2(0) + 1 = 1$$
$$f(1) = 2(1) + 1 = 3$$
$$f(2) = 2(2) + 1 = 5$$
$$f(3) = 2(3) + 1 = 7$$

Putting them into a table we see,

x	$f(x)$
-3	-5
-2	-3
-1	-1
0	1
1	3
2	5
3	7



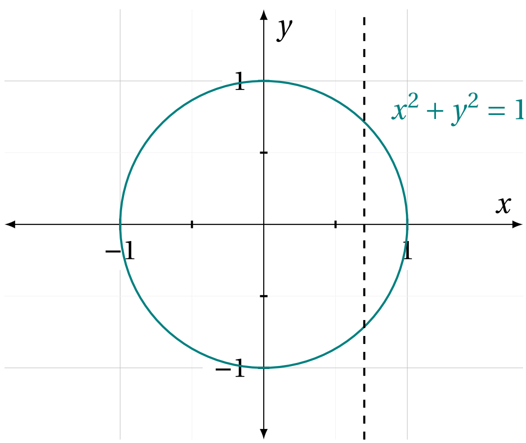
Now I will explain the concept of the vertical line test. The vertical line test is a useful tool that distinguish between functions and non-functions. Functions and equations share a lot some similarities. However, it is important to recognize their distinctions. For instance a circle (which is an equation), describes a relationship between variables but does not necessarily represent a function, as we will see in the following examples.

To determine whether a given graph represents a function, we can employ the vertical line test. This test involves drawing a vertical line and observing its intersection points with the graph. If the vertical line intersects the graph at more than one point, then the object being examined is not a function.

Every input (x-value) must correspond to a unique output (y-value) in order for the graph to represent a function. In other words, there should not be multiple outputs for a single input. If a vertical line intersects the graph in more than one location then it indicates that a particular x-value yields multiple y-values. Thus violating the definition of a function.

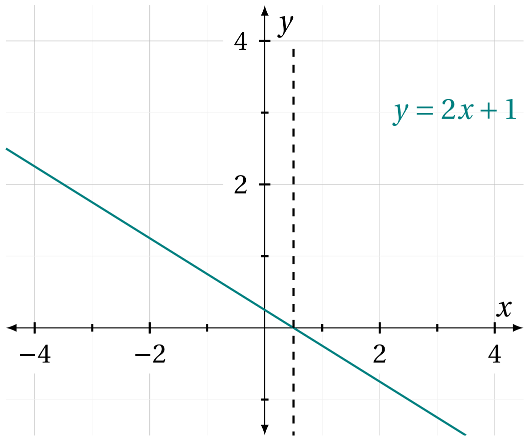
By employing the vertical line test, we can determine whether a given graph represents a valid function or not.

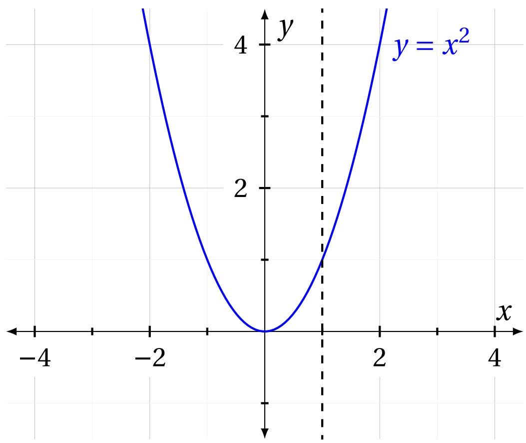
Example 3.25. Consider the circle $x^2 + y^2 = 1^2$.



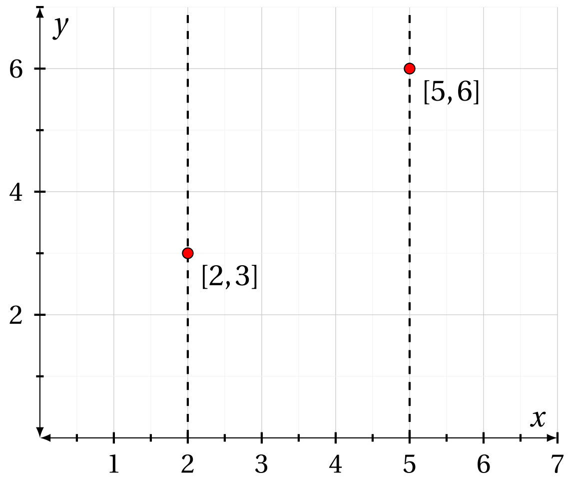
Notice that the dashed line intersects the circle twice and hence it is not a function.

Example 3.26. These two are functions. In fact, all lines are functions.



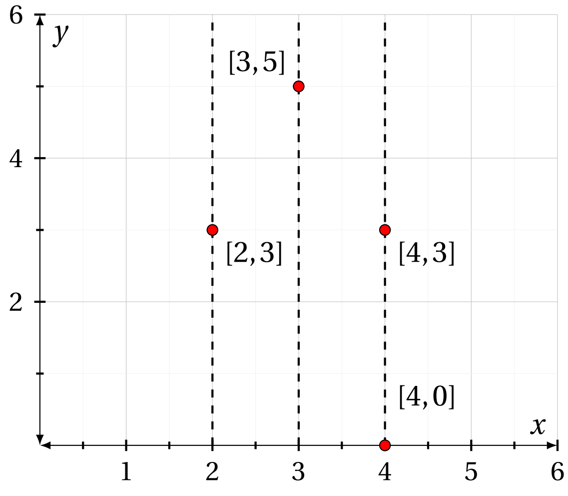


Example 3.27. Consider the points $[2, 3]$ and $[5, 6]$. Determine if they represent a function by using the vertical line test.



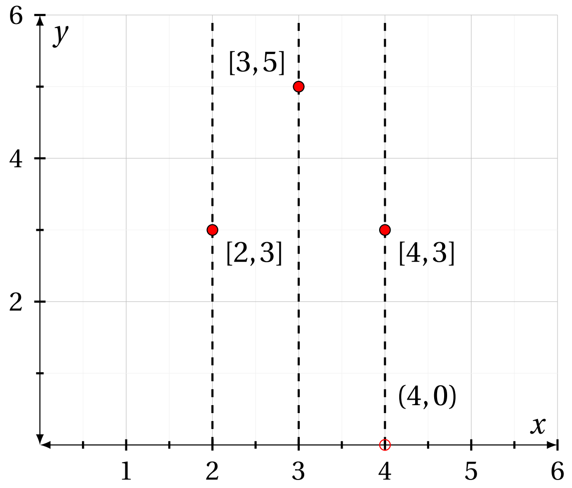
Notice that the set of points $[2,3]$ and $[5,6]$ passes the vertical line test, as each vertical line intersects the graph at only one point. Therefore, these points represent a function.

Example 3.28. Consider the points $[2,3]$, $[4,3]$, $[3,5]$, and $[4,0]$. Determine if they represent a function by using the vertical line test.



The set of points $[2,3]$, $[4,3]$, $[3,5]$, and $[4,0]$ does not pass the vertical line test, as the vertical line at $x = 4$ intersects the graph at two points $[4,0]$ and $[4,3]$. Therefore, these points do not represent a function.

Example 3.29. Consider the points $[2,3]$, $[4,3]$, $[3,5]$, and $(4,0)$. Determine if they represent a function by using the vertical line test.



The set of points $[2,3]$, $[4,3]$, $[3,5]$, and $(4,0)$ do pass the vertical line test. Notice that the point $(4,0)$ does not include the number 4. Hence the open circle. Thus the only point that includes 4 is $[4,3]$. That is, the vertical line only passes the point $[4,3]$ and not $(4,0)$.

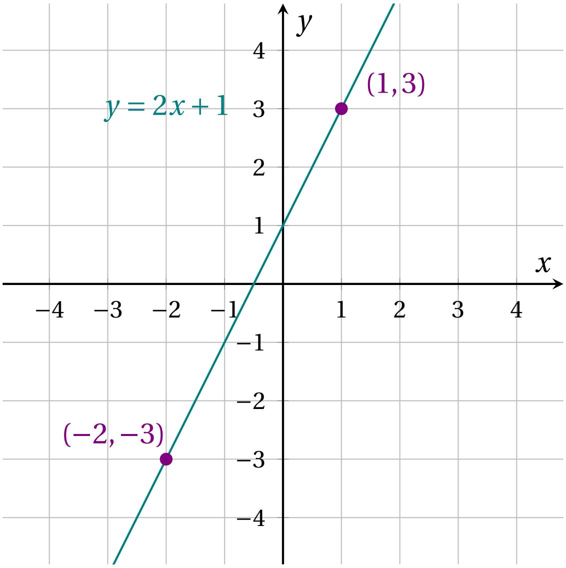
Definition

The ***domain*** of a function refers to all the possible input values, which are represented by the x-values. It determines how far the function extends horizontally from left to right.

Definition

The ***range*** of a function refers to all the possible out values, which are represented by the y-values. It determines how far the function extends vertically from up to down.

Example 3.30. Consider the function $y = 2x+1$ or $f(x) = 2x+1$.



1. What is the domain of f? Notice that the line extends left and right forever. Hence the domain is $(-\infty, \infty)$.

2. What is the range of f? Likewise, it extends up and down forever. Hence the range is $(-\infty, \infty)$.

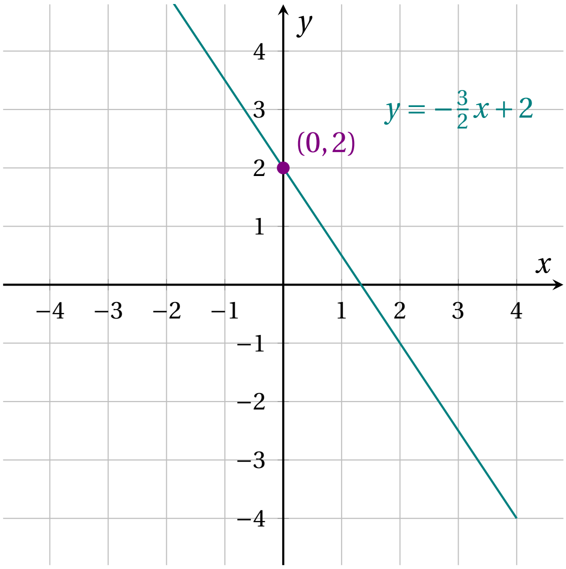
3. Find $f(1)$.

$$\begin{aligned} f(1) &= 2(1)+1 \\ &= 2+1 \\ &= 3 \end{aligned}$$

4. Find all values of x such that $f(x) = -3$.

$$\begin{aligned} -3 &= 2x+1 \\ -4 &= 2x \\ x &= -2 \end{aligned}$$

Example 3.31. Consider the function $y = -\frac{3}{2}x + 2$ or $f(x) = -\frac{3}{2}x + 2$.



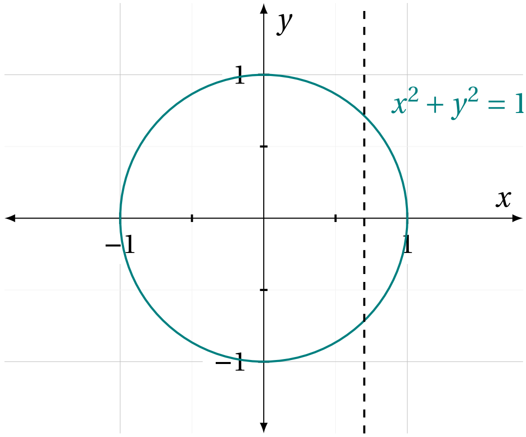
1. What is the domain of f? Since it's a linear function, the domain is all real numbers: $(-\infty, \infty)$.
2. What is the range of f? Similarly, as a linear function, the range is also all real numbers. That is, $(-\infty, \infty)$.
3. Find $f(0)$.

$$\begin{aligned} f(0) &= -\frac{3}{2}(0) + 2 \\ &= 0 + 2 \\ &= 2 \end{aligned}$$

Therefore, $f(0) = 2$.

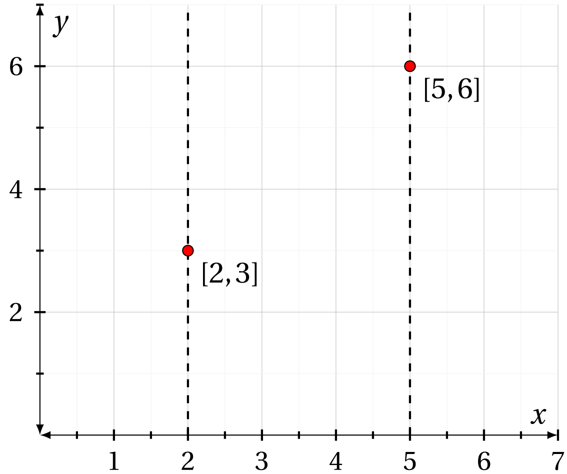
Let us revisit passed examples and decipher their domain and range.

Example 3.32. Consider the circle $x^2 + y^2 = 1^2$. Find and state its domain and range.



Notice that the circle extends from -1 to 1 along the x-axis, with both -1 and 1 included. Thus, establishing the **domain** as $[-1, 1]$. Likewise, the circle reaches from -1 to 1 along the y-axis, including these values, which sets the **range** as $[-1, 1]$.

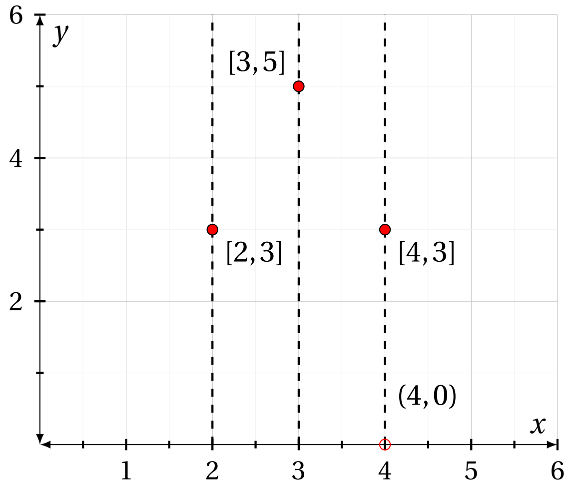
Example 3.33. Consider the points $[2, 3]$ and $[5, 6]$. Determine the domain and range of this function.



The **domain** of a function is the set of all possible input values (x-values). Therefore, the domain is $[2, 5]$. The furthest we can go left is at $x = 2$ and the furthest we can go right is at $x = 5$

The **range** of a function is the set of all possible output values (y-values). Therefore, the range is $[3,6]$. Again the highest up we can go is at $y = 6$ and the lowest downwards we can go is $y = 3$.

Example 3.34. Consider the points $[2,3]$, $[4,3]$, $[3,5]$, and $(4,0)$. Determine the domain and range of this function.



The **domain** of a function is the set of all possible input values (x-values). Therefore, the domain is $[2,4]$. The furthest we can go left is at $x = 2$ and the furthest we can go right is at $x = 4$. It is important to note that $(4,0)$ does not include the number 4. However, $[4,3]$ does include 4. Hence our domain includes it. The **range** of a function is the set of all possible output values (y-values). Therefore, the range is $(0,5]$. The highest up we can go is at $y = 5$. The lowest downwards we can go is $y = 0$, but since we have a parenthesis, we do not include 0.

3.4 Systems of Equations

We extend our concept of functions by adding another and seeing what information we can extract from the two. As mentioned, functions play a crucial role in mathematics; thus it's essential to have an efficient method for dealing with multiple functions simultaneously. Mathematicians use a powerful tool known as a system of equations to achieve this. Systems of equations are the most efficient way storing data. They are matrices in disguise, as you may soon find out.

When solving a system of equations, our goal is to find the values of x and y that simultaneously satisfy all the equations in the system. Let's explore the various possibilities that can occur when dealing with two linear equations.

Definition

Intersection: The two lines representing the equations intersect at a single point. This point of intersection corresponds to the solution of the system, where both equations are satisfied simultaneously.

Important

Whenever you have two lines defined by the same point (x, y), those lines ***intersect*** at that point.

Example 3.35. Find the point of intersection the system of equations.

$$\begin{cases} y = 2x + 7 \\ y = x + 4 \end{cases}$$

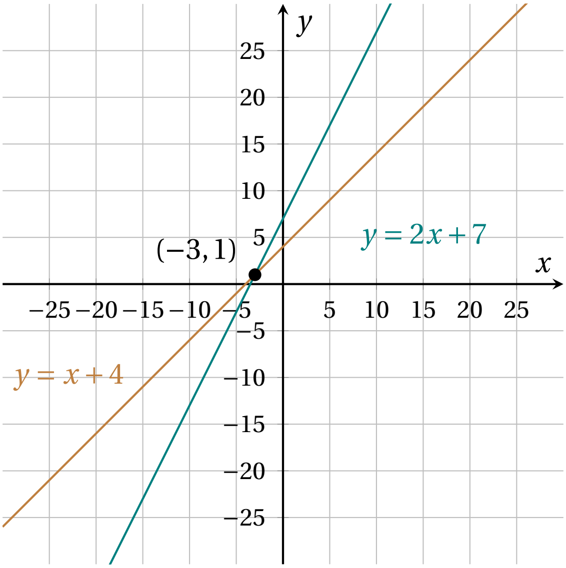
We can solve this system by noticing that the equations $y = 2x+7$ and $y = x+4$ are both equal to y. Thus, we can set these two equations equal to each other. Doing so, we have:

$$\begin{aligned} 2x + 7 &= x + 4 \\ 2x - x &= 4 - 7 \\ x &= -3 \end{aligned}$$

Now plug in $x = -3$ into any of the two equations.

$$y = x + 4$$
$$y = -3 + 4$$
$$y = 1$$

Hence our point of intersection is $(-3, 1)$. Graphing these two equations, we have the following.



Example 3.36. Find the point of intersection of the following system of equations.

$$\begin{cases} y = -x + 3 \\ y = 3x - 2 \end{cases}$$

Since both equations equal y, we set them equal to each other.

$$-x + 3 = 3x - 2$$

$$-x - 3x = -2 - 3$$

$$-4x = -5$$

$$x = \frac{-5}{-4}$$

$$x = \frac{5}{4}$$

Substitute $x = \frac{5}{4}$ into either equation to find y.

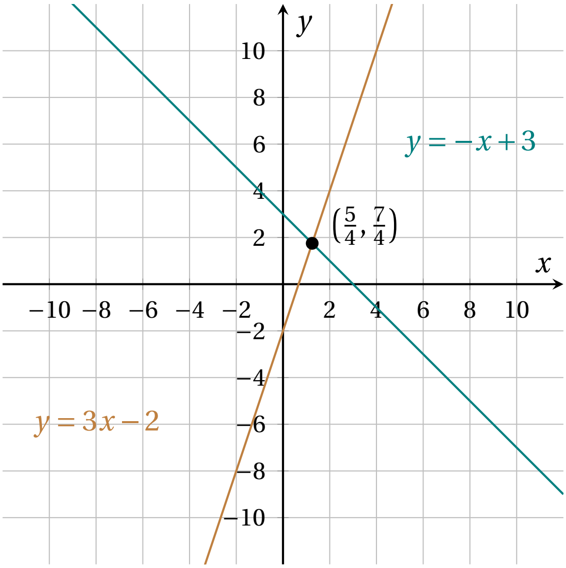
$$y = 3\left(\frac{5}{4}\right) - 2$$

$$y = \frac{15}{4} - 2$$

$$y = \frac{15}{4} - \frac{8}{4}$$

$$y = \frac{7}{4}$$

The point of intersection is $\left(\frac{5}{4}, \frac{7}{4}\right)$. When graphing these equations, it looks like this.



Important

By examining these different possibilities, we can gain insights into the nature of solutions when solving systems of two linear equations. Now let's discuss system of equations and how to solve them. There are two different approaches to these problems. Allow me to introduce both.

Example 3.37. Solve the following system of linear equations using either substitution or elimination.

By substitution

$$\begin{cases} 4x + 2y = 12 \\ 3x - y = 3 \end{cases}$$

First, express x from the first equation.

$$4x + 2y = 12$$

$$4x = 12 - 2y$$

$$x = \frac{12 - 2y}{4}$$

$$x = 3 - \frac{1}{2}y$$

Substitute $x = 3 - \frac{1}{2}y$ into the second equation.

$$3\left(3 - \frac{1}{2}y\right) - y = 3$$

$$9 - \frac{3}{2}y - y = 3$$

$$9 - \frac{5}{2}y = 3$$

$$-\frac{5}{2}y = 3 - 9$$

$$-\frac{5}{2}y = -6$$

$$y = \frac{-6 \cdot 2}{-5}$$

$$y = \frac{12}{5}$$

Substitute $y = \frac{12}{5}$ into $x = 3 - \frac{1}{2}y$.

$$x = 3 - \frac{1}{2}\left(\frac{12}{5}\right)$$

$$x = 3 - \frac{6}{5}$$

$$x = \frac{15}{5} - \frac{6}{5}$$

$$x = \frac{9}{5}$$

By elimination

$$\begin{cases} 4x + 2y = 12 \\ 3x - y = 3 \end{cases}$$

Multiply the second equation by 2 to align coefficients for elimination.

$$2 \cdot (3x - y) = 2 \cdot 3$$

$$6x - 2y = 6$$

Add this to the first equation.

$$(4x + 2y) + (6x - 2y) = 12 + 6$$

$$10x = 18$$

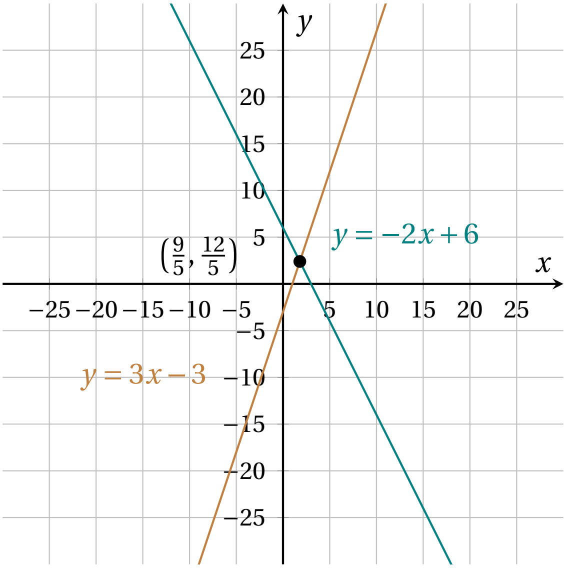
$$x = \frac{18}{10}$$

$$x = \frac{9}{5}$$

Substitute $x = \frac{9}{5}$ into either original equation (we'll use the first one).

$$4 \cdot \frac{9}{5} + 2y = 12$$
$$\frac{36}{5} + 2y = 12$$
$$2y = 12 - \frac{36}{5}$$
$$2y = \frac{60}{5} - \frac{36}{5}$$
$$2y = \frac{24}{5}$$
$$y = \frac{24}{5} \cdot \frac{1}{2}$$
$$y = \frac{12}{5}$$

Therefore, the solution to the system of equations is $x = \frac{9}{5}$ and $y = \frac{12}{5}$.



Definition

Coincident lines: The two lines representing the equations are identical and lie on top of each other. In this case, the system has infinitely "∞" many solutions since all points on the common line satisfy both equations.

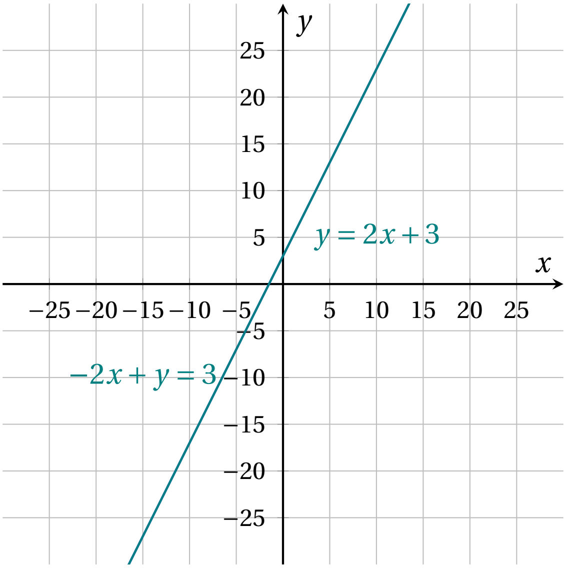
Important

Whenever you have two equal numbers equal to each other then there are ***infinitely many solutions***.

Example 3.38. Graph the system of equation.

$$\begin{cases} y = 2x + 3 \\ -2x + y = 3 \end{cases}$$

By graphing the two we have.



Notice, that they are essentially the same lines but written differently.

Example 3.39. Solve the following systems of linear equations using either substitution or elimination.

By substitution

$$\begin{cases} 2x+3y=8 \\ 6y+4x=16 \end{cases}$$

Solve for x in $2x+3y=8$

$$2x+3y=8$$

$$2x=-3y+8$$

$$x=-\frac{3}{2}y+4$$

Substitute $x=-\frac{3}{2}y+4$ into the x of $6y+4x=16$

$$6y+4\left(-\frac{3}{2}y+4\right)=16$$

$$\begin{aligned} 6y-6y+16&=16 \\ 16&=16 \end{aligned}$$

Hence, there is ∞ many solutions.
By elimination

$$\begin{cases} 2x+3y=8 \\ 6y+4x=16 \end{cases}$$

$$\begin{aligned} 2x+3y&=8 \\ 6y+4x&=16 \end{aligned}$$

Move $2x$ and $4x$ to opposites sides and then multiply $2x+3y=8$ by -2.

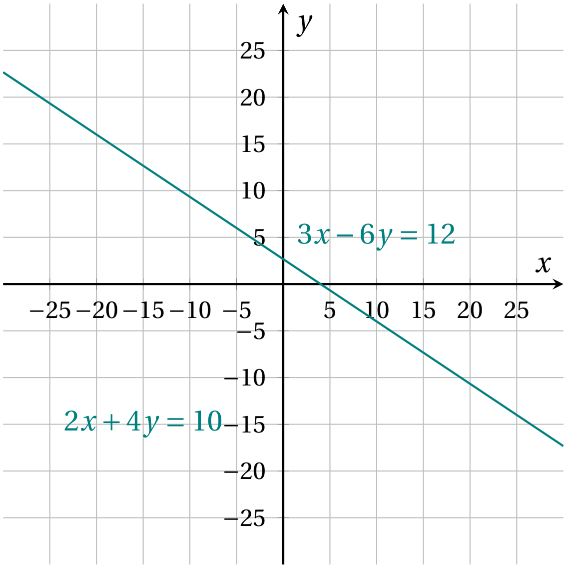
$$\begin{aligned} 2\cdot 3y&=2\cdot(8-2x) \\ 6y&=16-4x \end{aligned}$$

$$\begin{aligned} 6y&=16-4x \\ 6y&=16-4x \end{aligned}$$

Add these two equations.

$$0 = 0 + 0$$

Which is true and thus we arrive to the same conclusion.



Remember

Whenever you have two equal numbers equal to each other (such as the above example) then there are infinitely many solutions. That is, these two lines are stacked on top of each other.

Definition

Parallel lines: The two lines representing the equations do not intersect and are parallel to each other. In this scenario, there is no solution "∅" that satisfies both equations simultaneously. The system is considered inconsistent.

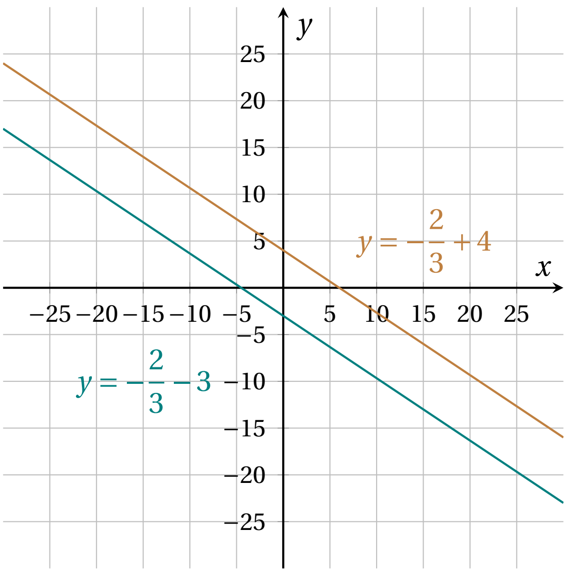
Important

Whenever you have two non equal numbers equal to each other then there is ***no solution***.

Example 3.40. Graph the system of equation.

$$\begin{cases} y = -\left(-\dfrac{2}{3}x - 3\right) \\ y = -\left(-\dfrac{2}{3}x + 4\right) \end{cases}$$

By graphing the two we have.



Notice that these two lines never touch. They essentially go on forever without touching each other. Therefore, there is no solution when comparing the lines.

Example 3.41. Graph the system of equations.

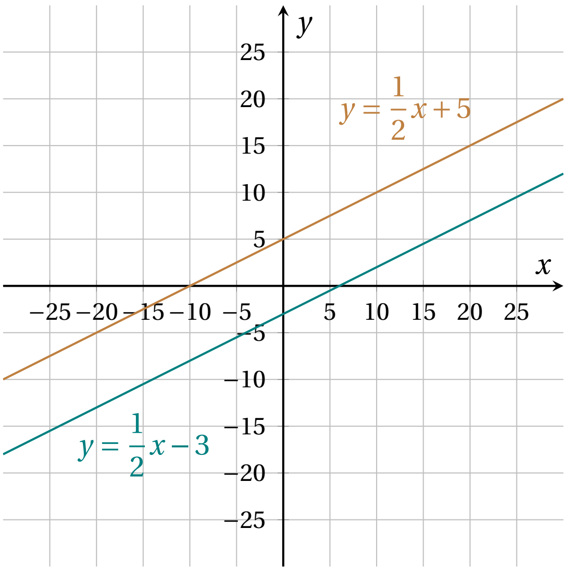
$$\begin{cases} 2y = x - 6 \\ y = \dfrac{1}{2}x + 5 \end{cases}$$

First, we can express both equations in terms of y.

$$y = \frac{1}{2}x - 3$$

$$y = \frac{1}{2}x + 5$$

By graphing these two equations, we have.



In this scenario, similar to your example, these two lines are parallel and will never intersect. They have the same slope but different y-intercepts, which is a characteristic of parallel lines. Again, we see that these two line have no solutions.

Example 3.42. Solve the following systems of linear equations using either substitution or elimination.

By substitution

$$\begin{cases} y = 2x + 4 \\ 8x - 4y = 7 \end{cases}$$

Substitute $y = 2x + 4$ into the y of $8x - 4y = 7$

$$\begin{aligned} 8x - 4(2x + 4) &= 7 \\ 8x - 8x - 16 &= 7 \\ -16 &\neq 7 \end{aligned}$$

No solution. Hence, $\emptyset$.

By elimination

$$\begin{cases} y = 2x + 4 \\ 8x - 4y = 7 \end{cases}$$

$$\begin{aligned} y &= 2x + 4 \\ 8x - 4y &= 7 \end{aligned}$$

Move $2x$ and y to opposites sides and then multiply $y = 2x + 4$ by 4.

$$\begin{aligned} -2x &= -y + 4 \\ 8x &= 7 + 4y \end{aligned}$$

$$\begin{aligned} 4(-2x) &= 4(-y + 4) \\ 8x &= 7 + 4y \end{aligned}$$

$$\begin{aligned} -8x &= -4y + 4 \\ 8x &= 4y + 7 \end{aligned}$$

Add these two equations.

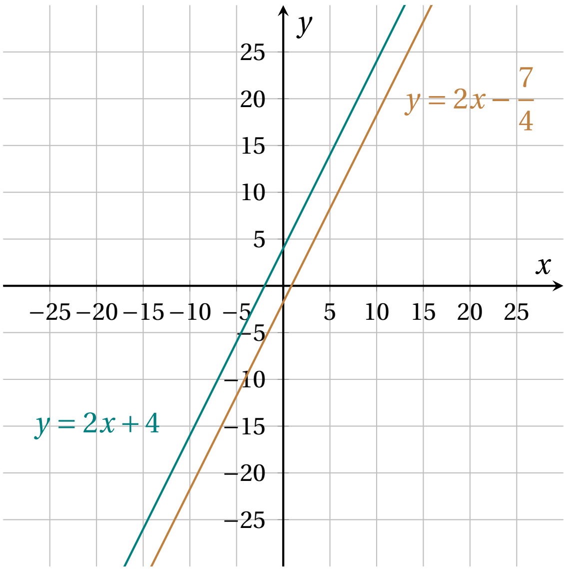
$$0 = 0 + 11$$

Which is obviously not true and hence we arrive to the same conclusion. Notice that,

$$8x - 4y = 7$$

$$\begin{aligned}
-4y &= 7-8x \\
\frac{-4y}{-4} &= \frac{7-8x}{-4} \\
y &= -\frac{7}{4}+2x \\
y &= 2x-\frac{7}{4}.
\end{aligned}$$

Using this information, we can graph the two lines.



Example 3.43. Solve the following system of linear equations using either substitution or elimination.

By substitution

$$\begin{cases} 3x+6y=12 \\ 2x+4y=10 \end{cases}$$

First, express x from the first equation.

$$\begin{aligned}
3x &= 12-6y \\
x &= 5-2y
\end{aligned}$$

Substitute $x = 5 - 2y$ into the first equation. You want to plug it back into the opposite equation and not itself.

$$\begin{gathered} 3(5-2y)+6y=12 \\ 15-6y+6y=12 \\ 15 \neq 12 \end{gathered}$$

No solution. Hence, $\emptyset$.

By elimination

$$\begin{cases} 3x+6y=12 \\ 2x+4y=10 \end{cases}$$

Multiply the first equation by 2 and the second by -3 to align coefficients for elimination.

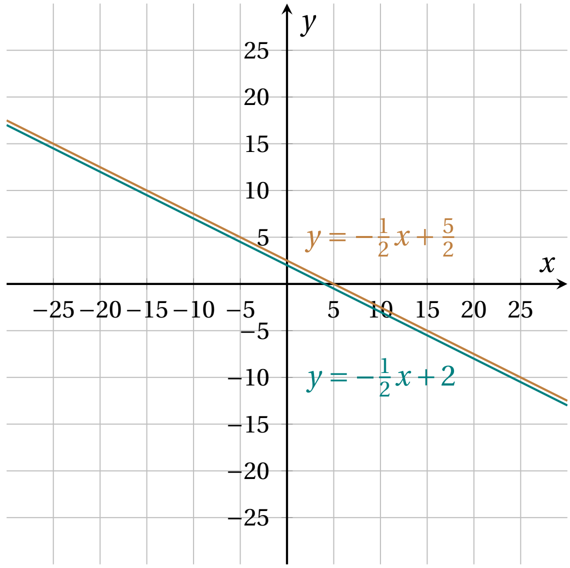
$$\begin{gathered} 2(3x+6y)=2\cdot 12 \\ -3(2x+4y)=-3\cdot 10 \end{gathered}$$

$$\begin{gathered} 6x+12y=24 \\ -6x-12y=-30 \end{gathered}$$

Subtract the second equation from the first.

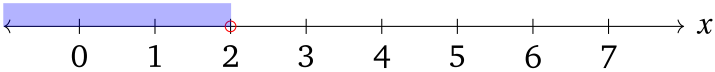
$$0=-6$$

This is a contradiction. Therefore, the system has no solution. Below are the two graphs.



3.5 Linear Inequalities in 2 Variables

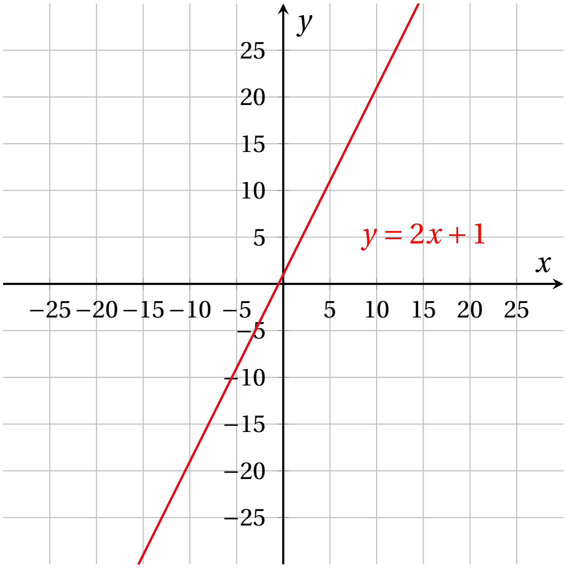
We have learned how to determine the solution set of a linear inequality when there is a single variable, such as $x < 2$, by representing it graphically on a number line. In this case, we choose a reference point on the number line, like $x = 2$, and shade the appropriate side of the point. In this case, we shade what is left of two and do not include 2. That is, we have an open circle.



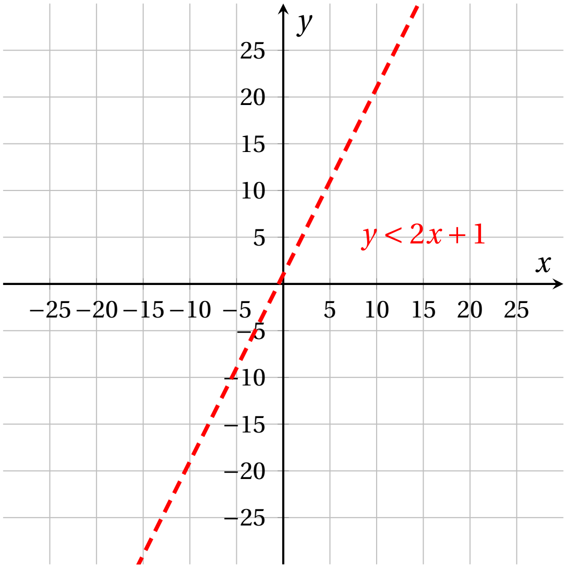
Now, let's extend this concept to linear inequalities with two variables. When dealing with two variables, we graph the solution set in two dimensions by identifying a reference line and shading the appropriate side of that line. It is literally the same thing; however we are adding in a new dimension.

To graph the solution set of a linear inequality with two variables, such as $y < 2x + 1$, we can follow these steps:

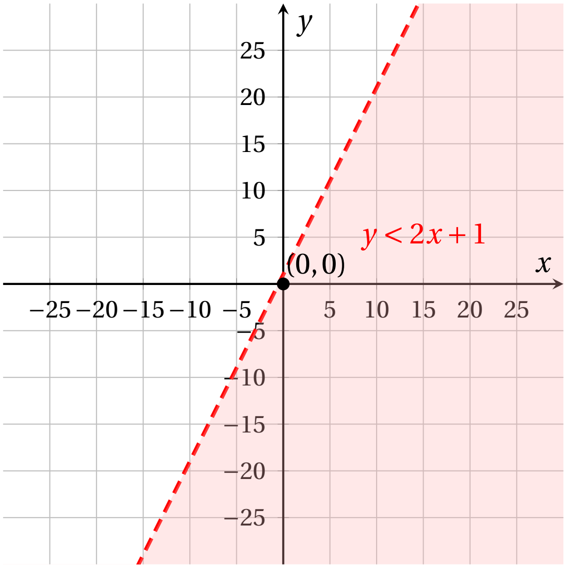
First, we graph the equation $y = 2x + 1$, which represents the boundary line of the inequality. Here $b = 1$ and thus we go up by 1. Our slope is $m = \frac{2}{1}$. This means that we go to up by 2 and to the right by 1. Then, we determine whether to shade above or below the line. We do this by selecting a test point that is not on the line and substituting its coordinates into the inequality. If the inequality is satisfied, we shade the corresponding region; otherwise, we shade the opposite region. Finally, we shade the appropriate side of the line to represent the solution set of the inequality. By applying these steps, we can graphically represent the solution set of a linear inequality with two variables. Let's start off by drawing the line for $y = 2x + 1$.



All graphs that have > or < have a dotted line.



We now need to determine which side of the line to shade, whether it is the left side or the right side. To determine this, we select points from either side. I always choose $(0,0)$ as it is the easiest to work with. Notice that in our case, $(0,0)$ is located to the right of the graph. We substitute $(0,0)$ into the equation $y < 2x + 1$, and if the statement is true, we shade to the right, which is where $(0,0)$ is located. If the statement is false, we shade the opposite side of the graph where $(0,0)$ is located.



In our example, when we substitute $(0,0)$ into the inequality.

$$0 < 2(0) + 1$$

Simplifying, we find that $0 < 1$, which is true. Therefore, we shade the side of the line where the test point $(0,0)$ is located, which, in this case, is the right side of the line.

By using this approach, we can determine which side of the line to shade when graphing a linear inequality. Remember, the test point $(0,0)$ is just one possible choice, and you can choose any other point from either side of the line to perform the same test. However, it is important not to choose a point that is on the line.

Definition

All lines that have $>$ or $<$ are dotted and all lines that are of the form $\geq$ or $\leq$ are just solid lines.

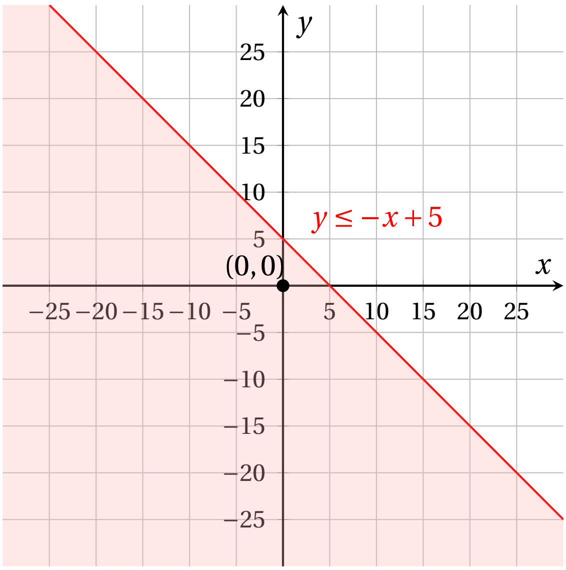
Example 3.44. Graph the system of equations.

$$\begin{cases} x + y \leq 5 \\ 2x - y < 4 \end{cases}$$

Notice that $x + y \leq 5$ is the same thing as $y \leq -x + 5$. Likewise notice that $2x - y < 4$ is the same as $2x - 4 < y$. Now, all that is needed to do is graph the lines.

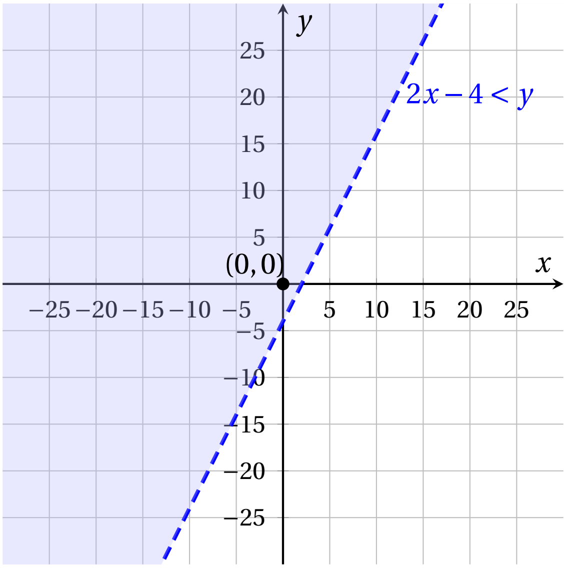
To find the shaded region for $y \leq -x + 5$ lets try out the points $(0, 0)$ for each of the lines.

$$y \leq -x + 5$$
$$0 \leq -0 + 5$$
$$0 \leq 5$$

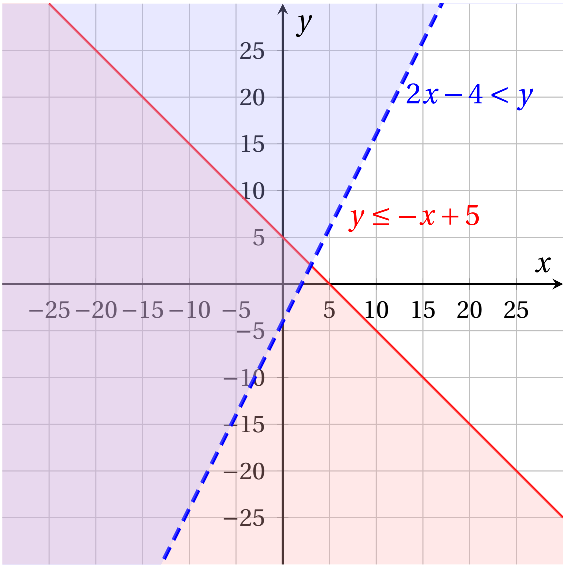


Hence, we shade to the left of the graph where $(0, 0)$ is located. Likewise, lets choose points $(0, 0)$ from the graph $2x - 4 < y$.

$$2x - 4 < y$$
$$2(0) - 4 \leq 0$$
$$-4 \leq 0$$



Which is again true and thus we shade to the left of the graph where $(0,0)$ is located. Together we have the following.



Our answer is the area that is shaded in twice. That is, the left side of the two graphs.

Example 3.45. Graph the system of equations.

$$\begin{cases} x - 2y \leq 3 \\ y < x + 1 \end{cases}$$

Notice that $x - 2y \leq 3$ is the same thing as $y \geq \frac{1}{2}x - \frac{3}{2}$. Likewise notice that $y < x + 1$ is already written in slope intercept form . Now, all that is needed to do is graph the lines.

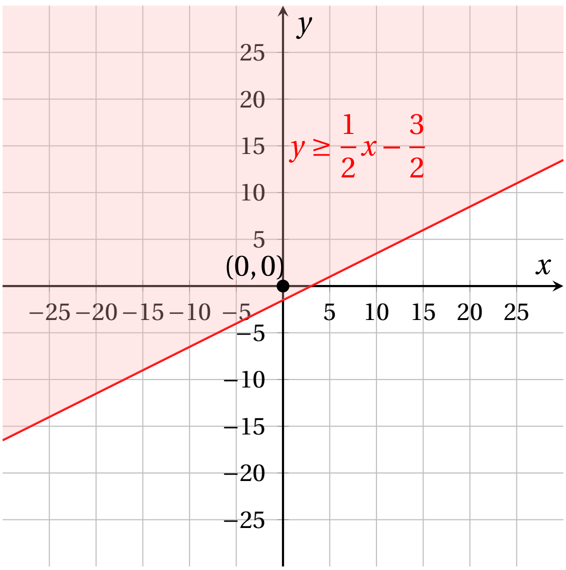
To find the shaded region for $2y \geq x - 3$ lets try out the point $(0,0)$ for each of the lines.

$$y \geq \frac{1}{2}x - \frac{3}{2}$$

$$0 \geq \frac{1}{2} \cdot 0 - 3$$

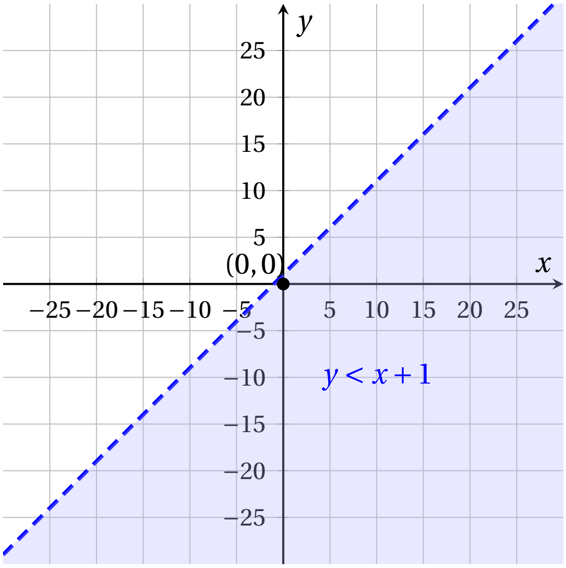
$$0 \geq -3$$

Which is true, thus we shade the top of the line where $(0,0)$ is located in.

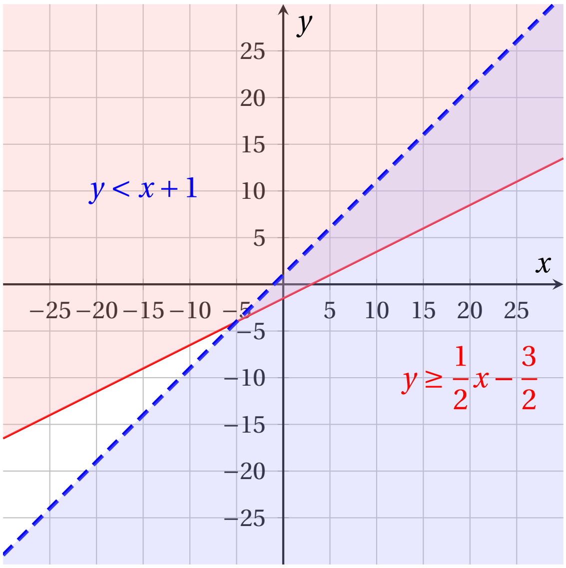


Likewise, let's choose the point $(0,0)$ from the graph $y < x + 1$.

$$y < x + 1$$
$$0 < 0 + 1$$
$$0 < 1$$



Which is again true, and thus we shade below the graph where $(0,0)$ is located. Together, we have the following.



Our answer is the area where both shaded regions overlap.

Example 3.46. Graph the system of inequalities.

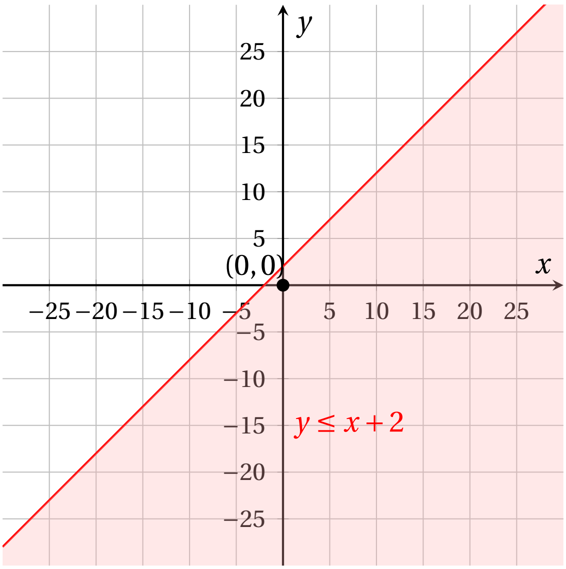
$$\begin{cases} y \le x + 2 \\ y > -x + 1 \\ y < \dfrac{1}{2}x \end{cases}$$

Notice that $y \le x+2$, $y > -x+1$, and $y < \frac{1}{2}x$ are already written in slope-intercept form. Now, all that is needed to do is graph the lines.

To find the shaded region for $y \le x+2$, let's try out the point $(0,0)$.

$$\begin{aligned} y &\le x + 2 \\ 0 &\le 0 + 2 \\ 0 &\le 2 \end{aligned}$$

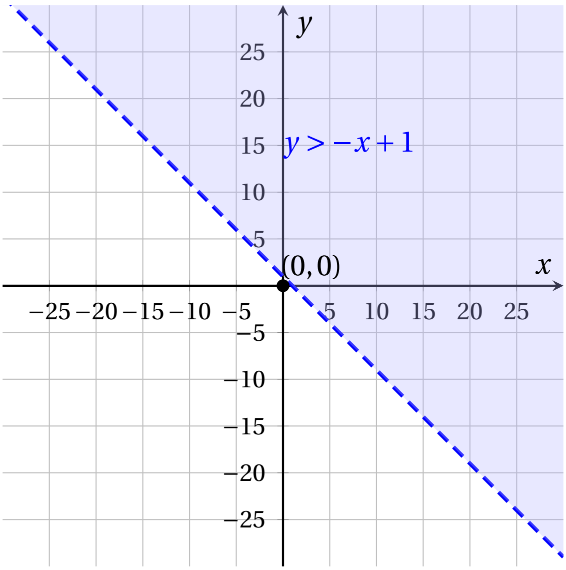
Which is true, thus we shade below the line where $(0,0)$ is located (the right side).



Likewise, let's choose the point $(0,0)$ from the graph $y > -x + 1$.

$$\begin{aligned} y &> -x + 1 \\ 0 &> -0 + 1 \\ 0 &> 1 \end{aligned}$$

Which is false, thus we shade above the line where $(0,0)$ is not located (the right side).



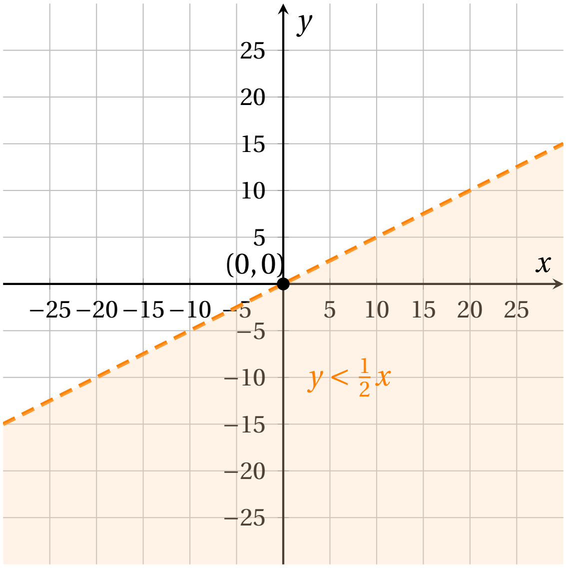
Finally, for the graph $y < \frac{1}{2}x$, let's test the point $(-1,1)$. Notice, that we do not choose $(0,0)$ since the line $y < \frac{1}{2}x$ passes through the origin. I.e it passes through the point $(0,0)$. We must choose points that are no on the line.

$$y < \frac{1}{2}x$$

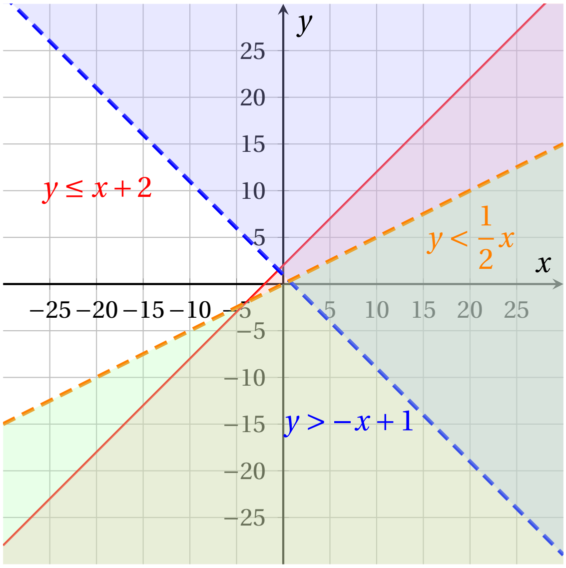
$$1 < \frac{1}{2}(-1)$$

$$1 < -\frac{1}{2}$$

Which is false, so we shade below the line where $(-1,1)$ is not located (the bottom).



To graph these inequalities, we will draw the corresponding lines and shade the appropriate regions.



Each line represents one of the inequalities, and the shaded regions indicate where each inequality is true. The solution to the system is the region where all shaded areas overlap.

3.6 Additional Problems

1. Identify the slope, and the $y-$intercept of each line.

   (a) $y = 4x - 4$
   (b) $2x - y = +x - 6$
   (c) $3y = 6x + 3$
   (d) $\frac{1}{6}y = 1 - 7x$
   (e) $y = -x$

2. Find the slope with respect to the given points.

   (a) $(-2, 1)$ and $(6, 1)$
   (b) $(7, -7)$ and $(8, 3)$
   (c) $(-3, 4)$ and $(3, -2)$
   (d) $(1, 5)$ and $(-1, -5)$
   (e) $(1, -1)$ and $(-3, 5)$
   (f) $(0, -4)$ and $(4, 0)$
   (g) $(-4, -2)$ and $(2, 2)$
   (h) $(0, 0)$ and $(7, 3)$

3. Which lines are parallel to the line $y = 7x - 3$?

   (a) $y = 4x - 3$
   (b) $y = -7x + 3$
   (c) $y = 7x + 3$
   (d) $y = -7x - 3$
   (e) $y = 7x$

4. Which lines are perpendicular to the line $y = -32x - 2024$?

   (a) $y + 2025 = -\dfrac{1}{32}x$
   (b) $y = -32x - 2$
   (c) $y = 32x + 37$
   (d) $y = \dfrac{1}{32}x$
   (e) $y = \dfrac{1}{32}x + 7276245$

5. Find the equation of each line below through the points $(-3, -6)$ and $(5, 2)$.

6. Find the equation of the line passing through the points $(1, -2)$ and $(3, 4)$.

7. Determine the equation of the line that passes through the points $(2, 3)$ and $(-4, -1)$.

8. A particle moves along a straight line with its position, $s(t)$ in meters, after t seconds given by the function

$$s(t) = 5 + 3t$$

   (a) How many seconds have passed if the particle is currently 26 meters from the starting point?

   (b) What will be the position of the particle after 10 seconds?

9. Consider the function $f(x) = \frac{3}{2}x - 1$.

   (a) What is the domain of f?

   (b) What is the range of f?

   (c) Find $f(2)$.

   (d) Find all values of x such that $f(x) = 4$.

10. Consider the function $f(x) = x + 3$.

   (a) What is the domain of f?

   (b) What is the range of f?

   (c) Find $f(0)$.

   (d) Find all values of x such that $f(x) = 5$.

11. Solve the following systems of linear equations using either substitution or elimination.

   (a) $\begin{cases} x = 3y - 1 \\ 2x - 4y = 2 \end{cases}$

   (b) $\begin{cases} 2x + y = 5 \\ 3x - 2y = -4 \end{cases}$

   (c) $\begin{cases} x - 2y = 3 \\ 4x + y = 1 \end{cases}$

12. The Sac State theatre group is hosting a musical evening and charges \$12 for each adult ticket and \$7 for each student ticket. During one of the performances, they sold a total of 50 tickets and collected a revenue of \$400. How many of each type of ticket was sold?

13. A park ranger is planting two types of trees in a new conservation area: oak trees and pine trees. Each oak tree requires 5 square meters of space, whereas each pine tree requires 3 square meters. The park ranger has allocated a total of 47 square meters for 13 trees. How many of each type of tree did the park ranger plant?

14. Graph the system of equations and give the shaded regions.

    (a) $\begin{cases} 3x - 2y \geq 6 \\ x + 2y < 4 \end{cases}$

    (b) $\begin{cases} 2x + y > 3 \\ x - y \leq 1 \end{cases}$

    (c) $\begin{cases} x + y \geq 2 \\ 3x - y \geq 4 \end{cases}$

    (d) $\begin{cases} \frac{1}{2}x + y \leq 5 \\ 4x + y > 2 \end{cases}$

    (e) $\begin{cases} y < 6x - 1 \\ 6y > 18 \end{cases}$

4 Exponents

4.1 Exponent Properties

Exponents are a fundamental concept in mathematics that continues to play a crucial role. They have been a part of mathematical exploration for a long time, and their significance remains unchanged. Thus far, we have encounter exponents with a degree of 1, it is important to recognize that exponents extend far beyond this limited scope. For instance, consider the linear equation $y = x + 2$. Although the degree of both x and y in this equation is 1, denoted as $x^1 = x$ and $y^1 = y$ respectively, it is worth noting that any constant number, such as 6, can also be viewed as having a degree of 1, represented as $6^1 = 6$. These instances of exponents with a degree of 1 may appear less prominent and be concealed to avoid unnecessary confusion. Nonetheless, exponents have consistently been an integral part of our mathematical journey and will continue to be so in the future. Let's begin this chapter with important results.

Theorem

For any real number a and any positive integer n,

$$\underbrace{a^n = a \cdot a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}$$

where there are n copies of a being multiplied.

Example 4.1. Simplify 2^4 using exponent rules.

$$2^4 = \underbrace{2 \cdot 2 \cdot 2 \cdot 2}_{4 \text{ times}}$$

$$= 16$$

There are nine new properties to learn. At first, these properties might seem familiar and can be confusing. As with any new material, it will take time to fully understand them. You might not completely grasp all the concepts by the end of this lesson, but you will improve with exponents as you continue your mathematical journey beyond this class.

Theorem

Product/Power Rules

(a) $x^m x^n = x^{m+n}$

(b) $(x^m)^n = x^{mn}$

(c) $(xy)^n = x^n y^n$

(d) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

In theory, all information can be stored via binary code. That is, all data is stored within the numbers 0 and 1. Using binary code, we can define any number that you think of. Let's look at the binary sequence for the number 1.

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

Essentially, we are looking at powers of 2. That is, to what power must 2 be raised to yield 1? From our definition, we know that any number raised to the power of 0 is 1. Hence, $2^0 = 1$.

Therefore, our binary code is determined by identifying the position above the 2's that we used. In this case, we only used 2^0. So, the binary code for 1 is $\ldots 0000001$.

Now, let's find the binary representation for the number 2. Again, to what power must 2 be raised to yield 2? Indeed, $2^1 = 2$. This means we are looking at the second position. Since we used the second position, we assign it the number 1, and all other positions are assigned 0 because they were not used. Thus, the binary sequence for 2 becomes $\ldots 0000010$.

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & 2^2 & 2^1 & 2^0 \end{array}$$

Let's skip ahead and find the number 5's binary sequence. How do we get 5 out of powers of 2 in which are not repeatable numbers? Notice that $4+1 = 2^2 + 2^0 = 5$. Hence we choose 1 in our sequence paired with 2^2 and 2^0.

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & \color{red}{1} & 0 & \color{red}{1} \\ 2^7 & 2^6 & 2^5 & 2^4 & 2^3 & \color{red}{2^2} & 2^1 & \color{red}{2^0} \end{array}$$

Our sequence then becomes $\ldots 0000101$.

Let's put the first 10 numbers and their binary sequence into a table.

Decimal (x)	**Binary Representation**
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010

Now, why did I go on about this?

In 1974, a radio message was sent from the Arecibo Observatory in Puerto Rico to the star cluster $M13$, which is approximately $25,000$ light-years away. This message was an attempt to communicate with potential extraterrestrial intelligence, sharing information about humanity and our scientific discoveries.

The message was encoded using **binary numbers** and included the following information:

- The numbers 1 to 10.
- The chemical elements that form DNA (hydrogen, carbon, nitrogen, oxygen, phosphorus).
- A diagram of the DNA double-helix structure.
- A figure of a human and Earth's population.
- A diagram of the solar system, showing Earth's position.
- A representation of the Arecibo radio telescope.

But how can information, such as DNA, be sent using just binary numbers? Let's consider a small DNA sequence, **ATCGGA**.

We can map each nucleotide to a binary code:

$$\mathrm{A} \to 00, \quad \mathrm{T} \to 01, \quad \mathrm{C} \to 10, \quad \mathrm{G} \to 11.$$

Using this mapping, the DNA sequence **ATCGGA** can be represented as:

$$\mathrm{ATCGGA} = 000110111100$$

This demonstrates how complex information, like DNA sequences, can be encoded and transmitted using simple binary numbers.

So yes, it is possible that some extraterrestrial intelligence has decrypted that message and now knows who we are and where we are. The potential intentions they might have with that information could be unsettling to consider.

Anyways, let's list the remaining four properties.

Theorem

Quotient Rule and Negative Exponents

(a) $\dfrac{x^m}{x^n} = x^{m-n}$

(b) $x^0 = 1$

(c) $x^{-n} = \dfrac{1}{x^n}$

(d) $\left(\dfrac{x}{y}\right)^{-n} = \dfrac{y^n}{x^n}$

Let's practice by doing some examples.

Example 4.2. Simplify using exponent rules.

$$= x^3 x^4 y^5 y^2$$

Rearrange the terms so like bases are together.

$$= x^{3+4} y^{5+2}$$

Apply the product rule: $x^m x^n = x^{m+n}$.

$$= x^7 y^7$$

Simplify the exponents.

Example 4.3. Simplify with positive exponents.

$$a^5 b^3 a^9 b^4 = a^5 a^9 b^4 b^3$$

Rearrange the terms so like bases are together.

$$= a^{5+9} b^{4+3}$$

Apply the product rule: $x^m x^n = x^{m+n}$.

$$= a^{14} b^7$$

Example 4.4. Simplify with positive exponents.

$$(-x)^2(y^4)^2x^4 = (-x)(-x)x^4(y^4)^2$$

Apply our first theorem and rearrange the terms to gather common bases.

$$= x^2x^4(y^4)^2$$
$$= x^{2+4}(y^{4\cdot 2})$$

Apply the product rule, $x^m x^n = x^{m+n}$.

$$= x^6y^8$$

Example 4.5. Simplify with positive exponents.

$$(-z^4)(y^3)^2z = -z^4 \cdot (y^3)^2 \cdot z$$

Use the product rule theorem $(x^m)^n = x^{mn}$

$$= -z^4 \cdot y^{3\cdot 2} \cdot z$$

Simplify the exponent.

$$= -z^4 \cdot y^6 \cdot z$$

Now, use the the product rule theorem : $x^m x^n = x^{m+n}$

$$= -z^{4+1} \cdot y^6$$

Simplify the exponent.

$$= -z^5 \cdot y^6$$

Example 4.6. Simplify with positive exponents.

$$(-2x)^6 = (-2x)(-2x)(-2x)(-2x)(-2x)(-2x)$$

Apply the first theorem. Notice that there are an even number of negatives.

$$\begin{aligned} &= (-2)(-2)(-2)(-2)(-2)(-2)xxxxxx \\ &= 2^6x^6 \\ &= 64x^6 \end{aligned}$$

Example 4.7. Simplify with positive exponents.

$$\left(\frac{a^{-3}}{b^{-2}}\right)^{-2} = \left(\frac{a^{(-3)(-1)}}{b^{(-2)(-1)}}\right)^2$$

We can distribute the negative sign into each term.

$$= \left(\frac{a^{(3)}}{b^{(2)}}\right)^2$$

Distribute the 2 using the power rule.

$$= \left(\frac{a^{3\cdot 2}}{b^{2\cdot 2}}\right)$$

Multiply the numbers.

$$= \left(\frac{a^6}{b^4}\right)$$

Example 4.8. Simplify with positive exponents.

$$\left(\frac{2x^3y^{-4}}{5x^{-2}y^3}\right)^{-2} = \left(\frac{2x^3x^2}{5y^3y^4}\right)^{-2}$$

Cancel the x^3 terms. When canceling, bring up the x^{-2} and change signs. Likewise, bring down the y^{-4} and change the sign.

$$= \left(\frac{2x^{3+2}}{5y^{3+4}}\right)^{-2}$$

Apply the product rule: $x^m x^n = x^{m+n}$.

$$= \left(\frac{2x^5}{5y^7}\right)^{-2}$$

Distribute the -2 into each term.

$$= \left(\frac{2^{-2}x^{-2\cdot 5}}{5^{-2}y^{-2\cdot 7}}\right)$$

Apply the negative exponent rule.

$$= \left(\frac{5^2 y^{14}}{2^2 x^{10}}\right)$$

Square each term in the fraction.

$$= \frac{25y^{14}}{4x^{10}}$$

Example 4.9. Simplify with positive exponents.

$$\left(\frac{3x^5y^2}{6x^5y^{-2}}\right)^{-4} = \left(\frac{3\cancel{x^5}y^2}{6\cancel{x^5}y^{-2}}\right)^{-4}$$

notice that we can cancel the x^5.

$$= \left(\frac{\cancel{3}y^2}{\cancel{3}\cdot 2y^{-2}}\right)^{-4}$$

down 6 and cancel the like terms.

$$= \left(\frac{y^2}{2y^{-2}}\right)^{-4}$$

Distribute the -4.

$$= \left(\frac{2^4 y^{-8}}{y^8}\right)$$

Bring down the y^{-8} and change sign.

$$= \left(\frac{16}{y^8 y^8}\right)$$

$$= \left(\frac{16}{y^{8+8}}\right)$$
$$= \left(\frac{16}{y^{16}}\right)$$

Example 4.10. Use the product rule to simplify.

$$\frac{(2p^4)^3}{(8p^3)^2} = \frac{(2^1 \cdot p^4)^3}{(2^3 \cdot p^3)^2}$$

Expanding the powers and breaking numbers into primes.

$$= \frac{2^3 \cdot p^{12}}{2^6 \cdot p^6}$$

Apply the cancellation.

$$= \frac{\cancel{2^3} \cdot p^{12}}{\cancel{2^3} \cdot 2^3 \cdot p^6}$$
$$= \frac{p^{12}}{2^3 \cdot p^6}$$

Further cancellation.

$$= \frac{\cancel{p^6} \cdot p^6}{2^3 \cdot \cancel{p^6}}$$
$$= \frac{p^6}{2^3}$$
$$= \frac{p^6}{8}$$

Example 4.11. Simplify with positive exponents.

$$(a^2b)^3(a^3b^2)^2 = (a^{2\cdot3}b^3)(a^{3\cdot2}b^{2\cdot2})$$

Apply the power rule to each term: $(x^m)^n = x^{mn}$.

$$= a^6b^3a^6b^4$$

Simplify the exponents.

$$= a^{6+6}b^{3+4}$$

Combine like bases using the product rule.

$$= a^{12}b^7$$

Example 4.12. Simplify completely with only positive exponents. In this problem, let $x+3=3$

$$\left(\frac{\sinh^7 3\theta \ln(a)}{(5b^7)^2 e^{i\cos(\pi)} - 1}\right)^{-x}$$

First let's solve for x.

$$\begin{aligned} x+3 &= 3 \\ x &= 3-3 \\ x &= 0 \end{aligned}$$

Let's substitute $x=0$ in the given expression..

$$\begin{aligned} \left(\frac{\sinh^7 3\theta \ln(a)}{(5b^7)^2 e^{i\cos(\pi)} - 1}\right)^{-x} &= \left(\frac{\sinh^7 3\theta \ln(a)}{(5b^7)^2 e^{i\cos(\pi)} - 1}\right)^{-0} \\ &= \left(\frac{\sinh^7 3\theta \ln(a)}{(5b^7)^2 e^{i\cos(\pi)} - 1}\right)^{0} \end{aligned}$$

Apply the rule: any non-zero number to the power of 0 is 1.

$$= 1$$

Example 4.13. Simplify completely with only positive exponents.

$$\left(\lim_{x\to\infty} 5\right)^{0\cdot 5}$$

First, notice that anything times 0 is 0.

$$\left(\lim_{x\to\infty} 5\right)^{0\cdot 5} = \left(\lim_{x\to\infty} 5\right)^{0}$$

Anything to the power of 0 is 1.

$$= 1$$

Yeah! The last two problems involve calculus properties. However, we do not need to know those properties to solve this problem. This is essentially an exponential problem, and understanding the exponential properties suffices.

4.2 Polynomial Addition, Subtraction, & Multiplication

Now, let's move on to polynomial operations. In this section we will focus on addition, subtraction, and multiplication, often referred to as distribution. Understanding the distinctions between these operations is important, as they form the foundation for various mathematical computations. Working with polynomials is similar to working with regular numbers. The only difference is the inclusion of variables. Remember that introducing variables automatically allows us venture into the realm of geometry, as they have geometric interpretations. Keep in mind that this is knowledge that will be expected in all your future classes.

Definition

A ***monomial*** is a single term consisting of a non-zero coefficient and variables raised to *non-negative* integer values.

Example 4.14. Which of the following are monomials?

$$3x^2$$

Yes, the exponent is not negative and non-zero coefficient (3 is not zero).

$$\frac{1}{2}xyz$$

Yes, the exponent is not negative and non-zero coefficient.

$$4a^{-2}b^3$$

No, -2 is a negative exponent.

$$\sqrt{2} = 2^{\frac{1}{2}}$$

Yes, the exponent is not negative and non-zero coefficient.

$$\frac{w^5u^2}{v} = w^5u^2v^{-1}$$

No,-1 is a negative exponent.

$$x^{1/2}y$$

No, notice that the exponent $\frac{1}{2}$ is a fraction thus not an integer.

Definition

A **polynomial** $P(x)$ in one variable x is an expression of the form:

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where:

- n is a non-negative integer called the *degree* of the polynomial.

- $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$ (unless $n = 0$).
- Each term $a_i x^i$ is called a *monomial*.

Notice that this is a sum (or difference) of monomials.

Example 4.15. Here are some examples of polynomial.

- $3x^2$
- $2y + 7\alpha + 3\lambda$
- $5x^2 + 2y - 4z - 2\phi$
- $7x^2y^3z + 2x^4y^3$
- $4a^2b^2c^2 - 2a^4b^3c + 10c$
- $\sqrt{7}$

Definition

The ***degree of a monomial*** term is the sum of the variable exponents. The ***degree of a polynomial*** is the highest of the degrees of the monomials that make up the polynomial.

Example 4.16. Here are some examples with polynomial.

- $3x^2$ (Deg: 2)
- $2y + 7$ (Deg: 1 - Binomial/linear)
- $5x^2 + 2y - 4z$ (Deg: 2 - Trinomial)
- $7x^2y^3z + 2x^4y^3$ (Deg: 7)
- $4a^2b^2c^2 - 2a^4b^3c + 10c$ (Deg: 8)
- $\sqrt{7}$ (Deg: 0 - Constant)

It's important to remember that we can only add or subtract variables if they have the same exponent. For example, in the expression $4y+5y^2$, we cannot simplify it further since the variables, y and y^2, are raised to different powers.

However, when the variables have the same exponent, we can combine them. For instance, $4y+5y$ simplifies to $9y$, as we add the coefficients and keep the variable term unchanged. This concept extends to numbers as well. For instance, $5+6$ equals 5^1+6^1, which simplifies to 11^1. This shows that adding numbers with the same exponent follows the same principles as adding variables.

Example 4.17. Simplify $(x^4-5x^3+4)-(3x^4-2x+1)$.

$$(x^4-5x^3+4)-(3x^4-2x+1) = (x^4-5x^3+4)-3x^4+2x-1$$

Distribute the negative sign.

$$= (x^4-3x^4)-5x^3+2x+(4-1)$$

Group like terms with the same exponents and then add.

$$= -2x^4-5x^3+2x+3$$

Example 4.18. Simplify $(2x^2+3x-5)+(x^2-x+1)-(4x^2+2x-3)$.

$$\begin{aligned}(2x^2+3x-5)+(x^2-x+1)-4x^2-2x+3 = {} & (2x^2+3x-5)+(x^2-x+1) \\ & -4x^2-2x+3\end{aligned}$$

Distribute the negative sign.

$$\begin{aligned}&= (2x^2 + x^2 - 4x^2) + (3x - x - 2x)\\&\quad + (-5 + 1 + 3)\end{aligned}$$

Group like terms with the same exponents and then add.

$$\begin{aligned}&= -x^2 + 0x - 1\\&= -x^2 - 1\end{aligned}$$

Example 4.19. Simplify $(\alpha^3 - 8\alpha^2 - 9) - (8\alpha^7 - 3\alpha^2 - 9) + (\alpha^2 + \alpha + 9)$.

$$\begin{aligned}(\alpha^3 - 8\alpha^2 - 9) - (8\alpha^7 - 3\alpha^2 - 9) + (\alpha^2 + \alpha + 9)\\&= (\alpha^3 - 8\alpha^2 - 9)\\&\quad - (8\alpha^7 - 3\alpha^2 - 9)\\&\quad + (\alpha^2 + \alpha + 9)\end{aligned}$$

Distribute the negative sign.

$$\begin{aligned}&= \alpha^3 - 8\alpha^2 - 9\\&\quad + (-8\alpha^7 + 3\alpha^2 + 9)\\&\quad + \alpha^2 + \alpha + 9\end{aligned}$$

Group like terms with the same exponents and then add.

$$\begin{aligned}&= -8\alpha^7 + \alpha^3 + (-8\alpha^2 + 3\alpha^2\\&\quad + \alpha^2) + \alpha + (-9 + 9 + 9)\\&= -8\alpha^7 + \alpha^3 + (-4\alpha^2) + \alpha + 9\end{aligned}$$

I give you permission to call α [13] fish.

[13] α is the Greek letter for alpha.

Example 4.20. Distribute and simplify.

$$\begin{aligned} 2\gamma(3\gamma^2-\gamma) &= 2\cdot 3\gamma^1\gamma^2 - 2\gamma^1\gamma^1 \\ &= 6\gamma^1\gamma^2 - 2\gamma^1\gamma^1 \\ &= 6\gamma^{1+2} - 2\gamma^{1+1} \\ &= 6\gamma^3 - 2\gamma^2 \end{aligned}$$

Example 4.21. Distribute and simplify.

$$\begin{aligned} 5\theta^2(2\theta^3-\theta^2) &= 5\cdot 2\theta^2\theta^3 - 5\theta^2\theta^2 \\ &= 10\theta^{2+3} - 5\theta^{2+2} \\ &= 10\theta^5 - 5\theta^4 \end{aligned}$$

Example 4.22. Distribute and simplify.

$$\begin{aligned} 3\alpha^2(2\alpha-\alpha^3)+\alpha^2 &= 3\cdot 2\alpha^2\alpha - 3\alpha^2\alpha^3 + \alpha^2 \\ &= 6\alpha^{2+1} - 3\alpha^{2+3} + \alpha^2 \\ &= 6\alpha^3 - 3\alpha^5 + \alpha^2 \\ &= -3\alpha^5 + 6\alpha^3 + \alpha^2 \end{aligned}$$

Example 4.23. Evaluate and simplify the following expression.

$$\frac{1}{2xy}(-8yx+8yx) = \frac{-8yx+8yx}{2xy}$$

Apply the distributive property.

$$= \frac{0}{2xy}$$

Since $-8yx + 8yx$ equals 0.

$$= 0$$

0 divided by any number is indeed 0.

Example 4.24. Distribute and simplify.

$$\begin{aligned} \delta(2\delta^2 + 3\delta^3) - \delta^2(4\delta - \delta^2) &= 2\delta^2\delta + 3\delta^3\delta - 4\delta\delta^2 - \delta^2(-\delta^2) \\ &= 2\delta^{2+1} + 3\delta^{3+1} - 4\delta^{1+2} + \delta^{2+2} \\ &= 2\delta^3 + 3\delta^4 - 4\delta^3 + \delta^4 \end{aligned}$$

Group terms that have the same exponent value.

$$= (3\delta^4 + \delta^4) + (2\delta^3 - 4\delta^3)$$

Add the number in front of the variable.

$$= 4\delta^4 - 2\delta^3$$

Example 4.25. Distribute and simplify.

$$2\beta(3\beta^2 - \beta^3) + 4\beta^2(2\beta^3 - 5\beta) + \beta^4 = 2 \cdot 3\beta\beta^2 - 2\beta\beta^3 + 4 \cdot 2\beta^2\beta^3 - 4 \cdot 5\beta^2\beta$$

$$\begin{aligned}&+\beta^4\\&=6\beta^{1+2}-2\beta^{1+3}+8\beta^{2+3}-20^{2+1}+\beta^4\\&=6\beta^3-2\beta^4+8\beta^5-20\beta^3+\beta^4\end{aligned}$$

Group terms that have the same exponent value.

$$=8\beta^5+(-2\beta^4+\beta^4)+(6\beta^3-20\beta^3)$$

Add the number in front of the variable.

$$=8\beta^5-\beta^4-14\beta^3$$

Example 4.26. Multiply and simplify the following polynomials.

$$(2x-3)(x^2+x-1)=2x(x^2+x-1)-3(x^2+x-1)$$

Distribute each term.

$$=2x\cdot x^2+2x\cdot x-2x\cdot 1-3x^2-3x+3$$

Perform multiplication.

$$=2x^3+2x^2-2x-3x^2-3x+3$$

Group like terms.

$$\begin{aligned}&=2x^3+(2x^2-3x^2)+(-2x-3x)+3\\&=2x^3-x^2-5x+3\end{aligned}$$

Example 4.27. Distribute and simplify.

$$\begin{aligned}(2x+3)(4x-1) &= 2x\cdot 4x + 2x\cdot(-1) + 3\cdot 4x + 3\cdot(-1)\\ &= 8x^2 - 2x + 12x - 3\\ &= 8x^2 + 10x - 3\end{aligned}$$

Example 4.28. Distribute and simplify.

$$\begin{aligned}(5a-2)(3a+4) &= 5a\cdot 3a + 5a\cdot 4 + (-2)\cdot 3a + (-2)\cdot 4\\ &= 15a^2 + 20a - 6a - 8\\ &= 15a^2 + 14a - 8\end{aligned}$$

Example 4.29. Distribute and simplify.

$$\begin{aligned}(\delta+3)(\delta+3) &= \delta\cdot\delta + \delta\cdot 3 + 3\cdot\delta + 3\cdot 3\\ &= \delta^2 + 3\delta + 3\delta + 9\\ &= \delta^2 + 6\delta + 9\end{aligned}$$

Example 4.30. Distribute and simplify.

$$\begin{aligned}(\delta+3)(\delta-3) &= \delta\cdot\delta + \delta\cdot(-3) + 3\cdot\delta + 3\cdot(-3)\\ &= \delta^2 - 3\delta + 3\delta - 9\\ &= \delta^2 - 9\end{aligned}$$

Example 4.31. Multiply and simplify $(2x+4)(x^4-3x^2+5x+4)$.

$$\begin{aligned}(2x+4)(x^4-3x^2+5x+4) &= 2x(x^4-3x^2+5x+4)\\ &\quad +4(x^4-3x^2+5x+4)\end{aligned}$$

Distribute each term.

$$\begin{aligned}&= 2x\cdot x^4+2x\cdot(-3x^2)+2x\cdot 5x+2x\cdot 4\\ &\quad +4\cdot x^4+4\cdot(-3x^2)+4\cdot 5x+4\cdot 4\end{aligned}$$

Perform multiplication.

$$\begin{aligned}&= 2x^5-6x^3+10x^2+8x\\ &\quad +4x^4-12x^2+20x+16\end{aligned}$$

Group like terms.

$$\begin{aligned}&= 2x^5+4x^4-6x^3+(10x^2-12x^2)\\ &\quad +(8x+20x)+16\\ &= 2x^5+4x^4-6x^3-2x^2+28x+16\end{aligned}$$

Example 4.32. Multiply and simplify $(x^2+2x-1)(2x^6+x+3)$.

$$\begin{aligned}(x^2+2x-1)(2x^6+x+3) &= x^2(2x^6+x+3)+2x(2x^6+x+3)\\ &\quad -1(2x^6+x+3)\end{aligned}$$

Distribute each term.

$$\begin{aligned}
&= x^2 \cdot 2x^6 + x^2 \cdot x + x^2 \cdot 3 \\
&\quad + 2x \cdot 2x^6 + 2x \cdot x + 2x \cdot 3 \\
&\quad - 2x^6 - x - 3
\end{aligned}$$

Perform multiplication.

$$\begin{aligned}
&= 2x^8 + x^3 + 3x^2 + 4x^7 + 2x^2 + 6x \\
&\quad - 2x^6 - x - 3
\end{aligned}$$

Group like terms.

$$\begin{aligned}
&= 2x^8 + 4x^7 - 2x^6 + x^3 \\
&\quad + (3x^2 + 2x^2) + (6x - x) - 3 \\
&= 2x^8 + 4x^7 - 2x^6 + x^3 + 5x^2 + 5x - 3
\end{aligned}$$

Let'a discuss a a topic that you will see extensively in this class and in your next. Take a minute to think if the two following expressions are logically equivalent. In other words, are these two equal?

$$(5+4)^2 = 5^2 + 4^2$$

You have the knowledge and the tools to answer this. To find the solution, compute both sides and then compare. Let's go ahead and do so. Notice that, $(5+4)^2 = (9)^2 = 81$ and $5^2 + 4^2 = 25 + 16 = 31$ which are not equivalent. Thus it is not true. In fact this is always false if we exclude 0.

Freshmen Dream

We Sac State mathematicians call this next statement the ***freshmen dream*** and it shall remain a dream. Thus, don't do it. This is important, and I expect you to know it.

$$(x+y)^2 \neq x^2+y^2$$

For x and y not equal to 0.

It is as simple as assigning $x=1$ and $y=1$ (we can use chose any two numbers.)

$$\begin{aligned}(x+y)^2 &\neq x^2+y^2\\(1+1)^2 &\neq 1^2+1^2\\(2)^2 &\neq 1+1\\4 &\neq 2\end{aligned}$$

Don't just take my word for it. It holds for all cases for $x \neq 0$ and $y \neq 0$.

Proof.

$$\begin{aligned}(x+y)^2 &\neq x^2+y^2\\(x+y)(x+y) &\neq x^2+y^2\\x\cdot x+x\cdot y+y\cdot x+y\cdot y &\neq x^2+y^2\\x^2+2xy+y^2 &\neq x^2+y^2\end{aligned}$$

□

Mathematicians usually put a square □ to indicate that the proof is done.

Example 4.33. Is the following true or false? Justify your claims.

$$(x+4)^2 = x^2+4^2$$

To justify, let's expand both sides of the equation:

$$\begin{aligned} \text{LHS: } (x+4)^2 &= (x+4)(x+4) \\ &= x \cdot x + x \cdot 4 + 4 \cdot x + 4 \cdot 4 \\ &= x^2 + 4x + 4x + 16 \\ &= x^2 + 8x + 16 \\ \text{RHS: } x^2 + 4^2 &= x^2 + 16 \end{aligned}$$

Comparing the two sides:

$$\text{LHS} = x^2 + 8x + 16 \neq \text{RHS} = x^2 + 16$$

Since the LHS (left-hand side of the equation) includes an additional $8x$ term that is not present on the RHS (right-hand side of the equation), the statement $(x+4)^2 = x^2 + 4^2$ is **false**. Thus, $(x+4)^2 \neq x^2 + 4^2$.

Example 4.34. Is the following true or false? Justify your claims.

$$(2x+3)^2 = (2x)^2 + (3)^2$$

To justify, let's expand both sides of the equation:

$$\begin{aligned} \text{LHS: } (2x+3)^2 &= (2x+3)(2x+3) \\ &= 2x \cdot 2x + 2x \cdot 3 + 3 \cdot 2x + 3 \cdot 3 \\ &= 4x^2 + 6x + 6x + 9 \\ &= 4x^2 + 12x + 9 \\ \text{RHS: } (2x)^2 + (3)^2 &= 4x^2 + 9 \end{aligned}$$

Comparing the two sides:

$$\text{LHS} = 4x^2 + 12x + 9 \neq \text{RHS} = 4x^2 + 9$$

Since the LHS (left-hand side of the equation) includes an additional $12x$ term that is not present on the RHS (right-hand side of the equation), the statement $(2x+3)^2 = (2x)^2 + (3)^2$ is **false**. Thus, $(2x+3)^2 \neq (2x)^2 + (3)^2$.

4.3 Polynomial Division

In this section, we will introduce polynomial long division, which extends the concept of long division to polynomials. You may already be familiar with long division using regular numbers. Here is an example.

	10	Remainder of 14
24	254	
	−240	
	14	

Hence the remainder is 14 since 24 does not go into 24 evenly and the quotient is 10

In a similar but more complicated manner, we can do the same, but with polynomials.

Example 4.35. Perform polynomial long division on $\dfrac{x^2+2x+6}{x-1}$.

		x	$+3$
$x-1$	x^2	$+2x$	$+6$
	$-(x^2$	$-x)$	
		$3x$	$+6$
		$-(3x$	$-3)$
			9

Step 1: Divide the leading term of the dividend, which is x^2, by the leading term of the divisor, which is x. This gives us the quotient term x.
Step 2: Multiply the divisor $x-1$ by the quotient term x, giving us x^2-x. Write this below the dividend and align like terms.
Step 3: Subtract the product obtained in Step 2 from the dividend.

$$x^2+2x+6-(x^2-x)=3x+6.$$

Step 4: Bring down the next term from the dividend, which is +6.

Step 5**:** We now have $3x+6$.
Step 6**:** Multiply the divisor $x-1$ by the new quotient term 3, giving us $3x-3$. Write this below the previous subtraction and align like terms.

$$(x-1)\cdot 3 = 3x-3.$$

Step 7**:** Subtract the product obtained in Step 6 from the previous result.

$$3x+6-(3x-3) = 9.$$

Hence the remainder is 9 and the quotient is $x+3$

Example 4.36. Perform polynomial long division on $\dfrac{4x^2-5x-21}{x-3}$.

$$\begin{array}{c|ccc}
 & & 4x & 7 \\
\hline
x-3 & 4x^2 & -5x & -21 \\
 & -(4x^2 & -12x) & \\
\hline
 & & 7x & -21 \\
 & & -(7x & -21) \\
\hline
 & & & 0
\end{array}$$

Step 1**:** Divide the leading term of the dividend, which is $4x^2$, by the leading term of the divisor, which is x. This gives us the quotient term $4x$.
Step 2**:** Multiply the divisor $x-3$ by the quotient term $4x$, giving us $4x^2-12x$. Write this below the dividend and align like terms.
Step 3**:** Subtract the product obtained in Step 2 from the dividend.

$$4x^2-5x-21-(4x^2-12x) = 7x-21.$$

Step 4**:** Bring down the next term from the dividend, which is -21.
Step 5**:** We now have $7x-21$.

Step 6: Multiply the divisor $x - 3$ by the new quotient term 7, giving us $7x - 21$. Write this below the previous subtraction and align like terms.
Step 7: Subtract the product obtained in Step 6 from the previous result.

$$7x - 21 - (7x - 21) = 0.$$

The remainder is 0, indicating that the division is exact.

> **Example 4.37.** Perform polynomial long division on $\dfrac{x^4 + 6x^2 + 2}{x^2 + 5}$.

$$
\begin{array}{c|cccc}
 & & & x^2 & +1 \\
\hline
x^2+5 & x^4 & 0x^3 & +6x^2 & +2 \\
 & -(x^4 & & +5x^2) & \\
\hline
 & & & +x^2 & +2 \\
 & & & -(x^2 & +5) \\
\hline
 & & & & -3
\end{array}
$$

Step 1: Divide the leading term of the dividend, which is x^4, by the leading term of the divisor, which is x^2. This gives us the quotient term x^2.
Step 2: Multiply the divisor $x^2 + 5$ by the quotient term x^2, giving us $x^4 + 5x^2$. Write this below the dividend and align like terms.
Step 3: Subtract the product obtained in Step 2 from the dividend.

$$x^4 + 6x^2 + 2 - (x^4 + 5x^2) = x^2 + 2$$

Step 4: Bring down the next term from the dividend, which is +2.
Step 5: We now have $x^2 + 2$.
Step 6: Multiply the divisor $x^2 + 5$ by the new quotient term 1, giving us $x^2 + 5$. Write this below the previous subtraction and align like terms.
Step 7: Subtract the product obtained in Step 6 from the previous result.

$$x^2 + 2 - (x^2 + 5) = -3$$

Hence, the remainder is -3 and the quotient is $x^2 + 1$.

4.4 Additional Problems

1. Simplify with positive exponents.

   (a) $\phi^7 \beta^3 \lambda^2 \beta \lambda^2$
   (b) $y^6 y^\pi \ y^\lambda y^4$
   (c) $(-1)^4$
   (d) $-(1)^4$
   (e) $a^5 b^2 c^3 bc^2$
   (f) $x^3 x^\pi x^\theta x^2$
   (g) $(-2)^3$
   (h) $-(2)^3$

2. Expand and simplify.

   (a) $-(x^2 + 5) - (x^2 - 3x - 3) - (x^2 + 4x + 1)$
   (b) $-(y^2 + 4) - (y^2 - 2y - 2) - (y^2 + 3y + 4)$
   (c) $(x^4 + x^2 - 1) + (-x^0 + 4x^4 - 1)$
   (d) $(-7x^3 - 3x) + (5x^2 + 6x)$

3. Expand and simplify.

   (a) $(6x + 1)(4x^2 + 2x + 7)$
   (b) $4\delta\beta^2(7\delta^3 z^\beta - 6\beta^4)$
   (c) $(\delta - 3)(\delta - 3)$
   (d) $3ab^3(5a^2 z^b - 4b^2)$
   (e) $(5x + 2)(3x^2 + x + 6)$
   (f) $(a - 2)(a - 4)$
   (g) $x^{1/2} + y^{1/3} - x^{1/2} + y^{2/6}$
   (h) $\left(\left(\frac{4c^{-1} r^6 s^{-5} h^0)}{16 r^6 s^7}\right)^{-1}\right)^{-3}$
   (i) $\left(\frac{2x^4 y^{-3}}{8x^4 y^5}\right)^2$
   (j) $\left(\frac{2x^{-1} y^2}{4x^3 1 y^{-4}}\right)^{-2}$
   (k) $\left(\frac{7x^{-2} s^0 y^3 z^{-1}}{14x^4 y^{-2} z^2}\right)^{-1}$
   (l) $\left(\frac{9x^2 y^{-4} z^3}{27x^{-3} y^2 z^{-1}}\right)^1$
   (m) $\left(\frac{3m^5 n^{-1} p^2}{9m^2 n^3 p^{-4}}\right)^{-0}$

4. Prove that the following are not true. Assume the variables are not 0.

(a) $7^2+3^2=(7+3)^2$

(b) $4^2-6^2=(4-6)^2$

(c) $10^2+1^2=(10+1)^2$

(d) $2^2+3^2=(2+3)^2$

(e) $9^2-5^2=(9+5)^2$

(f) $a^2+b^2=(a+b)^2$

(g) $2a^2+4b^2=(2a+4b)^2$

(h) $a^2-b^2=(a-b)^2$

5. Perform polynomial long division.

(a) $\dfrac{6x^3+11x^2-31x+1}{3x-2}$

5 Factoring

5.1 Greatest Common Factor

Factoring is the inverse operation of distribution, where instead of multiplying a group of numbers or variables together, we extract a common factor from them. Factoring plays a crucial role as it offers an efficient method of simplification. In fact, we have previously utilized factoring. Let's consider the following example.

Example 5.1. $\frac{114}{76} = \frac{\cancel{2} \cdot 3 \cdot \cancel{19}}{\cancel{2} \cdot 2 \cdot \cancel{19}} = \frac{3}{2}$

Notice that we factored out a 2, 3 and a 19 from 114. Likewise, we factored out two 2's and a 19 from 76. Doing so allowed us to simplify the fraction down to just $\frac{3}{2}$.

We will find out that this is also common for polynomials. Specific polynomial can be factored and furthered simplified. Although we haven't mastered factoring a trinomail consider the following.

Example 5.2. Factor and simplify.

$$\begin{aligned}
\frac{x^2+3x+2}{x+1} &= \frac{(x+1)(x+2)}{x+1} \\
&= \frac{\cancel{(x+1)}(x+2)}{\cancel{x+1}} \\
&= x+2
\end{aligned}$$

You have to trust me! x^2+3x+2 does equal $(x+1)(x+2)$... But it is much easier than long division ay?

Let's factor using the Greatest Common Factor (GCF).

Example 5.3. Factor and simplify.

$$15x^2 - 25x = \underline{5} \cdot 3\underline{x}x - \underline{5} \cdot 5\underline{x}$$

Both terms have a 5 and x in common. Thus, we can factor these out.

$$= 5x(3x - 5)$$

In this example, the greatest common factor (GCF) of 15 and 25 is 5. Both terms $15x^2$ and $25x$ share the variable x in common. We factor out $5x$, the highest degree of the common factor.

Example 5.4. Factor and simplify.

$$10y^3 + 5y^2 - 15y = \underline{5} \cdot 2\underline{y}y^2 + \underline{5} \cdot 1\underline{y}y - \underline{5} \cdot 3\underline{y}$$

Each term shares a 5 and y in common. Thus, we can factor these out.

$$= 5y(2y^2 + y - 3)$$

In this case, the greatest common factor (GCF) among 10, 5, and 15 is 5, and each term contains at least one factor of y. Therefore, we factor out $5y$.

Example 5.5. Factor and simplify.

$$6x^5y^3 - 8x^3y^6 = \underline{2} \cdot 3\underline{x^3}x^2\underline{y^3} - \underline{2} \cdot 4\underline{x^3}y^3\underline{y^3}$$

Notice that they each have a $2, x^3$ and y^3. Thus, we can factor these out.

$$= 2x^3y^3(3x^2 - 4y^3)$$

We can observe that the greatest common factor (GCF) of 6 and 8 is 2, as it is the largest whole number that divides both 6 and 8. Additionally, both terms $6x^5y^3$ and $8x^3y^6$ share the variables x and y in common. We can factor out the highest degree of each variable, which in this case is x^3 and y^3.

Example 5.6. Factor and simplify.

$$16a^5b - 8a^3b^4 = \underline{8} \cdot 2\underline{a^3}a^2\underline{b^1} - \underline{8} \cdot 1\underline{a^3}b^3\underline{b^1}$$

Notice that they each have a $8, a^3$ and b^1. Thus, we can factor these out.

$$= 8a^3b(a^2 - b^3)$$

Example 5.7. Factor and simplify.

$$\begin{aligned} 27z^3 + 12z^2 + 3z &= \underline{3} \cdot 9z^2\underline{z} + \underline{3} \cdot 4z \cdot \underline{z} + \underline{3} \cdot 1\underline{z} \\ &= 3z(9z^2 + 4z + 1) \end{aligned}$$

Notice that they each have a 3 and z in common. Thus, we can factor these out.

Example 5.8. Factor and simplify.

$$\begin{aligned} 12a^4b^3c^2 + 18a^2b^5c^3 - 24a^3b^2c^5 = {} & \underline{6a^2b^2c^2} \cdot 2a^2b \\ & + \underline{6a^2b^2c^2} \cdot 3b^3c - \underline{6a^2b^2c^2} \cdot 4ac^3 \end{aligned}$$

$$= 6a^2b^2c^2(2a^2b + 3b^3c - 4ac^3)$$

We can observe that the greatest common factor (GCF) of the numerical coefficients 12, 18, and 24 is 6, as it is the largest whole number that divides all three. Additionally, each term $12a^4b^3c^2$, $18a^2b^5c^3$, and $24a^3b^2c^5$ share the variables a, b, and c in common. We can factor out the lowest degree of each variable, which in this case is a^2 for a, b^2 for b, and c^2 for c.

Example 5.9. Factor and simplify.

$$\begin{aligned} 32x^5y - 16x^3y^2(x-4) &= \underline{16x^3y} \cdot 2x^2 - \underline{16x^3y} \cdot y(x-4) \\ &= 16x^3y(2x^2 - y(x-4)) \end{aligned}$$

We can observe that the greatest common factor (GCF) of the numerical coefficients 32 and 16 is 16, as it is the largest whole number that divides both. Additionally, both terms $32x^5y$ and $16x^3y^2(x-4)$ share the variables x and y in common. We can factor out the lowest degree of each variable, which in this case is x^3 for x and y for y.

Example 5.10. Factor and simplify.

$$\begin{aligned} -9m^4n^3 + 27m^2n^5 - 18m^3n^4 &= \underline{-9m^2n^3} \cdot m^2 - \underline{-9m^2n^3} \cdot 3n^2 \\ &\quad - \underline{9m^2n^3} \cdot 2mn \\ &= -9m^2n^3(m^2 - 3n^2 + 2mn) \end{aligned}$$

We can observe that the greatest common factor (GCF) of the numerical coefficients -9, 27, and -18 is 9, as it is the largest whole number that divides all three. Additionally, each term $-9m^4n^3$, $27m^2n^5$, and $-18m^3n^4$ share the variables m and n in common. We can factor out the lowest degree of each variable, which in this case is m^2 for m and n^3 for n.

Example 5.11. Factor and simplify.

$$\begin{aligned}(3C^2A^7T)-(C4AT^4) &= \underline{CAT}\cdot 3CA^6-\underline{CAT}\cdot 4T^3\\ &= CAT(3CA^6-4T^3)\end{aligned}$$

We can observe that the greatest common factor (GCF) of the numerical coefficients 3 and 4 is 1. Additionally, each term $3C^2A^7T$ and $C\cdot 4AT^4$ share the variables C, A, and T in common. We can factor out the lowest degree of each variable, which in this case is C for C, A for A, and T for T.

Notice that the following example can not be factored at all.

Example 5.12. Factor and simplify.

$$11xy^2+9\lambda$$

That is, there is no number or common variable that can be factored out of each.

Example 5.13. Factor and simplify.

$$\begin{aligned}(\alpha+w)u-(\alpha+w)v &= \underline{(\alpha+w)}u-\underline{(\alpha+w)}v\\ &= (u-v)(\alpha+w)\end{aligned}$$

In this example both u and v are both multiplied by $(\alpha+w)$. That said, we are able to factor out a $(\alpha+w)$.

5.2 Factoring by Grouping

It's always a good idea to start factoring by looking for a Greatest Common Factor (GCF) to factor out. However, in some cases, there might not be a (GCF) other than 1, so we need to employ alternative techniques like factoring by grouping. Let me explain this technique by using an example.

Example 5.14. Factor the expression $2x^3 + 4x^2 + 3x + 6$.

Firstly, look for a (GCF), but in this case, there isn't a (GCF) other than 1.

$$2x^3 + 4x^2 + 3x + 6 = (2x^3 + 4x^2) + (3x + 6)$$

Group the terms in pairs and factor out the (GCF) from each pair.

$$= 2x^2\underline{(x+2)} + 3\underline{(x+2)}$$

Observe that both terms now have a common factor of $(x+2)$

$$= (x+2)(2x^2+3)$$

Factoring by grouping is a technique used when the expression doesn't have a (GCF) other than 1. It involves grouping the terms, factoring out the (GCF) from each group, and then factoring out any common factors that emerge. This method allows us to simplify the expression and identify the common factors more efficiently. Lets look at other examples.

Example 5.15. Factor the expression $3x^3 - 8x^2 + 6x - 16$.

Firstly, look for a GCF, but in this case, there isn't a GCF other than 1.

$$3x^3 - 8x^2 + 6x - 16 = (3x^3 - 8x^2) + (6x - 16)$$

Group the terms in pairs and factor out the GCF from each pair.

$$= x^2\underline{(3x-8)} + 2\underline{(3x-8)}$$

Observe that both terms now have a common factor of $3x-8$

$$= (3x-8)(x^2+2)$$

Example 5.16. Factor the expression $4y^3 - 8y^2 + 9y - 18$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$4y^3 - 8y^2 + 9y - 18 = (4y^3 - 8y^2) + (9y - 18)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= 4y^2\underline{(y-2)} + 9\underline{(y-2)}$$

Observe that both terms now have a common factor of $(y-2)$

$$= (y-2)(4y^2+9)$$

Example 5.17. Factor the expression $3a^3 + 6a^2 - a - 2$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$3a^3 + 6a^2 - a - 2 = (3a^3 + 6a^2) - (a+2)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= 3a^2\underline{(a+2)} - 1\underline{(a+2)}$$

Observe that both terms now have a common factor of $(a+2)$

$$= (a+2)(3a^2-1)$$

Example 5.18. Factor the expression $5x^3+15x^2-2x-6$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$5x^3+15x^2-2x-6 = (5x^3+15x^2)-(2x+6)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= 5x^2\underline{(x+3)}-2\underline{(x+3)}$$

Observe that both terms now have a common factor of $(x+3)$

$$= (x+3)(5x^2-2)$$

Example 5.19. Factor the expression $6z^3-9z^2+4z-6$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$6z^3-9z^2+4z-6 = (6z^3-9z^2)+(4z-6)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= 3z^2\underline{(2z-3)}+2\underline{(2z-3)}$$

Observe that both terms now have a common factor of $(2z-3)$

$$= (2z-3)(3z^2+2)$$

Example 5.20. Factor the expression $7p^3 + 14p^2 - 5p - 10$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$7p^3 + 14p^2 - 5p - 10 = (7p^3 + 14p^2) - (5p + 10)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= 7p^2\underline{(p+2)} - 5\underline{(p+2)}$$

Observe that both terms now have a common factor of $(p+2)$

$$= (p+2)(7p^2-5)$$

Example 5.21. Factor the expression $x^3 + 5x^2 - 2x - 10$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1.

$$x^3 + 5x^2 - 2x - 10 = (x^3 + 5x^2) - (2x + 10)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$= x^2\underline{(x+5)} - 2\underline{(x+5)}$$

Observe that both terms now have a common factor of $(x+)$

$$= (x+5)(x^2-2)$$

Example 5.22. Factor the expression $a^3b-3a^2b^2+ab^3-3b^4$.

Firstly, look for a *GCF* (Greatest Common Factor). In this case, there isn't a *GCF* other than 1.

$$a^3b-3a^2b^2+ab^3-3b^4=(a^3b-3a^2b^2)+(ab^3-3b^4)$$

Group the terms in pairs and factor out the *GCF* from each pair.

$$=a^2b\underline{(a-3b)}+b^3\underline{(a-3b)}$$

Observe that both terms now have a common factor of $(a-3b)$

$$=(a^2b+b^3)(a-3b)$$

Notice that we can factor out a b from (a^2b+b^3)

$$=b(a^2+b^2)(a-3b)$$

Example 5.23. Factor the expression $x^2+4y-xy-4x$.

Firstly, rearrange the terms so that they have factorable variables.

$$\underline{x^2}+\underline{4y}-\underline{xy}-\underline{4x}=x^2-xy-4x+4y$$

Rearrange the terms to group factorable variables. Notice the red underline have an x in common. Likewise, the blue underlines have a 4 in common.

$$=(x^2-xy)+(-4x+4y)$$

Now, factor out the common terms from each pair.

$$= x\underline{(x-y)} - 4\underline{(x-y)}$$

Observe that both pairs now have a common factor of $(x-y)$.

$$= (x-y)(x-4)$$

Example 5.24. Factor the expression $7u + v^2 - 7v - uv$.

Firstly, rearrange the terms so that they have factorable variables.

$$\underline{7u} + \underline{v^2} - \underline{7v} - \underline{uv} = 7u - uv + v^2 - 7v$$

Rearrange the terms to group factorable variables. Notice the red underlines have a u in common. Likewise, the blue underlines have a v in common.

$$= (7u - uv) + (v^2 - 7v)$$

Now, factor out the common terms from each pair.

$$= u\underline{(7-v)} + v\underline{(v-7)}$$

Observe that both pairs now have a common factor of $(7-v)$, but in the second pair, it's in the reverse order. To make them identical, we factor out a -1 from the second pair.

$$= u(7-v) - v(7-v)$$

Now, factor out the common binomial $(7-v)$.

$$= (7 - v)(u - v)$$

Example 5.25. Factor the expression $mpx + mqx + npx + nqx$.

Factor out common terms from each group.

$$mpx + mqx + npx + nqx = (mpx + mqx) + (npx + nqx)$$

Factor out mx from the first two terms and nx from the last two terms.

$$= mx\underline{(p+q)} + nx\underline{(p+q)}$$

Observe that both groups now have a common factor of $(p+q)$.

$$= (mx + nx)(p + q)$$

Now, factor out x from the first factor.

$$= x(m + n)(p + q)$$

5.3 Difference of Squares

An other factoring technique is called the difference of squares.

Difference of Squares

Let x and y be any real number then the following holds.

$$(x + y)(x - y) = x^2 - y^2$$

Example 5.26. Factor the expression $x^2 - 1$.

$$
\begin{aligned}
x^2 - 1 &= x^2 - 1^2 \\
&= (x+1)(x-1)
\end{aligned}
$$

Example 5.27. Factor the expression $z^2 - 4$.

$$
\begin{aligned}
z^2 - 4 &= z^2 - 2^2 \\
&= (z-2)(z+2)
\end{aligned}
$$

Example 5.28. Factor the expression $b^2 - 9$.

$$
\begin{aligned}
b^2 - 9 &= b^2 - 3^2 \\
&= (b-3)(b+3)
\end{aligned}
$$

Example 5.29. Factor the expression $p^2 - 16$.

$$
\begin{aligned}
p^2 - 16 &= p^2 - 4^2 \\
&= (p-4)(p+4)
\end{aligned}
$$

Example 5.30. Factor the expression $s^2 - 25$.

$$
\begin{aligned}
s^2 - 25 &= s^2 - 5^2 \\
&= (s-5)(s+5)
\end{aligned}
$$

Example 5.31. Factor the expression $y^2 - 36$.

$$\begin{aligned} y^2 - 36 &= y^2 - 6^2 \\ &= (y+6)(y-6) \end{aligned}$$

Example 5.32. Factor the expression $t^2 - 49$.

$$\begin{aligned} t^2 - 49 &= t^2 - 7^2 \\ &= (t-7)(t+7) \end{aligned}$$

Example 5.33. Factor the expression $u^2 - 64$.

$$\begin{aligned} u^2 - 64 &= u^2 - 8^2 \\ &= (u-8)(u+8) \end{aligned}$$

Example 5.34. Factor the expression $v^2 - 81$.

$$\begin{aligned} v^2 - 81 &= v^2 - 9^2 \\ &= (v-9)(v+9) \end{aligned}$$

Example 5.35. Factor the expression $w^2 - 100$.

$$\begin{aligned} w^2 - 100 &= w^2 - 10^2 \\ &= (w-10)(w+10) \end{aligned}$$

Example 5.36. Factor the expression $f^2 - 121$.

$$
\begin{aligned}
f^2 - 121 &= f^2 - 11^2 \\
&= (f-11)(f+11)
\end{aligned}
$$

Example 5.37. Factor the expression $a^2 - 144$.

$$
\begin{aligned}
a^2 - 144 &= a^2 - 12^2 \\
&= (a+12)(a-12)
\end{aligned}
$$

I think that you get my point now. Inductively we can perform this to all perfect squares. Let's move on to more complicated examples.

Example 5.38. Factor the expression $4x^2 - 9$.

$$
\begin{aligned}
4x^2 - 9 &= (2x)^2 - 3^2 \\
&= (2x+3)(2x-3)
\end{aligned}
$$

Example 5.39. Factor the expression $9y^2 - 16$.

$$
\begin{aligned}
9y^2 - 16 &= (3y)^2 - 4^2 \\
&= (3y+4)(3y-4)
\end{aligned}
$$

Example 5.40. Factor the expression $25z^2 - 36$.

$$
\begin{aligned}
25z^2 - 36 &= (5z)^2 - 6^2 \\
&= (5z+6)(5z-6)
\end{aligned}
$$

Example 5.41. Factor the expression $x^4 - 1$.

$$\begin{aligned} x^4 - 1 &= (x^2)^2 - 1^2 \\ &= (x^2+1)(x^2-1) \end{aligned}$$

The expression $x^2 - 1$ is also a difference of squares, so we can further factor it as $(x+1)(x-1)$.

$$\begin{aligned} x^4 - 1 &= (x^2+1)(x^2-1) \\ &= (x^2+1)(x+1)(x-1) \end{aligned}$$

Example 5.42. Factor the expression $y^6 - 64$.

$$\begin{aligned} y^6 - 64 &= (y^3)^2 - 8^2 \\ &= (y^3+8)(y^3-8) \end{aligned}$$

Example 5.43. Factor the expression $2x^2 - 288$.

$$2x^2 - 288 = 2(x^2 - 144)$$

Factor out a 2 from each term.

$$= 2(x^2 - 144)$$

Next, we observe that $x^2 - 144$ is a difference of squares. We can express it as $(x+12)(x-12)$.

$$= 2(x+12)(x-12)$$

Example 5.44. Factor the expression $5x^3 - 125x$.

$$5x^3 - 125x = 5x(x^2 - 25)$$

Factor out a $5x$ from each term.

$$= 5x(x^2 - 5^2)$$

That is, $x^2 - 25$ is a difference of squares. We can express it as $(x+5)(x-5)$.

$$= 5x(x-5)(x+5)$$

Example 5.45. Factor the expression $x^3 + 2x^2 - x - 2$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1. Thus, we factor by grouping.

$$\begin{aligned} x^3 + 2x^2 - x - 2 &= (x^3 + 2x^2) - (x+2) \\ &= x^2\underline{(x+2)} - 1\underline{(x+2)} \\ &= (x+2)(x^2 - 1) \end{aligned}$$

Notice that $x^2 - 1$ is a difference of squares. We can express it as $(x+1)(x-1)$.

$$= (x+2)(x+1)(x-1)$$

Example 5.46. Factor the expression $2m^3 - m^2 - 8m + 4$.

Firstly, look for a *GCF*. In this case, there isn't a *GCF* other than 1. Therefore, let's first factor this by grouping.

$$\begin{aligned}2m^3 - m^2 - 8m + 4 &= (2m^3 - m^2) - (8m - 4)\\ &= m^2\underline{(2m-1)} - 4\underline{(2m-1)}\\ &= (2m-1)(m^2-4)\end{aligned}$$

Ah! $m^2 - 4$ is a difference of squares. We can express it as $(m+2)(m-2)$.

$$= (2m-1)(m+2)(m-2)$$

5.4 Factoring Trinomials

In this book, I will introduce you to the Tic Tac Toe Method [14]. This method can be intimidating at first and will need some getting used to. However, I found this to be the most effective way thus far. Without further ado, the Tic Tac Toe Method. Consider the following.

Important

To approach these rigorous problems effectively, it's essential to have an understanding beforehand. Diving in without preparation might lead to confusion, especially as my explanations can get intricate. Don't worry if certain details, like the choice of specific numbers, aren't immediately clear. Continue reading, and the future content should help clarify any uncertainties. In summation, don't get too fixated on any single aspect if you are baffled. Moving on without certain understatement is okay sometimes :).

Example 5.47. Factor the trinomial $x^2 - 9x + 20$.

[14] Shout out to Jason Broyles who taught me this method.

x^2	$-9x$	$+20$

The first step of factoring a trinomial is to break down x^2. Notice that $x^2 = x \cdot x$. We then put these two in the bottom two left boxes.

x^2	$-9x$	$+20$
x		
x		

The second step in factoring a trinomial involves determining the factors of 20. We can observe that the factors of 20 include $1, 2, 4, 5, 10$, and 20. Our objective now is to identify two factors of 20 that, when multiplied together is 20 and when added together result in -9.

$$2 \cdot 10 = 20 \text{ but } 2 + 10 = 12 \neq -9$$
$$-2 \cdot -10 = 20 \text{ but } -2 - 10 = -12 \neq -9$$
$$4 \cdot 5 = 20 \text{ but } 4 + 5 = 9 \neq -9$$
$$-4 \cdot -5 = 20 \text{ and } -4 - 5 = -9$$

Thus, we found our numbers -4 and -5. Place these in the bottom right boxes.

x^2	$-9x$	$+20$
x		-5
x		-4

To finish this, we multiply the x straight across with the -5 to its right and the x and the -4 and add the two. Notice how $-5x + -4x = -9x$ which is the second term of the polynomial (This step is important). They must add up to $-9x$.

x^2	$-9x$	$+20$
x		-5
x		-4

$\Rightarrow -5x + -4x = -9x$

Next we must group the x and the -4. Doing so we have.

$$(x-4)$$

Likewise we then group x and the -5. Hence,

$$(x-5)$$

Think about it as grouping diagonally. Finally we multiply the two.

$$(x-4)(x-5)$$

Notice that we can check our work by just multiplying the two.

$$\begin{aligned}(x-4)(x-5) &= x^2 - 4x - 5x + 20 \\ &= x^2 - 9x + 20\end{aligned}$$

The more that you practice these problems the better that you will get. They may seem confusing at first, but you will get the hang of them. Let's end this section with a more complected example.

Example 5.48. Factor the trinomial $x^2 - 2x + 1$.

x^2	$-2x$	$+1$

The first step of factoring a trinomial is to break down x^2. Notice that $x^2 = x \cdot x$. We then put these two in the bottom two left boxes.

x^2	$-2x$	$+1$
x		
x		

The second step in factoring a trinomial involves determining the factors of 1. We can observe that the factors of 1 include 1 and 1 only. Our objective now is to identify two factors of 1 that, when multiplied together is 1 and when added together result in -2.

$$1 \cdot 1 = 1 \text{ and } 1 + 1 = 2 \neq -2$$
$$-1 \cdot -1 = 1 \text{ and } -1 - 1 = -2$$

Thus, we found our numbers -1 and -1. Place these in the bottom right boxes.

x^2	$-2x$	$+1$
x		-1
x		-1

To finish this, we multiply the x straight across with the -1 to its right and the x and the -1 and add the two. Notice how $-1x + -1x = -2x$ which is the second term of the polynomial.

x^2	$-2x$	$+1$
x		-1
x		-1

$$\Rightarrow -1x - 1x = -2x$$

Next we must group the x and the -1. Doing so we have.

$$(x-1)$$

Likewise we then group x and the -1. Hence,

$$(x-1)$$

Multiply the two.

$$(x-1)(x-1)$$

We can check our work by just multiplying the two.

$$\begin{aligned}(x-1)(x-1) &= x^2 - x - x + 1 \\ &= x^2 - 2x + 1\end{aligned}$$

Example 5.49. Factor the trinomial $x^2 + 10x + 24$.

x^2	$+10x$	$+24$

The first step of factoring a trinomial is to break down x^2. Notice that $x^2 = x \cdot x$. We then put these two in the bottom two left boxes.

x^2	$+10x$	$+24$
x		
x		

The second step in factoring a trinomial involves determining the factors of 24. We can observe that the factors of 24 include $1, 2, 3, 4, 6, 8, 12$, and 24. Our objective now is to identify two factors of 24 that, when multiplied together is 24 and when added together result in 10.

$$\begin{gathered} 1 \cdot 24 = 24 \text{ but } 1 + 24 = 25 \neq 10 \\ 2 \cdot 12 = 24 \text{ but } 2 + 12 = 14 \neq 10 \\ 3 \cdot 8 = 24 \text{ but } 3 + 8 = 11 \neq 10 \\ 4 \cdot 6 = 24 \text{ and } 4 + 6 = 10 \end{gathered}$$

Thus, we found our numbers 4 and 6. Place these in the bottom right boxes.

x^2	$+10x$	$+24$
x		4
x		6

Now multiply the x straight across with the 4 to its right and the x and the 6 and add the two. Notice how $4x + 6x = 10x$ which is the second term of the polynomial.

x^2	$10x$	$+24$
x		4
x		6

$\Rightarrow 4x + 6x = 10x$

Let's group the x and the 6. Doing so we have.

$$(x+6)$$

Likewise we then group x and the 4. Hence,

$$(x+4)$$

Multiply the two.

$$(x+4)(x+6)$$

Lastly multiply the two.

$$\begin{aligned}(x+4)(x+6) &= x^2 + 6x + 4x + 24 \\ &= x^2 + 10x + 24\end{aligned}$$

Example 5.50. Factor the trinomial $x^2 + 6x + 8$.

x^2	$+6x$	$+8$

The first step of factoring a trinomial is to break down x^2. Notice that $x^2 = x \cdot x$. We then put these two in the bottom two left boxes.

x^2	$+6x$	$+8$
x		
x		

The second step in factoring a trinomial involves determining the factors of 8. We can observe that the factors of 8 include $1, 2, 4$, and 8. Our objective now is to identify two factors of 8 that, when multiplied together is 8 and when added together result in 6.

$$1 \cdot 8 = 8 \text{ but } 1 + 8 = 9 \neq 6$$
$$2 \cdot 4 = 8 \text{ and } 2 + 4 = 6$$

Hence our numbers are 2 and 4. Place these in the bottom right boxes.

x^2	$+6x$	$+8$
x		2
x		4

Next we multiply the x straight across with the 2 to its right and the x and the 4 and add the two. Notice how $2x + 4x = 6x$ which is the second term of the polynomial.

x^2	$6x$	$+8$
x		2
x		4

$$\Rightarrow 2x + 4x = 6x$$

Next we must group the x and the 4. Doing so we have.

$$(x + 4)$$

Likewise we then group x and the 2. Hence,

$$(x + 2)$$

Multiply the two together.

$$(x + 2)(x + 4)$$

Lastly, let's check our work.

$$\begin{aligned}(x + 2)(x + 4) &= x^2 + 4x + 2x + 8 \\ &= x^2 + 6x + 8\end{aligned}$$

Example 5.51. Factor the trinomial $2x^2 - 7x + 3$.

$2x^2$	$-7x$	$+3$

The first step of factoring the trinomial is to break down $2x^2$. Notice that $2x^2 = 2x \cdot x$. We then put these two in the bottom two left boxes.

$2x^2$	$-7x$	$+3$
$2x$		
x		

The second step in factoring a trinomial involves determining the factors of 3. We can observe that the factors of 3 include 1 and 3. -1 and -3 are also factors. Notice that $(-1)(-3) = 3$. Since the middle number is $-7x$, we pick -1 and -3. We now put -1 and -3 in the bottom right box.

$2x^2$	$-7x$	$+3$
$2x$		-3
x		-1

We purposely did this as $2x \cdot (-3) = -6x$ and $x \cdot (-1) = -x$ where $-6x - x = -7x$

More general notice that we have to multiply the $2x$ straight across with the -3 and the x and the -1 and add the two. Notice how $-6x + -x = -7x$ which is the second term of the polynomial (This step is important). They must add up to $-7x$.

$2x^2$	$-7x$	$+3$	
$2x$		-3	$\Rightarrow -6x - x = -7x$
x		-1	

Next we must group the $2x$ and the -1. Doing so we have.

$$(2x - 1)$$

Likewise we then group x and the -3. Hence,

$$(x-3)$$

Finally we multiply the two.

$$(2x-1)(x-3)$$

Notice that we can check our work by just multiplying the two.

$$(2x-1)(x-3) = 2x^2 - 6x - x + 3$$
$$(2x-1)(x-3) = 2x^2 - 7x + 3$$

Example 5.52. Factor the trinomial $6x^2 + x - 15$.

$6x^2$	$+x$	-15

The first step of factoring the trinomial is to break down $6x^2$. Notice that $6x^2 = 2x \cdot 3x$. We then put these two in the bottom two left boxes.

$6x^2$	$+x$	-15
$2x$		
$3x$		

The second step in factoring a trinomial involves determining the factors of -15. We can observe that the factors of -15 include 1 and -15, 3 and -5, -3 and 5, and -1 and 15. Since the middle number is x, we pick -3 and 5 because $-3 \cdot 5 = -15$. We now put 3 and -5 in the bottom right boxes.

$6x^2$	$+x$	-15
$2x$		5
$3x$		-3

Now, we multiply the $2x$ straight across with the 5 and the $3x$ and the -3 and add the two. Notice how $10x + -9x = x$ which is the second term of the polynomial (This step is important). They must add up to x.

$6x^2$	$-x$	-15
$2x$		5
$3x$		-3

$\Rightarrow -10x + 9x = -x$

Next we must group the $2x$ and the -3. Doing so we have.

$$(2x - 3)$$

Likewise we then group $3x$ and the -5. Hence,

$$(3x + 5)$$

Finally we multiply the two.

$$(2x - 3)(3x + 5)$$

Notice that we can check our work by just multiplying the two.

$$\begin{aligned}(2x-3)(3x+5) &= 6x^2 + 10x - 9x - 15 \\ &= 6x^2 + x - 15\end{aligned}$$

Example 5.53. Factor the trinomial $8x^2 - 10x + 3$.

$8x^2$	$-10x$	$+3$

The first step of factoring the trinomial is to break down $8x^2$. Notice that $8x^2 = 2x \cdot 4x$. We then put these two in the bottom two left boxes.

$8x^2$	$-10x$	$+3$
$2x$		
$4x$		

The second step in factoring a trinomial involves determining the factors of 3. We can observe that the factors of 3 include 1 and 3, -1 and -3. Since the middle number is $-10x$, we pick -1 and -3 because $-1 \cdot -3 = 3$. We now put -1 and -3 in the bottom right boxes.

$8x^2$	$-10x$	$+3$
$2x$		-3
$4x$		-1

Now, we multiply the $2x$ straight across with the -3 and the $4x$ and the -1 and add the two. Notice how $-6x + -4x = -10x$ which is different from the second term of the polynomial. Therefore, we need to rearrange the factors.

$8x^2$	$-10x$	$+3$	
$2x$		-3	$\Rightarrow -6x + -4x = -10x$
$4x$		-1	

Next we must group the $2x$ and the -1. Doing so we have.

$$(2x - 1)$$

Likewise we then group $4x$ and the -3. Hence,

$$(4x - 3)$$

Finally we multiply the two.

$$(2x - 1)(4x - 3)$$

Notice that we can check our work by just multiplying the two.

$$\begin{aligned}(2x-1)(4x-3) &= 8x^2 - 6x - 4x + 3\\ &= 8x^2 - 10x + 3\end{aligned}$$

5.5 Solving Quadratic Equations by Factoring

I begin this section with a definition.

Definition

Zero Factor Property: Assume a and b are real numbers.

$$\text{If } a \cdot b = 0, \text{ then } a = 0 \text{ or } b = 0.$$

Note: This is not limited to the product of two terms.

Important

Zero is the only number that has this property.

Example 5.54. Solve for w: $w^2 + 7w + 12 = 0$

$$w^2 + 7w + 12 = 0$$

w^2	$+7w$	$+12$
w		3
w		4

$$\Rightarrow 3w + 4w = 7w$$

$$\Rightarrow (w+3)(w+4) = 0$$

$$\Rightarrow w + 3 = 0 \quad \text{or} \quad w + 4 = 0$$

$$\Rightarrow \boxed{w = -3 \quad \text{or} \quad w = -4}$$

We call -3 and -4 the roots of the polynomial $w^2 + 7w + 12$.

Example 5.55. Solve for x: $x^2 + 5x + 6 = 0$

$$x^2 + 5x + 6 = 0$$

x^2	$+5x$	$+6$
x		2
x		3

$$\Rightarrow 2x + 3x = 5x$$

$$\Rightarrow (x+2)(x+3)=0$$

$$\Rightarrow (x+2)=0 \quad \text{or} \quad (x+3)=0$$

$$\Rightarrow \boxed{x=-2 \quad \text{or} \quad x=-3}$$

Example 5.56. Solve for y: $y^2-3y-10=0$

$$y^2-3y-10=0$$

y^2	$-3y$	-10
y		2
y		-5

$\Rightarrow 2y-5y=-3y$

$$\Rightarrow (y+2)(y-5)=0$$

$$\Rightarrow (y+2)=0 \quad \text{or} \quad (y-5)=0$$

$$\Rightarrow \boxed{y=-2 \quad \text{or} \quad y=5}$$

Example 5.57. Solve for y: $y^2-4y+4=0$

$$y^2-4y+4=0$$

y^2	$-4y$	$+4$
y		-2
y		-2

$\Rightarrow -2y-2y=-4y$

$$\Rightarrow (y-2)^2=0$$

$$\Rightarrow (y-2)=0$$

$$\Rightarrow \boxed{y=2}$$

Example 5.58. Solve for x: $x^3+x^2=0$

$$x^3+x^2=0$$

Factor out x^2 and then solve for each factor. That is, set x^2 and $x+1$ equal to 0.

$$\Rightarrow x^2(x+1)=0$$

$$\Rightarrow x^2=0 \quad \text{or} \quad (x+1)=0$$

$$\Rightarrow \boxed{x=0 \quad \text{or} \quad x=-1}$$

Example 5.59. Solve for y: $y^3-6y^2+9y=0$

$$y^3-6y^2+9y=0$$

Factor out y and then solve for each factor. That is, set y and y^2-6y+9 equal to 0.

$$\Rightarrow y(y^2-6y+9)=0$$

$$\Rightarrow y=0 \quad \text{or} \quad (y^2-6y+9)=0$$

y^2	$-6y$	$+9$
y		-3
y		-3

$$\Rightarrow -3y-3y=-6y$$

$$\Rightarrow y=0 \quad \text{or} \quad (y-3)^2=0$$

$$\Rightarrow \boxed{y=0 \quad \text{or} \quad y=3}$$

Example 5.60. Solve for z: $z^3-2z^2-3z=0$

$$z^3-2z^2-3z=0$$

Factor out z and then solve for each factor. That is, set z and z^2-2z-3 equal to 0.

$$\Rightarrow z(z^2-2z-3)=0$$

$$\Rightarrow z=0 \quad \text{or} \quad (z^2-2z-3)=0$$

z^2	$-2z$	-3
z		-3
z		1

$$\Rightarrow -3z+z=-2z$$

$$\Rightarrow z=0 \quad \text{or} \quad (z-3)(z+1)=0$$

$$\Rightarrow \boxed{z=0 \quad \text{or} \quad z=3 \quad \text{or} \quad z=-1}$$

Example 5.61. Solve for w: $(9w+18)(w^2-6w+8)=0$.

Let's see if any common factors that can be factored out. In this case, both terms $(9w+18)$ have a common factor of 9, so we can factor it out.

$$(9w+18)(w^2-6w+8)=0$$
$$9(w+2)(w^2-6w+8)=0$$

Next, we can try to factor the quadratic expression w^2-6w+8. Since the coefficient of w^2 is 1, we need to find two numbers that multiply to 8 and add up to -6.

w^2	$-6w$	$+8$
w		-4
w		-2

$$\Rightarrow -4w-2w=-6w$$

The numbers -2 and -4 satisfy these conditions, so we can factor the quadratic.

$$w^2-6w+8=(w-2)(w-4)$$

Let's put it all together.

$$(9w+18)(w^2-6w+8)=0$$
$$9(w+2)(w^2-6w+8)=0$$
$$9(w+2)(w-2)(w-4)=0$$

Setting $9(w+2)=0$ gives:

$$9(w+2)=0$$
$$w+2=0$$
$$w=-2$$

Setting $w-2=0$ gives:

$$w-2=0$$
$$w=2$$

Setting $w-4=0$ gives:

$$w-4=0$$
$$w=4$$

Therefore, the roots of the equation $3(3w+6)(w-2)(w-4)=0$ are $w=-2, w=2$, and $w=4$.

Example 5.62. Solve for v: $(5v+15)(v^2-4v+3)=0$.

Let's see if any common factors can be factored out. In this case, both terms $(5v+15)$ have a common factor of 5, so we can factor it out.

$$(5v+15)(v^2-4v+3)=0$$
$$5(v+3)(v^2-4v+3)=0$$

Next, we can try to factor the quadratic expression v^2-4v+3. Since the coefficient of v^2 is 1, we need to find two numbers that multiply to 3 and add up to -4.

v^2	$-4v$	$+3$
v		-1
v		-3

$\Rightarrow -1v - 3v = -4v$

The numbers -1 and -3 satisfy these conditions, so we can factor the quadratic.

$$v^2 - 4v + 3 = (v-1)(v-3)$$

Let's put it all together.

$$\begin{aligned}(5v+15)(v^2-4v+3) &= 0\\ 5(v+3)(v^2-4v+3) &= 0\\ 5(v+3)(v-1)(v-3) &= 0\end{aligned}$$

Setting $5(v+3) = 0$ gives:

$$\begin{aligned}5(v+3) &= 0\\ v+3 &= 0\\ v &= -3\end{aligned}$$

Setting $v-1=0$ gives:

$$\begin{aligned}v-1 &= 0\\ v &= 1\end{aligned}$$

Setting $v-3=0$ gives:

$$\begin{aligned}v-3 &= 0\\ v &= 3\end{aligned}$$

Therefore, the roots of the equation $5(v+3)(v-1)(v-3) = 0$ are $v = -3, v = 1$, and $v = 3$.

Let's do an application problem (word problems [15]). I know.

[15] I hate to break it to you, but all science majors end up primarily working with word problems. Whether it's mathematics, physics, chemistry, biology, or engineering, they all ultimately revolve around word problems. You can't get away from them as they are the only thing you will eventually do.

Example 5.63. If the length of each side of a square is increased by 4 inches, the area of the square becomes 9 times greater than the original area. Find the perimeter of the original square.

Let's assume the original side length of the square is denoted by x inches. According to the given information, if the length of each side is increased by 4 inches, the new side length becomes $x+4$ inches.

The area of the new square is then $(x+4)^2$ square inches. The problem states that the area of the new square is 9 times greater than the original area. Mathematically, we can express this as:

$$(x+4)^2 = 9(x^2)$$

Expanding the left side of the equation.

$$x^2 + 8x + 16 = 9x^2$$

Bringing all terms to one side.

$$\begin{aligned} 9x^2 - x^2 - 8x - 16 &= 0 \\ 8x^2 - 8x - 16 &= 0 \end{aligned}$$

Dividing both sides by 8.

$$x^2 - x - 2 = 0$$

x^2	$-x$	-2
x		-2
x		1

$\Rightarrow -2x - x = -x$

Now, we can factor the quadratic equation.

$$(x-2)(x+1)=0$$

Setting each factor equal to zero and solving for x.

$$x-2=0 \implies x=2$$
$$x+1=0 \implies x=-1$$

Since the length of a side cannot be negative, we discard the solution $x=-1$. Therefore, the original side length of the square is $x=2$ inches. The perimeter of the original square is given by $4\times$ side length, so the perimeter is $4\times 2=8$ inches.

Example 5.64. A bullet is shoot straight into the air. The height of the bullet H, in meters can be modeled by the following equation.

$$H(t)=120t-\frac{1}{2}\cdot 9.8t^2$$

1. After 10 seconds, how high off the ground is the bullet?
2. How long until it reaches the ground?

Question 1: Substituting $t=10$ into the equation, we get

$$\begin{aligned} H(10) &= 120\cdot 10-\frac{1}{2}\cdot 9.8\cdot 10^2 \\ &= 1200-4.9\cdot 100 \\ &= 1200-490 \\ &= 710m \end{aligned}$$

So, after 10 seconds, the bullet is 710 meters high off the ground.
Question 2: The bullet reaches the ground when $H(t)=0$.

$$0=120t-\frac{1}{2}\cdot 9.8t^2$$
$$0=120t-4.9t^2$$

Factor out a t.

$$\begin{aligned} 0 &= t(120-4.9t) \\ 0 &= 120-4.9t \\ 4.9t &= 120 \\ t &= \frac{120}{4.9} \\ t &= 24.4898s \end{aligned}$$

We find that the bullet hits the ground approximately at $t = 24.49$ seconds.

5.6 Additional Problems

1. Factor completely.

   (a) $9x^6y^4 - 3x^4y^6$
   (b) $12x^3y^2 - 4x^2y^5$
   (c) $5ab(a-b) + 3bc(a-b)$
   (d) $3xy(x+y) - 4xy(x+y)$
   (e) $xy + yc - uy + v$
   (f) $25v^2m^{\frac{1}{2}} - 45b^2vm^{\frac{1}{2}}$
   (g) $FAT + CATS$
   (h) $SAC - STATE$

2. Factor by grouping.

   (a) $a^3 + 3a^2b + 2ab^2 + 6b^3$
   (b) $x^3 + 2x^2y + x^2 + 2xy$
   (c) $x^3 - 2x^2 + 5x - 10$
   (d) $6v^3 + 3v^2 + 8v + 4$
   (e) $3x^2 + 2x + 12x + 8$
   (f) $x^2 + 4x + 3x + 12$
   (g) $2x^3 - 4x^2 + 8x - 16$
   (h) $x^3 + 6x^2 + 3x + 18$

3. Factor by using difference of square.

   (a) $9y^2 - 16$
   (b) $16a^2 - 25b^2$
   (c) $3y^2 - 48$
   (d) $\beta^2 - 9$
   (e) $4y^2 - 36$
   (f) $49z^2 - 64$
   (g) $x^8 - 1$
   (h) $x^4 - 16$

4. Factor the trinomial.

   (a) $3x^2 - 7x - 6$
   (b) $x^2 + 16x + 64$
   (c) $x^2 - 12x + 35$
   (d) $5a^2 - 30a + 45$
   (e) $6z^2 - 2z - 8$
   (f) $7b^2 - 23b + 18$
   (g) $15c^2 - 34c + 16$
   (h) $2d^2 - 5d + 2$

5. Solve for the given variable.

   (a) $8x^2 + 2x - 1 = 0$
   (b) $s^2 + 20s + 100 = 0$
   (c) $4x^2 - 8x + 4 = 0$
   (d) $x^2 + 3x - 28 = 0$
   (e) $(x^5)(5x + 4)(3x - 3) = 0$
   (f) $\left(4d - \frac{4}{5}\right)(x + 2) = 0$
   (g) $r^2 - 4 = 0$
   (h) $(g - 76)(g + 987) = 0$
   (i) $x^4 - 1 = 0$

6. A stone is dropped off a 64-foot cliff and falls into the ocean below. The height of the stone above sea level is given by the function

$$h(t) = -16t^2 + 64$$

   where t is the time in seconds since the stone was dropped. How long will it take for the stone to hit the water?

6 Rational Expressions & Fractions

6.1 Simplifying Rational Expressions

Now that we are more comfortable with factoring, we are ready to move on to rational expressions. On the surface, we are just simplifying, multiplying, dividing, adding, and subtracting fractions. However, instead of numbers, our terms will be polynomials. Note that all of the rules we know and love for rational numbers still apply here. But first, we look at restrictions and how to legally simplify rational expressions. Let's just begin this chapter with an example.

Example 6.1. Find the domains for the following rational function.

$$f(x) = \frac{3}{x-7}$$

Are there any numbers that when plugged in for x would give us an issue? The fastest way to solve these types of problems is to set the bottom not equal to 0 and then solve.

$$\begin{aligned} & x-7 \neq 0 \\ \Rightarrow \quad & x \neq 7 \end{aligned}$$

Notice that if we plug in 7 we have the following.

$$= f(x) = \frac{3}{x-7}$$

$$= f(7) = \frac{3}{7-7}$$

$$= f(7) = \frac{3}{0}$$

We can not divide by 0 [16] !!!!!

This means we have a restriction on our domain. In interval notation, we would say the domain of f is

$$(-\infty, 7) \cup (7, \infty).$$

Remember that the parentheses "),(" means that the number is excluded.

Knowing restrictions will become very important when we move from simplifying rational *expressions* to solving rational *equations*.

Example 6.2. Find the domains for the following rational function.

$$f(m) = \frac{5m+4}{8m+16}.$$

Are there any numbers when plugged in for m that would give us an issue? The fastest way to solve these types of problems is to set the denominator not equal to 0 and then solve.

$$\begin{aligned} & 8m + 16 \neq 0 \\ \Rightarrow \quad & m \neq -2 \end{aligned}$$

Notice that if we plug in -2, we have the following,

$$\begin{aligned} &= f(m) = \frac{5m+4}{8m+16} \\ &= f(-2) = \frac{5(-2)+4}{8(-2)+16} \\ &= f(-2) = \frac{-10+4}{-16+16} \\ &= f(-2) = \frac{-6}{0} = \text{Undefined} \end{aligned}$$

[16] 0h-we, 0h-we, 0h-0h.

We cannot divide by 0!

This means we have a restriction on our domain. In interval notation, we would say the domain of f is $(-\infty,-2)\cup(-2,\infty)$. Let's do some examples that require us to factor and cancel like terms.

Example 6.3. Find the domains for the following rational function.

$$f(x)=\frac{3x-9}{x^2-25}.$$

Are there any numbers when plugged in for x that would give us an issue? The fastest way to solve these types of problems is to set the denominator not equal to 0 and then solve.

$$\begin{aligned} & x^2-25\neq 0 \\ \Rightarrow\quad & (x-5)(x+5)\neq 0 \\ \Rightarrow\quad & x\neq 5 \text{ and } x\neq -5 \end{aligned}$$

Notice that if we plug in 5 or -5, we have the following:

$$\begin{aligned} &= f(x)=\frac{3x-9}{x^2-25} \\ &= f(5)=\frac{3(5)-9}{5^2-25} \\ &= f(5)=\frac{15-9}{25-25} \\ &= f(5)=\frac{6}{0}=\text{Undefined} \end{aligned}$$

$$\begin{aligned} &= f(x)=\frac{3x-9}{x^2-25} \\ &= f(-5)=\frac{3(-5)-9}{(-5)^2-25} \end{aligned}$$

$$= f(-5) = \frac{-15-9}{25-25}$$

$$= f(-5) = \frac{-24}{0} = \text{Undefined}$$

Thus our restrictions on our domain are -5 and 5. In interval notation, we would say the domain of f is $(-\infty,-5)\cup(-5,5)\cup(5,\infty)$. Notice there are two restrictions, thus we have two unions "$\cup$".

Example 6.4. Simplify the expression and find the domain.

$$\frac{x^2-9}{x^2-4x+3}.$$

To find its domain notice,

$$\begin{aligned} & x^2-4x+3 \neq 0 \\ \Rightarrow\quad & (x-1)(x-3) \neq 0 \\ \Rightarrow\quad & x \neq 1 \text{ and } x \neq 3 \end{aligned}$$

Note that $x \neq 1$ and $x \neq 3$. Hence, the domain is $(-\infty,1)\cup(1,3)\cup(3,\infty)$.

$$\begin{aligned} \frac{x^2-9}{x^2-4x+3} &= \frac{(x+3)(x-3)}{(x-1)(x-3)} \\ &= \frac{\cancel{(x-3)}(x+3)}{(x-1)\cancel{(x-3)}} \end{aligned}$$

The $(x-3)$'s only cancel when $(x-3) \neq 0$ holds true. If not, division by 0 occurs. Thus, we still exclude 3 from our domain.

$$= \frac{x+3}{x-1}$$

Example 6.5. Find the domain for the rational function

$$g(x) = \frac{x-4}{2x^2-8x}.$$

Set the denominator not equal to 0 and solve.

$$\begin{aligned} & 2x^2 - 8x \neq 0 \\ \Rightarrow \quad & 2x(x-4) \neq 0 \\ \Rightarrow \quad & x(x-4) \neq 0 \\ \Rightarrow \quad & x \neq 0 \text{ and } x \neq 4 \end{aligned}$$

Hence our domain of g is $(-\infty, 0) \cup (0,4) \cup (4,\infty)$.

Example 6.6. Find the domain of $\frac{2z+6}{z^2-36}$.

Set the denominator not equal to 0 and solve.

$$\begin{aligned} & z^2 - 36 \neq 0 \\ \Rightarrow \quad & (z-6)(z+6) \neq 0 \\ \Rightarrow \quad & z \neq 6 \text{ and } z \neq -6 \end{aligned}$$

Hence, the domain of the function is $(-\infty, -6) \cup (-6, 6) \cup (6, \infty)$.

Example 6.7. Simplify and then find the domain $\frac{y-8}{y^2-64}$.

$$\begin{aligned} \frac{y-8}{y^2-64} &= \frac{y-8}{y^2-8^2} \\ &= \frac{y-8}{(y-8)(y+8)} \end{aligned}$$

$$= \frac{\cancel{y-8}}{\cancel{(y-8)}(y+8)}$$

$$= \frac{1}{y+8}$$

Now let's find our domain.

$$\begin{aligned} & y^2 - 64 \neq 0 \\ \Rightarrow \quad & (y-8)(y+8) \neq 0 \\ \Rightarrow \quad & y \neq 8 \text{ and } y \neq -8 \end{aligned}$$

Notice that $y \neq -8$ and $y \neq 8$. Hence the domain is $(-\infty, -8) \cup (-8, 8) \cup (8, \infty)$.

Example 6.8. Simplify the following completely then find the domain

$$\frac{3x^2 - 12x}{6x^3 - 24x^2}.$$

$$\begin{aligned} \frac{3x^2 - 12x}{6x^3 - 24x^2} &= \frac{3x(x-4)}{6x^2(x-4)} \\ &= \frac{3x \cdot \cancel{(x-4)}}{6x^2 \cdot \cancel{(x-4)}} \\ &= \frac{3x}{6x^2} \\ &= \frac{1}{2x} \end{aligned}$$

Now let's find our domain.

$$\begin{aligned} & 6x^3 - 24x^2 \neq 0 \\ \Rightarrow \quad & 6x^2(x-4) \neq 0 \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad x^2(x-4) \neq 0 \\ &\Rightarrow \quad x \neq 0 \text{ and } x \neq 4 \end{aligned}$$

That is x cannot be equal to 0 or 4. The domain is $(-\infty, 0) \cup (0, 4) \cup (4, \infty)$.

Example 6.9. Simplify the following completely then find the domain

$$\frac{10-5x}{x^2-4x+4}.$$

$$\begin{aligned} \frac{10-5x}{x^2-4x+4} &= \frac{-5(x-2)}{(x-2)(x-2)} \\ &= \frac{-5\cdot\cancel{(x-2)}}{\cancel{(x-2)}(x-2)} \\ &= \frac{-5}{x-2} \end{aligned}$$

Now let's find our domain.

$$\begin{aligned} & x^2-4x+4 \neq 0 \\ \Rightarrow \quad & (x-2)(x-2) \neq 0 \\ \Rightarrow \quad & x \neq 2 \end{aligned}$$

Thus we can exclude 2. Hence, the domain is $(-\infty, 2) \cup (2, \infty)$.

Example 6.10. Simplify the following completely then find the domain

$$\frac{2x^2-18}{x^2-9}.$$

$$\frac{2x^2-18}{x^2-9} = \frac{2(x^2-9)}{x^2-9}$$

$$= \frac{2 \cdot \cancel{(x^2-9)}}{\cancel{(x^2-9)}}$$

$$= 2$$

Now let's find our domain.

$$\begin{aligned} & x^2 - 9 \neq 0 \\ \Rightarrow \quad & (x-3)(x+3) \neq 0 \\ \Rightarrow \quad & x \neq 3 \text{ and } x \neq -3 \end{aligned}$$

Given that x must not be 3 or -3 to prevent the denominator from being zero, the domain excludes these values. Thus, it is expressed as $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$.

Example 6.11. Simplify the following completely then find the domain

$$\frac{4x^2 - 16}{2x^2 - 8}.$$

$$\begin{aligned} \frac{4x^2-16}{2x^2-8} &= \frac{4(x^2-4)}{2(x^2-4)} \\ &= \frac{4 \cdot \cancel{(x^2-4)}}{2 \cdot \cancel{(x^2-4)}} \\ &= \frac{4}{2} \\ &= 2 \end{aligned}$$

Now let's determine the domain.

$$\begin{aligned} & 2x^2 - 8 \neq 0 \\ \Rightarrow \quad & x^2 - 4 \neq 0 \\ \Rightarrow \quad & (x-2)(x+2) \neq 0 \end{aligned}$$

$$\Rightarrow \quad x \neq 2 \text{ and } x \neq -2$$

The values 2 and -2 are excluded to avoid division by zero in the original expression, making the domain $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

Example 6.12. Is the following true $\dfrac{a-3b}{2b-a} = \dfrac{3b-a}{a-2b}$?

Yes it is! Notice

$$\begin{aligned} \frac{a-3b}{2b-a} &= \frac{a-3b}{2b-a} \\ &= (-1)\frac{-a+3b}{2b-a} \\ &= (-1)(-1)\frac{-a+3b}{-2b+a} \\ &= \frac{3b-a}{a-2b} \end{aligned}$$

6.2 Simplifying Complex Fractions

Next, I will introduce the flip and multiple technique. This technique has been employed throughout the course without a formal proof. As mathematicians and scientists, it is important that we provide a proof for all operations we utilize; thus allowing us full confidence in their applicability to any scenario. Allow me to give a comprehensive proof of this technique for every possible case.

Theorem

Flip and Multiply

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c}$$

Proof.

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b}}{\frac{c}{d}} \cdot \frac{\frac{d}{c}}{\frac{d}{c}}$$

Multiply by a convenient form of 1.

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}}$$

Multiply numerator and denominator of big fraction.

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{cd}{cd}}$$

Multiply denominator of big fraction and simplify.

$$= \frac{\frac{a}{b} \cdot \frac{d}{c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

□

Let's do some examples.

Example 6.13. Simplify the expression.

$$\frac{\frac{1}{3}-\frac{5}{2}}{\frac{3}{5}+\frac{1}{2}}$$

To simplify this expression, we need to find a common denominator for both the top and the bottom fractions.

$$\begin{aligned}
\frac{\frac{1}{3}-\frac{5}{2}}{\frac{3}{5}+\frac{1}{2}} &= \frac{\frac{1\cdot 2}{3\cdot 2}-\frac{5\cdot 3}{2\cdot 3}}{\frac{3\cdot 2}{5\cdot 2}+\frac{1\cdot 5}{2\cdot 5}} \\
&= \frac{\frac{2}{6}-\frac{15}{6}}{\frac{6}{10}+\frac{5}{10}}
\end{aligned}$$

Add straight across.

$$\begin{aligned}
&= \frac{\frac{2-15}{6}}{\frac{6+5}{10}} \\
&= \frac{\frac{-13}{6}}{\frac{11}{10}}
\end{aligned}$$

Flip and multiply.

$$= \frac{-13}{6}\cdot\frac{10}{11}$$

$$= \frac{-13 \cdot 10}{6 \cdot 11}$$

$$= \frac{-130}{66}$$

Factor out a 2 from each and simply.

$$= -\frac{65 \cdot \cancel{2}}{33 \cdot \cancel{2}}$$

$$= -\frac{65}{33}$$

Example 6.14. Simplify the expression.

$$\frac{7 - \frac{3}{2y}}{\frac{2}{2y} - 5}$$

To simplify this expression, let's find a common denominator. The common denominator is $2y$ for both fractions.

$$\frac{7 - \frac{3}{2y}}{\frac{2}{2y} - 5} = \frac{\frac{7(2y)}{2y} - \frac{3}{2y}}{\frac{2}{2y} - \frac{5(2y)}{2y}}$$

$$= \frac{\frac{14y}{2y} - \frac{3}{2y}}{\frac{2}{2y} - \frac{10y}{2y}}$$

We obtain common denominators.

$$= \frac{\dfrac{14y-3}{2y}}{\dfrac{2-10y}{2y}}$$

Add straight across.

$$= \frac{14y-3}{2y} \cdot \frac{2y}{2-10y}$$

Flip and multiply.

$$= \frac{(14y-3)\cdot\cancel{(2y)}}{\cancel{2y}\cdot(2-10y)}$$

$$= \frac{(14y-3)}{(2-10y)}$$

Example 6.15. Simplify the following expression.

$$\frac{\dfrac{x^5}{x+9}}{\dfrac{x+1}{x^2-81}}$$

$$\frac{\dfrac{x^5}{x+9}}{\dfrac{x+1}{x^2-81}} = \frac{x^5\cdot(x^2-81)}{(x+9)(x+1)}$$

Flip and multiply.

$$= \frac{x^5 \cdot (x-9)(x+9)}{(x+9)(x+1)}$$

Use completing the square on $(x^2 - 81)$

$$= \frac{x^5 \cdot (x-9)\cancel{(x+9)}}{\cancel{(x+9)}(x+1)}$$

$$= \frac{x^5 \cdot (x-9)}{(x+1)}$$

Example 6.16. Simplify the following expression.

$$\frac{\frac{x^4}{x-5}}{\frac{x-2}{x^2-25}}$$

$$\frac{\frac{x^4}{x-5}}{\frac{x-2}{x^2-25}} = \frac{x^4 \cdot (x^2-25)}{(x-5)(x-2)}$$

Flip and multiply.

$$= \frac{x^4 \cdot (x-5)(x+5)}{(x-5)(x-2)}$$

Use completing the square on $(x^2 - 25)$

$$= \frac{x^4 \cdot \cancel{(x-5)}(x+5)}{\cancel{(x-5)}(x-2)}$$

$$= \frac{x^4 \cdot (x+5)}{(x-2)}$$

Example 6.17. Simplify the following expression.

$$\frac{\dfrac{x^3}{x^2-5x+6}}{\dfrac{x+2}{x^2-4}}$$

$$\frac{\dfrac{x^3}{x^2-5x+6}}{\dfrac{x-1}{x^2-4}} = \frac{x^3 \cdot (x^2-4)}{(x^2-5x+6)(x+2)}$$

Flip and multiply.

$$= \frac{x^3 \cdot (x-2)\cancel{(x+2)}}{(x-2)(x-3)\cancel{(x+2)}}$$

Use completing the square on (x^2-4) and use tic tac toe on the bottom.

$$= \frac{x^3 \cdot \cancel{(x-2)}}{\cancel{(x-2)}(x-3)}$$

$$= \frac{x^3}{(x-3)}$$

Example 6.18. Simplify the following expression.

$$\frac{\dfrac{x^4-2x^3+x^2}{x^2-x-6}}{\dfrac{x^2-x}{x^2+5x+6}}$$

$$\frac{\dfrac{x^4-2x^3+x^2}{x^2-x-6}}{\dfrac{x^2-x}{x^2+5x+6}} = \frac{(x^4-2x^3+x^2)\cdot(x^2+5x+6)}{(x^2-x-6)\cdot(x^2-x)}$$

Flip and multiply.

$$= \frac{x^2(x^2-2x+1)(x+2)(x+3)}{(x-3)(x+2)x(x-1)}$$

Factor out an x^2 from the first term on the numerator and an x from the bottom right term. We can then use tic tac toe on the top and bottom trinomial.

$$= \frac{\cancel{x}\cdot x(x-1)\cancel{(x-1)}\cancel{(x+2)}(x+3)}{(x-3)\cancel{(x+2)}\cancel{x}\cancel{(x-1)}}$$

$$= \frac{x(x-1)(x+3)}{x-3}$$

Example 6.19. Simplify the following expression.

$$\frac{w^{-1}+z^{-1}}{w}$$

To simplify this expression, we can simplify the individual fractions first. Let's simplify the numerator.

$$w^{-1} + z^{-1} = \frac{1}{w} + \frac{1}{z}$$

Apply the exponential rules.

$$= \frac{1}{w} \cdot \frac{z}{z} + \frac{1}{z} \cdot \frac{w}{w}$$

Get common denominators.

$$= \frac{z}{wz} + \frac{w}{wz}$$

Add straight across.

$$= \frac{z+w}{wz}$$

Now, we can simplify the entire expression by dividing the numerator by the denominator.

$$\frac{w^{-1} + z^{-1}}{w} = \frac{\frac{z+w}{wz}}{\frac{w}{1}}$$

Notice that $\frac{w}{1} = w$.

$$= \frac{z+w}{wz} \cdot \frac{1}{w}$$

Flip and multiply.

$$= \frac{z+w}{wzw}$$

$$= \frac{z+w}{w^2 z}$$

Important

You might be tempted to cancel out a w from the numerator and the denominator. However, I promise you that it is already fully simplified. Cancellation can only occur when a factor is common to the entire numerator and the entire denominator. The numerator $z+w$ does not have a factor of w, since z does not have a w.

Example 6.20. Simplify the following expression.

$$\frac{\frac{\delta}{\beta} - \frac{\beta}{\delta}}{\frac{1}{\beta} - \frac{1}{\gamma}}$$

Firstly, we need to obtain common denominators.

$$\frac{\frac{\delta}{\beta} - \frac{\beta}{\delta}}{\frac{1}{\beta} - \frac{1}{\delta}} = \frac{\left(\frac{\delta}{\delta} \cdot \frac{\delta}{\beta}\right) - \left(\frac{\beta}{\delta} \cdot \frac{\beta}{\beta}\right)}{\left(\frac{\delta}{\delta} \cdot \frac{1}{\beta}\right) - \left(\frac{1}{\delta} \cdot \frac{\beta}{\beta}\right)}$$

We need to add the factions. Notice that $\frac{\delta}{\delta} = 1$ and $\frac{\beta}{\beta} = 1$.

$$= \frac{\frac{\delta^2}{\delta\beta} - \frac{\beta^2}{\delta\beta}}{\frac{\delta}{\delta\beta} - \frac{\beta}{\delta\beta}}$$

Add straight across.

$$= \frac{\frac{\delta^2 - \beta^2}{\delta\beta}}{\frac{\delta - \beta}{\delta\beta}}$$

Flip and multiply.

$$= \frac{\delta^2 - \beta^2}{\delta\beta} \cdot \frac{\delta\beta}{\delta - \beta}$$

$$= \frac{(\delta^2 - \beta^2) \cdot \cancel{\delta\beta}}{\cancel{\delta\beta} \cdot (\delta - \beta)}$$

Complete the square on $(\delta^2 - \beta^2)$

$$= \frac{(\delta + \beta)\cancel{(\delta - \beta)}}{\cancel{\delta - \beta}}$$

$$= \delta + \beta$$

6.3 Multiplication & Division

Definition

Expressions have no equal sign.

Example 6.21.

$$\frac{2\gamma^7 b}{c}$$

If you want to algebraically manipulate an expression, you are limited to multiplying by some convenient form of 1 (or adding some convenient form of 0, we literally did that in the last example).

Definition

Equations have an equal sign. This allows us the to add, subtract, multiply and divide long as we do the same operation to ***BOTH*** sides.

Example 6.22.

$$y = mx + b$$

We will be dealing with ***EXPRESSIONS*** in this chapter. This limits our problem solving techniques to only multiplying by 1. [17]

Important

Before you start multiplying straight across and preying for simplification, consider factoring each term.

Example 6.23. Simplify the following expression.

$$\frac{x^3 y^2}{x^2 y^5} \div \frac{x^4 y}{x^5 y^3}$$

To simplify this expression, we follow the rules of exponentiation and division of fractions. We start by flipping the second fraction and changing the division to multiplication.

$$\frac{x^3 y^2}{x^2 y^5} \div \frac{x^4 y}{x^5 y^3} = \frac{x^3 y^2}{x^2 y^5} \cdot \frac{x^5 y^3}{x^4 y}$$

[17] Do you have a favorite equation? Drake's equation is cool, although it is an approximation. Drake's equation is used to determine how many advanced civilizations are in our Milky Way Galaxy. Yeah, aliens.

Now, we multiply the fractions by multiplying numerators and denominators separately.

$$= \frac{x^3 x^5 y^2 y^3}{x^2 x^4 y^5 y}$$

Next, we simplify the expression by adding the exponents of like bases.

$$= \frac{x^{3+5} y^{2+3}}{x^{2+4} y^{5+1}}$$
$$= \frac{x^8 y^5}{x^6 y^6}$$

Notice that this can be rewritten as.

$$= \frac{\cancel{x^6} x^2 \cancel{y^5}}{\cancel{x^6} \cancel{y^5} y}$$
$$= \frac{x^2}{y}$$

Example 6.24. Simplify the following expression.

$$\frac{-x^2 y^{-2}}{x^{-1} y^{-3}} \div \frac{x^{-3} y^2}{x^4 y^{-1}}$$

To simplify this expression, we can use the rules of exponentiation and division of fractions. We start by flipping the second fraction and changing the division to multiplication.

$$\frac{-x^2 y^{-2}}{x^{-1} y^{-3}} \div \frac{x^{-3} y^2}{x^4 y^{-1}} = \frac{-x^2 y^{-2}}{x^{-1} y^{-3}} \cdot \frac{x^4 y^{-1}}{x^{-3} y^2}$$

Now, we multiply the fractions by multiplying numerators and denominators separately.

$$= \frac{-x^2 x^4 y^{-2} y^{-1}}{x^{-1} x^{-3} y^{-3} y^2}$$

Next, we simplify the expression by adding the exponents of like bases.

$$= \frac{-x^{2+4}y^{-2-1}}{x^{-1-3}y^{-3+2}}$$
$$= \frac{-x^{6}y^{-3}}{x^{-4}y^{-1}}$$

Finally, bring the denominator up and change the exponents sign. Simplify the expression by multiplying the exponents of like bases.

$$= -x^6y^{-3}x^4y$$
$$= -x^{6+4}y^{-3+1}$$
$$= -x^{10}y^{-2}$$
$$= -\frac{x^{10}}{y^2}$$

Example 6.25. Simplify the following expression.

$$\frac{a^4b^3}{a^2b^5c^2} \div \frac{a^3bc^4}{a^6b^2c^3}$$

To simplify this expression, we follow the rules of exponentiation and division of fractions. We start by flipping the second fraction and changing the division to multiplication.

$$\frac{a^4b^3}{a^2b^5c^2} \div \frac{a^3bc^4}{a^6b^2c^3} = \frac{a^4b^3}{a^2b^5c^2} \cdot \frac{a^6b^2c^3}{a^3bc^4}$$

Now, we multiply the fractions by multiplying numerators and denominators separately.

$$= \frac{a^4a^6b^3b^2c^3}{a^2a^3b^5b^2c^2c^4}$$

Next, we simplify the expression by adding the exponents of like bases.

$$= \frac{a^{4+6}b^{3+2}c^3}{a^{2+3}b^{5+2}c^{2+4}}$$

$$= \frac{a^{10}b^5c^3}{a^5b^7c^6}$$

Notice that this can be rewritten as.

$$= \frac{\cancel{a^5}a^5\cancel{b^5}\cancel{c^3}}{\cancel{a^5}\cancel{b^5}b^2\cancel{c^3}c^3}$$
$$= \frac{a^5}{b^2c^3}$$

Example 6.26. Simplify the following expression.

$$\frac{p^6q^4r}{p^3q^7r^5} \div \frac{p^2qr^3}{p^8q^3r^4}$$

To simplify this expression, we follow the rules of exponentiation and division of fractions. We start by flipping the second fraction and changing the division to multiplication.

$$\frac{p^6q^4r}{p^3q^7r^5} \div \frac{p^2qr^3}{p^8q^3r^4} = \frac{p^6q^4r}{p^3q^7r^5} \cdot \frac{p^8q^3r^4}{p^2qr^3}$$

Now, we multiply the fractions by multiplying numerators and denominators separately.

$$= \frac{p^6p^8q^4q^3rr^4}{p^3p^2q^7qq^3r^5r^3}$$

Next, we simplify the expression by adding the exponents of like bases.

$$= \frac{p^{6+8}q^{4+3}r^{1+4}}{p^{3+2}q^{7+1+3}r^{5+3}}$$
$$= \frac{p^{14}q^7r^5}{p^5q^{11}r^8}$$

Notice that this can be rewritten as.

$$= \frac{\cancel{p^5}p^9\cancel{q^7}r^5}{\cancel{p^5}\cancel{q^7}q^4\cancel{r^5}r^3}$$

$$= \frac{p^9}{q^4 r^3}$$

Example 6.27. Simplify the following expression.

$$\frac{m^{-2}n^3}{m^4 n^{-5}} \div \frac{m^{-3}n^{-1}}{m^2 n^6}$$

To simplify this expression, we follow the rules of exponentiation and division of fractions, particularly focusing on handling negative exponents.

$$\frac{m^{-2}n^3}{m^4 n^{-5}} \div \frac{m^{-3}n^{-1}}{m^2 n^6} = \frac{m^{-2}n^3}{m^4 n^{-5}} \cdot \frac{m^2 n^6}{m^{-3}n^{-1}}$$

Now, we multiply the fractions by multiplying numerators and denominators separately.

$$= \frac{m^{-2}m^2 n^3 n^6}{m^4 m^{-3} n^{-5} n^{-1}}$$

Next, we simplify the expression by adding the exponents of like bases.

$$= \frac{m^{-2+2}n^{3+6}}{m^{4-3}n^{-5-1}}$$
$$= \frac{m^0 n^9}{m^1 n^{-6}}$$

Since $m^0 = 1$ and to handle negative exponents, we move n^{-6} between numerator and denominator, changing their signs.

$$= \frac{n^9}{mn^{-6}}$$
$$= \frac{n^9 n^6}{m}$$
$$= \frac{n^{9+6}}{m}$$
$$= \frac{n^{15}}{m}$$

Example 6.28. Simplify the following expression.

$$\frac{x^3+2x^2+4x+8}{y^2-1}\cdot\frac{y^2+2y+1}{x^4-16}$$

Factor using the difference of squares.

$$\begin{aligned} y^2-1 &= y^2-1^2 \\ &= (y+1)(y-1) \end{aligned}$$

$$\begin{aligned} x^4-16 &= x^4-4^2 \\ &= (x^2+4)(x^2-4) \\ &= (x^2+4)(x-2)(x+2) \end{aligned}$$

Factor by tic tac toe.

$$y^2+2y+1=(y+1)(y+1)$$

Factor by grouping

$$\begin{aligned} x^3+2x^2+4x+8 &= x^2(x+2)+4(x+2) \\ &= (x^2+4)(x+2) \end{aligned}$$

And thus,

$$\begin{aligned} \frac{x^3+2x^2+4x+8}{y^2-1}\cdot\frac{y^2+2y+1}{x^4-16} &= \frac{(x^2+4)(x+2)}{(y+1)(y-1)}\cdot\frac{(y+1)(y+1)}{(x^2+4)(x-2)(x+2)} \\ &= \frac{\cancel{(x^2+4)}\cancel{(x+2)}}{\cancel{(y+1)}(y-1)}\cdot\frac{\cancel{(y+1)}(y+1)}{\cancel{(x^2+4)}(x-2)\cancel{(x+2)}} \\ &= \frac{(y+1)}{(y-1)(x-2)} \end{aligned}$$

Example 6.29. Simplify the following expression.

$$\frac{u^4-16}{v^2-4}\cdot\frac{v^2+4v+4}{u^4-16u^2-9u^2+144}$$

Factor using the difference of squares.

$$\begin{aligned} u^4-16 &= u^4-4^2 \\ &= (u^2+4)(u^2-4) \\ &= (u^2+4)(u+2)(u-2) \end{aligned}$$

$$\begin{aligned} v^2-4 &= v^2-2^2 \\ &= (v+2)(v-2) \end{aligned}$$

Factor as a perfect square trinomial.

$$v^2+4v+4=(v+2)^2$$

Factor by grouping.

$$\begin{aligned} u^4-16u^2-9u^2+144 &= u^2(u^2-16)-9(u^2-16) \\ &= (u^2-9)(u^2-16) \\ &= (u-3)(u+3)(u-4)(u+4) \end{aligned}$$

And thus,

$$\begin{aligned} \frac{u^4-16}{v^2-4}\cdot\frac{v^2+4v+4}{u^4-25u^2+144} &= \frac{(u^2+4)(u+2)(u-2)}{(v+2)(v-2)}\cdot\frac{(v+2)^2}{(u-3)(u+3)(u-4)(u+4)} \\ &= \frac{\cancel{(u+2)}\cancel{(u-2)}}{\cancel{(v+2)}(v-2)}\cdot\frac{\cancel{(v+2)}(v+2)}{(u-3)(u+3)\cancel{(u-2)}\cancel{(u+2)}} \\ &= \frac{(v+2)}{(v-2)(u-3)(u+3)} \end{aligned}$$

Example 6.30. Simplify the following expression.

$$\frac{p^4-36}{q^2-4}\cdot\frac{q^2+2q+1}{p^4-8p^2+16}$$

Factor using the difference of squares.

$$\begin{aligned}p^4-36&=p^4-6^2\\&=(p^2+6)(p^2-6)\end{aligned}$$

$$\begin{aligned}q^2-4&=q^2-2^2\\&=(q+2)(q-2)\end{aligned}$$

Factor as a perfect square trinomial.

$$q^2+2q+1=(q+1)^2$$

Factor by grouping.

$$\begin{aligned}p^4-8p^2+16&=p^2(p^2-8)+16\\&=p^2(p^2-8)+16\\&=(p^2-4)(p^2-4)\\&=(p-2)^2(p+2)^2\end{aligned}$$

And thus,

$$\begin{aligned}\frac{p^4-36}{q^2-4}\cdot\frac{q^2+2q+1}{p^4-8p^2+16}&=\frac{(p^2+6)(p^2-6)}{(q+2)(q-2)}\cdot\frac{(q+1)^2}{(p-2)^2(p+2)^2}\\&=\frac{\cancel{(p^2-6)}}{q-2}\cdot\frac{(q+1)^2}{(p+2)^2\cancel{(p^2-6)}}\\&=\frac{(q+1)^2}{(q-2)(p+2)^2}\end{aligned}$$

Example 6.31. Simplify the following expression.

$$\frac{3x^2-2x-8}{3x^2-5x-12}\cdot\frac{3-x}{x-2}$$

You can factor these via tic tac toe. However, I want to demonstrate that this can be done via grouping.

$$\begin{aligned}3x^2-2x-8 &= 3x^2-6x+4x-8\\ &= 3x(x-2)+4(x-2)\\ &= (x-2)(3x+4)\end{aligned}$$

$$\begin{aligned}3x^2-5x-12 &= 3x^2-9x+4x-12\\ &= 3x(x-3)+4(x-3)\\ &= (x-3)(3x+4)\end{aligned}$$

The term $\frac{3-x}{x-2}$ can be rewritten as $\frac{x-3}{x-2}(-1)$.
And thus,

$$\begin{aligned}\frac{3x^2-2x-8}{3x^2-5x-12}\cdot\frac{3-x}{x-2} &= \frac{(x-2)(3x+4)}{(x-3)(3x+4)}\cdot\frac{x-3}{x-2}(-1)\\ &= -1\cdot\frac{\cancel{(x-2)}\cancel{(3x+4)}}{\cancel{(3x+4)}\cancel{(x-3)}}\cdot\frac{\cancel{(x-3)}}{\cancel{(x-2)}}\\ &= -1\end{aligned}$$

Example 6.32. Simplify the following expression.

$$\frac{3x^2-2x-21}{3x^2-9x+7}\div\frac{2x^2-18}{2x^2-x-21}$$

Factor each polynomial.

$$3x^2-2x-21 = 3x^2-9x+7x-21$$

$$\begin{aligned}
&= 3x(x-3)+7(x-3)\\
&= (x-3)(3x+7)\\
3x^2-9x+7 &= 3x^2-6x-3x+7\\
&= 3x(x-2)-1(x-7)\\
&= (3x-1)(x-7)\\
2x^2-18 &= 2(x^2-9)\\
&= 2(x-3)(x+3)\\
2x^2-x-21 &= 2x^2-7x+6x-21\\
&= x(2x-7)+3(2x-7)\\
&= (2x-7)(x+3)
\end{aligned}$$

Perform the division, which is equivalent to multiplying by the reciprocal. That is, flip and multiply.

$$\begin{aligned}
\frac{3x^2-2x-21}{3x^2-9x+7} \div \frac{2x^2-18}{2x^2-x-21} &= \frac{3x^2-2x-21}{3x^2-9x+7}\cdot\frac{2x^2-x-21}{2x^2-18}\\
&= \frac{(x-3)(3x+7)}{(3x-1)(x-7)}\cdot\frac{(2x-7)(x+3)}{2(x-3)(x+3)}\\
&= \frac{\cancel{(x-3)}(3x+7)}{(3x-1)(x-7)}\cdot\frac{(2x-7)\cancel{(x+3)}}{2\cancel{(x-3)}\cancel{(x+3)}}\\
&= \frac{(3x+7)(2x-7)}{2(3x-1)(x-7)}
\end{aligned}$$

6.4 Addition & Subtraction

Example 6.33. Simplify the following expression.

$$\frac{5}{9\gamma}+\frac{2}{7\phi}.$$

$$\frac{5}{9\gamma}+\frac{2}{7\phi}=\frac{5}{9\gamma}\cdot\frac{7\phi}{7\phi}+\frac{2}{7\phi}\cdot\frac{9\gamma}{9\gamma}$$

We want a common denominator. That is we want the bottom to be the same in order to add the top.

$$=\frac{35\phi}{63\gamma\phi}+\frac{18\gamma}{63\gamma\phi}$$

Now that the bottom is the same we can add straight across.

$$=\frac{35\phi+18\gamma}{63\gamma\phi}$$

Example 6.34. Simplify the following expression.

$$\frac{3}{4\gamma}+\frac{5}{6\phi}$$

$$\frac{3}{4\gamma}+\frac{5}{6\phi}=\frac{3}{4\gamma}\cdot\frac{6\phi}{6\phi}+\frac{5}{6\phi}\cdot\frac{4\gamma}{4\gamma}$$

We want a common denominator. That is we want the bottom to be the same in order to add the top.

$$=\frac{18\phi}{24\gamma\phi}+\frac{20\gamma}{24\gamma\phi}$$

Now that the bottom is the same we can add straight across.

$$=\frac{18\phi+20\gamma}{24\gamma\phi}$$

Example 6.35. Simplify the following expression.

$$\frac{z}{3a}-\frac{z}{4b}.$$

$$\frac{z}{3a} - \frac{z}{4b} = \frac{z}{3a} \cdot \frac{4b}{4b} - \frac{z}{4b} \cdot \frac{3a}{3a}$$

To subtract these fractions, we need a common denominator. We do this by multiplying each fraction by an appropriate form of 1 to match the denominators.

$$= \frac{4bz}{12ab} - \frac{3az}{12ab}$$

With a common denominator, we can subtract the numerators directly.

$$= \frac{4bz - 3az}{12ab}$$
$$= \frac{z(4b - 3a)}{12ab}$$

Example 6.36. Simplify the following expression.

$$y - \frac{3}{x-1}$$

To simplify this expression, we can leave the second term as it is and find a common denominator for the first term. The common denominator is $x - 1$.

$$y - \frac{3}{x-1} = y \cdot \frac{x-1}{x-1} - \frac{3}{x-1}$$

Notice that $\frac{x-1}{x-1} = 1$ so we aren't technically changing the expression.

$$= \frac{(y \cdot (x-1))}{(x-1)} - \frac{3}{x-1}$$
$$= \frac{(yx - y)}{(x-1)} - \frac{3}{x-1}$$

Now that we have the same denominator, we can combine the numerators.

$$= \frac{(yx - y - 3)}{(x - 1)}$$

Example 6.37. Simplify the following expression.

$$\frac{x}{7x-9} - \frac{x+3}{8x}.$$

$$\frac{x}{7x-9} - \frac{x+3}{8x} = \frac{x \cdot 8x}{(7x-9)\cdot 8x} - \frac{(x+3)\cdot(7x-9)}{8x\cdot(7x-9)}$$

To subtract these fractions, we need a common denominator. Multiply each fraction by an appropriate form of 1 to match the denominators.

$$= \frac{8x^2}{56x^2 - 72x} - \frac{7x^2 - 9x + 21x - 27}{56x^2 - 72x}$$

Combine the numerators over the common denominator.

$$= \frac{8x^2 - (7x^2 + 12x - 27)}{56x^2 - 72x}$$
$$= \frac{8x^2 - 7x^2 - 12x + 27}{56x^2 - 72x}$$

Simplify the numerator.

$$= \frac{x^2 - 12x + 27}{56x^2 - 72x}$$

Example 6.38. Simplify the following expression.

$$\frac{6}{x-9} - \frac{7}{x^2 - 7x - 18}.$$

$$\frac{6}{x-9} - \frac{7}{x^2-7x-18} = \frac{6}{x-9} - \frac{7}{(x-9)(x+2)}$$

To subtract these fractions, we need a common denominator. The lowest common denominator here is $(x-9)(x+2)$.

$$= \frac{6(x+2)}{(x-9)(x+2)} - \frac{7}{(x-9)(x+2)}$$
$$= \frac{6x+12-7}{(x-9)(x+2)}$$

Combine the numerators over the common denominator.

$$= \frac{6x+5}{x^2-7x-18}$$

Example 6.39. Simplify the following expression.

$$\frac{2x}{x+4} - \frac{4x}{x-3}.$$

$$\frac{2x}{x+4} - \frac{4x}{x-3} = \frac{2x(x-3)}{(x+4)(x-3)} - \frac{4x(x+4)}{(x-3)(x+4)}$$

To subtract these fractions, we first find a common denominator, which is the product of the two denominators, $(x+4)(x-3)$.

$$= \frac{2x^2-6x-4x^2-16x}{(x+4)(x-3)}$$

Combine and simplify the numerators over the common denominator.

$$= \frac{-2x^2-22x}{x^2+x-12}$$

Example 6.40. Simplify the following expression.

$$\frac{3}{x-3} - \frac{4}{x^2+x-12}.$$

$$\frac{3}{x-3}-\frac{4}{x^2+x-12}=\frac{3}{x-3}-\frac{4}{(x-3)(x+4)}$$

To subtract these fractions, we find a common denominator. The lowest common denominator here is $(x-3)(x+4)$.

$$=\frac{3(x+4)}{(x-3)(x+4)}-\frac{4}{(x-3)(x+4)}$$
$$=\frac{3x+12-4}{(x-3)(x+4)}$$

Combine the numerators over the common denominator.

$$=\frac{3x+8}{x^2+x-12}$$

6.5 Solving Rational Equations

In this section, we will only deal with rational ***EQUATIONS***, we are free to use many other strategies so long as we do the same thing to both sides of the equation. We will be utilizing cross multiplication. However,

Theorem

Cross Multiplication

$$\text{If } \frac{a}{b}=\frac{c}{d}, \text{ then } ad=bc.$$

Proof. We start with the assumption that $\dfrac{a}{b}=\dfrac{c}{d}$

Note that the (LCD) of the two fractions here is bd. Multiplying both sides by the (LCD),

$$\frac{a}{b}=\frac{c}{d}$$

$$\Rightarrow bd\cdot\frac{a}{b}=\frac{c}{d}\cdot bd$$

$$\Rightarrow \frac{\cancel{b}d \cdot a}{\cancel{b}} = \frac{c \cdot b\cancel{d}}{\cancel{d}}$$
$$\Rightarrow ad = bc.$$

□

Example 6.41. Solve for c.

$$\frac{-12}{c+2} = \frac{-3}{2c}$$

$$\frac{-12}{c+2} = \frac{-3}{2c}$$

Using the cross multiply theorem.

$$(-12)(2c) = (c+2)(-3)$$
$$-24c = -3c - 6$$
$$6 = 24c - 3c$$
$$6 = 21c$$
$$\frac{6}{21} = c$$
$$\frac{2 \cdot 3}{7 \cdot 3} = c$$
$$\frac{2}{7} = c$$

Notice that $c \neq -2$ and $c \neq 0$ because

$$\frac{-12}{c+2} = \frac{-3}{2c}$$
$$\frac{-12}{0+2} \neq \frac{-3}{2 \cdot 0}$$
$$\frac{-12}{2} \neq \frac{-3}{0}$$

Likewise if $c = -2$ then,

$$\frac{-12}{-2+3} \neq \frac{-3}{2(-2)}$$
$$\frac{-12}{0} \neq \frac{-3}{4}$$

We call -2 and 0 **Extraneous Solutions**.

Example 6.42. Solve for x.

$$\frac{3}{2x-1} = \frac{5}{4x+3}$$

$$\frac{3}{2x-1} = \frac{5}{4x+3}$$

Using the cross multiply theorem.

$$3(4x+3) = 5(2x-1)$$

$$12x+9 = 10x-5$$

$$12x-10x = -5-9$$

$$2x = -14$$

$$x = -7$$

We can check our work by plugging back in $x = -7$ into our original equation.

$$3(4(-7)+3) = 5(2(-7)-1)$$
$$3(-28+3) = 5(-14-1)$$
$$3(-25) = 5(-15)$$
$$-75 = -75$$

Which is true thus our calculations were correct. The extraneous solutions are $\frac{1}{2}$ and $-\frac{3}{4}$ because when we plug these two values in we essentially are dividing by 0. To get the extraneous solutions, we just set the bottom of each denominator equal to 0 and solve for x.

Example 6.43. Solve for x.

$$\frac{4}{3x+2} = \frac{6}{5x-1}$$

We begin by using the cross multiply theorem.

$$\begin{aligned}\frac{4}{3x+2} &= \frac{6}{5x-1}\\ 4(5x-1) &= 6(3x+2)\\ 20x-4 &= 18x+12\\ 20x-18x &= 12+4\\ 2x &= 16\\ x &= 8\end{aligned}$$

We can check our work by plugging back in $x = 8$ into our original equation.

$$\begin{aligned}4(5(8)-1) &= 6(3(8)+2)\\ 4(40-1) &= 6(24+2)\\ 4(39) &= 6(26)\\ 156 &= 156\end{aligned}$$

Which is true, thus our calculations were correct. The extraneous solutions are $-\frac{2}{3}$ and $\frac{1}{5}$ because when we plug these two values in, we essentially are dividing by 0.

Example 6.44. Solve for x.

$$\frac{5}{x+4} + \frac{1}{x+4} = x-1$$

$$\frac{5}{x+4} + \frac{1}{x+4} = x-1$$

The extraneous solution is -4 since $\frac{5}{-4+4}+\frac{1}{-4+4}=\frac{5}{0}+\frac{1}{0}$

$$\begin{aligned}\frac{6}{x+4}&=x-1\\ \frac{6}{x+4}&=\frac{x-1}{1}\end{aligned}$$

Cross multiply.

$$\begin{aligned}6&=(x-1)(x+4)\\ 6&=x^2+3x-4\\ x^2+3x-4-6&=0\\ x^2+3x-10&=0\end{aligned}$$

By the tic tac toe.

$$(x+5)(x-2)=0$$

That is $x=-5$ or $x=2$.

Example 6.45. Solve for x.

$$\frac{2}{x}+\frac{1}{2}=\frac{7}{2x}$$

To eliminate the fractions, let's find a common denominator. The common denominator is $2x$.

$$\frac{2}{x}\cdot\frac{2}{2}+\frac{1}{2}\cdot\frac{x}{x}=\frac{7}{2x}$$

Simplifying the fractions.

$$\begin{aligned}\frac{4}{2x}+\frac{x}{2x}&=\frac{7}{2x}\\ \frac{4+x}{2x}&=\frac{7}{2x}\end{aligned}$$

Now we can cross multiply.

$$(4+x)\cdot(2x)=(2x)\cdot(7)$$

0 is an extraneous solution. It makes this equation untrue.

$$\begin{aligned}8x+2x^2&=14x\\ 2x^2+8x-14x&=0\\ 2x^2-6x&=0\\ 2x(x-3)&=0\\ x-3&=0\\ x&=3\end{aligned}$$

Example 6.46. Solve for y.

$$\frac{3}{y}-\frac{2}{3}=\frac{5}{3y}$$

To eliminate the fractions, let's find a common denominator. The common denominator is $3y$.

$$\frac{3}{y}\cdot\frac{3}{3}-\frac{2}{3}\cdot\frac{y}{y}=\frac{5}{3y}$$

Simplifying the fractions.

$$\frac{9}{3y} - \frac{2y}{3y} = \frac{5}{3y}$$
$$\frac{9-2y}{3y} = \frac{5}{3y}$$

Now we can cross multiply.

$$(9-2y)\cdot(3y) = (3y)\cdot(5)$$

0 is an extraneous solution. It makes this equation untrue.

$$27y - 6y^2 = 15y$$
$$6y^2 + 27y - 15y = 0$$
$$6y^2 + 12y = 0$$
$$6y(y+2) = 0$$
$$y+2 = 0$$
$$y = -2$$

Example 6.47. Solve for x and give all the extraneous solutions for
$$\frac{2}{x^2-x} = \frac{1}{x-1}.$$

An extraneous solutions is a solution to an equation that emerges from the process of solving the equation but is not a valid solution to the original equation.

$$\frac{2}{(x^2-x)} = \frac{1}{x-1}$$

Cross multiply to eliminate the fractions.

$$2(x-1) = (x^2-x)$$
$$2x-2 = x^2-x$$

$$0 = x^2 - x - 2$$

Factorizing the quadratic equation.

$$0 = (x-2)(x+1)$$

Hence, $x = 2$ or $x = -1$. Checking for extraneous solutions:

For $x = 2$,

$$\begin{aligned} \frac{2}{2^2-2} &= \frac{1}{2-1} \\ \frac{2}{2} &= \frac{1}{1} \\ 1 &= 1 \end{aligned}$$

This is true, so $x = 2$ is a valid solution.

For $x = -1$,

$$\begin{aligned} \frac{2}{(-1)^2-(-1)} &= \frac{1}{-1-1} \\ \frac{2}{1+1} &= -\frac{1}{2} \\ 1 &\neq -\frac{1}{2} \end{aligned}$$

This is not true, so $x = -1$ is an extraneous solution.

Example 6.48. A plant has two different potting crews. Crew 1 can plant a plant in 5 minutes and Crew 2 can plant the same type of plant in 6 minutes. How many days will it take if both crews are working together?

- Define x =the number of days it will take for both crews.
- $\frac{1}{5}$ is what crew 1 can complete in 1 minute.
- $\frac{1}{4}$ is what crew 2 can complete in 1 minute.

- $\frac{1}{x}$ is what both crews can complete together in 1 minute.

$$\frac{1}{5}+\frac{1}{6}=\frac{1}{x}$$

Get common denominator.

$$\begin{aligned}\frac{1}{5}\cdot\frac{6}{6}+\frac{1}{6}\cdot\frac{5}{5}&=\frac{1}{x}\\ \frac{6}{30}+\frac{5}{30}&=\frac{1}{x}\\ \frac{6+5}{30}&=\frac{1}{x}\\ \frac{11}{30}&=\frac{1}{x}\end{aligned}$$

Cross multiply.

$$\begin{aligned}11x&=30\\ x&=\frac{30}{11}\text{ days.}\end{aligned}$$

Example 6.49. Two pipes can fill a tank. Pipe A alone takes 8 hours to fill the tank, while Pipe B alone takes 10 hours. How long will it take for both pipes working together to fill the tank?

- Define $x =$ the number of hours it will take for both pipes working together.
- $\frac{1}{8}$ is the part of the tank Pipe A can fill in 1 hour.
- $\frac{1}{10}$ is the part of the tank Pipe B can fill in 1 hour.

- $\dfrac{1}{x}$ is the part of the tank both pipes can fill together in 1 hour.

$$\frac{1}{8}+\frac{1}{10}=\frac{1}{x}$$

Get common denominator.

$$\begin{aligned}\frac{1}{8}\cdot\frac{10}{10}+\frac{1}{10}\cdot\frac{8}{8}&=\frac{1}{x}\\ \frac{10}{80}+\frac{8}{80}&=\frac{1}{x}\\ \frac{10+8}{80}&=\frac{1}{x}\\ \frac{18}{80}&=\frac{1}{x}\end{aligned}$$

Simplify the fraction.

$$\frac{9}{40}=\frac{1}{x}$$

Cross multiply.

$$\begin{aligned}9x&=40\\ x&=\frac{40}{9}\text{ hours.}\end{aligned}$$

Example 6.50. Two observatories are tracking a comet. Observatory A, with its advanced equipment, would take 12 days to complete the tracking, while Observatory B, with less advanced equipment, would take 18 days. If both observatories collaborate from different locations, how many days will it take for them to complete the tracking together?

- Define $x =$ the number of days it will take for both observatories working together.
- $\frac{1}{12}$ is the part of the tracking Observatory A can complete in 1 day.
- $\frac{1}{18}$ is the part of the tracking Observatory B can complete in 1 day.
- $\frac{1}{x}$ is the part of the tracking both observatories can complete together in 1 day.

$$\frac{1}{12}+\frac{1}{18}=\frac{1}{x}$$

Get common denominator.

$$\begin{aligned}\frac{1}{12}\cdot\frac{18}{18}+\frac{1}{18}\cdot\frac{12}{12}&=\frac{1}{x}\\ \frac{18}{216}+\frac{12}{216}&=\frac{1}{x}\\ \frac{18+12}{216}&=\frac{1}{x}\\ \frac{30}{216}&=\frac{1}{x}\end{aligned}$$

Simplify the fraction.

$$\frac{5}{36}=\frac{1}{x}$$

Cross multiply.

$$5x=36$$

$$x = \frac{36}{5} \text{ days.}$$

Example 6.51. Astronomer Marco can analyze a set of star data twice as fast as astronomer Cliff. Working together, they can complete the analysis in 5 hours. How long would it take for Marco to complete the analysis working alone?

- Define $x =$ the number of hours it would take for Marco to complete the analysis alone.
- Since Marco can complete the task twice as fast as Cliff, it would take Cliff $2x$ hours to complete the same task alone.
- The part of the task Marco can complete in 1 hour is $\frac{1}{x}$.
- The part of the task Cliff can complete in 1 hour is $\frac{1}{2x}$.
- Together, they can complete the task in 5 hours, so the part they can complete together in 1 hour is $\frac{1}{5}$.

$$\frac{1}{x} + \frac{1}{2x} = \frac{1}{5}$$

Get common denominator.

$$\begin{aligned} \frac{2}{2x} + \frac{1}{2x} &= \frac{1}{5} \\ \frac{2+1}{2x} &= \frac{1}{5} \\ \frac{3}{2x} &= \frac{1}{5} \end{aligned}$$

Cross multiply.

$$
\begin{aligned}
3 \cdot 5 &= 2x \\
15 &= 2x \\
x &= \frac{15}{2} \text{ hours.}
\end{aligned}
$$

6.6 Additional Problems

1. Find the domains for the following rational function and simply when necessarily.

   (a) $\dfrac{9}{x^2+3x+2}$

   (b) $\dfrac{124356}{x^2+10x+25}$

   (c) $\dfrac{90}{2x^2+7x+6}$

   (d) $\dfrac{2x}{10x-2}$

   (e) $\dfrac{6x^{\frac{1}{2}}}{x^2-64}$

   (f) $\dfrac{x^4}{\frac{1}{2}x-7}$

2. Simplify the following completely.

   (a) $\dfrac{10-5x}{x^2-4x+4}$

   (b) $\dfrac{6-3y}{y^2-4y+4}$

3. Simplify the following completely by flipping and multiplying when necessity. Additionally, find the domain.

   (a) $\dfrac{9x^2-3x-2}{2x-6} \div \dfrac{9x^2-4}{x-3}$

   (b) $\dfrac{\frac{x+2}{x-5}}{\frac{x+2}{x}}$

   (c) $\dfrac{\frac{4x^2-4x-3}{3x-6}}{\frac{4x^2-9}{x+2}}$

   (d) $\dfrac{x^2-1}{x^2+10x+25} \div \dfrac{x-1}{2x+10}$

4. Simplify the following expression.

   (a) $\dfrac{\alpha}{\alpha^2-2\alpha+1} + \dfrac{\alpha}{\alpha^2-1}$

   (b) $\dfrac{7\phi^0-8\lambda^0+1^0}{(a-b)^0}$

   (c) $\dfrac{6}{x-5} + \dfrac{x-3}{x-2}$

   (d) $\dfrac{8}{5-6x} - \dfrac{x}{6x-5}$

5. Solve for the variable.

   (a) $\dfrac{-10}{x-2} = \dfrac{-6x}{x-4}$

   (b) $\dfrac{50}{t-2} - \dfrac{16}{t} = \dfrac{30}{t}$

   (c) $\dfrac{x^2+4}{x-1} = \dfrac{5}{x-1}$

   (d) $\dfrac{7}{y^2-8y+15} = \dfrac{1}{y-3} + \dfrac{2}{y-5}$

6. In the field of optics, the formula $\dfrac{1}{r} = \dfrac{1}{r_1} + \dfrac{1}{r_2}$ is used to find the effective focal length (r) of a system of lenses. The terms r_1 and r_2 represent the focal lengths of individual lenses in the system. This formula is derived from the thin lens equation and is used when two lenses are in close contact.

   (a) Solve for the effective focal length (r) in terms of r_1 and r_2.

7. A new type of blood analyzer can process a batch of samples in 3 hours. An older analyzer can process the same batch of samples in 4 hours. How long will it take to process the batch of samples if both analyzers are working together at the same time?

8. A new beaver can build a dam in 5 days. An older beaver can build the same dam in 7 days. How long will it take to build the dam if both beavers are working together at the same time?

7 Radical Expressions & Equations

7.1 *n*th Roots

Symbols are a brilliant means of assigning significance to seemingly unrelated objects, such as squiggles on a surface. Therefore, as humans, we possess a fondness for establishing universality and abstraction in these symbols. In fact, these words that you are reading have meaning. The alphabet is something that is meaningless [18], however we humans give those letters "symbols" meaning. Note, mathematical notion does not just spring into existence. Just like with any other notion, there was a mathematical process.

$$abc\ldots z$$

Here are some other symbols.

[18]"It is clear that we are just an advanced breed of primates on a minor planet orbiting around a very average star, in the outer suburb of one among a hundred billion galaxies. BUT, ever since the dawn of civilization people have craved for an understanding of the underlying order of the world. There ought to be something very special about the boundary conditions of the universe. And what can be more special than that there is no boundary? And there should be no boundary to human endeavor. We are all different. However bad life may seem, there is always something you can do, and succeed at. While there is life, there is hope. " - Stephen Hawking. But how truly special are we? How significant is Earth in the vast expanse of the universe? From an astronomical standpoint, if we were to eliminate our solar system, the Milky Way galaxy would remain unaffected. Nonetheless, it is the significance we attach to our existence during our limited time that holds meaning. I think that the meaning to life is what we choose it to be. Ay?

Example 7.1.

$\bigstar$ = Star
$\diamond$ = Diamond
$\checkmark$ = Check Mark
© = Copyright
\$ = Money
 = Ghost!!
 = Cloud
 = You should know
♌ = Leo sign, because I am a Leo
☽ = Moon
$\ShootingStar$ = Shooting Star
⚛ = Atom

Mathematicians have assigned their own meaning to symbols. We know that + means addition, − means subtract, etc. Let's introduce a new symbol.

Definition

The Square Root of x

For a real number x, $\sqrt[2]{x} = \sqrt{x} = x^{\frac{1}{2}}$ is the number that when squared is x. We call it "the square root of x".

However, this definition arises to some complications [19]. Let me show you.

[19] I wonder what aliens would think of our mathematics. Mathematics shares a strong correlation with logic, so if our logic were slightly off, then it would undermine everything. Perhaps there is a barrier in our mathematics that hinders our success in certain aspects. I say this because all subjects stem from math, and if our math is not as efficient, then it could impede our ability to accomplish certain things or certainly set us back.

The idea that aliens would have a perspective on our mathematics is an interesting concept. Mathematics is considered a universal language that is based

Example 7.2. $\sqrt{36} = \sqrt{6^2} = 6$. But the following also holds true $\sqrt{36} = \sqrt{(-6)^2} = -6$.

Now, why is this a problem? Remember that we love to work with functions, but this is not a function as it has two outputs. However, we can tweak it to become a function by only taking its positive output.

Definition

Square Root functions

$$\sqrt{x^2} = |x| = x$$

Hence, we only take the positive value making it a well define function. How does $f(x) = \sqrt[2]{x}$ look like when we graph it? Lets plot points.

Example 7.3. Consider the function $y = \sqrt{x}$ or $f(x) = x^{\frac{1}{2}}$.

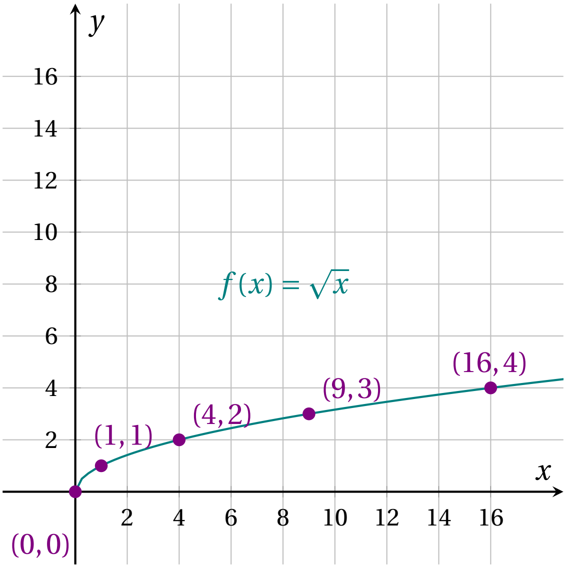
Notice that we can plug in x values and get an output "y or $f(x)$" values. Together we can plot points a create a curve.

$$\begin{aligned} f(-9) &= \text{Undefined} \\ f(-4) &= \text{Undefined} \\ f(-1) &= \text{Undefined} \\ f(0) &= \sqrt{0} = \sqrt{0^2} = 0 \\ f(1) &= \sqrt{1} = \sqrt{1^2} = 1 \\ f(4) &= \sqrt{4} = \sqrt{2^2} = 2 \\ f(9) &= \sqrt{9} = \sqrt{3^2} = 3 \end{aligned}$$

Putting them into a table we see.

on logical principles and concepts that are independent of culture or language. If aliens were to encounter our mathematics, they might find similarities or differences compared to their own mathematical systems. This could provide insights into their level of logical understanding. Okay, I will stop talking.

x	$f(x)$
-9	Undefined
-4	Undefined
-1	Undefined
0	0
1	1
4	2
9	3



Notice that the domain is $[0, \infty)$. The curve starts at the origin $(0,0)$ and goes to the right all the way to infinity.

Example 7.4. Use the square root property.

$$\sqrt{36}$$

$$\sqrt{36} = \sqrt{6^2}$$

By the definition of square root functions.

$$= 6$$

Example 7.5. Use the square root property.

$$\sqrt{16x^2}$$

$$\begin{aligned}\sqrt{16x^2} &= \sqrt{(4x)^2}\\ &= \sqrt{4^2x^2}\end{aligned}$$

By the definition of square root functions.

$$\begin{aligned}&= |4x|\\ &= 4|x|\end{aligned}$$

Example 7.6. Use the square root property.

$$\sqrt{x^2+6x+9}$$

$$\sqrt{x^2+6x+9} = \sqrt{(x+3)(x+3)}$$

By the definition of square root functions.

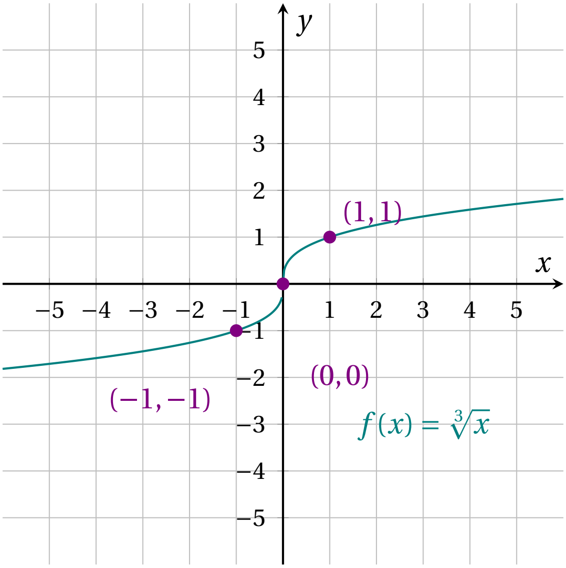
$$\begin{aligned}&= \sqrt{(x+3)^2}\\ &= |x+3|\end{aligned}$$

We can extend this property to any nth root. Lets try the cube root.

Definition

The Cube Root of x

For a real number x, $\sqrt[3]{x} = x^{\frac{1}{3}}$ is the number that when cubed is x. We call it "the cube root of x".



$\sqrt[3]{x}$ is already a function because every value both negative and positive has a signal output.

Example 7.7. Consider $\sqrt[3]{8} = \sqrt[3]{2^3} = 2$. You might think that -2 is a possible answer. It is not and we can see this via a table. The only way to possibly get -2 is if $f(-8) = \sqrt[3]{-8} = \sqrt[3]{(-2)(-2)(-2)} = \sqrt[3]{(-2)^3} = -2$.

x	$f(x)$
-27	-3
-8	-2
-1	-1
0	0
1	1
8	2
27	3
64	4
125	5

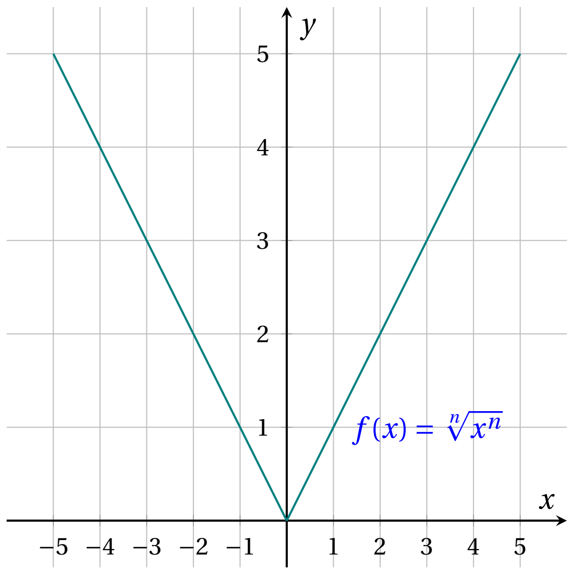
Now let's discuss the nth possibility.

Definition

The nth Root of x^n

$$\sqrt[n]{x^n} = \begin{cases} |x| & \text{if } n \text{ is even} \\ x & \text{if } n \text{ is odd.} \end{cases}$$

There are two specific graphs for this function. When x is an ***odd*** number the the graph looks something like this.



Lets use what we know to simplify the following.

Example 7.8. Use the definition to simplify the following. Assume all variables are positive.

$$\begin{aligned}\sqrt{100} &= \sqrt{10^2} \\ &= 10\end{aligned}$$

Example 7.9. Use the definition to simplify the following. Assume all variables are positive.

$$\begin{aligned}-\sqrt{100} &= -\sqrt{10^2} \\ &= -10\end{aligned}$$

Example 7.10. Use the definition to simplify the following. Assume all variables are positive.

$$\sqrt{-100} = \text{ No real solution.}$$

There is a way to solve these using complex numbers. We will talk more about those in the future.

Example 7.11. Use the definition to simplify the following. Assume all variables are positive.

$$\begin{aligned}\sqrt[3]{27} &= \sqrt[3]{3\cdot 3\cdot 3}\\ &= \sqrt[3]{3^3}\\ &= 3\end{aligned}$$

Example 7.12. Use the definition to simplify the following. Assume all variables are positive.

$$\begin{aligned}\sqrt{\frac{x^4}{16}} &= \sqrt{\frac{x^2\cdot x^2}{4\cdot 4}}\\ &= \sqrt{\frac{(x^2)^2}{4^2}}\\ &= \frac{x^2}{4}\end{aligned}$$

Example 7.13. Use the definition to simplify the following. Assume all variables are positive.

$$\sqrt[n]{x^n} = |x|$$

7.2 Radical Functions

Now, lets investigate radical functions. In specifically we will look at their domains and graphs.

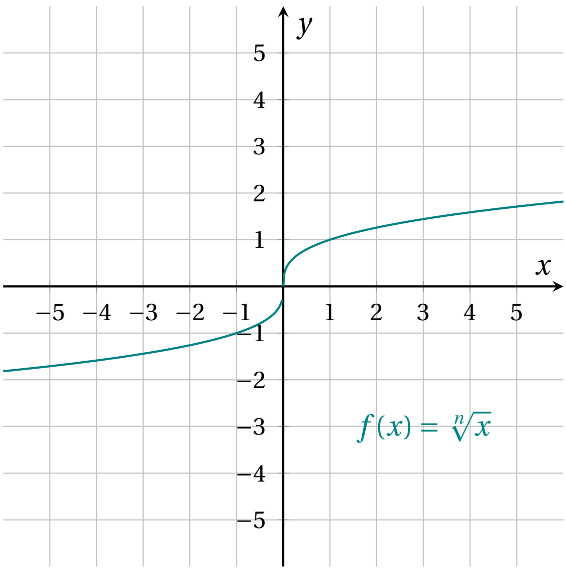
Before I begin, I would like to introduce two definitions.

Definition

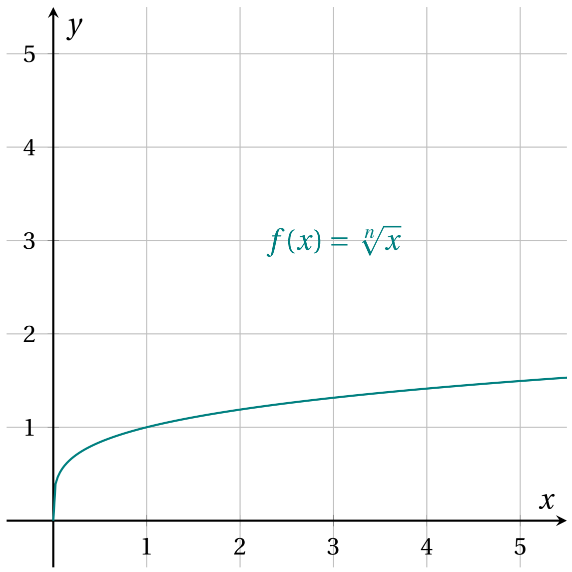
The nth Root of x

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

You get two types of graphs depending on whether n is ***even*** or ***odd***. When n is ***odd*** we get something similar to this graph.



Important: The domain for any odd n is always $(-\infty, \infty)$. For example, $f(x) = \sqrt[3]{x}$ graph is very similar to this and has the domain of $(-\infty, \infty)$. Now, if n is ***even*** we get the following type of graph.



There are two ways of graphing radicals. One way, as demonstrate in the last section is via a table. There is an easier but less accurate way. This method known as the Horizontal Shift also depends on less complicated functions. However, you can always sketch a radical graph once taught. Allow me to explain why by an example.

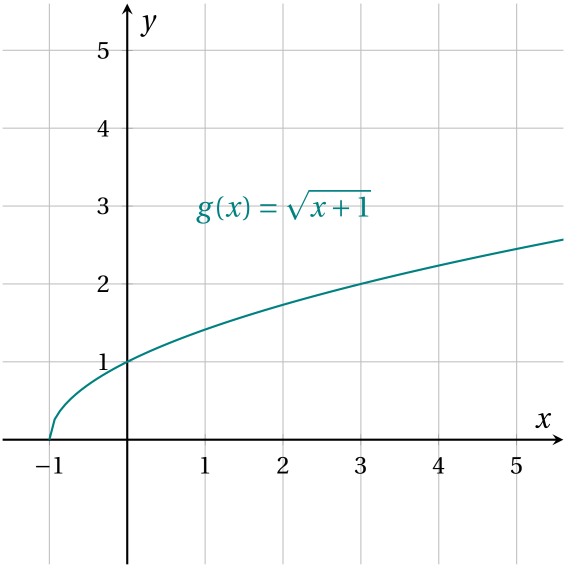
Example 7.14. Consider the function $g(x) = \sqrt{x+1}$

$$x + 1 \geq 0$$

We can claim this because we defined square roots to always be positive.

$$x \geq -1$$

This also shows that our domain will be $[-1, \infty)$.

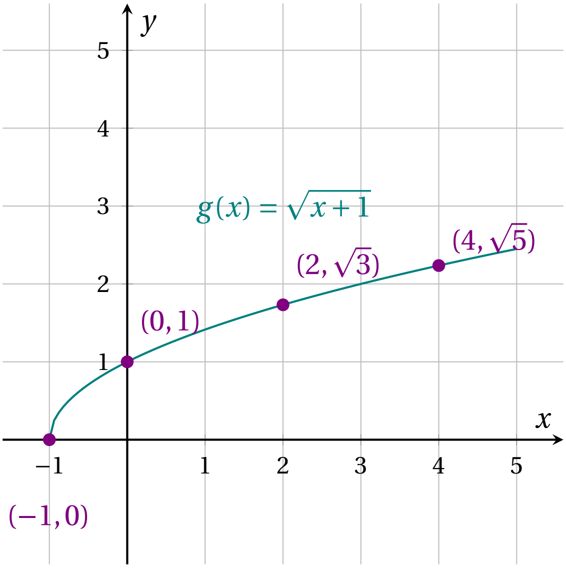


Notice that this looks exactly like the function $f(x) = \sqrt{x}$. What is the difference? The only difference is that it begins at $x = -1$. This will work for any $g(x) = \sqrt{x+a}$ where a is any number. If we had $g(x) = \sqrt{x+7}$ then we would start our graph at $x = -7$ and draw a $g(x) = \sqrt{x}$ starting at $x = -7$.

To get the exact points we would need to utilize a table.

x	$g(x)$
-1	$\sqrt{0} = 0$
0	$\sqrt{1} = 1$
1	$\sqrt{2}$
2	$\sqrt{3}$
3	$\sqrt{4} = 2$
4	$\sqrt{5}$
5	$\sqrt{6}$

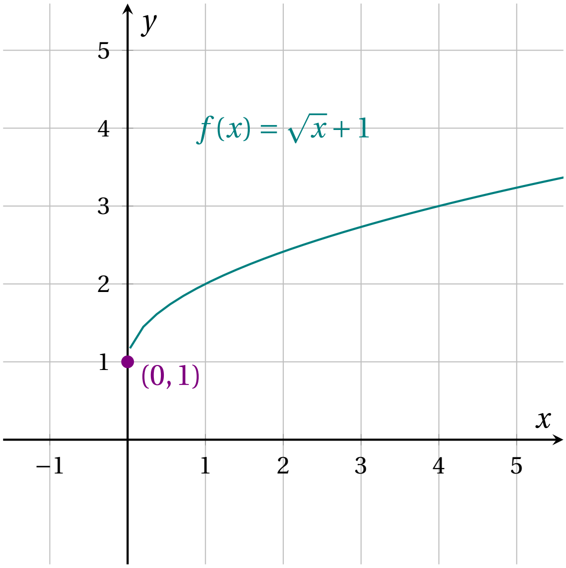
Lets now re-graph the function $g(x) = \sqrt{x+1}$ using the table.



It turns out that the range is $[0, \infty)$.

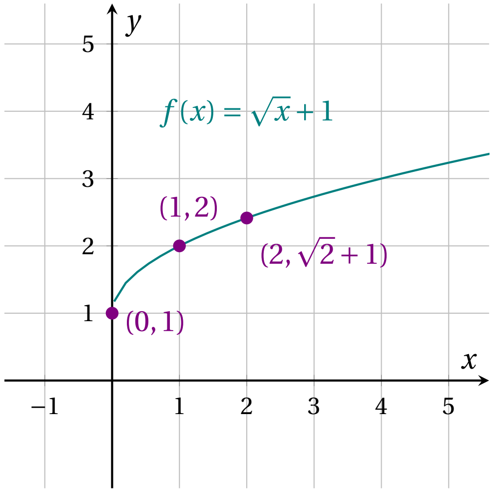
Example 7.15. Graph, give the domain and range of the following function $f(x) = \sqrt{x} + 1$

There is an easy way to do this that is again less accurate. Let's go ahead and try that. We already know how the function $\sqrt{x}$ looks like. However, our function also includes a $+1$. Do you remember the $y = mx + b$ formula? Notice that $y = 1 \cdot \sqrt{x} + 1$. That is, $b = 1$ and we just move up 1 on the y axis. After this we can just draw a $\sqrt{x}$ starting at the point $(0, 1)$.



As mentioned the only guaranteed points are $(0,1)$. To be more accurate, lets do a table. Here is the finalized table.

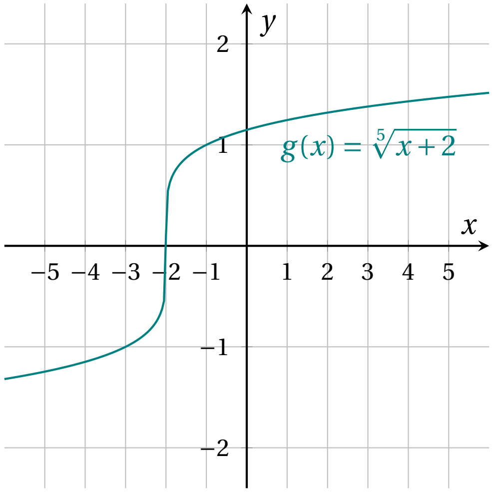
x	$f(x)$
0	$\sqrt{0}+1=1$
1	$\sqrt{1}+1=2$
2	$\sqrt{2}+1$
3	$\sqrt{3}+1$
4	$\sqrt{4}+1=3$



Let's do some quick one without the table. Do understand that, we do loose preciseness when doing this.

Example 7.16. Graph and give the domain of $g(x) = \sqrt[5]{x+2}$.

Here is the short cut to these problems. Inside the radical we have $x+2$. Set this equal to 0 and we get $x = -2$. Therefore, we start at $(-2, 0)$ and draw a graph similar to 7.2 to get something like this.



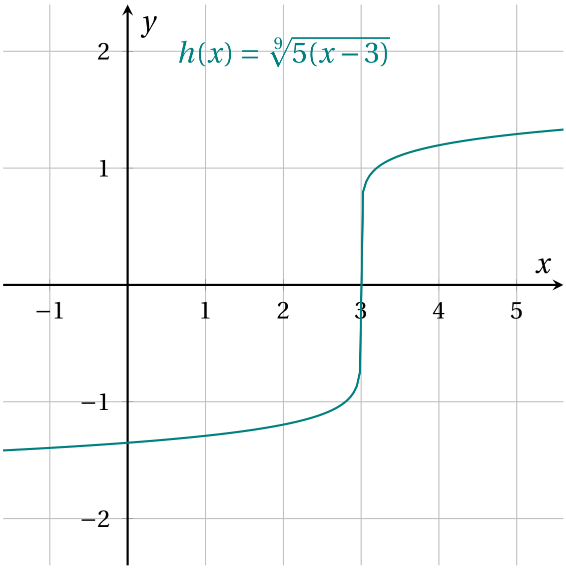
As mention previously, the domain for any function with odd root is $(-\infty, \infty)$

Example 7.17. Graph and give the domain of $h(x) = \sqrt[9]{(5x-15)}$.

To analyze this function, we first recognize that it is a ninth root function. T We find the starting point of the graph by setting the inside of the radical to zero.

$$\begin{aligned} 5x - 15 &= 0 \\ 5(x-3) &= 0 \\ x - 3 &= 0 \\ x &= 3 \end{aligned}$$

So, the graph will start at the point $(3,0)$. The shape of the graph will be similar to other odd root functions, smoothly curving and extending to all real numbers in both directions from $(3,0)$.



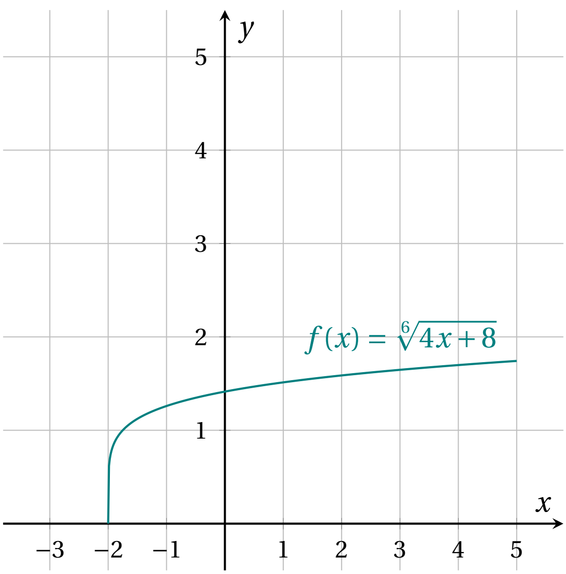
The domain of such a function includes all real numbers, as a ninth root can be taken of any real number $h(x)$ is $(-\infty, \infty)$.

Example 7.18. Graph and give the domain of $f(x) = \sqrt[6]{4x+8}$.

To analyze this function, we recognize that it is a sixth root function. Unlike odd root functions, even root functions like this one are defined only for values that make the expression inside the root non-negative.

$$\begin{aligned}
4x+8 &\geq 0 \\
4(x+2) &\geq 0 \\
x+2 &\geq 0 \\
x &\geq -2
\end{aligned}$$

This inequality indicates that the graph will start at the point $(-2, 0)$ and extend to the right, as the function is only defined for $x \geq -2$.



The domain of $f(x)$ is $[-2, \infty)$, as the sixth root is only defined for non-negative values inside the radical.

7.3 Radical Exponents

Remember that we defined x^n to be the following.

Theorem

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{\text{n times.}}$$

In the last sections we defined some important properties of radical exponents. For example, we used properties of exponents to conclude that $x^0 = 1$ for any x.

Example 7.19. Show that $x^{\frac{1}{2}} = \sqrt{x}$.

We will show that this is true by squaring both and showing that they are both x.

$$\sqrt{x} \stackrel{?}{=} x^{\frac{1}{2}}$$

$$(\sqrt{x})^2 \stackrel{?}{=} (x^{\frac{1}{2}})^2$$

$$\sqrt{x} \cdot \sqrt{x} \stackrel{?}{=} x^{\frac{1}{2} \cdot 2}$$

$$\sqrt{x^2} \stackrel{?}{=} x^{\frac{2}{2}}$$

$$x = x^1$$

$$x = x$$

Therefore, $x^{\frac{1}{2}} = \sqrt{x}$. Similarly this works for all n.

Theorem

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$

For any integer n. Hence, this holds true for any whole number.

Example 7.20. Change from rational exponent to radical notation, or vice versa, and simplify where possible for $\alpha^{\frac{1}{7}}$

$$\alpha^{\frac{1}{7}} = \sqrt[7]{\alpha}$$

Example 7.21. Change from rational exponent to radical notation, or vice versa, and simplify where possible for $9^{\frac{1}{4}}$

$$9^{\frac{1}{4}} = \sqrt[4]{9}$$

Thus far, we have made sense of having an exponent of $\frac{1}{n}$. However, what do you think happens if the numerator isn't 1? What if we have something like $x^{m/n}$? Let's extend this property.

Theorem

$$x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$$

Example 7.22. Find $\sqrt[4]{16^2}$.

$$\sqrt[4]{16^2} = \sqrt[4]{2^8}$$

Using the above theorem.

$$\begin{aligned} &= 2^{\frac{8}{4}} \\ &= 2^2 \\ &= 4 \end{aligned}$$

Example 7.23. Find $64^{1/3}$

$$64^{1/3} = (2^6)^{1/3}$$

Expressing 64 as a power of 2.

$$= 2^{6\cdot\frac{1}{3}}$$

Applying the exponent rule for powers.

$$= 2^2$$

Simplifying the exponent.

$$= 4$$

Example 7.24. Find $27^{2/3}$

$$27^{2/3} = (3^3)^{2/3}$$

Expressing 27 as a power of 3.

$$= 3^{3\cdot\frac{2}{3}}$$

Applying the exponent rule for powers.

$$= 3^2$$

Simplifying the exponent.

$$= 9$$

Lets do some more exponential problem. However, these are a bit different. We won't be getting a number because we will be working with variables.

Example 7.25. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

$$\begin{aligned} x^2 \cdot x^3 &= x^{2+3} \\ &= x^5 \end{aligned}$$

Example 7.26. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

$$\frac{w^9}{w^3} = w^{9-3}$$

Subtracting the exponents.

$$= w^6$$

Example 7.27. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

$$\frac{w^{5/6}}{w^{3/2}} = w^{5/6-3/2}$$

Subtracting the exponents.

$$= w^{\frac{5}{6}-\frac{3}{2}\cdot\frac{3}{3}}$$

Rewriting with a common denominator.

$$= w^{\frac{5}{6}-\frac{9}{6}}$$

Calculating the fractions.

$$= w^{-\frac{4}{12}}$$

Simplifying the fractions.

$$\begin{aligned} &= w^{-\frac{1}{3}} \\ &= \frac{1}{w^{\frac{1}{3}}} \\ &= \frac{1}{\sqrt[3]{w}} \end{aligned}$$

Example 7.28. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

$$(a^2)^5 = a^{2\cdot 5}$$

Applying the exponent to the base and exponent.

$$= a^{10}$$

Example 7.29. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

$$\left(a^{5/2}\right)^{11/7} = a^{\frac{5}{2}\cdot\frac{11}{7}}$$

Applying the exponent to the base and exponent.

$$= a^{\frac{55}{14}}$$

7.4 Simplifying Radical Expressions

Let's begin with an important property that we will practice in this section.

Theorem

Multiplication Property of Radicals

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \qquad (\text{if } \sqrt[n]{a},\ \sqrt[n]{b} \in \mathbb{R})$$

Of course this is a fact, but don't just take my word for it. Here is the proof.

Proof.

$$\begin{aligned}\sqrt[n]{ab} &= (ab)^{1/n}\\ &= a^{1/n} \cdot b^{1/n}\\ &= \sqrt[n]{a} \cdot \sqrt[n]{b}\end{aligned}$$

□

Here is another property. The proof is similar to the last one.

Theorem

Division Property of Radicals

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} \qquad \text{(if } \sqrt[n]{a},\ \sqrt[n]{b} \in \mathbb{R}\text{)}$$

Example 7.30. Simplify the radical expression.

$$\sqrt{75}$$

$$\sqrt{75} = \sqrt{5 \cdot 5 \cdot 3}$$

Break 75 into a product of primes.

$$= \sqrt{5^2 \cdot 3}$$

Multiplication Property of Radicals.

$$= \sqrt{5^2}\sqrt{3}$$
$$= 5\sqrt{3}$$

Example 7.31. Simplify the radical expression.

$$\sqrt{48}$$

$$\sqrt{48} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3}$$

Break 48 into a product of primes.

$$= \sqrt{2^4 \cdot 3}$$

Multiplication Property of Radicals.

$$= \sqrt{2^4}\sqrt{3}$$

Product Property of Radicals.

$$= \sqrt{2^2 \cdot 2^2}\sqrt{3}$$
$$= \sqrt{2^2}\sqrt{2^2}\sqrt{3}$$

Multiplication Property of Radicals.

$$= 2 \cdot 2\sqrt{3}$$
$$= 4\sqrt{3}$$

Example 7.32. Simplify the radical expression.

$$\sqrt{60x^5}$$

$$\sqrt{60x^5} = \sqrt{2 \cdot 2 \cdot 3 \cdot 5 \cdot x^5}$$

Break $60x^5$ into a product of primes and factors of x.

$$= \sqrt{2^2 \cdot 3 \cdot 5 \cdot x^4 \cdot x}$$

Rewrite using powers.

$$= \sqrt{2^2}\sqrt{3}\sqrt{5}\sqrt{x^4}\sqrt{x}$$

Product property of radicals.

$$= 2\sqrt{3}\sqrt{5} \cdot x^2\sqrt{x}$$

Combine and simplify the radicals.

$$= 2x^2\sqrt{15x}$$

Example 7.33. Simplify the radical expression.

$$\sqrt[3]{\frac{54}{16}}$$

$$\sqrt[3]{\frac{54}{16}} = \frac{\sqrt[3]{54}}{\sqrt[3]{16}}$$

Rewriting the cube root as a fraction.

$$= \frac{\sqrt[3]{27 \cdot 2}}{\sqrt[3]{2^4}}$$

Prime factorizing numerator and denominator.

$$= \frac{\sqrt[3]{3^3} \cdot \sqrt[3]{2}}{\sqrt[3]{2^3} \cdot \sqrt[3]{2}}$$

Factoring out a $\sqrt[3]{2}$.

$$= \frac{\sqrt[3]{3^3} \cdot \cancel{\sqrt[3]{2}}}{\sqrt[3]{2^3} \cdot \cancel{\sqrt[3]{2}}}$$

Canceling like terms.

$$= \frac{\sqrt[3]{3}}{\sqrt[3]{2}}$$

Simplifying cube roots.

$$= \frac{3}{2}$$

There are radicals that can not be simplified.

Example 7.34. Simplify the radical expression.

$$\sqrt[3]{\frac{128}{27}}$$

$$\sqrt[3]{\frac{128}{27}} = \frac{\sqrt[3]{128}}{\sqrt[3]{27}}$$

Rewriting the cube root as a fraction.

$$= \frac{\sqrt[3]{2^7}}{\sqrt[3]{3^3}}$$

Prime factorizing numerator and denominator.

$$= \frac{\sqrt[3]{2^3} \cdot \sqrt[3]{2^3} \cdot \sqrt[3]{2}}{\sqrt[3]{3^3}}$$

Simplifying cube roots.

$$= \frac{2 \cdot 2 \cdot \sqrt[3]{2}}{3}$$

$$= \frac{4 \cdot \sqrt[3]{2}}{3}$$

Example 7.35. Simplify the radical expression.

$$\sqrt[3]{\frac{125x^3}{8y^3}}$$

$$\sqrt[3]{\frac{125x^3}{8y^3}} = \frac{\sqrt[3]{125x^3}}{\sqrt[3]{8y^3}}$$

Rewriting the cube root as a fraction.

$$= \frac{\sqrt[3]{5^3x^3}}{\sqrt[3]{2^3y^3}}$$

Prime factorizing numerator and denominator, and rewriting variables as powers.

$$= \frac{5x\sqrt[3]{1}}{2y\sqrt[3]{1}}$$

Simplifying cube roots.

$$= \frac{5x}{2y}$$

Example 7.36. Simplify the radical expression.

$$\sqrt[3]{\frac{16x^3}{50x}}$$

$$\sqrt[3]{\frac{16x^3}{50x}} = \frac{\sqrt[3]{16x^3}}{\sqrt[3]{50x}}$$

Rewriting the cube root as a fraction.

$$= \frac{\sqrt[3]{2^4x^3}}{\sqrt[3]{2\cdot 5^2x}}$$

Prime factorizing numerator and denominator.

$$= \frac{\sqrt[3]{2^3}\cdot \cancel{\sqrt[3]{2}}\cdot \sqrt[3]{x^3}}{\cancel{\sqrt[3]{2}}\cdot \sqrt[3]{5^2x}}$$

Cross out like terms and simplify.

$$= \frac{2x}{\sqrt[3]{25x}}$$

Example 7.37. Simplify the radical expression.

$$\sqrt[5]{27}$$

$\sqrt[5]{27} = \sqrt[5]{3^3}$ is already simplified because the power is less than the index. LIKEWISE, $\sqrt[5]{x^3}$ is simplified because the power is less than the index.

Example 7.38. Simplify the radical expression.

$$\sqrt[3]{x^7}$$

$$\sqrt[3]{x^7} = \sqrt[3]{(x^2)^3\cdot x}$$

Rewriting x^7 as $(x^2)^3 \cdot x$

$$= \sqrt[3]{(x^2)^3} \cdot \sqrt[3]{x}$$

Using the product property of cube roots.

$$\begin{aligned} &= (x^2) \cdot \sqrt[3]{x} \\ &= x^2 \sqrt[3]{x} \end{aligned}$$

Example 7.39. Simplify the radical expression.

$$\sqrt[5]{-2^5 \cdot \delta^5}$$

$$\sqrt[5]{-2^5 \cdot \delta^5} = \sqrt[5]{-2^5} \sqrt[5]{\delta^5}$$

Since 5 is odd we can take the negative outside.

$$\begin{aligned} &= -\sqrt[5]{2^5} \sqrt[5]{\delta^5} \\ &= -2 \sqrt[5]{\delta^5} \\ &= -2\delta \end{aligned}$$

Example 7.40. Simplify the radical expression.

$$\sqrt[3]{6^3 x^3 x^2 x^2 u^3 u^3 u}$$

$$\sqrt[3]{6^3 x^3 x^2 x^2 u^3 u^3 u} = \sqrt[3]{6^3 x^3 x^3 x u^3 u^3 u}$$

We want to group exponents with powers of 3 to cancel with the radical.

$$= \sqrt[3]{6^3}\sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{x}\sqrt[3]{u^3}\sqrt[3]{u^3}\sqrt[3]{u}$$
$$= 6xxuu\sqrt[3]{xu}$$
$$= 6x^2u^2\sqrt[3]{xu}$$

Example 7.41. Simplify the radical expression.

$$\sqrt{1000f^5ff^2fx^2\delta^2\delta^2\delta^2}$$

$$\begin{aligned}
\sqrt{1000f^5ff^2fx^2\delta^2\delta^2\delta^2} &= \sqrt{1000f^6f^2fx^2\delta^2\delta^2\delta^2}\\
&= \sqrt{1000f^2f^2f^2f^2fx^2\delta^2\delta^2\delta^2}\\
&= \sqrt{1000}\sqrt{f^2}\sqrt{f^2}\sqrt{f^2}\sqrt{f^2}\sqrt{f}\sqrt{x^2}\sqrt{\delta^2}\sqrt{\delta^2}\sqrt{\delta^2}\\
&= ffffx\delta\delta\delta\sqrt{10^2 10f}\\
&= 10f^4x\delta^3\sqrt{10f}
\end{aligned}$$

Example 7.42. Simplify the radical expression.

$$\sqrt{9x^2-12x+4}+\sqrt{9x^2-24x+16}$$

$$\begin{aligned}
\sqrt{9x^2-12x+4}+\sqrt{9x^2-24x+16} &= \sqrt{(3x-2)^2}+\sqrt{(3x-4)^2}\\
&= (3x-2)+(3x-4)\\
&= 6x-6
\end{aligned}$$

Example 7.43. Simplify the radical expression.

$$\sqrt{w^2-2nw+n^2}+\sqrt{4\gamma^2+4\gamma\delta+\alpha^2}$$

$$\sqrt{w^2-2nw+n^2}+\sqrt{4\gamma^2+4\gamma\delta+\alpha^2} = \sqrt{(w-n)^2}+\sqrt{(2\gamma+\alpha)^2}$$

$$= (w - n) + (2\gamma + \alpha)$$
$$= w - n + 2\gamma + \alpha$$

7.5 Radical Operations

Now we discuss Radical Addition, Subtraction, and Multiplication.

Example 7.44. Simplify the radical expression.

$$\sqrt{27} - \sqrt{12} + \sqrt{50}$$

$$\sqrt{27} - \sqrt{12} + \sqrt{50} = \sqrt{9 \cdot 3} - \sqrt{4 \cdot 3} + \sqrt{25 \cdot 2}$$

Factorize each number under the square root into a product of a perfect square and another number.

$$= \sqrt{9} \cdot \sqrt{3} - \sqrt{4} \cdot \sqrt{3} + \sqrt{25} \cdot \sqrt{2}$$

Take the square root of the perfect squares 9, 4, and 25 and keep the other factors under the radical.

$$= 3\sqrt{3} - 2\sqrt{3} + 5\sqrt{2}$$

Perform the multiplication inside each term.

$$= (3 - 2)\sqrt{3} + 5\sqrt{2}$$

Combine like terms.

$$= \sqrt{3} + 5\sqrt{2}.$$

Example 7.45. Simplify the radical expression.

$$4\sqrt{63} + \sqrt{28} + 10\sqrt{7}$$

$$\begin{aligned}4\sqrt{63}+\sqrt{28}+10\sqrt{7} &= 4\sqrt{7\cdot 9}+\sqrt{4\cdot 7}+10\sqrt{7}\\ &= 4\sqrt{7\cdot 3^2}+\sqrt{2^2\cdot 7}+10\sqrt{7}\end{aligned}$$

Break the numbers into a product of primes.

$$= 4\sqrt{7}\sqrt{3^2}+\sqrt{2^2}\sqrt{7}+10\sqrt{7}$$

Using the product property of square roots.

$$\begin{aligned}&= 4\cdot 3\sqrt{7}+2\sqrt{7}+10\sqrt{7}\\ &= 12\underline{\sqrt{7}}+2\underline{\sqrt{7}}+10\underline{\sqrt{7}}\end{aligned}$$

Notice that they are all multiplied by the same radicand. Thus, we can add them all together.

$$\begin{aligned}&= (12+2+10)\sqrt{7}\\ &= 24\sqrt{7}.\end{aligned}$$

Example 7.46. Simplify the radical expression.

$$2\sqrt{18}-3\sqrt{8}+4\sqrt{50}$$

In this example, we have three terms with square roots, but we can't combine them further since they have different radicands (numbers inside the square roots).

$$2\sqrt{18}-3\sqrt{8}+4\sqrt{50} = 2\sqrt{9\cdot 2}-3\sqrt{4\cdot 2}+4\sqrt{25\cdot 2}$$

Break down each radical into prime factors.

$$= 2\sqrt{3^2}\cdot\sqrt{2}-3\sqrt{2^2}\cdot\sqrt{2}+4\sqrt{5^2}\cdot\sqrt{2}$$

Simplify the square roots of perfect squares.

$$= 2 \cdot 3 \cdot \sqrt{2} - 3 \cdot 2 \cdot \sqrt{2} + 4 \cdot 5 \cdot \sqrt{2}$$

Perform multiplication inside each term.

$$= 6\sqrt{2} - 6\sqrt{2} + 20\sqrt{2}$$

Combine like terms.

$$= 0\sqrt{2} + 20\sqrt{2}$$

Simplify the expression.

$$= 20\sqrt{2}.$$

Example 7.47. Simplify the radical expression.

$$4\sqrt{48} + \sqrt{12} - \sqrt{27}$$

$$4\sqrt{48} + \sqrt{12} - \sqrt{27} = 4\sqrt{16 \cdot 3} + \sqrt{4 \cdot 3} - \sqrt{9 \cdot 3}$$

Factorize each number under the square root into a product of a perfect square and another number.

$$= 4\sqrt{16} \cdot \sqrt{3} + \sqrt{4} \cdot \sqrt{3} - \sqrt{9} \cdot \sqrt{3}$$

Take the square root of the perfect squares 16, 4, and 9 and keep the other factors under the radical.

$$= 4 \cdot 4 \cdot \sqrt{3} + 2 \cdot \sqrt{3} - 3 \cdot \sqrt{3}$$

Perform the multiplication inside each term.

$$= 16\sqrt{3} + 2\sqrt{3} - 3\sqrt{3}$$

Combine like terms.

$$= (16 + 2 - 3)\sqrt{3}$$

$$= 15\sqrt{3}.$$

Example 7.48. Simplify the radical expression.

$$\sqrt{54} - \sqrt{75}$$

$$\sqrt{54} - \sqrt{75} = \sqrt{9 \cdot 6} - \sqrt{25 \cdot 3}$$

Factorize each number under the radical into a product of a perfect square and another number.

$$= \sqrt{9} \cdot \sqrt{6} - \sqrt{25} \cdot \sqrt{3}$$

Take the square root of the perfect squares (9) and (25) and keep the other factors under the radical.

$$= 3 \cdot \sqrt{6} - 5 \cdot \sqrt{3}$$

Simplify the expression to its final form.

Example 7.49. Simplify the radical expression.

$$3xy\sqrt{12x} - 2\sqrt{27y^2}$$

To simplify this expression, we can simplify the numbers inside the square roots.

$$3xy\sqrt{12x} - 2\sqrt{27y^2} = 3xy\sqrt{4 \cdot 3x} - 2\sqrt{9 \cdot 3y^2}$$

Factorize the numbers under the radicals into perfect squares and their products.

$$= 3xy\sqrt{4} \cdot \sqrt{3x} - 2\sqrt{9} \cdot \sqrt{3y^2}$$

Take the square roots of the perfect squares 4 and 9 and keep the other factors under the radical.

$$= 3xy \cdot 2 \cdot \sqrt{3x} - 2 \cdot 3 \cdot \sqrt{3y^2}$$

Simplify each term by performing the multiplication.

$$= 6xy\sqrt{3x} - 6\sqrt{3}y$$

The final simplified form of the expression.

Example 7.50. Simplify the radical expression.

$$\sqrt{6} \cdot \sqrt{8} \cdot \sqrt{8}$$

$$\sqrt{6} \cdot \sqrt{8} \cdot \sqrt{8} = \sqrt{2 \cdot 3} \cdot \sqrt{2^3} \cdot \sqrt{2^3}$$

Break down each radical into its prime factors.

$$= \sqrt{2 \cdot 3} \cdot \sqrt{2^6}$$

Combine the radicals by multiplying their insides.

$$= \sqrt{2 \cdot 3} \cdot \sqrt{2^2 2^2 2^2}$$

Rewrite $\sqrt{2^6}$ as the product of $\sqrt{2^2}$ terms.

$$= \sqrt{2 \cdot 3} \cdot \sqrt{2^2} \cdot \sqrt{2^2} \cdot \sqrt{2^2}$$

Take the square roots of the perfect squares each 2^2.

$$= \sqrt{2} \cdot \sqrt{3} \cdot 2 \cdot 2 \cdot 2$$

Simplify the radicals and perform the multiplications.

$$= 2 \cdot 2 \cdot 2 \cdot \sqrt{2} \cdot \sqrt{3}$$

Rearrange the terms for clarity.

$$= 8\sqrt{2} \cdot \sqrt{3}$$

Combine the square roots into one.

$$= 8\sqrt{6}.$$

Example 7.51. Simplify the radical expression.

$$\left(\sqrt{\gamma} + \sqrt{\delta}\right)\left(\sqrt{\gamma} - \sqrt{\delta}\right)$$

$$\begin{aligned}
\left(\sqrt{\gamma} + \sqrt{\delta}\right)\left(\sqrt{\gamma} - \sqrt{\delta}\right) &= \sqrt{\gamma} \cdot \sqrt{\gamma} + \sqrt{\gamma} \cdot \left(-\sqrt{\delta}\right) + \sqrt{\delta} \cdot \sqrt{\gamma} + \sqrt{\delta} \cdot \left(-\sqrt{\delta}\right) \\
&= \gamma - \sqrt{\gamma \cdot \delta} + \sqrt{\gamma \cdot \delta} - \delta \\
&= \gamma - \delta.
\end{aligned}$$

Therefore, the simplified expression is $\gamma - \delta$. The result is the same as when using the difference of squares formula.

Example 7.52. Simplify the radical expression.

$$\sqrt[3]{24x^8} - \sqrt[3]{3x^8} + \sqrt[3]{81x^8}$$

$$\sqrt[3]{24x^8} - \sqrt[3]{3x^8} + \sqrt[3]{81x^8} = \sqrt[3]{8 \cdot 3x^3x^3x^2} - \sqrt[3]{3x^3x^3x^2} + \sqrt[3]{3 \cdot 3 \cdot 3 \cdot 3x^3x^3x^2}$$

Factorize the expressions inside the cube roots.

$$= \sqrt[3]{8 \cdot 3x^3x^3x^2} - \sqrt[3]{3x^3x^3x^2} + \sqrt[3]{3^3 \cdot 3x^3x^3x^2}$$

Rewrite the expressions as products of cube roots.

$$\begin{aligned}
&= \sqrt[3]{8}\sqrt[3]{3}\sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{x^2} - \sqrt[3]{3}\sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{x^2} \\
&+ \sqrt[3]{3^3}\sqrt[3]{3}\sqrt[3]{x^3}\sqrt[3]{x^3}\sqrt[3]{x^2}
\end{aligned}$$

Simplify each cube root.

$$= 2\sqrt[3]{3}\cdot x\cdot x\sqrt[3]{x^2} - \sqrt[3]{3}\cdot x\cdot x\sqrt[3]{x^2} + 3x\cdot x\sqrt[3]{3}\sqrt[3]{x^2}$$

Combine the terms.

$$= 2\cdot x\cdot x\sqrt[3]{3x^2} - x\cdot x\sqrt[3]{3x^2} + 3x\cdot x\sqrt[3]{3x^2}$$

Simplify the expression further.

$$= 2\cdot x^2\sqrt[3]{3x^2} - x^2\sqrt[3]{3x^2} + 3x^2\sqrt[3]{3x^2}$$

Combine like terms.

$$\begin{aligned} &= (2-1+3)x^2\sqrt[3]{3x^2} \\ &= 4x^2\sqrt[3]{3x^2} \end{aligned}$$

Example 7.53. Simplify the radical expression.

$$\sqrt[3]{3a}\cdot\sqrt[3]{9a^4}$$

$$\sqrt[3]{3a}\cdot\sqrt[3]{9a^4} = \sqrt[3]{(3a)\cdot(9a^4)}$$

Combine the cube roots by multiplying their insides.

$$= \sqrt[3]{27a^5}$$

Factorize the expression inside the cube root.

$$= \sqrt[3]{3^3\cdot a^3\cdot a^2}$$

Simplify the cube roots of perfect cubes.

$$= 3\cdot a\cdot\sqrt[3]{a^2}$$

Express the final simplified form of the expression.

$$= 3a\sqrt[3]{a^2}.$$

7.6 Rationalizing Denominators

As mathematicians, we deal with numbers in various forms (integers and real numbers.) Among these forms, radicals are just one type. However, when it comes to division, we have a preference: we prefer having integers in the denominator. Integers are of the form $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$ To get an integer in the denominator, we use a technique called rationalizing the denominator.

Example 7.54. Perform division aka rationalize each denominator.

$$\frac{1}{\sqrt{2}}$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

Remember that $\frac{\sqrt{2}}{\sqrt{2}} = 1$. Hence, we are just multiplying by 1.

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{2}$$

It is easy to make mistakes. The following is **NOT TRUE**.

False

$$\frac{1}{\sqrt[5]{5}} = \frac{1}{\sqrt[5]{5}} \cdot \frac{\sqrt[5]{5}}{\sqrt[5]{5}}$$

$$= \frac{\sqrt[5]{5}}{5}$$

This is how to actually solve this problem.

True

$$\frac{1}{\sqrt[5]{5}} = \frac{1}{\sqrt[5]{5}} \cdot \frac{\sqrt[5]{5^4}}{\sqrt[5]{5^4}}$$

Multiply by a form of $1 = \frac{\sqrt[5]{5^5}}{\sqrt[5]{5^4}}$ to rationalize the denominator.

$$= \frac{\sqrt[5]{5^4}}{\sqrt[5]{5 \cdot 5^4}}$$

Combine the terms in the denominator under a single fifth root.

$$= \frac{\sqrt[5]{5^4}}{\sqrt[5]{5^5}}$$

Simplify the denominator as the fifth root of 5^5 equals 5.

$$= \frac{\sqrt[5]{5^4}}{5}$$

Example 7.55. Rationalize each denominator.

$$\frac{3}{\sqrt[3]{2}}$$

$$\frac{3}{\sqrt[3]{2}} = \frac{3}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{2^3}}{\sqrt[3]{2^2}}$$

Multiply by a form of $1 = \frac{\sqrt[3]{2^3}}{\sqrt[3]{2^2}}$ to rationalize the denominator.

$$= \frac{3 \cdot \sqrt[3]{2^2}}{\sqrt[3]{2 \cdot 2^2}}$$

Combine the terms in the denominator under a single cube root.

$$= \frac{3 \cdot \sqrt[3]{2^2}}{\sqrt[3]{2^3}}$$

Simplify the denominator as the cube root of 2^3 equals 2.

$$= \frac{3 \cdot \sqrt[3]{2^2}}{2}$$

Example 7.56. Rationalize each denominator.

$$\frac{1}{\sqrt{3}-2}$$

To simplify this expression, we can again use the technique of rationalizing the denominator. We'll multiply the numerator and denominator by the conjugate of the denominator, which is $\sqrt{3}+2$.

$$\frac{1}{\sqrt{3}-2} = \frac{1}{\sqrt{3}-2} \cdot \frac{\sqrt{3}+2}{\sqrt{3}+2}$$

$$= \frac{1 \cdot (\sqrt{3}+2)}{(\sqrt{3}-2)\cdot(\sqrt{3}+2)}$$

$$= \frac{\sqrt{3}+2}{(\sqrt{3})^2-(2)^2}$$

$$= \frac{\sqrt{3}+2}{3-4}$$

$$= \frac{\sqrt{3}+2}{-1}$$

$$= -(\sqrt{3}+2)$$

Example 7.57. Rationalize each denominator.

$$\frac{4}{\sqrt[4]{3x}}$$

$$\frac{4}{\sqrt[4]{3x}} = \frac{4}{\sqrt[4]{3x}} \cdot \frac{\sqrt[4]{(3x)^4}}{\sqrt[4]{(3x)^3}}$$

Multiply by a form of $1 = \frac{\sqrt[4]{(3x)^4}}{\sqrt[4]{(3x)^3}}$ to rationalize the denominator.

$$= \frac{4 \cdot \sqrt[4]{(3x)^3}}{\sqrt[4]{3x \cdot (3x)^3}}$$

Combine the terms in the denominator under a single fourth root.

$$= \frac{4 \cdot \sqrt[4]{(3x)^3}}{\sqrt[4]{(3x)^4}}$$

Simplify the denominator as the fourth root of $(3x)^4$ equals $3x$.

$$= \frac{4 \cdot \sqrt[4]{(3x)^3}}{3x}$$

Example 7.58. Perform each division aka rationalize each denominator.

$$\sqrt{\frac{x}{y}}$$

$$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$$

Applying the square root property.

$$= \frac{\sqrt{x}}{\sqrt{y}} \cdot \frac{\sqrt{y}}{\sqrt{y}}$$

Multiplying numerator and denominator by $\sqrt{y}$.

$$= \frac{\sqrt{x}\sqrt{y}}{\sqrt{y}\sqrt{y}}$$

Multiplying the square roots.

$$= \frac{\sqrt{xy}}{y}$$

Example 7.59. Rationalize each denominator.

$$\frac{5}{\sqrt{2x} + \sqrt{3y}}$$

$$\frac{5}{\sqrt{2x} + \sqrt{3y}} = \frac{5}{\sqrt{2x} + \sqrt{3y}} \cdot \frac{\sqrt{2x} - \sqrt{3y}}{\sqrt{2x} - \sqrt{3y}}$$

Multiply by the conjugate $\frac{\sqrt{2x}-\sqrt{3y}}{\sqrt{2x}-\sqrt{3y}}$ to rationalize the denominator.

$$= \frac{5(\sqrt{2x}-\sqrt{3y})}{(\sqrt{2x}+\sqrt{3y})(\sqrt{2x}-\sqrt{3y})}$$

Apply the difference of squares formula in the denominator.

$$= \frac{5(\sqrt{2x}-\sqrt{3y})}{2x-3y}$$

Simplify the expression.

$$= \frac{5\sqrt{2x}-5\sqrt{3y}}{2x-3y}$$

Express the final simplified form of the expression.

Example 7.60. Perform division aka rationalize each denominator.

$$\frac{\sqrt{y}+\sqrt{3}}{\sqrt{y}-\sqrt{3}}$$

$$\frac{\sqrt{y}+\sqrt{3}}{\sqrt{y}-\sqrt{3}} = \frac{(\sqrt{y}+\sqrt{3})}{(\sqrt{y}-\sqrt{3})} \cdot \frac{(\sqrt{y}+\sqrt{3})}{(\sqrt{y}+\sqrt{3})}$$

$$= \frac{(\sqrt{y}+\sqrt{3})^2}{(\sqrt{y}-\sqrt{3})(\sqrt{y}+\sqrt{3})}$$

$$= \frac{\sqrt{y}^2+2\sqrt{y}\sqrt{3}+\sqrt{3}^2}{\sqrt{y}^2-\sqrt{3}^2}$$

$$= \frac{y + 2\sqrt{3y} + 3}{y - 3}$$

$$= \frac{y + 3 + 2\sqrt{3y}}{y - 3}$$

Example 7.61. Perform division aka rationalize each denominator.

$$\frac{6}{\sqrt[6]{x^4 y^3}}$$

$$\frac{6}{\sqrt[6]{x^4 y^3}} = \frac{6}{\sqrt[6]{x^4 y^3}} \cdot \frac{\sqrt[6]{x^2 y^3}}{\sqrt[6]{x^2 y^3}}$$

$$= \frac{6\sqrt[6]{x^2 y^3}}{\sqrt[6]{x^4 y^3 x^2 y^3}}$$

$$= \frac{6\sqrt[6]{x^2 y^3}}{\sqrt[6]{x^4 x^2 y^3 y^3}}$$

$$= \frac{6\sqrt[6]{x^2 y^3}}{\sqrt[6]{x^{4+2} y^{3+3}}}$$

$$= \frac{6\sqrt[6]{x^2 y^3}}{\sqrt[6]{x^6 y^6}}$$

$$= \frac{6\sqrt[6]{x^2 y^3}}{xy}$$

7.7 Radical Equations

We are no longer working with expressions. In this section we will only work with equations. Remember, in algebraic manipulations, you are allowed to perform operations on equations as long as you apply the same manipulation to both sides.

Students often get confused with squaring both sides. Remember that following is false.

False

$$x + y = 4$$

Square both sides.

$$x^2 + y^2 = 4^2$$
$$x^2 + y^2 = 4^2$$

Remember the ***freshmen dream*** 4.2? If you square $x + y$, then you have two square them as a whole, not individually.

Here is the correct way to square both sides of an equation.

True

$$x + y = 4$$

Square both sides.

$$\begin{aligned}(x+y)^2 &= 4^2\\ (x+y)(x+y) &= 4^2\\ x^2 + yx + xy + y^2 &= 4^2\\ x^2 + 2xy + y^2 &= 4^2\end{aligned}$$

Example 7.62. Solve for x and give all the extraneous solutions for $\sqrt{x+6} = 2$

$$\sqrt{x+6} = 2$$
$$(\sqrt{x+6})^2 = 2^2$$

$$x+6=4$$
$$x=-2$$

To see if $x=-2$ is extraneous or not, we plug -2 back into the original equation. If it is true then -2 is not extraneous.

$$\sqrt{-2+6}=2$$
$$\sqrt{4}=2$$
$$2=2$$

Which is of course true. Therefore, $x=-2$ is not extraneous.

Example 7.63. Solve for x in the equation $\sqrt{x+4}=3$.

$$\sqrt{x+4}=3$$
$$(\sqrt{x+4})^2=3^2$$
$$x+4=9$$
$$x=9-4$$
$$x=5$$

To check if $x=5$ is a valid solution, substitute it back into the original equation.

$$\sqrt{5+4}=3$$
$$\sqrt{9}=3$$
$$3=3$$

This confirms that $x=5$ is the correct solution.

Example 7.64. Solve for y and give all the extraneous solutions for $y-6=\sqrt{3y}$.

$$y-6=\sqrt{3y}$$

$$
\begin{aligned}
(y-6)^2 &= (\sqrt{3y})^2 \\
y^2 - 12y + 36 &= 3y \\
y^2 - 15y + 36 &= 0
\end{aligned}
$$

Using the tic tac toe method.

$$(y-3)(y-12) = 0$$

Hence, $y = 3$ or $y = 12$. Plugging 3 back into the equation gives us,

$$
\begin{aligned}
3 - 6 &= \sqrt{3 \cdot 3} \\
-3 &= 3
\end{aligned}
$$

Which is not true hence 3 is an extraneous solution. Now lets plug in 12

$$
\begin{aligned}
12 - 6 &= \sqrt{3 \cdot 12} \\
6 &= \sqrt{36} \\
6 &= 6
\end{aligned}
$$

Which is indeed true. Hence $y = 6$.

Example 7.65. Solve for x and give all the extraneous solutions for $x - 2 = \sqrt{14 - x}$.

$$
\begin{aligned}
x - 2 &= \sqrt{14 - x} \\
(x-2)^2 &= (\sqrt{14 - x})^2 \\
x^2 - 4x + 4 &= 14 - x \\
x^2 - 3x - 10 &= 0
\end{aligned}
$$

Factorizing the quadratic equation.

$$(x-5)(x+2)=0$$

Hence, $x=5$ or $x=-2$. Checking for extraneous solutions.

For $x=5$,

$$\begin{aligned}5-2&=\sqrt{14-5}\\3&=\sqrt{9}\\3&=3\end{aligned}$$

This is true, so $x=5$ is a valid solution.

For $x=-2$,

$$\begin{aligned}-2-2&=\sqrt{14-(-2)}\\-4&=\sqrt{16}\\-4&\neq 4\end{aligned}$$

This is not true, so $x=-2$ is an extraneous solution.

Example 7.66. Solve for x and give all the extraneous solutions for $\sqrt{3x+1}+1=x$

$$\begin{aligned}\sqrt{3x+1}+1&=x\\\sqrt{3x+1}&=x-1\\(3x+1)&=(x-1)^2\end{aligned}$$

Squaring both sides.

$$\begin{aligned}3x+1&=x^2-2x+1\\1-1&=x^2-2x-3x\\0&=x^2-5x\\0&=x(x-5)\end{aligned}$$

Now we have two possible solutions: $x=0$ and $x=5$. To check if any of these solutions are extraneous, we substitute them back into the original equation.

For $x=0$,

$$\begin{aligned}\sqrt{3(0)+1}+1&=0\\ \sqrt{1}+1&=0\\ 1+1&=0\end{aligned}$$

The equation $2=0$ is not true, so $x=0$ is an extraneous solution.

For $x=5$,

$$\begin{aligned}\sqrt{3(5)+1}+1&=5\\ \sqrt{16}+1&=5\\ 4+1&=5\end{aligned}$$

The equation $5=5$ is true, so $x=5$ is a valid solution. Therefore, the only solution to the equation $\sqrt{3x+1}+1=x$ is $x=5$.

Example 7.67. Solve for x and give all the extraneous solutions for $\sqrt{7x-24}+2=x$.

$$\begin{aligned}\sqrt{7x-24}+2&=x\\ \sqrt{7x-24}&=x-2\\ (\sqrt{7x-24})^2&=(x-2)^2\\ 7x-24&=x^2-4x+4\\ 0&=x^2-11x+28\end{aligned}$$

Factorizing the quadratic equation.

$$0=(x-4)(x-7)$$

Hence, $x=4$ or $x=7$. Checking for extraneous solutions:

For $x=4$,

$$\sqrt{7\cdot 4-24}+2=4$$

$$\begin{aligned}\sqrt{4}+2&=4\\2+2&=4\end{aligned}$$

This is true, so $x=4$ is a valid solution.

For $x=7$,

$$\begin{aligned}\sqrt{7\cdot7-24}+2&=7\\\sqrt{25}+2&=7\\5+2&=7\end{aligned}$$

This is true, so $x=7$ is also a valid solution.

Example 7.68. Solve for x and give all the extraneous solutions for $x=6+\sqrt{18-3x}$.

$$\begin{aligned}x&=6+\sqrt{18-3x}\\x-6&=\sqrt{18-3x}\\(x-6)^2&=(\sqrt{18-3x})^2\\x^2-12x+36&=18-3x\\x^2-9x+18&=0\end{aligned}$$

Factorizing the quadratic equation.

$$(x-3)(x-6)=0$$

Hence, $x=3$ or $x=6$. Checking for extraneous solutions.

For $x=3$,

$$\begin{aligned}3-6&=\sqrt{18-3\cdot3}\\-3&=\sqrt{9}\\-3&\neq3\end{aligned}$$

This is not true, so $x=3$ is an extraneous solution.

For $x = 6$,

$$\begin{aligned} 6 - 6 &= \sqrt{18 - 3 \cdot 6} \\ 0 &= \sqrt{0} \\ 0 &= 0 \end{aligned}$$

This is true, so $x = 6$ is a valid solution.

7.8 Additional Problems

1. Use the definition to simplify the following. Assume all variables are positive.

   (a) $\sqrt{x^6}$
   (b) $\sqrt{x^{18}}$
   (c) $\sqrt{x^{500}}$
   (d) $\sqrt[3]{x^7}$
   (e) $\sqrt[4]{x^9}$
   (f) $\sqrt{60x^5}$
   (g) $\sqrt[5]{-2^5 x^5}$
   (h) $\sqrt[3]{-5^3 x^3}$
   (i) $\sqrt[5]{32c^0 b^8 m^{15}}$
   (j) $\sqrt[3]{64x^0 t^6 y^{11}}$
   (k) $\sqrt{4x^4} + \sqrt{16x^4}$
   (l) $\sqrt{64x^2} - \sqrt{81x^2}$
   (m) $\sqrt[3]{\frac{54}{16}}$
   (n) $\sqrt[3]{\frac{128}{27}}$

2. Change from rational exponent to radical notation, or vice versa, and simplify where possible

   (a) $\dfrac{1}{27}^{\frac{1}{3}}$
   (b) $\dfrac{1}{125}^{\frac{1}{3}}$
   (c) $16^{\frac{1}{4}}$
   (d) $64^{\frac{1}{6}}$
   (e) $125^{\frac{1}{3}}$
   (f) $1000^{\frac{1}{3}}$
   (g) $-16^{\frac{1}{4}}$
   (h) $(-16)^{\frac{1}{4}}$
   (i) $\sqrt{-25}$

3. Use properties of exponents to simplify the following completely with only positive exponents. Assume all variables are positive.

(a) $x^{\frac{3}{8}} \cdot x^{\frac{2}{3}}$
(b) $x^{\frac{5}{7}} \cdot x^{\frac{1}{2}}$
(c) $\left(\frac{1}{32}\right)^{1/5}$
(d) $8^{\frac{1}{3}} + 9^{\frac{1}{2}}$
(e) $125^{\frac{1}{3}} + \sqrt{144} + 81^{\frac{1}{4}}$
(f) $\frac{x^9}{x^3}$
(g) $\frac{7^{\frac{1}{3}}}{7^{\frac{2}{5}}}$
(h) $\frac{w^9}{w^3}$
(i) $\frac{w^{\frac{5}{6}}}{w^{\frac{3}{2}}}$
(j) $\frac{w^{\frac{2}{5}}}{w^{\frac{3}{4}}}$
(k) $(a^{\frac{5}{2}})^{\frac{11}{7}}$

4. Simplify each term in a sum or difference and then combine like terms to simplify.

(a) $2\sqrt{18} - \sqrt{8} + 5\sqrt{2}$
(b) $-9\sqrt{24} + 3\sqrt{54} - 12\sqrt{6}$
(c) $\sqrt{25} + \sqrt{49} - \sqrt{99^2}$
(d) $\sqrt{36} + \sqrt{81} + \sqrt{1^0}$
(e) $\sqrt{4} + \sqrt{49} + \left(\sqrt{1}\right)^2$
(f) $\sqrt[3]{8} - \sqrt[3]{125} + \left(\sqrt[3]{100}\right)^3$
(g) $4^{\frac{1}{2}} + 27^{\frac{1}{3}}$
(h) $(\sqrt{2} + \sqrt{10})(\sqrt{2} - \sqrt{10})$
(i) $(\sqrt{5} + 2)(\sqrt{3} - 4)$
(j) $8^{\frac{1}{3}} + 9^{\frac{1}{2}}$

5. Solve for the variable and give all the extraneous solutions.

(a) $\sqrt[3]{\beta - 99} = 10$
(b) $\sqrt[5]{x - 2} = 2$
(c) $\sqrt[100]{y - 45} = 1$
(d) $\sqrt{3y} - y = -6$
(e) $3\sqrt{x} + 4 = x$

6. Rationalize the denominator.

(a) $\sqrt{\frac{2x}{5y}}$
(b) $\sqrt[5]{\frac{3}{8x^2}}$
(c) $\frac{3}{1 + \sqrt{2}}$
(d) $\frac{4}{2 + \sqrt{3}}$
(e) $\frac{5}{3 - \sqrt{2}}$
(f) $\frac{2}{\sqrt{5} + \sqrt{7}}$
(g) $\frac{7}{\sqrt{3} - 1}$

7. Give the domains of the following functions.

   (a) $\sqrt[3]{x+1}$

   (b) $\sqrt[5]{x-12}$

   (c) $\sqrt{-10x+15}$

   (d) $\sqrt[6]{21-7x}$

8. Given the function

$$f(x) = \sqrt{x-2}$$

   (a) Find $f(4)$

   (b) Is $f(-2)$ a real number?

   (c) What can we conclude about $f(-5)$

9. Given the function

$$f(x) = \sqrt{x+1}$$

   (a) Find $f(3)$

   (b) Is $f(-2)$ a real number?

   (c) What can we conclude about $f(-1)$

10. Given the function

$$f(x) = \sqrt[3]{2x-4}$$

   (a) Find $f(6)$

11. Given the function

$$f(x) = \sqrt[3]{x}+3$$

   (a) Find $f(27)$

12. Given the function

$$f(x) = x^{\frac{1}{4}}+1$$

   (a) Find $f(16)$

13. Are the following true or false. Explain or show why.

(a) $(\sqrt{5}+x)^2 = \sqrt{5}^2 + x^2$

(b) $(\sqrt{9}-x)^2 = \sqrt{9}^2 - x^2$

(c) $(x-\sqrt{2})^3 = x^3 - (\sqrt{2})^3$

(d) $(8^{\frac{1}{2}} + x)^2 = 8 + x^2$

8 The Complex Field

8.1 Complex Numbers

Complex analysis constitutes an entire field of study. Mathematicians devote themselves to exploring and extracting valuable insights from this subject. Once done, they subsequently apply it to the practical world. This pattern holds true for various other "abstract" [20] disciplines as well.

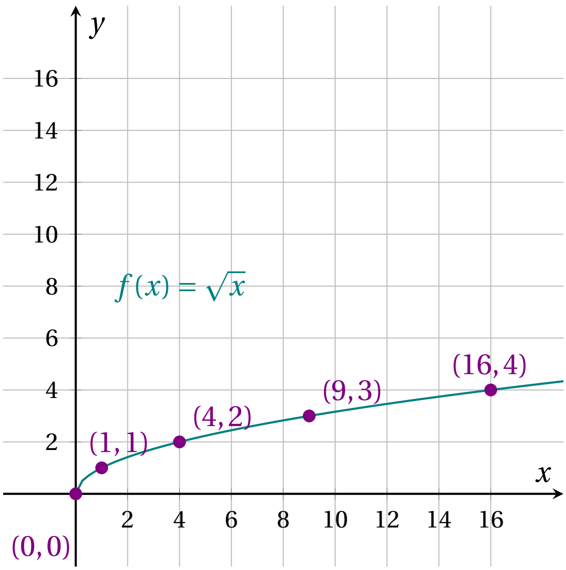
Thus far we have studied functions and observed their behaviors. In chapter 7 example 7.3 we found that all negative inputs were undefined for the function $f(x) = \sqrt{x}$.

$$\begin{gathered} f(-9) = \text{Undefined} \\ f(-4) = \text{Undefined} \\ f(-1) = \text{Undefined} \\ f(0) = \sqrt{0} = 0 \\ f(1) = \sqrt{1} = \sqrt{1^2} = 1 \\ f(4) = \sqrt{4} = \sqrt{2^2} = 2 \\ f(9) = \sqrt{9} = \sqrt{3^2} = 3 \end{gathered}$$

Putting them into a table we see

x	$f(x)$
-9	Undefined
-4	Undefined
-1	Undefined
0	0
1	1
4	2
9	3

[20] HOT TAKE! Abstract math is in theory, applied. What? You want to extract all the useful information in abstract math, use it in engineering, biology, physics, computer science, etc and disregard all the "useless" material and deem it pure? Um? Make it make sense.



This result holds in the real number field. Allow me to explain the issue. Consider the polynomial

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \end{aligned}$$

This statement claims that there exists a number such that when it is squared, the result is -1. However, we all know that this is not true. Thus, we created a new field, the complex field. Let's introduce it. We will not go to in dept with complex numbers. There is a whole field on complex numbers called complex analysis.

Definition

$$i = \sqrt{-1}$$

$$-or-$$

$$i^2 = -1$$

Definition

A ***complex number*** is of the form

$$a + bi$$

where a and b are real numbers.

Example 8.1. Is 102477272 a complex number?

Yes it is. Using definition.

$$102477272 + 0i = 102477272$$

We know that anything to the power of 0 is 1. Thus, $i^0 = 1$. Additionally, we know that anything to the power of 1 is just itself. Therefore, $i^1 = i$. The definition states that $i^2 = -1$. If we multiply both side by an i then,

$$\begin{aligned} i^2 &= -1 \\ i^2 \cdot i &= -1 \cdot i \\ i^3 &= -i. \end{aligned}$$

If we go on, we can construct a table.

i^n	**Value**	i^n	**Value**	i^n	**Value**	i^n	**Value**
i^0	1	i^5	i	i^{10}	-1	i^{15}	$-i$
i^1	i	i^6	-1	i^{11}	$-i$	i^{16}	1
i^2	-1	i^7	$-i$	i^{12}	1	i^{17}	i
i^3	$-i$	i^8	1	i^{13}	i	i^{18}	-1
i^4	1	i^9	i	i^{14}	-1	i^{19}	$-i$

As you can see, the pattern repeats every four exponentiations, cycling through the values 1, i, -1, and $-i$. This pattern continues indefinitely for any positive integer exponent.

None of the previously methods change in the complex field. Our division, multiplication, addition, subtraction and exponentials properties still work. Let's do a couple examples.

Example 8.2. Simplify $\sqrt{-49}$.

$$\begin{aligned}\sqrt{-49} &= \sqrt{49}\sqrt{-1}\\ &= \sqrt{7^2}\cdot i\end{aligned}$$

By definition.

$$= 7i$$

Example 8.3. Simplify $\sqrt{-64}$.

$$\begin{aligned}\sqrt{-64} &= \sqrt{64}\sqrt{-1}\\ &= \sqrt{8^2}\cdot i\end{aligned}$$

By definition.

$$= 8i$$

Example 8.4. Evaluate $\sqrt{-25}+2\sqrt{-9}$.

$$\sqrt{-25}+2\sqrt{-9} = \sqrt{25}\sqrt{-1}+2(\sqrt{9}\sqrt{-1})$$

$$= \sqrt{5^2} \cdot i + 2(\sqrt{3^2} \cdot i)$$

By definition.

$$\begin{aligned} &= 5i + 2(3i) \\ &= 5i + 6i \\ &= 11i \end{aligned}$$

Example 8.5. Find the value of $\sqrt{-81} - 3\sqrt{-16}$.

$$\begin{aligned} \sqrt{-81} - 3\sqrt{-16} &= \sqrt{81}\sqrt{-1} - 3(\sqrt{16}\sqrt{-1}) \\ &= \sqrt{9^2} \cdot i - 3(\sqrt{4^2} \cdot i) \end{aligned}$$

By definition.

$$\begin{aligned} &= 9i - 3(4i) \\ &= 9i - 12i \\ &= -3i \end{aligned}$$

Example 8.6. Simplify $(2 - i) + 4i - (2 - 8i)$.

$$(2 - i) + 4i - (2 - 8i) = 2 - i + 4i - 2 + 8i$$

Distributing the negative sign.

$$= 2 - 2 - i + 4i + 8i$$

Rearranging terms.

$$= (2-2) + (-i + 4i + 8i)$$

Grouping like terms and combining them.

$$= 0 + 11i$$

Simplifying.

$$= 11i$$

Example 8.7. Simplify $(3+2i) - 5i + (1+4i)$.

$$(3+2i) - 5i + (1+4i) = 3 + 2i - 5i + 1 + 4i$$

Expanding the expression.

$$= 3 + 1 + 2i - 5i + 4i$$

Rearranging terms.

$$= (3+1) + (2i - 5i + 4i)$$

Grouping like terms and combining them.

$$= 4 + 1i$$

Simplifying.

$$= 4 + i$$

Example 8.8. Simplify $(3+5i)(6-6i)$.

$$(3+5i)(6-6i) = 3(6)+3(-6i)+5i(6)+5i(-6i)$$

Applying distributive property to expand the expression.

$$= 18-18i+30i-30i^2$$

Multiplying each term and simplifying.

$$= 18-18i+30i-30(-1)$$

Using the identity $i^2 = -1$ to simplify $30i^2$ to -30.

$$= 18-18i+30i+30$$

Substituting -30 for $30i^2$ and simplifying.

$$= 48+12i$$

Combining like terms to reach the final result.

Example 8.9. Simplify $(3+5i)(6-6i)$.

$$(3+5i)(6-6i) = 3(6)+3(-6i)+5i(6)+5i(-6i)$$

Applying distributive property to expand the expression.

$$= 18 - 18i + 30i - 30i^2$$

Multiplying each term and simplifying.

$$= 18 - 18i + 30i - 30(-1)$$

Using the identity $i^2 = -1$ to simplify $30i^2$ to -30.

$$= 18 - 18i + 30i + 30$$

Substituting -30 for $30i^2$ and simplifying.

$$= 48 + 12i$$

Combining like terms to reach the final result.

Example 8.10. Evaluate the expression $(2+3i)^2 - (1-4i)(1+4i)$

$$(2+3i)^2 - (1-4i)(1+4i) = (2+3i)(2+3i) - (1-4i)(1+4i)$$

Expanding the expressions.

$$= (4 + 6i + 6i + 9i^2) - (1 - 4i + 4i - 16i^2)$$

Applying distributive law.

$$= (4 + 12i + 9(-1)) - (1 - 16(-1))$$

Since $i^2 = -1$.

$$= (4 + 12i - 9) - (1 + 16)$$
$$= -5 + 12i - 17$$

Combining like terms.

$$= -22 + 12i$$

Example 8.11. Rationalizing the denominator $\dfrac{7}{4+i}$.

$$\frac{7}{4+i} = \frac{7}{4+i} \cdot \frac{4-i}{4-i}$$

Since our denominator is $4+i$, we change the middle sign. That is, we multiply the top and the bottom by $4-i$

$$= \frac{7(4-i)}{(4+i)(4-i)}$$

$$= \frac{28-7i}{16-i^2}$$

$$= \frac{28-7i}{16-(-1)}$$

$$= \frac{28-7i}{17}$$

$$= \frac{28}{17} - \frac{7}{17}i$$

Example 8.12. Rationalize the denominator $\dfrac{5}{3-2i}$.

$$\frac{5}{3-2i} = \frac{5}{3-2i} \cdot \frac{3+2i}{3+2i}$$

To rationalize the denominator of $3-2i$, we multiply the numerator and the denominator by the conjugate $3+2i$.

$$\begin{aligned} &= \frac{5(3+2i)}{(3-2i)(3+2i)} \\ &= \frac{15+10i}{9-4i^2} \\ &= \frac{15+10i}{9-(4(-1))} \end{aligned}$$

Using the identity $i^2 = -1$ to simplify the denominator.

$$\begin{aligned} &= \frac{15+10i}{9+4} \\ &= \frac{15+10i}{13} \\ &= \frac{15}{13} + \frac{10}{13}i \end{aligned}$$

Example 8.13. Divide and simplify $\frac{4i}{2+5i}$.

$$\frac{4i}{2+5i} = \frac{4i}{2+5i} \cdot \frac{2-5i}{2-5i}$$

To simplify, we multiply the numerator and the denominator by the conjugate of the denominator, which is $2-5i$.

$$= \frac{4i(2-5i)}{(2+5i)(2-5i)}$$

$$= \frac{8i-20i^2}{4-25i^2}$$

Expanding both the numerator and the denominator.

$$= \frac{8i-20(-1)}{4-25(-1)}$$

Using the identity $i^2 = -1$.

$$= \frac{8i+20}{4+25}$$

$$= \frac{8i+20}{29}$$

$$= \frac{8}{29}i + \frac{20}{29}$$

Example 8.14. Simplify i^{242}

To calculate the value of i^{242}, we can use the pattern of powers of i. The powers of i repeat in a cycle. i, -1, $-i$, 1. Every fourth power of i results in 1. Let's find the remainder when 242 is divided by 4. $242 \div 4 = 60$ with a remainder of 2. Since the remainder is 2, we can say that i^{242} is the same as i^2.

$$\begin{aligned} i^{242} &= i^2 \\ &= -1 \end{aligned}$$

Example 8.15. Determine the value of i^{153}.

The powers of the imaginary unit i cycle in a repeating pattern. i, -1, $-i$, 1. This cycle repeats every four powers. To simplify i^{153}, we can find the equivalent power in the first cycle by calculating the remainder when 153 is divided by 4. $153 \div 4 = 38$ with a remainder of 1. The remainder determines the equivalent power of i in the first cycle. Since the remainder is 1, i^{153} simplifies to the same as i^1.

$$\begin{aligned} i^{153} &= i^1 \\ &= i \end{aligned}$$

Example 8.16. Compute the value of i^{196}.

The powers of the imaginary unit i follow a repeating cycle: i, -1, $-i$, 1. This cycle repeats every four powers. To compute i^{196}, we consider the remainder when 196 is divided by 4. $196 \div 4 = 49$ with a remainder of 0. When the remainder is 0, it corresponds to the fourth power in the cycle, which is 1.

$$\begin{aligned} i^{196} &= i^{4\times 49} \\ &= (i^4)^{49} \\ &= 1^{49} \\ &= 1 \end{aligned}$$

9 Quadratic Equations

9.1 Square Root Property

Thus far, we have learned several factoring techniques. Here is an other important one.

Theorem

Square Root Property

$$\text{If } x^2 = c \text{ then } x = -\sqrt{c}, \sqrt{c}$$

Note: The main take away here is that there are ***two*** solutions. This is because there are ***two*** numbers that when squared equal c.

Proof.

$$x^2 = c$$

Subtracting c from both sides to set the equation to zero.

$$x^2 - c = 0$$

Factorizing the equation into the product of two binomials.

$$\left(x + \sqrt{c}\right)\left(x - \sqrt{c}\right) = 0$$

Applying the zero product property, where if the product of two factors is zero, at least one of the factors must be zero.

$$\left(x + \sqrt{c}\right) = 0 \quad \text{or} \quad \left(x - \sqrt{c}\right) = 0$$

Solving each equation for x gives the two possible solutions.

$$x = -\sqrt{c} \quad \text{or} \quad x = \sqrt{c}$$

□

Example 9.1. Use the square root property.

$$x^2 = 1$$

$$\sqrt{x^2} = \pm\sqrt{1}$$

Apply the square root to both sides. Note that taking the square root introduces a plus/minus (±) sign.

$$x = -\sqrt{1} \text{ or } \sqrt{1}$$

Simplify the square root of x^2 and 1, leading to two possible solutions.

$$x = -1 \text{ or } 1$$

Example 9.2. Use the square root property.

$$4x^2 = 36$$

$$x^2 = \frac{36}{4}$$

Divide both sides by 4.

$$x^2 = 9$$

Simplify the division.

$$\sqrt{x^2} = \pm\sqrt{9}$$

Apply the square root to both sides. Note that taking the square root introduces a plus/minus (±) sign.

$$x = -\sqrt{9} \text{ or } \sqrt{9}$$

Simplify the square root of x^2 and 9, leading to two possible solutions.

$$x = -3 \text{ or } 3$$

Example 9.3. Use the square root property.

$$x^2 - 20 = 0$$

Start with the given equation.

$$x^2 = 20$$

Add 20 to both sides to isolate the term with x.

$$\sqrt{x^2} = \pm\sqrt{20}$$

Apply the square root to both sides of the equation.

$$x = \pm\sqrt{20}$$

Simplify the square root of x^2 to x.

$$x = \pm\sqrt{4 \cdot 5}$$

Factorize the square root of 20 into $\sqrt{4}$ and $\sqrt{5}$ for easier simplification.

$$x = \pm\sqrt{4} \cdot \sqrt{5}$$

Break down the square root of the product into the product of square roots.

$$x = \pm 2\sqrt{5}$$

Example 9.4. Use the square root property.

$$3x^2 - 75 = 0$$

Start with the given equation.

$$3x^2 = 75$$

Add 75 to both sides to isolate the term with x.

$$x^2 = \frac{75}{3}$$

Divide both sides by 3.

$$x^2 = 25$$

Simplify the division.

$$\sqrt{x^2} = \pm\sqrt{25}$$

Apply the square root to both sides of the equation.

$$x = \pm\sqrt{25}$$

Simplify the square root of x^2 to x.

$$x = \pm 5$$

Example 9.5. Use the square root property.

$$(\delta - 4)^2 = 16$$

Start with the given equation.

$$\sqrt{(\delta - 4)^2} = \pm\sqrt{16}$$

Apply the square root to both sides, noting that the square root of a square can be positive or negative.

$$\delta - 4 = \pm 4$$

Simplify the square roots.

$$\delta = 4 \pm 4$$

Isolate δ by adding 4 to both sides.

$$\delta = 4 + 4 \quad \text{or} \quad \delta = 4 - 4$$

Consider both cases, adding and subtracting the value.

$$\delta = 8 \quad \text{or} \quad \delta = 0$$

Example 9.6. Use the square root property.

$$(y+6)^2 = 49$$

Start with the given equation.

$$\sqrt{(y+6)^2} = \pm\sqrt{49}$$

Apply the square root to both sides, noting that the square root of a square can be positive or negative.

$$y + 6 = \pm 7$$

Simplify the square roots.

$$y = -6 \pm 7$$

Isolate y by subtracting 6 from both sides.

$$y = -6 + 7 \quad \text{or} \quad y = -6 - 7$$

Consider both cases: adding and subtracting the value.

$$y = 1 \quad \text{or} \quad y = -13$$

Example 9.7. Use the square root property.

$$9x^2 + 25 = 0$$

Start with the given equation.

$$9x^2 = -25$$

Subtract 25 from both sides.

$$x^2 = -\frac{25}{9}$$

Divide both sides by 9.

$$x = \pm\sqrt{-\frac{25}{9}}$$

Apply the square root to both sides. Note that the square root of a negative number introduces an imaginary unit i.

$$x = \pm\frac{5i}{3}$$

Example 9.8. Use the square root property.

$$(x+3)^2 = 20$$

Start with the given equation.

$$\sqrt{(x+3)^2} = \pm\sqrt{20}$$

Apply the square root to both sides.

$$x+3 = \pm\sqrt{20}$$

Simplify the square root on the left side.

$$x = -3 \pm \sqrt{20}$$

Isolate x by subtracting 3 from both sides.

$$x = -3 \pm 2\sqrt{5}$$

Example 9.9. Use the square root property.

$$(2x-5)^2 = 36$$

Start with the given equation.

$$\sqrt{(2x-5)^2} = \pm\sqrt{36}$$

Apply the square root to both sides.

$$2x - 5 = \pm 6$$

Simplify the square root on the left side.

$$2x = 5 \pm 6$$

Isolate $2x$ by adding 5 to both sides.

$$x = \frac{5 \pm 6}{2}$$

Solve for x by dividing both sides by 2.

9.2 Pythagorean Theorem

As I previously mentioned, we are utilizing algebra to solve complected geometrical problems. The Pythagorean Theorem is a brilliant way to calculate distance. However, what shape does this theorem employ? The correct answer is a right triangle.

Pythagorean Theorem

This is a theorem about right triangles and can be summarised in the next equation

$$a^2 + b^2 = c^2$$

To find the distance between two points (x_1, y_1) and (x_2, y_2).

Theorem

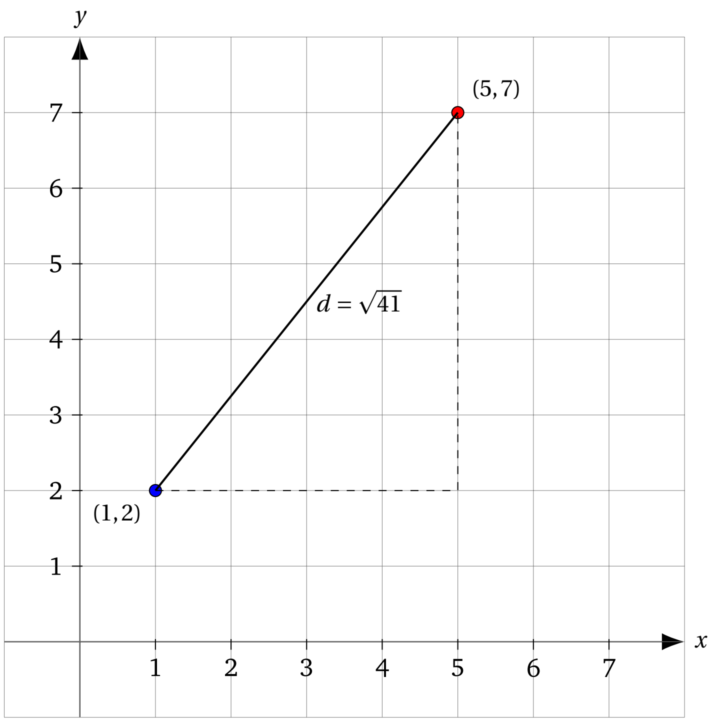
Distance of Two Points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

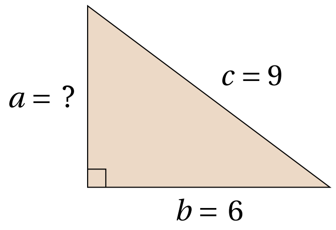
Example 9.10. Find the distance between the points $(5,7)$ and $(1,2)$.

To find the distance of two points, we use the definition.

$$
\begin{aligned}
d &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \\
&= \sqrt{(1-5)^2 + (2-7)^2} \\
&= \sqrt{(-4)^2 + (-5)^2} \\
&= \sqrt{16+25} \\
&= \sqrt{41}
\end{aligned}
$$



Example 9.11. A 9-ft ladder leans against the side of a house. The bottom of the ladder is 6 ft from the side of the house. How high is the top of the ladder from the ground? If necessary, round your answer to the nearest tenth.



$$a^2 + b^2 = c^2$$

$$b^2 + 6^2 = 9^2$$

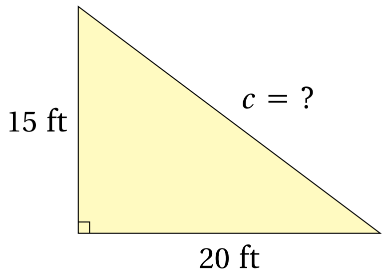
$$a^2 = 9^2 - 6^2$$

$$a^2 = 81 - 36$$

$$a^2 = 45$$

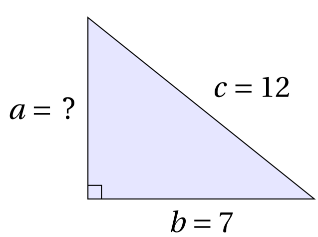
$$a = \pm\sqrt{45} \text{ ft}$$

Example 9.12. In a rectangular park, a diagonal walking path connects two opposite corners. If the length of the park is 20 ft and the width is 15 ft, find the length of the diagonal walking path. Round your answer to the nearest tenth if necessary.



$$
\begin{aligned}
a &= 20 \text{ ft} \\
b &= 15 \text{ ft} \\
a^2 + b^2 &= c^2 \\
20^2 + 15^2 &= c^2 \\
400 + 225 &= c^2 \\
625 &= c^2 \\
c &= \sqrt{625} \text{ ft} \\
c &= 25 \text{ ft}
\end{aligned}
$$

Example 9.13. A 12-ft ladder leans against the side of a house. The bottom of the ladder is 7 ft from the side of the house. How high is the top of the ladder from the ground? If necessary, round your answer to the nearest tenth.



$$a^2 + b^2 = c^2$$

$$a^2 + 7^2 = 12^2$$

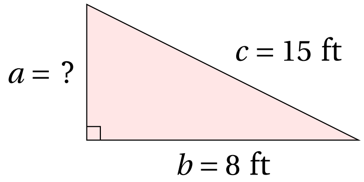
$$a^2 = 12^2 - 7^2$$

$$a^2 = 144 - 49$$

$$a^2 = 95$$

$$a = \sqrt{95} \text{ ft}$$

Example 9.14. A 15-ft plank is placed to cross over a ditch. The ends of the plank rest on the ground on either side of the ditch. If the plank touches the ground 8 ft from one end, how wide is the ditch? Round your answer to the nearest tenth if necessary.



$$a^2 + b^2 = c^2$$

$$a^2 + 8^2 = 15^2$$

$$a^2 = 15^2 - 8^2$$

$$a^2 = 225 - 64$$

$$a^2 = 161$$

$$a = \sqrt{161} \text{ ft}$$

9.3 Completing the Square

There are numerous techniques for factoring trinomials. So far, you have encountered the tic-tac-toe method, as well as the diamond method. Another method for solving for variables is completing the square, which can be somewhat more tedious. Certain methods are more suitable than others in specific situations and thus, we will learn them all.

Steps to Complete the Square Continue:

- The coefficient of x^2 must be 1. If it is not, then divide by the number that is multiplying x^2.

$$2x^2 - x - 1 = 0$$
$$\frac{1}{2} \cdot (2x^2 - x - 1) = \frac{1}{2} \cdot 0$$
$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0$$

Steps to Complete the Square:

- Divide the number in front of x by 2 the square it.

  Let's look at $-\frac{1}{2}x$. That is, just $-\frac{1}{2}$

$$\left(\frac{\frac{1}{2}}{2}\right)^2 = \frac{1}{16}$$

- Add this number to both sides.

$$x^2 - \frac{1}{2}x = \frac{1}{2}$$
$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{1}{2} + \frac{1}{16}$$
$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{9}{16}$$

- Factor the trinomial and combine like terms.

$$x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{9}{16}$$
$$\left(x - \frac{1}{4}\right)^2 = \frac{9}{16}$$

- Solve the equation using the square-root property.

$$\sqrt{\left(x - \frac{1}{4}\right)^2} = \pm\sqrt{\frac{9}{16}}$$
$$\left(x - \frac{1}{4}\right) = \pm\frac{3}{4}$$
$$x = 1 \text{ or } x = -\frac{1}{2}$$

This example was difficult, but it does not get more challenging then this. Let's tone it down a bit and do a less rigorous question.

Example 9.15. Solve for x by completing the square of $x^2 + 10x = 0$.

- The coefficient of x^2 is already 1, so no need for division.
- Make sure the constant term is on the right side.

$$x^2 + 10x = 0$$

- Divide the coefficient of x by 2 and square it.

$$\left(\frac{10}{2}\right)^2 = 25$$

- Add this number to both sides.

$$x^2 + 10x + 25 = 0 + 25$$
$$x^2 + 10x + 25 = 25$$

- Factor the trinomial and simplify.

$$(x+5)^2 = 25$$

- Take the square root of both sides.

$$x + 5 = \pm\sqrt{25}$$

- Solve for x.

  When we take the positive square root.

$$x + 5 = \pm 5$$
$$x = -5 \pm 5$$

This gives us two solutions: $x = 0$ and $x = -10$.

When we take the negative square root.

$$x + 5 = \pm(-5)$$
$$x = -5 \pm 5$$

This also gives us two solutions: $x = -10$ and $x = 0$.

Example 9.16. Determine the values of x by completing the square for $x^2 - 6x - 7 = 0$.

- The coefficient of x^2 is 1, which means division is unnecessary.
- Ensure the constant term is on the opposite side.

$$x^2 - 6x = 7$$

- Halve the coefficient of x, then square the result.

$$\left(\frac{-6}{2}\right)^2 = 9$$

- Add this square to both sides of the equation.

$$x^2 - 6x + 9 = 7 + 9$$
$$x^2 - 6x + 9 = 16$$

- Factorize the left-hand side and simplify the equation.

$$(x - 3)^2 = 16$$

- Extract the square root on both sides.

$$x - 3 = \pm\sqrt{16}$$

- Isolate x.

  For the positive square root.

$$\begin{aligned} x - 3 &= 4 \\ x &= 4 + 3 \end{aligned}$$

  This results in $x = 7$.

  For the negative square root.

$$\begin{aligned} x - 3 &= -4 \\ x &= -4 + 3 \end{aligned}$$

  This results in $x = -1$.

Example 9.17. Find the roots of x by completing the square for the equation $x^2 - 12x + 27 = 0$.

- The coefficient in front of x^2 is already 1, so dividing is not required.
- Move the constant term to the right side of the equation.

$$x^2 - 12x = -27$$

- Divide the coefficient of x by 2, then square it.

$$\left(\frac{-12}{2}\right)^2 = 36$$

- Add this value to each side of the equation.

$$\begin{aligned} x^2 - 12x + 36 &= -27 + 36 \\ x^2 - 12x + 36 &= 9 \end{aligned}$$

- Factor the left side and simplify.

$$(x-6)^2 = 9$$

- Take the square root of both sides.

$$x-6 = \pm\sqrt{9}$$

- Isolate x.

  With the positive square root.

$$\begin{aligned} x-6 &= 3 \\ x &= 3+6 \end{aligned}$$

  This yields $x = 9$.

  With the negative square root.

$$\begin{aligned} x-6 &= -3 \\ x &= -3+6 \end{aligned}$$

  This yields $x = 3$.

Example 9.18. Solve the quadratic equation $x^2 + 8x - 5 = 0$ by completing the square.

- Since the coefficient of x^2 is 1, there's no need to divide the entire equation.
- First, rearrange the equation to move the constant term to the right side.

$$x^2 + 8x = 5$$

- Take half of the coefficient of x, and then square it.

$$\left(\frac{8}{2}\right)^2 = 16$$

- Add this squared number to both sides of the equation.

$$x^2 + 8x + 16 = 5 + 16$$
$$x^2 + 8x + 16 = 21$$

- Factor the left side of the equation.

$$(x + 4)^2 = 21$$

- Now, take the square root of both sides.

$$x + 4 = \pm\sqrt{21}$$

- Finally, solve for x.

  For the positive square root.

$$x + 4 = \sqrt{21}$$
$$x = \sqrt{21} - 4$$

  For the negative square root.

$$x + 4 = -\sqrt{21}$$
$$x = -\sqrt{21} - 4$$

Example 9.19. Solve the equation $3x^2 - 18x + 15 = 0$ by completing the square.

- Start by dividing the entire equation by the coefficient of x^2, which is 3, to normalize the quadratic term.

$$x^2 - 6x + 5 = 0$$

- Move the constant term to the other side.

$$x^2 - 6x = -5$$

- Find the number to complete the square: take half of the coefficient of x, and square it.

$$\left(\frac{-6}{2}\right)^2 = 9$$

- Add this number to both sides of the equation.

$$x^2 - 6x + 9 = -5 + 9$$
$$x^2 - 6x + 9 = 4$$

- Factorize the left-hand side and simplify.

$$(x-3)^2 = 4$$

- Extract the square root on both sides.

$$x - 3 = \pm\sqrt{4}$$

- Solve for x.

  For the positive square root:

$$x - 3 = 2$$
$$x = 2 + 3$$

  This gives $x = 5$.

  For the negative square root:

$$x - 3 = -2$$
$$x = -2 + 3$$

  This gives $x = 1$.

Example 9.20. Find the solutions for x in the equation $4x^2 - 16x + 7 = 0$ by completing the square.

- First, divide the entire equation by the coefficient of x^2, which is 4.

$$x^2 - 4x + \frac{7}{4} = 0$$

- Next, isolate the quadratic and linear terms.

$$x^2 - 4x = -\frac{7}{4}$$

- To complete the square, take half of the coefficient of x, square it, and add it to both sides.

$$\left(\frac{-4}{2}\right)^2 = 4$$

$$x^2 - 4x + 4 = -\frac{7}{4} + 4$$

$$x^2 - 4x + 4 = \frac{9}{4}$$

- Factorize the left side and simplify the right side.

$$(x-2)^2 = \frac{9}{4}$$

- Extract the square root on both sides.

$$x - 2 = \pm\frac{3}{2}$$

- Finally, solve for x.

  For the positive square root:

$$x - 2 = \frac{3}{2}$$
$$x = \frac{3}{2} + 2$$

  This yields $x = \frac{7}{2}$.

  For the negative square root:

$$x - 2 = -\frac{3}{2}$$
$$x = -\frac{3}{2} + 2$$

  This yields $x = \frac{1}{2}$.

9.4 Quadratic Formula

As we witnessed, completing the square is indeed a powerful technique for solving quadratic equations. It allows us to manipulate the equation and express it in a more convenient form. However, mathematicians constantly explore various methods and properties to extract the maximum amount of information and insights from mathematical equations and concepts. We know that the generic version of any trinomial set equal to 0 looks something like this.

$$ax^2 + bx + c = 0.$$

As it turns out, we get an extremely useful formula when completing the square of the trinomial above.

Theorem

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. Derive the quadratic formula by completing the square on a general quadratic.

Start with the general form of a quadratic equation.

$$ax^2 + bx + c = 0$$

Divide the entire equation by a to normalize the coefficient of x^2.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Isolate the quadratic and linear terms.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Complete the square by adding $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

Rewrite the left side as a perfect square and simplify the right side.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square root of both sides.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

Isolate x.

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

Simplify to get the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

□

Using this formula, we can factor any trinomial given we have information on a, b and c. Okay, okay, okay, okay, okay, okay, o-.

Example 9.21. Solve for x by using the quadratic formula.

$$x^2 + 5x + 6$$

We know that $x^2 + 5x + 6$ factors into $(x+3)(x+2)$. Therefore, $x = -3$ or $x = -2$. Notice that

$$x^2 + 5x + 6 = 1 \cdot x^2 + 5x + 6,$$

hence $a = 1$, $b = 5$ and $c = 6$. Check this out.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(6)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 24}}{2}$$

$$x = \frac{-5 \pm \sqrt{1}}{2}$$

$$x = \frac{-5 \pm 1}{2}$$

When we take the positive square root.

$$x = \frac{-5 + 1}{2} = \frac{-4}{2} = -2$$

When we take the negative square root.

$$x = \frac{-5 - 1}{2} = \frac{-6}{2} = -3$$

Example 9.22. Use the quadratic formula to find the values of x for the equation below.

$$x^2 - 3x - 4$$

$$x^2 - 3x - 4 = 1 \cdot x^2 - 3x - 4,$$

thus we identify $a = 1$, $b = -3$, and $c = -4$. Let's proceed with the calculation.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$$

$$x = \frac{3 \pm \sqrt{9 + 16}}{2}$$

$$x = \frac{3 \pm \sqrt{25}}{2}$$

$$x = \frac{3 \pm 5}{2}$$

Considering the positive square root.

$$x = \frac{3 + 5}{2} = \frac{8}{2} = 4$$

Considering the negative square root.

$$x = \frac{3 - 5}{2} = \frac{-2}{2} = -1$$

Example 9.23. Determine the roots of the equation $x^2 + 4x - 21 = 0$ using the quadratic formula.

$$x^2 + 4x - 21 = 1 \cdot x^2 + 4x - 21,$$

indicating that $a = 1$, $b = 4$, and $c = -21$. Let's apply the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-4 \pm \sqrt{4^2 - 4(1)(-21)}}{2(1)}$$

$$x = \frac{-4 \pm \sqrt{16 + 84}}{2}$$

$$x = \frac{-4 \pm \sqrt{100}}{2}$$

$$x = \frac{-4 \pm 10}{2}$$

Taking the positive square root.

$$x = \frac{-4 + 10}{2} = \frac{6}{2} = 3$$

Taking the negative square root.

$$x = \frac{-4 - 10}{2} = \frac{-14}{2} = -7$$

Example 9.24. Find the solutions for the quadratic equation by employing the quadratic formula.

$$2x^2 - 10x + 12 = 2 \cdot x^2 - 10x + 12,$$

implying $a = 2$, $b = -10$, and $c = 12$. Now, let's proceed with the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(2)(12)}}{2(2)}$$

$$x = \frac{10 \pm \sqrt{100 - 96}}{4}$$

$$x = \frac{10 \pm \sqrt{4}}{4}$$

$$x = \frac{10 \pm 2}{4}$$

Considering the positive square root.

$$x = \frac{10+2}{4} = \frac{12}{4} = 3$$

Considering the negative square root.

$$x = \frac{10-2}{4} = \frac{8}{4} = 2$$

Example 9.25. Apply the quadratic formula to solve the equation $6x^2 - 17x - 20 = 0$.

$$6x^2 - 17x - 20$$

$$6x^2 - 17x - 20 = 6 \cdot x^2 - 17x - 20,$$

indicating that $a = 6$, $b = -17$, and $c = -20$. Let's solve it using the formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-17) \pm \sqrt{(-17)^2 - 4(6)(-20)}}{2(6)}$$

$$x = \frac{17 \pm \sqrt{289 + 480}}{12}$$

$$x = \frac{17 \pm \sqrt{769}}{12}$$

Considering both the positive and negative square roots.

$$x_1 = \frac{17 + \sqrt{769}}{12}$$

and

$$x_2 = \frac{17 - \sqrt{769}}{12}$$

It would be impossible to find the roots of this using the tic tac toe. Understand that this is saying that

$$6x^2 - 17x - 20 = 6\left(x - \frac{17 + \sqrt{769}}{12}\right)\left(x - \frac{17 - \sqrt{769}}{12}\right)$$

> **Example 9.26.** Solve the quadratic equation $5x^2 + 6x + 8 = 0$ using the quadratic formula.

$$5x^2 + 6x + 8 = 5 \cdot x^2 + 6x + 8,$$

thus we have $a = 5$, $b = 6$, and $c = 8$. Let's calculate the solutions.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{6^2 - 4(5)(8)}}{2(5)}$$

$$x = \frac{-6 \pm \sqrt{36 - 160}}{10}$$

$$x = \frac{-6 \pm \sqrt{-124}}{10}$$

Since we have a negative number under the square root, the solutions will be complex numbers.

$$x = \frac{-6 \pm \sqrt{-124}}{10}$$

$$x = \frac{-6 \pm 2i\sqrt{31}}{10}$$

Separating into real and imaginary parts.

$$x_1 = \frac{-6 + 2i\sqrt{31}}{10}$$

and

$$x_2 = \frac{-6 - 2i\sqrt{31}}{10}$$

Example 9.27. Solve for x by using the quadratic formula.

$$x^2 + 5x + 10$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1,\ b = 5$ and $c = 10$.

$$x = \frac{-5 \pm \sqrt{5^2 - 4(1)(10)}}{2(1)}$$

$$x = \frac{-5 \pm \sqrt{25 - 40}}{2}$$

$$x = \frac{-5 \pm \sqrt{-15}}{2}$$

$$x = \frac{-5 \pm \sqrt{-15}}{2}$$

$$x = \frac{-5 \pm i\sqrt{15}}{2}$$

Since we have the $\pm$ lets split the roots.

$$x = \frac{-5 + i\sqrt{15}}{2} \text{ or } \frac{-5 - i\sqrt{15}}{2}$$

Example 9.28. Solve for x by using the quadratic formula.

$$2x^2 + 7x + 3$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 2,\ b = 7$ and $c = 3$.

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{-7 \pm \sqrt{49 - 24}}{4}$$

$$x = \frac{-7 \pm \sqrt{25}}{4}$$

$$x = \frac{-7 \pm 5}{4}$$

$$x = \frac{-7 + 5}{4} \text{ or } \frac{-7 - 5}{4}$$

$$x = -\frac{2}{4} \text{ or } -\frac{12}{4}$$

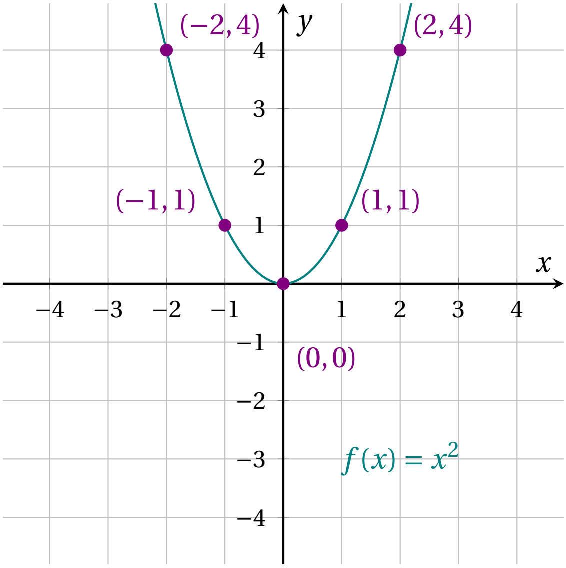
$$x = -\frac{1}{2} \text{ or } -3$$

9.5 Graphing Quadratics

Graphing functions is a crucial technique to have in calculus. Although functions come in all shapes and sizes, we will be focusing on one of the more popular functions: $f(x) = x^2$. We start with a t-table.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
2	4

Plotting the points gives us the following graph.



The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.

It turns out that any function that is of the form $ax^2 + bx + c$ looks something like this. We call this type of graph a parabola.

Definition

Parabolas are of the form $f(x) = ax^2 + bx + c$. The ***vertex*** can be found using the formula: $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$

Let me list the steps to graph a quadratic including the vertex.

Graphing $ax^2 + bx + c$

1. Determine which way the parabola goes.
   (a) If $(a > 0)$ then the parabola opens up.
   (b) If $(a < 0)$ then the parabola opens down.
2. Find the vertex and axis of symmetry.
   (a) Find the x value of the vertex by using $\frac{-b}{2a}$.
   (b) Find the y value of the vertex by plugging the x

value back into the original function: $f\left(\frac{-b}{2a}\right)$

(c) The axis of symmetry is the vertical line through the vertex: $x = \frac{-b}{2a}$

(d) Our vertex is then (x, y).

3. Find the y-intercept using $f(0)$. It is the point $(0, c)$.

4. Find the x-intercept(s) values using $f(x) = 0$. These should also be written as points.

5. Plot any other points needed by using the axis of symmetry with any other known points (usually the y-intercept).

6. Draw a smooth curve through the points.

Example 9.29. Graph the following quadratic $f(x) = -x^2 + 4x - 3$.

***x*-vertex:**

$$\begin{aligned} \frac{-b}{2a} &= \frac{-4}{2(-1)} \\ &= 2 \end{aligned}$$

***y*-vertex:**

$$\begin{aligned} f(2) &= -(2)^2 + 4(2) - 3 \\ f(2) &= -4 + 8 - 3 \\ f(2) &= 1 \end{aligned}$$

Hence our vertex is $(2, 1)$

***x*-intercept:** We set $y = 0$.

$$0 = -x^2 + 4x - 3$$

Move everything to the left.

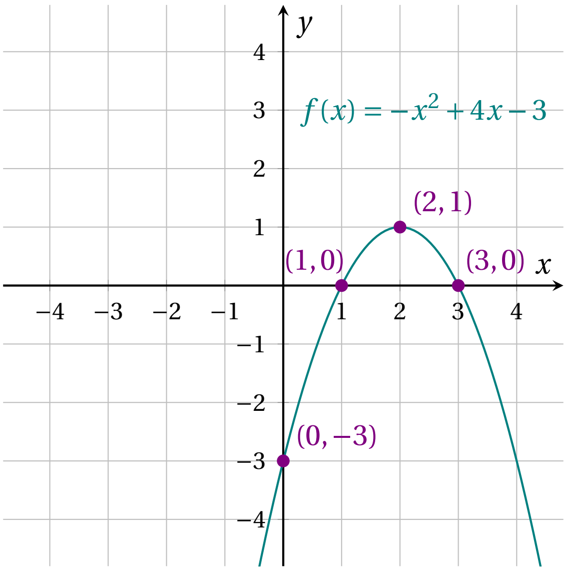
$$\begin{aligned} x^2 - 4x + 3 &= 0 \\ (x-1)(x-3) &= 0 \\ x &= 1 \text{ or } 3 \end{aligned}$$

Our second pair of points are $(1,0)$ and $(3,0)$. Notice that we set $y = 0$ and arrived at two $x's$. Therefore, the points $(1,0)$ and $(3,0)$ make sense.

***y*-intercept:** We set $x = 0$.

$$\begin{aligned} y &= -(0)^2 + 4(0) - 3 \\ &= -3 \end{aligned}$$

Our last point is then $(0,-3)$. Lets graph all these points.



The domain is $(-\infty, \infty)$ and the range is $(-\infty, 1]$.

Example 9.30. Plot the quadratic function $f(x) = x^2 - 6x + 5$.

Vertex on the x-axis:

$$\frac{-b}{2a} = \frac{-(-6)}{2(1)}$$
$$= \frac{6}{2(1)}$$
$$= 3$$

Vertex on the y-axis:

$$f(3) = (3)^2 - 6(3) + 5$$
$$f(3) = 9 - 18 + 5$$
$$f(3) = -4$$

Therefore, the vertex is at $(3, -4)$.

Intercepts on the x-axis: Set $y = 0$.

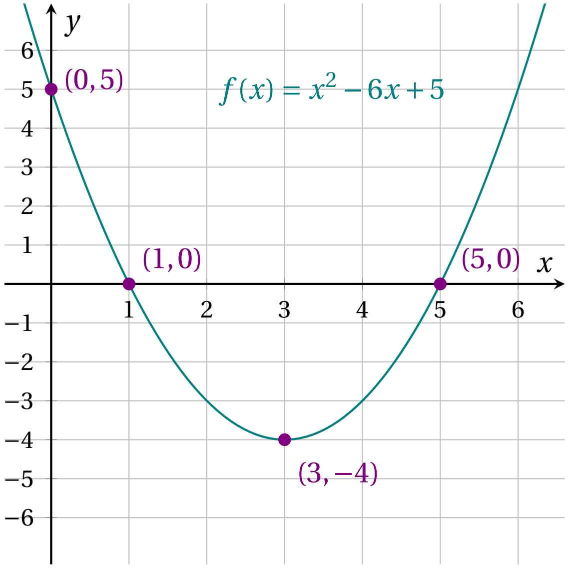
$$0 = x^2 - 6x + 5$$
$$0 = x^2 - 6x + 5$$
$$0 = (x-5)(x-1)$$
$$x = 5 \text{ or } 1$$

This yields the points $(5, 0)$ and $(1, 0)$ on the x-axis.

Intercept on the y-axis: Set $x = 0$.

$$y = (0)^2 - 6(0) + 5$$
$$= 5$$

The point on the y-axis is then $(0, 5)$. Time to graph these points.



The domain is all real numbers $(-\infty, \infty)$, and the range is $[-4, \infty)$.

Example 9.31. Graph the following quadratic $y = x^2 + 4x - 5$.

***x*-vertex:**

$$\frac{-b}{2a} = \frac{-4}{2(1)}$$

$$= -2$$

***y*-vertex:**

$$\begin{aligned} f(-2) &= (-2)^2 + 4(-2) - 5 \\ &= 4 - 8 - 5 \\ &= -9 \end{aligned}$$

Hence, our vertex is $(-2, -9)$.

***x*-intercepts:** We set $y = 0$.

$$\begin{aligned} x^2 + 4x - 5 &= 0 \\ (x-1)(x+5) &= 0 \end{aligned}$$

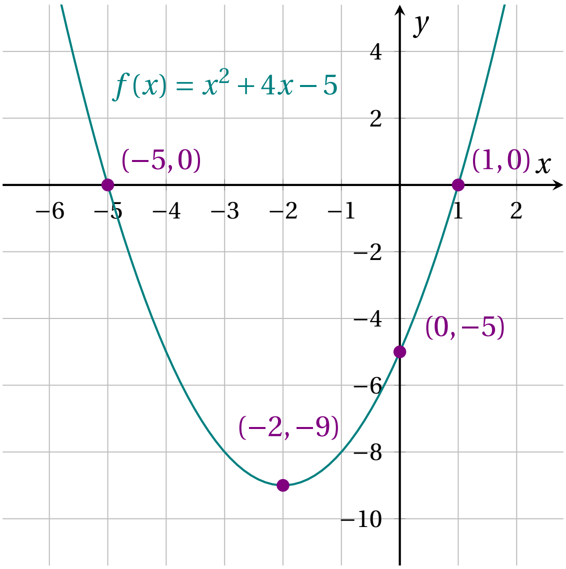
$$x = 1 \quad \text{or} \quad x = -5$$

Our second pair of points are $(1,0)$ and $(-5,0)$.

y-intercept: We set $x = 0$.

$$\begin{aligned} y &= (0)^2 + 4(0) - 5 \\ &= -5 \end{aligned}$$

Our last point is $(0,-5)$. Let's graph all these points.



The domain is $(-\infty,\infty)$ and the range is $[-9,\infty)$.

Example 9.32. Graph the quadratic function $f(x) = x^2 - 4x + 3$.

Vertex on the x-axis:

$$\frac{-b}{2a} = \frac{-(-4)}{2(1)}$$

$$= \frac{4}{2(1)}$$

$$= 2$$

Vertex on the y-axis:

$$\begin{aligned} f(2) &= (2)^2 - 4(2) + 3 \\ f(2) &= 4 - 8 + 3 \\ f(2) &= -1 \end{aligned}$$

Thus, the vertex is at $(2, -1)$.

x-intercepts: Set $y = 0$.

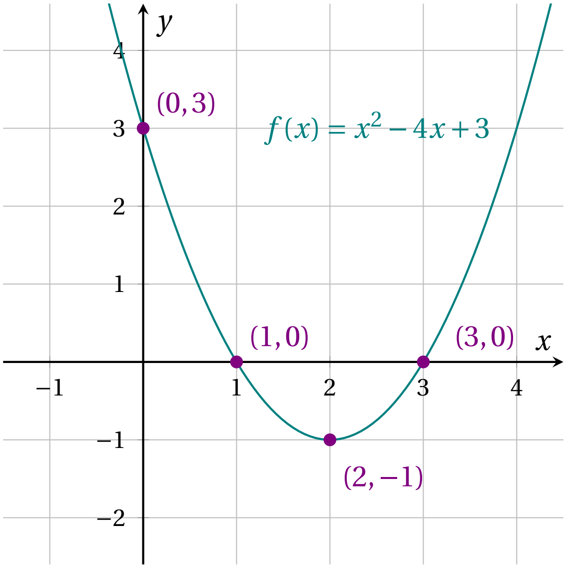
$$\begin{aligned} 0 &= x^2 - 4x + 3 \\ 0 &= (x-1)(x-3) \\ x &= 1 \text{ or } 3 \end{aligned}$$

We find the intercepts $(1, 0)$ and $(3, 0)$ on the x-axis.

y-intercept: Set $x = 0$.

$$\begin{aligned} y &= (0)^2 - 4(0) + 3 \\ &= 3 \end{aligned}$$

The intercept on the y-axis is $(0, 3)$. Let's graph these points.



The domain is $(-\infty,\infty)$ and the range is $[-1,\infty)$.

Example 9.33. Graph the quadratic function $f(x) = x^2-2x-3$.

Vertex on the x-axis:

$$\begin{aligned}\frac{-b}{2a} &= \frac{-(-2)}{2(1)}\\ &= \frac{2}{2(1)}\\ &= 1\end{aligned}$$

Vertex on the y-axis:

$$\begin{aligned}f(1) &= (1)^2-2(1)-3\\ f(1) &= 1-2-3\\ f(1) &= -4\end{aligned}$$

Therefore, the vertex is at $(1,-4)$.

x-intercepts: Set $y=0$.

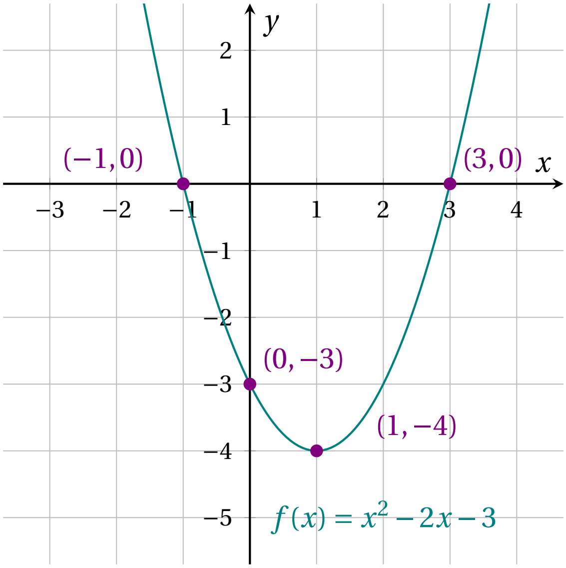
$$\begin{aligned}0 &= x^2-2x-3\\ 0 &= (x+1)(x-3)\\ x &= -1 \text{ or } 3\end{aligned}$$

The intercepts $(-1,0)$ and $(3,0)$ are found on the x-axis.

y-intercept: Set $x=0$.

$$\begin{aligned}y &= (0)^2-2(0)-3\\ &= -3\end{aligned}$$

The intercept on the y-axis is $(0,-3)$. Now, let's plot these points.



The domain is $(-\infty, \infty)$, and the range is $[-4, \infty)$.

Example 9.34. Graph the quadratic function $f(x) = 2x^2 - 8x + 6$.

Vertex on the x-axis:

$$\begin{aligned}
\frac{-b}{2a} &= \frac{-(-8)}{2(2)} \\
&= \frac{8}{2(2)} \\
&= 2
\end{aligned}$$

Vertex on the y-axis:

$$\begin{aligned}
f(2) &= 2(2)^2 - 8(2) + 6 \\
f(2) &= 8 - 16 + 6 \\
f(2) &= -2
\end{aligned}$$

Thus, the vertex is at $(2, -2)$.

***x*-intercepts:** Set $y = 0$.

$$\begin{aligned} 0 &= 2x^2 - 8x + 6 \\ 0 &= 2(x^2 - 4x + 3)2 \\ 0 &= x^2 - 4x + 3 \end{aligned}$$

By factoring out at 2 and then dividing both sides by 2.

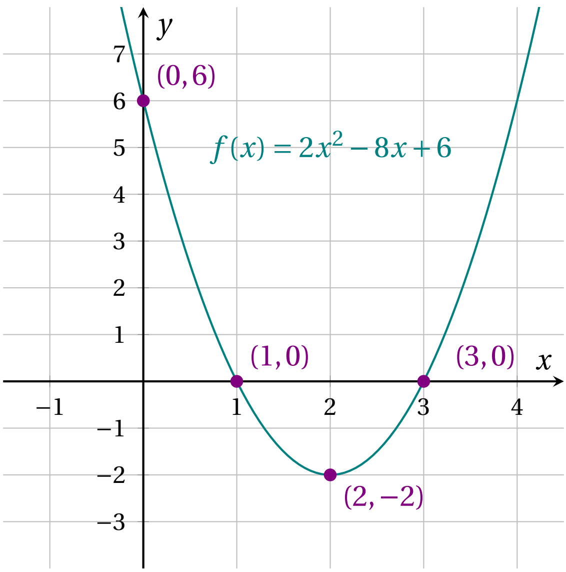
$$\begin{aligned} 0 &= (x-1)(x-3) \\ x &= 1 \text{ or } 3 \end{aligned}$$

We find the intercepts $(1,0)$ and $(3,0)$ on the x-axis.

***y*-intercept:** Set $x = 0$.

$$\begin{aligned} y &= 2(0)^2 - 8(0) + 6 \\ &= 6 \end{aligned}$$

The intercept on the y-axis is $(0,6)$. Let's graph these points.



The domain is $(-\infty, \infty)$ and the range is $[-2, \infty)$.

Example 9.35. Graph the quadratic function $f(x) = -2x^2 + 8x - 6$.

Vertex on the x-axis:

$$\frac{-b}{2a} = \frac{-8}{2(-2)} \\ = 2$$

Vertex on the y-axis:

$$f(2) = -2(2)^2 + 8(2) - 6$$
$$f(2) = -8 + 16 - 6$$
$$f(2) = 2$$

The vertex is therefore located at $(2,2)$.

x-intercepts: We set $y = 0$.

$$0 = -2x^2 + 8x - 6$$

Rearrange the equation.

$$2x^2 - 8x + 6 = 0$$
$$x^2 - 4x + 3 = 0$$

Factor out a 2 and divide it by both sides.

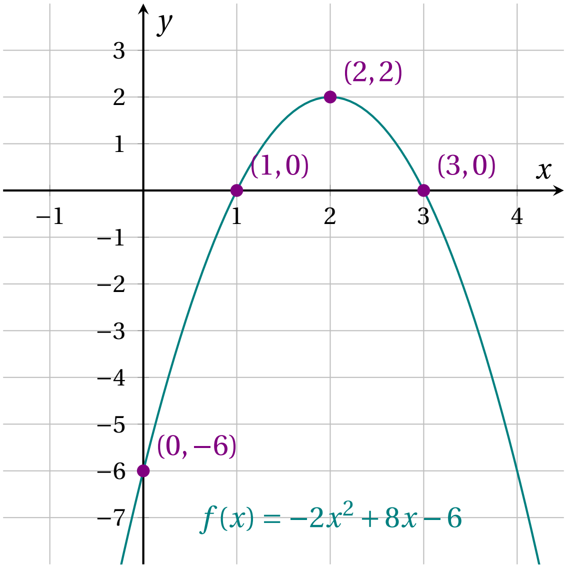
$$(x-1)(x-3) = 0$$
$$x = 1 \text{ or } 3$$

This gives us the intercepts $(1,0)$ and $(3,0)$ on the x-axis.

y-intercept: We set $x = 0$.

$$y = -2(0)^2 + 8(0) - 6$$
$$= -6$$

The intercept on the y-axis is $(0,-6)$. Now, let's plot these points.



The domain is the set of all real numbers $(-\infty, \infty)$, and the range is $(-\infty, 2]$.

9.6 Additional Problems

1. Use the square root property.

   (a) $(x+9)^2 = 75$
   (b) $(y-7)^2 = 64$
   (c) $(z+5)^2 = 49$
   (d) $(w-4)^2 = 100$
   (e) $3(c-8)^3 - 24 = 101$
   (f) $2^2(5c-1)^2 - 1^3 = 63$

2. Solve for x by completing the square.

   (a) $x^2 - 4x + 5 = 0$
   (b) $x^2 - 6x + 8 = 0$
   (c) $x^2 + 14x = -24$
   (d) $x^2 - 3x = 18$

3. Solve for x by using the quadratic formula.

   (a) $3x^2 - 2x - 1 = 0$
   (b) $2x^2 - 5x + 3 = 0$
   (c) $4x^2 + x - 2 = 0$
   (d) $x^2 - 4x + 4 = 0$

(e) $-5x^2+7x+2=0$

(f) $-4x^2-6x-1=0$

(g) $4x^2-5x=-2$

(h) $2x^2=-3x-6$

4. Graph the quadratic function.

(a) $y=x^2+2x+1$

(b) $y=x^2-4x+3$

(c) $y=x^2+6x+9$

(d) $y=-x^2+6x-5$

10 Functions & Their Operations

10.1 Function Operation

Similar to numbers, we can combine functions together using addition, subtraction, multiplication and division.

Theorem

Function Operations

1. $(f+g)(x)=f(x)+g(x)$
2. $(f-g)(x)=f(x)-g(x)$
3. $(fg)(x)=f(x)g(x)$
4. $\left(\frac{f}{g}\right)(x)=\dfrac{f(x)}{g(x)}$ (provided $g(x)\neq 0$)

Example 10.1. Consider the functions $f(x)=2x^2+1$ and $g(x)=5x-3$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$ and find the value(s) not in the domain of $\frac{f}{g}$. Lastly, Evaluate $(f+g)(-1)$.

$(f+g)(x)=f(x)+g(x)$

$$\begin{aligned}(f+g)(x)&=(2x^2+1)+(5x-3)\\&=2x^2+5x+1-3\\&=2x^2+5x-2\end{aligned}$$

$(f-g)(x)=f(x)-g(x)$

$$\begin{aligned}(f-g)(x)&=(2x^2+1)-(5x-3)\\&=2x^2+1-5x+3\\&=2x^2-5x+1+3\\&=2x^2-5x+4\end{aligned}$$

$(fg)(x) = f(x)g(x)$

$$\begin{aligned}(fg)(x) &= (2x^2+1)(5x-3)\\ &= (2x^2)(5x)+(2x^2)(-3)+(1)(5x)+(1)(-3)\\ &= 10x^3-6x^2+5x-3\end{aligned}$$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\begin{aligned}\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)}\\ &= \frac{2x^2+1}{5x-3}\end{aligned}$$

To find out what x can not be, set the denominator equal to 0 and solve for x.

$$\begin{aligned}5x-3 &= 0\\ 5x &= 3\\ x &= \frac{3}{5}\end{aligned}$$

Hence $x \neq \frac{3}{5}$ since

$$\frac{2x^2+1}{5\left(\frac{3}{5}\right)-3}$$

gives us division by 0.

To find $(f+g)(-1)$, first notice that $(f+g)(x) = 2x^2+5x-2$. Now, replace all the $x's$ by -1.

$$\begin{aligned}(f+g)(-1) &= 2(-1)^2+5(-1)-2\\ &= 2(-1)(-1)-5-2\\ &= 2-5-2\end{aligned}$$

$$= -5$$

Example 10.2. Consider the functions $h(x) = 3x^2 - 4$ and $k(x) = 2x + 1$. Find $(h+k)(x)$, $(h-k)(x)$, $(hk)(x)$, $\left(\frac{h}{k}\right)(x)$ and determine the value(s) not in the domain of $\frac{h}{k}$. Lastly, Evaluate $(h+k)(2)$.

$(h+k)(x) = h(x) + k(x)$

$$\begin{aligned}(h+k)(x) &= (3x^2 - 4) + (2x + 1)\\ &= 3x^2 - 4 + 2x + 1\\ &= 3x^2 + 2x - 3\end{aligned}$$

$(h-k)(x) = h(x) - k(x)$

$$\begin{aligned}(h-k)(x) &= (3x^2 - 4) - (2x + 1)\\ &= 3x^2 - 4 - 2x - 1\\ &= 3x^2 - 2x - 5\end{aligned}$$

$(hk)(x) = h(x)k(x)$

$$\begin{aligned}(hk)(x) &= (3x^2 - 4)(2x + 1)\\ &= (3x^2)(2x) + (3x^2)(1) - (4)(2x) - (4)(1)\\ &= 6x^3 + 3x^2 - 8x - 4\end{aligned}$$

$\left(\frac{h}{k}\right)(x) = \frac{h(x)}{k(x)}$

$$\begin{aligned}\left(\frac{h}{k}\right)(x) &= \frac{h(x)}{k(x)}\\ &= \frac{3x^2 - 4}{2x + 1}\end{aligned}$$

To determine what x cannot be, set the denominator equal to zero and solve for x.

$$\begin{aligned} 2x+1&=0 \\ 2x&=-1 \\ x&=-\frac{1}{2} \end{aligned}$$

Hence $x \neq -\frac{1}{2}$ since

$$\frac{3x^2-4}{2\left(-\frac{1}{2}\right)+1}$$

would result in division by zero.

To find $(h+k)(2)$, first notice that $(h+k)(x) = 3x^2+2x-3$. Now, replace all the x's with 2.

$$\begin{aligned} (h+k)(2) &= 3(2)^2+2(2)-3 \\ &= 3(4)+4-3 \\ &= 12+4-3 \\ &= 13 \end{aligned}$$

Example 10.3. Consider the functions $p(x) = x^2+3x-1$ and $q(x) = x-2$. Calculate $(p+q)(x)$, $(p-q)(x)$, $(pq)(x)$, $\left(\frac{p}{q}\right)(x)$, and identify the value(s) not in the domain of $\frac{p}{q}$. Lastly, Evaluate $(p+q)(3)$.

$(p+q)(x) = p(x)+q(x)$

$$\begin{aligned} (p+q)(x) &= (x^2+3x-1)+(x-2) \\ &= x^2+3x-1+x-2 \\ &= x^2+4x-3 \end{aligned}$$

$(p-q)(x) = p(x)-q(x)$

$$(p-q)(x) = (x^2+3x-1)-(x-2)$$

$$\begin{aligned} &= x^2 + 3x - 1 - x + 2 \\ &= x^2 + 2x + 1 \end{aligned}$$

$(pq)(x) = p(x)q(x)$

$$\begin{aligned} (pq)(x) &= (x^2 + 3x - 1)(x - 2) \\ &= x^2(x-2) + 3x(x-2) - 1(x-2) \\ &= x^2 \cdot x + x^2 \cdot (-2) + 3x \cdot x + 3x \cdot (-2) - 1 \cdot x + 1 \cdot 2 \\ &= x^3 - 2x^2 + 3x^2 - 6x - x + 2 \\ &= x^3 - 2x^2 + 3x^2 - 6x - x + 2 \end{aligned}$$

Now, combine like terms.

$$\begin{aligned} &= x^3 + (3x^2 - 2x^2) + (-6x - x) + 2 \\ &= x^3 + x^2 - 7x + 2 \end{aligned}$$

$\left(\frac{p}{q}\right)(x) = \frac{p(x)}{q(x)}$

$$\begin{aligned} \left(\frac{p}{q}\right)(x) &= \frac{p(x)}{q(x)} \\ &= \frac{x^2 + 3x - 1}{x - 2} \end{aligned}$$

To identify the value x cannot take, set the denominator of $\frac{p}{q}$ equal to zero:

$$\begin{aligned} x - 2 &= 0 \\ x &= 2 \end{aligned}$$

Hence $x \neq 2$ as it would lead to division by zero in $\frac{p}{q}$.

To evaluate $(p+q)(3)$, replace x with 3 in $(p+q)(x) = x^2 + 4x - 3$.

$$(p+q)(3) = 3^2 + 4(3) - 3$$

$$\begin{aligned} &= 9+12-3 \\ &= 18 \end{aligned}$$

Example 10.4. Let $a(x) = 3x^2 - x + 4$ and $b(x) = 2x - 5$. Determine $(a+b)(x)$, $(a-b)(x)$, $(ab)(x)$, $\left(\frac{a}{b}\right)(x)$, and find the value(s) that are not in the domain of $\frac{a}{b}$. Also, calculate $(a+b)(-2)$.

$(a+b)(x) = a(x) + b(x)$

$$\begin{aligned} (a+b)(x) &= (3x^2 - x + 4) + (2x - 5) \\ &= 3x^2 - x + 4 + 2x - 5 \\ &= 3x^2 + x - 1 \end{aligned}$$

$(a-b)(x) = a(x) - b(x)$

$$\begin{aligned} (a-b)(x) &= (3x^2 - x + 4) - (2x - 5) \\ &= 3x^2 - x + 4 - 2x + 5 \\ &= 3x^2 - 3x + 9 \end{aligned}$$

$(ab)(x) = a(x)b(x)$

$$\begin{aligned} (ab)(x) &= (3x^2 - x + 4)(2x - 5) \\ &= (3x^2)(2x) - (3x^2)(5) - x(2x) + x(5) + 4(2x) - 4(5) \\ &= 6x^3 - 15x^2 - 2x^2 + 5x + 8x - 20 \\ &= 6x^3 - 17x^2 + 13x - 20 \end{aligned}$$

$\left(\dfrac{a}{b}\right)(x) = \dfrac{a(x)}{b(x)}$

$$\begin{aligned} \left(\frac{a}{b}\right)(x) &= \frac{a(x)}{b(x)} \\ &= \frac{3x^2 - x + 4}{2x - 5} \end{aligned}$$

To identify values not in the domain of $\frac{a}{b}$, set the denominator equal to zero.

$$\begin{aligned} 2x - 5 &= 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Hence, $x \neq \frac{5}{2}$ as it results in division by zero.

To evaluate $(a+b)(-2)$.

$$\begin{aligned} (a+b)(-2) &= 3(-2)^2 + (-2) - 1 \\ &= 3(4) - 2 - 1 \\ &= 12 - 2 - 1 \\ &= 9 \end{aligned}$$

Example 10.5. Consider the functions $f(x) = x^2 - 4x + 3$ and $g(x) = 2x + 1$. Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, and find the value(s) not in the domain of $\frac{f}{g}$. Lastly, evaluate $(f+g)(2)$.

$(f+g)(x) = f(x) + g(x)$

$$\begin{aligned} (f+g)(x) &= (x^2 - 4x + 3) + (2x + 1) \\ &= x^2 - 4x + 3 + 2x + 1 \\ &= x^2 - 2x + 4 \end{aligned}$$

$(f-g)(x) = f(x) - g(x)$

$$\begin{aligned} (f-g)(x) &= (x^2 - 4x + 3) - (2x + 1) \\ &= x^2 - 4x + 3 - 2x - 1 \\ &= x^2 - 6x + 2 \end{aligned}$$

$(fg)(x) = f(x)g(x)$

$$(fg)(x) = (x^2 - 4x + 3)(2x + 1)$$

$$\begin{aligned} &= 2x^3 + x^2 - 8x^2 - 4x + 6x + 3 \\ &= 2x^3 - 7x^2 + 2x + 3 \end{aligned}$$

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$$\left(\frac{f}{g}\right)(x) = \frac{x^2 - 4x + 3}{2x + 1}$$

To find the values not in the domain of $\frac{f}{g}$, we set the denominator equal to zero and solve for x.

$$\begin{aligned} 2x + 1 &= 0 \\ 2x &= -1 \\ x &= -\frac{1}{2} \end{aligned}$$

Therefore, $x \neq -\frac{1}{2}$ since division by zero is not defined. To evaluate $(f+g)(2)$, we substitute $x = 2$ into the expression $(f+g)(x)$.

$$\begin{aligned} (f+g)(2) &= 2^2 - 2(2) + 4 \\ &= 4 - 4 + 4 \\ &= 4 \end{aligned}$$

Hence, $(f+g)(2) = 4$.

Example 10.6. Consider the functions $f(x) = x^3$ and $g(x) = x + 6$. Find $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, and find the value(s) not in the domain of $\frac{f}{g}$.

$(f-g)(x) = f(x) - g(x)$

$$\begin{aligned} (f-g)(x) &= x^3 - (x+6) \\ &= x^3 - x - 6 \end{aligned}$$

$(fg)(x) = f(x)g(x)$

$$(fg)(x) = x^3 \cdot (x+6)$$

$$= x^4 + 6x^3$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

$$\left(\frac{f}{g}\right)(x) = \frac{x^3}{x+6}$$

To find the values not in the domain of $\frac{f}{g}$, we set the denominator equal to zero and solve for x.

$$\begin{aligned} x + 6 &= 0 \\ x &= -6 \end{aligned}$$

Therefore, $x \neq -6$ since division by zero is not defined.

10.2 Function Composition

Definition

Given two functions f and g, the composition function, $f \circ g$, is defined

$$(f \circ g)(x) = f(g(x))$$

Important

"∘" is an operation called composition. It is not multiplication. "·" is an operation as well. As you know, it is called multiplication.

Function composition is challenging. It is better to explain via an example.

Example 10.7. Let $f(x) = 2x + 1$ and $g(x) = x - 4$. Compute: $(f \circ g)(x)$, $(g \circ f)(x)$ and $(f \circ g)(-2)$

Let's start off with $(f \circ g)(x) = f(g(x))$ (by definition).

$$f(g(x)) = f(x-4)$$

Plug in $(x-4)$ into every x in $2x+1$.

$$\begin{aligned} &= 2(x-4)+1 \\ &= 2x-8+1 \\ &= 2x-7 \end{aligned}$$

Now lets do Let's start off with $(g \circ f)(x) = g(f(x))$ (by definition).

$$g(f(x)) = g(2x+1)$$

Plug in $2x+1$ into every x in $(x-4)$.

$$\begin{aligned} &= (2x+1)-4 \\ &= 2x+1-4 \\ &= 2x-3 \end{aligned}$$

We know that $(f \circ g)(x) = f(g(x)) = 2x-7$. Therefore,

$$\begin{aligned} (f \circ g)(-2) &= 2(-2)-7 \\ &= -11 \end{aligned}$$

Example 10.8. Let $m(x) = 3x-2$ and $n(x) = x^2+5$. Compute: $(m \circ n)(x)$, $(n \circ m)(x)$ and $(m \circ n)(1)$.

Let's start with $(m \circ n)(x) = m(n(x))$ (by definition).

$$m(n(x)) = m(x^2+5)$$

Plug x^2+5 into every x in $3x-2$.

$$
\begin{aligned}
&= 3\underline{(x^2+5)}-2 \\
&= 3x^2+15-2 \\
&= 3x^2+13
\end{aligned}
$$

Again we will do $(n\circ m)(x) = n(m(x))$ (by definition).

$$n(\underline{m(x)}) = n(\underline{3x-2})$$

Plug $3x-2$ into every x in x^2+5.

$$
\begin{aligned}
&= \underline{(3x-2)}^2+5 \\
&= (3x-2)^2+5 \\
&= 9x^2-12x+4+5 \\
&= 9x^2-12x+9
\end{aligned}
$$

We know that $(m\circ n)(x) = m(n(x)) = 3x^2+13$. Therefore,

$$
\begin{aligned}
(m\circ n)(1) &= 3(1)^2+13 \\
&= 3+13 \\
&= 16
\end{aligned}
$$

Example 10.9. Let $\underline{u(x) = x^2-3x}$ and $\underline{v(x) = 2-x}$. Calculate $(u\circ v)(x)$, $(v\circ u)(x)$, and $(u\circ v)(3)$.

Begin with $(u\circ v)(x) = u(v(x))$ (by definition).

$$u(\underline{v(x)}) = u(\underline{2-x})$$

Substitute $2-x$ for every x in x^2-3x.

$$\begin{aligned}&=(2-x)^2-3(2-x)\\&=4-4x+x^2-6+3x\\&=x^2-x-2\end{aligned}$$

Next, calculate $(v\circ u)(x)=v(u(x))$ (by definition).

$$v(\underline{u(x)})=v(\underline{x^2-3x})$$

Substitute x^2-3x for every x in $2-x$.

$$\begin{aligned}&=2-(x^2-3x)\\&=2-x^2+3x\\&=-x^2+3x+2\end{aligned}$$

To evaluate $(u\circ v)(3)$,

$$\begin{aligned}(u\circ v)(3)&=(3)^2-(3)-2\\&=9-3-2\\&=4\end{aligned}$$

Example 10.10. Let $\underline{f(x)=x^2+5}$ and $\underline{g(x)=\sqrt{x+7}}$. Compute: $(f\circ g)(x)$, $(g\circ f)(x)$, and $(f\circ g)(2)$.

First, calculate $(f\circ g)(x)=f(g(x))$ (by definition).

$$f(\underline{g(x)})=f(\underline{\sqrt{x+7}})$$

Substitute $\sqrt{x+7}$ into every x in x^2+5.

$$= (\sqrt{x+7})^2 + 5$$
$$= x + 7 + 5$$
$$= x + 12$$

Next, calculate $(g \circ f)(x) = g(f(x))$ (by definition).

$$g(f(x)) = g(x^2 + 5)$$

Substitute $x^2 + 5$ into every x in $\sqrt{x+7}$.

$$= \sqrt{x^2 + 5 + 7}$$
$$= \sqrt{x^2 + 12}$$

To evaluate $(f \circ g)(2)$, replace x with 2 in $(f \circ g)(x) = x + 12$.

$$(f \circ g)(2) = 2 + 12$$
$$= 14$$

Example 10.11. Let $q(x) = -2x - 3$ and $r(x) = -5x + 3$. Compute: $(q \circ r)(x)$, $(r \circ q)(x)$, and $(q \circ r)(-2)$.

Let's start with $(q \circ r)(x) = q(r(x))$ (by composition).

$$q(r(x)) = q(-5x + 3)$$

Plug in $(-5x + 3)$ into every x in $-2x - 3$.

$$= -2(-5x + 3) - 3$$
$$= 10x - 6 - 3$$
$$= 10x - 9$$

Now let's do $(r \circ q)(x) = r(q(x))$ (by composition).

$$r(\underline{q(x)}) = r(\underline{-2x-3})$$

Plug in $(-2x-3)$ into every x in $-5x+3$.

$$\begin{aligned} &= -5(\underline{-2x-3})+3 \\ &= 10x+15+3 \\ &= 10x+18 \end{aligned}$$

We know that $(q \circ r)(x) = q(r(x)) = 10x-9$. Therefore,

$$\begin{aligned} (q \circ r)(-2) &= 10(-2)-9 \\ &= -20-9 \\ &= -29 \end{aligned}$$

Example 10.12. Let $\underline{f(x) = \sqrt{3x}}$ and $\underline{g(x) = x^2-6x+9}$. Determine if the function composition is commutative by computing $(f \circ g)(x)$ and $(g \circ f)(x)$.

Begin with $(f \circ g)(x) = f(g(x))$ (by definition).

$$f(\underline{g(x)}) = f(\underline{x^2-6x+9})$$

Substitute x^2-6x+9 for every x in $\sqrt{3x}$.

$$\begin{aligned} &= \sqrt{3(\underline{x^2-6x+9})} \\ &= \sqrt{3x^2-18x+27} \end{aligned}$$

Now compute $(g \circ f)(x) = g(f(x))$ (by definition).

$$g(\underline{f(x)}) = g(\underline{\sqrt{3x}})$$

Plug $\sqrt{3x}$ into every x in $x^2 - 6x + 9$.

$$\begin{aligned} &= (\sqrt{3x})^2 - 6(\sqrt{3x}) + 9 \\ &= 3x - 6\sqrt{3x} + 9 \end{aligned}$$

Compare $(f \circ g)(x)$ and $(g \circ f)(x)$.

$$(f \circ g)(x) = \sqrt{3x^2 - 18x + 27}$$

and

$$(g \circ f)(x) = 3x - 6\sqrt{3x} + 9$$

Since $(f \circ g)(x)$ and $(g \circ f)(x)$ are not equal, the function composition is not commutative.

Example 10.13. Let $h(x) = 4x - 1$ and $j(x) = x^2 + 2x + 1$. Investigate if the function composition is commutative by computing $(h \circ j)(x)$ and $(j \circ h)(x)$.

Start with $(h \circ j)(x) = h(j(x))$ (by definition).

$$h(j(x)) = h(x^2 + 2x + 1)$$

Substitute $x^2 + 2x + 1$ for every x in $4x - 1$.

$$\begin{aligned} &= 4(x^2 + 2x + 1) - 1 \\ &= 4x^2 + 8x + 4 - 1 \\ &= 4x^2 + 8x + 3 \end{aligned}$$

Now compute $(j \circ h)(x) = j(h(x))$ (by definition).

$$j(h(x)) = j(4x - 1)$$

Plug $4x-1$ into every x in x^2+2x+1.

$$\begin{aligned}&= \underline{(4x-1)}^2 + 2\underline{(4x-1)} + 1\\ &= (16x^2-8x+1)+(8x-2)+1\\ &= 16x^2-8x+1+8x-2+1\\ &= 16x^2+0x\end{aligned}$$

Compare $(h\circ j)(x)$ and $(j\circ h)(x)$.

$$(h\circ j)(x) = 4x^2+8x+3$$

and

$$(j\circ h)(x) = 16x^2$$

Since $(h\circ j)(x)$ and $(j\circ h)(x)$ are not equal, the function composition is not commutative.

Example 10.14. Let $q(x) = \underline{2x}$ and $r(x) = \underline{-x^2-1}$. Compute: $(r\circ q)(4)$ and $(q\circ r)(4)$.

Let's start with $(r\circ q)(x) = r(q(x))$ (by composition).

$$r\underline{(q(x))} = r\underline{(2x)}$$

Plug in $(2x)$ into every x in $-x^2-1$.

$$\begin{aligned}&= -\underline{(2x)}^2-1\\ &= -4x^2-1\end{aligned}$$

Now let's do $(q\circ r)(x) = q(r(x))$ (by composition).

$$q(\underline{r(x)}) = q(\underline{-x^2 - 1})$$

Plug in $(-x^2 - 1)$ into every x in $2x$.

$$\begin{aligned} &= 2\underline{(-x^2 - 1)} \\ &= -2x^2 - 2 \end{aligned}$$

Therefore, we have,

$$\begin{aligned} (r \circ q)(4) &= -4(4)^2 - 1 \\ &= -4(16) - 1 \\ &= -64 - 1 \\ &= -65 \end{aligned}$$

$$\begin{aligned} (q \circ r)(4) &= -2(4)^2 - 2 \\ &= -2(16) - 2 \\ &= -32 - 2 \\ &= -34 \end{aligned}$$

Hence, $(r \circ q)(4) = -65$ and $(q \circ r)(4) = -34$.

10.3 Additional Problems

1. Consider the functions $f(x) = 9x^3 - x^2 - 1$ and $g(x) = 2x^2$.
   (a) Find $(f+g)(x)$, $(f-g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$, and find the value(s) not in the domain of $\frac{f}{g}$. Lastly, evaluate $(f+g)(-1)$.
2. Consider the functions $h(x) = 4x^3 + 3x^2 - 5$ and $k(x) = 5x^2 - 2$.

(a) Find $(h+k)(x)$, $(h-k)(x)$, $(hk)(x)$, $\left(\frac{h}{k}\right)(x)$, and find the value(s) not in the domain of $\frac{h}{k}$. Lastly, evaluate $(h+k)(2)$.

3. Consider the functions $p(x) = x^2 - 3$ and $q(x) = \sqrt{x-2}$.

    (a) Compute: $(q \circ p)(4)$ and $(p \circ q)(4)$.

4. Consider the functions $f(x) = x^2 + 7$ and $g(x) = \sqrt{x+4}$.

    (a) Compute: $(g \circ f)(5)$ and $(f \circ g)(5)$.

11 Exponential Functions

11.1 Graphing Exponentials

For those of you who are going into business, finance, or any major that requires statistics, this section will probably be the most important one. In this section, the concepts and applications are the most challenging parts. Most of the problems will require a formula where we just plug in numbers to get an answer. However, understanding the variables and how to manipulate them is crucial. With many formulas, it can easily become overwhelming. As you may have realized with all my yapping that this chapter is more applied. We already understand the power rules. In a way, we are doing the same thing, but now we are focusing on exact numbers. Enough of the mumbo jumbo, let's get to the math, shall we?

Let's look at a less complected family tree that only involves you. You have a mom and a dad and they too had parents. Inductively, we have a table that looks something like this.

$n=0$			You		
$n=1$		Mom		Dad	
$n=2$	Mom's dad	Mom's mom		Dad's dad	Dad's mom
$n=\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$

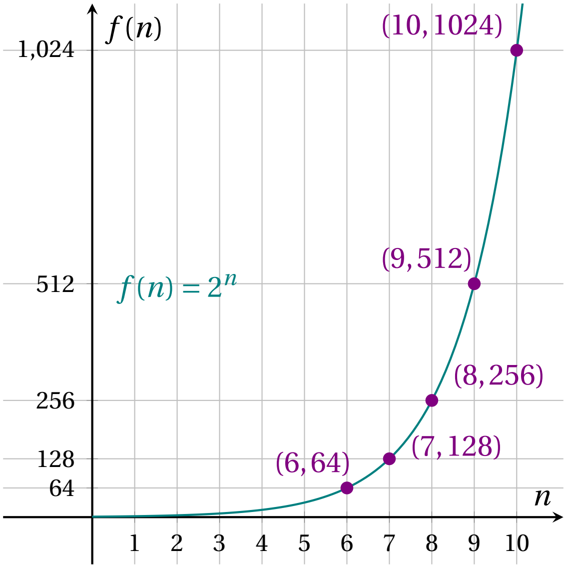
If I go any longer the table will exceed the margins of the page. However, you get it. Your family tree is an exponential, in fact this one is $f(n) = 2^n$. There is an important significance for each line. Within them, lies of numerous love stories that had to unfold precisely as they did for you to come to existence. Life itself is a constant battle against probability, yet despite the odds, it is we who stand here today.

Consider the function $f(n) = 2^n$. Let's create a t-table for this function.

x	$f(n) = 2^n$	**Family**
0	$2^0 = 1$	You
1	$2^1 = 2$	Mom and Dad
2	$2^2 = 4$	Mom's dad, Mom's mom, Dad's dad, Dad's mom
3	$2^3 = 8$	$\vdots$
4	$2^4 = 16$	$\vdots$
5	$2^5 = 32$	$\vdots$
6	$2^6 = 64$	$\vdots$
7	$2^7 = 128$	$\vdots$
8	$2^8 = 256$	$\vdots$
9	$2^9 = 512$	$\vdots$
10	$2^{10} = 1024$	$\vdots$
11	$2^{11} = 2048$	$\vdots$
12	$2^{12} = 4096$	$\vdots$
13	$2^{13} = 8192$	$\vdots$
14	$2^{14} = 16,384$	$\vdots$
15	$2^{15} = 32,768$	$\vdots$
16	$2^{16} = 65,536$	$\vdots$

You see how fast this increases. If you look closely, you can observer that the numbers double. This is an other way of viewing exponetials. When we consider the family tree up to $n = 10$ (10 generations ago), there are a total of $\frac{1024}{2} = 512$ pairs of love stories that must've unfolded for your existence [21].
We will conclude this example by graphing $f(n) = 2^n$.

[21] Can you tell that this is my favorite section?



Consider the point $(10, 1024)$, what does this really mean? This says that if you go back 10 generations in your family tree, then there are 1024 people that eventually link up to you. Note, that this is excluding siblings. It is just you, your parents, their parents, etc.

Let's properly define what an exponential function is.

Definition

An ***exponential function*** with base b is defined by the equation:

$$f(x) = b^x$$

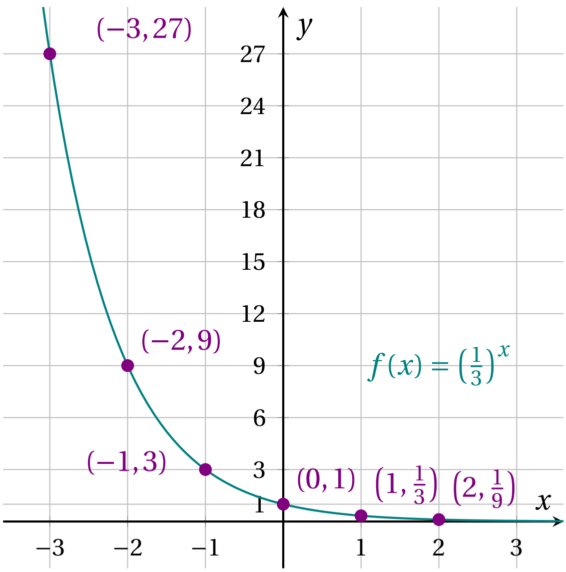
where $b > 0, b \neq 1$ and x is a real number.

Example 11.1. Graph the following exponential function $f(x) = \left(\frac{1}{3}\right)^x$

The t-table for the function $f(x) = \left(\frac{1}{3}\right)^x$.

x	$f(x) = \left(\frac{1}{3}\right)^x$
-3	27
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$
3	$\frac{1}{27}$

In this table, each value of x is plugged into the function $f(x) = \left(\frac{1}{3}\right)^x$ to determine the corresponding output value. Using this information, let's graph $f(x) = \left(\frac{1}{3}\right)^x$.

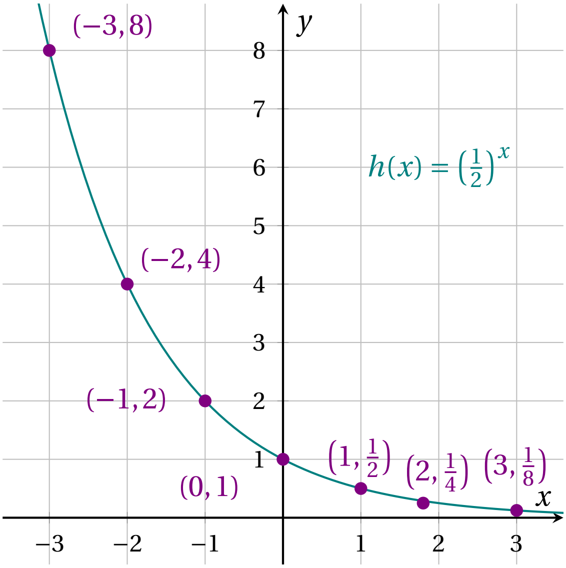


Example 11.2. Graph the following exponential function $h(x) = \left(\frac{1}{2}\right)^x$

The t-table for the function $h(x) = \left(\frac{1}{2}\right)^x$.

x	$h(x) = \left(\frac{1}{2}\right)^x$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$

This table presents the outcome of inserting different x values into the function $h(x) = \left(\frac{1}{2}\right)^x$. Armed with these calculated values, we will now proceed to plot the graph of $h(x) = \left(\frac{1}{2}\right)^x$.

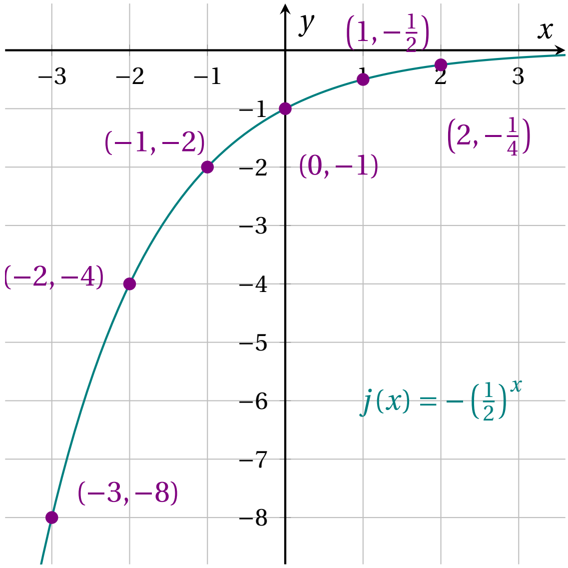


Example 11.3. Graph the following exponential function $j(x) = -\left(\frac{1}{2}\right)^x$

The t-table for the function $j(x) = -\left(\frac{1}{2}\right)^x$.

x	$j(x) = -\left(\frac{1}{2}\right)^x$
-3	-8
-2	-4
-1	-2
0	-1
1	$-\frac{1}{2}$
2	$-\frac{1}{4}$

In this table, we input various values of x into the function $j(x) = -\left(\frac{1}{2}\right)^x$ to obtain the respective result for each. With these results at hand, we will now proceed to construct the graph of the function $j(x) = -\left(\frac{1}{2}\right)^x$.

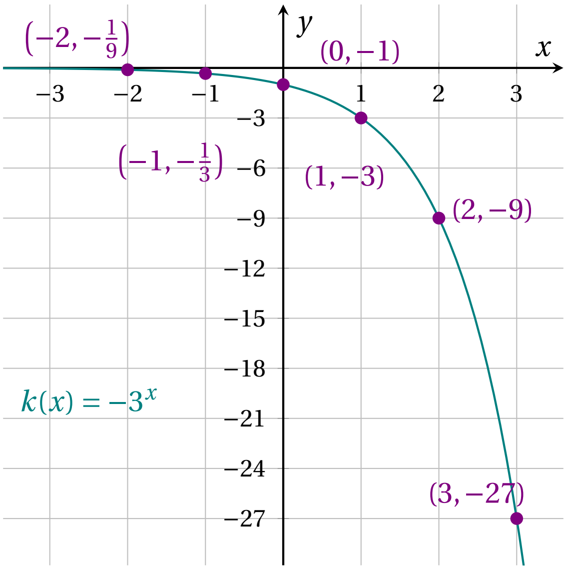


Example 11.4. Graph the following exponential function $k(x) = -3^x$

The t-table for the function $k(x) = -3^x$.

x	$f(x)$
-9	Undefined
-4	Undefined
-1	Undefined
0	0
1	1
4	2
9	3

In this table, we calculate the output values by substituting each x value into the function $k(x) = -3^x$. With these computed values, we will proceed to plot the graph of $k(x) = -3^x$.



11.2 Application

As we have observed, exponential functions provide an excellent means to describe rapid growth. Examples of such natural structures include population growth, the spread of infectious diseases like COVID-19, temperature increase and decay, and more. Exponential functions offer a precise and effective way to characterize various natural activities. Using them leads to numerous applications in different fields. In this chapter, we will focus on thoroughly analyzing the properties of exponential functions.

Theorem

Analyzing Exponential Functions A function in the following form models exponential growth or decay (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

- $f(t)$ is the final amount.
- The constant a is the initial amount when no time has passed i.e. $t = 0$.
- The constant b tells whether the function models exponential growth or decay. From this value we can also find the rate of growth/decay.
  - If $b > 1$, the function models growth; $b = 1 + r$ where r is the rate of growth.
  - If $0 < b < 1$, then the function models decay; $b = 1 - r$ where r is the rate of decay.

Let's give some more concrete [22] examples using this application.

[22] The Chernobyl accident that occurred in 1986 was caused by a flawed reactor design and a combination of other factors. The accident boiled down to inadequate training, engineering and poor management. Many individuals in close proximity to the explosion suffered from radiation sickness and passed to their injuries. Due to the high levels of radiation, the bodies were buried in cemented coffins as a precautionary measure.

Example 11.5. A car is purchased for $24,500. After each year, the resale value of the car decreases by 35%. What is the resale value be after 4 years?

Notice that this is an exponentially decaying problem since the car value is decreasing. We are given that,

$$a = \$24,000$$
$$r = .35$$

By definition,

$$b = 1 - r \quad \text{Since } r = .35$$
$$b = 1 - .35$$
$$b = .65$$

By definition, A function in the following form models exponential growth or decay (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

Thus,

$$f(t) = a(b)^t$$
$$f(t) = 24500(.65)^t$$
$$f(t) = 24500(.65)^4$$
$$f(t) = \$4373$$

Example 11.6. A student takes out a loan for $26,000 for 10 years at 8.5% interest, compounded annually. If the loan is paid back in full at the end of the 10 year period, how much must be paid back in total?

Notice that this is an exponentially growth problem since the loan value is increasing. We know that,

$$a = \$26,000$$
$$r = .085$$

By definition,

$$\begin{gathered} b = 1 + r \quad \text{Since } r = .085. \\ b = 1 + .085 \\ b = 1.085 \end{gathered}$$

By definition, A function in the following form models exponential growth or decay (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

Thus,

$$\begin{gathered} f(t) = a(b)^t \\ f(t) = 26000(1.085)^t \\ f(t) = 26000(1.085)^{10} \\ f(t) = \$58785.57 \end{gathered}$$

Example 11.7. A sample of a radioactive isotope has an initial mass of 50 grams. The isotope has a decay rate of 20% per year. What will be the mass of the isotope after 5 years?

Notice that this is an example of exponential decay, as the mass of the radioactive isotope decreases over time. We are given that,

$$\begin{aligned} a &= 50 \text{ grams (initial mass of the isotope)} \\ r &= 0.20 \text{ (decay rate per year)} \end{aligned}$$

By definition,

$$\begin{aligned} b &= 1 - r, \text{ since } r = 0.20 \\ b &= 1 - 0.20 \\ b &= 0.80 \end{aligned}$$

By definition, a function in the following form models exponential growth or decay (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

Thus,

$$f(t) = 50(0.80)^t$$
$$f(t) = 50(0.80)^5$$
$$f(t) \approx 20.48 \text{ grams}$$

Example 11.8. North Sacramento starts with 150 stray cats. Due to the lack of effective animal control measures, the population of stray cats is increasing by 12% every year. What will be the population of stray cats after 5 years?

This scenario illustrates exponential growth, as the stray cat population increases each year. The initial conditions and growth rate are given as.

$$a = 150 \text{ cats (initial population)}$$
$$r = 0.12 \text{ (annual growth rate)}$$

By definition,

$$b = 1 + r, \text{ since } r = 0.12$$
$$b = 1 + 0.12$$
$$b = 1.12$$

According to the exponential growth model (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

Applying this model to our problem:

$$f(t) = 150(1.12)^t$$
$$f(t) = 150(1.12)^5$$
$$f(t) \approx 264.35 \text{ cats}$$

Example 11.9. A certain species of bird has a population of 5,000 individuals. Due to habitat loss, poaching and other environmental factors, their population is decreasing by 8% each year. What will be the population of this bird species after 10 years?

This problem demonstrates exponential decay, as the bird population decreases annually. We start with.

$$a = 5000 \text{ birds (initial population)}$$
$$r = 0.08 \text{ (annual decrease rate)}$$

By definition,

$$\begin{aligned} b &= 1 - r, \text{ since } r = 0.08 \\ b &= 1 - 0.08 \\ b &= 0.92 \end{aligned}$$

In exponential decay, the function takes the form $a(b)^t$.

$$f(t) = a(b)^t$$

Calculating the population after 10 years.

$$\begin{aligned} f(t) &= 5000(0.92)^t \\ f(t) &= 5000(0.92)^{10} \\ f(t) &\approx 2315 \text{ birds} \end{aligned}$$

Example 11.10. A petri dish starts with 100 bacteria. The number of bacteria doubles every 3 hours. How many bacteria will be in the petri dish after 24 hours?

Note that this is an example of exponential growth, as the bacteria population is doubling at regular intervals. In this case,

$$\begin{aligned} a &= 100 \text{ initial number of bacteria} \\ t &= 24 \text{ hours} \\ \text{Doubling time} &= 3 \text{ hours} \end{aligned}$$

Since the population doubles every 3 hours, the rate of growth per hour can be expressed as.

$$b = 2^{\frac{1}{3}} \text{ (because the population doubles every 3 hours)}$$

By definition, an exponential growth function has the form $f(t) = a(b)^t$. Applying this to our problem.

$$\begin{aligned}
f(t) &= 100\left(2^{\frac{1}{3}}\right)^t \\
f(t) &= 100\left(2^{\frac{1}{3}}\right)^{24} \\
f(t) &= 100\left(2^{\frac{24}{3}}\right) \\
f(t) &= 100(2^8) \\
f(t) &= 25600
\end{aligned}$$

Example 11.11. A small town has an initial population of 3,000 people. The population is increasing at a rate of 5% per year. What will be the population after 7 years?

Notice that this is an example of exponential growth, as the population of the town is increasing over time. We are given that,

$$\begin{aligned}
a &= 3000 \text{ people (initial population)} \\
r &= 0.05 \text{ (growth rate per year)}
\end{aligned}$$

By definition,

$$\begin{aligned}
b &= 1 + r, \text{ since } r = 0.05 \\
b &= 1 + 0.05 \\
b &= 1.05
\end{aligned}$$

By definition, a function in the following form models exponential growth or decay (where $a, b > 0, b \neq 1$).

$$f(t) = a(b)^t$$

Thus,

$$f(t) = 3000(1.05)^t$$

$$f(t) = 3000(1.05)^7$$
$$f(t) \approx 4200 \text{ people}$$

Example 11.12. As dead sun (star) cools, its temperature $C(t)$ in degrees Celsius after t decades is given by the exponential function

$$C(t) = 24,726(0.99)^t.$$

1. Find the initial temperature of the star.

$$24,726°C$$

2. Does this function represent growth or decay?

Decay, the star is dead, thus cooling. Likewise, $b = .99 < 1$

3. By what percent does the temperature change each decade?

$$1 - .99 = .01$$

Hence, the temperature changes by 1% each decade.

Example 11.13. A certain species of bacteria in a lab culture grows at a rate proportional to its current population. The population $P(t)$ of the bacteria, in thousands, after t hours is given by the exponential function

$$P(t) = 2(1.05)^t.$$

1. Find the initial population of the bacteria.

2 thousands bacteria (since $P(0) = 2 \cdot 1.05^0 = 2$)

2. Does this function represent growth or decay?

Growth, as the population of bacteria is increasing. The base of the exponential, $b = 1.05$, is greater than 1.

3. By what percent does the population change each hour?

$$1.05 - 1 = 0.05$$

Hence, the population grows by 5% each hour.

Theorem

Formula For Compound Interest: If $\$P$ is deposited in an account and interest is paid k times a year at an annual rate r, the amount A in the account t years is given by

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

Example 11.14. Parents invest \$12,000 in a mutual fund for their newborn's college fund, expecting a 10% annual return with quarterly reinvestment. Calculate the fund's value after 18 years.

$$P = 12{,}000$$
$$r = 0.10$$
$$t = 18$$

In this situation, we use the compound interest formula where P is the principal amount, r is the annual interest rate, and t is the time in years. Since the interest is compounded quarterly, the number of compounding periods per year, k, is 4.

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

Plug in the variables respectively.

$$= 12,000\left(1+\frac{0.10}{4}\right)^{4\cdot 18}$$

Work with what is inside the parenthesis before you calculate everything else.

$$\begin{aligned}
&= 12,000(1+0.025)^{72}\\
&= 12,000(1.025)^{72}\\
&\approx \$71,006.74
\end{aligned}$$

Therefore, after 18 years, the value of the account will be approximately $71,006.74.

Example 11.15. An individual invests 20,000 in a retirement account, expecting a 7% annual return with monthly reinvestment. Calculate the account's value after 25 years.

$$\begin{aligned}
P &= 20,000\\
r &= 0.07\\
t &= 25
\end{aligned}$$

In this scenario, we apply the compound interest formula where P is the principal amount, r is the annual interest rate, and t is the time in years. Since the interest is compounded monthly, the number of compounding periods per year, k, is 12.

$$A = P\left(1+\frac{r}{k}\right)^{kt}$$

Plug in the variables respectively.

$$= 20,000\left(1 + \frac{0.07}{12}\right)^{12\cdot 25}$$

Work with what is inside the parenthesis before you calculate everything else.

$$\begin{aligned} &= 20,000\left(1 + \frac{0.07}{12}\right)^{300} \\ &= 20,000(1.0058333)^{300} \\ &\approx \$108,366.77 \end{aligned}$$

Therefore, after 25 years, the value of the account will be approximately 108,366.77.

Example 11.16. A scientist observes that a certain type of bacteria doubles in number every 3 hours in a controlled lab environment. If the initial population is 500 bacteria, calculate the population after 24 hours.

$$P_0 = 500$$

where P_0 is the initial population of bacteria.

$$r = \text{doubling every 3 hours}$$

where r represents the growth rate, here implied by doubling.

$$t = 24$$

where t is the total number of hours for observation.

$$k = \frac{24}{3} = 8$$

since the population doubles every 3 hours, there will be 8 doubling periods in 24 hours.

$$P = P_0 \cdot 2^k$$

using the formula for exponential growth in a population

$$= 500 \cdot 2^8$$

substituting the values into the formula.

$$= 500 \cdot 256$$
$$= 128,000$$

calculating the final population after 24 hours.

Therefore, after 24 hours, the bacterial population will be 128,000.

Theorem

Formula For Exponential Growth: Let the quantity P increase or decrease at an annual rate r, compounded continuously, then the amount A after t years is

$$A = Pe^{rt}$$

The number $e = 2.718281828...$ is an important number. We will discuss more about this number in the next section. For, now blindly accept my claims.

Let's do any example. In fact, lets redo our previous example 11.14.

Example 11.17. Parents invest 12,000 in a mutual fund for their newborn's college fund, expecting a 10% annual return with quarterly reinvestment. Calculate the fund's value after 18 years.

Recall that the quality $P = 12{,}000$, $e = 2.718$, rate of increase $r = .10\%$ and time $t = 18$.

$$A = Pe^{rt}$$

Plug in the values.

$$A = 12{,}000e^{0.10 \cdot 18}$$
$$A = 12{,}000e^{1.8}$$
$$A \approx 72{,}595.77$$

Notice that this is around \$1,589 more than the result in the last example. This is due to the last example relaying on interest that was compounds quarterly. Haaaaa!

Example 11.18. A company invests 15,000 in new technology, expecting an average annual growth rate of 12%. Calculate the value of the investment after 10 years using continuous compounding.

Recall that the principal amount $P = 15{,}000$, $e = 2.718$ (the base of the natural logarithm), rate of increase $r = 0.12$, and time $t = 10$.

$$A = Pe^{rt}$$

Apply the formula for continuous compounding.

$$= 15{,}000e^{0.12 \cdot 10}$$

Substitute the values into the formula and calculate.

$$= 15{,}000e^{1.2}$$
$$\approx 49{,}209.34$$

Hence, after 10 years, the value of the investment, with continuous compounding, will be approximately 49,209.34.

Example 11.19. A small lake's fish population is estimated to be 3,000 at the beginning of the year. Biologists predict that the population will increase at a continuous annual rate of 20%. Calculate the fish population at the end of 5 years.

Given that the initial population $P = 3{,}000$, $e = 2.718$, the rate of increase $r = 0.20$, and time $t = 5$.

$$N = Pe^{rt}$$

Apply the formula for continuous growth.

$$= 3{,}000e^{0.20 \cdot 5}$$

Substitute the values into the formula and simplify.

$$= 3{,}000e^{1}$$
$$\approx 22{,}255$$

Therefore, after 5 years, the estimated fish population in the lake will be approximately 22,255.

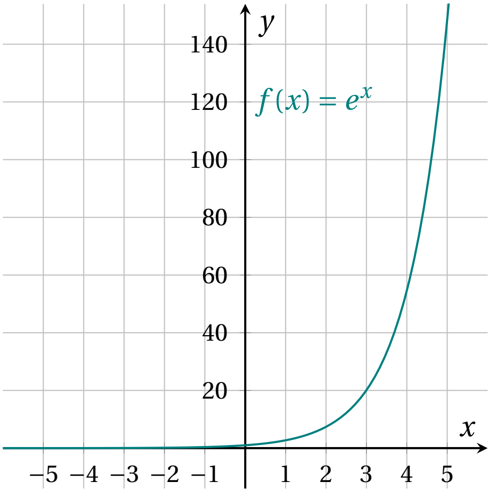
11.3 e^x

Hands down, my favorite function is $f(x) = e^x$. I am fascinated with the concept of evaluation and extinction. Human extinction is not something that we usually think about. However, extinction is a natural process of life. Cosmology is my favorite field of study in physics. This field explores the evolution of the universe. It is proven that the celestial bodies of immense energy stars (suns) have a finite lifespan. The duration of a star's lifespan boils down to the size of the star. The life span varies from millions to billions of years and this depends on its specific characteristics. However, the demise of a star involves various variables and stages. Nevertheless, it is a fact that all entities within the vast of our entire universe will meet their end. Biologically speaking, humans will meet extinction one day. Whether that be in a few hundred years or thousands of years. No one truly knows. Additionally, no one knows how humans will meet their end. There are endless ways in which such devastations may occur. Nuclear warfare, climate change, asteroid impacts, aliens, AI, and volcanism just to name a few. I know this sounds a bit nutty; however, it is estimated that 99.9 percent of all plant and animal species that ever lived have gone extinct. Think about that probability; it is inevitable. This is a topic that I always find myself thinking about. It is what I do not know that catches my attention. The fact that I will most likely die without knowing the fate of humans makes me want to know more. Likewise, humans have a high influence on extinction. Both on ourselves and other species. Each year dozens of plant and animal species go extinct due to poaching and other anthropogenic stresses.

Definition

The function $f(x) = e^x$ is a unique exponential function. Here $e = 2.718281828...$ is a continuous decimal. I.e an irrational number.

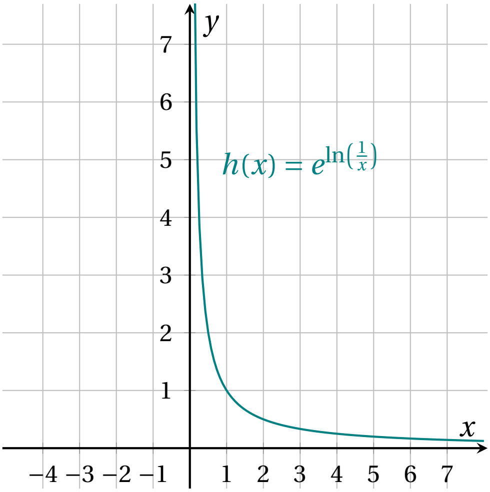
Lets graph $f(x) = e^x$



Food for thought. Currently, there are around 8.7 million species that exist on Earth. Out of these, it is estimated that 1 million are facing threats of extinction due to human activities. Ridiculously, 11.5 percent of species are going extinct because of us. This is caused by factors such as pesticides, poaching, climate changes, and deforestation, just to name a few. The most saddening aspect is that we, as humans, are aware that this is happening, yet we do little to nothing about it. Anyways, back to the material.

If e^x grows continuously, then why does extinction occur? Notice, that extinction occurs when the graph approaches 0. Once there are 0 humans, extinction [23] occurs since there are no humans to reproduce. How do we get this graph to approach 0 though? How does e^x converge to 0? Let $x = \ln\left(\frac{1}{x}\right)$. Where the $\lim_{x\to\infty} e^{\ln\left(\frac{1}{x}\right)} = 0$ (This is calculus gibberish. Thus, fret not, you needn't need to know this). That said, you can expect an e^x and a $\ln(x)$ function being of importance when talking about extinction and population growth. This is but a silly example, the study of extinction is more rigorous than this. Which brings us to our last chapter. The inverses of e^x, the function $\ln(x)$.

Let's now graph $h(x) = e^{\ln\left(\frac{1}{x}\right)}$.



[23]We are going to die, and that makes us the lucky ones. Most people are never going to die because they are never going to be born. The potential people who could have been here in my place but who will in fact never see the light of day outnumber the sand grains of Arabia. - Richard Dawkins

Hence, approaching 0. Thus, one of the ways that extinction can be represented is by e^x and $\ln(x)$.

You see! Mathematics is an incredibly powerful tool that can help us better understand the universe around us, providing answers to questions about how and why things work the way they do. While it may be possible to study certain subjects like biology without using math, it's important to consider how far we can really go without the rigor that math provides. Ultimately, the use of math can greatly enhance our ability to conduct research and deepen our understanding of the world we live in. Yeah!

11.4 Additional Problems

For questions $1-3$ use the **Analyzing Exponential Functions theorem.**

1. Obi deposits $15,000 into a high-yield savings account that earns 4.6% interest annually. What will the value of John's savings account be after 6 years?

2. You buy $2,000 worth of stocks. After each year, the value of the stocks decreases by 15%. What will the value of your stocks be after 3 years?

3. A scalper travels to walmart and purchases Pokémon packs for an initial price of $45 each. After each year, the price of the Pokémon packs increases by 65%. What will the price of a Pokémon pack be after 3 years?

For questions $4-5$ use the **Formula For Compound Interest theorem**.

4. A population of a specific fish species is placed in an environment with an initial population of 1,000 fish. The population is expected to grow at an annual rate of 8% with quarterly reproduction cycles (once every four months). Calculate the fish population after 10 years.

5. An investor places \$500 in stocks, and the value of the stocks increases by 48% in single day. Calculate the value of the stocks after one day.

Lastly, for questions $6-7$ use the **Formula For Exponential Growth theorem.**

6. According to a Google search, around a century ago, there were roughly 2 billion people in the world. Assuming an annual growth rate of 1.4%, calculate the world population today.

7. Pops deposits $\$5,000$ into a high-yield savings account that earns an annual interest rate of 3%. Calculate the value of her savings account after 25 years, assuming the interest is compounded monthly.

8. Researchers have shown that cancer cells can divide into up to three or more daughter cells. Assume that a cancer cell divides into thirds. What is the function that models this cell growth? Give your function in terms of time t.

9. You are offered two choices: either you are given $\$1,000,000$ flat or you are given a penny that doubles every day for a whole month (31 days). What option provides you the most money? Provide a table that supports your claim.

10. A certain type of mushroom releases spores that grow into new mushrooms. Each new mushroom releases its own spores the next day, doubling the total number of mushrooms each day. If you start with one mushroom, how many mushrooms will there be after 15days?

12 Logarithmic Functions

12.1 Logarithmic Properties

Let's go out with a bang, and what better way to go out than to talk about logarithmic functions. As previously mentioned, logarithms are mathematical operations that relate to exponents. Logarithmic functions find various applications in different fields. As functional inverses of exponentials, they play a significant role in the natural process related to decay. Additionally, logarithms are proven valuable when testing pH levels and determining the magnitudes of earthquakes. Furthermore, by employing magnitude scales derived from logarithmic functions, we can effectively measure the brightness of stars.

The function $\log_b(x)$ tells us what power of b will give us x. Thus, For every positive base b, there exists a corresponding logarithmic function.

Definition

If $b > 0$ and $b \neq 1$, then the logarithmic function with base b is defined by

$$y = \log_b(x) \iff x = b^y$$

The domain of the log function is $(0, \infty)$. To see this, consider this example.

Example 12.1. The expression $\log_3(-27)$ says "What power of 3 would give me -27?" That is what y will give you $-27 = 3^y$? No such negative x exist.

Notice that any number that you plug in for y, both negative and positive, result in a positive number.

$$3^{-3} = 3^{-1 \cdot 3}$$
$$3^{-3} = \frac{1}{3^3}$$

$$3^{-3} = \frac{1}{27}$$

It turns out that all negative values of y give a positive fraction.

Let's do a bunch of examples using the above example.

Example 12.2. Use the definition of log to simplify $\log_6(36)$. (Hint: Set each $\log_b$ equal to x.)

$$y = \log_6(36) \Longleftrightarrow 36 = 6^y$$

Therefore, y must be 2.

Example 12.3. Simplify the expression $\log_2(8)$.

We set $\log_2(8)$ equal to x.

$$x = \log_2(8)$$

By using the definition of logarithm, we can rewrite the equation.

$$2^x = 8$$

Since 8 can be expressed as 2^3, the equation becomes:

$$2^x = 2^3$$

Equating the exponents, we find that x must be 3. We will define this technique as a theorem in the last section. For now, trust in me.

Example 12.4. Use the definition of log to solve for x $\log_7(x) = 2$.

$$\log_7(x) = 2$$

Applying the definition of the logarithmic.

$$\begin{aligned} 7^2 &= x \\ 49 &= x \end{aligned}$$

Example 12.5. Solve the equation $\log_{36}(x) = \dfrac{1}{2}$.

$$\log_{36}(x) = \frac{1}{2}$$

Applying the definition of the logarithmic.

$$\begin{aligned} 36^{\frac{1}{2}} &= x \\ \sqrt{36} &= x \\ 6 &= x \end{aligned}$$

Example 12.6. Solve the equation $\log_{27}(y) = \frac{1}{3}$.

$$\log_{27}(y) = \frac{1}{3}$$

Applying the definition of the logarithm.

$$\begin{aligned} 27^{\frac{1}{3}} &= y \\ \sqrt[3]{27} &= y \\ \sqrt[3]{3^3} &= y \\ 3 &= y \end{aligned}$$

Example 12.7. Solve the equation $\log_6(x) = 0$.

$$\log_6(x) = 0$$

Applying the definition of the logarithmic.

$$\begin{aligned} 6^0 &= x \\ 1 &= x \end{aligned}$$

Example 12.8. Solve the equation $\log_{33820191}(x) = 0$.

$$\log_{33820191}(x) = 0$$

Applying the definition of the logarithmic.

$$\begin{aligned} 33820191^0 &= x \\ 1 &= x \end{aligned}$$

That is when $y = 0$, then $x = 1$ for any b we choose. Let's generalize this example.

Example 12.9. Solve the equation $\log_b(x) = 0$.

$$\log_b(x) = 0$$

Applying the definition of the logarithmic.

$$b^0 = x$$

$$1 = x$$

Example 12.10. Solve the equation $\log_x\left(\frac{25}{16}\right) = 2$.

$$\log_x\left(\frac{25}{16}\right) = 2$$

Applying the definition of the logarithmic.

$$x^2 = \frac{25}{16}$$
$$x^2 = \left(\frac{5}{4}\right)^2$$
$$x = \frac{5}{4}$$

Remember that the base x can not be negative.

Example 12.11. Evaluate $y = \log_2\left(\frac{1}{8}\right)$.

$$\log_2\left(\frac{1}{8}\right) = \log_2\left(\frac{1}{2^3}\right)$$

Bring up the 2^3 and reverse the exponent's sign.

$$= \log_2\left(2^{-3}\right)$$

Applying the logarithmic property $\log_b(x) = y$ then $x = b^y$.

$$2^{-3} = 2^y$$

Hence $y = -3$

Example 12.12. Solve the equation $\log_2(x+3) = 4$.

$$\log_2(x+3) = 4$$

Applying the definition of the logarithmic.

$$\begin{aligned} 2^4 &= x+3 \\ 16 &= x+3 \\ 16-3 &= x \\ 13 &= x \end{aligned}$$

Example 12.13. Solve the equation $\log_{10}(2x-1) = 3$.

$$\log_{10}(2x-1) = 3$$

Applying the definition of the logarithmic.

$$\begin{aligned} 10^3 &= 2x-1 \\ 1000 &= 2x-1 \\ 1000+1 &= 2x \\ 1001 &= 2x \\ \frac{1001}{2} &= x \end{aligned}$$

Example 12.14. Solve the equation $\log_5(3x+2) = 2$.

$$\log_5(3x+2) = 2$$

Apply the logarithmic definition to convert the equation.

$$5^2 = 3x + 2$$

Rewriting in exponential form.

$$25 = 3x + 2$$

Solving for x.

$$\begin{aligned} 25 - 2 &= 3x \\ 23 &= 3x \\ \frac{23}{3} &= x \end{aligned}$$

Example 12.15. Evaluate the following.

1. $\log_3(1)$

$$\log_3(1) = y$$

Since this is exactly $3^y = 1$ then $y = 0$.

$$\begin{aligned} \log_3(1) &= 0 \\ 3^0 &= 1 \\ 1 &= 1 \end{aligned}$$

2. $\log_3(3)$

$$\log_3(3) = y$$

Since this is exactly $3^y = 3$ then $y = 1$.

$$\begin{aligned} \log_3(3) &= 1 \\ 3^1 &= 3 \\ 3 &= 3 \end{aligned}$$

3. $\log_3(9)$

$$\log_3(9) = y$$

Since this is exactly $3^y = 9$ then $y = 2$.

$$\log_3(9) = 2$$
$$3^2 = 9$$
$$9 = 9$$

4. $\log_3(27)$

$$\log_3(27) = y$$

Since this is exactly $3^y = 27$ then $y = 3$

$$\log_3(27) = 3$$
$$3^3 = 27$$
$$27 = 27$$

5. $\log_3(1/3)$

$$\log_3\left(\frac{1}{3}\right) = y$$

Since this is exactly $3^y = \frac{1}{3}$ then $y = -1$.

$$\log_3\left(\frac{1}{3}\right) = -1$$
$$3^{-1} = \frac{1}{3}$$
$$\frac{1}{3} = \frac{1}{3}$$

6. $\log_3(1/9)$

$$\log_3\left(\frac{1}{9}\right) = y$$

Since this is exactly $3^y = \frac{1}{9}$ then $y = -2$.

$$\log_3\left(\frac{1}{9}\right) = -2$$

$$3^{-2} = \frac{1}{9}$$

$$\frac{1}{9} = \frac{1}{9}$$

7. $\log_3(1/27)$

$$\log_3\left(\frac{1}{27}\right) = y$$

Since this is exactly $3^y = \frac{1}{27}$ then $y = -3$.

$$\log_3\left(\frac{1}{27}\right) = -3$$

$$3^{-3} = \frac{1}{27}$$

$$\frac{1}{27} = \frac{1}{27}$$

Yeah!

12.2 Natural Logarithms

In this section, I will discuss the key properties of two important mathematical functions: exponential function $f(x) = e^x$ and natural logarithm function $f(x) = \ln(x)$. I believe it is essential to dedicate a separate section to highlight their significance.

Theorem

$$\ln(x) = \log_e(x)$$

We usually denote the natural logarithm function $\ln(x)$ as $\log(x)$ in higher mathematics. As mentioned earlier, logarithms can be thought of as exponents.

Theorem

$$e^{\ln(x)} = x$$

In general, $\log_b(x)$ represents the exponent to which b must be raised to obtain x.

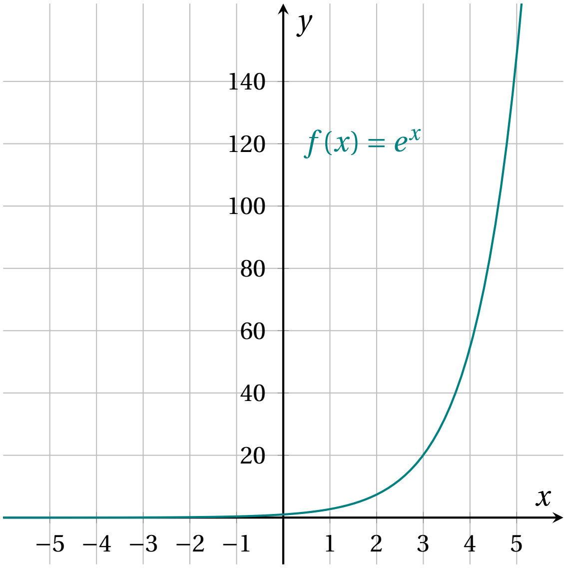
Theorem

$$\ln(e^x) = x$$

Vice-versa, $\ln(x)$ represents the exponent to which the base e must be raised to yield x. This says that the exponential function and natural log function are undoing each other. In other words, they are inverses of each other.

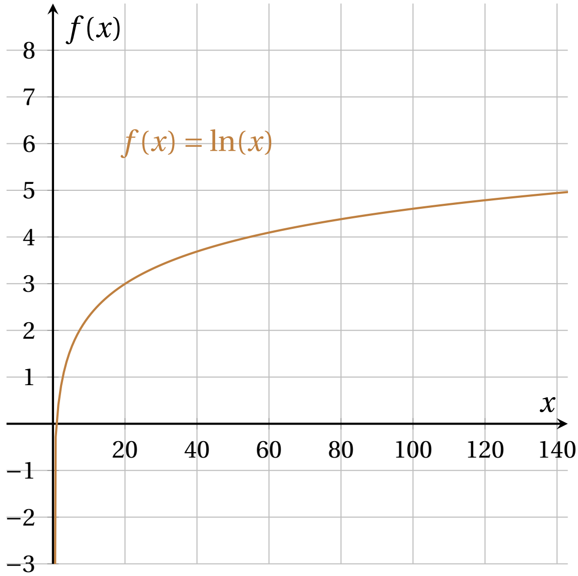
12.3 Graphing Logarithmic & More Properties

We know how the function $f(x) = e^x$ looks like. We also claimed that $\ln(x)$ is the inverse function of $f(x) = e^x$. Thus, the question arises, how does $\ln(x)$ look like? We can do a t-table as shown below, but you'll see that the numbers do not come out pretty. First, lets re-graph e^x.



x	$f(x) = \ln(x)$
0.1	−2.3026
20	3.0445
40	3.6889
60	4.0943
80	4.3820
100	4.6052
120	4.7875
140	4.9416

Using these points we get the following graph which represents $f(x) = \ln(x)$.



Example 12.1 demonstrated that when dealing with logarithms, obtaining a negative value for x is impossible. This is because $x = b^y$ is positive for all values of y. Here are some other important properties about logarithmics.

Theorem

If b is a positive, $b \neq 1$, then

$$\begin{aligned} \log_b(1) = 0 &\text{ because } b^0 = 1 \\ \log_b(b) = 1 &\text{ because } b^1 = b \\ \log_b(b^x) = x &\text{ because } b^x = b^x \end{aligned}$$

$$b^{\log_b(x)} = x$$

because $\log_b(x)$ is the exponent to which b is raised to get x.

The following properties are of utmost importance as they may unpredictably appear in your future classes. Hence, it is crucial to allocate time to memorize and comprehend them now.

Theorem

If M, N and b are positive numbers ($b \neq 1$) then

$$\log_b(M \cdot N) = \log_b(M) + \log_b(N)$$

Theorem

If M, N and b are positive numbers $(b \neq 1)$ then

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$$

Theorem

If M, N, p and b are positive numbers $(b \neq 1)$ then

$$\log_b(M^p) = p\log_b(M)$$

Example 12.16. Expand $\log_3(8xy)$.

$$\log_3(8xy) = \log_3(2^3 \cdot x \cdot y)$$

Expressing 8 as 2^3 and applying the property $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$.

$$= \log_3(2^3) + \log_3(x) + \log_3(y)$$

Applying the property $\log_b(M^p) = p\log_b(M)$.

$$= 3\log_3(2) + \log_3(x) + \log_3(y)$$

Example 12.17. Expand $\log_5(32x^2y^3)$.

$$\log_5(32x^2y^3) = \log_5(2^5 \cdot x^2 \cdot y^3)$$

Expressing 32 as 2^5 and applying the property $\log_b(M \cdot N \cdot P) = \log_b(M) + \log_b(N) + \log_b(P)$.

$$= \log_5(2^5) + \log_5(x^2) + \log_5(y^3)$$

Applying the property $\log_b(M^p) = p\log_b(M)$.

$$= 5\log_5(2) + 2\log_5(x) + 3\log_5(y)$$

Example 12.18. Simplify and expand the expression $\log_4\left(\dfrac{64x^5y}{z^2}\right)$.

$$\log_4\left(\frac{64x^5y}{z^2}\right) = \log_4(64x^5y) - \log_4(z^2)$$

Using the property $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$.

$$= \log_4(2^6x^5y) - \log_4(z^2)$$

Expressing 64 as 2^6.

$$= \log_4(2^6) + \log_4(x^5) + \log_4(y) - \log_4(z^2)$$

Applying the property $\log_b(MN) = \log_b(M) + \log_b(N)$.

$$= 6\log_4(2) + 5\log_4(x) + \log_4(y) - 2\log_4(z)$$

Using the power rule $\log_b(M^p) = p\log_b(M)$.

Example 12.19. Expand $\log_2\left(\frac{16}{x^2}\right)$

$$\log_2\left(\frac{16}{x^2}\right) = \log_2(16) - \log_2(x^2)$$

Applying the property $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

$$= \log_2(2^4) - \log_2(x^2)$$

Expressing 16 as 2^4 and applying the property $\log_b(M^p) = p\log_b(M)$

$$= 4\log_2(2) - 2\log_2(x)$$

Example 12.20. Expand $\log_5\left(\sqrt{\frac{25x}{y^3}}\right)$.

$$\log_5\left(\sqrt{\frac{25x}{y^3}}\right) = \log_5\left(\frac{25x}{y^3}\right)^{\frac{1}{2}}$$

Applying the property $\log_b(M^p) = p\log_b(M)$.

$$= \frac{1}{2}\log_5\left(\frac{25x}{y^3}\right)$$

Using $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$

$$= \frac{1}{2}\left(\log_5(25x) - \log_5(y^3)\right)$$

Using $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$

$$= \frac{1}{2}\left(\log_5(25) + \log_5(x) - 3\log_5(y)\right)$$

$$= \frac{1}{2}\left(\log_5(5^2) + \log_5(x) - 3\log_5(y)\right)$$

$$= \frac{1}{2}\left(2 + \log_5(x) - 3\log_5(y)\right)$$

$$= 1 + \frac{1}{2}\log_5(x) - \frac{3}{2}\log_5(y)$$

Example 12.21. Expand using properties of logarithms. Assume that all variables are positive. $\log_3\left(\frac{xy^3}{z^2}\right)$

$$\log_3\left(\frac{xy^3}{z^2}\right) = \log_3(xy^3) - \log_3(z^2)$$

Applying the quotient rule of logarithms $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$.

$$= \log_3(x) + \log_3(y^3) - \log_3(z^2)$$

Applying the product rule to the first term $\log_b(M \cdot N) = \log_b(M) + \log_b(N)$.

$$= \log_3(x) + 3\log_3(y) - 2\log_3(z)$$

Applying the power rule to the second and third terms $\log_b(M^p) = p\log_b(M)$.

Example 12.22. Combine into a single logarithm $\log_b(x) + \log_b(x+5) - \log_b(6)$.

$$\log_b(x) + \log_b(x+5) - \log_b(6) = \log_b(x \cdot (x+5)) - \log_b(6)$$

Using the property $\log_b(M) + \log_b(N) = \log_b(M \cdot N)$ to combine the first two terms.

$$= \log_b\left(\frac{x \cdot (x+5)}{6}\right)$$

Using the property $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ to simplify the expression further

Example 12.23. Combine into a single logarithm $\log_a(y-3) + 2\log_a(y+1) - \frac{1}{2}\log_a(4)$.

$$\log_a(y-3) + 2\log_a(y+1) - \frac{1}{2}\log_a(4) = \log_a(y-3) + \log_a((y+1)^2) - \log_a(\sqrt{4})$$

Applying the power rule $\log_b(M^p) = p\log_b(M)$ to the second and third terms.

$$= \log_a(y-3) + \log_a((y+1)^2) - \log_a(2)$$

Simplifying $\sqrt{4}$ to 2.

$$= \log_a\left((y-3) \cdot (y+1)^2\right) - \log_a(2)$$

Using the property $\log_b(M) + \log_b(N) = \log_b(M \cdot N)$ to combine the first two terms.

$$= \log_a\left(\frac{(y-3)\cdot(y+1)^2}{2}\right)$$

Using the property $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ to simplify the expression further.

Example 12.24. Combine into a single logarithm $\log_c(3x) - \log_c(x-2) + \frac{1}{3}\log_c(27)$.

$$\log_c(3x) - \log_c(x-2) + \frac{1}{3}\log_c(27) = \log_c(3x) - \log_c(x-2) + \log_c(27^{\frac{1}{3}})$$

Applying the power rule $\log_b(M^p) = p\log_b(M)$ to the third term.

$$= \log_c(3x) - \log_c(x-2) + \log_c(3)$$

Simplifying $27^{\frac{1}{3}}$ to 3.

$$= \log_c(3x) + \log_c(3) - \log_c(x-2)$$

Reordering the terms for clarity.

$$= \log_c(9x) - \log_c(x-2)$$

Using the property $\log_b(M) + \log_b(N) = \log_b(M\cdot N)$ to combine the first two terms.

$$= \log_c\left(\frac{9x}{x-2}\right)$$

Using the property $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ to simplify the expression further.

Example 12.25. Combine $\log_2(a)-\log_2(b)+\log_2(c)$ into a single logarithm.

$$\log_2(a)-\log_2(b)+\log_2(c)=\log_2\left(\frac{a}{b}\right)+\log_2(c)$$

Using $\log_b(M)-\log_b(N)=\log_b\left(\frac{M}{N}\right)$.

$$=\log_2\left(\frac{a}{b}\cdot c\right)$$

Using $\log_b(M)+\log_b(N)=\log_b(M\cdot N)$

$$=\log_2\left(\frac{ac}{b}\right)$$

Example 12.26. Combine into a single logarithm $\frac{1}{4}\log_2(z)+2\log_2(x)-\log_2(y)$.

$$\frac{1}{4}\log_2(z)+2\log_2(x)-\log_2(y)=\log_2(z^{\frac{1}{4}})+\log_2(x^2)-\log_2(y)$$

Using the power rule $\log_b(M^p)=p\log_b(M)$ for the first and second terms.

$$=\log_2\left(z^{\frac{1}{4}}\cdot x^2\right)-\log_2(y)$$

Using the property $\log_b(M)+\log_b(N)=\log_b(M\cdot N)$ to combine the first two terms.

$$=\log_2\left(\frac{z^{\frac{1}{4}}\cdot x^2}{y}\right)$$

Using the property $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ to simplify the expression further.

Example 12.27. Combine into a single logarithm $3\log_m(u^2) - \frac{1}{2}\log_m(v) + \log_m(w)$.

$$3\log_m(u^2) - \frac{1}{2}\log_m(v) + \log_m(w) = \log_m(u^{2\cdot 3}) - \log_m(v^{\frac{1}{2}}) + \log_m(w)$$

Applying the power rule $\log_b(M^p) = p\log_b(M)$ to the first and second terms.

$$= \log_m(u^6) - \log_m(\sqrt{v}) + \log_m(w)$$

Simplifying $v^{\frac{1}{2}}$ to $\sqrt{v}$.

$$= \log_m(u^6) + \log_m(w) - \log_m(\sqrt{v})$$

Reordering the terms for clarity.

$$= \log_m(u^6 \cdot w) - \log_m(\sqrt{v})$$

Using the property $\log_b(M) + \log_b(N) = \log_b(M \cdot N)$ to combine the first two terms.

$$= \log_m\left(\frac{u^6 \cdot w}{\sqrt{v}}\right)$$

Using the property $\log_b(M) - \log_b(N) = \log_b\left(\frac{M}{N}\right)$ to simplify the expression further.

Important

$$\frac{\log_b(A)}{\log_b(B)} \neq \log_b(A) - \log_b(B) = \log_b\left(\frac{A}{B}\right)$$

Important

$$\log(A-B) \neq \frac{\log_b(A)}{\log_b(B)} = \log_b\left(\frac{A}{B}\right)$$

12.4 Exponential & Logarithmic Equations

In this section, we will explore and analyze a significant property. We then follow by a series of problem-solving exercises related to it.

Theorem

If $a^n = a^m$, then $n = m$.

Example 12.28. If $5^x = 5^3$, then $x = 3$.

Example 12.29. Solve for x.

$$9^{5x} = 9^{20}$$

$a^n = a^m$, then $n = m$

$$\begin{aligned} 5x &= 20 \\ x &= \frac{20}{5} \\ x &= 4 \end{aligned}$$

Example 12.30. Solve for x.

$$
\begin{aligned}
8^{2x} &= 64^5 \\
\left(2^3\right)^{2x} &= \left(2^6\right)^5 \\
2^{6x} &= 2^{30}
\end{aligned}
$$

$a^n = a^m$, then $n = m$

$$
\begin{aligned}
6x &= 30 \\
x &= \frac{30}{6} \\
x &= 5
\end{aligned}
$$

Example 12.31. Solve for x.

$$
\begin{aligned}
27^{x+1} &= 9^{2x-1} \\
\left(3^3\right)^{x+1} &= \left(3^2\right)^{2x-1} \\
3^{3x+3} &= 3^{4x-2}
\end{aligned}
$$

If $a^n = a^m$, then $n = m$.

$$
\begin{aligned}
3x+3 &= 4x-2 \\
3 &= x-2 \\
x &= 5
\end{aligned}
$$

Example 12.32. Solve for y.

$$125^{y-2} = 25^{2y+1}$$

$$\begin{aligned}
\left(5^3\right)^{y-2} &= \left(5^2\right)^{2y+1}\\
5^{3y-6} &= 5^{4y+2}
\end{aligned}$$

If $a^n = a^m$, then $n = m$.

$$\begin{aligned}
3y - 6 &= 4y + 2\\
-6 - 2 &= 4y - 3y\\
-8 &= y
\end{aligned}$$

Example 12.33. Solve for x.

$$\begin{aligned}
\left(\frac{1}{4}\right)^{3x+1} &= 64\\
\left(4^{-1}\right)^{3x+1} &= 2^6\\
4^{-3x-1} &= 2^6\\
\left(2^2\right)^{-3x-1} &= 2^6\\
2^{-6x-2} &= 2^6
\end{aligned}$$

$a^n = a^m$, then $n = m$

$$\begin{aligned}
-6x - 2 &= 6\\
-6x &= 8\\
x &= -\frac{4}{3}
\end{aligned}$$

Example 12.34. Solve for x. State any extraneous solutions if they exist.

$$\ln(x-1) - \ln(5) = \ln(2)$$

Combine the logarithms on the left side using the property $\ln(a) - \ln(b) = \ln\left(\frac{a}{b}\right)$.

$$\ln\left(\frac{x-1}{5}\right) = \ln(2)$$

Raise both side to the e to cancel the ln's.

$$\begin{aligned} e^{\ln\left(\frac{x-1}{5}\right)} &= e^{\ln(2)} \\ \frac{x-1}{5} &= 2 \end{aligned}$$

Solve for x.

$$\begin{aligned} x-1 &= 10 \\ x &= 11 \end{aligned}$$

Check for extraneous solutions. Since $x-1>0$, the original logarithmic equation is valid. Therefore, $x=11$ is the solution and there are no extraneous solutions.

Remember the following theorem. We will utilize this theorem in the following examples.

Theorem

If b is a positive, $b \neq 1$, then

$$\log_b(b^x) = x$$

Example 12.35. Solve for x.

$$10^x = 25$$

$$\log_{10}(10^x) = \log_{10}(25)$$

Taking the logarithm with base 10 of both sides. We use 10 instead of 25 since 10 is raised to the x.

$$x = \log_{10}(25)$$

Since $\log_{10}(25)$ represents the exponent to which we need to raise 10 to obtain 25, we can use a calculator to evaluate it.

$$x \approx 1.39794$$

Example 12.36. Solve for x.

$$2^x = 10$$

Taking the logarithm with base 2 of both sides. We use 2 since 2 is raised to the x and we wish to isolate x.

$$\begin{aligned} \log_2(2^x) &= \log_2(10) \\ x &= \log_2(10) \end{aligned}$$

Example 12.37. Solve for y.

$$3^y = 27$$

Taking the logarithm with base 3 of both sides. We use 3 since 3 is raised to the y and we wish to isolate y.

$$\log_3(3^y) = \log_3(27)$$

$$y = \log_3(27)$$

Since 27 is a power of 3, namely 3^3, this simplifies to:

$$y = 3$$

Example 12.38. Determine the value of x.

$$5^{2x-1} = 125$$

Apply logarithms to both sides to facilitate solving for x. Here, we choose base 5 logarithm because the equation involves powers of 5.

$$\log_5\left(5^{2x-1}\right) = \log_5(125)$$

Simplifying the right-hand side as 125 is 5^3.

$$\log_5\left(5^{2x-1}\right) = \log_5(5^3)$$

Apply the property of logarithms that equates $\log_b(b^y)$ to y.

$$2x - 1 = 3$$

Now solve the linear equation for x.

$$2x = 4$$
$$x = 2$$

Example 12.39. Determine the value of z.

$$4^{3z+2} = 1024$$

To isolate z, apply logarithms to both sides. We select base 4 logarithm due to the presence of a power of 4 in the equation.

$$\log_4\left(4^{3z+2}\right) = \log_4(1024)$$

Recognize that 1024 is 4^5.

$$\log_4\left(4^{3z+2}\right) = \log_4(4^5)$$

Utilize the logarithmic property that equates $\log_b(b^y)$ with y.

$$3z + 2 = 5$$

Proceed to solve the linear equation for z.

$$3z = 3$$
$$z = 1$$

Here we use the following theorem which is the last in this section.

Theorem

If b is a positive, $b \neq 1$, then

$$b^{\log_b(x)} = x$$

Example 12.40. Solve for x.

$$\log_3(x+2) = 4$$

Applying the logarithm with base 3 to both sides.

$$3^{\log_3(x+2)} = 3^4$$

Using the last theorem.

$$\begin{aligned} x+2 &= 3^4 \\ x+2 &= 81 \\ x &= 79 \end{aligned}$$

Example 12.41. Solve for x.

$$\log_2(5x-3) = 2$$

Applying the logarithm with base 2 to both sides, we have.

$$2^{\log_2(5x-3)} = 2^2$$

Using the last theorem.

$$\begin{aligned} 5x-3 &= 2^2 \\ 5x-3 &= 4 \\ 5x &= 7 \\ x &= \frac{7}{5} \end{aligned}$$

Example 12.42. Solve for y.

$$\log_3(7y+1) = 3$$

Convert the equation from logarithmic form to exponential form using base 3.

$$3^{\log_3(7y+1)} = 3^3$$

Using the last theorem.

$$\begin{aligned} 7y+1 &= 3^3 \\ 7y+1 &= 27 \\ 7y &= 26 \\ y &= \frac{26}{7} \end{aligned}$$

Example 12.43. Solve for x.

$$\log_2(3x) = 4$$

Convert the equation from logarithmic form to exponential form using base 2.

$$\begin{aligned} 2^{\log_2(3x)} &= 2^4 \\ 3x &= 2^4 \\ 3x &= 16 \\ x &= \frac{16}{3} \end{aligned}$$

Example 12.44. Combine the logarithms using logarithmic properties and solve for x.

$$\log_3(x^2) - \log_3(x-4) = 2 + \log_3(2)$$

Move all the logs to one side.

$$\log_3(x^2) - \log_3(x-4) - \log_3(2) = 2$$

Apply the quotient rule of logarithms.

$$\log_3\left(\frac{x^2}{2(x-4)}\right) = 2$$
$$\log_3\left(\frac{x^2}{2x-8}\right) = 2$$

Applying the logarithm with base 3 to both sides, we have.

$$3^{\log_3\left(\frac{x^2}{2x-8}\right)} = 3^2$$
$$\frac{x^2}{2x-8} = 3^2$$

Cross multiply.

$$x^2 = (2x-8)\cdot 9$$
$$x^2 = 18x - 72$$

Move all terms to one side to form a quadratic equation.

$$x^2 - 18x + 72 = 0$$

Factorize or use the quadratic formula to solve for x. In this case, we can factor the quadratic equation.

$$(x-6)(x-12) = 0$$

Setting each factor to zero, we have two possible solutions.

$$x - 6 = 0 \quad \Rightarrow \quad x = 6$$
$$x - 12 = 0 \quad \Rightarrow \quad x = 12$$

12.5 Additional Problems

1. Find the value of x in the equation.

   (a) $\log_{10}(1000) = x$

   (b) $\log_4(x^2) = 3$

   (c) $\log_{22}(1) = x$

   (d) $\log_4\left(\frac{1}{16}\right) = x$

   (e) $\log_x\left(\frac{1}{64}\right) = -6$

   (f) $\log_7(x) = -1$

2. Fill in the blanks.

   (a) $\log_{10}(1000) + \log_{10}(3) = \log_{10}(\qquad)$

   (b) $\log_3(8) - \log_3(\qquad) = \log_3(2)$

3. Expand.

   (a) $\log_b\left(\sqrt{\beta\delta}\right)$

   (b) $\log_a(xy^2)$

   (c) $\log_c\left(\frac{x}{y}\right)$

   (d) $\log_m\left(\sqrt[3]{x^2 y}\right)$

   (e) $\log_k(x^3 y\sqrt{z})$

   (f) $\log_c\left(\frac{x^4 y^3}{z^2}\right)$

   (g) $\log_9\left(\frac{r+1}{n\sqrt[3]{c}}\right)$

   (h) $\log_2\left(\sqrt[6]{\frac{x^7 cb^2}{d^2 n^0}}\right)$

4. Combine into a single logarithm.

   (a) $\log_3(x+2) + 2\log_3(y) - \log_3(z)$

   (b) $\log_a(x) + \log_a(y) - \frac{1}{2}\log_a(z)$

   (c) $3\log_b(m) - \log_b(n) + \log_b(p)$

   (d) $2\log_c(x+1) - \log_c(x-1) + \frac{1}{3}\log_c(y^2)$

   (e) $\log_d(u+v) - \log_d(u-v) + \log_d(w)$

   (f) $\frac{1}{4}\log_7(x) + 3\log_7(z) - \log_7(y)$

   (g) $\ln(x) - 3\ln(x) + 4\ln(x)$

5. Solve for x.

(a) $4^x = 64$

(b) $3^x = 9$

(c) $9^{1-2x} = 3^{-2x}$

(d) $2^{2x+3} = 32$

(e) $5^{2x+3} = 125$

(f) $e^{2x} = e^4$

(g) $\ln(5^x) = \ln(7^2)$

(h) $\ln(2x+1) = 3$

(i) $4^{x-2} = 64$

(j) $e^{3x} = e^6$

(k) $\log(3^x) = \log(5^3)$

(l) $\log(2x-3) = 2$

13 The End

Outroduction

I began this book by stating a fact. I am still awaiting a contradiction. Mathematics is the foundation of everything. Whether we are aware of it or not our dependence on math is inevitable. We rely on individuals who have studied mathematics in countless ways. Whether you're scrolling through these words on a screen or flipping through pages, the technologies and systems that brought them to you are governed by code that follows mathematical algorithm. It is through the lens of mathematics that we unlock the inter-connectedness of the sciences. As your proficiency in mathematics grows, you will gradually recognize its immense significance. With your creative abilities, I hope you can harness the mathematics alongside the science and contribute to the betterment of our world. I would like to conclude with these final words. As daunting as it may seem, never stop chasing your dreams, no matter how impossible it may seem. That is, much is uncertain in this world, but one thing that is certain is my faith in you, now and always. I sincerely appreciate your ongoing support and attention.

Jose Alfredo Alanis

Index