Bachelor Thesis

Perfect State Transfer in Qudit Networks via Dynamical Decoupling

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Theoretical Quantum Physics
Prof. Dr. Gernot Alber

Perfect State Transfer in Qudit Networks via Dynamical Decoupling
Transport von Quantenzuständen in Qudit-Netzwerken mit Hilfe von dynamischer Entkopplung

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Darmstadt, im Februar 2014

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1 Introduction

In the field of quantum information theory the ambition is to be able to implement quantum processors using the principles of quantum physics to perform calculations and algorithms not accessible with classical methods. In contrast to a conventional computer, which is based on classical bits, a quantum computer consists of qubits (i.e. quantum bits). A qubit is capable of assuming superpositions between its two states $|0\rangle$ and $|1\rangle$. In addition, entanglement between two or more qubits is possible and desired. A collection of qubits can form a quantum register on which the quantum processor can perform its calculations. It is necessary to protect quantum registers from undesired couplings within the system and with the environment and also to be able to use certain couplings to our benefit.

One particular technique to control quantum systems, which has previously been studied [1], is called *dynamical decoupling*. It is essentially a generalisation of a method which is used in the *nuclear magnetic resonance* research to protect prepared ensembles of nuclear spin states against decoherence. *Dynamical decoupling* can additionally be used to manipulate an existing interaction between multiple qubits within a network [2]. In this thesis we want to generalise this often used application of *dynamical decoupling*, to work also on qudit networks. Qudits are quantum objects with d distinguishable states. They offer the property to store quantum information even under inevitable decoherence effects if a non-classical subspace (i.e. of dimension $d \geq 2$) can be protected from such effects [3, p. 425 ff].

To demonstrate the effectiveness of *dynamical decoupling* in simulations our goal is to perform a certain operation on a qudit network, which is called *perfect state transfer* (PST). This operation is necessary when we want to transport an arbitrary qudit state within the network, for example to use it for calculations. In conventional computers state transfers are accomplished through redundancy: A state is read, cloned, and the copies are sent. However, an arbitrary quantum state can neither be read nor be cloned [4]. In order to perform such a PST we apply a Hamiltonian interaction to the qudit network. We will show that such a Hamiltonian can be accomplished by manipulating an existing nearest neighbour coupling between the involved qudits via *dynamical decoupling*.

Chapter 2 of this thesis will introduce the used PST Hamiltonian and we will study it in detail before we introduce the principles of *dynamical decoupling* in chapter 3. We will see how it can be used to manipulate an acting Hamiltonian responsible for the time evolution of the system. In chapter 4 we will present a special case studied in detail and provide numerical simulation results.

Before we delve into the theory of *dynamical decoupling* let us first establish the basic concepts required to give meaning to the introduced methods.

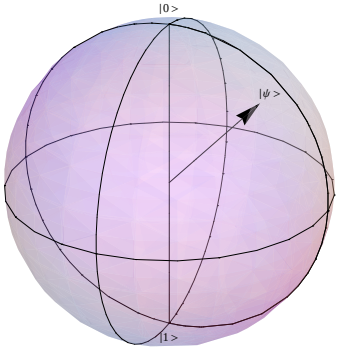

Figure 1.1: A state vector $|\Psi\rangle$ of a qubit depicted in the *Bloch sphere*.

1.1 Qudits

A qudit is an arbitrary quantum object living on the Hilbert space $\mathcal{H}_q = \mathbb{C}^d$. A state vector $|\Psi\rangle$ represents a one-dimensional subspace of $\mathcal{H}_q$. Given an orthonomal basis $\{|i\rangle\}_{i=0}^{d-1}$ of $\mathcal{H}_q$ we can write an arbitrary pure state vector as

$$|\Psi\rangle = \sum_{i=0}^{d-1} a_i |i\rangle \quad \text{with} \quad a_i \in \mathbb{C} \quad \text{and} \quad \sum_{i=0}^{d-1} |a_i|^2 = 1. \tag{1.1}$$

The state vector is unique up to a phase factor $e^{\mathrm{i}\varphi}$, this means that $|\Psi\rangle$ and $e^{\mathrm{i}\varphi}|\Psi\rangle$ represent the same state since they lie in the same subspace of $\mathcal{H}_q$. The phase factor is not observable because the physical quantities we can measure in experiments are squared absolute values $|a_i|^2$.

For the qubit case $d = 2$ we can find a geometrical representation called the *Bloch sphere*. We express the state vector with two real numbers θ and ϕ as

$$|\Psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{\mathrm{i}\phi}\sin\frac{\theta}{2}|1\rangle\,. \tag{1.2}$$

Then we can interpret θ and ϕ as spherical coordinates depicting a point on the surface of a sphere with unit radius. The *Bloch sphere* visualises the state of a qubit, whereby the basis states $|0\rangle$ and $|1\rangle$ represent the north and the south pole of the sphere (as shown in figure 1.1).

1.2 Linear Operators on Hilbert Spaces

Let $\mathcal{B}(\mathcal{H}_q)$ be the space of all linear operators acting on the Hilbert space $\mathcal{H}_q$. In two dimensions the Pauli operators σ_i, where $i \in \{1,2,3\}$, together with the identity operator $\mathbb{1}$, in this context also referred to as σ_0, form a unitary and Hermitian basis of $\mathcal{B}(\mathcal{H}_q)$. The Pauli operators are defined by their action on the basis states, however, if we choose an eigenbasis, for example of σ_3, we can express the state vectors as $|0\rangle = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $|1\rangle = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and thus derive a 2×2 matrix representation of $\mathcal{B}(\mathcal{H}_q)$ with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathbb{i} \\ \mathbb{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.3}$$

1.2.1 Generalised Pauli Matrices

For $d > 2$ we can still construct bases of $\mathcal{H}_q$, however, for the common generalisations we find them not to be both Hermitian and unitary. For arbitrary dimension d we can construct a matrix representation of a unitary basis with the so-called *clock matrix* Σ and *shift matrix* Ξ as introduced in [5]:

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{d-1} \end{pmatrix} \qquad \Xi = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{1.4}$$

with $\omega = e^{\frac{1}{d}2\pi\mathbb{i}}$, a d-th root of unity. Note that $Tr(\Sigma) = Tr(\Xi) = 0$. The d^2 basis matrices for fixed dimension d are composed via

$$\sigma_{d-1-k,j} = \Xi^{d-1-k}\Sigma^j \quad \text{with} \quad j,k \in \{0,1,\ldots,d-1\}.$$

These matrices are then labelled *generalised Pauli matrices*. Let us take a closer look at the three dimensional Hilbert space. A unitary basis is given by

$$\begin{aligned}
\sigma_{0,0} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \sigma_{0,1} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} & \sigma_{0,2} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \\
\sigma_{1,0} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} & \sigma_{1,1} &= \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} & \sigma_{1,2} &= \begin{pmatrix} 0 & \omega^2 & 0 \\ 0 & 0 & \omega \\ 1 & 0 & 0 \end{pmatrix} \\
\sigma_{2,0} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} & \sigma_{2,1} &= \begin{pmatrix} 0 & 0 & \omega^2 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix} & \sigma_{2,2} &= \begin{pmatrix} 0 & 0 & \omega \\ 1 & 0 & 0 \\ 0 & \omega^2 & 0 \end{pmatrix}.
\end{aligned}$$

1.2.2 Generalised Gell-Mann Matrices

In order to derive a Hermitian basis representation of $\mathcal{H}_q$ we can construct the *generalised Gell-Mann matrices* [6] for arbitrary dimension d as follows:

Let $E_{j,k}$ be the $d \times d$-matrix with 1 in the jk-th entry and 0 elsewhere. Then we define the following matrices:

$$\begin{aligned}
&\text{For } k < j: && \eta^d_{k,j} = E_{j,k} + E_{k,j},\\
&\text{for } k > j: && \eta^d_{k,j} = -\mathbb{i}(E_{j,k} - E_{k,j}),\\
&\text{for } k = j = 1: && \eta^d_{k,j} = \mathbb{1}_d, \text{ the identity matrix},\\
&\text{for } 1 < k = j < d: && \eta^d_{k,j} = \eta^{d-1}_{k,j} \oplus 0_d,\\
&\text{for } k = j = d: && \eta^d_{k,j} = \sqrt{\frac{2}{d(d-1)}}(\mathbb{1}_{d-1} \oplus (1-d)E_{d,d}),
\end{aligned}$$

with 0_d the $d \times d$-matrix with no non-zero elements. From now on we omit the upper index d, which is clear from context. In three dimensions the basis is given by

$$\eta_{1,1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \eta_{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \eta_{3,3} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$\eta_{1,2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \eta_{1,3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \eta_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\eta_{2,1} = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \eta_{3,1} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \eta_{3,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}.$$

As bases are equivalent, we can find a transformation between the unitary and the Hermitian basis. Each matrix of the Hermitian basis can be expressed as a linear combination of the unitary matrices.

1.3 Networks of Qudits

We call an ensemble of n distinguishable qudits an n-qudit network. Experimentally there are multiple ways to implement such networks. A common method is to trap single atoms in dipole traps, for example in a composition of lasers, along a line or a grid (for example in [7]). Another implementation would be ions in a doted solid body as shown in [8]. However, the space separations of the qudits will usually form some kind of regular grid structure. In the following we consider a one-dimensional qudit chain. From the basis states of one qudit we can construct the basis states of the network via the tensor product. The set of vectors

$$\{|e_j\rangle\}_{j=1}^{d^2} \quad \text{with} \quad |e_j\rangle = |b_j^{(1)}\rangle \otimes |b_j^{(2)}\rangle \otimes \ldots \otimes |b_j^{(n)}\rangle \tag{1.5}$$

forms an orthonormal basis for the network. Each $|b_j^{(i)}\rangle$ can thereby be each basis state of a single qudit such that the set of vectors $|e_j\rangle$ contains all possible combinations of single qudit states.

Similarly the basis for any linear operator on the quantum network can be constructed via tensor products of the *generalised Pauli* or *Gell-Mann matrices* or other basis representations. The basis is then given as one of the following sets of operators

$$\{S_l\}_{l=1}^{d^{2n}} \quad \text{with} \quad S_l = \sigma_{k_l,j_l}^{(1)} \otimes \sigma_{k_l,j_l}^{(2)} \otimes \ldots \otimes \sigma_{k_l,j_l}^{(n)}, \tag{1.6}$$

$$\{E_l\}_{l=1}^{d^{2n}} \quad \text{with} \quad E_l = \eta_{k_l,j_l}^{(1)} \otimes \eta_{k_l,j_l}^{(2)} \otimes \ldots \otimes \eta_{k_l,j_l}^{(n)}, \tag{1.7}$$

where each $\sigma_{k_l,j_l}^{(i)}$ is one of the d^2 *generalised Pauli matrices* and each $\eta_{k_l,j_l}^{(i)}$ one of the d^2 *generalised Gell-Mann matrices*. The bases can be used equivalently.

1.4 Perfect State Transfer

As the *No Cloning Theorem* [4] states it is not possible to clone or copy an arbitrary qudit state. However, we have the option to transfer the state from one qudit to another within a network. The state of the original qudit will change.

Let us consider an n-qudit chain with arbitrary numbering. Without loss of generality we want to transfer the arbitrary state $|\varphi\rangle$ of qudit 1 onto qudit n. Formally, the qudit network is in the state

$$|\Psi\rangle = |\varphi\rangle \otimes |\Phi\rangle\,, \tag{1.8}$$

where $|\Phi\rangle$ describes the state of the network without qudit 1. We now apply some Hamiltonian operator H for a fixed time T, so that

$$|\Psi'\rangle = e^{-\mathrm{i}HT/\hbar}\,|\Psi\rangle = |\Phi'\rangle \otimes |\varphi\rangle\,, \tag{1.9}$$

where $|\Phi'\rangle$ describes the state of the network without qudit n. From now on we set $\hbar = 1$. We call this process *perfect state transfer*. The choice of the Hamiltonian H and the interaction time T depend on the particular network and possibly also on the chosen input state $|\Phi\rangle$.

The scenario we will use in this thesis to show the effectiveness of the presented *dynamical decoupling* techniques is a *perfect state transfer* Hamiltonian that will perform a swap of the qudit states 1 and n. If the qudits 2 through $n-1$ are in the ground state $|0\rangle$, they remain untouched. For the qubit case we find [9–11]

$$H_{PST}^{Qubit} = \lambda \sum_{i=1}^{n-1} J_i \frac{1}{2}\left(\sigma_1^{(i)}\sigma_1^{(i+1)} + \sigma_2^{(i)}\sigma_2^{(i+1)}\right), \tag{1.10}$$

with $J_i = \frac{1}{2}\sqrt{i(n-i)}$, an interaction strength λ, and the *Pauli operator* $\sigma_j^{(i)}$ applied to the i-th qudit as in

$$\sigma_j^{(i)}\sigma_j^{(i+1)} = \mathbb{1}^{(1)} \otimes \ldots \mathbb{1}^{(i-1)} \otimes \sigma_j^{(i)} \otimes \sigma_j^{(i+1)} \otimes \mathbb{1}^{(i+2)} \otimes \ldots \otimes \mathbb{1}^{(n)}. \tag{1.11}$$

After being active for the time $\frac{\pi}{\lambda}$ this Hamiltonian swaps the arbitrary states of the first and last qubit of the chain. Depending on the length of the chain, i.e. the number of qubits in the network, the Hamiltonian introduces a relative phase change φ. To correct this we can either apply a quantum phase gate or add an appropriately chosen static magnetic field to the Hamiltonian, which will prevent the phase shift.

A generalised Hamiltonian for the qudit case will be introduced in chapter 2.

1.5 Time-reversal of Couplings

Let $\mathcal{B}(\mathcal{H}_q)$ be the set of all linear operators acting on the Hilbert space $\mathcal{H}_q$. Consider an arbitrary linear operator $L \in \mathcal{B}(\mathcal{H}_q)$ governing a coupling between qudits. We want to find a unitary operator $I \in \mathcal{B}(\mathcal{H}_q)$ so that

$$I^\dagger \; L \; I = -L. \tag{1.12}$$

If I meets this condition we call I the *time-reversing* or *inverting operator* for L. In general I is dependent on L. However it is possible to find an inverting operator applicable on specific types of operators as we will see in chapter 4.

1.6 Entanglement Fidelity and State Fidelity

We want to measure the performance of quantum system manipulation. One quantity to do so is the *entanglement fidelity* F_e as shown in [12]. In this case we consider the auxiliary quantum system identical to the prepared system under the influence of the desired time evolution $U_{id}(t)$ and compute the overlap between both systems whilst the actual system is under the time evolution $U(t)$.

$$F_e = \left| \frac{1}{d^n} \mathrm{Tr}\left(U_{id}^\dagger(t) U(t) \right) \right|^2. \tag{1.13}$$

For a more specific look on the performance we will also use the *state fidelity* F_s, where we consider a certain input state $|\Psi\rangle$ under a time evolution $U(t)$ and measure the overlap with a desired output state $|\Psi'\rangle$.

$$F_s = \left| \langle \Psi' | U(t) | \Psi \rangle \right|^2. \tag{1.14}$$

2 Perfect State Transfer Hamiltonian for Qudits

In section 1.4 we have introduced a Hamiltonian that can perform a *perfect state transfer* on a qubit chain. We want to generalise the scenario and derive a Hamiltonian that works on qudits. This has been done but not yet been published by H. Frydrych and we have studied it in collaboration. For a chain composed of n qudits we find

$$H_{PST} = \lambda \sum_{i=1}^{n-1} \sum_{j=2}^{d} J_i \frac{1}{2} \left(\eta_{1,j}^{(i)} \eta_{1,j}^{(i+1)} + \eta_{j,1}^{(i)} \eta_{j,1}^{(i+1)} \right), \tag{2.1}$$

with $J_i = \frac{1}{2}\sqrt{i(n-i)}$, an interaction strength λ, and the *generalized Gell-Mann operator* $\eta_{k,j}^{(i)}$ applied to the i-th qudit. After being active for the time $\frac{\pi}{\lambda}$ this Hamiltonian swaps the arbitrary states of the first and last qudit of the chain, just like H_{PST}^{Qubit} did in the qubit case. In addition we derive a $\sqrt{\text{swap}}$ operation, which is fully entangling two qudits, if we halve the interaction time of the swap operation. Note that again a relative phase change φ will be introduced, which has to be corrected with a quantum phase gate or a magnetic field. For the full swap the phase change is of the form

$$e^{\mathrm{i}\varphi} \quad \text{with} \quad \varphi = \frac{\pi}{2} k \quad \text{for} \quad k = n \mod 4. \tag{2.2}$$

If we compare this general Hamiltonian H_{PST} to the Hamiltonian of the qubit case from section 1.4

$$H_{PST}^{Qubit} = \lambda \sum_{i=1}^{n-1} J_i \frac{1}{2} \left(\sigma_1^{(i)} \sigma_1^{(i+1)} + \sigma_2^{(i)} \sigma_2^{(i+1)} \right), \tag{2.3}$$

we notice that the latter consists of components taken from the *Pauli matrices*, whereas for the qudit case we find components taken from the *generalised Gell-Mann matrices*. However, if we consider the *generalised Gell-Mann matrices* on a 2-dimensional Hilbert space, we find indeed that they coincide with the *Pauli matrices*.

2.1 Nearest Neighbour Coupling

The H_{PST} is very similar to another Hamiltonian H_{NNC}, which will gain importance in chapter 4. This Hamiltonian governs a nearest neighbour interaction with constant coupling strength Λ between the qudits.

$$H_{NNC} = \Lambda \sum_{i=1}^{n-1} \sum_{j=2}^{d} \frac{1}{2} \left(\eta_{1,j}^{(i)} \eta_{1,j}^{(i+1)} + \eta_{j,1}^{(i)} \eta_{j,1}^{(i+1)} \right). \tag{2.4}$$

2.2 Interpretation of the Hamiltonian's Components

Consider the operator $\eta_{k,j}$, with $k \neq j$. From its matrix elements we suspect that it constitutes an interaction between the basis states $|k\rangle$ and $|j\rangle$ of the qudit. When we examine the time evolution of such an operator, indeed we find that it represents a rotation between the relevant states. For example

$$e^{-\mathbb{i}\eta_{2,1}\tau} = \begin{pmatrix} \cos\tau & -\sin\tau & 0 & \cdots & 0 \\ \sin\tau & \cos\tau & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{2.5}$$

Inspecting the *perfect state transfer* Hamiltonian in equation (2.1) we find that it governs a nearest neighbour interaction between the qudits with an interaction strength λJ_i which is strongest in the middle of the chain and decreases symmetrically trending to qudit 1 and n.

2.3 Pauli Expansion

In order for us to construct a decoupling scheme in chapter 3 it is necessary to expand the PST Hamiltonian via the Pauli basis, which was introduced in section 1.2.1. Since the Hamiltonian is represented by a sum of two-qudit couplings, we can find the expansion by solving a linear equation system that contains all possible two-qudit operations from the Pauli basis elements:

$$\sum_{k,l,m,n=0}^{d-1} \xi_{k,l,m,n} \; \sigma_{k,l}^{(i)} \sigma_{m,n}^{(i+1)} = \sum_{j=2}^{d} \frac{1}{2} \left(\eta_{1,j}^{(i)} \eta_{1,j}^{(i+1)} + \eta_{j,1}^{(i)} \eta_{j,1}^{(i+1)} \right) \tag{2.6}$$

with the coefficients $\xi_{k,l,m,n}$ that we need to find. Solving the equations system we derive

$$\xi_{k,l} = \frac{1}{d^2} \left(e^{\mathbb{i}2\pi \frac{-k}{d}} + e^{\mathbb{i}2\pi \frac{l}{d}} \right). \tag{2.7}$$

The coefficients $\xi_{k,l}$ from the basis transformation are shown in figure 2.1. For the Hamiltonians we find

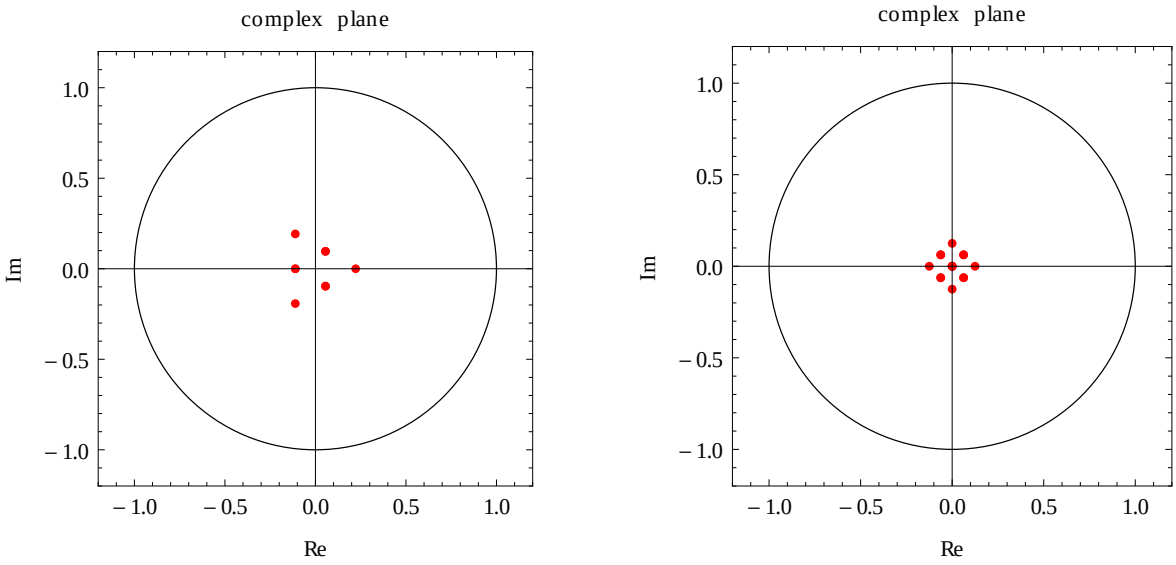

Figure 2.1: The coefficients $\xi_{k,l}$ for the expansion of the PST Hamiltonian in the unitary basis of the *generalised Pauli matrices* in $d = 3$ (left) and $d = 4$ dimensions. With growing dimension d the average absolute value decreases. For dimension d we find $2d$ points.

$$H_{PST} = \lambda \sum_{i=1}^{n-1} \sum_{\substack{k,l=0 \\ j=1}}^{d-1} J_i \; \xi_{jk,jl} \; \sigma_{j,k}^{(i)} \sigma_{d-j,l}^{(i+1)}, \tag{2.8}$$

$$H_{NNC} = \Lambda \sum_{i=1}^{n-1} \sum_{\substack{k,l=0 \\ j=1}}^{d-1} \xi_{jk,jl} \; \sigma_{j,k}^{(i)} \sigma_{d-j,l}^{(i+1)}. \tag{2.9}$$

3 Dynamical Decoupling

The technique we use to protect a quantum network and to design Hamiltonians that act onto the network is called *dynamical decoupling*. We use a set of very fast local control operations on the system to manipulate it. As a first approximation we assume that these control operations are instantaneous (as in bang-bang control [2]) and can thus be represented by a set of unitary operators.

With the chosen operators we perform mathematically a rotation in the state space of the quantum system. When we apply those operators in a certain order it is possible that the effects of unwanted Hamiltonian interactions cancel over time. In general, the couplings only cancel in first order which means that *dynamical decoupling* is an approximate technique and therefore only suppresses errors, but does not achieve perfect error correction. Over the years, however, *dynamical decoupling* has been successfully implemented in experimental setups to protect different realisations of a qubit state [13–16].

Most of the provided theory is inspired by and partly and humbly taken from [17,18].

3.1 Dynamical Decoupling - Basics

Consider a quantum system on a Hilbert space

$$\mathcal{H} = \mathcal{H}_q \otimes \mathcal{H}_{env} \tag{3.1}$$

$$\mathcal{H}_q = \bigotimes_{i=1}^{n} \mathbb{C}^d \tag{3.2}$$

where $\mathcal{H}_q$ describes the quantum system consisting of n qudits in an arbitrary environment $\mathcal{H}_{env}$. The system living in this state space is governed by some Hamiltonian H. If we assume that we are able to instantaneously apply unitary operators υ, which will be more closely discussed below, to all of the qudits individually with a time delay Δt between two successive operators, this leads to a time evolution

$$U(m\Delta t) = p_m e^{-\mathbb{i}H\Delta t} p_{m-1} e^{-\mathbb{i}H\Delta t} p_{m-2} \cdots p_1 e^{-\mathbb{i}H\Delta t} p_0, \tag{3.3}$$

with the control pulses

$$p_k = \upsilon_{k_1} \otimes \upsilon_{k_2} \otimes \cdots \otimes \upsilon_{k_n} \otimes \mathbb{1}_{env}, \tag{3.4}$$

a set of unitary operators applied to the qudits and no interaction with the environment. This time evolution can be rewritten by introducing the control operators

$$g_k = p_k \cdot p_{k-1} \cdot \ldots \cdot p_0. \tag{3.5}$$

Note that

$$p_k = g_k g_{k-1}^\dagger. \tag{3.6}$$

Continuing

$$\begin{aligned} U(m\Delta t) &= g_m g_{m-1}^\dagger e^{-\mathbb{i}H\Delta t} g_{m-1} g_{m-2}^\dagger e^{-\mathbb{i}H\Delta t} g_{m-2} \cdots g_0^\dagger e^{-\mathbb{i}H\Delta t} g_0 && (3.7)\\ &= g_m e^{-\mathbb{i}g_{m-1}^\dagger H g_{m-1}\Delta t} e^{-\mathbb{i}g_{m-2}^\dagger H g_{m-2}\Delta t} \cdots e^{-\mathbb{i}g_0^\dagger H g_0 \Delta t} && (3.8)\\ &\overset{t=m\Delta t}{=} g_m e^{-\mathbb{i}\frac{1}{m} g_{m-1}^\dagger H g_{m-1} t} e^{-\mathbb{i}\frac{1}{m} g_{m-2}^\dagger H g_{m-2} t} \cdots e^{-\mathbb{i}\frac{1}{m} g_0^\dagger H g_0 t}. && (3.9) \end{aligned}$$

Because the g_k are unitary we can move them into the exponents in the second step. The operator g_m is arbitrary and we can choose it to be $\mathbb{1}$, the identity. We identify this expression with the time evolution of an average Hamiltonian $\bar{H}$:

$$U(t) = e^{-\mathbb{i}\bar{H}t}, \quad \text{with} \quad t = m\Delta t. \tag{3.10}$$

To derive a suitable expression for $\bar{H}$ we compare the time derivatives of these time evolutions:

$$\begin{aligned} \frac{d}{dt}U(t) &= -\mathbb{i}\bar{H}U(t) && (3.11)\\ &= -\mathbb{i}\frac{1}{m} g_{m-1}^\dagger H g_{m-1}\ U(t)\\ &\quad + e^{-\mathbb{i}g_{m-1}^\dagger H g_{m-1}\Delta t}\ (-\mathbb{i}\frac{1}{m} g_{m-2}^\dagger H g_{m-2})\ e^{\mathbb{i}g_{m-1} H g_{m-1}^\dagger \Delta t}\ U(t)\\ &\quad + \ldots\\ &\quad + e^{-\mathbb{i}g_{m-1}^\dagger H g_{m-1}\Delta t} \ldots e^{-\mathbb{i}g_0^\dagger H g_0 \Delta t}\ (-\mathbb{i}\frac{1}{m} g_0^\dagger H g_0)\\ &\quad\ e^{\mathbb{i}g_0 H g_0^\dagger \Delta t} \ldots e^{\mathbb{i}g_{m-1} H g_{m-1}^\dagger \Delta t}\ U(t) && (3.12)\\ &\xrightarrow{\Delta t \to 0} -\mathbb{i}\frac{1}{m}\sum_{k=0}^{m-1} g_k^\dagger H g_k\ U(t). && (3.13) \end{aligned}$$

In the last step we identify the lowest order of a series expansion of the average Hamiltonian $\bar{H}$ in respect to Δt with the result. This is equivalent to the *Magnus expansion* [19]

$$\bar{H} = \bar{H}^{(0)} + \bar{H}^{(1)} + \bar{H}^{(2)} + \ldots. \tag{3.14}$$

We find:

$$\bar{H}^{(0)} = \frac{1}{m}\sum_{k=0}^{m-1} g_k^\dagger H g_k. \tag{3.15}$$

Our goal is to find a suitable sequence of control pulses that turns the acting average Hamiltonian into a desired H_{id}. This can include the elimination of couplings between the environment and the quantum network and also the suppression of inner couplings between single qudits:

$$\bar{H}^{(0)} = \frac{1}{m}\sum_{k=0}^{m-1} g_k^\dagger H g_k = \frac{1}{D} H_{id}. \tag{3.16}$$

Here $\frac{1}{D}$ is a possible scaling factor, which can be compensated by adjusting the interaction time accordingly. Usually $D \geq 1$.

Dynamical decoupling is an approximate method because in general higher orders of $\bar{H}$ do not vanish. However, since $\bar{H}^{(k)} \sim (\Delta t)^k$, with sufficiently small delay times Δt between the control pulses we are able to improve the fidelity of the system significantly, depending on the particular scenario.

3.2 Existence and Design of Decoupling Schemes

The control pulses are represented by a set of unitary operators. From chapter 1.3 we know that the set of unitary operators $\{S_l\}_{l=1}^{d^{2n}}$ forms a basis for all linear operators acting on $\mathcal{H}_q$. We can therefore expand H and H_{id} in this basis with coefficients μ_l and ν_l: (Note that we could find alternative linearly independent unitary basis decompositions that are appropriate.)

$$H = \sum_{l=1}^{d^{2n}} \mu_l \; S_l \otimes O_l + H_{env} \tag{3.17}$$

$$H_{id} = \sum_{l=1}^{d^{2n}} \nu_l \; S_l \otimes \mathbb{1}_{env} + H_{env}, \tag{3.18}$$

with arbitrary linear operators O_l on $\mathcal{H}_{env}$ and the Hamiltonian H_{env} only acting on the environment and thus of no concern to us.

Without loss of generality H and H_{id} can be chosen to be traceless: If $Tr(H) \neq 0$ we choose a new Hamiltonian $H' = H - c\mathbb{1}$ with $Tr(c\mathbb{1}) = Tr(H)$. Knowing $[\mathbb{1}, H] = 0$ this leads to a new time propagation $U'(t) = e^{-\mathbb{i}c\mathbb{1}t}e^{-\mathbb{i}Ht}$ that is identical to the original time propagation $U(t) = e^{-iHt}$ up to a global phase, which is of no concern to us since it is not observable. Thus the new Hamiltonian describes the same physics as the original one, however, in addition $Tr(H') = 0$. Thus we can omit the identity operator $S_1 = \mathbb{1}$ from the expansion. With the decoupling condition (3.16) this leads to

$$\frac{1}{m}\sum_{l=2}^{d^{2n}}\sum_{k=1}^{m-1} \mu_l \; g_k^\dagger S_l g_k \otimes O_l = \frac{1}{D}\sum_{l=2}^{d^{2n}} \nu_l \; S_l \otimes \mathbb{1}. \tag{3.19}$$

If we choose the operators g_k from the same set as in the expansion of the Hamiltonian we can argue that

$$\sum_{k=1}^{m-1} g_k^\dagger S_l g_k = \sum_{k=2}^{d^n} c_k \; S_k^\dagger S_l S_k, \tag{3.20}$$

with the number of occurrence $c_k \in \mathbb{N}$ of the operator S_k in the scheme. Consider the properties of the unitary basis operators $\sigma_{k,j}$ forming the basis $\{S_l\}_{l=1}^{d^{2n}}$ with the involved unity roots $\omega = e^{\frac{1}{d}2\pi\mathbb{i}}$:

$$\begin{aligned}
\sigma_{k,j}^\dagger &= \omega^{kj}\sigma_{d-k,d-j} \\
\sigma_{k,j}\sigma_{s,t} &= \omega^{ks}\sigma_{j+s,k+t} \\
\sigma_{s,t}^\dagger\sigma_{k,j}\sigma_{s,t} &= \omega^{js-kt}\sigma_{k,j}
\end{aligned}$$

Thus it is valid that

$$S_k^\dagger S_l S_k = a_{lk} S_l, \tag{3.21}$$

where a_{lk} is one of the unity roots ω^j with $j \in \{0, 1, \ldots, d-1\}$.

We can rewrite the decoupling condition to

$$\frac{1}{m}\sum_{l=2}^{d^{2n}}\sum_{k=1}^{d^{2n}} \mu_l \; c_k \; a_{lk} \; S_l \otimes O_l = \frac{1}{D}\sum_{l=2}^{d^{2n}} \nu_l \; S_l \otimes \mathbb{1}. \tag{3.22}$$

When we deconstruct the occurring sum over l and examine the coefficients μ_l and ν_l individually, we derive a linear equation system consisting of $d^{2n}-1$ equations. This is possible because the operators S_l are linearly independent. We can easily see that for $\mu_l = 0$ and for $O_l \neq \mathbb{1}_{env}$ we require $\nu_l \stackrel{!}{=} 0$. This means that our decoupling scheme can not create coupling terms in our Hamiltonian H_{id} that do not already exist in H and that any operator S_l that has a coupling with the environment can only be suppressed completely since it can not be separated from the environmental term O_l.

For $l \in L$ with $L = \{l \in \mathbb{N} \cap [1, d^{2n}-1] : \mu_l \neq 0\}$ this is our equation system:

$$\frac{D}{m}\sum_{k=1}^{d^{2n}} a_{lk} \; c_k = \frac{\nu_l}{\mu_l}. \tag{3.23}$$

We do not want to show here that this equation system has suitable solutions. This can be read exhaustively in [17]. However, we give a short outline of the proof: The coefficients a_{lk} can be considered a $d^{2n} \times d^{2n}$-matrix A consisting of unity roots. We find that all rows and columns are linearly independent and that A is a complex *Hadamard matrix* [20]. We can therefore use one of the well known techniques to solve inhomogeneous linear equation systems. However, an infinite number of solutions exists and it is not trivial to choose the best possible decoupling scheme. The solutions differ in size m of the scheme and in the scaling factor D. Considering a practical realisation we prefer

short schemes with a scaling close to 1, because we can assume those schemes to be easier to implement experimentally. Those solutions can be found via linear programming. However, we can give no guarantee to find the best possible performing scheme. Looking at numerical simulations in [17, 18] we see that the found schemes perform sufficiently well.

3.3 Protecting Two-Qubit Interactions from Decoherence and Errors

We can now use the introduced method to solve specific problems in quantum information theory. For example it is necessary for the construction of a quantum processor to implement an arbitrary non-trivial unitary two-qubit gate in order to perform quantum computations. Such a gate in addition with unitary single qubit gates is then universal for quantum computation [3, p. 188 ff], which means that any unitary two-qubit gate can be constructed with the former. With *dynamical decoupling* we want to control even a two-qudit gate and also protect it from errors. We will take a specific look at the *perfect state transfer* Hamiltonian, which implements actually a swap operation at the ends of a qudit chain. If at least some kind of nearest neighbour coupling exists throughout the entire system we can stabilise the Hamiltonian and suppress decoherence. If that holds true the implementation of a two-qudit swap gate including protection from any kind of unwanted interaction and decoherence is theoretically accomplished.

3.3.1 Case Study

Consider the *perfect state transfer* Hamiltonian expanded in the *generalised Pauli matrices* introduced in section 2.3

$$H_{PST} = \lambda \sum_{i=1}^{n-1} \sum_{\substack{k,l=0 \\ j=1}}^{d-1} J_i \; \xi_{jk,jl} \; \sigma_{j,k}^{(i)} \sigma_{d-j,l}^{(i+1)}, \tag{3.24}$$

We saw that a coupling between the qudits is mandatory to exist in the quantum system in order for us to manipulate it. *Dynamical decoupling* can not create such couplings. Since usually the free space between qudits in a network is equal throughout the system, it is reasonable to consider a constant nearest neighbour coupling (*NNC*) as introduced in section 2.1

$$H_{NNC} = \Lambda \sum_{i=1}^{n-1} \sum_{\substack{k,l=0 \\ j=1}}^{d-1} \xi_{jk,jl} \; \sigma_{j,k}^{(i)} \sigma_{d-j,l}^{(i+1)}, \tag{3.25}$$

with the constant coupling strength Λ.

A single qudit error could look like this

$$H_{Error} = \sum_{i=1}^{n} \sum_{k,j=0}^{d-1} s_{k,j}^{(i)} \sigma_{k,j}^{(i)} \otimes O_{k,j}, \tag{3.26}$$

where $O_{k,j}$ represents some arbitrary Hamiltonian acting on the environment and with the coupling strength $s_{k,j}$. Furthermore any additional coupling or error could exist summed up in the expression H_{Add}. We want to suppress them all, which leads us to the Hamiltonians relevant for our method:

$$H = H_{NNC} + H_{Error} + H_{Add}, \tag{3.27}$$

$$H_{id} = H_{PST}. \tag{3.28}$$

This means that the $d^2(d-1)$ operators for $n-1$ qudits included in H_{PST} may remain and therefore carry a non-zero coefficient in the equation system. We derive

$$\frac{\nu_l}{\mu_l} = \begin{cases} \lambda \; J_i \; \Lambda^{-1} & \text{for the } d^2(d-1)(n-1) \text{ occuring coupling terms,} \\ 0 & \text{for all other terms, particularly single qudit terms.} \end{cases}$$

We have not solved the arisen equation system but it can in principle be done with a computer. However, we can also find a more straight-forward approach to the contrived problem as will be considered in the following chapter.

4 Alternative Design of Decoupling Schemes

The previous section contains an instruction manual to set up decoupling schemes in order to attain the desired Hamiltonian. However, for certain problems there is a more direct way to construct such schemes. We found that it is not possible to create but only to manipulate and suppress couplings in a quantum system via *dynamical decoupling*. In this sense manipulating means to alter the interaction strength between the coupled qudits. If we can find a unitary operator that time-reverses certain couplings, we can modify the interaction strength by inserting the operator via pulses into the system for a short period of time and then let the system evolve freely again. This procedure is utterly effective as we want to show in this chapter exemplary for the *perfect state transfer* Hamiltonian. The technique has previously been used on qubits [18]. In the following we will see that it can be generalised and thus also work on qudits.

4.1 Time-reversal of the involved Couplings

Consider the *perfect state transfer* Hamiltonian derived in chapter 2

$$H_{PST} = \lambda \sum_{i=1}^{n-1} \sum_{j=2}^{d} J_i \frac{1}{2} \left(\eta_{1,j}^{(i)} \eta_{1,j}^{(i+1)} + \eta_{j,1}^{(i)} \eta_{j,1}^{(i+1)} \right), \tag{4.1}$$

Our goal is to time-reverse the contained operators in order to weaken the coupling between certain qudits in the system. That achieved we can include this operator in pulses inserted into the quantum network and therefore manipulate the interaction strength to our benefit.

We need to find an inverting operator ι for the components of H_{PST}, so that for $j \in 2, \ldots, d$

$$\begin{aligned} \iota^\dagger \eta_{1,j} \iota &= -\eta_{1,j}, \\ \iota^\dagger \eta_{j,1} \iota &= -\eta_{j,1}. \end{aligned} \tag{4.2}$$

For odd dimension $\frac{d-1}{2} \in \mathbb{N}$ we can construct

the inverting operator	$\iota = e^{-\mathrm{i}p\varphi}$	(4.3)
via the phase factor	$\varphi = \pi\sqrt{d-1}$	(4.4)
and the Hermitian operator	$p = \frac{1}{\sqrt{d-1}} \sum_{i=1}^{d} \sigma_{0,i}.$	(4.5)

The phase factor $\varphi = \lambda\tau$ consists of an interaction strength λ and an interaction time τ, thus the latter has to be chosen appropriately according to the former. From the matrix representation of the unitary basis we can identify

$$\sum_{i=1}^{d} \sigma_{0,i} = \begin{pmatrix} 1-d & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{4.6}$$

We calculate

$$\begin{aligned} \iota &= e^{-\mathrm{i}p\varphi} \qquad (4.7) \\ &= e^{-i\frac{1}{\sqrt{1-d}}\sum_{i=1}^{d}\sigma_{0,i}\pi\sqrt{d-1}} \\ &= e^{i\pi\sum_{i=1}^{d}\sigma_{0,i}} \\ &= \begin{pmatrix} e^{-i\pi(d-1)} & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix}. \qquad (4.8) \end{aligned}$$

The last step holds true because $\frac{d-1}{2} \in \mathbb{N}$.

Knowing this we can construct ι for arbitrary dimension d via a sum of unitary pulses taken from the unitary basis representation of the given Hilbert space. An arbitrary sum of unitary matrices itself is not necessarily unitary. However, this is the case as we can see that ι has only non-zero elements on its diagonal which are either 1 or -1.

$$\iota = \frac{2-d}{d}\sigma_{0,0} + \sum_{i=1}^{d-1}\frac{2}{d}\sigma_{0,i} \tag{4.9}$$

If we consider a matrix M with arbitrary entries in the first line and first column except for a zero on the first diagonal element and zeros elsewhere and we can proof that $\iota^\dagger M \iota = -M$ holds true, our requirement (4.2) for ι holds also true because the components of H_{PST} are special cases of a matrix M.

$$
\begin{aligned}
&\iota^\dagger M \iota \\
&= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 0 & m_{1,2} & m_{1,3} & \cdots & m_{1,d} \\ m_{2,1} & 0 & 0 & \cdots & 0 \\ m_{3,1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{d,1} & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{pmatrix} \begin{pmatrix} 0 & -m_{1,2} & -m_{1,3} & \cdots & -m_{1,d} \\ m_{2,1} & 0 & 0 & \cdots & 0 \\ m_{3,1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{d,1} & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -m_{1,2} & -m_{1,3} & \cdots & -m_{1,d} \\ -m_{2,1} & 0 & 0 & \cdots & 0 \\ -m_{3,1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -m_{d,1} & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= - \begin{pmatrix} 0 & m_{1,2} & m_{1,3} & \cdots & m_{1,d} \\ m_{2,1} & 0 & 0 & \cdots & 0 \\ m_{3,1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{d,1} & 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= -M.
\end{aligned}
$$

4.2 Composition of Pulses

Consider the control operator g_k as introduced in chapter 3.1 equation (3.5). We define particular operators of that kind

$$
\begin{aligned}
g_a &= \mathbb{1}^{(i)} \otimes \mathbb{1}^{(i+1)}, \\
g_b &= \mathbb{1}^{(i)} \otimes \iota^{(i+1)}, \\
g_c &= \iota^{(i)} \otimes \mathbb{1}^{(i+1)}, \\
g_d &= \iota^{(i)} \otimes \iota^{(i+1)}.
\end{aligned} \tag{4.10}
$$

The operators act onto the entire chain composing the qudit network. The not displayed qudits, i.e. every qudit except qudit i and $i+1$, evolve freely under the unity operation $\mathbb{1}$. Upon calculation we find for the components of H_{PST} $c_{PST} = \eta^{(i)}_{1,j}\eta^{(i+1)}_{1,j} + \eta^{(i)}_{j,1}\eta^{(i+1)}_{j,1}$

$$
\begin{aligned}
g_a^\dagger \; c_{PST} \; g_a &= c_{PST}, \\
g_b^\dagger \; c_{PST} \; g_b &= -c_{PST}, \\
g_c^\dagger \; c_{PST} \; g_c &= -c_{PST}, \\
g_d^\dagger \; c_{PST} \; g_d &= c_{PST}.
\end{aligned} \tag{4.11}
$$

From these operators we construct $m = \lfloor \frac{n}{2} \rfloor$ operators g_j depending on the length n of the qudit chain. This works as follows: The coupling strength in the PST-Hamiltonian is strongest in the middle where $\Lambda = \lambda J_{\lfloor n/2 \rfloor}$. We do not want to change the coupling there, but use it as a reference, and dampen the coupling to the chain ends symmetrically. Therefore we construct the unity operator g_0 and operators that dampen the out-most coupling g_1, the two out-most couplings g_2 and so on leading to $m = \lfloor \frac{n}{2} \rfloor$ operators.

$$
\begin{aligned}
g_0 &= \mathbb{1} \otimes \ldots \otimes \mathbb{1}, \\
g_1 &= g_c \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g_c, \\
g_2 &= g_b \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g_b, \\
g_3 &= g_c \otimes g_c \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g_c \otimes g_c, \\
g_4 &= g_b \otimes g_b \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes g_b \otimes g_b, \\
&\vdots
\end{aligned} \tag{4.12}
$$

Now we have to calculate how often each of these operators needs to be radiated into the system in order to arrive from H_{NNC} to H_{PST}. We define the relative frequency of occurrence of each operator c_j in correspondence with g_j, and $c_0 = 1$ as a reference value. The relative coupling strength must equal the difference between positive and negative signs divided by the total length of the recoupling scheme [18]:

$$
\frac{J_j}{J_m} = \frac{\sum_{k=0}^{j-1} c_k - \sum_{k=j}^{m-1} c_k}{\sum_{k=0}^{m-1} c_k} \quad \text{with} \quad j \in \{1, \ldots, m-1\} \tag{4.13}
$$

After solving this set of equations for the c_j we need to find a common multiple such that all c_j are approximately integers. Trying-out has shown that a gap of 0.1 or even 0.2 to the nearest actual integer is sufficient to produce well working decoupling schemes. However, the closer each c_j is to an integer, the better the PST Hamiltonian is reproduced. The then found result denotes the absolute number of occurrences of each operator in the decoupling scheme.

4.3 Exemplary Application

4.3.1 Perfect State Transfer in a 4-Qutrit Chain

We consider a particular scenario in order to gain a better understanding of the introduced methods. For 4 qudits in a chain-shaped network we derive the control operators

$$g_0 = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1},$$
$$g_1 = \iota \otimes \mathbb{1} \otimes \mathbb{1} \otimes \iota. \tag{4.14}$$

Solving the equation system (4.13) this leads to a decoupling scheme using g_0 14 times and then once g_1. We have implemented this set-up with the software package *Wolfram Mathematica v8.0* and ran a simulation for different time-delays Δt between the pulses.

$$\frac{1}{\Delta t_1} = 15,$$
$$\frac{1}{\Delta t_2} = 30,$$
$$\frac{1}{\Delta t_3} = 37.5,$$
$$\frac{1}{\Delta t_4} = 60.$$

To measure the *state fidelity* we need to define an input state $|\Psi\rangle$ and an output state $|\Psi'\rangle$. We swap an arbitrary excited single-qutrit state of the first qutrit $|\Psi_1\rangle = a\,|0\rangle + b\,|1\rangle + c\,|2\rangle \equiv \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $|a|^2 + |b|^2 + |c|^2 = 1$ onto the n-th qutrit $|\Psi_n\rangle$, so that

$$|\Psi\rangle = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.15}$$
$$|\Psi'\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{4.16}$$

In the simulation we transported each basis state separately due to computational handiness. The ground state $|0\rangle$ is an eigenstate of the used Hamiltonians and has maximum fidelity at any time. In the plots we see the fidelities for the states $|1\rangle$ and $|2\rangle$, which are completely identical.

4.3.2 Simulation Results

Figures 4.1, 4.2, and 4.3 show the simulation results of the time evolution of the quantum system. In figure 4.1 we have plotted the *entanglement fidelity* over the the number of scheme applications m for different repetition rates $\frac{1}{\Delta t}$ between two pulses of our scheme consisting of 15 operators.

$$F_e = \left| \frac{1}{3^4} \mathrm{Tr}\left((e^{\mathrm{i}\pi H_{PST}})^m (g_1^\dagger e^{-\mathrm{i}\pi \frac{1}{15}\Delta t H_{NNC}} g_1 e^{-\mathrm{i}\pi \frac{14}{15}\Delta t H_{NNC}})^{\frac{m}{\Delta t}} \right) \right|^2. \tag{4.17}$$

A consistent entanglement fidelity at $F_e = 1$ will assure a perfect state transfer. We find that the repetition rate $\frac{1}{\Delta t_2}$ does not suffice. Such a procedure can not keep the

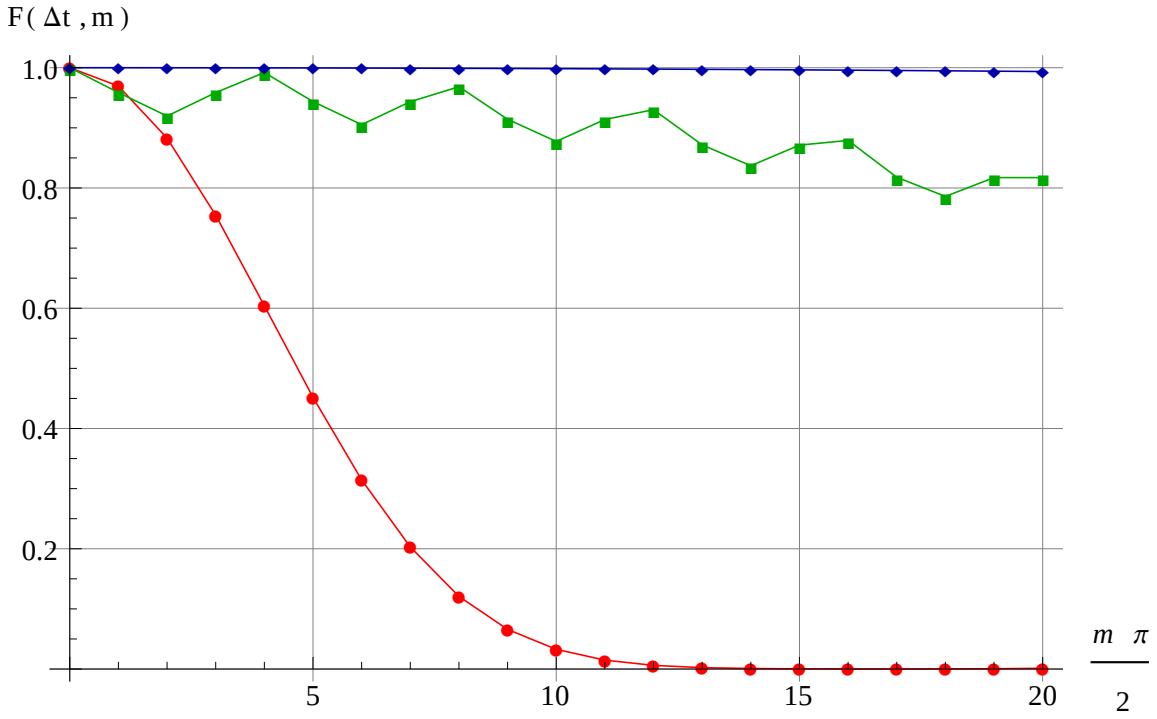

Figure 4.1: A plot of the simulated *entanglement fidelity* between the quantum system under the influence of our decoupling scheme with different repetition rate $\frac{1}{\Delta t}$ and the ideal time evolution under the *perfect state transfer* Hamiltonian. The scaling on the x-axis is in respect to the repetition rate $\frac{1}{\Delta t_2}$. The red graph shows the system with repetition rate $\frac{1}{\Delta t_2}$ between the control pulses. The green and blue graph show the same but with higher repetition rate $\frac{1}{\Delta t_3}$ and $\frac{1}{\Delta t_4}$ respectively, which is sufficient.

entanglement fidelity at an adequate level. However, for the repetition rate $\frac{1}{\Delta t_4}$ this is the case.

In figure 4.2 we have plotted the *entanglement fidelity* after a fixed time π in dependence of $\frac{1}{\Delta t}$. We find that shortening beyond Δt_4 of the time delay is not profitable.

Figure 4.3 shows the *state fidelity* of a *perfect state transfer*.

$$F_s = \left| \langle \Psi' | (g_1^\dagger e^{-\mathrm{i}\pi \frac{14}{15} \Delta t H_{NNC} t} g_1 g_0^\dagger e^{-\mathrm{i}\pi \frac{1}{15} \Delta t H_{NNC} t} g_0)^{\frac{1}{\Delta t}} | \Psi \rangle \right|^2$$

We find that for sufficiently high repetition rates the proposed decoupling scheme can indeed change a H_{NNC} into an effective H_{PST}.

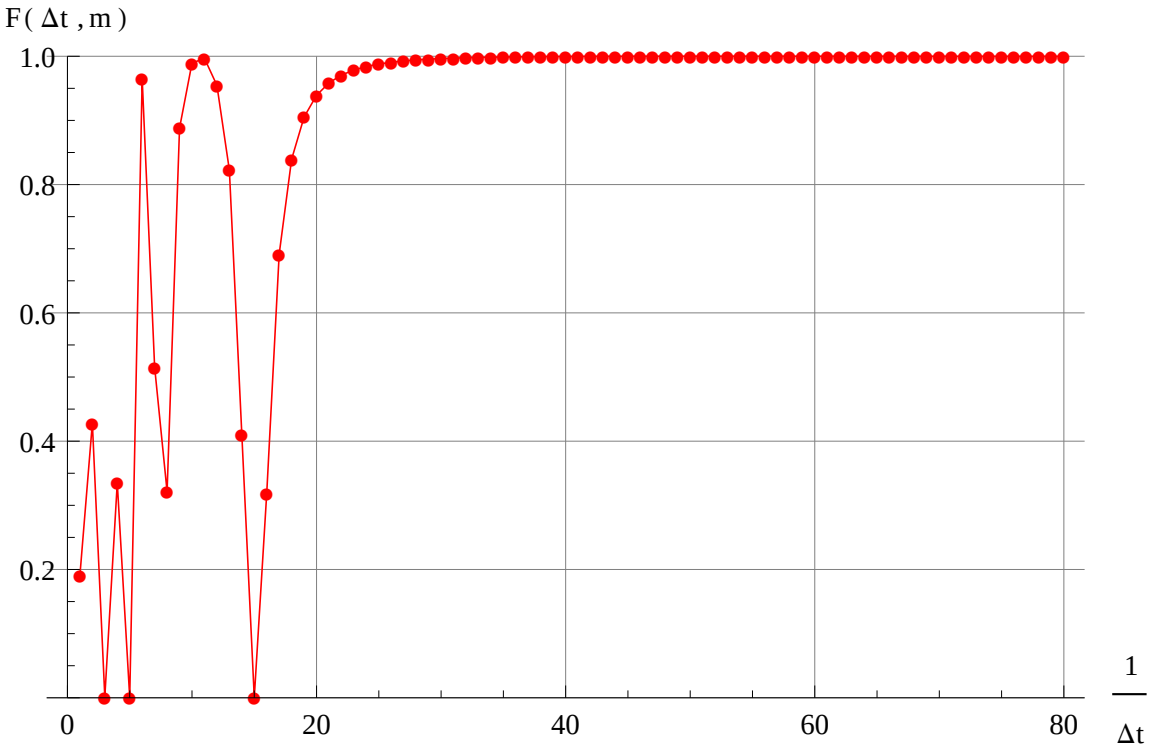

Figure 4.2: If the level of the *entanglement fidelity* is persistent, is highly dependent on the time delay Δt of two successive pulses radiated into the quantum system. We have plotted the *entanglement fidelity* after a fixed time $t = \pi$ in dependence of $\frac{1}{\Delta t}$. We now find that for $\frac{\pi}{\Delta t} \geq 40$ the fidelity stabilises, and we will derive good results.

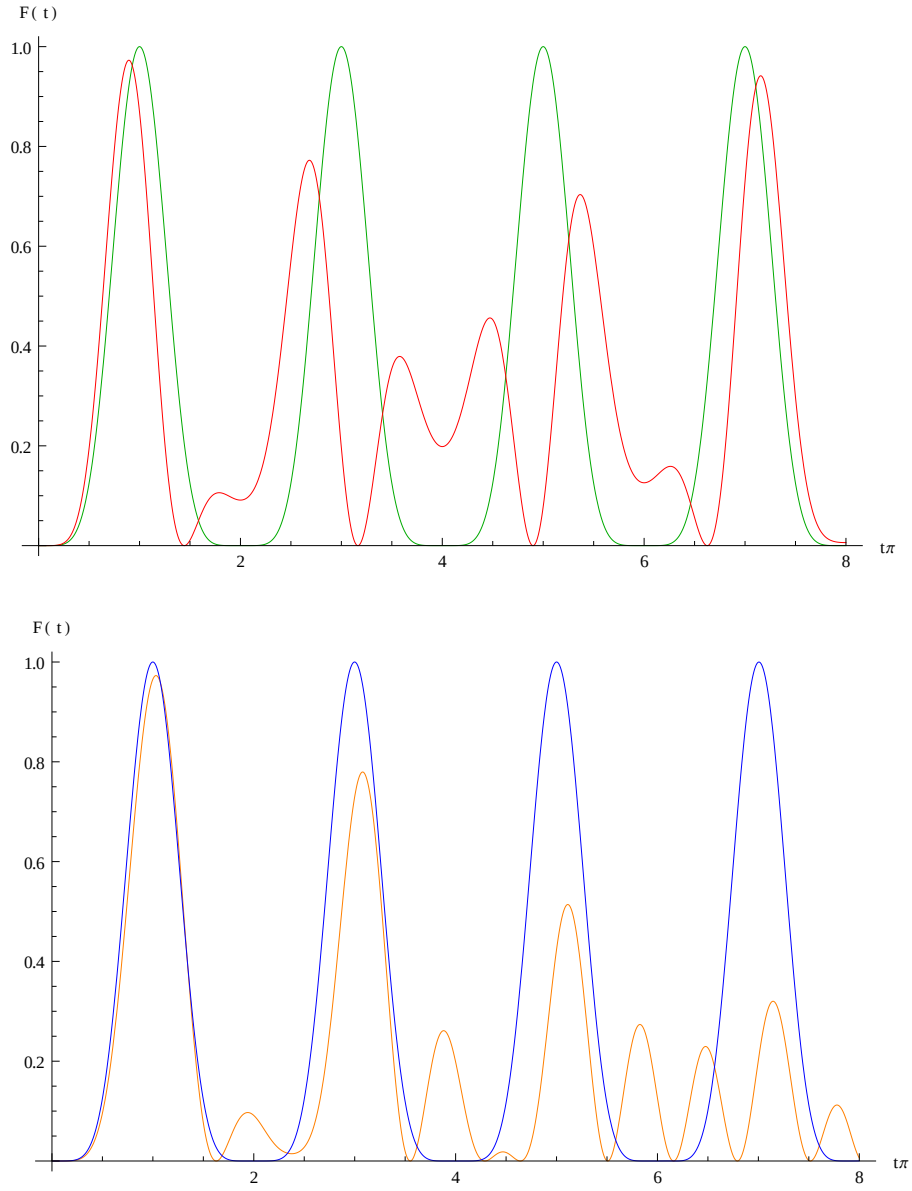

Figure 4.3: A plot of the simulated *state fidelity* for a *perfect state transfer*. Under the influence of H_{PST} (green) a swap operation is accomplished after the time $t = k\pi, k \in \mathbb{N}$ and the input state of the last qutrit of the chain is transferred onto the first qutrit. Since the compared output state $|\Psi\rangle$ includes a swap once, we find zero fidelity after the time $t = 2k\pi, k \in \mathbb{N}$, as then two swaps are done and the situation is the same as the initial situation. Under the influence of a nearest neighbour coupling H_{NNC} (red) we find no reliable fidelity. Using the proposed decoupling scheme we find for small repetition rate $\frac{1}{\Delta t_1}$ (orange) no increase in performance, but for a higher rate $\frac{1}{\Delta t_4}$ (blue) significant improvement.

5 Summary

In this thesis we studied possibilities to manipulate and control quantum systems. We require the system to be build of elementary quantum objects with distinguishable states, so-called qudits that are equidistantly arranged in a chain. If we want to use such a qudit-network as a quantum register we need to protect it from decoherence and all kinds of errors, be it entanglement with the environment, unwanted couplings in-between the qudits, or single qudit errors, for example from external magnetic fields. The method we employ to accomplish this capability is called *dynamical decoupling*. To this end we radiate pulses into the quantum system to gain control over it. Mathematically we use unitary operators g_k and apply them to the Hamiltonian H governing the system to receive a wanted Hamiltonian H_{id}:

$$\frac{1}{m}\sum_{k=0}^{m-1} g_k^\dagger H g_k = \frac{1}{D} H_{id}.$$

We saw that this equation leads to an equation system, whose solutions will give us a scheme of unitary operators to use on the system. The complexity of the equation system is dependent on the expansion we choose for the involved Hamiltonians. Due to certain requirements to the used operators we found the suitable basis to be the one expanded by the *generalised Pauli matrices*.

For the special case to perform a *perfect state transfer* we need to apply a Hamiltonian to the system:

$$H_{PST} = \lambda \sum_{i=1}^{n-1} \sum_{\substack{k,l=0 \\ j=1}}^{d-1} J_i \; \xi_{jk,jl} \; \sigma_{j,k}^{(i)} \sigma_{d-j,l}^{(i+1)},$$
$$\text{with} \quad \xi_{k,l} = \frac{1}{d^2}\left(e^{\mathrm{i}2\pi\frac{-k}{d}} + e^{\mathrm{i}2\pi\frac{l}{d}}\right).$$

If a constant first neighbour interaction is active in the quantum system we can then find a decoupling scheme that constitutes the desired Hamiltonian and protects the system against decoherence.

An alternate approach to design Hamiltonian interactions in qudit networks is to construct a scheme of pulses using a time-reversing operator ι for the involved couplings. Successively applied and alternating with a free time evolution of the system, we can then dampen or suppress these couplings. For the special case of a *perfect state transfer* Hamiltonian we found

$$\iota = \frac{2-d}{d}\sigma_{0,0} + \sum_{i=1}^{d} \frac{2}{d}\sigma_{0,i}.$$

With a simulation of the quantum system we determined the lower boundary for the repetition rate $\frac{1}{\Delta t}$ between two successive pulses in a special case exemplarily studied.

5.1 Review and Prospects

It is reasonable to consider the assumption of instantaneously induced unitary pulses most critically. We know that experimentally we are most likely to use electro-magnetic fields to influence a quantum system, be it laser pulses, radio pulses, magnetic fields etcetera. What we have to consider is a unitary time evolution which has the form

$$U = e^{\mathbb{i}A\lambda\tau},$$

with a given linear operator A, the interaction strength λ, and the interaction time τ. The only option we have is to increase λ gradually so that $\tau \to 0$, but at least $\tau \ll \Delta t$. Experimentally speaking we have to increase the intensities of the induced pulses to a level that the assumption becomes a valid approximation. If this can be accomplished has to be proofed experimentally. A theoretical approach describing non-vanishing pulse durations can be found in [21, section 3.4] using Euler paths.

In addition, if we want to use the method of *dynamical decoupling* to design Hamiltonian interactions and implement qudit gates the existence of two-qudit couplings is essential. It is, however, the main task of experimental quantum information to establish such couplings. Until then *dynamical decoupling* for that use remains theory. However, it is still a powerful method for the suppression of decoherence and errors.

Furthermore, the alternative procedure introduced in chapter 4 relies on the realisation of the unitary operator ι. It is an experimental task, whether it is possible to design pulses that have this special form. For odd dimensions d it is, however, possible to implement it as the unitary time evolution of an operator p constructed from the *generalised Pauli matrices*.

For the future dynamical decoupling has to proof its theoretical potential in actual experiments. This thesis provides an instruction manual to find decoupling schemes for a given scenario, but with growing number of qudits it becomes computationally expensive to find them. We have also seen that an adequate expansion of the given Hamiltonians can significantly simplify the problem. It remains that decoupling schemes have to be constructed manually for each particular scenario.

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