

# Propositions of solutions for *Analysis II* by Terence Tao

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**Remarks.** The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

## 12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence  $(a_n)_{n=m}^{\infty}$  defined by  $a_n := d(x_n, x) = |x_n - x|$  for all  $n \geq m$ . We have to prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if and only if  $\lim_{n \rightarrow \infty} x_n = x$ .

- Let be  $\varepsilon > 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then there exists an  $N \geq m$  such that  $|a_n| < \varepsilon$  whenever  $n \geq N$ . Thus, there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ , which means that  $\lim_{n \rightarrow \infty} x_n = x$ .
- Let be  $\varepsilon > 0$ . Conversely, if  $\lim_{n \rightarrow \infty} x_n = x$ , then there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . But since  $|a_n| := |x_n - x|$ , it means that  $\lim_{n \rightarrow \infty} a_n = 0$ , as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric  $d(x, y) := |x - y|$  is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function. With respect to Definition 12.1.2, give an example of a pair  $(X, d)$  which...*

- (a) obeys the axioms (bcd) but not (a).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, x) = 1$  and  $d(x, y) = 5$  for all  $x \neq y \in \mathbb{R}$ .

- (b) obeys the axioms (acd) but not (b).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = 0$  for all  $x, y \in \mathbb{R}$ .

- (c) obeys the axioms (abd) but not (c).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = \max(x - y, 0)$  for all  $x, y \in \mathbb{R}$ .

- (d) obeys the axioms (abc) but not (d).

Consider the finite set  $X := \{1, 2, 3\}$  and the application  $d$  defined by  $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$ , and  $d(1, 3) = d(3, 1) := 5$ , and  $d(x, x) = 0$  for all  $x \in X$ .

EXERCISE 12.1.4. — *Show that the pair  $(Y, d|_{Y \times Y})$  defined in Example 12.1.5 is indeed a metric space.*

By definition, since  $Y \subseteq X$ , we have  $x, y \in X$  whenever  $x, y \in Y$ . And furthermore, since  $d|_{Y \times Y}(x, y) := d(x, y)$ , then the application  $d|_{Y \times Y}$  obeys all four statements (a)–(d) of Definition 12.1.2. Thus,  $(Y, d|_{Y \times Y})$  is indeed a metric space.

EXERCISE 12.1.5. — Let  $n \geq 1$ , and let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Verify the identity  $(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$ , and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.

Let's prove these three statements.

(i) To prove the first identity, let's use induction on  $n$ .

The base case  $n = 1$  is obvious: on the left-hand side, we just get  $(a_1 b_1)^2$ , and on the right-hand side, we get  $a_1^2 b_1^2$ , hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer  $n \geq 1$ , and let's prove that it is still true for  $n + 1$ . We have to prove that

$$\underbrace{\left( \sum_{i=1}^{n+1} a_i b_i \right)^2}_{:=A} + \underbrace{\frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right)}_{:=C} \quad (12.1)$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for  $A$ , we have

$$\begin{aligned} A &:= \left( \sum_{i=1}^{n+1} a_i b_i \right)^2 \\ &= \left( a_{n+1} b_{n+1} + \sum_{i=1}^n a_i b_i \right)^2 \\ &= (a_{n+1} b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1} b_{n+1}) \sum_{i=1}^n a_i b_i \end{aligned}$$

- for  $B$ , we have

$$\begin{aligned} B &:= \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{j=1}^{n+1} (a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2} \sum_{i=1}^n (a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2 \times S} + \underbrace{\frac{1}{2} \sum_{j=1}^n (a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2 \times S} \\ &\quad + \underbrace{\frac{1}{2} (a_{n+1} b_{n+1} - b_{n+1} a_{n+1})^2}_{=0} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for  $A + B$ , we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A + B &:= (a_{n+1}b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2 \sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number  $n$ .

- (ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2}. \quad (12.2)$$

Indeed, since  $B \geq 0$  in the identity (12.1), we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$\begin{aligned}
\sum_{i=1}^n (a_i^2 + b_i^2) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \\
&\leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{by eq. (12.2)}) \\
&\leq \left( \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

EXERCISE 12.1.6. — *Show that  $(\mathbb{R}^n, d_{l^2})$  in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^2}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $(x_i - y_i)^2 > 0$ , and  $d_{l^2}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(y, x) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_{l^2}(x, y)$$

as expected.

- (d) Triangle inequality: for all  $x, y, z \in \mathbb{R}^n$ , we have

$$\begin{aligned}
d_{l^2}(x, z) &:= \left( \sum_{i=1}^n (x_i - z_i)^2 \right)^{1/2} \\
&= \left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \quad \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{Exercise 12.1.5(iii)}) \\
&\leq \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} + \left( \sum_{i=1}^n (y_i - z_i)^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) + d_{l^2}(y, z)
\end{aligned}$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^2})$  is indeed a metric space.

EXERCISE 12.1.7. — *Show that  $(\mathbb{R}^n, d_{l^1})$  in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^1}(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $|x_i - y_i| > 0$ , and  $d_{l^1}(x, y) = \sum_{i=1}^n |x_i - y_i| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^1}(y, x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x, y)$$

as expected.

- (d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality  $|a - c| \leq |a - b| + |b - c|$  for all  $a, b, c \in \mathbb{R}$ . Thus, for all  $x, y, z \in \mathbb{R}^n$ , we have

$$d_{l^1}(x, z) := \sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x, y) + d_{l^1}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y) \quad (12.3)$$

- The first inequality, since everything is non-negative, is equivalent to  $d_{l^2}(x, y)^2 \leq d_{l^1}(x, y)^2$ , and we will prove it in this form.

Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l^1}(x, y)^2 &:= \left( \sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \left( \sum_{i=1}^n |x_i - y_i| \right) \times \left( \sum_{i=1}^n |x_i - y_i| \right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leq i, j \leq n; i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geq 0} \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x, y)^2 \end{aligned}$$

as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x, y) &:= \sum_{i=1}^n |x_i - y_i| \\
&= \left| \sum_{i=1}^n |x_i - y_i| \times 1 \right| \\
&\leq \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left( \sum_{i=1}^n 1^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) \times \sqrt{n}
\end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair  $(\mathbb{R}^n, d_{l^\infty})$  in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we clearly have  $d_{l^\infty}(x, x) = \sup\{|x_i - x_i| : 1 \leq i \leq n\} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq j \leq n$  such that  $x_j \neq y_j$ . Thus  $|x_j - y_j| > 0$ , and  $d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_j - y_j| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} = \sup\{|y_i - x_i| : 1 \leq i \leq n\} = d_{l^\infty}(y, x)$$

as expected.

- (d) Triangle inequality. Let be  $x, y, z \in \mathbb{R}^n$ . We have  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$  for all  $1 \leq i \leq n$ , by Proposition 4.3.3(g). But, by definition of the supremum, we have  $|x_i - y_i| \leq d_{l^\infty}(x, y)$  and  $|y_i - z_i| \leq d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ . Thus, we have  $|x_i - z_i| \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ ; i.e.,  $d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  is an upper bound of the set  $\{|x_i - z_i| : 1 \leq i \leq n\}$ . By definition of the supremum, it implies that

$$d_{l^\infty}(x, z) := \sup\{|x_i - z_i| : 1 \leq i \leq n\} \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ ,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y).$$

First, a preliminary remark. By definition, we have  $d_{l^\infty}(x, y) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}$  for all  $x, y \in \mathbb{R}^n$ . Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a  $1 \leq m \leq n$  such that  $d_{l^\infty}(x, y) = |x_m - y_m|$  (the supremum belongs to the set). The index “ $m$ ” will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}
\frac{1}{\sqrt{n}}d_{l^2}(x, y) &:= \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - y_i)^2} \\
&\leq \sqrt{\frac{1}{n} \sum_{i=1}^n (x_m - y_m)^2} \\
&\leq \sqrt{\frac{n}{n} (x_m - y_m)^2} \\
&= |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}
d_{l^2}(x, y) &:= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \\
&= \sqrt{(x_m - y_m)^2 + \sum_{1 \leq i \leq n; i \neq m} (x_i - y_i)^2} \\
&\geq \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

EXERCISE 12.1.11. — *Show that the discrete metric  $(X, d_{\text{disc}})$  in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in X$ , we have  $d_{\text{disc}}(x, x) := 0$  by definition, so that there is nothing to prove here.
- (b) Positivity: for all  $x \neq y \in X$ , we have  $d_{\text{disc}}(x, y) := 1 > 0$  by definition, so that there's still nothing to prove.
- (c) Symmetry: for all  $x, y \in X$ , we have  $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x) = 1$ , so that  $d_{\text{disc}}$  obeys the symmetry property.
- (d) Triangle inequality. Let be  $x, y, z \in X$ , and let's consider  $d_{\text{disc}}(x, z)$ .
  - If  $x = z$ , then  $d_{\text{disc}}(x, z) = 0$ . And since  $d_{\text{disc}}$  is a non-negative application, we clearly have  $0 =: d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$  for all  $y \in X$ .
  - If  $x \neq z$ , then we cannot have both  $x = y$  and  $y = z$  (it would be a clear contradiction with  $x \neq z$ ). Thus, at least one of the propositions " $x \neq y$ ", " $y \neq z$ " is true. Another way to say that is  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq 1$ . But since  $d_{\text{disc}}(x, z) := 1$ , we have actually  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq d_{\text{disc}}(x, z)$ , as expected.



EXERCISE 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all  $x, y \in \mathbb{R}^n$ , we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y).$$

Note that  $n$  is a real constant here.

- Let's prove that (a)  $\implies$  (b). If  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , then by the limit laws, the sequence  $t_k := \sqrt{n} d_{l^2}(x^{(k)}, x)$  also converges to 0 as  $k \rightarrow \infty$ , since  $\sqrt{n}$  is a constant real number. Thus, we have

$$d_{l^2}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x) \leq \sqrt{n} d_{l^2}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x)$  as expected.

- Let's prove that (b)  $\implies$  (c). If  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x) = 0$ , then we have

$$0 \leq d_{l^\infty}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x)$  as expected.

- Let's prove that (c)  $\implies$  (d). Suppose that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x) = 0$ . Then, for all  $1 \leq j \leq n$ , we have  $0 \leq |x_j^{(k)} - x_j| \leq d_{l^\infty}(x^{(k)}, x)$ . Still by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} |x_j^{(k)} - x_j| = 0$ , i.e. that  $(x_j^{(k)})_{k=m}^\infty$  converges to  $x_j$  as  $k \rightarrow \infty$  (by Lemma 12.1.1), as expected.
- Finally, let's prove that (d)  $\implies$  (a). Using the definition of convergence is more appropriate here. Let be  $\varepsilon > 0$  a positive real number, and let be  $1 \leq j \leq n$ . By definition, there exists a natural number  $N \geq m$  such that  $|x_j^{(k)} - x_j| \leq \varepsilon/\sqrt{n}$  whenever  $k \geq N$ . Thus, if  $k \geq N$ , we have

$$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^n (x_j^{(k)} - x_j)^2} \leq \sqrt{\sum_{j=1}^n \frac{\varepsilon^2}{n}} \leq \varepsilon$$

so that  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , i.e.,  $(x^{(k)})_{k=m}^\infty$  converges to  $x$  as  $k \rightarrow \infty$  in the  $l^2$  metric (by Lemma 12.1.1), as expected.

EXERCISE 12.1.13. — *Prove Proposition 12.1.19.*

Let be  $(x^{(n)})_{n=m}^\infty$  a sequence of elements of a set  $X$ .

- First suppose that  $(x^{(n)})_{n=m}^\infty$  is eventually constant. Thus, by definition, there exists an  $N \geq m$  and an element  $x \in X$  such that  $(x^{(n)})_{n=m}^\infty = x$  for all  $n \geq N$ . This implies that we have  $d_{\text{disc}}(x^{(n)}, x) = 0$  for all  $n \geq N$ . In particular, for all  $\varepsilon > 0$ , we have  $d_{\text{disc}}(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N$ , so that  $(x^{(n)})_{n=m}^\infty$  indeed converges to  $x$  with respect to  $d_{\text{disc}}$ .
- Conversely, suppose that  $(x^{(n)})_{n=m}^\infty$  converges to  $x$  with respect to  $d_{\text{disc}}$ . Let be  $\varepsilon = 1/2$ . By definition, there exists an  $N \geq m$  such that  $d_{\text{disc}}(x^{(n)}, x) \leq 1/2$  whenever  $n \geq N$ . Since  $d_{\text{disc}}(x^{(n)}, x)$  cannot be 1, it is necessarily equal to 0, so that  $x^{(n)} = x$  whenever  $n \geq N$ . Thus, the sequence  $x^{(n)}$  is indeed eventually constant.

EXERCISE 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have  $\lim_{n \rightarrow \infty} d(x^{(n)}, x) = 0$  and  $\lim_{n \rightarrow \infty} d(x^{(n)}, x') = 0$ . Suppose, for the sake of contradiction, that we have  $x \neq x'$ . Thus, the real number  $\varepsilon := \frac{d(x, x')}{3}$  is positive.

Since  $x^{(n)}$  converges to  $x$ , there exists a  $N_1 \geq m$  such that  $d(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N_1$ .

Similarly, since  $x^{(n)}$  converges to  $x'$ , there exists a  $N_2 \geq m$  such that  $d(x^{(n)}, x') \leq \varepsilon$  whenever  $n \geq N_2$ .

By the triangle inequality, we thus have, for all  $n \geq \max(N_1, N_2)$ ,

$$d(x, x') \leq d(x, x^{(n)}) + d(x^{(n)}, x') \leq \varepsilon + \varepsilon = \frac{2}{3}d(x, x')$$

which is a contradiction (since  $d(x, x') > 0$  by hypothesis).

Thus, the limit is unique, and we must have  $x = x'$ .

EXERCISE 12.1.15. — *Let be  $X := \{(a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty\}$ . We define on this space the metrics  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sum_{n=0}^\infty |a_n - b_n|$ , and  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |a_n - b_n|$ . Then...*

We have to prove the following statements.

1.  $d_{l^1}$  is a metric on  $X$ .

We have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^1}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^1}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |b_n - a_n| = \sum_{n=0}^\infty |a_n - b_n| = d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately

$$\begin{aligned} d_{l^1}((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty) &:= \sum_{n=0}^\infty |a_n - c_n| \\ &\leq \sum_{n=0}^\infty (|a_n - b_n| + |b_n - c_n|) \quad (\text{consequence of Prop. 7.1.11(h)}) \\ &\leq \sum_{n=0}^\infty |a_n - b_n| + \sum_{n=0}^\infty |b_n - c_n| \quad (\text{by Proposition 7.2.14(a)}) \\ &\leq d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) + d_{l^1}((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty). \end{aligned}$$

Thus,  $d_{l^1}$  is indeed a metric on  $X$ .

2.  $d_{l^\infty}$  is a metric on  $X$ .

Once again, we have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |b_n - a_n| = \sup_{n \in \mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately  $|a_m - c_m| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$  for all  $m \in \mathbb{N}$ , by definition of the supremum. In other words,  $(\sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|)$  is an upper bound for the set  $\{|a_m - c_m| : m \in \mathbb{N}\}$ . Thus we have, still by definition of the supremum,  $\sup_{n \in \mathbb{N}} |a_n - c_n| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$ , as expected.

Thus,  $d_{l^\infty}$  is indeed a metric on  $X$ .

3. There exist sequences  $x^{(1)}, x^{(2)}, \dots$ , of elements of  $X$  (i.e., sequences of sequences) which are convergent with respect to  $d_{l^\infty}$ , but are not convergent with respect to  $d_{l^1}$ .

Here we are dealing with sequences of sequences: we have a sequence  $(x^{(k)})_{k=1}^\infty$  where each  $x^{(k)}$  is itself a sequence of real numbers. Thus, let's define  $(x^{(k)})_{k=1}^\infty$  as follows:

$$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

Just to make things clearer, we have for instance

$$\begin{aligned} x^{(1)} &:= \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \\ x^{(2)} &:= \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots \\ x^{(3)} &:= \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots \end{aligned}$$

Also, let be the null sequence  $(a_n)_{n=0}^\infty$  defined by  $a_n := 0$  for all  $n \in \mathbb{N}$ . Thus:

- $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ . Indeed, if we consider a given positive integer  $k$  (fixed), we have

$$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

so that  $d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$ .

Thus,  $\lim_{k \rightarrow \infty} d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) = 0$ , or in other words,  $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$  in  $X$ .

- But  $(x_n^{(k)})_{n=0}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ . Indeed, we have, for each given (fixed)  $k$ ,

$$d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = \sum_{n=0}^k \frac{1}{k+1} = 1$$

Thus, we clearly do not have  $\lim_{k \rightarrow \infty} d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = 0$ , i.e.,  $(x_n^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

4. Conversely, any sequence which converges with respect to  $d_{l^1}$  also converges with respect to  $d_{l^\infty}$ .

Suppose, for the sake of contradiction, that  $(x_n^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ , but does converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

In this case, there exists a  $\varepsilon > 0$  such that, for all  $k \geq 1$ , we have  $(\sup_{n \geq 0} |x_n^{(k)} - a_n|) > \varepsilon$ . In particular, for all  $k \geq 1$  and all  $n \geq 0$ , we have  $|x_n^{(k)} - a_n| > \varepsilon$ . Thus,  $\sum_{n=0}^\infty |x_n^{(k)} - a_n|$  is not even a convergent series, and we cannot have  $\lim_{k \rightarrow \infty} \left( \sum_{n=0}^\infty |x_n^{(k)} - a_n| \right) = 0$ .

Note that this exercise actually shows that in this space  $X$ , the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

EXERCISE 12.1.16. — Let  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  be two sequences in a metric space  $(X, d)$ . Suppose that  $(x_n)_{n=1}^\infty$  converges to a point  $x \in X$ , and  $(y_n)_{n=1}^\infty$  converges to a point  $y \in X$ . Show that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ .

On the one hand, the triangle inequality applied two times to  $d$  gives us

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leq d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

Let be  $\varepsilon > 0$ . By hypothesis, there exists a  $N_1 \geq 1$  such that  $d(x_n, x) \leq \varepsilon/2$  whenever  $n \geq N_1$ . Similarly, there exists a  $N_2 \geq 1$  such that  $d(y_n, y) \leq \varepsilon/2$  whenever  $n \geq N_2$ . Thus, if we set  $N := \max(N_1, N_2)$ , then for all  $n \geq N$  we have

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \leq 2\varepsilon/2 = \varepsilon$$

which shows that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ , as expected.

EXERCISE 12.2.1. — *Verify the claims in Example 12.2.8*

Let be  $(X, d_{\text{disc}})$  a metric space, and  $E$  a subset of  $X$ .

- Let be  $x \in E$ . Then  $x$  is an interior point of  $E$ . Indeed, we have  $B(x, 1/2) = \{x\} \subseteq E$ .
- Let be  $y \notin E$ . Then  $y$  is an exterior point of  $E$ . Indeed, we have  $B(y, 1/2) \cap E = \{y\} \cap E = \emptyset$ .
- Finally, there are no boundary points of  $E$  in  $(X, d_{\text{disc}})$ . Indeed, let be  $r > 0$  and any  $x \in X$ . We will always have  $B(x, r) = \{x\}$  by definition of the discrete metric  $d_{\text{disc}}$ . Thus, we have either  $x \in E$  and then  $x \in \text{int}(E)$ , or  $x \notin E$  and then  $x \in \text{ext}(E)$ . Thus,  $E$  has no boundary points.

EXERCISE 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that  $(a) \implies (b)$ . We will use the contrapositive, assuming that  $x_0$  is neither an interior point of  $E$ , nor a boundary point of  $E$ . By definition, it means that  $x_0$  is an exterior point of  $E$ , i.e. that there exists  $r > 0$  such that  $B(x_0, r) \cap E = \emptyset$ . This is precisely the negation of  $x_0$  being an adherent point of  $E$ . Thus, we have showed that if  $x_0$  is an adherent point of  $E$ , it is either an interior point or a boundary point.
- Let's show that  $(b) \implies (c)$ . Let be a positive integer  $n > 0$ , and suppose that  $x_0$  is either an interior point of  $E$ , or a boundary point of  $E$ . In either case, the set  $A_n := B(x_0, 1/n) \cap E$  is non empty, i.e., there exists  $a_n \in X$  such that  $d(a_n, x_0) < 1/n$ . By the (countable) axiom of choice, we can define a sequence  $(a_n)_{n=1}^\infty$  such that  $a_n \in A_n$  for all  $n \geq 1$ .

Let be  $\varepsilon > 0$ . There exists  $N > 0$  such that  $1/N < \varepsilon$  (Exercise 5.4.4). Thus, for all  $n \geq N$ , we have

$$d(a_n, x_0) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

i.e., the sequence  $(a_n)_{n=1}^\infty$  converges to  $x_0$  with respect to the metric  $d$ , as expected.

- Finally, let's show that  $(c) \implies (a)$ . Let be  $r > 0$ . If  $(a_n)_{n=1}^\infty$  in  $E$  converges to  $x_0$  with respect to  $d$ , then there exists a  $n$  such that  $d(x_0, a_n) < r$ . But since  $a_n \in E$ , it means that  $B(x_0, r) \cap E$  is non empty, i.e. that  $x_0$  is an adherent point of  $E$ .