Propositions of solutions for *Analysis II* by Terence Tao

Frédéric Santos

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Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

Exercise 12.1.4. — *Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y\times Y}(x, y) := d(x, y)$, then the application $d|_{Y\times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y\times Y})$ is indeed a metric space.

Exercise 12.1.5. — *Let $n \geqslant 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Verify the identity $\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.*

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n.

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geqslant 1$, and let's prove that it is still true for $n+1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i\right)^2}_{:=A} + \underbrace{\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right)}_{:=C} \tag{12.1}$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A, we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i\right)^2 \\ &= \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2 \\ &= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \end{aligned}$$

- for B, we have

$$\begin{aligned} B &:= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n+1}(a_{n+1}b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2\times S} + \underbrace{\frac{1}{2}\sum_{j=1}^{n}(a_{n+1}b_j - a_j b_{n+1})^2}_{:=1/2\times S} \\ &\quad + \underbrace{\frac{1}{2}(a_{n+1}b_{n+1} - b_{n+1}a_{n+1})^2}_{=0} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for $A+B$, we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A+B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2\sum_{i=1}^{n} a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^{n}(a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) + \sum_{i=1}^{n}(a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number n.

(ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}. \tag{12.2}$$

Indeed, since $B \geqslant 0$ in the identity (12.1), we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality. We have

$$
\begin{aligned}
\sum_{i=1}^{n}(a_i^2+b_i^2) &= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \\
&\leqslant \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(by eq. (12.2))} \\
&\leqslant \left(\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\right)^2
\end{aligned}
$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

Exercise 12.1.6. — *Show that $(\mathbb{R}^n, d_{l^2})$ in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $(x_i - y_i)^2 > 0$, and $d_{l^2}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$
d_{l^2}(y,x) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_{l^2}(x,y)
$$

as expected.

(d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$
\begin{aligned}
d_{l^2}(x,z) &:= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} && \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(Exercise 12.1.5(iii))} \\
&\leqslant \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2} \\
&\leqslant d_{l^2}(x,y) + d_{l^2}(y,z)
\end{aligned}
$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^2})$ is indeed a metric space.

Exercise 12.1.7. — *Show that* $(\mathbb{R}^n, d_{l^1})$ *in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x,x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x,y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y,x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x,y)$$

as expected.

(d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a-c| \leqslant |a-b| + |b-c|$ for all $a, b, c \in \mathbb{R}$. Thus, for all $x, y, z \in \mathbb{R}^n$, we have

$$d_{l^1}(x,z) := \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x,y) + d_{l^1}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x,y)^2 \leqslant d_{l^1}(x,y)^2$, and we will prove it in this form.

  Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l_1}(x,y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \left(\sum_{i=1}^n |x_i - y_i|\right) \times \left(\sum_{i=1}^n |x_i - y_i|\right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leqslant i,j \leqslant n;\, i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geqslant 0} \\ &\geqslant \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x,y)^2 \end{aligned}$$

  as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x,y) &:= \sum_{i=1}^{n} |x_i - y_i| \\
&= \left| \sum_{i=1}^{n} |x_i - y_i| \times 1 \right| \\
&\leqslant \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} \\
&\leqslant d_{l^2}(x,y) \times \sqrt{n}
\end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair* $(\mathbb{R}^n, d_{l^\infty})$ *in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we clearly have $d_{l^\infty}(x,x) = \sup\{|x_i - x_i| : 1 \leqslant i \leqslant n\} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant j \leqslant n$ such that $x_j \neq y_j$. Thus $|x_j - y_j| > 0$, and $d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} \geqslant |x_j - y_j| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} = \sup\{|y_i - x_i| : 1 \leqslant i \leqslant n\} = d_{l^\infty}(y,x)$$

as expected.

(d) Triangle inequality. Let be $x, y, z \in \mathbb{R}^n$. We have $|x_i - z_i| \leqslant |x_i - y_i| + |y_i - z_i|$ for all $1 \leqslant i \leqslant n$, by Proposition 4.3.3(g). But, by definition of the supremum, we have $|x_i - y_i| \leqslant d_{l^\infty}(x,y)$ and $|y_i - z_i| \leqslant d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$. Thus, we have $|x_i - z_i| \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$; i.e., $d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ is an upper bound of the set $\{|x_i - z_i| : 1 \leqslant i \leqslant n\}$. By definition of the supremum, it implies that

$$d_{l^\infty}(x,z) := \sup\{|x_i - z_i| : 1 \leqslant i \leqslant n\} \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y).$$

First, a preliminary remark. By definition, we have $d_{l^\infty}(x,y) := \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\}$ for all $x, y \in \mathbb{R}^n$. Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a $1 \leqslant m \leqslant n$ such that $d_{l^\infty}(x,y) = |x_m - y_m|$ (the supremum belongs to the set). The index "m" will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}\frac{1}{\sqrt{n}} d_{l^2}(x,y) &:= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2}\\ &\leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_m - y_m)^2}\\ &\leqslant \sqrt{\frac{n}{n}(x_m - y_m)^2}\\ &= |x_m - y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}d_{l^2}(x,y) &:= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}\\ &= \sqrt{(x_m - y_m)^2 + \sum_{1\leqslant i\leqslant n;\, i\neq m}(x_i - y_i)^2}\\ &\geqslant \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

Exercise 12.1.11. — *Show that the discrete metric (X, d_{disc}) in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in X$, we have $d_{\mathrm{disc}}(x,x) := 0$ by definition, so that there is nothing to prove here.

(b) Positivity: for all $x \neq y \in X$, we have $d_{\mathrm{disc}}(x,y) := 1 > 0$ by definition, so that there's still nothing to prove.

(c) Symmetry: for all $x, y \in X$, we have $d_{\mathrm{disc}}(x,y) = d_{\mathrm{disc}}(y,x) = 1$, so that d_{disc} obeys the symmetry property.

(d) Triangle inequality. Let be $x, y, z \in X$, and let's consider $d_{\mathrm{disc}}(x,z)$.

  - If $x = z$, then $d_{\mathrm{disc}}(x,z) = 0$. And since d_{disc} is a non-negative application, we clearly have $0 =: d_{\mathrm{disc}}(x,z) \leqslant d_{\mathrm{disc}}(x,y) + d_{\mathrm{disc}}(y,z)$ for all $y \in X$.
  - If $x \neq z$, then we cannot have both $x = y$ and $y = z$ (it would be a clear contradiction with $x \neq z$). Thus, at least one of the propositions "$x \neq y$", "$y \neq z$" is true. Another way to say that is $d_{\mathrm{disc}}(x,y) + d_{\mathrm{disc}}(y,z) \geqslant 1$. But since $d_{\mathrm{disc}}(x,z) := 1$, we have actually $d_{\mathrm{disc}}(x,y) + d_{\mathrm{disc}}(y,z) \geqslant d_{\mathrm{disc}}(x,z)$, as expected.

Exercise 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all $x, y \in \mathbb{R}^n$, we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y).$$

Note that n is a real constant here.

- Let's prove that $(a) \implies (b)$. If $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, then by the limit laws, the sequence $t_k := \sqrt{n}\, d_{l^2}(x^{(k)}, x)$ also converges to 0 as $k \to \infty$, since $\sqrt{n}$ is a constant real number. Thus, we have

  $$d_{l^2}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x) \leqslant \sqrt{n}\, d_{l^2}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x)$ as expected.

- Let's prove that $(b) \implies (c)$. If $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x) = 0$, then we have

  $$0 \leqslant d_{l^\infty}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x)$ as expected.

- Let's prove that $(c) \implies (d)$. Suppose that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x) = 0$. Then, for all $1 \leqslant j \leqslant n$, we have $0 \leqslant |x_j^k - x_j| \leqslant d_{l^\infty}(x^{(k)}, x)$. Still by the squeeze test, this implies that $\lim_{k\to\infty} |x_j^k - x_j| = 0$, i.e. that $(x_j^k)_{k=m}^\infty$ converges to x_j as $k \to \infty$ (by Lemma 12.1.1), as expected.

- Finally, let's prove that $(d) \implies (a)$. Using the definition of convergence is more appropriate here. Let be $\varepsilon > 0$ a positive real number, and let be $1 \leqslant j \leqslant n$. By definition, there exists a natural number $N \geqslant m$ such that $|x_j^{(k)} - x_j| \leqslant \varepsilon/\sqrt{n}$ whenever $k \geqslant N$. Thus, if $k \geqslant N$, we have

  $$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^{n} (x_j^{(k)} - x_j)^2} \leqslant \sqrt{\sum_{j=1}^{n} \frac{\varepsilon^2}{n}} \leqslant \varepsilon$$

  so that $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, i.e., $(x^k)_{k=m}^\infty$ converges to x as $k \to \infty$ in the l^2 metric (by Lemma 12.1.1), as expected.

Exercise 12.1.13. — *Prove Proposition 12.1.19.*

Let be $(x^{(n)})_{n=m}^\infty$ a sequence of elements of a set X.

- First suppose that $(x^{(n)})_{n=m}^\infty$ is eventually constant. Thus, by definition, there exists an $N \geqslant m$ and an element $x \in X$ such that $(x^{(n)})_{n=m}^\infty = x$ for all $n \geqslant N$. This implies that we have $d_{\text{disc}}(x^{(n)}, x) = 0$ for all $n \geqslant N$. In particular, for all $\varepsilon > 0$, we have $d_{\text{disc}}(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N$, so that $(x^{(n)})_{n=m}^\infty$ indeed converges to x with respect to d_{disc}.

- Conversely, suppose that $(x^{(n)})_{n=m}^{\infty}$ converges to x with respect to d_{disc}. Let be $\varepsilon = 1/2$. By definition, there exists an $N \geqslant m$ such that $d_{\text{disc}}(x^{(n)}, x) \leqslant 1/2$ whenever $n \geqslant N$. Since $d_{\text{disc}}(x^{(n)}, x)$ cannot be 1, it is necessarily equal to 0, so that $x^{(n)} = x$ whenever $n \geqslant N$. Thus, the sequence $x^{(n)}$ is indeed eventually constant.

Exercise 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have $\lim_{n\to\infty} d(x^{(n)}, x) = 0$ and $\lim_{n\to\infty} d(x^{(n)}, x') = 0$. Suppose, for the sake of contradiction, that we have $x \neq x'$. Thus, the real number $\varepsilon := \frac{d(x,x')}{3}$ is positive.

Since $x^{(n)}$ converges to x, there exists a $N_1 \geqslant m$ such that $d(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N_1$.

Similarly, since $x^{(n)}$ converges to x', there exists a $N_2 \geqslant m$ such that $d(x^{(n)}, x') \leqslant \varepsilon$ whenever $n \geqslant N_2$.

By the triangle inequality, we thus have, for all $n \geqslant \max(N_1, N_2)$,

$$d(x, x') \leqslant d(x, x^{(n)}) + d(x^{(n)}, x') \leqslant \varepsilon + \varepsilon = \frac{2}{3} d(x, x')$$

which is a contradiction (since $d(x, x') > 0$ by hypothesis).

Thus, the limit is unique, and we must have $x = x'$.

Exercise 12.1.15. — *Let be $X := \{(a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty\}$. We define on this space the metrics $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|$, and $d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n\in\mathbb{N}} |a_n - b_n|$. Then...*

We have to prove the following statements.

1. d_{l^1} is a metric on X.

   We have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^{\infty} \in X$. We have $d_{l^1}((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^1}((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |b_n - a_n| = \sum_{n=0}^{\infty} |a_n - b_n| = d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately

   $$\begin{aligned} d_{l^1}((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}) &:= \sum_{n=0}^{\infty} |a_n - c_n| \\ &\leqslant \sum_{n=0}^{\infty} (|a_n - b_n| + |b_n - c_n|) \quad \text{(consequence of Prop. 7.1.11(h))} \\ &\leqslant \sum_{n=0}^{\infty} |a_n - b_n| + \sum_{n=0}^{\infty} |b_n - c_n| \quad \text{(by Proposition 7.2.14(a))} \\ &\leqslant d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) + d_{l^1}((b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}). \end{aligned}$$

Thus, d_{l^1} is indeed a metric on X.

2. d_{l^∞} is a metric on X.

   Once again, we have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

$$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |b_n - a_n| = \sup_{n\in\mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately $|a_m - c_m| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$ for all $m \in \mathbb{N}$, by definition of the supremum. In other words, $(\sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|)$ is an upper bound for the set $\{|a_m - c_m| : m \in \mathbb{N}\}$. Thus we have, still by definition of the supremum, $\sup_{n\in\mathbb{N}} |a_n - c_n| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$, as expected.

   Thus, d_{l^∞} is indeed a metric on X.

3. There exist sequences $x^{(1)}$, $x^{(2)}$, ..., of elements of X (i.e., sequences of sequences) which are convergent with respect to d_{l^∞}, but are not convergent with respect to d_{l^1}.

   Here we are dealing with sequences of sequences: we have a sequence $(x^{(k)})_{k=1}^\infty$ where each $x^{(k)}$ is itself a sequence of real numbers. Thus, let's define $(x^{(k)})_{k=1}^\infty$ as follows:

$$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

   Just to make things clearer, we have for instance

$$\begin{aligned} x^{(1)} &:= \frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ 0,\ \ldots \\ x^{(2)} &:= \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0,\ 0,\ \ldots \\ x^{(3)} &:= \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 0,\ \ldots \end{aligned}$$

   Also, let be the null sequence $(a_n)_{n=0}^\infty$ defined by $a_n := 0$ for all $n \in \mathbb{N}$. Thus:

   - $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞}. Indeed, if we consider a given positive integer k (fixed), we have

$$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

so that $d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) := \sup_{n\in\mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$.

Thus, $\lim_{k\to\infty} d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = 0$, or in other words, $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} in X.

- But $(x^{(k)})_{k=1}^\infty$ does not converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1}. Indeed, we have, for each given (fixed) k,

$$d_{l^1}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = \sum_{n=0}^{k} \frac{1}{k+1} = 1$$

Thus, we clearly do not have $\lim_{k\to\infty} d_{l^1}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = 0$, i.e., $(x^{(k)})_{k=1}^\infty$ does not converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1}.

4. Conversely, any sequence which converges with respect to d_{l^1} also converges with respect to d_{l^∞}.

Suppose, for the sake of contradiction, that $(x^{(k)})_{k=1}^\infty$ does not converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞}, but does converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1}.

In this case, there exists a $\varepsilon > 0$ such that, for all $k \geqslant 1$, we have $(\sup_{n\geqslant 0} |x_n^{(k)} - a_n|) > \varepsilon$. In particulier, for all $k \geqslant 1$ and all $n \geqslant 0$, we have $|x_n^{(k)} - a_n| > \varepsilon$. Thus, $\sum_{n=0}^\infty |x_n^{(k)} - a_n|$ is not even a convergent series, and we cannot have $\lim_{k\to\infty}\left(\sum_{n=0}^\infty |x_n^{(k)} - a_n|\right) = 0$.

Note that this exercise actually shows that in this space X, the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

Exercise 12.1.16. — *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be two sequences in a metric space (X, d). Suppose that $(x_n)_{n=1}^\infty$ converges to a point $x \in X$, and $(y_n)_{n=1}^\infty$ converges to a point $y \in X$. Show that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.*

On the one hand, the triangle inequality applied two times to d gives us

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leqslant d(x_n, y_n) - d(x, y) \leqslant d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y).$$

Let be $\varepsilon > 0$. By hypothesis, there exists a $N_1 \geqslant 1$ such that $d(x_n, x) \leqslant \varepsilon/2$ whenever $n \geqslant N_1$. Similarly, there exists a $N_2 \geqslant 1$ such that $d(y_n, y) \leqslant \varepsilon/2$ whenever $n \geqslant N_2$. Thus, if we set $N := \max(N_1, N_2)$, then for all $n \geqslant N$ we have

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y) \leqslant 2\varepsilon/2 \; leq \varepsilon$$

which shows that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$, as expected.

Exercise 12.2.1. — *Verify the claims in Example 12.2.8*

Let be (X, d_{disc}) a metric space, and E a subset of X.

- Let be $x \in E$. Then x is an interior point of E. Indeed, we have $B(x, 1/2) = \{x\} \subseteq E$.
- Let be $y \notin E$. Then y is an exterior point of E. Indeed, we have $B(y, 1/2) \cap E = \{y\} \cap E = \varnothing$.
- Finally, there are no boundary points of E in (X, d_{disc}). Indeed, let be $r > 0$ and any $x \in X$. We will always have $B(x, r) = \{x\}$ by definition of the discrete metric d_{disc}. Thus, we have either $x \in E$ and then $x \in \text{int}(E)$, or $x \notin E$ and then $x \in \text{ext}(E)$. Thus, E has no boundary points.

Exercise 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that $(a) \implies (b)$. We will use the contrapositive, assuming that x_0 is neither an interior point of E, nor a boundary point of E. By definition, it means that x_0 is an exterior point of E, i.e. that there exists $r > 0$ such that $B(x_0, r) \cap E = \varnothing$. This is precisely the negation of x_0 being an adherent point of E. Thus, we have showed that if x_0 is an adherent of of E, it is either an interior point of a boundary point.
- Let's show that $(b) \implies (c)$. Let be a positive integer $n > 0$, and suppose that x_0 is either an interior point of E, or a boundary point of E. In either case, the set $A_n := B(x_0, 1/n) \cap E$ is non empty, i.e., there exists $a_n \in X$ such that $d(a_n, x_0) < 1/n$. By the (countable) axiom of choice, we can define a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A_n$ for all $n \geqslant 1$.

  Let be $\varepsilon > 0$. There exists $N > 0$ such that $1/N < \varepsilon$ (Exercise 5.4.4). Thus, for all $n \geqslant N$, we have
  $$d(a_n, x_0) < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$
  i.e., the sequence $(a_n)_{n=1}^{\infty}$ converges to x_0 with respect to the metric d, as expected.
- Finally, let's show that $(c) \implies (a)$. Let be $r > 0$. If $(a_n)_{n=1}^{\infty}$ in E converges to x_0 with respect to d, then there exists a n such that $d(x_0, a_n) < r$. But since $a_n \in E$, it means that $B(x_0, r) \cap E$ is non empty, i.e. that x_0 is an adherent point of E.

Exercise 12.2.3. — *Prove Proposition 12.2.5.*

Let be (X, d) a metric space.

(a) Let be $E \subseteq X$. First suppose that E is open; this means that $E \cap \partial E = \varnothing$. Let be $x \in E$, then we have $x \notin \partial E$. But since $x \in E$, we have $x \in \overline{E}$, and thus $x \in \text{int}(E)$ by Proposition 12.2.10(b). We have shown that $x \in E \implies x \in \text{int}(E)$, and since the converse implication is trivial (Remark 12.2.6), we have $E = \text{int}(E)$ as expected.

Now suppose that $E = \text{int}(E)$. Let be $x \in E$. We thus have $x \in \text{int}(E)$. By definition, x is thus not a boundary point of E, i.e. x $\notin \partial E$. This means that $E \cap \partial E = \varnothing$, i.e. that E is open, as expected.

(b) Let be $E \subseteq X$. First suppose that E is closed; i.e. that $\partial E \subseteq E$. Let be $x \in \overline{E}$. By Proposition 12.2.10, we have $\overline{E} = \text{int}(E) \cup \partial E$; such that $\overline{E}$ is the union of two subsets of E, and thus is itself a subset of E, as expected.

Conversely, suppose that $\overline{E} \subseteq E$. It means that $\text{int}(E) \cup \partial E \subseteq E$, and in particular that $\partial E \subseteq E$, i.e. that E is closed, as expected.

(c) Let be $x_0 \in X$, $r > 0$ and $E := B(x_0, r)$. To show that E is open, we must show that $E = \text{int}(E)$ (by Proposition 1.2.15(a)), and in particular that $E \subseteq \text{int}(E)$ (the converse inclusion being trivial). Let be $x \in E$, and let's show that $x \in \text{int}(E)$. By definition, we have $d(x, x_0) < r$, so that $\varepsilon := r - d(x, x_0)$ is a positive real number. Thus, let be $y \in B(x, \varepsilon)$. By the triangle inequality, we have

$$\begin{aligned} d(x_0, y) &< d(x, x_0) + d(x, y) \\ &< d(x, x_0) + \varepsilon \\ &< d(x, x_0) + r - d(x, x_0) = r \end{aligned}$$

so that $y \in E$. Thus, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq E$, i.e., x is an interior point of E. This shows that $E \subseteq \text{int}(E)$, as expected.

Now let be $F := \{x \in X : d(x, x_0) \leqslant r\}$, and let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in F. To show that F is closed, we have to show that $\ell := \lim_{n \to \infty} a_n$ lies in F (Proposition 12.2.15(b)). Suppose, for the sake of contradiction, that $\ell \notin F$. We thus have $d(\ell, x_0) > r$, so that $\varepsilon := d(\ell, x_0) - r$ is a positive real number. Since $(a_n)_{n=1}^{\infty}$ converges to ℓ, there exists a $N > 0$ such that $d(a_n, \ell) < \varepsilon$ whenever $n \geqslant N$. By the triangle inequality, for $n \geqslant N$, we have

$$\begin{aligned} d(x_0, \ell) &\leqslant d(x_0, a_n) + d(a_n, \ell) \\ d(x_0, a_n) &\geqslant d(x_0, \ell) - d(a_n, \ell) \\ &\geqslant d(x_0, \ell) - \varepsilon \\ &\geqslant d(x_0, \ell) + r - d(\ell, x_0) \\ &\geqslant r \end{aligned}$$

and thus, $a_n \notin B(x_0, r)$, a contradiction. Thus, we must have $\ell \in F$, so that F is indeed a closed set.

(d) Let be $\{x_0\}$ a singleton with $x_0 \in X$. To show that E is closed, we may use Proposition 12.2.15(b), and show that $\{x_0\}$ contains all its adherent points. Let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in $\{x_0\}$; it can only be the constant sequence $x_0, x_0, \ldots$. Since it is a constant sequence, its limit can only be x_0 itself, and this limit belongs to $\{x_0\}$. Thus, $\{x_0\}$ is a closed set.

(e) First we can form a lemma: for any subset $E \subseteq X$, we have $\text{int}(E) = \text{ext}(X \backslash E)$. This is a direct consequence of Definition 12.2.5. Indeed, $x \in \text{int}(E)$ iff there exists a $r > 0$ such that $B(x, r) \subseteq E$, which is equivalent to "$\exists r > 0 : B(x, r) \cap (X \backslash E) = \varnothing$", which is equivalent to $x \in \text{ext}(X \backslash E)$.

This implies that the interior points of E are the exterior points of $X \backslash E$, and conversely, that the exterior points of E are the interior points of $X \backslash E$. Thus, in particular, we have this useful fact:

$$\partial E = \partial (X \backslash E). \tag{12.4}$$

Now we go back to the main proof. First suppose that E is open. Thus, by Definition 12.2.12, we have $E \cap \partial E = \varnothing$, so that $\partial E \subseteq X \backslash E$, which means that $X \backslash E$ is a closed set. The converse also applies: if we suppose that $X \backslash E$ is closed, then $\partial(X \backslash E) \subseteq X \backslash E$. By equation (12.4) above, this is equivalent to $\partial E \subseteq X \backslash E$, and thus we have $\partial E \cap E = \varnothing$. This means that E is open, as expected.[1]

(f) Let $E_1, \ldots, E_n$ be open sets. Thus, for all $1 \leqslant i \leqslant n$, if $x \in E_i$, there exists a $r_i > 0$ such that $B(x, r_i) \subseteq E_i$. Let's define $r := \min(r_1, \ldots, r_n)$. We have $B(x, r) \subseteq B(x, r_i) \subseteq E_i$ for all $1 \leqslant i \leqslant n$, i.e. $B(x, r) \subseteq E_1 \cap \ldots \cap E_n$. Thus, $E_1 \cap \ldots \cap E_n$ is an open set.

Also, let $F_1, \ldots, F_n$ be closed sets. By the previous result (e), the complementary sets $X \backslash F_1, \ldots X \backslash F_n$ are open sets. Thus, we have just proved that $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n)$ is an open set. But we have $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n) = X \backslash (F_1 \cup \ldots \cup F_n)$, and this set is open. Thus, by (e), its complementary set, $F_1 \cup \ldots \cup F_n$, is closed, as expected.

(g) Let $(E_\alpha)_{\alpha \in I}$ be open sets. Suppose that we have $x \in \bigcup_{\alpha \in I} E_\alpha$. By definition, there exists a $i \in I$ such that $x \in E_i$. Since E_i is an open set, there exists $r_i > 0$ such that $B(x, r_i) \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$. Thus, by (a), $\bigcup_{\alpha \in I} E_\alpha$ is an open set, as expected.

Now let be $(F_\alpha)_{\alpha \in I}$ be closed sets. Suppose that we have a convergent sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \in \bigcap_{\alpha \in I} F_\alpha$ for all $n \geqslant 1$. Thus, for all $\alpha \in I$, the sequence $(x_n)_{n=1}^{\infty}$ entirely belongs to the closed set F_α, so that its limit ℓ also lies in F_α according to (b). Thus, $\ell \in \bigcup_{\alpha \in I} F_\alpha$, so that $\bigcap_{\alpha \in I} F_\alpha$ is a closed set, as expected.

(h) Let be $E \subseteq X$.

- Let's show that $\operatorname{int}(E)$ is the largest open set included in E. It has not clearly be proved in the main text that $\operatorname{int}(E)$ is an open set, so we begin by proving it. Let be $x \in \operatorname{int}(E)$. By definition, there exists $r > 0$ so that $B(x, r) \subseteq E$. But by (c), we know that $B(x, r)$ is an open set, so that any point y of $B(x, r)$ is an interior point of this open ball, and thus an interior point of E. Thus, $\operatorname{int}(E)$ is open.

  Now consider another open set $V \subseteq E$, and let's show that $V \subseteq \operatorname{int}(E)$. If $x \in \operatorname{int}(V)$, then there exists $r > 0$ such that $B(x, r) \subseteq V \subseteq E$, so that $x \in \operatorname{int}(E)$. This shows that $V \subseteq \operatorname{int}(E)$, as expected.

- Similarly, let's show that $\overline{E}$ is the smallest closed set that contains E. First we show that $\overline{E}$ is closed, i.e. that $\overline{\overline{E}} \subseteq \overline{E}$. (Hint: see Exercise 9.1.6 for an intuition.) Let be $x \in \overline{\overline{E}}$. By definition, for all $r > 0$, $B(x, r) \cap \overline{E} \neq \varnothing$. Thus, there exists $y \in B(x, r)$ such that $y \in \overline{E}$. Thus, because $B(x, r)$ is an open set and y is adherent to E, there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq B(x, r)$ and $B(y, \varepsilon) \cap E \neq \varnothing$; i.e., there exists $z \in B(y, \varepsilon) \subseteq B(x, r)$ such that $z \in E$. We have showed that whenever $x \in \overline{\overline{E}}$, we have $B(x, r) \cap E \neq \varnothing$ for all $r > 0$, i.e. that x is an adherent point of E, as expected. Thus, $\overline{E}$ is closed.

  Now we consider a closed set K such that $E \subseteq K$, and we have to show that $\overline{E} \subseteq K$. Let be $x \in \overline{E}$. By definition, for all $r > 0$, we have $B(x, r) \cap E \neq \varnothing$. But since $E \subseteq K$, we also have $B(x, r) \cap K \neq \varnothing$ for all $r > 0$. Thus, x is an adherent point of K, i.e., $x \in \overline{K}$. But since K is closed, we have $K = \overline{K}$, and thus $x \in K$. This shows that $\overline{E} \subseteq K$, as expected.

[1] This important result will be used in future proofs to turn any statement on closed sets into a statement on open sets.

Exercise 12.2.4. — *Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leqslant r\}$.*

Let's prove the following claims:

(a) Show that $\overline{B} \subseteq C$.

Let be $x \in \overline{B}$. By definition, since x is an adherent point of B, for all $\varepsilon > 0$ we have $B(x, \varepsilon) \cap B \neq \varnothing$. In other words, there exists y such that we have both $d(x, y) < \varepsilon$ and $d(x_0, y) < r$. Thus, by the triangle inequality, we have

$$\begin{aligned} d(x, x_0) &\leqslant d(x, y) + d(y, x_0) \\ &\leqslant \varepsilon + r \quad \text{for all } \varepsilon > 0 \end{aligned}$$

which is equivalent (as a quick proof by contradiction would show) to $d(x, x_0) \leqslant r$. Thus, $x \in C$.

We have indeed proved that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

Let's take $X = \mathbb{R}$, $d = d_{\text{disc}}$, $x = 0$ and $r = 1$. One the one hand, we have $B := \{0\}$ and $C := \mathbb{R}$. Now let's work out $\overline{B}$. By Proposition 12.2.15(bd), B is closed, so that we have $\overline{B} = B$. Thus, we clearly do not have $\overline{B} \neq C$ here. (Note however that any $x_0 \in \mathbb{R}$ would be convenient here; there is nothing special about 0.)

Exercise 12.3.1. — *Prove Proposition 12.3.4(b).*

Let's show each direction of the equivalence.

- First suppose that E is relatively closed w.r.t. Y, and let's show that there exists a closed subset $K \subseteq X$ such that $E = K \cap Y$.

  Since E is closed w.r.t. Y, the set $Y \backslash E$ is open w.r.t. Y (by Proposition 12.2.15(e)). Thus, by (a), there exists an open subset $V \subseteq X$ such that $Y \backslash E = V \cap Y$.

  Let be $K := X \backslash V$; this subset $K \subseteq X$ is closed w.r.t. X by Proposition 12.2.15(e) since it is the complementary set of an open set. We have to show that $E = K \cap Y$.

  - Let be $x \in E$. Thus, we have $x \in Y$, since $E \subseteq Y$. And since $x \in E$, by definition, we have $x \notin Y \backslash E$. Thus, $x \notin V \cap Y$, which implies that $x \notin V$ (since $x \in Y$). Thus, by definition, $x \in K$, and thus, $x \in K \cap Y$.
  - Conversely, let be $x \in K \cap Y$. By definition, $x \in Y$ and $x \notin V$. Thus, $x \notin V \cap Y$, or, in other words, $x \notin Y \backslash E$. We finally get $x \in E$, as expected.

  Thus, we have indeed $E = K \cap Y$, for some closed subset $K \subseteq X$, as expected.

- Now let's prove the converse implication: suppose that $E = K \cap Y$ for some closed subset $K \subseteq X$, and let's prove that E is relatively closed w.r.t. Y.

  Still by Proposition 12.2.15(e), we know that the subset $V := X \backslash K$ is open w.r.t. X. Thus, by the previous result from this exercise, $V \cap Y$ is relatively open w.r.t. Y. Thus, its complementary set $Y \backslash (V \cap Y) = Y \backslash V$ is relatively closed w.r.t. Y. Now we want to show that $E = Y \backslash V$ to close the proof.

– First suppose that $x \in E$. Since $E = K \cap Y$, we thus have $x \in Y$ and $x \in K$, i.e. $x \notin V$. Thus, $x \in Y \backslash V$.
– Now suppose that $x \in Y \backslash V$. We thus have $x \in X$ (since $Y \subseteq X$) and $x \notin V$, so that we necessarily have $x \in K$. Thus $x \in Y \cap K$, i.e. $x \in E$.

Thus $E = Y \backslash V$ is relatively closed w.r.t. Y, as expected.

Exercise 12.4.1. — *Prove Lemma 12.4.3.*

We have to prove that any subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of a convergent sequence $(x^{(n)})_{n=m}^{\infty}$ converges to the same limit as the whole sequence itself.

Suppose that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to x_0. Let be $\varepsilon > 0$. By definition, we have a positive integer $N \geqslant m$ such that $n \geqslant N \implies d(x^{(n)}, x_0) \leqslant \varepsilon$. Our aim here is to show that there exists a positive integer $J \geqslant 1$ such that $j \geqslant J \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon$.

By Definition 12.4.1, we know that we have $m \leqslant n_1 < n_2 < n_3 < \ldots$. Thus, as a quick induction would show, we have $n_j \geqslant m + j - 1$ for all $j \geqslant 1$. Let's take $J := N$. In this case, if $j \geqslant J$, i.e. if $j \geqslant N$, we have $n_j \geqslant m + N - 1 \geqslant N$. Thus:

$$j \geqslant J \implies n_j \geqslant N \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it means that $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.2. — *Prove Proposition 12.4.5.*

Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space. We have to prove that the following two statements are equivalent:

(a) L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.

(b) There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence which converges to L.

We will prove the two implications, but first, note that (with the notations from Definition 12.4.1) if we have $1 \leqslant m \leqslant n_1 < n_2 < n_3 < \ldots$, then a quick induction shows that we have $n_j \geqslant j$ for all $j \geqslant 1$.

- First we prove that (b) implies (a). If some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to L, then we have by definition:

$$\forall \varepsilon > 0, \exists J \geqslant 1 : j \geqslant J \implies d(x^{(n_j)}, L) \leqslant \varepsilon \tag{12.5}$$

  Now, consider any $\varepsilon > 0$ and any $N \geqslant m$. For this particular choice of ε, consider the corresponding real number J given by equation (12.5), and let's define $p := \max(N, J)$. Thus, we have $n_p \geqslant p \geqslant J$, and by equation (12.5), we thus have $d(x^{(n_p)}, L) \leqslant \varepsilon$. If we set $n := n_p$, we have indeed found an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, as required.

- Now we prove that (a) implies (b). Suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, there exists a natural number $n_1 \geqslant m$ such that $d(x^{(n_1)}, L) \leqslant 1$. Now, for $j > 1$, let's define inductively $n_j := \min\{n > n_{j-1} : d(x^{(n)}, L) \leqslant 1/j\}$. This set is non-empty (by definition of a limit point), so that the well-ordering principle

(Proposition 8.1.4) ensures that it has a (unique) minimal element, i.e. that n_j indeed exists. Let's define the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ obtained following this process. We thus have $d(x^{(n_j)}, L) \leqslant 1/j$ for all $j \geqslant 1$, by construction.

Thus, let be $\varepsilon > 0$. There exists a $j \geqslant 1$ such that $0 < 1/j < \varepsilon$ (Exercise 5.4.4). Thus, for this positive integer j, we have $d(x^{(n_j)}, L) \leqslant 1/j < \varepsilon$. By construction, for all other natural numbers $k \geqslant j+1$, we have $d(x^{(n_k)}, L) \leqslant 1/k \leqslant 1/j \leqslant \varepsilon$.

In summary, for our arbitrary choice of ε, we have showed that there exists $j \geqslant 1$ such that, for all $k \geqslant j$, we have $d(x^{(n_k)}, L) \leqslant \varepsilon$. Thus, the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ constructed in this way converges to L, as expected.

Exercise 12.4.3. — *Prove Lemma 12.4.7.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a convergent sequence of points in a metric space (X, d), and that its limit is x_0. Let's show that it is a Cauchy sequence.

By the triangle inequality, we know that for all $j, k \geqslant m$, we have:

$$d(x^{(j)}, x^{(k)}) \leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0).$$

Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, there exists an $N \geqslant m$ such that we have $d(x^{(n)}, x_0) \leqslant \varepsilon/3$ for all $n \geqslant N$. Thus, if we take $j, k \geqslant N$, we have:

$$\begin{aligned} d(x^{(j)}, x^{(k)}) &\leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \\ &\leqslant \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, as expected.

Exercise 12.4.4. — *Prove Lemma 12.4.9.*

Let be an arbitrary $\varepsilon > 0$. Since the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, there exists a $J \geqslant 1$ such that $d(x^{(n_j)}, x_0) \leqslant \varepsilon/3$ whenever $j \geqslant J$.

But the whole sequence $(x^{(n)})_{n=m}^{\infty}$ is supposed to be a Cauchy sequence. Thus, there also exists a $N \geqslant m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon/3$ whenever $j, k \geqslant N$.

Now, let be $K := \max(J, N)$. If $k \geqslant K$, we have

$$\begin{aligned} d(x^{(k)}, x_0) &\leqslant d(x^{(k)}, x^{(n_k)}) + d(x^{(n_k)}, x_0) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.5. — *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d) and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\{x^{(n)} : n \geqslant m\}$. Is the converse true?*

First suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, it means that

$$\forall \varepsilon > 0,\ \forall N \geqslant m,\ \exists n \geqslant N \ :\ d(x^{(n)}, L) \leqslant \varepsilon \tag{12.6}$$

Let be an arbitrary $\varepsilon > 0$, and let's take $N = m$. By formula (12.6) above, there exists an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, this $x^{(n)}$ belongs to both sets $\{x^{(n)} : n \geqslant m\}$ and $B(L, \varepsilon)$. We have just proved that for all $\varepsilon > 0$, the intersection $B(L, \varepsilon) \cap \{x^{(n)} : n \geqslant m\}$ is always non-empty. In other words, L is thus an adherent point of $\{x^{(n)} : n \geqslant m\}$.

However, the converse is not true. Indeed, consider the sequence $(x^{(n)})_{n=1}^{\infty}$ defined in $(\mathbb{R}, d)$ by $x^{(1)} = 1$ and $x^{(n)} = 0$ for all $n \geqslant 2$, i.e. the sequence $1, 0, 0, 0, \ldots$. It is clear that $L := 1$ is an adherent point of $\{x^{(n)} : n \geqslant 1\}$ (which is just the set $\{0, 1\}$). But 1 is not a limit point of $(x^{(n)})_{n=1}^{\infty}$, since we have $d(x^{(n)}, 1) > 1/2$ for all $n \geqslant 2$.

Exercise 12.4.6. — *Show that every Cauchy sequence can have at most one limit point.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in a metric space (X, d), such that L, L' are limit points. Then we have $L = L'$. We will give two different proofs for this fact.

- **Proof 1** (short proof using previous results). By Proposition 12.4.5, since L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, there exists a subsequence that converges to L. But by Lemma 12.4.9, it means that the whole original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to L. The same argument can be used to show that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to L'. But by uniqueness of limits (Proposition 12.1.20), we must have $L = L'$, as expected.

- **Proof 2** (a more "manual" proof). Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, there exists $N \geqslant m$ such that $d(x^{(p)}, x^{(q)}) \leqslant \varepsilon/3$ for all $p, q \geqslant N$.

  If L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, then for this $N \geqslant m$, there exists $p \geqslant N$ such that $d(x^{(p)}, L) \leqslant \varepsilon/3$. Similarly, there exists $q \geqslant N$ such that $d(x^{(q)}, L') \leqslant \varepsilon/3$.

  We thus have, by triangle inequality:

$$\begin{aligned} d(L, L') &\leqslant d(L, x^{(p)}) + d(x^{(p)}, x^{(q)}) + d(x^{(q)}, L') \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leqslant \varepsilon \end{aligned}$$

  Thus, $d(L, L') \leqslant \varepsilon$ for all $\varepsilon > 0$, which implies $L = L'$.

Exercise 12.4.7. — *Prove Proposition 12.4.12.*

For statement (a), consider a convergent sequence $(y^{(n)})_{n=m}^{\infty}$ of elements of $Y \subseteq X$. Since it is convergent, it is a Cauchy sequence (Lemma 12.4.7). Saying that $(Y, d_{Y \times Y})$ is complete means that $(y^{(n)})_{n=m}^{\infty}$ converges in $(Y, d_{Y \times Y})$. Thus, every convergent sequence in Y has its limit in Y: this is exactly the characterization of closed sets given by Proposition 12.2.15(b).

For statement (b), consider a Cauchy sequence $(y^{(n)})_{n=m}^{\infty}$ of elements of a given closed subset $Y \subseteq X$. Since (X, d) is complete, $(y^{(n)})_{n=m}^{\infty}$ must converge to some value $L \in X$. But since Y is closed, we have $L \in Y$ by Proposition 12.2.15(b). Thus, every Cauchy sequence in Y converges in Y. This means that $(Y, d_{Y \times Y})$ is complete, as expected.

Exercise 12.4.8. — *The following construction generalizes the construction of the reals from the rationals in Chapter 5. In what follows, we let (X, d) be a metric space.*

We have to prove the following statements. Note that this is a generalization of the process of construction of the real numbers, so that we can use all results relative to the real numbers below.

(a) Given any Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ in X, we denote $\mathrm{LIM}_{n\to\infty}x_n$ its formal limit. We say that two formal limits $\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n$ are equal iff $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then, this equality relation obeys the reflexive, symmetry and transitive axioms.

- This relation is reflexive: for every Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$, we have $d(x_n, x_n) = 0$ for all $n \geqslant 1$, by definition of a metric. Thus, $d(x_n, x_n)$ is constant and equal to zero, so that $\lim_{n\to\infty} d(x_n, x_n) = 0$.
- By the property of symmetry of the metric d, we have $d(x_n, y_n) = d(y_n, x_n)$ for all $n \geqslant 1$ and all Cauchy sequence $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$. Thus, $\mathrm{LIM}_{n\to\infty}x_n = \mathrm{LIM}_{n\to\infty}y_n$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$, iff $\lim_{n\to\infty} d(y_n, x_n) = 0$, which is equivalent to $\mathrm{LIM}_{n\to\infty}y_n = \mathrm{LIM}_{n\to\infty}x_n$.
- For transitivity, suppose that $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ and $(z^{(n)})_{n=1}^{\infty}$ are Cauchy sequences in X. If $\mathrm{LIM}_{n\to\infty}x_n = \mathrm{LIM}_{n\to\infty}y_n$ and $\mathrm{LIM}_{n\to\infty}y_n = \mathrm{LIM}_{n\to\infty}z_n$, then by definition we have $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. Let be $\varepsilon > 0$. By definition, there exists $N_1 \geqslant 1$ such that $d(x_n, y_n) \leqslant \varepsilon/2$ whenever $n \geqslant N_1$. Similarly, there exists $N_2 \geqslant 1$ such that $d(y_n, z_n) \leqslant \varepsilon/2$ whenever $n \geqslant N_2$. Thus, if $n \geqslant N := \max(N_1, N_2)$, we have by the triangle inequality $d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n) \leqslant \varepsilon$. It means that $\lim_{n\to\infty} d(x_n, z_n)$, i.e. that $\mathrm{LIM}_{n\to\infty}x_n = \mathrm{LIM}_{n\to\infty}z_n$, as expected.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, with the above equality relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to \mathbb{R}^+$ by setting

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Then this function is well-defined and gives $\overline{X}$ the structure of a metric space.

- First we have to show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists (in $\mathbb{R}^+$) for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$. We already know that $\mathbb{R}$ is complete, thus $\mathbb{R}^+$ is complete as a closed subset of the complete space $\mathbb{R}$ (Proposition 12.4.12(b)).

  Let be the sequence defined by $u_n := d(x_n, y_n)$ for all $n \geqslant 1$. Obviously, this sequence is in $\mathbb{R}^+$, which is a complete space. Thus, to show that it converges, we just have to show that it is a Cauchy sequence.

  Consider the usual metric on $\mathbb{R}^+$. We have, for all $p, q \geqslant 1$,

$$\begin{aligned} |u_p - u_q| &= |d(x_p, y_p) - d(x_q, y_q)| \\ &\leqslant |d(x_p, x_q) + d(x_q, y_q) + d(y_q, y_p) - d(x_q, y_q)| \\ &\leqslant |d(x_p, x_q)| + |d(y_p, y_q)|. \end{aligned}$$

  Now let be $\varepsilon > 0$. Since $(x^{(n)})_{n=1}^{\infty}$ and $(y^{(n)})_{n=1}^{\infty}$ are Cauchy sequences, there exists $N_1, N_2 \geqslant 1$ such that $d(x_p, x_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_1$, and $d(y_p, y_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_2$. Thus, if $p, q \geqslant N := \max(N_1, N_2)$, we have

$$|u_p - u_q| \leqslant |d(x_p, x_q)| + |d(y_p, y_q)| \leqslant \varepsilon.$$

  This shows that $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence, and thus, that $\lim_{n\to\infty} d(x_n, y_n)$ exists.

- Now we must show that the axiom of substitution is obeyed. In other words, consider a Cauchy sequence $(z^{(n)})_{n=1}^{\infty}$ in (X,d) such that $\mathrm{LIM}_{n\to\infty} z_n = \mathrm{LIM}_{n\to\infty} x_n$. We must show that $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} z_n, \mathrm{LIM}_{n\to\infty} y_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n)$, i.e. that

$$\lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) \tag{12.7}$$

  By the previous bullet point, we know that both limits in (12.7) do exist. Thus, the limit laws apply. We have:

$$d(z_n, y_n) \leqslant d(z_n, x_n) + d(x_n, y_n)$$

  but since $\lim_{n\to\infty} d(z_n, x_n) = 0$ by definition, we obtain

$$\lim_{n\to\infty} d(z_n, y_n) \leqslant \lim_{n\to\infty} d(x_n, y_n)$$

  if we take the limits of both sides in the previous inequality.
  But similarly, we have $d(x_n, y_n) \leqslant d(x_n, z_n) + d(z_n, y_n)$, so that a similar argument gives

$$\lim_{n\to\infty} d(x_n, y_n) \leqslant \lim_{n\to\infty} d(z_n, y_n).$$

  Thus, we have indeed $\lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_n, y_n)$, as expected.
- Finally, we must show that $d_{\overline{X}}$ is a metric on $\overline{X}$. To prove this statement, we must show that $d_{\overline{X}}$ obeys all four axioms that define a metric.
  - First, it is clear that $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$ for all Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ in (X,d).
  - Now let be two Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ in X, such that $\mathrm{LIM}_{n\to\infty} x_n \neq \mathrm{LIM}_{n\to\infty} y_n$. This latest property implies that $\lim_{n\to\infty} d(x_n, y_n) > 0$, by definition. Thus, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) > 0$.
  - Symmetry: we have

$$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} d(y_n, x_n) \text{ (symmetry of } d \text{ on } \mathbb{R}^+) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} x_n) \end{aligned}$$

    for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$.
  - Triangle inequality: by the limit laws, we have

$$\begin{aligned} & d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} z_n) = \lim_{n\to\infty} d(x_n, z_n) \\ & \leqslant \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) \\ & \leqslant \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ & \leqslant d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) + d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} y_n, \mathrm{LIM}_{n\to\infty} z_n) \end{aligned}$$

    for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ and $(z^{(n)})_{n=1}^{\infty}$.

Thus, $d_{\overline{X}}$ is indeed a metric on $\overline{X}$.

(c) The metric space $(\overline{X}, d_{\overline{X}})$ is complete.

To prove this statement, consider a Cauchy sequence $(u_n)_{n=1}^{\infty}$ in $\overline{X}$: we have to prove that this sequence converges in $(\overline{X}, d_{\overline{X}})$.

By definition, $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence of formal limits of Cauchy sequences that take their values in X; i.e., for all $k \geqslant 1$, there exists a Cauchy sequence $(x_n^{(k)})_{n=1}^{\infty}$ of elements of X such that $u_k := \mathrm{LIM}_{n\to\infty} x_n^{(k)}$.

Since all $(x_n^{(k)})_{n=1}^{\infty}$ are Cauchy sequences, then for all $k \geqslant 1$, there exists a threshold N_k such that $d(x_n^{(k)}, x_{N_k}^{(k)}) < 1/k$ whenever $n \geqslant N_k$. Thus, (using the countable axiom of choice) we can build a sequence $(z_k)_{k=1}^{\infty}$ defined by

$$z_k := \left(x_{N_k}^{(k)}\right)$$

for all $k \geqslant 1$. Now:

- We claim that $(z_k)_{k=1}^{\infty}$ is itself a Cauchy sequence. Indeed, consider an arbitrary positive real number $\varepsilon > 0$. We must prove that $d(z_p, z_q) := d(x_{N_p}^{(p)}, x_{N_q}^{(q)})$ is eventually lesser than ε.

  Since $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\overline{X}$, there exists a $N \geqslant 1$ such that, if $p, q \geqslant N$, we have $d_{\overline{X}}(u_p, u_q) < \varepsilon/3$, i.e.:

$$\begin{aligned}
\varepsilon/3 &> d_{\overline{X}}(u_p, u_q) \\
&\geqslant d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n^{(p)}, \mathrm{LIM}_{n\to\infty} x_n^{(q)}) \\
&\geqslant \lim_{n\to\infty} d(x_n^{(p)}, x_n^{(q)})
\end{aligned}$$

  Thus, there exists a $N' \geqslant 1$ such that, if $n \geqslant N'$, we have $d(x_n^{(p)}, x_n^{(q)}) \leqslant \varepsilon/3$[2]. Also, by Exercise 5.4.4, there exists a $k > 0$ such that $1/k \leqslant \varepsilon/3$. Thus, if $n, p, q \geqslant \max(k, N', N_p, N_q)$, we have

$$\begin{aligned}
d(z_p, z_q) &= d(x_{N_p}^{(p)}, x_{N_q}^{(q)}) \\
&\leqslant \underbrace{d(x_{N_p}^{(p)}, x_n^{(p)})}_{\leqslant 1/p \leqslant \varepsilon/3} + \underbrace{d(x_n^{(p)}, x_n^{(q)})}_{\leqslant \varepsilon/3} + \underbrace{d(x_n^{(q)}, x_{N_q}^{(q)})}_{\leqslant 1/q \leqslant \varepsilon/3} \\
&\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&\leqslant \varepsilon
\end{aligned}$$

  Thus, $(z_k)_{k=1}^{\infty}$ is indeed a Cauchy sequence in X.

- Consequently, we can take the formal limit $L := \mathrm{LIM}_{n\to\infty} z_n$, and this formal limit L lies in $\overline{X}$ by definition. We claim that $\lim_{n\to\infty} u_n = L \in \overline{X}$; proving this claim will close the proof of (c).

  Let be $\varepsilon > 0$. Since $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence in X, there exists a $N_1 \geqslant 1$ such that $d(z_p, z_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_1$.

[2]Indeed, for any sequence $(v_n)_{n=1}^{\infty}$ that converges to ℓ, if we have $0 \leqslant \ell < \varepsilon$, then there exists an $N \geqslant 1$ such that $v_n \leqslant \varepsilon$ whenever $n \geqslant N$ (why? use a proof by contradiction.).

Once again, by Exercise 5.4.4, there exists a $K' \geqslant 1$ such that $1/K' < \varepsilon/2$. Thus, if $k \geqslant K$ and $n > N_k$, we have

$$d(x_n^{(k)}, z_k) := d(x_n^{(k)}, x_{N_k}^{(k)}) < \frac{1}{k} \leqslant \frac{1}{K} < \frac{\varepsilon}{2}.$$

Thus, by the triangle inequality, we have, for all $n > \max(N_k, N_1)$,

$$d(x_n^{(k)}, z_n) \leqslant d(x_n^{(k)}, z_k) + d(z_k, z_n) \leqslant \varepsilon/2 + \varepsilon/2 \leqslant \varepsilon.$$

Consequently, we have, for all $k > K'$,

$$d_{\overline{X}}(u_k, L) := \lim_{n\to\infty} d(x_n^{(k)}, b_n) < \varepsilon.$$

This shows that $(u_n)_{n=1}^{\infty} \to L$ in $(\overline{X}, d_{\overline{X}})$, which closes the proof.

(d) We identify an element $x \in X$ with the corresponding formal limit $\mathrm{LIM}_{n\to\infty} x$ in $\overline{X}$.

- This is legitimate since we have $x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$.
  Indeed, it is clear that if $x = y$, then we have $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$ by definition. Conversely, if $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$, then we have $\lim_{n\to\infty} d(x,y) = 0$, i.e. $d(x,y) = 0$, i.e. $x = y$. Thus, this identification is legitimate.
- With this identification, we have $d(x,y) = d_{\overline{X}}(x,y)$. Indeed:

$$\begin{aligned} d_{\overline{X}}(x,y) &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x, \mathrm{LIM}_{n\to\infty} y) \\ &= \lim_{n\to\infty} d(x,y) \\ &= d(x,y). \end{aligned}$$

Thus, (X,d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) The closure of X in $\overline{X}$ is $\overline{X}$.

Indeed, let be C the closure of X in $\overline{X}$. We clearly have $C \subseteq \overline{X}$, by definition. Thus we just have to show that $\overline{X} \subseteq C$.

Let be $x \in \overline{X}$, and let's show that $x \in C$. By definition, $x \in C$ means that x is an adherent point of X in $\overline{X}$, i.e. that for all $\varepsilon > 0$, $B_{(\overline{X}, d_{\overline{X}})}(x, \varepsilon) \cap X \neq \varnothing$. In other words, for all $\varepsilon > 0$, we must show that there exists a $y \in X$ such that $d_{\overline{X}}(x,y) < \varepsilon$.

Thus, let be $\varepsilon > 0$. By definition, x is the formal limit of a Cauchy sequence $(x_n)_{n=1}^{\infty}$ of elements of X, so that $x := \mathrm{LIM}_{n\to\infty} x_n$. Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists an $N \geqslant 1$ such that $d(x_n, x_N) < \varepsilon/2$ whenever $n \geqslant N$. Thus:

$$\begin{aligned} d_{\overline{X}}(x, x_N) &:= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_N) \\ &= \lim_{n\to\infty} d(x_n, x_N) \\ &\leqslant \varepsilon/2 < \varepsilon \end{aligned}$$

so that $y := x_N$ is a convenient choice. This shows that x is an adherent point of X in $\overline{X}$, as expected.

(f) Finally, the formal limit agrees with the actual limit, i.e., $\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \in \overline{X}$ for all Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X.

Indeed, let be $(x_n)_{n=1}^{\infty}$ a Cauchy sequence of elements of X. We know that (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$, so that $(x_n)_{n=1}^{\infty}$ can be thought of as a sequence of elements of $\overline{X}$. But we have showed that $(\overline{X}, d_{\overline{X}})$ is complete. Thus, the sequence $(x_n)_{n=1}^{\infty}$ converges in $\overline{X}$ to a certain limit $L \in \overline{X}$; i.e., we have $\lim_{n\to\infty} x_n = L$ for some $L \in \overline{X}$.

Consider this limit L. By definition of $\overline{X}$, there exists a Cauchy sequence $(a_n)_{n=1}^{\infty}$ of elements of X such that $L := \mathrm{LIM}_{n\to\infty} a_n$. What we need to prove is that we have

$$L = \lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n \tag{12.8}$$

and thus, it is sufficient to show that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n$, since we already have the other equalities. And, by definition of the equality relation established in (a), in order to prove that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n$, we just have to show that $\lim_{n\to\infty} d(x_n, a_n) = 0$. Or, in yet another equivalent way, we have to show that for all $\varepsilon > 0$, there exists an $N \geqslant 1$ such that $d(x_n, a_n) \leqslant \varepsilon$ whenever $n \geqslant N$.

Thus, let be an arbitrary $\varepsilon > 0$. Let's unfold our hypotheses.

- We know that the sequence $(x_n)_{n=1}^{\infty}$ converges to L in $\overline{X}$. Thus, by definition, there exists a $N_1 \geqslant 1$ such that $d_{\overline{X}}(x_k, L) \leqslant \varepsilon/2$ whenever $k \geqslant N_1$. In other words, $\lim_{n\to\infty} d(x_k, a_n) \leqslant \varepsilon/3 < \varepsilon/2$ whenever $k \geqslant N_1$.

  Thus, there exists a N_2 such that $d(x_k, a_n) \leqslant \varepsilon/2$ whenever $k \geqslant N_1$ and $n \geqslant N_2$ (see footnote 2 p. 22 from the present document).
- We also know that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. It means that there exists a $N_3 \geqslant 1$ such that $d(x_p, x_q) \leqslant \varepsilon/2$ for all $p, q \geqslant N_3$.

Let be $N := \max(N_1, N_2, N_3)$. Using the triangle inequality, we finally get, for all $n \geqslant N$,

$$\begin{aligned} d(x_n, a_n) &\leqslant d(x_n, x_N) + d(x_N, a_n) \\ &\leqslant \varepsilon/2 + \varepsilon/2 \\ &\leqslant \varepsilon \end{aligned}$$

This closes the proof.

EXERCISE 12.5.1. — *Show that Definitions 9.1.22 and 12.5.3 match when talking about subsets of the real line with the standard metric.*

Consider $Y \subseteq \mathbb{R}$ and the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. We have to show that both definitions of boundedness are equivalent in this case.

- First, suppose that Y is bounded in the sense of Definition 12.5.3. Thus, there exists a real number x and a positive real number $r > 0$ such that $Y \subseteq B(x, r)$. In other words, we have $Y \subseteq]x - r, x + r[\subseteq [x - r, x + r]$. Let be $M := |x| + |r|$. We clearly have $x + r \leqslant M$, and $-M \leqslant x - r$. Thus, we have $Y \subseteq [-M, M]$, and Y is bounded in the sense of Definition 9.1.22.

- Conversely, suppose that Y is bounded in the sense of Definition 9.1.22. Thus, there exists a positive real $M > 0$ such that $Y \subseteq [-M, M] \subset]-2M, 2M[$. But this later interval is simply $B(0, 2M)$, so that Y is bounded in the sense of Definition 12.5.1, taking $x := 0$ and $r := 2M$.

Exercise 12.5.2. — *Prove Proposition 12.5.5.*

We must prove that any compact space (X, d) is both complete and bounded. In both cases, we will use a proof by contradiction.

- First, let's prove completeness. Suppose, for the sake of contradiction, that the compact space (X, d) is not complete. Since it is not complete, there exists a Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X which does not converge in (X, d). But since it is compact, there exists a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ of this Cauchy sequence, which converges in (X, d). But, by Lemma 12.4.9, if a Cauchy sequence has a convergent subsequence, then it is convergent itself; thus $(x^{(n)})_{n=1}^{\infty}$ converges. It is a clear contradiction. Thus, (X, d) must be complete.

- Now we show boundedness. Similarly, suppose for the sake of contradiction that (X, d) is not bounded. It means that, for all positive real $r > 0$ and all $x \in X$, we have $X \nsubseteq B(x, r)$. In particular, for any positive natural number $n \geqslant 1$ and an arbitrary $x \in X$, the set $X \backslash B(x, n)$ is not empty. Thus, using the (countable) axiom of choice, we can build a sequence $(x^{(n)})_{n=1}^{\infty}$ such that $x^{(n)} \in X \backslash B(x, n)$ for all positive integer $n \geqslant 1$. Or, in other words, we have $d(x, x^{(n)}) \geqslant n$ for all $n \geqslant 1$.

  But recall that (X, d) is compact. Thus, there must exist a convergent subsequence $(x^{(n_k)})_{k=1}^{\infty}$ of the original sequence. Say that this subsequence converges to some value L. Thus, by definition,

$$\forall \varepsilon > 0, \exists K \geqslant 1 \,:\, k \geqslant K \implies d(x^{(n_k)}, L) \leqslant \varepsilon.$$

  Let's take $\varepsilon := 1$ (there is nothing special about this value; this is just any arbitrary ε to obtain a contradiction). There must exist a $K_1 \geqslant 1$ such that $d(x^{(n_k)}, L) \leqslant 1$ whenever $k \geqslant K_1$. But, at the same time, we have by the triangle inequality

$$\begin{aligned} & d(x^{(n_k)}, x) \leqslant d(x^{(n_k)}, L) + d(L, x) \\ \implies & d(x^{(n_k)}, L) \geqslant d(x^{(n_k)}, x) - d(L, x) \end{aligned}$$

  For instance by the Archimedean principle, there exists an $N \in \mathbb{N}$ such that $N \geqslant d(L, x) + 3$. Let be $K_2 := \min\{k \in \mathbb{N} : n_k \geqslant N\}$ (this natural number exists simply because $n_N \geqslant N$, so that the set is not empty). We thus have

$$d(x, x^{(n_k)}) \geqslant n_k \geqslant N \geqslant d(L, x) + 3$$

  for all $k \geqslant K_2$.

  Thus, for all $k \geqslant \max(K_1, K_2)$, we have both $d(x^{(n_k)}, x) \leqslant 1$ (because $k \geqslant K_1$), and $d(x^{(n_k)}, L) \geqslant d(x^{(n_k)}, x) - d(L, x) \geqslant d(L, x) + 3 - d(L, x) \geqslant 3$ (because $k \geqslant K_2$). This is a contradiction. Thus, (X, d) is bounded.

Exercise 12.5.3. — *Prove Theorem 12.5.7.*

Let be $(\mathbb{R}^n, d)$ an Euclidean space, where d is either the Euclidean, taxicab or sup norm metric. Also, let be $E \subseteq \mathbb{R}^n$. We have to prove that E is compact iff E is closed and bounded. By Corollary 12.5.6, we already know that if E is compact, then it is closed and bounded. We thus have to prove the converse implication.

Suppose that E is both closed and bounded. Since E is a subset of $\mathbb{R}^n$, we can write $E := E_1 \times \ldots \times E_n$, where $E_j \subseteq \mathbb{R}$ for all $1 \leqslant j \leqslant n$.

We have to prove that any sequence $(x^{(k)})_{k=1}^{\infty}$ in E has a convergent subsequence in (E, d). This sequence can be written as a sequence of vectors of length n, i.e., we have $x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$, where $x_j^{(k)} \in E_j$ for all $k \geqslant 1$ and all $1 \leqslant j \leqslant n$.

We will first need a lemma:

Lemma. If E is bounded, then each $E_j \subseteq \mathbb{R}$ is also bounded.

Sketch of proof. Suppose that d is the sup norm metric. If E is bounded, we have $E \subseteq B(x, r)$ for some $x \in \mathbb{R}^n$ and some $r > 0$ (Definition 12.5.3). In other words, we have $d(x, y) < r$ for all $y \in E$. Since d is the sup norm metric, this implies that

$$\forall j \in [\![1, n]\!],\ |x_j - y_j| \leqslant \max_{j=1,\ldots,n} |x_j - y_j| := d(x, y) < r.$$

Thus, $E_j \subseteq B(x_j, r)$, i.e. E_j is bounded for all $1 \leqslant j \leqslant n$.

The proof is similar if d is the Euclidean metric, or the taxicab metric. $\square$

Now we go back to the main proof. Since each sequence $(x_j^{(k)})_{k=1}^{\infty}$ is a sequence of real numbers in the bounded subset $E_j \subseteq \mathbb{R}$, then by Theorem 9.1.24 this sequence has a convergent subsequence $(x_j^{(k_l)})_{l=1}^{\infty}$, which converges to $L_j \in \mathbb{R}_j$. But by Proposition 12.1.18, this implies that the whole subsequence $(x^{(k_l)})_{l=1}^{\infty}$ converges to $(L_1, \ldots, L_n)$ (since it converges component-wise).

Thus, $(x_j^{(k)})_{k=1}^{\infty}$ indeed has a convergent subsequence, as expected; and E is compact.

Exercise 12.5.4. — *Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and an open set $V \subseteq \mathbb{R}$, such that the image $f(V) := \{f(x) : x \in V\}$ of V is not open.*

As a simple example, consider the constant function $f(x) = 0$ defined on $V :=]-1, 1[$. The interval V is clearly open, but we have $f(V) = \{0\}$. This singleton (or more generally, any singleton) is not open in $(\mathbb{R}, d)$, since for all $r > 0$, there always exists a real number x such that $x \in B(0, r) \backslash \{0\}$.

Exercise 12.5.5. — *Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and closed set $F \subseteq \mathbb{R}$, such that $f(F)$ is not closed.*

One can give the example of the function $\tan^{-1}(x)$ defined on the closed set $F := \mathbb{R}$, but this function has not really been defined so far in the book. So, let's use a simpler example.

Consider the closed set $F := [1, +\infty[$ and the function $f(x) = 1/x$. We have $f(F) =]0, 1]$, which is not a closed set.

Exercise 12.5.6. — *Prove Corollary 12.5.9.*

Consider a sequence $K_1 \supset K_2 \supset K_3 \supset \ldots$ of non-empty compact sets in a metric space (X, d). We have to show that $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Let's work in the space $(K_1, d_{K_1 \times K_1})$. We define the sets $V_n := K_1 \backslash K_n$ for all $n \geqslant 1$, i.e.,

$$\begin{aligned} V_1 &:= K_1 \backslash K_1 = \varnothing \\ V_2 &:= K_1 \backslash K_2 \\ V_3 &:= K_1 \backslash K_3 \\ &\ldots \end{aligned}$$

so that the V_n clearly constitute an increasing sequence:

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \ldots,$$

so that $\bigcup_{k=1}^{n} V_k = V_n$ for all $n \geqslant 1$.

Furthermore, each set V_n is open in $(K_1, d_{K_1 \times K_1})$, since it is the complementary set of a compact (and then closed) set (Proposition 12.2.15 (e)).

Suppose, for the sake of contradiction, that we have $\bigcap_{n=1}^{\infty} K_n = \varnothing$. We would thus have:

$$\begin{aligned} \bigcup_{n=1}^{\infty} V_n &= \bigcup_{n=1}^{\infty} (K_1 \backslash K_n) \\ &= K_1 \backslash \left(\bigcap_{n=1}^{\infty} K_n \right) \text{ (Exercise 3.4.11)} \\ &= K_1 \backslash \varnothing \text{ (by hypothesis)} \\ &= K_1. \end{aligned}$$

But since K_1 is compact, then by Theorem 12.5.8, there exists a finite open cover of K_1, i.e., there exists a finite number k of indices $n_1 < \ldots < n_k\}$ such that

$$\bigcup_{n \in \{n_1, \ldots, n_k\}} V_n = K_1.$$

But since the V_n form an increasing sequence, this implies $V_{n_k} = K_1$, i.e., $K_1 \backslash K_{n_k} = K_1$, so that we finally get $K_{n_k} = \varnothing$.

But all the sets K_n were supposed to be non empty: this is thus a contradiction, and we must have $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Exercise 12.5.7. — *Prove Theorem 12.5.10.*

Let be (X, d) a metric space.

(a) Let be $Z \subseteq Y \subseteq X$, with Y compact. We have to show that Z is closed iff it is compact. We already know that if Z is compact, then it is closed (Corollary 12.5.6); so that we just have to show the converse implication.

Suppose that Z is closed, and let be $(z^{(n)})_{n=1}^{\infty}$ a sequence of elements of Z. Since $Z \subseteq Y$, $(z^{(n)})_{n=1}^{\infty}$ is also a sequence of elements of Y; and since Y is compact, there exists a subsequence $(z^{(n_k)})_{k=1}^{\infty}$ that converges to some $z \in Y$. But since Z is closed, we must have $z \in Z$ (by Proposition 12.2.15(b)). Thus, any sequence of elements of Z has a subsequence that converges in Z, i.e., Z is indeed compact.

(b) Let be $Y_1, \dots, Y_n$ be n compact subsets of X; we have to show that the finite union $Y_1 \cup \dots \cup Y_n$ is compact. Let's use the topological characterization of compact sets: suppose that we have an open cover $\bigcup_{\alpha \in I} V_\alpha$ (possibly uncountable), i.e. that

$$Y_1 \cup \dots \cup Y_n \subseteq \bigcup_{\alpha \in I} V_\alpha.$$

Clearly, we have $Y_1 \subseteq \bigcup_{\alpha \in I} V_\alpha$, and since V_1 is compact, there exists a finite open cover, i.e. $Y_1 \subseteq \bigcup_{i=1}^{s_1} V_{a_i}$. Similarly, there exist finite open covers for each other subset Y_i, i.e.,

$$Y_2 \subseteq \bigcup_{i=1}^{s_2} V_{b_i}$$
$$\dots$$
$$Y_n \subseteq \bigcup_{i=1}^{s_n} V_{n_i}.$$

Thus, there exists a finite open cover

$$Y_1 \cup \dots \cup Y_n \subseteq \bigcup_{\alpha \in \{a_1, \dots, a_{s_1}, b_1, \dots, b_{s_2}, \dots, n_1, \dots, n_{s_n}\}} V_\alpha$$

so that $Y_1 \cup \dots \cup Y_n$ is indeed compact.

(c) Let be Y a finite subset of X; we have to show that Y is compact.

First, suppose that Y is a singleton $\{a\}$. By definition, any sequence of elements of Y can only be the constant sequence $a, a, a, \dots$. Thus, any subsequence of this sequence is still the constant sequence $a, a, \dots$, and still converges to a. Thus, any sequence of elements of Y has a subsequence that converges in Y, i.e., Y is compact.

Now suppose that Y is a finite subset of cardinality n. Let's write $Y := \{y_1, \dots, y_n\}$. This can also be written $Y := \{y_1\} \cup \dots \cup \{y_n\}$, so that we are back in the previous case (b): Y is the finite union of compact subsets of X. Thus, Y is itself compact.

Note that for the limit case $Y = \varnothing$, we can say that the empty set is just a closed[3] subset of the compact set $\{a\}$, so that by the previous case (a), $Y = \varnothing$ is compact.

Exercise 12.5.8. — *Let (X, d_{l^1}) be the metric space from Exercise 12.1.15. For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^\infty$ be the sequence in X such that $e_j^{(n)} := 1$ when $n = j$ and $e_j^{(n)} := 0$ when $n \neq j$. Show that the set $\{e^{(n)} : n \in \mathbb{N}\}$ is a closed and bounded subset of X, but is not compact.*

Recall that (X, d_{l^1}) is the metric space of absolutely convergent sequences, with the metric defined by $d_{l^1}((a^{(n)}), (b^{(n)})) := \sum_{n=0}^\infty |a_n - b_n|$. Hereafter, we denote $E := \{e^{(n)} : n \in \mathbb{N}\}$, with

$$e^{(0)} := 1, 0, 0, 0, \dots$$
$$e^{(1)} := 0, 1, 0, 0, \dots$$
$$e^{(2)} := 0, 0, 1, 0, \dots$$
$$\dots$$

[3] See Remark 12.2.14.

- First, we show that E is not compact. To prove this statement, we just have to find one sequence of elements of E that has no convergent subsequence in E.

  Consider the "canonical" sequence of elements of E defined by $e^{(0)}, e^{(1)}, e^{(2)}, \ldots$. The distance between any two distinct elements of this sequence is

$$d_{l^{(1)}}(e^{(j)}, e^{(k)}) := \sum_{i=0}^{\infty} |e_i^{(j)} - e_i^{(k)}| = 2 > 1.$$

  Thus, this sequence is not a Cauchy sequence itself, and it is clear that no subsequence can be a Cauchy sequence either. Thus, no subsequence of this sequence can converge in E, i.e., E is not compact.

- However, E is a closed subset of X. To prove this property, consider a convergent sequence of elements of E; we have to prove that its limit lies in E. We've just shown that the distance between any two distinct terms $e^{(j)}, e^{(k)}$ for $j \neq k$ is equal to 2. Thus, if a sequence of elements of E converges, it must be eventually 0.5-stable, and the only possibility for that is to be eventually constant. In other words, it must be eventually equal to $e^{(n_0)}$ for $n_0 \in \mathbb{N}$, so that it necessarily converges to $e^{(n_0)}$, which is an element of E. This shows that E is closed.

- Furthermore, E is bounded. To show the boundedness of E, we have to show that $E \subseteq B_{(X,d_{l^1})}((x_j)_{j=0}^{\infty}, r)$ for some $r > 0$ and some sequence $(x_j)_{j=0}^{\infty} \in X$. Consider the zero sequence $(z_j)_{j=0}^{\infty} := 0, 0, 0, \ldots$. This is clearly a sequence in X (since it converges to 0), and we have

$$d_{l^1}\left((z_j)_{j=0}^{\infty}, (e_j^{(n)})_{j=0}^{\infty}\right) = \sum_{j=0}^{\infty} |z_j - e_j^{(n)}| = 1 < 2$$

  for all $n \in \mathbb{N}$. Thus, we have $E \subseteq B_{(X,d_{l^1})}((z_j)_{j=0}^{\infty}, 2)$, which shows that E is bounded.

Thus, the case of the subset E of the metric space (X, d_{l^1}) shows that the Heine-Borel theorem (stated for the metric space $(\mathbb{R}^n, d)$) is not valid in more general metric spaces.

Exercise 12.5.9. — *Show that a metric space (X, d) is compact if and only if every sequence in X has at least one limit point.*

A metric space (X, d) is compact iff any sequence of elements of X has a subsequence that converges in (X, d). Thus, the statement is a direct consequence of Proposition 12.4.5, which says basically that "having a convergent subsequence" and "having a limit point" are synonymous.

Exercise 12.5.13. — *Let E and F be two compact subsets of $\mathbb{R}$ (with the standard metric $d(x, y) = |x - y|$). Show that the Cartesian product $E \times F := \{(x, y) : x \in E, y \in F\}$ is a compact subset of $\mathbb{R}^2$ (with the Euclidean metric d_{l^2}).*

To prove that $E \times F$ is compact, we will show that it is both closed and bounded (by Heine-Borel theorem).

- First we show that $E \times F$ is bounded.

  Since E and F are compact, they are themselves bounded (by Heine-Borel theorem). Thus, there exist $a \in E$, $b \in F$ and $r_1, r_2 > 0$ such that $E \subseteq B_d(a, r_1)$ and $F \subseteq B_d(b, r_2)$, by Definition 12.5.3. In other words, we have:

  $$\forall x \in E, |x - a| < r_1$$
  $$\forall y \in F, |y - b| < r_2.$$

  Thus, let be $(x, y) \in E \times F$. We have:

  $$\begin{aligned} d_{l^2}\left((x,y),(a,b)\right) &= \sqrt{(x-a)^2 + (y-b)^2} \\ &< \sqrt{r_1^2 + r_2^2}. \end{aligned}$$

  This means that each $(x, y) \in E \times F$ lies in $B_{d_{l^2}}\left((a,b), \sqrt{r_1^2 + r_2^2}\right)$. Thus, $E \times F$ is indeed bounded.

- Now let's show that $E \times F$ is closed.

  Since E and F are compact, they are themselves closed (by Heine-Borel theorem). Consider a sequence $((x^{(n)}, y^{(n)}))_{n=1}^{\infty}$ of elements of $E \times F$ which converges to (x_0, y_0) with respect to d_{l^2}. By Proposition 12.1.18, this means that this sequence converges component-wise, i.e. that $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, and $(y^{(n)})_{n=1}^{\infty}$ converges to y_0. By definition, we have $x_0 \in E$ and $y_0 \in F$, since E and F are closed. Thus, $(x_0, y_0) \in E \times F$. This shows that $E \times F$ is indeed bounded.

Thus, $E \times F$ is compact, as expected.

13. Continuous functions on metric spaces

Exercise 13.1.1. — *Prove Theorem 13.1.4.*

Since the implication $(b) \implies (c)$ may be slightly more difficult to write, we will prove the implications $(a) \implies (c)$, $(c) \implies (b)$ and $(b) \implies (a)$ in this order.

Let be $f : (X, d_X) \to (Y, d_Y)$, and $x_0 \in X$.

- First let's prove $(a) \implies (c)$. Suppose that f is continuous at x_0, and let be $V \subseteq Y$ an open set that contains $f(x_0)$. By Proposition 12.2.15(a), there exists a $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq V$. But since f is continuous at x_0, we know that there exists a $\delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$. Thus, if we set $U := B_X(x_0, \delta)$, we have found an open set $U \subseteq X$ such that $f(U) \subseteq B_Y(f(x_0), \varepsilon) \subseteq V$, as required.

- Now we prove $(c) \implies (b)$. Consider a sequence $(x^{(n)})_{n=1}^{\infty}$ in X which converges to x_0 with respect to d_X. Let be an arbitrary $\varepsilon > 0$; we set $V_\varepsilon := B_Y(f(x_0), \varepsilon)$. By (c), we know that there exists an open set $U \subseteq X$ containing x_0 and such that $f(U) \subseteq V_\varepsilon$. But since U is open set, by Proposition 12.2.15(a), there exists a $\delta > 0$ such that $B_X(x_0, \delta) \subseteq U$.

  Since $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, there exists a natural number $N \geqslant 1$ such that $d_X(x^{(n)}, x_0) < \delta$ whenever $n \geqslant N$. Or, in other words, we have $x^{(n)} \in B_X(x_0, \delta) \subseteq U$ whenever $n \geqslant N$.

  But since $f(U) \subseteq V$ by hypothesis, we thus have $f(x^{(n)}) \in V_\varepsilon$ whenever $n \geqslant N$. Since this is true for any arbitrary $\varepsilon > 0$, this shows that the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to d_Y, as expected.

- Finally, we prove $(b) \implies (a)$. Suppose that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ whenever $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, and let's show that f is continuous at x_0.

  Suppose, for the sake of contradiction, that f is *not* continuous at x_0. Thus, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x \in X$ such that $d_Y(f(x), f(x_0)) \geqslant \varepsilon$ although $d_X(x, x_0) < \delta$.

  Thus, using the (countable) axiom of choice, we build a sequence $(x^{(n)})_{n=1}^{\infty}$ such that, for all $n \geqslant 1$, we have $d_Y(f(x^{(n)}), f(x_0)) \geqslant \varepsilon$ although $d_X(x^{(n)}, x_0) < \frac{1}{n}$. It is thus clear that $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, but that $(f(x^{(n)}))_{n=1}^{\infty}$ does not converge to $f(x_0)$, since $f(x^{(n)})$ and $f(x_0)$ are never $\varepsilon/2$-close. This is a contradiction with (c). Thus, f must be continuous at x_0, as expected.

Exercise 13.1.2. — *Prove Theorem 13.1.5.*

We already know from Theorem 13.1.4 that (a) and (b) are equivalent. Let's prove the other implications.

- First we prove that $(a) \implies (c)$. Let be V an open set in Y. We must show that $f^{-1}(V)$ is an open set in X. Thus, if we take an arbitrary $x_0 \in f^{-1}(V)$, we must show that there exists an $r_0 > 0$ such that $B_X(x_0, r_0) \subseteq f^{-1}(V)$ (cf. Theorem 12.2.15(a)).

  Consider this arbitrary $x_0 \in f^{-1}(V)$. By definition, we have $f(x_0) \in V$. But since V is an open set, there exists an $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq V$.

But f is continuous: for this $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $x \in X$, we have $d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon$. In other words, we have $x \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon) \subseteq V$.

Thus, if we set $r_0 := \delta$, we are done: for all $x \in B_X(x_0, r_0)$, we have $f(x) \in V$, i.e. $x \in f^{-1}(V)$. This shows that $B_X(x_0, \delta) \subseteq f^{-1}(V)$, and thus that $f^{-1}(V)$ is an open set, as expected.

- Now we show that $(c) \implies (d)$. By Theorem 12.2.15(e), we know that $F \subseteq X$ is closed iff $X \backslash F$ is open. Thus, consider $F \subseteq Y$ a closed set in Y. Let be $V := Y \backslash F$ its complementary set, which is thus an open set. By (c), the set $f^{-1}(V)$ is an open set in X. But we have :

$$\begin{aligned} f^{-1}(F) &= \{x \in X : f(x) \in F\} \\ &= \{x \in X : f(x) \in Y \backslash V\} \\ &= \{x \in X : f(x) \notin V\} \end{aligned}$$

  so that $f^{-1}(F) = X \backslash f^{-1}(V)$. Since $f^{-1}(F)$ is the complementary set of the open set $f^{-1}(V)$, it is closed in X, as expected.

- The implication $(d) \implies (c)$ can be shown in exactly the same way as above.

- Finally, let's show that $(c) \implies (a)$. Let be $\varepsilon > 0$, let be $x_0 \in X$. Consider $V := B_Y(f(x_0), \varepsilon)$, which is an open set in Y. By (c), the set $f^{-1}(V)$ is open in X. Thus, by Theorem 12.2.15(a), there exists a $\delta > 0$ such that $B_X(x_0, \delta) \subseteq f^{-1}(V)$. Thus, if $x \in B_X(x_0, \delta)$, we have $f(x) \in V$.

  In other words, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x0)) < \varepsilon$. This shows that f is continuous at x_0, for any arbitrary $x_0 \in X$, as expected.

Exercise 13.1.3. — *Use Theorem 13.1.4 and Theorem 13.1.5 to prove Corollary 13.1.7.*

To show (a), consider $(x^{(n)})_{n=1}^{\infty}$ a sequence of elements of X that converges to $x_0 \in X$. Since f is continuous at x_0, then by Theorem 13.1.4(b), we know that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0) \in Y$. But $(f(x^{(n)}))_{n=1}^{\infty}$ is a sequence of elements of Y. Since g is continuous at $f(x_0)$, then still by Theorem 13.1.4(b), we know that $(g(f(x^{(n)})))_{n=1}^{\infty}$ converges to $g(f(x_0)) \in Z$.

Thus, we have proved that for any sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X that converges to $x_0 \in X$, the sequence $(g \circ f(x^{(n)}))_{n=1}^{\infty}$ converges to $g \circ f(x_0)$. This shows that $g \circ f$ is continuous at x_0, as expected.

Once (a) is proved, the result (b) is clear, since it is just (a) at any arbitrary $x_0 \in X$.

Exercise 13.1.4. — *Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that (a) f is not continuous, but g and $g \circ f$ are continuous; (b) g is not continuous, but f and $g \circ f$ are continuous; (c) f and g are not continuous, but $g \circ f$ is continuous. Explain briefly why these examples do not contradict Corollary 13.1.7.*

Here, the simplest way is to use piecewise constant functions, at least for one of the functions f, g.

(a) Let be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

and the constant function $g(x) := 3$. We thus have $g \circ f(x) = 3$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

(b) Let be, for instance,

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

and $f(x) := x^2 + 1$. We thus have $g \circ f(x) = 1$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

(c) Let be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 3 & \text{if } x \geqslant 0 \end{cases}$$

and

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

We thus have $g \circ f(x) = 1$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

This does not contradict Corollary 13.1.7, since the initial hypothesis of this corollary is that both functions f, g are continuous, and it says nothing about non discontinuous functions.

Exercise 13.1.5. — *Let (X, d) be a metric space, and let $(E, d|_{E \times E})$ be a subspace of (X, d). Let $\iota_{E \to X} : E \to X$ be the inclusion map, defined by setting $\iota_{E \to X}(x) := x$ for all $x \in E$. Show that $\iota_{E \to X}$ is continuous.*

Let be $x_0 \in E$ an arbitrary point in E, and let be $\varepsilon > 0$ a positive real number. Note that we have, for all $x \in E$,

$$d(\iota_{E \to X}(x_0), \iota_{E \to X}(x)) = d_{E \times E}(x_0, x).$$

Thus, if we take $\delta := \varepsilon$ in Definition 13.1.1 of continuity, we are done: if $d_{E \times E}(x_0, x) < \varepsilon$, we automatically have $d(\iota_{E \to X}(x_0), \iota_{E \to X}(x)) < \varepsilon$, so that $\iota_{E \to X}$ is continuous at any arbitrary $x_0 \in E$, as expected.

Exercise 13.1.6. — *Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E \times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus $f|_E(x) := f(x)$ when $x \in E$. If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.) Conclude that if f is continuous, then $f|_E$ is continuous.*

Let's use Exercise 13.1.5. First we note that we have $f|_E = f \circ \iota_{E \to X}$. Indeed, $f \circ \iota_{E \to X}$ is a function from E to Y just like f, and for all $x \in E$, we clearly have $f \circ \iota E \to X(x) = f(x) = f|_E(x)$.

We have shown in Exercise 13.1.5 that $\iota_{E \to X}$ is continuous at any $x_0 \in E$, and f is supposed to be continuous at $x_0 \in E$. Thus, by Corollary 13.1.7, $f|_E$ is continuous at x_0 since it is the composition of two continuous functions. Since this is true for any arbitrary $x_0 \in E$, the function $f|_E$ is continuous on E.

The converse statement is not true: consider the piecewise constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geqslant 0$ for all $x \in \mathbb{R}$; and let be $E := [0, +\infty[$. The restriction $f|_E$ is clearly continuous (as a constant function) at 0, but the function f itself is clearly not continuous at 0.

Exercise 13.2.1. — *Prove Lemma 13.2.1.*

Here we just have to prove the statement (a), since the statement (b) is essentially (a) applied to any arbitrary $x_0 \in X$.

- First suppose that f and g are both continuous at $x_0 \in X$, and let be $(x^{(n)})_{n=1}^{\infty}$ a sequence of elements of X. Then, by Theorem 13.1.4(b), the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ in $\mathbb{R}$, and the sequence $(g(x^{(n)}))_{n=1}^{\infty}$ converges to $g(x_0)$ in $\mathbb{R}$.

  Thus, by Theorem 12.1.18, the sequence $((f(x^{(n)}), g(x^{(n)})))_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$ converges to $(f(x_0), g(x_0))$ in $\mathbb{R}^2$ with respect to the metric d_{l^2} (since it converges componentwise). In other words, for any arbitrary sequence $(x^{(n)})_{n=1}^{\infty}$ that converges to x_0, the sequence $(f \oplus g(x^{(n)}))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0)) = f \oplus g(x_0)$. Thus, $f \oplus g$ is continuous at x_0, by Theorem 13.1.4.

- Conversely, if $f \oplus g$ is continuous at x_0, then for any sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X, the sequence $(f \oplus g(x^{(n)}))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0))$, by Theorem 13.1.4. Thus, by Theorem 12.1.18, $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ in $\mathbb{R}$, and $(g(x^{(n)}))_{n=1}^{\infty}$ converges to $g(x_0)$ in $\mathbb{R}$. Thus, f and g are both continuous at x_0.

Exercise 13.2.2. — *Prove Lemma 13.2.2.*

First we prove that the addition function, $f : (x, y) \mapsto x + y$, defined from $\mathbb{R}^2$ to $\mathbb{R}$, is continuous. Consider a sequence $(x_n, y_n)_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$, that converges to (x_0, y_0) with respect to the metric d_{l^2}. In particular, $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ are both sequences of real numbers, and we know from Proposition 12.1.8 that $(x_n)_{n=1}^{\infty}$ converges to x_0, and $(y_n)_{n=1}^{\infty}$ converges to y_0. But we have $f(x_n, y_n) = x_n + y_n$, and we know by the limit laws (Proposition 6.1.19(a)) that $\lim_{n\to\infty}(x_n + y_n) = x_0 + y_0 =: f(x_0, y_0)$. Thus, for any sequence $((x_n, y_n))_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$ which converges to (x_0, y_0), we have proved that $(f(x_n, y_n))_{n=1}^{\infty}$ converges to $f(x_0, y_0)$. Thus, the addition function f is continuous at any $(x_0, y_0) \in \mathbb{R}^2$, and thus is continuous on $\mathbb{R}^2$.

A similar proof applies for the substraction function, the multiplication function, and the maximum and minimum functions.

An additional precaution is required for the division function, $f : (x, y) \mapsto x/y$ defined from $\mathbb{R} \times \mathbb{R}\backslash\{0\}$ to $\mathbb{R}$. Let be $(x_0, y_0) \in \mathbb{R} \times \mathbb{R}\backslash\{0\}$, and let be $(x_n, y_n)_{n=1}^{\infty}$ a sequence of elements in $\mathbb{R} \times \mathbb{R}\backslash\{0\}$ which converges to (x_0, y_0). In the proof above, we can indeed apply Proposition 6.1.19(f) since $y_n \neq 0$ for all $n \geqslant 1$.

Exercise 13.2.3. — *Show that if $f : X \to \mathbb{R}$ is a continuous function, so is the function $|f| : X \to \mathbb{R}$ defined by $|f|(x) := |f(x)|$.*

The function $|f|$ is the composition of the functions f and $x \mapsto |x|$. We already know (from Proposition 9.4.12) that the function $x \mapsto |x|$ is continuous on $\mathbb{R}$ (basically because

$|x| := \max(x, -x)$, and thus is the composition of two continuous functions). Thus, by Corollary 13.1.7, $|f|$ is continuous on X.

Exercise 13.2.4. — *Let $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_2 : \mathbb{R}^2 \to \mathbb{R}$ be the functions $\pi_1(x, y) := x$ and $\pi_2(x, y) := y$. Show that π_1 and π_2 are continuous. Conclude that if $f : \mathbb{R} \to X$ is any continuous function into a metric space (X, d), then the functions $g_1 : \mathbb{R}^2 \to X$ and $g_2 : \mathbb{R}^2 \to X$ defined by $g_1(x, y) := f(x)$ and $g_2(x, y) := f(y)$ are also continuous.*

Let be a sequence of elements of $\mathbb{R}^2$, that we will note as $((x_n, y_n))_{n=1}^{\infty}$, and suppose that this sequence converges to $(x_0, y_0) \in \mathbb{R}^2$. By Proposition 12.1.18, we know that this sequence converges component-wise, i.e. that the sequence of real numbers $(x_n)_{n=1}^{\infty}$ converges to x_0, and the sequence of real numbers $(y_n)_{n=1}^{\infty}$ converges to y_0. But we have $x_n = \pi_1(x_n, y_n)$, $x_0 = \pi_1(x_0, y_0)$, $y_n = \pi_2(x_n, y_n)$, and $y_0 = \pi_2(x, y_0)$. To summarise, for any sequence $((x_n, y_n))_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$ that converges to (x_0, y_0), the sequence $\pi_1(x_n, y_n)$ converges to $\pi_1(x_0, y_0)$, so that π_1 is continuous. A similar argument shows that π_2 is continuous.

Furthermore, the function g_1 defined above can be expressed as $g_1 := f \circ \pi_1$, and is thus continuous as the composition of two continuous functions (Corollary 13.1.7). Similarly, $g_2 = f \circ \pi_2$ is continuous.

Exercise 13.2.5. — *Let $n, m \geqslant 0$ be integers. Suppose that for every $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$ we have a real number c_{ij}. Form the function $P : \mathbb{R}^2 \to \mathbb{R}$ defined by*

$$P(x, y) := \sum_{i=0}^{n} \sum_{j=0}^{m} c_{ij} x^i y^j.$$

Show that P is continuous. Conclude that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are continuous functions, then the function $P(f, g) : X \to \mathbb{R}$ defined by $P(f, g)(x) := P(f(x), g(x))$ is also continuous.

By Exercise 13.2.4, we know that if $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the function $f \circ \pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is also continuous. It is clear that the function $f : x \mapsto x^i$ is continuous for all integer $i \geqslant 0$ (as a product of i continuous functions: apply Corollary 13.2.3 i times to the identity function $x \mapsto x$). Thus, the function $f_1 = f \circ \pi_1$ defined by $f_1(x, y) = x^i$ is continuous on $\mathbb{R}^2$. Similarly, the function $f_2(x, y) = y^j$ is continuous on $\mathbb{R}^2$ for all $j \geqslant 0$.

By Corollary 13.2.3 another time, the function $(x, y) \mapsto x^i y^j$ is thus continuous (since it is the product of two continuous functions), and still by Corollary 13.2.3, the function $(x, y) \mapsto c_{ij} x^i y^j$ is also continuous, for all real constant c_{ij}. And thus, $P(x, y)$ is also continuous as a sum of continuous functions.

Now consider the function $H := P(f, g)$, such that $H(x) := P(f(x), g(x))$. We have $H = P \circ (f \oplus g)$. But we have just showed that P is continuous, and we already know that $f \oplus g$ is continuous whenever f and g are continuous (Lemma 2.2.1). Thus, H continuous (as the composition of two continuous functions), as expected.

Exercise 13.2.6. — *Let $\mathbb{R}^m$ and $\mathbb{R}^n$ be Euclidean spaces. If $f : X \to \mathbb{R}^m$ and $g : X \to \mathbb{R}^n$ are continuous functions, show that $f \oplus g : X \to \mathbb{R}^{m+n}$ is also continuous, where we have identified $\mathbb{R}^m \times \mathbb{R}^n$ with $\mathbb{R}^{m+n}$ in the obvious manner. Is the converse statement true?*

This exercise generalizes the result of Lemma 13.2.1. The proof will thus be very close to the approach adopted in Exercise 13.2.1.

- Let be an arbitrary $x_0 \in X$, and let's show that $f \oplus g$ is continuous at x_0.

  Let be $(x^{(n)})_{n=1}^{\infty}$ a sequence of elements of X that converges to x_0. We will use below the following notations:

$$\begin{aligned} f(x_0) &:= (x_1, \ldots, x_m) \\ g(x_0) &:= (y_1, \ldots, y_m) \\ f(x^{(k)}) &:= (x_1^{(k)}, \ldots, x_m^{(k)}) \text{ for all } k \geqslant 1 \\ g(x^{(k)}) &:= (y_1^{(k)}, \ldots, y_n^{(k)}) \text{ for all } k \geqslant 1 \end{aligned}$$

  First, note that by Proposition 12.1.18, the sequence $(x^{(n)})_{n=1}^{\infty}$ converges component-wise, i.e., we have $\lim_{k\to\infty} x_p^{(k)} = x_p$ for all $1 \leqslant p \leqslant m$.

  Since f is continuous, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0) \in \mathbb{R}^m$. Similarly for g, the sequence $(g(x^{(n)}))_{n=1}^{\infty}$ converges to $g(x_0) \in \mathbb{R}^n$.

  By definition, we have $f \oplus g(x_0) = (x_1, \ldots, x_m, y_1, \ldots, y_n) \in \mathbb{R}^{m+n}$.

  But we also have, for all $k \geqslant 1$,

$$\begin{aligned} f \oplus g(x^{(k)}) &:= \left(f(x^{(k)}), g(x^{(k)})\right) \\ &= \left(x_1^{(k)}, \ldots, x_m^{(k)}, y_1^{(k)}, \ldots, y_n^{(k)}\right) \end{aligned}$$

  And since we already know that this sequence converges component-wise, we have by Proposition 12.1.18

$$\lim_{k\to\infty} f \oplus g(x^{(k)}) = (x_1, \ldots, x_m, y_1, \ldots, y_m) =: f \oplus g(x_0).$$

  Thus, $f \oplus g$ is continuous at x_0. And since this is true for any arbitrary $x_0 \in X$, $f \oplus g$ is continuous on X.

- The converse statement is also true; this can also be proved in a very similar fashion as Exercise 13.2.1.

Exercise 13.2.7. — *Let $k \geqslant 1$, let I be a finite subset of $\mathbb{N}^k$, and let $c : I \to \mathbb{R}$ be a function. Form the function $P : \mathbb{R}^k \to \mathbb{R}$ defined by*

$$P(x_1, \ldots, x_k) := \sum_{(i_1,\ldots,i_k)\in I} c(i_1, \ldots, i_k) x_1^{i_1} \ldots x_k^{i_k}.$$

Show that P is continuous.

Let's use induction on k.

- First, if $k = 1$, we have $I \subset \mathbb{N}$, and P is simply of the form $P(x) := \sum_{i\in I} c_i x^i$. Such a function is clearly continuous by Lemma 13.2.2, as the product and sum of continuous functions.

- We can also note that, for $k = 2$, the result corresponds exactly to Exercise 13.2.5.

- Now suppose that the property is true for a given positive integer k, and let's show that it is still true for $k+1$.

  Let be I a finite subset of $\mathbb{N}^{k+1}$, and a function $P(x_1, \ldots, x_k, x_{k+1})$ as defined above. Note that (for example by Corollary 3.6.14(e)), since I is supposed to be finite, we have $I = I_1 \times \ldots \times I_k \times I_{k+1}$ where each I_j is also a finite subset of $\mathbb{N}$. In particular, I_{k+1} is finite.

  We thus have:

$$\begin{aligned} P(x_1, \ldots, x_k, x_{k+1}) &:= \sum_{(i_1, \ldots, i_k, i_{k+1}) \in I} c(i_1, \ldots, i_k, i_{k+1})\, x_1^{i_1} \ldots x_k^{i_k} x_{k+1}^{i_{k+1}} \\ &= \sum_{i_{k+1} \in I_{k+1}} \left(\sum_{(i_1, \ldots, i_k) \in I_1 \times \ldots \times I_k} c(i_1, \ldots, i_k, i_{k+1}) x_1^{i_1} \ldots x_k^{i_k} \right) x_{k+1}^{i_{k+1}} \end{aligned}$$

  By the induction hypothesis, the expression enclosed in the parentheses is a continuous function. Thus, $P(x_1, \ldots, x_{k+1})$ is also continuous, as a (finite) sum and product of continuous functions.

Exercise 13.2.8. — *Let (X, d_X) and (Y, d_Y) be metric spaces. Define the metric $d_{X\times Y} : (X \times Y) \times (X \times Y) \to [0, \infty[$ by the formula*

$$d_{X\times Y}\left((x, y), (x', y')\right) := d_X(x, x') + d_Y(y, y').$$

Show that $(X \times Y, d_{X\times Y})$ is a metric space, and deduce an analogue of Proposition 12.1.18 and Lemma 13.2.1.

1. First we prove that $d_{X\times Y}$ is indeed a metric. In all the proofs below, we simply use the fact that d_X and d_Y are themselves metrics.
   - For all $(x, y) \in X \times Y$, we have:
   $$d_{X\times Y}((x, y), (x, y)) = \underbrace{d_X(x, x)}_{=0} + \underbrace{d_Y(y, y)}_{=0} = 0.$$
   - For all distinct points $(x, y), (x', y') \in X \times Y$, we have
   $$d_{X\times Y}\left((x, y), (x', y')\right) = \underbrace{d_X(x, x')}_{>0} + \underbrace{d_Y(y, y')}_{>0} > 0.$$
   - Symmetry: for all $(x, y), (x', y') \in X \times Y$, we have
   $$\begin{aligned} d_{X\times Y}\left((x, y), (x', y')\right) &= d_X(x, x') + d_Y(y, y') \\ &= d_X(x', x) + d_Y(y', y) \\ &= d_{X\times Y}((x', y'), (x, y)). \end{aligned}$$
   - Triangle inequality: for all $(x, y), (x', y'), (x'', y'') \in X \times Y$, we have
   $$\begin{aligned} & d_{X\times Y}\left((x, y), (x', y')\right) + d_{X\times Y}\left((x', y'), (x'', y'')\right) \\ &= d_X(x, x') + d_X(x', x'') + d_Y(y, y') + d_Y(y', y'') \\ &\leqslant d_X(x, x'') + d_Y(y, y'') \\ &\leqslant d_{X\times Y}\left((x, x''), (y, y')\right). \end{aligned}$$

Thus, $d_{X\times Y}$ is indeed a metric on $X \times Y$.

2. Now we give an analogue of Proposition 12.1.18. Note that X and Y are "abstract" metric spaces here, and d_X, d_Y also are "abstract" distances, so that we cannot really give an analogue to the notion of the equivalence of metrics $d_{l^1}, d_{l^2}, d_{l^\infty}$. However, we can give an analogue of the notion of "component-wise" convergence. Indeed, let be $(a^{(n)})_{n=1}^\infty$ a sequence of elements of $X \times Y$ (so that we can write $a^{(n)} := (x^{(n)}, y^{(n)})$ for all $n \geqslant 1$ with obvious notations), and let be (x_0, y_0) a point in $X \times Y$. The following two statements are equivalent:

   (i) $(a^{(n)})_{n=1}^\infty := (x^{(n)}, y^{(n)})_{n=1}^\infty$ converges to (x_0, y_0) with respect to the metric $d_{X\times Y}$

   (ii) $(x^{(n)})_{n=1}^\infty$ converges to x_0 with respect to d_X and $(y^{(n)})_{n=1}^\infty$ converges to y_0 with respect to d_Y.

   Proof. First we show that (i) $\implies$ (ii). If $(x^{(n)}, y^{(n)})_{n=1}^\infty$ converges to (x_0, y_0) with respect to $d_{X\times Y}$, then by definition we have $\lim_{n\to\infty} d_{X\times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) = 0$. Thus, for any arbitrary $\varepsilon > 0$, there exists $N \geqslant 1$ such that for all $n \geqslant N$ we have

   $$\begin{aligned} d_{X\times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) &< \varepsilon \\ d_X(x^{(n)}, x_0) + d_Y(y^{(n)}, y_0) &< \varepsilon. \end{aligned}$$

   In particular, we have both $d_X(x^{(n)}, x_0) < \varepsilon$ and $d_Y(y^{(n)}, y_0) < \varepsilon$ whenever $n \geqslant N$. Thus, we have $\lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0$ and $\lim_{n\to\infty} d_Y(y^{(n)}, y_0) = 0$, which is precisely the statement (ii).

   Now we show that (ii) $\implies$ (i). Let be $\varepsilon > 0$. Since we have both $\lim_{n\to\infty} d_X(x^{(n)}, x_0) = 0$ and $\lim_{n\to\infty} d_Y(y^{(n)}, y_0) = 0$, there exists $N_1, N_2 \geqslant 1$ such that $d_X(x^{(n)}, x_0) < \varepsilon/2$ whenever $n \geqslant N_1$, and $d_Y(y^{(n)}, y_0) < \varepsilon/2$ whenever $n \geqslant N_2$. Thus, for all $n \geqslant \max(N_1, N_2)$, we have

   $$\begin{aligned} d_{X\times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) &= d_X(x^{(n)}, x_0) + d_Y(y^{(n)}, y_0) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

   which means that $\lim_{n\to\infty} d_{X\times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) = 0$. This is the statement (i). □

3. Similarly, we can give an analogue for Lemma 13.2.1:

   Let be S an arbitrary domain for the functions $f : S \to X$ and $g : S \to Y$. Consider the metric spaces (X, d_X), (Y, d_Y) and $(X \times Y, d_{X\times Y})$. Then the function $f \oplus g : S \to X \times Y$ is continuous at $s_0 \in S$ iff both f and g are continuous at s_0.

   Proof. Suppose that both f and g are continuous at $s_0 \in S$. Let be $\varepsilon > 0$, and let be $(s^{(n)})_{n=1}^\infty$ a sequence of elements of S that converges to s_0. By definition, there exists $N_1 \geqslant 1$ such that $d_X(f(s^{(n)}), f(s_0)) < \varepsilon/2$ whenever $n \geqslant N_1$; and there exists $N_2 \geqslant 1$ such that $d_Y(g(s^{(n)}), g(s_0)) < \varepsilon/2$ whenever $n \geqslant N_2$. Thus, for $n \geqslant N := \max(N_1, N_2)$,

   $$d_{X\times Y}(f \oplus g(s^{(n)}), f \oplus g(s_0)) := \underbrace{d_X(f(s^{(n)}), f(s_0))}_{<\varepsilon/2} + \underbrace{d_Y(g(s^{(n)}), g(s_0))}_{<\varepsilon/2} < \varepsilon$$

which means that $f \oplus g$ is continuous at s_0, as expected.

The converse statement can be proved similarly (see also the proof of 2. above). $\square$

Exercise 13.2.10. — *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Show that for each $x \in \mathbb{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbb{R}$, and for each $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbb{R}$. Thus a function $f(x, y)$ which is jointly continuous in (x, y) is also continuous in each variable x, y separately.*

Let be $x_0 \in \mathbb{R}$ a real number. We will just prove the result for the function $g : y \mapsto f(x_0, y)$, since the result for the other function can be shown in the same way. We will work below with the metric space $(\mathbb{R}^2, d_{l^2})$ but the proof can easily be adapted for the metrics d_{l^1} or d_{l^∞}.

Let be $y_0 \in \mathbb{R}$ an arbitrary real number: we have to prove that g is continuous at y_0. Let be $\varepsilon > 0$ an arbitrary positive real number. Since f is continuous on $\mathbb{R}^2$, it is continuous in particular at $(x_0, y_0) \in \mathbb{R}^2$. Thus, there exists a $\delta > 0$ such that $|f(x, y) - f(x_0, y_0)| < \varepsilon$ whenever $d((x, y), (x_0, y_0)) < \delta$.

If we set $x := x_0$ and choose any $y \in \mathbb{R}$ such that $|y - y_0| < \delta$, we will have $d((x_0, y), (x_0, y_0)) = |y - y_0| < \delta$. Thus, we will have $\varepsilon > |f(x_0, y) - f(x_0, y_0)| = |g(y) - g(y_0)|$. This means that g is continuous at y_0, as expected.

Exercise 13.2.11. — *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) := \frac{xy}{x^2+y^2}$ when $(x, y) \neq (0, 0)$, and $f(x, y) = 0$ otherwise. Show that for each fixed $x \in \mathbb{R}$, the function $y \mapsto f(x, y)$ is continuous on $\mathbb{R}$, and that for each fixed $y \in \mathbb{R}$, the function $x \mapsto f(x, y)$ is continuous on $\mathbb{R}$, but that the function $f : \mathbb{R} \mapsto \mathbb{R}$ is not continuous on $\mathbb{R}$. This shows that the converse to Exercise 13.2.10 fails; it is possible to be continuous in each variable separately without being jointly continuous.*

First, let be $x_0 \in \mathbb{R}$ an arbitrary real number, and let's consider the function $g : \mathbb{R} \to \mathbb{R}$ such that $g(y) = f(x_0, y)$ for all $y \in \mathbb{R}$. We have two cases:

- If $x_0 = 0$, then g is the constant null function, i.e. $g(y) = 0$ for all $y \in \mathbb{R}$. Thus (for example by Exercise 9.4.2), g is continuous on $\mathbb{R}$.
- If $x_0 \neq 0$, then $x_0^2 + y^2 \neq 0$ for all $y \in \mathbb{R}$. Thus, $g(y) = \frac{xy}{x^2+y^2}$ is the ratio between the continuous function $y \mapsto xy$ and the continuous, positive function $y \mapsto x_0^2 + y^2$ (these functions can be shown to be continuous by applying Lemma 13.2.2 several times). Thus, by Corollary 13.2.3, g is continuous on $\mathbb{R}$.

Of course, the same demonstration shows that the function $h : x \mapsto f(x, y)$ is also continuous on $\mathbb{R}$.

Now let's show that f is not jointly continuous on $\mathbb{R}$, and more precisely that it is not continuous at the point $(0, 0)$. By Theorem 13.1.5, to show that a function f is not continuous, it is enough to find a sequence $(x^{(n)})_{n=1}^\infty$ that converges to x_0, but such that $(f(x^{(n)}))_{n=1}^\infty$ does not converge to $f(x_0)$. Consider the sequence $(1/n, 1/n)_{n=1}^\infty$ of elements of $\mathbb{R}^2$. It is clear that this sequence converges to $(0, 0)$. However, we have $f(1/n, 1/n) = \frac{1/n^2}{2/n^2} = \frac{1}{2}$. Thus, $(f(1/n, 1/n))_{n=1}^\infty$ does not converge to $f(0, 0) = 0$, which means that f is not continuous at $(0, 0)$.

Exercise 13.2.12. — *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x, y) := x^2/y$ when $y \neq 0$, and $f(x, y) := 0$ when $y = 0$. Show that $\lim_{t \to 0} f(tx, ty) = f(0, 0)$ for every $(x, y) \in \mathbb{R}^2$, but* *that f is not continuous at the origin. Thus being continuous on every line through the origin is not enough to guarantee continuity at the origin!*

By definition, we first note that we have $f(0,0) = 0$. Furthermore, for all $t \in \mathbb{R}$, we have $f(tx, ty) = \frac{t^2x^2}{ty} = \frac{tx^2}{y}$. Thus, considering the function $t \mapsto \frac{tx^2}{y}$ with x, y fixed real numbers, we have immediately $\lim_{t\to 0} f(tx, ty) = \lim_{t\to 0} \frac{tx^2}{y} = 0 = f(0,0)$, as expected.

However, f is not continuous at $(0,0)$. Indeed, consider the sequence $(1/n, 1/n^2)_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$. This sequence converges (component-wise, and thus globally) to $(0,0)$ when $n \to \infty$. However, we have $f(1/n, 1/n^2) = \frac{1/n^2}{1/n^2} = 1$ for all $n \in \mathbb{N}$, so that it is clear that $(f(1/n, 1/n^2))_{n=1}^{\infty}$ does not converge to $f(0,0) = 0$. Thus, f is not continuous at the origin.

Exercise 13.3.1. — *Prove Theorem 13.3.1.*

We have to prove that continuous maps preserve compactness. Thus, let's consider a function $f : X \to Y$ from one metric space (X, d_X) to another (Y, d_Y), and let be $K \subseteq X$ a compact subset of X. We have to prove that the image $f(K)$ is also compact.

To prove that $f(K)$ is compact, let's use Definition 12.5.1. Consider a sequence $(y^{(n)})_{n=1}^{\infty}$ of elements of $f(K)$, and let's show that there exists a convergent subsequence.

By definition, for all positive integer $n \geqslant 1$, there exists an element $x^{(n)} \in K$ such that $y^{(n)} = f(x^{(n)})$. For example by using the countable axiom of choice, we thus get a sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of K. Since K is compact by hypothesis, there exists a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ that converges in K to some $x_0 \in K$ (relative to d_X). Since f is continuous on X, it is in particular continuous at x_0; and using Theorem 13.1.4 (i.e., sequential characterization of continuity), we immediately can say that the subsequence $(y^{(n_k)})_{k=1}^{\infty} = (f(x^{(n_k)}))_{k=1}^{\infty}$ converges to $f(x_0)$ in $f(K)$.

This shows that $f(K)$ is compact, as expected.

Exercise 13.3.2. — *Prove Proposition 13.3.2.*

We have to prove the maximum principle in the general setting of metric spaces. Let be $f : X \to \mathbb{R}$ a continuous function on a compact metric space (X, d).

- First we prove that f is bounded. We can simply modify the proof of Lemma 9.6.3. Suppose, for the sake of contradiction, that f is not bounded. Thus, for all $n \geqslant 1$, the set $X_n := \{x \in X : |f(x)| > n\}$ is non-empty, and contains at least one element $x^{(n)}$. Thus, using the countable axiom of choice, we can form a sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X such that $x^{(n)} \in X_n$ for all $n \geqslant 1$. But since X is compact, the sequence $(x^{(n)})_{n=1}^{\infty}$ has a convergent subsequence, say $(x^{(n_k)})_{k=1}^{\infty}$, which converges to L in X. Since f is continuous, the sequence $(f(x^{(n_k)}))_{k=1}^{\infty}$ converges to $f(L)$. Since this sequence is convergent, it is necessarily bounded. But this is a contradiction with the fact that we have $|f(x^{(n_k)})| > n_k \geqslant k$ for all $k \geqslant 1$. Thus, f must be bounded.

- Now let's prove that f attains its maximum on X, i.e. that there exists a point $x_{max} \in X$ such that $f(x) \leqslant f(x_{max})$ for all $x \in X$. We have shown that f is bounded on X, thus there exists an $M \in \mathbb{R}$ such that $M := \sup_{x\in X} f(x)$. By definition of a supremum, for all $n \geqslant 1$, the set $G_n := \{x \in X : M - 1/n < f(x) \leqslant M\}$ is non-empty and contains at least one element $x^{(n)}$. Thus, using the countable axiom of choice, we can form a sequence $(x^{(n)})_{n=1}^{\infty}$ such that $x^{(n)} \in G_n$ for all $n \geqslant 1$. Because X is compact, there exists a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ that converges to some point $x_{max} \in X$. Since f is continuous, the sequence $(f(x^{(n_k)}))_{k=1}^{\infty}$ also converges (to $f(x_{max})$).

By construction, we have $f(x) \leqslant M$ for all $x \in X$, and thus we have in particular $f(x_{max}) \leqslant M$.

On the other hand, we have $M - 1/k \leqslant M - 1/n_k < f(x^{(n_k)})$ for all $k \geqslant 1$, so that taking the limit as $k \to \infty$ on both sides leads to $f(x_{max}) \geqslant M$.

Thus, there indeed exists a $x_{max} \in X$ such that $f(x_{max}) = M = \sup_{x \in X} f(x)$, as required.

- A similar proof would show that f also attains its minimum on X.

Exercise 13.3.3. — *Show that every uniformly continuous function is continuous, but give an example that shows that not every continuous function is uniformly continuous.*

Below, we consider two metric spaces (X, d_X) and (Y, d_Y).

- Let be $f : X \to Y$ a function that is uniformly continuous on X. Let be $\varepsilon > 0$. By definition, there exists a $\delta > 0$ such that for all $x, x' \in X$, we have $d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$.

  Consider an arbitrary $x' \in X$ and let's show that f is continuous at X. For this same $\varepsilon > 0$, we already have a $\delta > 0$ such that $d_Y(f(x), f(x')) < \varepsilon$ whenever $d_X(x, x') < \delta$. Thus, f is continuous at x'.

- Now we give an example of function $f : X \to Y$ that is continuous but not uniformly continuous. A simple example would be to consider $X = [0, +\infty[$ and $Y = [0, +\infty[$, with $f(x) = x^2$. Consider $\varepsilon := 1$. We have to show that, for all $\delta > 0$, there exist $x, y \in \mathbb{R}$ such that $|x^2 - y^2| \geqslant 1$ although $|x - y| < \delta$. If we consider an arbitrary $\delta > 0$ and set $y := x + \delta/2$, we clearly have $|x - y| < \delta$. However, the inequality $|x^2 - (x + \delta/2)^2| \geqslant 1$ always has a solution, in $[0, \infty[$; more precisely, any $x \geqslant 1/\delta - \delta/4$ will do the trick. Thus, f is not uniformly continuous.

Exercise 13.3.4. — *Let $(X, d_X), (Y, d_Y), (Z, d_Z)$ be metric spaces, and let $f : X \to Y$ and $g : Y \to Z$ be two uniformly continuous functions. Show that $g \circ f : X \to Z$ is also uniformly continuous.*

Let be $\varepsilon > 0$ an arbitrary real number.

- Since g is uniformly continuous, there exists a $\delta > 0$ such that, for all $y, y' \in Y$, we have $d_Y(y, y') < \delta \implies d_Z(g(y), g(y')) < \varepsilon$.

- For this $\delta > 0$, since f is uniformly continuous, there exists a $\delta' > 0$ such that, for all $x, x' \in X$, we have $d_X(x, x') < \delta' \implies d_Y(f(x), f(x')) < \delta$.

- Combining the two previous results, there exists a $\delta' > 0$ such that, for all $x, x' \in X$,

$$d_X(x, x') < \delta' \implies d_Y(f(x), f(x')) < \delta \implies d_Z(g \circ f(x), g \circ f(x')) < \varepsilon.$$

  This shows that $g \circ f$ is uniformly continuous, as expected.

Exercise 13.3.5. — *Let (X, d_X) be a metric space, and let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be uniformly continuous functions. Show that the direct sum $f \oplus g : X \to \mathbb{R}^2$ defined by $f \oplus g(x) := (f(x), g(x))$ is uniformly continuous.*

First note that the exercise does not specify which metric (d_{l^1}, d_{l^2} or d_{l^∞}) should be considered on $\mathbb{R}^2$. However, we will see that one choice will come naturally, and that the two others cases can be deduced immediately from that choice.

Let be $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta_1 > 0$ such that, for all $x, x' \in X$, we have $d_X(x, x') < \delta_1 \implies |f(x) - f(x')| < \varepsilon/2$.

Similarly, since g is uniformly continuous, there exists a $\delta_2 > 0$ such that, for all $x, x' \in X$, we have $d_X(x, x') < \delta_2 \implies |g(x) - g(x')| < \varepsilon/2$.

Thus, if we set $\delta := \min(\delta_1, \delta_2)$, then for all $x, x' \in X$ we have

$$d_X(x, x') < \delta \implies |f(x) - f(x')| + |g(x) - g(x')| < \varepsilon/2 + \varepsilon/2$$

i.e.,

$$d_{l^1}\left((f(x), g(x)), (f(x'), g(x'))\right) < \varepsilon$$

or finally

$$d_{l^1}\left(f \oplus g(x), f \oplus g(x')\right) < \varepsilon. \tag{13.1}$$

This means that $f \oplus g$ is indeed uniformly continuous if we consider the metric space $(\mathbb{R}^2, d_{l^1})$. However, remember from Exercise 12.1.8 and 12.1.10 that we have $d_{l^\infty}(x, y) \leqslant d_{l^2}(x, y) \leqslant d_{l^1}(x, y)$ for all $x, y \in \mathbb{R}^2$. Thus, the inequality (13.1) remains true if we consider the metric spaces $(\mathbb{R}^2, d_{l^2})$ or $(\mathbb{R}^2, d_{l^\infty})$ instead. This shows that $f \oplus g$ is indeed uniformly continuous in all cases.

Exercise 13.3.6. — *Show that the addition function $(x, y) \mapsto x + y$ and the subtraction function $(x, y) \mapsto x - y$ are uniformly continuous from $\mathbb{R}^2$ to $\mathbb{R}$, but the multiplication function $(x, y) \mapsto xy$ is not. Conclude that if $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ are uniformly continuous functions on a metric space (X, d), then $f + g : X \to \mathbb{R}$ and $f - g : X \to \mathbb{R}$ are also uniformly continuous. Give an example to show that $fg : X \to \mathbb{R}$ need not be uniformly continuous. What is the situation for* $\max(f, g)$, $\min(f, g)$, *f/g, and cf for a real number c?*

Because it is by far the most common case, we will consider below that we work on the metric space $(\mathbb{R}, d_{l^2})$. Similar proofs would show the results for other choices of metrics on $\mathbb{R}^2$.

- First let's show that $f : (x, y) \mapsto x + y$ is uniformly continuous. Let be $\varepsilon > 0$. We have to show that there exists a $\delta > 0$ such that, for all $(x, y), (x', y') \in \mathbb{R}^2$, we have $d_{l^2}((x, y), (x', y')) < \delta \implies |f(x, y) - f(x', y')| < \varepsilon$, i.e.

  $$\sqrt{(x - x')^2 + (y - y')^2} < \delta \implies |(x + y) - (x' + y')| < \varepsilon.$$

  First note that we always have $|x - x'| \leqslant \sqrt{(x - x')^2 + (y - y')^2}$, and $|y - y'| \leqslant \sqrt{(y - y')^2 + (y - y')^2}$. Consequently, if we set $\delta := \varepsilon/3$, and if we suppose that $\sqrt{(x - x')^2 + (y - y')^2} < \delta$, we will have $|x - x'| \leqslant \varepsilon/3$ and $|y - y'| \leqslant \varepsilon/3$. Thus, we will also have, by triangle inequality, $|(x + y) - (x' + y')| \leqslant |x + y| + |x' + y'| \leqslant \varepsilon/3 + \varepsilon/3 < \varepsilon$, as expected. Thus, the addition function is indeed uniformly continuous.

- Only a very small modification in the previous proof is needed to handle the case of the substraction function, so that we will not give the detailed proof here.

- However, the multiplication function $f : (x, y) \mapsto xy$ is not uniformly continuous. Indeed, let's take $\varepsilon := 1$. We have to show that for all $\delta > 0$, we will always be able to find $(x, y, (x', y') \in \mathbb{R}^2$ such that $|xy - x'y'| \geqslant 1$ although $d_{l^2}((x, y), (x', y')) < \delta$.

  Consider an arbitrary $\delta > 0$, and an arbitrary $(x, y) \in \mathbb{R}^2$. If we set $(x', y') := (x+\delta/2, y)$, we clearly have $d_{l^2}((x, y), (x', y')) < \delta$. On the other hand, we have

$$\begin{aligned} |xy - x'y'| &= |xy - (x + \delta/2)y| \\ &= |xy - xy - y\delta/2| \\ &= \frac{\delta}{2}|y| \end{aligned}$$

  Thus, we just have to choose $y \geqslant \frac{2}{\delta}$ to get $|xy - x'y'| \geqslant 1$, although $d_{l^2}((x, y), (x', y')) < \delta$. This shows that f is not uniformly continuous.

- Now consider $f, g : X \to \mathbb{R}$ two uniformly continuous functions. We can note that $f + g : X \to \mathbb{R}$ is simply the composition of the function $f \oplus g$ with the addition function. But we have just shown that the addition function is uniformly continuous, and we have shown in Exercise 13.3.5 that $f \oplus g$ is also uniformly continuous. Thus, $f + g$ is uniformly continuous, by Exercise 13.3.4. The same argument applies to $f - g$.

- On the other hand, if $f, g : X \to \mathbb{R}$ are two uniformly continuous functions, the function fg is not necessarily continuous. Indeed, we have a "free" example provided by Exercise 13.3.3: if we take $f(x) = g(x) = x$, we have $(fg)(x) = x^2$ and we have already shown that this function is not uniformly continuous.

- If $f, g : X \to \mathbb{R}$ are two uniformly continuous functions, the function $\max(f, g)$ is uniformly continuous, since it is simply the composition of $f \oplus g$ and of the function $(x, y) \mapsto \max(x, y)$. The first one is continuous by Exercise 13.3.5, the second one is continuous by Proposition 13.3.2. The same argument applies to $\min(f, g)$.

- Finally, the function f/g is not necessarily continuous: we can simply consider the case of $f(x) = 1$ and $g(x) = x$, which are clearly uniformly continuous, whereas $(f/g)(x) = 1/x$ is not (see for instance Example 9.9.1).

Exercise 13.4.1. — *Let (X, d_{disc}) be a metric space with the discrete metric. Let E be a subset of X which contains at least two elements. Show that E is disconnected.*

By definition, there exists at least two elements $x, y \in E$. Thus, we have $\{x\} \subseteq E$, and the set $X \backslash \{x\}$ contains at least the element y, and is thus non empty. In other words, $\{x\}$ is a proper subset of X, i.e. $\{x\} \subsetneq X$.

By Remark 12.2.14, we know that for the discrete metric d_{disc}, every set is both open and closed. Thus, the set $\{x\}$ is a proper subset of X that is non-empty, open, and closed. This shows that (X, d_{disc}) is connected (Definition 13.4.1).

Exercise 13.4.2. — *Let $f : X \to Y$ be a function from a connected metric space (X, d) to a metric space (Y, d_{disc}) with the discrete metric. Show that f is continuous if and only if it is constant. (Hint: use Exercise 13.4.1.)*

(Remark: this exercise is much easier to solve if we assume that we already know Theorem 13.4.6 and/or Corollary 13.4.7. We will proceed accordingly.)

- First, it is clear that if f is constant, then it is continuous (since $d_{\text{disc}}(f(x_0), f(x)) = 0$, and is thus inferior to any $\varepsilon > 0$, no matter how close are $x, x_0 \in X$).
- Furthermore, if f is continuous, then it is constant. Indeed, X is connected, thus $f(X)$ must be connected (by Theorem 13.4.6). By Exercise 13.4.1, $f(X)$ can only have one element, i.e., f is constant.

Exercise 13.4.3. — *Prove the equivalence of statements (b) and (c) in Theorem 13.4.5.*

First we prove that (b) $\implies$ (c). X is supposed to be a non-empty set of the real line, and thus has a supremum $b \in \overline{\mathbb{R}}$ and an infimum $a \in \overline{\mathbb{R}}$ (by Definition 5.5.10). Consider a real number $z \in]a, b[$. Because $z < b$, then by definition of the supremum of X, there exists a real number $y \in X$ such that $z < y \leqslant b$. Similarly, because $a < z$, there must exist a $x \in X$ such that $a \leqslant x < z$. Thus, there exist $x, y \in X$ such that $a \leqslant x < z < y \leqslant b$. By property (b), this implies that $z \in X$. Since this applies to any $z \in [a, b]$, we actually have $]a, b[\subseteq X$. Thus, X can be either $]a, b[$, $[a, b[$, $]a, b]$ or $[a, b]$, but in each case it is an interval in the sense of Definition 9.1.1.

Now let's prove that (c) $\implies$ (b). Suppose that X is an interval in the sense of Definition 9.1.1, for instance that $X = [a, b]$. Then it is clear that for all $a \leqslant x < y \leqslant b$ we have $[x, y] \subseteq X$. This is just as clear for any other type of interval listed in Definition 9.1.1.

Exercise 13.4.4. — *Prove Theorem 13.4.6. (Hint: the formulation of continuity in Theorem 13.1.5(c) is the most convenient to use.)*

Given that connectedness is a "negative" notion (i.e., not being separable into two disjoint open sets), it may seem more natural to give a proof by contradiction. Thus, suppose that $f(E)$ is *not* connected. In this case, we can split $f(E)$ into two disjoint open (non-empty) sets V and W, i.e. we have $f(E) = V \sqcup W$.

- Since f is continuous on X, it is in particular continuous on E. Thus, by Theorem 13.1.5(c), $f^{-1}(V)$ and $f^{-1}(W)$ are open in X, and the sets $A := E \cap f^{-1}(V)$ and $B := E \cap f^{-1}(W)$ are open in E[4].
- Of course, since V and W are non-empty, A and B are also non-empty.
- Furthermore, A and B are disjoint. Indeed, suppose that we have $x \in A \cap B$. Thus we would have $f(x) \in V \cap W$, which is a contradiction with the fact that V and W are disjoint.
- Finally, we have $E = A \sqcup B$. Indeed, the inclusion $A \sqcup B \subseteq E$ is clear (since A, B are subsets of E by definition). Conversely, the inclusion $E \subseteq A \sqcup B$ can be proved easily:

[4] Note that we use here a relative topology in the metric space $(E, d|_{E \times E})$.

if $x \in E$, then $f(x) \in f(E)$, i.e. $f(x) \in V \sqcup W$. If $f(x) \in V$, then $x \in A$; and if $x \in W$, then $x \in B$. In both cases, $x \in A \sqcup B$, which proves the second inclusion. Thus, we have $E = A \sqcup B$.

Thus, the metric space $(E, d|_{E\times E})$ is not connected, since E can be split into two open disjoint and non-empty subsets. This is a contradiction with our initial hypothesis, as expected; and $f(E)$ must be connected.

(Note that for this exercise, we could have re-used some statements proved earlier, for example in Exercises 3.4.2 or 3.4.4, to make the proof a bit shorter.)

Exercise 13.4.5. — *Use Theorem 13.4.6 to prove Corollary 13.4.7.*

Let be $f : X \to \mathbb{R}$ a continuous map, let be $E \subseteq X$ a connected set, and let be $a, b \in E$. Suppose that we have $f(a) \leqslant f(b)$ (the proof can easily be adapted if $f(b) > f(a)$), and let be $f(a) \leqslant y \leqslant f(b)$.

By hypothesis, E is a connected subset of X, so that $f(E)$ must also be connected, by Theorem 13.4.6. And since $a, b \in E$, we clearly have $f(a), f(b) \in f(E)$, by definition. Recall that we suppose here $f(a) \leqslant f(b)$, and that $f(E) \subseteq \mathbb{R}$ by definition.

- (Trivial case) If $f(a) = f(b)$, then we necessarily have $f(a) = y = f(b)$, so that we just have to pick $c := a$ to exhibit a number $c \in E$ such that $f(c) = y$.
- (General case) If $f(a) < f(b)$, then by Theorem 13.4.5, since $f(E)$ is a connected subset of $\mathbb{R}$, the interval $[f(a), f(b)]$ is contained in $f(E)$. Thus, for any $y \in [f(a), f(b)]$, we have $y \in f(E)$, and there exists a $c \in E$ such that $f(c) = y$, as expected.

Exercise 13.4.6. — *Let (X, d) be a metric space, and let $(E_\alpha)_{\alpha\in I}$ be a collection of connected sets in X (we suppose that I is non-empty). Suppose also that $\bigcap_{\alpha\in I} E_\alpha$ is non-empty. Show that $\bigcup_{\alpha\in I} E_\alpha$ is connected.*

As in Exercise 13.4.4, it may seem more natural to give a proof by contradiction to prove connectedness. Thus, let's suppose for the sake of contradiction that $\bigcup_{\alpha\in I} E_\alpha$ is disconnected. It means that there exists two open, disjoint and non empty sets V and W in X such that $\bigcup_{\alpha\in I} E_\alpha = V \sqcup W$.

Let's take an arbitrary E_α. We clearly have $E_\alpha \subseteq V \sqcup W$. But since E_α is connected, we must have either $E_\alpha \subseteq V$, or $E_\alpha \subseteq W$. Indeed, if it were not the case, E_α could be split into two open (in E_α), disjoint and non-empty sets (namely $V \cap E_\alpha$ and $W \cap E_\alpha$), which is a contradiction. Thus, for all $\alpha \in I$, we have either $E_\alpha \subseteq V$, or $E_\alpha \subseteq W$.

Furthermore, there exists at least one $\alpha_1 \in I$ such that $E_{\alpha_1} \subseteq V$ (otherwise V would be empty, which is excluded); and similarly there exists at least one $\alpha_2 \in I$ such that $E_{\alpha_2} \subseteq W$. Thus, $E_{\alpha_1} \cap E_{\alpha_2} = \varnothing$, which implies that $\bigcap_{\alpha\in I} E_\alpha$ is empty. This contradicts out initial hypothesis.

Thus, $\bigcup_{\alpha\in I} E_\alpha$ is necessarily connected.

Exercise 13.4.7. — *Let (X, d) be a metric space, and let E be a subset of X. We say that E is path-connected iff, for every $x, y \in E$, there exists a continuous function $\gamma : [0, 1] \to E$ from the unit interval $[0, 1]$ to E such that $\gamma(0) = x$ and $\gamma(1) = y$. Show that every non-empty path-connected set is connected.*

As for the previous exercises, we will use a proof by contradiction here. Thus, let be E a path-connected set, and suppose for the sake of contradiction that E is not connected. Thus, there exist two open (in E), disjoint and non empty sets $V, W \subseteq X$ such that $E = V \sqcup W$.

Consider two points, $x \in V$ and $y \in W$. Since E is path connected, there exists a continuous map $\gamma : [0,1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$. We know (for instance by Theorem 13.4.5) that the interval $[0,1]$ is a connected set. Thus, the image set $S := \gamma([0,1])$ is also a connected set, since γ is continuous (Theorem 13.4.6).

But since V is open in E, then $V \cap S$ is open in S (Proposition 12.3.4). Similarly, $W \cap S$ is open in S. These sets are non-empty since they contain x and y respectively; and they are clearly disjoint since V and W are disjoint. Thus, we have $S = (S \cap V) \sqcup (S \cap W)$, so that S is disconnected. This is a contradiction.

Thus, E must be a connected set.

Exercise 13.4.8. — *Let (X, d) be a metric space, and let E be a subset of X. Show that if E is connected, then the closure $\overline{E}$ of E is also connected. Is the converse true?*

Suppose, for the sake of contradiction, that E is connected but $\overline{E}$ is not connected. In this case, there exist two open (in $\overline{E}$), disjoint and non-empty sets V and W such that $\overline{E} = V \sqcup W$. Intuitively, we will show that this partition of $\overline{E}$ will provide a partition of E into two open, non-empty sets, which will be a contradiction with the connectedness of E. Thus, let be $V' := V \cap E$ and $W' := W \cap E$.

- These sets V', W' are open in E, by Proposition 12.3.4.
- These sets are also non-empty[5]. Indeed, let be $x \in V$. Since $V \subset \overline{E}$, x is adherent to E. Thus, x belongs either to E, or to ∂E. If $x \in E$, then clearly, $x \in V' := V \cap E$. If $x \in \partial E$, remember that V is open in $\overline{E}$, so that there exists an open ball $B_{\overline{E}}(x, \varepsilon) \subseteq V$. But this open ball has a non-empty intersection with E, because $x \in \overline{E}$. Thus, in both cases, V' is non-empty. The same argument applies to W and W'.
- Furthermore, since $E \subseteq \overline{E}$, we have[6]

$$\begin{aligned} V' \sqcup W' &= (V \cap E) \sqcup (W \cap E) \\ &= (V \sqcup W) \cap E \\ &= \overline{E} \cap E \\ &= E. \end{aligned}$$

Thus, E is disconnected, which is a contradiction.

However, the converse statement is not true. Indeed, consider the set $E :=]-1, 0[\sqcup]0, 1[$ in $\mathbb{R}$. This set is clearly disconnected, since it is natively the disjoint union of two open intervals. But the closure $\overline{E} = [-1, 1]$ is connected, since it is an interval.

[5]This is, quite paradoxically, the key point of this proof. Or, more precisely, this is the point that will not necessarily be true if we consider a superset of E that is larger that $\overline{E}$ (can you find an example?). The general idea here is that $\overline{E}$ is "so close to E" that $\overline{E}$ has "no room to be disconnected" if E is connected.

[6]See for instance Proposition 3.1.28(f).

14. Uniform convergence

Exercise 14.1.1. — *Let (X, d_X) and (Y, d_Y) be metric spaces, let E be a subset of X, let $f : E \to Y$ be a function, and let x_0 be an element of E. Show that the limit $\lim_{x\to x_0;x\in E} f(x)$ exists if and only if the limit $\lim_{x\to x_0;x\in E\backslash\{x_0\}} f(x)$ exists and is equal to $f(x_0)$. Also, show that if the limit $\lim_{x\to x_0;x\in E} f(x)$ exists at all, then it must equal $f(x_0)$.*

We will denote (i) and (ii) the two following statements:

(i) $\lim_{x\to x_0;x\in E} f(x) = f(x_0)$;

(ii) $\lim_{x\to x_0;x\in E\backslash\{x_0\}} f(x) = f(x_0)$.

First, it is clear that we have (i) $\implies$ (ii), since $E\backslash\{x_0\} \subseteq E$.

Now let's show that (ii) $\implies$ (i). It means that for any arbitrary $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_Y(f(x), f(x_0)) < \varepsilon$ whenever $x \in E\backslash\{x_0\}$ and $d_X(x, x_0) < \delta$. Note that, obviously, if $x = x_0$, then $f(x) = f(x_0)$, so that the previous statement can be extended over E (it is true by hypothesis if $x \in E\backslash\{x_0\}$ and it remains true if $x = x_0$). Thus, we have indeed (ii), as expected.

Finally, suppose that the limit $\lim_{x\to x_0;x\in E} f(x)$ exists at all, and let's show that it must equal $f(x_0)$. Suppose that the limit exists, and suppose for the sake of contradiction that this limit is some $L \neq f(x_0)$. Consider $\varepsilon := d_Y(L, f(x_0))/2$. By hypothesis, we have $\varepsilon > 0$, and there exists a $\delta > 0$ such that $d_Y(f(x), L) < \varepsilon$ whenever $x \in E$ and $d_X(x, x_0) < \delta$. At $x = x_0$, we clearly have $x_0 \in E$ and $d(x, x_0) = 0 < \delta$, but we have $d_Y(f(x), L) = 2\varepsilon > \varepsilon$. This is a contradiction, and thus we must have $L = f(x_0)$.

Exercise 14.1.2. — *Prove Proposition 14.1.5.*

In this Proposition, the three statements (a), (b), (c) are the exact counterparts of Theorem 13.1.4, whereas statement (d) is sort of a whole new statement. We can thus prove it separately, showing first that (a) $\implies$ (c) $\implies$ (b) $\implies$ (a), and then that (a) $\iff$ (d).

- First we show that (a) $\implies$ (*c*). Suppose that $\lim_{x\to x_0;x\in E} f(x) = L$, and let be $V \subseteq Y$ an open set such that $L \in V$. By definition of an open set (Proposition 12.2.15(a)), there exists $\varepsilon > 0$ such that $B_Y(L, \varepsilon) \subseteq V$. By hypothesis (a), for this $\varepsilon > 0$, there exists a $\delta > 0$ such that:

  $$(x \in E; x \in B_X(x_0, \delta) \implies f(x) \in B_Y(L, \varepsilon) \subseteq V.$$

  If we set $U := B_X(x_0, \delta)$, we have indeed found an open set U such that $f(U \cap E) \subseteq V$, as expected.

- Now we show that (c) $\implies$ (b). Consider a sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of E that converges to x_0. Let be $\varepsilon > 0$. It is clear that $V := B_Y(L, \varepsilon)$ is an open set containing L. By (c), there exists an open set $U \subseteq X$ containing x_0 such that $f(U \cap E) \subseteq V$. Since U is open, there exists an open ball $B_X(x_0, \delta) \subseteq U$; and since $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, there exists an $N \geqslant 1$ such that $n \geqslant N \implies x^{(n)} \in B_X(x_0, \delta)$. In particular, $n \geqslant N \implies x^{(n)} \in U \cap E$, so that by (c), we have $f(x^{(n)}) \in V$. Unfolding these statements, for all $\varepsilon > 0$, there exists $N \geqslant 1$ such that $n \geqslant N \implies d_Y(f(x^{(n)}), L) < \varepsilon$. Thus, $(f(x^{(n)}))_{n=1}^{\infty}$ converges to L, as expected.

- Now let's show that (b) $\implies$ (a). Suppose, for the sake of contradiction, that we have $\lim_{x\to x_0;x\in E} f(x) \neq L$. Thus, there exists an $\varepsilon > 0$ such that, for all $\delta > 0$, we have an $x \in E$ with $d_Y(f(x), L) \geqslant \varepsilon$ although $d_X(x, x_0) < \delta$. In particular, for all $n \geqslant 1$, there exists a $x^{(n)} \in E$ such that $d_X(x^{(n)}, x_0) < 1/n$ but $d_Y(f(x^{(n)}), L) \geqslant \varepsilon$. By the (countable) axiom of choice, we can form the corresponding sequence $(x^{(n)})_{n=1}^{\infty}$, which clearly converges to x_0. However, the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ does not converge to L, a contradiction with (b). Thus, we must have $\lim_{x\to x_0;x\in E} f(x) = L$.

- Now let's show that (a) $\implies$ (d). Suppose that $\lim_{x\to x_0;x\in E} f(x) = L = g(x_0)$. We have indeed $\lim_{x\to x_0;x\in E\cup\{x_0\}} g(x) = L = g(x_0)$, because

  – if $x \in E\backslash\{x_0\}$, we have $\lim_{x\to x_0;x\in E\backslash\{x_0\}} g(x) = \lim_{x\to x_0;x\in E\backslash\{x_0\}} f(x) = L = g(x_0)$ by definition (using the fact that $E\backslash\{x_0\} \subseteq E$);

  – if $x = x_0$, we have of course $g(x) = g(x_0)$, i.e. $d_Y(g(x), g(x_0)) = 0$.

  Thus, in both cases, for all $\varepsilon > 0$, we can indeed find a $\delta > 0$ such that $d_Y(g(x), g(x_0)) < \varepsilon$ whenever $x \in E \cup \{x_0\}$ and $d_X(x, x_0) < \delta$. In other words, g is continuous at x_0.

  Furthermore, if $x_0 \in E$, we already know by Exercise 14.1.1 that we must have $f(x_0) = L$.

- Finally, let's show that (d) $\implies$ (a). By hypothesis, we have $\lim_{x\to x_0;x\in E\cup\{x_0\}} g(x) = g(x_0) = L$, and if $x_0 \in E$, we have $f(x_0) = g(x_0) = L$. We can split into two cases:

  – if $x_0 \notin E$, then $E\backslash\{x_0\}$ is simply E, so that the result is trivial: we have $L = \lim_{x\to x_0;x\in E\backslash\{x_0\}} g(x) = \lim_{x\to x_0;x\in E} f(x)$;

  – if $x_0 \in E$, then $E \cup \{x_0\}$ is simply E, and g is simply f, so that (a) is trivial.

Exercise 14.1.5. — *Let (X, d_X), (Y, d_Y), (Z, d_Z) be metric spaces, and let $x_0 \in X$, $y_0 \in Y$, $z_0 \in Z$. Let $f : X \to Y$ and $g : Y \to Z$ be functions, and let E be a subset of X. If we have $\lim_{x\to x_0;x\in E} f(x) = y_0$ and $\lim_{y\to y_0;y\in f(E)} g(y) = z_0$, conclude that $\lim_{x\in x_0;x\in E} g \circ f(x) = z_0$.*

Let be $\varepsilon > 0$.

- Since g converges to z_0 at y_0 in $f(E)$, there exists a $\delta > 0$ such that $d_Z(g(y), z_0) < \varepsilon$ whenever $y \in f(E)$ and $d_Y(y, y_0) < \delta$.

- Consider this same number $\delta > 0$. Since f converges to y_0 at x_0 in E, there exists an $\alpha > 0$ such that $d_Y(f(x), y_0) < \delta$ whenever $x \in E$ and $d_X(x, x_0) < \alpha$.

Thus, for any $\varepsilon > 0$, we can find an $\alpha > 0$ such that:

$$(x \in E,\ d_X(x, x_0) < \alpha) \implies (f(x) \in f(E),\ d_Y(f(x), y_0) < \delta) \implies d_Z(g \circ f(x), z_0) < \varepsilon.$$

Thus, we have indeed $\lim_{x\in x_0;x\in E} g \circ f(x) = z_0$.

Exercise 14.1.6. — *State and prove an analogue of the limit laws in Proposition 9.3.14 when X is now a metric space rather than a subset of $\mathbb{R}$. (Hint: use Corollary 13.2.3.)*

Proposition. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f, g : X \to Y$ be functions. Let be $E \subseteq X$ and x_0 an adherent point of E. Suppose that $\lim_{x\to x_0;x\in E} f(x) = L$

and $\lim_{x\to x_0;x\in E} g(x) = M$. Then $f+g$ has limit $L+M$ at x_0; $f-g$ has limit $L-M$ at x_0; $\max(f,g)$ and $\min(f,g)$ have limit $\max(L,M)$ and $\min(L,M)$ at x_0 respectively; fg has limit LM at x_0; and cf has limit cL at x_0 for any constant c. Furthermore, if $g(x) \neq 0$ for all $x \in X$ and $M \neq 0$, then f/g has limit L/M at x_0.

Proof. We just prove the first claim, i.e. that $\lim_{x\to x_0;x\in E}(f+g)(x) = L+M$, since all the other claims can be proved similarly.

The hint suggests to use Corollary 13.2.3, but this result applies to continuous functions and we do not know know whether f, g are continuous at x_0 (we can possibly even have $x_0 \notin E$ here, if x_0 is only a boundary point of E). However, we can actually “build” continuous counterparts for f, g, using Proposition 14.1.5(d). Indeed, let be $F : E \cup \{x_0\} \to Y$ such that $F(x) = f(x)$ if $x \neq x_0$, and $F(x_0) = L$. Similarly, let be $G : E \cup \{x_0\} \to Y$ defined by $G(x) = g(x)$ if $x \neq x_0$ and $G(x_0) = M$. This time, it is clear by Proposition 14.1.5(d) that F, G are continuous at x_0.

Thus, by Corollary 13.2.3, $F+G$ is continuous at x_0. Consequently, we have

$$\lim_{x\to x_0;x\in E\cup\{x_0\}}(F+G)(x) = (F+G)(x_0) = L+M.$$

This function $(F+G) : E \cup \{x_0\} \to Y$ is a continuous function such that $(F+G)(x_0) = L+M$, and $(F+G)(x) = (f+g)(x)$ for all $x \in E\backslash\{x_0\}$. The function $F+G$ satisfies Proposition 14.1.5(d)[7], and thus we also have Proposition 14.1.5(a), i.e. $\lim_{x\to x_0;x\in E}(f+g)(x) = L+M$, as expected. □

Exercise 14.2.1. — *The purpose of this exercise is to demonstrate a concrete relationship between continuity and pointwise convergence, and between uniform continuity and uniform convergence. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. For any $a \in \mathbb{R}$, let $f_a : \mathbb{R} \to \mathbb{R}$ be the shifted function $f_a(x) := f(x-a)$.*

(a) Show that f is continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge pointwise to f.

(b) Show that f is uniformly continuous if and only if, whenever $(a_n)_{n=0}^\infty$ is a sequence of real numbers which converges to zero, the shifted functions f_{a_n} converge uniformly to f.

Let be $a \in \mathbb{R}$, and $(a_n)_{n=0}^\infty$ a sequence of real numbers that converges to 0.

(a) Suppose that f is continuous. Let be $\varepsilon > 0$, and let be an arbitrary $x \in \mathbb{R}$. In particular, since f is continuous at x, there exists a $\delta_x > 0$ such that $|x-y| < \delta_x \implies |f(x) - f(y)| < \varepsilon$. Furthermore, since $(a_n)_{n=0}^\infty$ converges to 0, there exists a $N \geqslant 0$ such that $n \geqslant N \implies |a_n| < \delta_x$. Thus, for any $x \in \mathbb{R}$ and any $\varepsilon > 0$ we have a $N \geqslant 0$ such that

$$n \geqslant N \implies |x - (x-a_n)| < \delta_x \implies \underbrace{|f(x) - f(x-a_n)|}_{=|f(x)-f_{a_n}(x)|} < \varepsilon$$

which means that $(f_{a_n})_{n=0}^\infty$ converges pointwise to f.

[7] We also have the part “if $x_0 \in E$, then $(f+g)(x_0) = L+M$” because we have this part for f, g taken separately.

Conversely, suppose that $(f_{a_n})_{n=0}^{\infty}$ converges pointwise to f for any sequence $(a_n)_{n=0}^{\infty}$, but suppose, for the sake of contradiction, that f is not continuous. Thus, there exists some $\varepsilon > 0$ such that, for all $n \geqslant 0$, we have a $y_n \in \mathbb{R}$ with $|x - y_n| < \frac{1}{n+1}$ but $|f(x) - f(y_n)| \geqslant \varepsilon$. By the (countable) axiom of choice, we can form the corresponding sequence $(y_n)_{n=0}^{\infty}$. And if we define another sequence $a_n := x - y_n$, then it is clear that $(a_n)_{n=0}^{\infty}$ converges to 0 (since $|a_n| < \frac{1}{n+1}$ for all $n \geqslant 0$); but that the functions f_{a_n} do not converge pointwise to f, since $|f(x) - f_{a_n}(x)| \geqslant \varepsilon$ for all $n \geqslant 0$.

(b) Suppose that f is uniformly continuous. Let be $\varepsilon > 0$. By definition, there exists a $\delta > 0$ such that, for all $x, y \in \mathbb{R}$, we have $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$. Furthermore, since $(a_n)_{n=0}^{\infty}$ converges to 0, there exists a $N \geqslant 0$ such that $n \geqslant N \implies |a_n| < \delta$. Thus, for any $\varepsilon > 0$ we have a $N \geqslant 0$ such that, for all $x \in \mathbb{R}$,

$$n \geqslant N \implies |x - (x - a_n)| < \delta \implies \underbrace{|f(x) - f(x - a_n)|}_{=|f(x) - f_{a_n}(x)|} < \varepsilon$$

which means that $(f_{a_n})_{n=0}^{\infty}$ converges uniformly to f.

Conversely, suppose that $(f_{a_n})_{n=0}^{\infty}$ converges uniformly to f for any sequence $(a_n)_{n=0}^{\infty}$, but suppose, for the sake of contradiction, that f is not uniformly continuous. Thus, there exists some $\varepsilon > 0$ such that, for all $n \geqslant 0$, we have some $x_n, y_n \in \mathbb{R}$ with $|x_n - y_n| < \frac{1}{n+1}$ but $|f(x_n) - f(y_n)| \geqslant \varepsilon$. By the (countable) axiom of choice, we can form the corresponding sequences $(x_n)_{n=0}^{\infty}$ and $(y_n)_{n=0}^{\infty}$. If we define a new sequence $a_n := x_n - y_n$, then $(a_n)_{n=0}^{\infty}$ clearly converges to 0. However, $(f_{a_n})_{n=0}^{\infty}$ does not converge uniformly to f, since for all $n \geqslant 0$, we have

$$\varepsilon \leqslant |f(x_n) - f(y_n)| = |f(x_n) - f_{a_n}(x_n)|.$$

This is thus a contradiction, and f must be uniformly continuous.

Exercise 14.2.2. — *(a) Let $(f^{(n)})_{n=1}^{\infty}$ be a sequence of functions from one metric space (X, d_X) to another (Y, d_Y), and let $f : X \to Y$ be another function from X to Y. Show that if $f^{(n)}$ converges uniformly to f, then $f^{(n)}$ also converges pointwise to f.*
(b) For each integer $n \geqslant 1$, let $f^{(n)} :]-1, 1[\to \mathbb{R}$ be the function $f^{(n)}(x) := x^n$. Prove that $f^{(n)}$ converges pointwise to the zero function 0, but does not converge uniformly to any function $f :]-1, 1[\to \mathbb{R}$.
(c) Let $g : (-1, 1) \to \mathbb{R}$ be the function $g(x) := x/(1 - x)$. With the notation as in (b), show that the partial sums $\sum_{n=1}^{N} f(n)$ converges pointwise as $N \to \infty$ to g, but does not converge uniformly to g, on the open interval $]-1, 1[$. (Hint: use Lemma 7.3.3.) What would happen if we replaced the open interval $]-1, 1[$ with the closed interval $[-1, 1]$?

(a) Let be $\varepsilon > 0$. Since the $f^{(n)}$ converge uniformly to f, there exists a $N \geqslant 0$ such that, for all $x \in X$, we have $d_X(f^{(n)}(x), f(x)) < \varepsilon$. Thus, for any fixed $x \in \mathbb{R}$, this same $N \geqslant 0$ is such that $d_X(f^{(n)}(x), f(x)) < \varepsilon$ whenever $n \geqslant N$, i.e., the $f^{(n)}$ converge pointwise to f.

(b) By Lemma 6.5.2, we know that for all $x \in]-1, 1[$, we have $\lim_{n\to\infty} x^n = 0$. Thus, $f^{(n)}$ converges pointwise to the zero function.

However, $f^{(n)}$ does not converge uniformly to the zero function[8], as stated in the main text. More formally, we can give a proof in the same spirit as what we did in Exercises

[8]If the uniform limit exists, it can only be equal to the pointwise limit, by (a) and the uniqueness of limits.

13.3.3 and 13.3.5. Indeed, let be $\varepsilon = 0.1$, and suppose that there exists $N \geqslant 0$ such that, for all $x \in]-1,1[$, $n \geqslant N \implies |x^n| < 0.1$. Actually, we are always able to find an $x \in]-1,1[$ such that $|x^n| \geqslant 0.1$ although $n \geqslant N$. For instance, for $n = N$ and $x = 0.1^{1/N}$, we have $|x^n| = \left|((1/10)^{1/N})^N\right| = 0.1$.

(c) Here we will denote $S_N(x) := \sum_{n=1}^{N} f^{(n)}(x)$ the partial sums. We already know, by Lemma 7.3.3, that we have, for all $x \in]-1,1[$,

$$\lim_{N\to\infty} \sum_{n=0}^{N} f^{(n)}(x) = \frac{1}{1-x},$$

so that we have

$$\lim_{N\to\infty} \sum_{n=1}^{N} f^{(n)}(x) = \lim_{N\to\infty} \left(\sum_{n=0}^{N} f^{(n)}(x) - 1 \right) = \frac{1}{1-x} - 1 = \frac{x}{1-x},$$

thus meaning that S_N converges pointwise to $x/(1-x)$ as $N \to \infty$, as expected.

However, S_N does not converge uniformly to $x/(1-x)$ as $N \to \infty$. The explicit expression of the partial sums is $S_N := \frac{1-x^{N+1}}{1-x} - 1 = \frac{x(1-x^N)}{1-x}$, for all $N \geqslant 1$, and for all $x \in]-1,1[$. In the same spirit as question (b), let be $\varepsilon := 0.1$, and suppose that there exists $N_0 \geqslant 0$ such that, for all $x \in]-1,1[$, we have

$$N \geqslant N_0 \implies \left| \frac{x(1-x^N)}{1-x} - \frac{x}{1-x} \right| = \left| \frac{-x^{N+1}}{1-x} \right| = \frac{x^{N+1}}{1-x} < 0.1$$

Actually, we are always able to find an $x \in]-1,1[$ such that $\frac{x^{N+1}}{1-x} \geqslant 0.1$ although $N \geqslant N_0$. For instance, for $N = N_0$ and $x = (1/10)^{\frac{1}{N_0+1}}$, we have $\frac{x^{N+1}}{1-x} = 1/9 > 0.1$ although $N \geqslant N_0$. Thus, we do not have uniform convergence here.

Finally, if we replace $]-1,1[$ by $[-1,1]$, then $S_N(x)$ does not even converge pointwise (see Lemma 7.3.3 when $x = 1$ or $x = -1$).

Exercise 14.2.3. — *Let be (X, d_X) a metric space, and for every integer $n \geqslant 1$, let $f_n : X \to \mathbb{R}$ be a real-valued function. Suppose that f_n converges pointwise to another function $f : X \to \mathbb{R}$ on X (in this question we give $\mathbb{R}$ the standard metric $d(x,y) := |x-y|$). Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that the functions $h \circ f_n$ converge pointwise to $h \circ f$ on X, where $h \circ f_n : X \to \mathbb{R}$ is the function $h \circ f_n(x) := h(f_n(x))$, and similarly for $h \circ f$.*

This exercise is a trivial application of the equivalence (a) $\iff$ (b) in Theorem 13.1.4. Indeed, let be an arbitrary $x_0 \in X$. Since f_n converges pointwise to f on X, the sequence $(f_n(x_0))_{n=1}^{\infty}$ converges to $f(x_0)$ in $(\mathbb{R}, d)$, by definition. And since h is continuous on $\mathbb{R}$, then by Theorem 13.2.4(b), we have $\lim_{n\to\infty} h(f_n(x_0)) = h(f(x_0))$, which means exactly that $(h \circ f_n)_{n=1}^{\infty}$ converges pointwise to $h \circ f$ on X.

Exercise 14.2.4. — *Let $f_n : X \to Y$ be a sequence of bounded functions from one metric space (X, d_X) to another metric space (Y, d_Y). Suppose that f_n converges uniformly to another function $f : X \to Y$. Suppose that f is a bounded function; i.e., there exists a ball $B_{(Y,d_Y)}(y_0, R)$ in Y such that $f(x) \in B_{(Y,d_Y)}(y_0, R)$ for all $x \in X$. Show that the sequence f_n is* *uniformly bounded; i.e. there exists a ball $B_{(Y,d_Y)}(y_0,R)$ in Y such that $f_n(x) \in B_{(Y,d_Y)}(y_0,R)$ for all $x \in X$ and all positive integers n.*

Let be $\varepsilon := 1$, for instance. Since $(f_n)_{n=1}^{\infty}$ converges to f, there exists $N \geqslant 1$ such that, for all $x \in X$, we have $d_Y(f_n(x), f(x)) < 1$ whenever $n > N$.

Furthermore, f is supposed to be bounded. Thus, there exists a $y_0 \in Y$ and a real $R > 0$ such that $d_Y(f(x), y_0) < R$ for all $x \in X$.

Thus, if $n > N$, we have (by triangle inequality) for all $x \in X$,

$$d_Y(f_n(x), y_0) \leqslant d_Y(f_n(x), f(x)) + d_Y(f(x), y_0) < 1 + R.$$

In other words, the sequence $(f_n)_{n=N+1}^{\infty}$ is uniformly bounded. Now, we still have to extend this result to the whole sequence $(f_n)_{n=1}^{\infty}$.

Recall that each f_n is bounded. Thus, there exists in particular a finite sequence of centers $y_1, \ldots, y_N \in Y$ and a finite sequence of radii $R_1, \ldots, R_N > 0$ such that, for all $x \in X$,

$$f_1(x) \in B_Y(y_1, R_1) \ ; \ \ldots \ ; \ f_N(x) \in B_Y(y_N, R_N).$$

Let be

$$S := \left(\max_{1 \leqslant i \leqslant N} d_Y(y_i, y_0)\right) + \left(\max_{1 \leqslant i \leqslant N} R_i\right) + (R+1).$$

We have thus by triangle inequality, for all $1 \leqslant i \leqslant N$ and for all $x \in X$,

$$\begin{aligned} d_Y(f_i(x), y_0) &\leqslant d_Y(f_i(x), y_i) + d_Y(y_i, y_0) \\ &\leqslant R_i + d_Y(y_i, y_0) \\ &\leqslant \left(\max_{1 \leqslant i \leqslant N} R_i\right) + \left(\max_{1 \leqslant i \leqslant N} d_Y(y_i, y_0)\right) \\ &\leqslant S. \end{aligned}$$

Thus, we have indeed $d_Y(f_n(x), y_0) < S$ for all $x \in X$ and for all $n \geqslant 1$, which means that $(f_n)_{n=1}^{\infty}$ is uniformly bounded.

Exercise 14.3.1. — *Prove Theorem 14.3.1. Explain briefly why your proof requires uniform convergence, and why pointwise convergence would not suffice.*

Let be $\varepsilon > 0$ an arbitrary real number. By our initial hypothesis, the sequence $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to f. Thus, there exists an $N \geqslant 1$ such that:

$$(n > N) \text{ and } (x \in X) \implies d_Y(f^{(n)}(x), f(x)) < \varepsilon/3. \tag{14.1}$$

Furthermore, each function in the sequence $(f^{(n)})_{n=1}^{\infty}$ is continuous at x_0. In particular, the function f^{N+1} is continuous at x_0: there exists a $\delta > 0$ such that

$$(x \in X) \text{ and } (d_X(x, x_0) < \delta) \implies d_Y(f^{(N+1)}(x), f^{(N+1)}(x_0)) < \varepsilon/3. \tag{14.2}$$

Since $N + 1 > N$, we have, for all $x \in X$ such that $d_X(x, x_0) < \delta$,

$$\begin{aligned} d_Y\left(f(x), f(x_0)\right) &\leqslant \underbrace{d_Y\left(f(x), f^{(N+1)}(x)\right)}_{<\varepsilon/3 \text{ by } (14.1)} + \underbrace{d_Y\left(f^{(N+1)}(x), f^{(N+1)}(x_0)\right)}_{<\varepsilon/3 \text{ by } (14.2)} \\ &\quad + \underbrace{d_Y\left(f^{(N+1)}(x_0), f(x_0)\right)}_{<\varepsilon/3 \text{ by } (14.1)} \\ &< \varepsilon. \end{aligned}$$

Since this is true for any arbitrary $\varepsilon > 0$, the function f is indeed continuous at x_0.

Note that the hypothesis of *uniform* convergence was indeed necessary. In the previous inequalities, one single $N \geqslant 1$ was convenient to have both

$$d_Y\left(f(x), f^{(N+1)}(x)\right) < \varepsilon/3 \quad \text{and} \quad d_Y\left(f^{(N+1)}(x_0), f(x_0)\right) < \varepsilon/3,$$

at the same time at x and at x_0. Example 14.2.4 is a counter-example when we only suppose pointwise convergence.

Exercise 14.3.2. — *Prove Proposition 14.3.3*

Let be $\varepsilon > 0$, and x_0 an adherent point to E. Hereafter, we denote by convenience $\ell_n := \lim_{x \to x_0; x \in E} f^{(n)}(x)$, for all $n \geqslant 1$. We thus have to prove that $\lim_{n \to \infty} \ell_n = \lim_{x \to x_0; x \in E} f(x)$.

However, we must begin by proving that the sequence $(\ell_n)_{n=1}^{\infty}$ is indeed convergent. Since this sequence is entirely composed of elements of Y, and since Y is complete, it is sufficient to prove that $(\ell_n)_{n=1}^{\infty}$ is a Cauchy sequence.

To that end, consider two integers $j, k \geqslant 1$ and an element $x \in E$. We note that we can always write

$$\begin{aligned} d_Y(\ell_j, \ell_k) \leqslant\ & d_Y(\ell_j, f^{(j)}(x)) + d_Y(f^{(j)}(x), f(x)) \\ & + d_Y(f(x), f^{(k)}(x)) + d(f^{(k)}(x), \ell_k). \end{aligned}$$

Let's prove that each term of this sum is inferior to $\varepsilon/4$ under certain conditions.

First, the sequence $(f^{(n)})_{n=1}^{\infty}$ is supposed to converge uniformly to f. Thus, there exists an $N_1 \geqslant 1$ such that

$$(n > N) \text{ and } (x \in E) \implies d_Y(f^{(n)}(x), f(x)) < \varepsilon/4. \tag{14.3}$$

Let be two integers $j, k > N_1$. By definition of ℓ_n, there exist two real numbers $\delta_1, \delta_2 > 0$ such that

$$(x \in E) \text{ and } (d_X(x, x_0) < \delta_1) \implies d_Y(f^{(j)}(x), \ell_j) < \varepsilon/4, \tag{14.4}$$

$$(x \in E) \text{ and } (d_X(x, x_0) < \delta_2) \implies d_Y(f^{(k)}(x), \ell_k) < \varepsilon/4. \tag{14.5}$$

Now we set $\delta := \min(\delta_1, \delta_2)$. Thus, for all $j, k > N_1$ and for all $x \in E$ such that $d_X(x, x_0) < \delta$, we have

$$\begin{aligned} d_Y(\ell_j, \ell_k) \leqslant\ & \underbrace{d_Y(\ell_j, f^{(j)}(x))}_{<\varepsilon/4 \text{ by } (14.4)} + \underbrace{d_Y(f^{(j)}(x), f(x))}_{<\varepsilon/4 \text{ by } (14.3)} \\ & + \underbrace{d_Y(f(x), f^{(k)}(x))}_{<\varepsilon/4 \text{ by } (14.3)} + \underbrace{d(f^{(k)}(x), \ell_k)}_{<\varepsilon/4 \text{ by } (14.5)} \\ <\ & \varepsilon. \end{aligned}$$

Thus, $(\ell_n)_{n=1}^{\infty}$ is a Cauchy sequence, and thus converges in Y; in other words, there exists a $\ell \in Y$ such that $\ell = \lim_{n \to \infty} \ell_n$.

To close the proof, we still have to show that $\lim_{x \to x_0; x \in E} f(x) = \ell$. Since $\ell = \lim_{n\to\infty} \ell_n$, there exists an integer $N_2 \geqslant 1$ such that $d_Y(\ell, \ell_n) < \varepsilon/4$ for all $n \geqslant N_2$. Let be $N := \max(N_1, N_2) + 1$. We thus have, for all $x \in E$ such that $d_X(x, x_0) < \delta$, and all $n \geqslant N$,

$$\begin{aligned} d_Y(f(x), \ell) &\leqslant d_Y(f(x), f^{(n)}(x)) + d_Y(f^{(n)}(x), \ell_n) + d_Y(\ell_n, \ell) \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 \\ &< \varepsilon. \end{aligned}$$

This shows that $\lim_{x \to x_0; x \in E} f(x) = \ell$, and thus closes the proof:

$$\begin{aligned} \lim_{x \to x_0; x \in E} \lim_{n\to\infty} f^{(n)}(x) &= \lim_{x \to x_0; x \in E} f(x) = \ell = \lim_{n\to\infty} \ell_n \\ &= \lim_{n\to\infty} \lim_{x \to x_0; x \in E} f^{(n)}(x). \end{aligned}$$

Exercise 14.3.5. — *Give an example to show that Proposition 14.3.4 fails if the phrase "converges uniformly" is replaced by "converges pointwise". (Hint: some of the examples already given earlier will already work here.)*

Consider once again the sequence of functions $(f^{(n)})_{n=1}^{\infty}$, such that for all $n \geqslant 1$, the function $f^{(n)} : [0,1] \to \mathbb{R}$ is defined by $f^{(n)}(x) := x^n$. This sequence converges pointwise to the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) := 0$ when $x \in [0,1[$, and $f(1) := 1$. Furthermore, consider the sequence $(x^{(n)})_{n=1}^{\infty}$ defined by $x^{(n)} := (1/2)^{1/n}$.

By Lemma 6.5.3, we have $\lim_{n\to\infty} x^{(n)} = 1$. However, for all $n \geqslant 1$, we have $f^{(n)}(x^{(n)}) = 1/2$. Thus, it is clear that this constant sequence $1/2$ does not converge to $f(1) = 1$.

Exercise 14.3.6. — *Prove Proposition 14.3.6. Discuss how this proposition differs from Exercise 14.2.4*

Consider for instance $\varepsilon := 1$. Since the functions $f^{(n)}$ converge uniformly to f, there exists an integer $N \geqslant 1$ such that, for all $n > N$ and all $x \in X$, we have $d_Y(f^{(n)}(x), f(x)) < 1$.

Furthermore, since all the functions $f^{(n)}$ are bounded, then in particular $f^{(N+1)}$ is bounded: there exists a ball $B_Y(y_{N+1}, R_{N+1})$ such that $f^{(N+1)}(x) \in B_Y(y_{N+1}, R_{N+1})$ for all $x \in X$. In other words, we have $d_Y(f^{(N+1)}(x), y_{N+1}) < R_{N+1}$ for all $x \in X$.

Thus, by triangle inequality, we have for all $x \in X$,

$$\begin{aligned} d_Y(f(x), y_{N+1}) &\leqslant d_Y(f(x), f^{(N+1)}(x)) + d_Y(f^{(N+1)}, y_{N+1}) \\ &< 1 + R_{N+1}. \end{aligned}$$

We thus have $f(x) \in B_Y(y_{N+1}, 1 + R_{N+1})$ for all $x \in X$, which shows f is bounded on X.

This provides some sort of reciprocal to Exercise 13.2.4, where we supposed that the uniform limit f was bounded to show that the $f^{(n)}$ were themselves bounded.

Exercise 14.3.7. — *Give an example to show that Proposition 14.3.6 fails if "converge uniformly" is replaced by "converges pointwise".*

Consider once again the functions from Exercise 14.2.2(c), i.e., a sequence of functions $g^{(n)} := \sum_{k=1}^{n} x^k$ defined on $(-1, 1)$ and that converge pointwise to $g(x) := x/(1-x)$. It is clear that each $g^{(n)}$ is bounded on $(-1,1)$: we have $g^{(n)}(x) \in B(0, n)$ for all n. However, the function $g(x)$ is not bounded on $(-1,1)$.

Exercise 14.3.8. — *Let (X, d) be a metric space, and for every positive integer n, let $(f^{(n)})_{n=1}^{\infty} : X \to \mathbb{R}$ and $(g^{(n)})_{n=1}^{\infty} : X \to \mathbb{R}$ be functions. Suppose that $(f^{(n)})_{n=1}^{\infty}$ converges uniformly to another function $f : X \to \mathbb{R}$, and that $(g^{(n)})_{n=1}^{\infty}$ converges uniformly to another function $g : X \to \mathbb{R}$. Suppose also that the functions $(f^{(n)})_{n=1}^{\infty}$ and $(g^{(n)})_{n=1}^{\infty}$ are uniformly bounded. Prove that the functions $f_n g_n : X \to \mathbb{R}$ converge uniformly to $fg : X \to \mathbb{R}$.*

Let be an arbitrary $\varepsilon > 0$. To show that $(f_n g_n)_{n=1}^{\infty}$ converge uniformly to fg, we must show that there exists an integer $N \geqslant 1$ such that $|f_n(x)g_n(x) - f(x)g(x)| < \varepsilon$ for all $n > N$ and all $x \in \mathbb{R}$. Note that

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
&= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leqslant |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\
&\leqslant |f_n(x)| \times |g_n(x) - g(x)| \;+\; |g(x)| \times |f_n(x) - f(x)|.
\end{aligned}$$

We can find an upper bound for each term in this expression.

- First, $(f_n)_{n=1}^{\infty}$ is uniformly bounded, so that there exists a $M_1 \geqslant 0$ such that $|f_n(x)| \leqslant M_1$ for all $n \geqslant 1$ and all $x \in \mathbb{R}$.
- Similarly, $(g_n)_{n=1}^{\infty}$ is uniformly bounded. Thus, by Proposition 14.3.6, the uniform limit g is also bounded: there exists a $M_2 \geqslant 0$ such that $|g(x)| \leqslant M_2$ for all $x \in \mathbb{R}$.
- Furthermore, $(f_n)_{n=1}^{\infty}$ converges uniformly to f: there exists an integer $N_1 \geqslant 1$ such that $|f_n(x) - f(x)| < \varepsilon/2M_2$ for all $n > N_1$ and all $x \in \mathbb{R}$.
- Finally, $(g_n)_{n=1}^{\infty}$ converges uniformly to g: there exists an integer $N_2 \geqslant 1$ such that $|g_n(x) - g(x)| < \varepsilon/2M_1$ for all $n > N_2$ and all $x \in \mathbb{R}$.

Let be $N := \max(N_1, N_2)$. We thus have, for all $x \in \mathbb{R}$ and all $n > N$,

$$\begin{aligned}
&|f_n(x)g_n(x) - f(x)g(x)| \\
&\leqslant |f_n(x)| \times |g_n(x) - g(x)| \;+\; |g(x)| \times |f_n(x) - f(x)| \\
&< M_1 \times \frac{\varepsilon}{2M_1} + M_2 \times \frac{\varepsilon}{2M_1} \\
&< \varepsilon.
\end{aligned}$$

It means that $(f_n g_n)_{n=1}^{\infty}$ converges uniformly to fg, as expected.

Exercise 14.4.1. — *Let (X, d_X) and (Y, d_Y) be metric spaces. Show that the space $B(X \to Y)$ defined in Definition 14.4.2, with the metric $d_{B(X \to Y)}$, is indeed a metric space.*

Here we must prove that d_∞ satisfies all four properties of a distance on $B(X \to Y)$, as stated in Definition 12.1.2.

(a) For all $f \in B(X \to Y)$ and all $x \in X$, we clearly have $d_Y(f(x), f(x)) = 0$, since d_Y is a distance on Y. Thus it is clear that

$$d_\infty(f, f) = \sup_{X \in X} d_Y(f(x), f(x)) = \sup\{0 \,:\, x \in X\} = 0.$$

(b) For all distinct $f, g \in B(X \to Y)$, there exists an $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. We thus have $d_Y(f(x_0), g(x_0)) > 0$, since d_Y is a distance on Y. Thus, by Definition 5.5.5 of a supremum, we have $0 < d_Y(f(x_0), g(x_0)) \leqslant \sup_{x \in X} d_Y(f(x), g(x))$, which implies $d_\infty(f, g) > 0$.

(c) Since d_Y is a distance on Y, we have $d_Y(f(x), g(x)) = d_Y(g(x), f(x))$ for all $f, g \in B(X \to Y)$ and all $x \in X$. We thus have

$$d_\infty(f, g) := \sup_{x \in X} d_Y(f(x), g(x)) = \sup_{x \in X} d_Y(g(x), f(x)) =: d_\infty(g, f).$$

(d) Let $f, g, h \in B(X \to Y)$ be functions. Since d_Y is a distance on Y, it satisfies the triangle inequality on Y: for all $x \in X$, we have $d_Y(f(x), h(x)) \leqslant d_Y(f(x), g(x)) + d_Y(g(x), h(x))$.

In particular, for all $x \in X$, we have

$$d_Y(f(x), h(x)) \leqslant \sup_{X \in X} d_Y(f(x), g(x)) + \sup_{x \in X} d_Y(g(x), h(x)),$$

or in other words, $d_\infty(f, g) + d_\infty(g, h)$ is an upper bound of the set $\{d_Y(f(x), h(x)) : x \in X\}$. Thus, by Definition 5.5.5 of a supremum, we have

$$d_\infty(f, h) := \sup_{x \in X}\{d_Y(f(x), h(x)) : x \in X\} \leqslant d_\infty(f, g) + d_\infty(g, h),$$

as expected.

Exercise 14.5.1. — *Let $f^{(1)}, \ldots, f^{(n)}$ be a sequence of bounded functions from a metric space (X, d_X) to $\mathbb{R}$. Show that $\sum_{i=1}^n f^{(i)}$ is also bounded. Prove a similar claim when "bounded" is replaced by "continuous". What if "continuous" was replaced by "uniformly continuous"?*

Let $F : X \to \mathbb{R}$ be the finite sum defined by $F(x) := \sum_{i=1}^n f^{(i)}(x)$.

- If the functions $f^{(1)}, \ldots, f^{(n)}$ are bounded, then there exist n positive real numbers $M_1, \ldots, M_n$ such that $|f^{(i)}(x)| \leqslant M_i$ for all $x \in X$ and all $1 \leqslant i \leqslant n$. Let be $M := \sum_{i=1}^n M_i$.

  We thus have, by Lemma 7.1.4(e),

  $$|F(x)| = \left|\sum_{i=1}^n f^{(i)}(x)\right| \leqslant \sum_{i=1}^n \left|f^{(i)}(x)\right| \leqslant \sum_{i=1}^n M_i = M.$$

  The finite sum F is thus bounded (by M).

- Suppose that the functions $f^{(1)}, \ldots, f^{(n)}$ are continuous at some point $x_0 \in X$, and let be an arbitrary $\varepsilon > 0$. By definition, for all $1 \leqslant i \leqslant n$, there exists a $\delta_i > 0$ such that $d_X(x, x_0) \leqslant \delta_i \implies |f^{(i)}(x) - f^{(i)}(x_0)| \leqslant \varepsilon/n$. Let be $\delta := \min(\delta_1, \ldots, \delta_n)$.

For all $x \in X$ such that $d_X(x, x_0) \leqslant \delta$, we thus have

$$\begin{aligned}
|f(x) - f(x_0)| &= \left| \sum_{i=1}^{n} f^{(i)}(x) - \sum_{i=1}^{n} f^{(i)}(x_0) \right| \\
&\leqslant \sum_{i=1}^{n} \left| f^{(i)}(x) - f^{(i)}(x_0) \right| && \text{(Lemma 7.1.4(e))} \\
&\leqslant \sum_{i=1}^{n} \frac{\varepsilon}{n} \\
&\leqslant \varepsilon.
\end{aligned}$$

The finite sum F is thus also continuous at x_0. And since x_0 was arbitrary, we have the more general result that if $f^{(1)}, \dots, f^{(n)}$ are continuous on X, then F is continuous on X.

- Similarly, if $f^{(1)}, \dots, f^{(n)}$ are uniformly continuous on X, then F is also uniformly continuous on X (The proof is almost identical to the previous one.)

Exercise 14.5.2. — *Prove Theorem 3.5.7.*

Let $(f^{(n)})_{n=1}^{\infty}$ be a sequences of functions of $C(X \to \mathbb{R})$, and suppose that the series (of non-negative real numbers) $\sum_{n=1}^{\infty} \|f^{(n)}\|_\infty$ converges. Let be $\varepsilon > 0$ arbitrary. By Proposition 7.2.5, there exists a $K \geqslant 1$ such that $\sum_{n=i}^{j} \|f^{(n)}\|_\infty \leqslant \varepsilon$ for all $i, j \geqslant K$.

For $N \geqslant 1$, let be $F^{(N)}$ the partial sums defined by $F^{(N)}(x) := \sum_{n=1}^{N} f^{(n)}(x)$ for all $x \in X$. By Exercise 14.5.1, for all $N \geqslant 1$, the partial sum $F^{(N)}$ is itself an element of $C(X \to \mathbb{R})$. Furthermore, let's show that $(F^{(N)})_{N=1}^{\infty}$ is a Cauchy sequence. For all $x \in X$ and all $p > q \geqslant K$, we have

$$\begin{aligned}
\left| F^{(p)}(x) - F^{(q)}(x) \right| &= \left| \sum_{n=1}^{p} f^{(n)}(x) - \sum_{n=1}^{q} f^{(n)}(x) \right| \\
&= \left| \sum_{n=q+1}^{p} f^{(n)}(x) \right| \\
&\leqslant \sum_{n=q+1}^{p} \left| f^{(n)}(x) \right| && \text{(Lemma 7.1.4(e))} \\
&\leqslant \sum_{n=q+1}^{p} \|f^{(n)}\|_\infty \\
&\leqslant \varepsilon.
\end{aligned}$$

In particular, we thus have for all $p > q \geqslant K$,

$$\|F^{(p)} - F^{(q)}\|_\infty \leqslant \varepsilon.$$

The sequence $(F^{(N)})_{N=1}^{\infty}$ is thus a Cauchy sequence in $C(X \to Y)$. By Theorem 14.4.5, this sequence converges to a function $F \in C(X \to Y)$, as expected, and this convergence is uniform by Proposition 14.4.4.