Propositions of solutions for *Analysis I* by Terence Tao

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1. Introduction

No exercises in this chapter.

2. The natural numbers

Exercise 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a + b) + 0 = a + (b + 0)$. The left hand side is equal to $(a + b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b + 0)$ part, we get $a + (b + 0) = a + b$. Both sides are equal to $a + b$, and the base case is thus done.
- Now let's suppose inductively that $(a + b) + c = a + (b + c)$: we have to prove that $(a + b) + c{++} = a + (b + c{++})$. Using Lemma 2.2.3 on the right hand side leads to $a + (b + c){++}$. Now consider the left hand side. Using still the same lemma, we get $(a + b) + c{++} = ((a + b) + c){++}$. By the inductive hypothesis, this is also equal to $(a + (b + c)){++}$. And, using the lemma 2.2.3 again, this also leads to $a + b + c{++}$. Therefore, both sides are equal to $a + b + c{++}$, and we have closed the induction.

Exercise 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b{++} = a$.*

Let's use induction on a.

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0{++} = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b{++} = 1$. Then, we would have $b{++} = 0{++}$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.
- Let's suppose inductively that there is exactly one natural number b such as $b{++} = a$. We have to prove that there is exactly one natural number b' such as $b'{++} = a{++}$. By the induction hypothesis, and taking $b' = b{++}$, we have $b'{++} = (b{++}){++} = a{++}$. So there exists a solution, with $b' = b{++} = a$. Uniqueness is given by Axiom 2.4.: if $b'{++} = a{++}$, then we necessarily have $b' = a$.