

Propositions of solutions for *Analysis I* by Terence Tao

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1. Introduction

No exercises in this chapter.

2. The natural numbers

EXERCISE 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c , we have $(a + b) + c = a + (b + c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a + b) + 0 = a + (b + 0)$. The left hand side is equal to $(a + b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b + 0)$ part, we get $a + (b + 0) = a + b$. Both sides are equal to $a + b$, and the base case is thus done.
- Now let's suppose inductively that $(a + b) + c = a + (b + c)$: we have to prove that $(a + b) + c++ = a + (b + c++)$. Using Lemma 2.2.3 on the right hand side leads to $a + (b + c)++$. Now consider the left hand side. Using still the same lemma, we get $(a + b) + c++ = ((a + b) + c)++$. By the inductive hypothesis, this is also equal to $(a + (b + c))++$. And, using the lemma 2.2.3 again, this also leads to $a + b + c++$. Therefore, both sides are equal to $a + b + c++$, and we have closed the induction.

EXERCISE 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b++ = a$.*

Let's use induction on a .

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0++ = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b++ = 1$. Then, we would have $b++ = 0++$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.
- Let's suppose inductively that there is exactly one natural number b such that $b++ = a$. We have to prove that there is exactly one natural number b' such that $b'++ = a++$. By the induction hypothesis, and taking $b' = b++$, we have $b'++ = (b++)++ = a++$. So there exists a solution, with $b' = b++ = a$. Uniqueness is given by Axiom 2.4.: if $b'++ = a++$, then we necessarily have $b' = a$.

EXERCISE 2.2.3. — Let a, b, c be natural numbers. Prove the following properties of order for natural numbers:

- (a) Reflexivity: $a \geq a$. This is true since $a = 0 + a$ by Definition 2.2.1. By commutativity of addition, we can also write $a = a + 0$. So there is indeed a natural number n (with $n = 0$) such that $a = a + n$, i.e. $a \geq a$.
- (b) Transitivity: if $a \geq b$ and $b \geq c$, then $a \geq c$. From the part $a \geq b$, there exists a natural number n such that $a = b + n$ according to Definition 2.2.11. A similar consideration for the part $b \geq c$ leads to $b = c + m$, m being a natural number. Combining together those two equalities, we can write $a = b + n = (c + m) + n = c + (m + n)$ by associativity (see Exercise 2.2.1). Then, $n + m$ being a natural number¹, the transitivity is demonstrated.
- (c) Anti-symmetry: if $a \geq b$ and $b \geq a$, then $a = b$. From the part $a \geq b$, there exists a natural number n such that $a = b + n$. Similarly, there exists a natural number m such that $b = a + m$. Combining those two equalities leads to $a = b + n = (a + m) + n = a + (m + n)$. By cancellation law (Proposition 2.2.6), we can conclude that $0 = m + n$. According to Corollary 2.2.9, this leads to $m = n = 0$. Therefore, both m and n are null, meaning that $a = b + 0 = b$.
- (d) Preservation of order: $a \geq b$ iff $a + c \geq b + c$. First, let's prove that $a + c \geq b + c \implies a \geq b$. If $a + c \geq b + c$, there exists a natural number n such that $a + c = b + c + n$. By cancellation law (Proposition 2.2.6)², we conclude that $a = b + n$, i.e. $a \geq b$, thus demonstrating the first implication. Conversely, let's suppose that $a \geq b$. There exists a natural number m such that $a = b + m$. Therefore, $a + c = b + m + c$ for any natural number c . Still by associativity and commutativity, we can rewrite this as $a + c = (b + c) + m$, i.e. $a + c \geq b + c$.
- (e) $a < b$ iff $a ++ \leq b$. First, let's prove that $a ++ \leq b \implies a < b$. By definition of ordering, there exists a natural number n such that $b = (a++) + n$. By definition of addition, we can re-write: $b = (a++ + n)++$. Then, by commutativity and yet again by definition of addition, $b = (n + a++)++ = (n++) + (a++)$. Thus, there exists a natural number $n++$ such that $b = n++ + a$, which means that $b \geq a$. But we still have to prove that $a \neq b$. Let's suppose that $a = b$: in this case, by cancellation law, we would have $n++ = 0$, which is impossible according to Axiom 2.3 (0 is not the successor of any natural number). Thus, $a \neq b$ et $b \geq a$: we have showed that $a < b$.
Conversely, let's prove that $a < b \implies a ++ \leq b$. Starting from that strict inequality, there exists a *positive*³ natural number n such that $b = a + n$. By Lemma 2.2.10, since n is positive, it has one unique antecessor m , so that n can be written $m++$. Thus, $b = a + (m++) = (a + m)++ = (m + a)++ = m + (a++) = (a++) + m$. And, m being a natural number, this corresponds to the statement $b \geq a$.
- (f) $a < b$ iff $b = a + d$ for some positive number d . First, let's prove the first implication, $a < b \implies b = a + d$ with $d \neq 0$. Since $a < b$, we have in particular $a \leq b$, and

¹This is a trivial induction from the definition of addition.

²And also associativity and commutativity that we do not detail explicitly here.

³We make use here of the statement (f) demonstrated below. There is no circularity here, since proving (f) will not make use of (e).

there exists a natural number d such that $b = a + d$. For the sake of contradiction, let's suppose that $d = 0$. We would have $b = a$, which would contradict the condition $a \neq b$ of the strict inequality. Thus, d is a positive number, which demonstrates the left-to-right implication.

Conversely, let's suppose that $b = a + d$, with $d \neq 0$. This expression gives immediately $a \leq b$. But if $a = b$, by cancellation law, this would lead to $0 = d$, a contradiction with the fact that d is a positive number. Thus, $a \neq b$ and $a \leq b$, which demonstrates $a < b$.

EXERCISE 2.2.4. — *Demonstrate three lemmas used to prove to trichotomy or order for natural numbers.*

- (a) Show that we have $0 \leq b$ for any natural number b . This is obvious since, by definition of addition, $0 + b = b$ for any natural number b . This is precisely the definition of $0 \leq b$.
- (b) Show that if $a > b$, then $a++ > b$. If $a > b$, then $a = b + d$, d being a positive natural number. Let's recall that $a++ = a + 1$. Thus, $a++ = a + 1 = b + d + 1 = b + (d + 1)$ by associativity of addition. Furthermore, $d+1$ is a positive natural number (by Proposition 2.2.8). Thus, $a++ > b$.
- (c) Show that if $a = b$, then $a++ > b$. Once again, let's use the fact that $a++ = a + 1$. Thus, $a++ = a + 1 = b + 1$, and 1 is a positive natural number. This is the definition of $a++ > b$.