Propositions of solutions for *Analysis I* by Terence Tao

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Contents

1. Introduction

No exercises in this chapter.

2. Starting at the beginning: the natural numbers

Exercise 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a + b) + 0 = a + (b + 0)$. The left hand side is equal to $(a + b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b + 0)$ part, we get $a + (b + 0) = a + b$. Both sides are equal to $a + b$, and the base case is thus done.

- Now let's suppose inductively that $(a + b) + c = a + (b + c)$: we have to prove that $(a + b) + c{+\!+} = a + (b + c{+\!+})$. Using Lemma 2.2.3 on the right hand side leads to $a + (b + c){+\!+}$. Now consider the left hand side. Using still the same lemma, we get $(a + b) + c{+\!+} = ((a + b) + c){+\!+}$. By the inductive hypothesis, this is also equal to $(a + (b + c)){+\!+}$. And, using the lemma 2.2.3 again, this also leads to $a + b + c{+\!+}$. Therefore, both sides are equal to $a + b + c{+\!+}$, and we have closed the induction.

Exercise 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b{+\!+} = a$.*

Let's use induction on a.

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0{+\!+} = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b{+\!+} = 1$. Then, we would have $b{+\!+} = 0{+\!+}$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.

- Let's suppose inductively that there is exactly one natural number b such that $b{+\!+} = a$. We have to prove that there is exactly one natural number b' such that $b'{+\!+} = a{+\!+}$. By the induction hypothesis, and taking $b' = b{+\!+}$, we have $b'{+\!+} = (b{+\!+}){+\!+} = a{+\!+}$. So there exists a solution, with $b' = b{+\!+} = a$. Uniqueness is given by Axiom 2.4.: if $b'{+\!+} = a{+\!+}$, then we necessarily have $b' = a$.

Exercise 2.2.3. — *Let a, b, c be natural numbers. Prove the following properties of order for natural numbers:*

(a) Reflexivity: $a \geqslant a$. This is true since $a = 0 + a$ by Definition 2.2.1. By commutativity of addition, we can also write $a = a + 0$. So there is indeed a natural number n (with $n = 0$) such that $a = a + n$, i.e. $a \geqslant a$.

(b) Transitivity: if $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$ according to Definition 2.2.11. A similar consideration for the part $b \geqslant c$ leads to $b = c + m$, m being a natural number. Combining together those two equalities, we can write $a = b + n = (c + m) + n = c + (m + n)$ by associativity (see Exercise 2.2.1). Then, $n + m$ being a natural number[1], the transitivity is demonstrated.

[1] This is a trivial induction from the definition of addition.

(c) Anti-symmetry: if $a \geqslant b$ and $b \geqslant a$, then $a = b$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$. Similarly, there exists a natural number m such that $b = a + m$. Combining those two equalities leads to $a = b + n = (a + m) + n = a + (m + n)$. By cancellation law (Proposition 2.2.6), we can conclude that $0 = m + n$. According to Corollary 2.2.9, this leads to $m = n = 0$. Therefore, both m and n are null, meaning that $a = b + 0 = b$.

(d) Preservation of order: $a \geqslant b$ iff $a+c \geqslant b+c$. First, let's prove that $a+c \geqslant b+c \Longrightarrow a \geqslant b$. If $a+c \geqslant b+c$, there exists a natural number n such that $a+c = b+c+n$. By cancellation law (Proposition 2.2.6)[2], we conclude that $a = b+n$, i.e. $a \geqslant b$, thus demonstrating the first implication. Conversely, let's suppose that $a \geqslant b$. There exists a natural number m such that $a = b + m$. Therefore, $a + c = b + m + c$ for any natural number c. Still by associativity and commutativity, we can rewrite this as $a + c = (b + c) + m$, i.e. $a + c \geqslant b + c$.

(e) $a < b$ iff $a{+\!+} \leqslant b$. First, let's prove that $a{+\!+} \leqslant b \Longrightarrow a < b$. By definition of ordering, there exists a natural number n such that $b = (a{+\!+}) + n$. By definition of addition, we can re-write: $b = (a{+\!+} + n){+\!+}$. Then, by commutativity and yet again by definition of addition, $b = (n + a{+\!+}){+\!+} = (n{+\!+}) + (a{+\!+})$. Thus, there exists a natural number $n{+\!+}$ such that $b = n{+\!+} + a$, which means that $b \geqslant a$. But we still have to prove that $a \neq b$. Let's suppose that $a = b$: in this case, by cancellation law, we would have $n{+\!+} = 0$, which is impossible according to Axiom 2.3 (0 is not the successor of any natural number). Thus, $a \neq b$ et $b \geqslant a$: we have showed that $a < b$.

Conversely, let's prove that $a < b \Longrightarrow a{+\!+} \leqslant b$. Starting from that strict inequality, there exists a *positive*[3] natural number n such that $b = a + n$. By Lemma 2.2.10, since n is positive, it has one unique antecessor m, so that n can be written $m{+\!+}$. Thus, $b = a + (m{+\!+}) = (a + m){+\!+} = (m + a){+\!+} = m + (a{+\!+}) = (a{+\!+}) + m$. And, m being a natural number, this corresponds to the statement $b \geqslant a$.

(f) $a < b$ iff $b = a + d$ for some positive number d. First, let's prove the first implication, $a < b \Longrightarrow b = a + d$ with $d \neq 0$. Since $a < b$, we have in particular $a \leqslant b$, and there exists a natural number d such that $b = a + d$. For the sake of contradiction, let's suppose that $d = 0$. We would have $b = a$, which would contradict the condition $a \neq b$ of the strict inequality. Thus, d is a positive number, which demonstrates the left-to-right implication.

Conversely, let's suppose that $b = a + d$, with $d \neq 0$. This expression gives immediately $a \leqslant b$. But if $a = b$, by cancellation law, this would lead to $0 = d$, a contradiction with the fact that d is a positive number. Thus, $a \neq b$ and $a \leqslant b$, which demonstrates $a < b$.

Exercise 2.2.4. — *Demonstrate three lemmas used to prove the trichotomy of order for natural numbers.*

(a) Show that we have $0 \leqslant b$ for any natural number b. This is obvious since, by definition of addition, $0 + b = b$ for any natural number b. This is precisely the definition of $0 \leqslant b$.

[2] And also associativity and commutativity that we do not detail explicitly here.

[3] We make use here of the statement *(f)* demonstrated below. There is no circularity here, since proving *(f)* will not make use of *(e)*.

(b) Show that if $a > b$, then $a{+\!+} > b$. If $a > b$, then $a = b + d$, d being a positive natural number. Let's recall that $a{+\!+} = a + 1$. Thus, $a{+\!+} = a + 1 = b + d + 1 = b + (d + 1)$ by associativity of addition. Furthermore, $d+1$ is a positive natural number (by Proposition 2.2.8). Thus, $a{+\!+} > b$.

(c) Show that if $a = b$, then $a{+\!+} > b$. Once again, let's use the fact that $a{+\!+} = a + 1$. Thus, $a{+\!+} = a + 1 = b + 1$, and 1 is a positive natural number. This is the definition of $a{+\!+} > b$.

Exercise 2.2.5. — *Prove the strong principle of induction, formulated as follows: Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geqslant m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leqslant m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geqslant m_0$.*

First let's introduce a small lemma (similar to Proposition 2.2.12*(e)*).

Lemma. *For any natural number a and b, $a < b{+\!+}$ iff $a \leqslant b$.*

Proof. If $a < b{+\!+}$, then $b{+\!+} = a + n$ for a given positive natural n. By Lemma 2.2.10, there exists one natural number m such as $n = m{+\!+}$. Thus $b{+\!+} = a + m{+\!+}$, which can be rewritten $b{+\!+} = (a + m){+\!+}$ by Lemma 2.2.3[4]. By Axiom 2.4., this is equivalent to $b = a + n$, which can also be written $a \leqslant b$.

Conversely, if $a \leqslant b$, there exists a natural number m such as $b = a + m$. Thus, $b{+\!+} = (a + m){+\!+} = a + (m{+\!+})$ by Definition of addition (2.2.1). And, $m{+\!+}$ being a positive number, this means that $b > a$ according to Proposition 2.2.12*(f)*. □

Now we can prove the main proposition. Let $Q(n)$ be the property "$P(m)$ is true for all m such that $m_0 \leqslant m < n$". Let's induct on n.

- (Although this is not necessary,) we could consider two types of base cases. If $n < m_0$, $Q(n)$ is the proposition "$P(m)$ is true for all m such that $m_0 \leqslant m < n$", but there is no such natural number m. Thus, $Q(n)$ is vacuously true. If $n = m_0$, $P(m_0)$ is true by hypothesis, thus $Q(m_0)$ is also true.

- Now let's suppose inductively that $Q(n)$ is true, and show that $Q(n{+\!+})$ is also true. If $Q(n)$ is true, $P(m)$ is true for all m such that $m_0 \leqslant m < n$. By hypothesis, this implies that $P(n)$ is true. Thus, $P(m)$ is true for any natural number m such that $m_0 \leqslant m \leqslant n$, i.e. such that $m_0 \leqslant m < n{+\!+}$ according to the lemma introduced above. This is precisely $Q(n{+\!+})$, and this closes the induction.

Thus, $Q(n)$ is true for all natural numbers n, which means in particular that $P(m)$ is true for any natural number $m \geqslant m_0$. This demonstrates the principle of strong induction.

Exercise 2.2.6. — *Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+\!+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leqslant n$; this is known as the* principle of backwards induction.

[4] We could also rewrite $b + 1 = a + m + 1$ and then use the cancellation law.

Terence Tao suggests to use induction on n. So let $Q(n)$ be the following property: "if $P(n)$ is true, then $P(m)$ is true for all $m \leqslant n$. The goal is to prove $Q(n)$ for all natural numbers n.

- Base case $n = 0$: here, $Q(n)$ means that if $P(0)$ is true, then $P(m)$ is true for any $m \leqslant 0$. By Definition 2.2.11, if $m \leqslant 0$, there exists a natural number d such that $0 = m + d$. But, by Corollary 2.2.9, this implies that both $m = 0$ and $d = 0$. Thus, the only number m such that $m \leqslant 0$ is 0 itself. Therefore, $Q(0)$ is simply the tautology "if $P(0)$ is true, then $P(0)$ is true"— a statement that we can safely accept. The base case is the, demonstrated.

- Let's suppose inductively that $Q(n)$ is true: we must show that $Q(n{+}{+})$ is also true. If $P(n{+}{+})$ is true, then by definition of P, $P(n)$ is also true. Then, by induction hypothesis, $P(m)$ is true for all $m \leqslant n$. We have showed that $P(n{+}{+})$ implies $P(m)$ for all $m \leqslant n{+}{+}$[5], which is precisely $Q(n{+}{+})$. This closes the induction.

Exercise 2.3.1. — *Show that multiplication is commutative, i.e., if n and m are natural numbers, show that $n \times m = m \times n$.*

We will use an induction of n while keeping m fixed. However, this is not a trivial result, and even the base case is not straightforward. We will first introduce some lemmas.

Lemma. *For any natural number n, $n \times 0 = 0$.*

Proof. Let's induct on n. For the base case $n = 0$, we know by Definition 2.3.1 of multiplication that $0 \times 0 = 0$, since $0 \times m = 0$ for any natural number m.

Now let's suppose that $n \times 0 = 0$. Thus, $n{+}{+} \times 0 = (n \times 0) + 0$ by Definition 2.3.1. But by induction hypothesis, $n \times 0 = 0$, so that $n{+}{+} \times 0 = 0 + 0 = 0$. This closes the induction. □

Lemma. *For all natural numbers m and n, we have $m \times n{+}{+} = (m \times n) + m$.*

Proof. Let's induct on m. The base case $m = 0$ is easy to prove: $0 \times n{+}{+} = 0$ by Definition 2.3.1 of multiplication, and $(0 \times n) + 0 = 0$.

Now suppose inductively that $m \times n{+}{+} = (m \times n) + m$, and we must show that

$$m{+}{+} \times n{+}{+} = (m{+}{+} \times n) + m{+}{+} \tag{2.1}$$

We begin by the left hand side: by Definition 2.3.1, $m{+}{+} \times n{+}{+} = (m \times n{+}{+}) + n{+}{+}$. By induction hypothesis, this is equal to $(m \times n) + m + n{+}{+}$.

Then, apply the definition of multiplication to the right hand side: $(m{+}{+} \times n) + m{+}{+} = (m \times n) + n + m{+}{+}$. The Lemma 2.2.3 and the commutativity of addition leads to $(m \times n) + n + m{+}{+} = (m \times n) + (n + m){+}{+} = (m \times n) + (m + n){+}{+} = (m \times n) + m + n{+}{+}$, which is equal to the left hand side.

Thus, both sides of equation (2.1) are equal, and we can close the induction. □

Now it is easier to prove the main result ($n \times m = m \times n$), by an induction on n.

[5] Actually, we use here yet another lemma, similar to the one introduced for the previous exercise. We use the fact that $m \leqslant n{+}{+}$ is equivalent to $m = n{+}{+}$ or $m \leqslant n$, which is easy to prove, but is not part of the "standard" results presented in the textbook.

- Base case $n = 0$: we already know by Definition 2.3.1 that $0 \times m = 0$. The first lemma introduced in this exercise also provides $m \times 0 = 0$. Thus, the base case is proved, since $0 \times m = m \times 0 \ (= 0)$.

- Now we suppose inductively that $n \times m = m \times n$, and we must prove that:

$$n\text{++} \times m = m \times n\text{++} \tag{2.2}$$

  By Definition 2.3.1 of multiplication, the left hand side is equal to $(n \times m) + m$.

  Using the lemma introduced above, the right hand side is equal to $(m \times n) + m$. By induction hypothesis, this is also equal to $(n \times m) + m$, which closes the induction.

Exercise 2.3.2. — *Show that positive natural numbers have no zero divisors, i.e. that $nm = 0$ iff $n = 0$ or $m = 0$. In particular, if n and m are both positive, then nm is also positive.*

We will prove the second statement first. Suppose, for the sake of contradiction, that $nm = 0$ and that both n and m are positive numbers. Since they are positive, by Lemma 2.2.10, there exists two (unique) natural numbers a and b such that $n = a\text{++}$ and $m = b\text{++}$. Thus, the hypothesis $nm = 0$ can also be written $(a\text{++}) \times (b\text{++}) = 0$. But, by Definition 2.3.1 of multiplication, $(a\text{++}) \times (b\text{++}) = (a \times b\text{++}) + b\text{++}$. Thus, we should have $(a \times b\text{++}) + b\text{++} = 0$. By Corollary 2.2.9, this implies that both $(a \times b\text{++}) = 0$ and $b\text{++} = 0$, which is impossible since zero is the successor of no natural number (Axiom 2.3).

Thus, we have proved that if n and m are both positive, then nm is also positive. The main statement can now be proved more easily.

- The right-to-left implication is straightforward: if $n = 0$, then by Definition of multiplication, $n \times m = 0 \times m = 0$. Since multiplication is commutative, we have the same result if $m = 0$.

- The left-to-right implication is exactly the contrapositive of the statement we have just proved above.

Exercise 2.3.3. — *Show that multiplication is associative, i.e., for any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.*

We will induct on c while keeping a and b fixed.

- Base case: for $c = 0$, we must prove that $(a \times b) \times 0 = a \times (b \times 0)$. The left hand side is equal to 0 by definition (and commutativity) of multiplication[6]. The right hand side is equal to $a0$, which is also 0. Both sides are null, and the base case is proved.

- Suppose inductively that $(a \times b) \times c = a \times (b \times c)$, and let's prove that $(a \times b) \times c\text{++} = a \times (b \times c\text{++})$. By definition (and commutativity) of multiplication, the left hand side is equal to $(a \times b) \times c + (a \times b)$. The right hand side is equal to $a \times (b \times c + b)$, and by distributive law (i.e., Proposition 2.3.4), this is also $a \times (b \times c) + a \times b$. But then, by inductive hypothesis, this can be rewritten $(a \times b) \times c + a \times b$, which is equal to the left hand side. The induction is closed.

[6] Actually, we use the second lemma introduced for the resolution of Exercise 2.3.1.

Exercise 2.3.4. — *Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.*

By distribution law (i.e., Proposition 2.3.4) and commutativity of multiplication, we have:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) = (a+b)a + (a+b)b \\ &= a \times a + b \times a + a \times b + b \times b \\ &= a^2 + a \times b + a \times b + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

(For the last step, we recall that, by Definition 2.3.1, $2 \times m = m + m$ for any natural number m.)

Exercise 2.3.5. — *Euclidean algorithm. Let n be a natural number, and let q be a positive number. Prove that there exists natural numbers m, r such that $0 \leqslant r < q$ and $n = mq + r$.*

We will induct on n while remaining q fixed.

- Base case: if $n = 0$, there exists an obvious solution, namely $m = 0$ and $r = 0$.
- Suppose inductively that there exists m, r such that $n = mq + r$ with $0 \leqslant r < q$, and let's prove that there exists m', r' such that $n + 1 = m'q + r'$, with $0 \leqslant r' < q$.

  By the induction hypothesis, we have $n+1 = mq+r+1$. Since $r < q$, we have $r+1 \leqslant q$ (this is Proposition 2.2.12). Thus, we have two cases here:

  1. If $r + 1 < q$, then $n + 1 = mq + (r + 1)$, with $0 \leqslant r + 1 < q$, so that choosing $m' = m$ and $r' = r + 1$ is convenient.
  2. If $r + 1 = q$, then $n + 1 = mq + q = (m + 1)q$ according to the distributive law (Proposition 2.3.4). Thus, choosing $m' = m + 1$ and $r' = 0$ is convenient.

This closes the induction.

3. Set theory

Exercise 3.1.2. — *Using only Definition 3.1.4, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets $\varnothing$, $\{\varnothing\}$, $\{\{\varnothing\}\}$, and $\{\varnothing, \{\varnothing\}\}$ are all distinct (i.e., no two of them are equal to each other).*

As a general reminder, we recall that sets are objects (Axiom 3.1) and the empty set $\varnothing$ is such that no object is an element of $\varnothing$, thus $\varnothing \notin \varnothing$.

1. First let's show that $\varnothing$ is different from all other sets. $\varnothing$ is an element of $\{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$, and $\{\varnothing\}$ is an element of $\{\{\varnothing\}\}$. But none of those two objects are elements of $\varnothing$ (by Axiom 3.2), thus $\varnothing$ is different from all three other sets.

2. Then let's show that $\{\varnothing\} \neq \{\{\varnothing\}\}$. By Axiom 3.3, the singleton $\{\varnothing\}$ is such that $x \in \{\varnothing\} \iff x = \varnothing$. Similarly, the singleton $\{\{\varnothing\}\}$ is such that $x \in \{\{\varnothing\}\} \iff x = \{\varnothing\}$. But we already know that $\varnothing \neq \{\varnothing\}$ so there exists an object, $\varnothing$, which is a element of $\{\varnothing\}$ but not an element of $\{\{\varnothing\}\}$. Those sets are not equal.

3. Now let's show that $\{\varnothing\} \neq \{\varnothing, \{\varnothing\}\}$. By Axiom 3.3, the pair $\{\varnothing, \{\varnothing\}\}$ is such that x is an element of this set iff $x = \varnothing$ or $x = \{\varnothing\}$. Thus, $\{\varnothing\}$ is an element of $\{\varnothing, \{\varnothing\}\}$, but is not an element of $\{\varnothing\}$ (if it was, we should have $\varnothing = \{\varnothing\}$, which would be a contradiction with the first point of this proof). Those two sets are thus different.

4. Finally, we also have $\{\{\varnothing\}\} \neq \{\varnothing, \{\varnothing\}\}$. Indeed, we have $\varnothing \in \{\varnothing, \{\varnothing\}\}$ by Axiom 3.3. However, $\varnothing \in \{\{\varnothing\}\} \iff \varnothing = \{\varnothing\}$ by definition of a singleton, and we know this latest statement is false by the first point of this proof. Those two sets are also different.

Exercise 3.1.3. — *Prove the remaining claims in Lemma 3.1.13.*

Those claims are the following:

1. $\{a, b\} = \{a\} \cup \{b\}$. By Axiom 3.3, the pair $\{a, b\}$ is such that $x \in \{a, b\} \iff x = a$ or $x = b$. Let's consider three cases:

   - if $x = a$, $x \in \{a\}$ by Axiom 3.3, thus $x \in \{a\} \cup \{b\}$ by Axiom 3.4
   - if $x = b$, $x \in \{b\}$ by Axiom 3.3, thus $x \in \{a\} \cup \{b\}$ by Axiom 3.4
   - if $x \neq a$ and $x \neq b$, $x \notin \{a\}$ and $x \notin \{b\}$ by Axiom 3.3, so that $x \notin \{a\} \cup \{b\}$

   Thus, $\{a, b\}$ and $\{a\} \cup \{b\}$ have the same elements, and are equal.

2. $A \cup B = B \cup A$ for all sets A and B. Indeed, $x \in A \cup B \iff x \in A$ or $x \in B$. If $x \in A$, then $x \in B \cup A$ by Axiom 3.4. A similar argument holds if $x \in B$. Thus, in both cases, $x \in B \cup A$. We can show in a similar fashion that any element of $B \cup A$ is in $A \cup B$.

3. $A \cup \varnothing = \varnothing \cup A = A$. Since we've just showed that union is commutative, proving $A \cup \varnothing = A$ is sufficient. If $x \in A$, then $x \in A \cup \varnothing$. The converse is also true: if $x \in A \cup \varnothing$, then $x \in A$ or $x \in \varnothing$. But there is no element in $\varnothing$, so that we have necessarily $x \in A$. Thus, $A \cup \varnothing$ and A have the same elements: they are equal.

EXERCISE 3.1.4. — *Prove the remaining claims from Proposition 3.1.18.*

Let A, B, C be sets. Those claims are the following:

1. If $A \subseteq B$ and $B \subseteq A$, then $B = A$. According to Definition 3.1.4, two sets A and B are equal iff every element of A is an element of B, and vice versa. This is precisely the present claim.

2. If $A \subsetneq B$ and $B \subsetneq C$, then $A \subsetneq C$. Let x be an element of A. Since $A \subsetneq B$, x is also an element of B. And since $B \subsetneq C$, x is also an element of C. This holds for any x in A, and thus it demonstrates that $A \subset C$. Furthermore, since $A \subsetneq B$, there exists an element $y \in B$ which is not an element of A. As $B \subsetneq C$, y is also an element of C. Thus we have y, an element of C which is not in A. Combined to the previous result $A \subset C$, this demonstrates $A \subsetneq C$.

EXERCISE 3.1.5. — *Let A, B be sets. Show that the three statements $A \subseteq B$, $A \cup B = B$ and $A \cap B = A$ are logically equivalent (i.e., any one of them implies the other two).*

1. First, we prove that $A \subseteq B \implies A \cup B = B$. The first inclusion $B \subseteq A \cup B$ is trivial, since any element of a set B is always either in A or B. For the converse inclusion, let x be an element of $A \cup B$, and let's prove that $x \in B$. By Axiom 3.4, we have $x \in A$ or $x \in B$. If $x \in B$, the result holds. If $x \in A$, then we also have $x \in B$ since $A \subseteq B$. Thus, any element of $A \cup B$ is an element of B, which demonstrates the equality $A \cup B = B$.

2. Then, we prove that $A \cup B = B \implies A \cap B = A$. The first inclusion is trivial: if $x \in A \cap B$, then we always have $x \in A$. Now let's prove the converse inclusion: let x be an element of A; we must show that $x \in A \cap B$. If $x \in A$, then $x \in A \cup B$. But, by hypothesis, $A \cup B = B$, thus $x \in B$. So, $x \in A$ and $x \in B$, i.e. $x \in A \cap B$. This demonstrates the implication.

3. Finally, we prove that $A \cap B = A \implies A \subseteq B$. Let $x \in A$. Since $A \cap B = A$, we have $x \in A \cap B$. It follows that $x \in B$. We have proved that any element $x \in A$ is also an element of B, i.e. $A \subseteq B$.

EXERCISE 3.1.8. — *Let A, B be sets. Prove the absorption laws $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$.*

1. The first inclusion $A \cap (A \cup B) \subseteq A$ is trivial: if $x \in A \cap (A \cup B)$ then in particular $x \in A$ by Definition 3.1.23 of an intersection[7]. Thus, we have $A \cap (A \cup B) \subseteq A$.

   For the converse inclusion, let x be an element of A. Then by definition $x \in A$, and we have also $x \in A \cup B$ since $x \in A$. Thus, $x \in A \cap (A \cup B)$, which proves the converse inclusion.

   Consequently, $A = A \cap (A \cup B)$.

2. First we show that $A \cup (A \cap B) \subseteq A$. Let $x \in A \cup (A \cap B)$. By Definition of an union, we have either $x \in A$, or $x \in A \cap B$. In both cases[8], we have $x \in A$, so that the inclusion is proved.

[7] This intersection is not empty since A and $A \cup B$ are not disjoint.

[8] If A and B are disjoint, then the first case $x \in A$ necessarily holds, since $x \in A \cup B$ is impossible.

Conversely, let $x \in A$. Then in particular, we have $x \in A \cup (A \cap B)$ by Definition of an union, because $x \in A$. Thus, $x \in A \cup (A \cap B)$.

We have proved that $A \cup (A \cap B) = A$.

EXERCISE 3.1.9. — *Let A, B, X be sets such that $A \cup B = X$ and $A \cap B = \varnothing$. Show that $A = X \backslash B$ and $B = X \backslash A$.*

The two sets A and B play a symmetrical role here, so that proving one of these two assertions is sufficient. For instance, we prove that $A = X \backslash B$.

- Let x be an element of A. Since $x \in A$, we also have $x \in A \cup B$ by definition of an union. But $A \cup B = X$, and then $x \in X$. On the other hand, we cannot have $x \in B$, because $x \in A$ and the sets A, B are disjoint. Thus, $x \in X$ and $x \notin B$, which means that $x \in X \backslash B$. We have proved that $A \subseteq X \backslash B$.
- Conversely, let x be an element of $X \backslash B$. By definition, this means that $x \in X$, i.e. $x \in A \cup B$, and $x \notin B$. Since $x \in A \cup B$, we have either $x \in A$ or $x \in B$, but we know that the latter is impossible. Thus, we have necessarily $x \in A$. We have proved that $X \backslash B \subseteq A$.
- We can conclude that $X \backslash B = A$.

EXERCISE 3.1.11. — *Prove that the axiom of replacement (Axiom 3.6) implies the axiom of specification (Axiom 3.5).*

Let's recall the axiom of replacement. Let A be a set. For every $x \in A$, and for every (abstract) object y, let $P(x, y)$ be a statement pertaining to both x and y, such that for any $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$, such that for any object z,

$$z \in \{y : P(x, y) \text{ is true for some } x \in A\} \iff P(x, z) \text{ is true for some } x \in A$$

Now, let A be a set, x an element of A, and y an object. We accept the axiom of replacement, and show that it implies the axiom of specification.

Let $Q(x, y)$ be the property "$x = y$ and $P(x)$". According to the axiom of replacement, there exists a set $\{y : Q(x, y) \text{ is true for some } x \in A\}$ such that:

$$\begin{aligned} & z \in \{y : Q(x, y) \text{ is true for some } x \in A\} \\ \iff & Q(x, z) \text{ is true for some } x \in A \\ \iff & x = z \text{ and } P(x) \text{ is true for some } x \in A \\ \iff & x = z \text{ and } P(z) \text{ is true for some } x \in A \text{ (by axiom of substitution)} \\ \iff & z \in A \text{ and } P(z) \text{ is true} \end{aligned}$$

Thus, we have proved the existence of a set (the set $\{y : Q(x, y) \text{ is true for some } x \in A\}$) satisfying the axiom of specification: z belongs to this set iff $z \in A$ and $P(z)$ is true.

EXERCISE 3.3.1. — *Show that the definition of equality in Definition 3.3.7 is reflexive, symmetric and transitive. Also verify the substitution property: if $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ are functions such that $f_1 f_2$ and $g_1 = g_2$, then $g_1 \circ f_1 = g_2 \circ f_2$.*

1. Definition 3.3.7 says that two functions f and g are equal if they have same domain X and range Y, and if, for all $x \in X$, $f(x) = g(x)$. This definition of equality is obviously reflexive, symmetric and transitive if we assume that the objects in the domain X and the range Y verify themselves the axioms of equality.

2. Since $f_1 = f_2$, they have same domain X and same range Y. This is also the case for g_1 and g_2, with domain Y and range Z. Thus, $g_1 \circ f_1$ has domain X and range Z, and so has $g_2 \circ f_2$. Furthermore, we have, for all $x \in X$:

$$\begin{aligned} g_2 \circ f_2(x) &= g_2 \circ f_1(x) \text{ (since } f_1 = f_2) \\ &= g_1 \circ f_1(x) \text{ (since } g_1 = g_2) \end{aligned}$$

   which closes the demonstration.

Exercise 3.3.2. — *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$. Similarly, show that if f and g are both surjective, then so is $g \circ f$.*

First let's note that $g \circ f : X \to Z$.

1. Suppose that f and g are both injective, and let $x, x' \in X$. We have successively :

$$\begin{array}{rcl} g \circ f(x) &=& g \circ f(x') \\ g(f(x)) &=& g(f(x')) \\ f(x) &=& f(x') \text{ because } g \text{ is injective} \\ x &=& x' \text{ because } f \text{ is injective} \end{array}$$

   We have showed that $g \circ f(x) = g \circ f(x') \to x = x'$ for all $x, x' \in X$, i.e. that $g \circ f$ is injective.

2. Suppose that f and g are both surjective, and let be $z \in Z$. Since g is surjective, there exists $y \in Y$ such that $z = g(y)$. And since f is surjective, there exists $x \in X$ such that $y = f(x)$. Thus, combining those two results, there exists $x \in X$ such that $z = g(f(x))$. This means precisely that $g \circ f$ is surjective.

Exercise 3.3.3. — *When is the empty function injective? surjective? bijective?*

Let f be the empty function, i.e. $f : \varnothing \to Y$ for a certain range Y.

1. f is injective iff $x \neq x' \to f(x) \neq f(x')$. This can be considered as vacuously true since there are no such x and x'. f can be considered as always injective, for any range Y.

2. f is surjective iff for any $y \in Y$, there exists $x \in \varnothing$ such that $y = f(x)$. We can clearly see that this assertion is false if $Y \neq \varnothing$, since any $y \in Y$ will have no antecedent in $\varnothing$. Conversely, if $Y = \varnothing$, the assertion is vacuously true, since there is no element in Y. Thus, f is surjective iff $Y = \varnothing$.

3. Since f is always injective, and is surjective iff $Y = \varnothing$, it is clear that f is bijective iff $Y = \varnothing$.

Exercise 3.3.4. — *Let $f : X \to Y$, $\tilde{f} : X \to Y$, $g : Y \to Z$, $\tilde{g} : Y \to Z$ be functions. Show that if $g \circ f = g \circ \tilde{f}$ and g is injective, then $f = \tilde{f}$. Is this statement true if g is not injective? Also, show that if $g \circ f = \tilde{g} \circ f$ and f is surjective, then $g = \tilde{g}$. Is this statement true if f is not surjective?*

This exercise introduces some cancellation laws for composition.

1. First, note that f and $\tilde{f}$ have same domain and range, which is the first condition for two functions to be equal (by Definition 3.3.7). Then, suppose that $g \circ f = g \circ \tilde{f}$ and g is injective. For the sake of contradiction, suppose that there exists $x \in X$ such that $f(x) \neq \tilde{f}(x)$. Since g is injective, we would thus have $g(f(x)) \neq g(\tilde{f}(x))$, which would be a contradiction to the hypothesis $g \circ f = g \circ \tilde{f}$. Thus, there is no x such that $f(x) = \tilde{f}(x)$, or in other words, $f = \tilde{f}$.

   This property is false if g is not injective. As a counterexample, one can think of $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x$, $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with $\tilde{f}(x) = -x$, and $g : \mathbb{R} \to \mathbb{R}_+$ with $g(x) = |x|$.

2. As previously, first note that g and $\tilde{g}$ have same domain and range. Let be $y, y' \in Y$. Since f is surjective, there exist $x, x' \in X$ such that $y = f(x)$ and $y' = f(x')$ respectively. Since $g \circ f = g \circ \tilde{f}$, we have $g(f(x)) = g(f(x'))$, i.e. $g(y) = g(y')$. We have showed that, for any $y, y' \in Y$, we have $g(y) = g(y')$, which means that $g = \tilde{g}$.

   This statement is false if f is not surjective. For instance, let f be a constant function, e.g. $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 1$ for all x. Let $g, \tilde{g} : \mathbb{R} \to \mathbb{R}$ with $g(x) = 0$ and $\tilde{g}(x) = -x + 1$. We have $g(1) = \tilde{g}(1)$, i.e. $g(f(x)) = \tilde{g}(x)$ for all $x \in X$, but we obviously do not have $g = \tilde{g}$.

Exercise 3.3.5. — *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if $g \circ f$ is injective, then f must be injective. Is it true that g must also be injective? Show that if $g \circ f$ is surjective, then g must be surjective. Is it true that f must be surjective?*

1. If $g \circ f$ is injective, then for any given objects $x, x' \in X$, we have $g(f(x)) = g(f(x')) \implies x = x'$. For the sake of contradiction, suppose that f is not injective. In this case, there exist two elements $a, a' \in X$ such that $a \neq a'$ and $f(a) = f(a')$. We would thus have $g(f(a)) = g(f(a'))$ (axiom of substitution) and $a \neq a'$, which is incompatible with the hypothesis that $g \circ f$ is injective.

   Thus, $g \circ f$ injective implies that f is injective.

   However, g does not need to be injective. For instance, let's consider $X = \{1, 2\}$ and $Y = Z = \{1, 2, 3\}$. Let's define the function f as the mapping $f(1) = 1$, $f(2) = 2$. Let's define the function g as the mapping $g(1) = 1$, $g(2) = 2$, $g(3) = 2$. Here, f is injective, so is $g \circ f$, but g is not injective.

2. If $g \circ f$ is surjective, then for all $z \in Z$, there exists $x \in X$ such that $z = g(f(x))$. For the sake of contradiction, suppose that g is not surjective: then, there exists $z \in Z$ such that for all $y \in Y$, $z \neq g(y)$. In particular, for all $x \in X$, since $f(x) \in Y$, we would have $g(f(x)) \neq z$, which would be a contradiction with $g \circ f$ surjective.

   However, f does not need to be surjective. For instance, let's consider $X = Y = \{1, 2\}$ and $Z = \{1\}$. Let f be the mapping $f(1) = f(2) = 1$, and g be the mapping $g(1) = g(2) = 1$. Here, $g \circ f$ is surjective, but f is not.

Exercise 3.3.6. — *Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Verify the cancellation laws $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$. Conclude that f^{-1} is also invertible and has f as its inverse.*

Recall that, by definition, for all $y \in Y$, $f^{-1}(y)$ is the only element $x \in X$ such that $f(x) = y$.

1. Let a be an element of X, we thus have $f(a) \in Y$. Let's apply the definition to the element $y = f(a) \in Y$: by definition, $f^{-1}(f(a))$ is the only element $x \in X$ such that $f(x) = f(a)$. Since f is bijective, this implies $x = a$. We thus have proved that $f^{-1}(f(a)) = a$.

2. The proof for $f(f^{-1}(y)) = y$ is similar.

3. To prove that f^{-1} is also invertible, we need to prove that f^{-1} is bijective, i.e. injective and surjective.

   For any given $y \in Y$, since f is bijective, there exists exactly one $x \in X$ such that $y = f(x)$. Similarly, for any given $y' \in Y$, there exists exactly one $x' \in X$ such that $y' = f(x')$. In other words, $f^{-1}(y) = x$ and $f^{-1}(y') = x'$. Suppose that $f^{-1}(y) = f^{-1}(y')$. This can be written $x = x'$, which necessarily implies $f(x) = f(x')$ since f is a function (and by axiom of substitution). And this can also be written $y = y'$. We thus have proved that for any $y, y' \in Y$, $f^{-1}(y) = f^{-1}(y') \Longrightarrow y = y'$. Thus, f^{-1} is injective.

   Furthermore, for any given $x \in X$, let's denote $y = f(x)$. Since f is bijective, this means that $f^{-1}(y) = x$. Thus, any $x \in X$ has a predecessor $y \in Y$ for f^{-1}, i.e. f^{-1} is surjective.

Exercise 3.3.7. — *Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and we have $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

The first point is an immediate consequence of Exercise 3.3.2. We just have to show that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Let be any given element $z \in Z$. Since g is bijective, there exists one single element $y \in Y$ such that $z = g(y)$, i.e. $y = g^{-1}(z)$. And since f is also bijective, there exists exactly one single element $x \in X$ such that $y = f(x)$, i.e. $x = f^{-1}(y) = f^{-1}(g^{-1}(z))$.

Thus, for every $z \in Z$, there exists exactly one $x \in X$ such that $g \circ f(x) = z$, and this element is $f^{-1}(g^{-1}(z))$. This means exactly that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Exercise 3.4.1. — *Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same as the inverse image of V under f; thus the fact that both sets are denoted as f^{-1} will not lead to any inconsistency.*

Since "$f^{-1}(V)$" may refer to two different things here, let's first introduce some notations to avoid any confusion :

- Let F be the *forward image* of V under f^{-1}, i.e. $F = \{f^{-1}(y) \mid y \in V\}$.
- Let I be the *inverse image* of V under f, i.e. $I = \{x \in X \mid f(x) \in V\}$.

In this exercise we must show that $F = I$, so as to ensure that the two definitions of f^{-1} are equivalent. So, we will prove that $F \subseteq I$ and $I \subseteq F$.

1. Let be $x \in F$. Thus, there exists $y \in V$ such that $x = f^{-1}(y)$. By definition, this is equivalent to $f(x) = y$. But since $y \in V$, we can say that $f(x) \in V$. Thus, we have both $x \in X$ (beacuse $F \subseteq X$) and $f(x) \in V$, which means that $x \in I$.

2. Conversely, let be $x \in I$. By definition, this means that $x \in X$ and that $f(x) \in V$, i.e. there exists a certain element $y \in V$ such that $y = f(x) \in V$. This latter statement is equivalent to $x = f^{-1}(y)$. Thus, we have $x \in X$ and $x = f^{-1}(y)$ for a certain $y \in V$, which means that $x \in F$.

Exercise 3.4.2. — *Let $f : X \to Y$ be a function, let S be a subset of X and let U be a subset of Y. What, in general, can one say about $f^{-1}(f(S))$ and S? What about $f(f^{-1}(U))$ and U?*

This exercise gives a first taste of Exercise 3.4.5 below.

1. First consider $f^{-1}(f(S))$ and S.
   - Do we have $f^{-1}(f(S)) \subset S$? Generally, no. As an counterexample, let's consider $f(x) = x^2$ with $X = Y = \mathbb{R}$ and $S = \{0, 2\}$. We have $f^{-1}(f(S)) = f^{-1}(\{0, 4\}) = \{-2, 0, 2\}$. In this set, we have an element, -2, which is not an element of S.
   - Do we have $S \subset f^{-1}(f(S))$? Yes. Let be $x \in S$. Then, by definition, $f(x) \in f(S)$. So, $x \in X$ and is such that $f(x) \in f(S)$: this is precisely the definition of $x \in f^{-1}(f(S))$.
   - Conclusion: generally speaking, S and $f^{-1}(f(S))$ are not equal, but $S \subset f^{-1}(f(S))$.

2. Now consider $f(f^{-1}(U))$ and U.
   - Do we have $U \subset f(f^{-1}(U))$? Generally, no. As a counterexample, let's consider $f(x) = \sqrt{x}$ with $X = \mathbb{R}_+$, $Y = \mathbb{R}$ and $U = [-1, 1]$. We have $f(f^{-1}(U)) = f([0, 1]) = [0, 1]$, which is clearly not a subset of U.
   - Do we have $f(f^{-1}(U)) \subset U$? Yes. Let be $y \in f(f^{-1}(U))$. By definition, there exists $x \in f^{-1}(U)$ such that $y = f(x)$. But if $x \in f^{-1}(U)$, we have $f(x) \in U$. And since $y = f(x)$, this means that $y \in U$.
   - Conclusion: generally speaking, $U \neq f(f^{-1}(U))$, but $f(f^{-1}(U)) \subset U$.

Exercise 3.4.3. — *Let A, B be two subsets of X, and let be $f : X \to Y$. Show that $f(A \cap B) \subseteq f(A) \cap f(B)$, that $f(A) \backslash f(B) \subseteq f(A \backslash B)$, and $f(A \cup B) = f(A) \cup f(B)$. Is it true that, for the first two statements, the $\subseteq$ relation can be improved to $=$?*

Let's prove the three statements successively:

1. If $y \in f(A \cap B)$, then there exists $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$, we have both $x \in A$ and $x \in B$, which implies $y = f(x) \in f(A)$ and $y = f(x) \in B$ respectively. Thus, $y \in f(A) \cap f(B)$, and we have proved that $f(A \cap B) \subseteq f(A) \cap f(B)$.

   However, the converse inclusion is false in general. For instance, let's consider the two sets $A = \{1, 2\}$, $B = \{2, 3\}$ and the (non injective) function f defined as the mapping $f(1) = 1$, $f(2) = 2$, $f(3) = 1$. We have $f(A) = \{1, 2\}$, $f(B) = \{1, 2\}$, thus $f(A) \cap f(B) = \{1, 2\}$. This is not a subset of $f(A \cap B) = f(\{2\}) = \{2\}$.

2. If $y \in f(A)\backslash f(B)$, then there exists $x_0 \in A$ such that $y = f(x_0)$, but we have $f(b) \neq y$ for all $b \in B$. Suppose that $x_0 \in B$: in this case, $f(x_0) \neq y$, a contradiction. Thus, $y = f(x_0)$ with $x_0 \in A\backslash B$, which proves that $f(A)\backslash f(B) \subseteq f(A\backslash B)$.

   However, the converse inclusion is false in general. For instance, let's consider the two sets $A = \{1, 2, 3\}$, $B = \{3\}$ and the function f defined as the mapping $f(1) = 1$, $f(2) = 2$, $f(3) = 1$. We have $f(A\backslash B) = \{1, 2\}$ but $f(A)\backslash f(B) = \{2\}$.

3. If $y \in f(A \cup B)$, then there exists $x \in A \cup B$ such that $y = f(x)$. If $x \in A$, then $f(x) \in f(A)$, which implies $x \in f(A) \cup f(B)$. There is an identical result if $x \in B$. Thus, $f(A \cup B) \subseteq f(A) \cup f(B)$.

   Conversely, if $y \in f(A) \cup f(B)$, then we have either $y \in f(A)$ or $y \in f(B)$ (or both). In the first case, there exists $x \in A$ such that $y = f(x)$. But since $x \in A$, we also have $x \in A \cup B$, so that $y \in f(A \cup B)$. The same result holds if $y \in B$. Thus, in both cases, $y \in f(A \cup B)$.

Exercise 3.4.4. — *Let be $f : X \to Y$ a function, and let A, B be subsets of Y. Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$, that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$, and that $f^{-1}(A\backslash B) = f^{-1}(A)\backslash f^{-1}(B)$.*

We prove only the first statement here; since only very small adjustments are required in its proof to prove the last two ones.

- Let be $x \in f^{-1}(A \cup B)$. By definition, $f(x) \in A \cup B$, so that we have either $f(x) \in A$ or $f(x) \in B$.

  If $f(x) \in A$, then $x \in f^{-1}(A)$ by definition. This implies that $x \in f^{-1}(A) \cup f^{-1}(B)$.

  The same conclusion holds if $f(x) \in B$. Thus, we have demonstrated that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

- For the conserve inclusion, let be $x \in f^{-1}(A) \cup f^{-1}(B)$. We have either $x \in \inf f(A)$ or $x \in f^{-1}(B)$.

  If $x \in f^{-1}(A)$, then $f(x) \in A$, and since $A \subset A \cup B$, we have $f(x) \in A \cup B$. This implies $x \in f^{-1}(A \cup B)$.

  The same conclusion holds if $x \in f^{-1}(B)$. Thus, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

- This proves the equality $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

Exercise 3.4.5. — *Let $f : X \to Y$ be a function. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq Y$ iff f is surjective. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq X$ iff f is injective.*

This exercise is a continuation of Exercise 3.4.2. Let's recall its results, that will reduce the amount of things to be proven here:

- we always have $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq Y$, thus we just have to prove that f is surjective iff $S \subseteq f^{-1}(f(S))$ for every $S \subseteq Y$.

- we always have $S \subseteq f(f^{-1}(S))$ for every $S \subseteq X$, thus we just have to prove that f is injective iff then $f(f^{-1}(S)) \subseteq S$ for every $S \subseteq X$.

Let's prove those two statements.

1. Let f be surjective: let's show that $S \subseteq f(f^{-1}(S))$ for all $S \subseteq Y$. Let S be a subset of Y, and $y \in S$[9]. Since f is surjective, there exists $x \in X$ such that $y = f(x)$. Recall that $y \in S$: this means that$f(x) \in S$, i.e. $x \in f^{-1}(S)$. Thus, $y = f(x) \in f(f^{-1}(S))$. We have proved that, if f is surjective, $y \in S \to y \in f(f^{-1}(S))$, i.e. $S \subseteq f(f^{-1}(S))$.

   Conversely, suppose that $S \subseteq f(f^{-1}(S))$ for every $S \subseteq Y$ and let's show that f is surjective. Let's choose $S = Y$: by hypothesis, we have $Y \subseteq f(f^{-1}(Y))$. Then, let be $y \in Y$. There exists $x \in f^{-1}(Y) \subseteq X$ such that $y = f(x)$. This means precisely that f is surjective.

   The first equivalence is proved.

2. Let f be injective, and $S \subseteq X$. Let be $x \in f^{-1}(f(S))$. Thus, by definition, $f(x) \in f(S)$. This means that there exists $x' \in f(S)$ such that $f(x) = f(x')$. And since f is injective, $x = x' \in S$. Thus, if f is injective, $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$.

   Conversely, suppose that $f^{-1}(f(S)) \subseteq S$ for every $S \subseteq X$. In particular, this is true for any singleton $S = \{x_0\}$, with $x_0 \in S$. In such a case, we obtain $f^{-1}(f(\{x_0\})) = \{x_0\}$. For any element $x \in X$, if $x \neq x_0$, we have $x \notin \{x_0\}$ by definition of a singleton, thus $x \notin f^{-1}(f(\{x_0\}))$, and thus $f(x) \neq f(x_0)$. This means that f is injective.

   The second equivalence is proved.

EXERCISE 3.4.6. — *Prove lemma 3.4.9. (Hint: start with the set $\{0,1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$.)*

First, let's recall the main propositions involved in this exercise:

- **Replacement axiom**. Let A be a set. For any object $x \in A$, and any object y, suppose we have a statement $P(x,y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x,y)$ is true. Then there exists a set $\{y \mid P(x,y) \text{ is true for some } x \in A\}$, such that for any object z,

$$z \in \{y \mid P(x,y) \text{ is true for some } x \in A\} \iff P(x,z) \text{ is true for some } x \in A$$

- **Power set axiom**. Let X and Y be sets. There there exists a set, denoted Y^X, which consists of all the function from X to Y:

$$f \in Y^X \iff (f \text{ is a function from } X \text{ to } Y\}$$

- **Lemma 3.4.9**. Let X be a set. Then the set $\{Y \mid Y \text{ is a subset of X}\}$ is a set.

The aim is to prove this lemma using the two axioms recalled here.

1. Let X be a set, and $Y = \{0,1\}$. Per the power set axiom, $\{0,1\}^X$ is a set, and it contains all the functions $f : X \to \{0,1\}$.

2. Let A be a subset of X. One can define the function $f_A : X \to \{0,1\}$, such that for all $x \in X$, $f(x) = 1$ if $x \in A$, and $f(x) = 0$ otherwise. We can even say more:

[9]If S is empty, the statement is vacuously true, so that we can suppose $S \neq \varnothing$.

- If A is a subset of X, then there exists an element $f \in \{0,1\}^X$ such that $A = f^{-1}(\{1\})$: this is precisely f_A as defined above.
- Conversely, if $f \in \{0,1\}^X$, then $A = f^{-1}(\{1\})$ is by definition a subset of X.

Thus, the two statements "A is a subset of X" and "there exists $f \in \{0,1\}^X$ such that $A = f^{-1}(\{1\})$" are equivalent.

3. Finally, let be $A \subset X$ and $f \in \{0,1\}^X$. Let's define $P(f, A)$ the statement "$A = f^{-1}(\{1\})$". For each f, there is at most one A (in fact, *exactly* one A) such that $P(f, A)$ is true. Thus, per the axiom of replacement, there exists a set:

$$\mathcal{P} = \{A \mid A = f^{-1}(\{1\}) \text{ for some } f \in \{0,1\}^X\}$$

And, thanks to the equivalence demonstrated in 2.:

$$\mathcal{P} = \{A \mid A \text{ is a subset of } X\}$$

is a well-defined set, which we wanted to prove.

Exercise 3.4.7. — *Let X, Y be sets. Define a partial function from X to Y to be any function $f : X' \to Y'$ with $X' \subseteq X$ and $Y' \subseteq Y$. Show that the collection of all partial functions from X to Y is itself a set.*

- Let be $X' \subseteq X$ and $Y' \subseteq Y$. If both X' and Y' are fixed, then per the power set axiom (3.10), there exists a set $Y'^{X'}$ which consists of all the functions from X' to Y'.

- By lemma 3.4.9, there exist a set 2^X which consists of all the subsets of X, and a set 2^Y which consists of all the subsets of Y.

- Now we *fix* an element X' of 2^X. Let be Y' an element of the set 2^Y, A a set, and P the property "$P(Y', A)$: $A = Y'^{X'}$". Per the replacement axiom, there exists exactly one (and thus, at most one) set:

$$\begin{aligned}\{A \,|\, P(Y', A) \text{ is true for some } Y' \in 2^Y\} &= \{A \,|\, A = Y'^{X'} \text{ for some } Y' \in 2^Y\} \\ &= \{Y'^{X'} \,|\, Y' \in 2^Y\}\end{aligned}$$

- Each element of this set is itself a set. Thus we can apply the axiom of union (3.11): there exists a set $\bigcup\{Y'^{X'} \,|\, Y' \in 2^Y\}$ whose elements are those objects which are elements of elements of $\{Y'^{X'} \,|\, Y' \in 2^Y\}$, i.e.:

$$\bigcup\{Y'^{X'} \,|\, Y' \in 2^Y\} = \{f | f : X' \to Y' \text{ for some } Y' \in 2^Y\}$$

This set is obtained for one given *fixed* subset $X' \subseteq X$, so let's denote this set:

$$S_{X'} = \{f | f : X' \to Y' \text{ for some } Y' \in 2^Y\}$$

- Now we apply once again the union set (3.11), especially in its second formulation. If we denote $I = 2^X$, then for each element $X' \in I$ we do have one set $S_{X'}$, which is defined above. Thus, there exists a set $\bigcup_{X' \in 2^X} S_{X'} := \bigcup\{S_{X'} \,|\, X' \in 2^X\}$. And, for every function f, we have $f \in \bigcup\{S_{X'} \,|\, X' \in 2^X\}$ iff there exists $X' \in 2^X$ such that $f \in S_{X'}$, i.e. if there exists $X' \subset X$ and $Y' \subset Y$ such that $f : X' \to Y'$.

- Consequently, we have proved that there exists a set which consists of the collection of all partial functions from X to Y.

Exercise 3.4.8. — *Prove that Axiom 3.4 can be deduced from Axiom 3.1, Axiom 3.3 and Axiom 3.11.*

Let's recall very briefly the four axioms involved here :

- Axiom 3.4 (to be proved) says that if A and B are sets, then there exists a union set $A \cup B$ such that $x \in A \cup B$ iff $x \in A$ or $x \in B$.
- Axiom 3.1 essentially says that sets are objects.
- Axiom 3.3 says that singletons are pair sets do exist.
- Axiom 3.11: let A be a set, whose all elements are themselves sets. Then there exists a set $\bigcup A$ whose elements are those objects which are elements of elements of A, i.e., $x \in \bigcup A$ iff $x \in S$ for some $S \in A$.

Here is a sketch of proof for Axiom 3.4. Let A and B be sets. According to Axiom 3.1, A and B are themselves objects: they can be elements of other sets. Consequently, according to Axiom 3.3, it makes sense to talk about the singleton sets $\{A\}$ and $\{B\}$, and the set $\{A, B\}$.

Now we consider this latter set, which we denote $\mathcal{A} = \{A, B\}$. According to axiom 3.11, there exists a set $\bigcup \mathcal{A}$ whose elements those objects which are the elements of $\mathcal{A}$, i.e., $x \in \mathcal{A}$ iff there exists $S \in \mathcal{A}$ such that $x \in S$. But $\mathcal{A}$ is a pair set with only two elements, so that S must necessarily be equal to A or B.

This leads to the following conclusion: if A and B are sets, then there exists a set $\mathcal{A}$ such that $x \in \mathcal{A}$ iff $x \in A$ or $x \in B$. This is precisely the Axiom 3.4.

Exercise 3.5.1. — *Suppose we define the ordered pair (x, y) for any objects x and y by the formula $(x, y) := \{\{x\}, \{x, y\}\}$. Show that this definition obeys the property (3.5), and also whenever X and Y are sets, the cartesian product $X \times Y$ is also a set.*

Recall that property (3.5) says that $(x, y) = (x', y')$ iff $x = x'$ and $y = y'$. The proof below is heavily inspired by the sketch given by Paul Halmos in his famous book, *Naive Set Theory.* (The proof below is just immensely more verbose.)

1. First, we go back to Remark 3.1.9 by Terence Tao (page 37). In this remark, Tao says that for any object a, the pair set $\{a, a\}$ is in fact the singleton $\{a\}$. Tao asks *why?* to the reader. This is easy to prove using the Definition 3.1.4 (equality of sets): both sets have the same elements, thus they are equal. This fact is a crucial point for the current proof.

   Indeed, first note that for any object x, the ordered pair (x, x) will be (by definition) equal to $\{\{x\}, \{x, x\}\}$. Applying twice the Remark 3.1.9 made by Terence Tao, we can conclude that $(x, x) = \{\{x\}\}$ for any object x.

   Conversely, if any ordered pair (x, y) is a singleton, this means that $\{\{x\}, \{x, y\}\}$ is a singleton. This implies that both elements of this pair set are equal, i.e. $\{x\} = \{x, y\}$. Thus, (by Definition 3.1.4,) $y \in \{x\}$, i.e. $x = y$.

   This gives a handy conclusion, that we can write as a lemma:

Lemma. *An ordered pair (x, y) is a singleton if and only if $x = y$ (and in this case, this singleton is $\{\{x\}\} = \{\{y\}\}$).*

We can now prove more easily that property (3.5) is met.

2. Let's prove that the property (3.5) is satisfied.

   - First, let be two ordered pairs $(a, b) = \{\{a\}, \{a, b\}\}$ and $(x, y) = \{\{x\}, \{x, y\}\}$. If $a = x$ and $y = b$, then we obviously have $\{\{a\}, \{a, b\}\} = \{\{x\}, \{x, y\}\}$.
   - For the reciprocal, suppose that $(a, b) = (x, y)$, and let's show that $a = x$ and $b = y$. We will consider two distinct cases.

     (a) First consider the case where $a = b$ (note that this also covers the case $x = y$, since they play symmetrical roles). Thus $(a, b) = \{\{a\}\}$. Since $(a, b) = (x, y)$, we have $\{\{x\}, \{x, y\}\} = \{\{a\}\}$.
     This means that $\{x\} \in \{\{a\}\}$, i.e. $a = x$.
     But we also have $\{x, y\} \in \{\{a\}\}$, i.e. $\{x, y\} = \{a\}$. This means in particular that $\{x, y\}$ is a singleton, which is only possible if $x = y$ according to the lemma introduced above.
     Thus, $a = b$ by hypothesis, and $a = x$, and $x = y$. This finally means that all four elements are equal: $a = b = x = y$.
     We can insist: if we have either $a = b$ or $x = y$, then all four elements are equal, and property (3.5) is met.

     (b) The other case is $a \neq b$ (which also implies $x \neq y$, otherwise all four elements would be equal). Since $(a, b) = (x, y)$, we have $\{a\} \in \{\{x\}, \{x, y\}\}$, so that we have either $\{a\} = \{x\}$ or $\{a\} = \{x, y\}$. The latter case can be excluded: $\{a\} = \{x, y\}$ would mean that $\{x, y\}$ is a singleton, thus $x = y$, a contradiction with our hypothesis. Thus, the only possibility is $\{a\} = \{x\}$, i.e. $a = x$.
     We also have $\{a, b\} \in \{\{x\}, \{x, y\}\}$, and for the same reason, the only possibility is $\{a, b\} = \{x, y\}$. But we have showed that $a = x$, so that $\{a, b\} = \{a, y\}$. The conclusion is $y = b$.
   - Conclusion: in both cases, (a, b) implies both $a = x$ and $y = b$, which is our initial goal. Property (3.5) is met.

3. Finally, if we adopt this definition, $X \times Y$ is a set. Indeed, for every $x \in X$ and $y \in Y$, both x and y are elements of $X \cup Y$. Thus, the singleton $\{x\}$ and the pair set $\{x, y\}$ are elements of the power set of $X \cup Y$ (which is indeed a set, by Lemma 3.4.9: see Exercise 3.4.6). In other words, if $\mathcal{P}(A)$ denotes the power set of a set A, we have $\{x\} \in \mathcal{P}(X \cup Y)$ and $\{x, y\} \in \mathcal{P}(X \cup Y)$.

   Thus, for every objects $x \in X$ and $y \in Y$, $\{\{x\}, \{x, y\}\} \subset \mathcal{P}(X \cup Y)$. This latter statement is equivalent to $\{\{x\}, \{x, y\}\} \in \mathcal{P}(\mathcal{P}(X \cup Y))$, which is also a well-defined set by a (recursive) application of Lemma 3.4.9.

   Then, for any element $S \in \mathcal{P}(\mathcal{P}(X \cup Y))$, let $P(S)$ be the property "There exists $x \in X$ and $y \in Y$ such that $S = \{\{x\}, \{x, y\}\}$". By the axiom of specification (Axiom 3.5), there exists a set $\{S \in \mathcal{P}(\mathcal{P}(X \cup Y)) \mid P(S) \text{ is true}\}$: this set is precisely the cartesian product $X \times Y$ we were looking for.

Exercise 3.6.3. — *Let n be a natural number, and let $f : \{i \in \mathbb{N} : 1 \leqslant i \leqslant n\} \to \mathbb{N}$ be a function. Show that there exists a natural number M such that $f(i) \leqslant M$ for all $1 \leqslant i \leqslant n$. Thus finite subsets of the natural numbers are bounded.*

Let's induct on n.

- Let's take $n = 1$ for the base case. f is thus a function from the singleton $\{1\}$ to $\mathbb{N}$. If we choose the natural number $M := f(1)$, we have indeed $f(i) \leqslant M$ for all $i \in \{1\}$.

- Now suppose inductively that the proposition is true for a natural number n, and let's show that it is still true for $n + 1$.

  Let be $f : \{i \in \mathbb{N} : 1 \leqslant i \leqslant n + 1\} \to \mathbb{N}$. By the induction hypothesis, we already know that there exists an M such that $f(i) \leqslant M$ for $1 \leqslant i \leqslant n$. Now let be $M' := M + f(n+1)$. Since $f(n+1)$ is a natural number, we have $M \leqslant M'$, so that $f(i) \leqslant M'$ for all $1 \leqslant i \leqslant n$ by transitivity.

  Also, we have $f(n + 1) \leqslant f(n + 1) + M = M'$ because M is a natural number.

  Thus, we have found a natural number M' such that $f(i) \leqslant M'$ for all $i \in [\![1, n + 1]\!]$, as required. This closes the proof.

Exercise 3.6.7. — *Let A and B be sets. Let us say that A has* lesser or equal cardinality *to B if there exists an injection $f : A \to B$ from A to B. Show that if A and B are finite sets, then A has lesser or equal cardinality to B if and only if $\#(A) \leqslant \#(B)$.*

Since A is finite, there exists a natural number n such that $\#A = n$; and thus a bijection $g : [\![1, n]\!] \to A$.

Similarly, there exists $m \in \mathbb{N}$ such that $\#B = m$; and thus a bijection $h : [\![1, m]\!] \to B$.

- First suppose that $n \leqslant m$, and let's show that there exists a bijection $f : A \to B$. Let's define a function $f : A \to B$, such that for all $i \in [\![1, n]\!]$, we have $f(g(i)) = h(i)$. We claim that f is injective. Indeed, let's suppose that we have $f(x) = f(x')$ for $x, x' \in A$. Since g is bijective, there exists $i, j \in [\![1, n]\!]$ such that $f(g(i)) = f(g(j))$. By definition of f, this means that $h(i) = h(j)$. Since h is also bijective, it means that $i = j$. And since $i = i$, we have $g(i) = g(j)$, and thus $x = x'$.

  We have showed that $f(x) = f(x')$ for all $x, x' \in A$, i.e., that f is injective, as required.

- Now suppose that there exists an injection $f : A \to B$, and let's show that $\#A \leqslant \#B$. Obviously, f is a bijection from A to $f(A)$. It means that $f(A)$ and A have the same cardinality, n.

  But $f(A) \subseteq B$, so that by Proposition 3.6.14(c), we have $\#B \geqslant \#(f(A)) = n$, which closes the proof.

4. Integers and rationals

Exercise 4.1.1. — *Verify that the definition of equality on the integers is both reflexive and symmetric.*

Recall the Definition 4.1.1 of equality on integers: two integers $a —— b$ and $c —— d$ are equal iff $a + d = c + b$. This defines a binary relation on $\mathbb{Z}$, denoted "=". Let's show that this relation is reflexive and symmetric.

- Reflexivity: let a and b be natural numbers, so that $a —— b$ is an integer. We know that, on natural numbers, equality is reflexive, i.e. $a + b = a + b$. This equality means precisely that $a —— b = a —— b$.
- Symmetry: let a, b, c, d be natural numbers. If $a —— b = c —— d$, do we also have $c —— d = a —— b$?

$$\begin{array}{rrcl} & a —— b & = & c —— d \\ \iff & a + d & = & c + b \\ \iff & c + b & = & a + d \text{ (equality is symmetric on naturals)} \\ \iff & c —— d & = & a —— b \end{array}$$

Exercise 4.1.2. — *Show that the definition of negation on the integers is well-defined in the sense that if $(a —— b) = (a' —— b')$, then $-(a —— b) = -(a' —— b')$ (so equal integers have equal negations).*

Since $a —— b = a' —— b'$, we have $a + b' = a' + b$.
Also, by Definition 4.1.4 of negation, we have:

$$-(a —— b) = b —— a$$
$$-(a' —— b') = b' —— a'$$

Then, we have successively:

$$\begin{aligned} b + a' &= a' + b \text{ (addition is commutative on naturals, Prop. 2.2.4)} \\ &= a + b' \text{ (initial hypothesis)} \\ &= b' + a \text{ (by commutativity on naturals once again)} \end{aligned}$$

and this equality $b + a' = b' + a$ precisely means that $b —— a = b' —— a'$, i.e. that $-(a —— b) = -(a' —— b')$.

Exercise 4.1.3. — *Show that $(-1) \times a = -a$ for every integer a.*

Since a is an integer, there exist two natural numbers n and m such that $a = m —— n$.
On the one hand, by Definition 4.1.4, $-a = n —— m$.
On the other hand, using once again Definition 4.1.4 and Definition 4.1.2,

$$\begin{aligned} (-1) \times a &= (0 —— 1) \times (m —— n) \\ &= (0 \times m + 1 \times n) —— (0 \times n + 1 \times m) \\ &= n —— m \end{aligned}$$

Thus, we have indeed $(-1) \times a = -a$.

Exercise 4.1.4. — *Prove the remaining identities in Proposition 4.1.6.*

Let $x = a —— b$, $y = c —— d$ and $z = e —— f$ be three integers. Those identities are the following:

1. $x + y = y + x$, i.e., we must prove that addition is commutative on the integers. We have:

$$\begin{aligned} x + y &= (a —— b) + (c —— d) \\ &= (a + c) —— (b + d) \text{ (by Definition 4.1.2)} \\ &= (c + a) —— (d + b) \text{ (addition is commutative on naturals)} \\ &= (c —— d) + (a —— b) \text{ (by Definition 4.1.2 again)} \\ &= y + x \end{aligned}$$

2. $(x + y) + z = x + (y + z)$, i.e. prove that addition is associative on the integers. This is a very similar proof, and this is a direct consequence of associativity of addition on the naturals.

3. $x + 0 = 0 + x = x$. We have already showed that addition is commutative on the integers, so we just have to prove that $x + 0 = x$.

$$\begin{aligned} x + 0 &= (a —— b) + (0 —— 0) \\ &= (a + 0) —— (b + 0) \\ &= a —— b = x. \end{aligned}$$

4. $x+(-x) = (-x)+x = 0$. Once again, thanks to the previous result about commutativity, we just have to prove that $x + (-x) = 0$.

$$\begin{aligned} x + (-x) &= (a —— b) + (b —— a) \text{ (by Definition 4.1.4)} \\ &= (a + b) —— (b + a) \text{ (by Definition 4.1.2)} \\ &= (a + b) —— (a + b) \text{ (addition is commutative on naturals)} \\ &= 0 \text{ (because } m —— m = 0 —— 0 \text{ for all integer } m\text{)} \end{aligned}$$

5. $xy = yx$, i.e. multiplication is commutative on the integers.

$$\begin{aligned} xy &= (a —— b) \times (c —— d) \\ &= (ac + bd) —— (ad + bc) \text{ (by Definition 4.1.2)} \\ &= (ca + db) —— (da + cb) \text{ (multiplication is commutative on the naturals)} \\ &= yx \text{ (by Definition 4.1.2)} \end{aligned}$$

6. $(xy)z = x(yz)$, i.e. multiplication is associative on the integers. This is actually the only identity proved in the main text by Terence Tao.

7. $x1 = 1x = x$. The equality between the first two terms is a direct consequence of commutativity of multiplication on the integers. We just have to prove that $x1 = x$. And indeed, $x1 = (a \text{——} b) \times (1 \text{——} 0) = (a1 + b0) \text{——} (b1 + a0) = a \text{——} b = x$.

8. $x(y + z) = xy + xz$, i.e. show distributivity on the integers. On the left side, we have:

$$
\begin{aligned}
x(y + z) &= (a \text{——} b)\,((c \text{——} d) + (e \text{——} f)) \\
&= (a \text{——} b)\,((c + e) \text{——} (d + f)) \\
&= (a(c + e) + b(d + f)) \text{——} (a(d + f) + b(c + e)) \\
&= ((ac + ae + bd + bf)) \text{——} ((ad + af + bc + be))
\end{aligned}
$$

   and then on the left side:

$$
\begin{aligned}
xy + xz &= (a \text{——} b)(c \text{——} d) + (a \text{——} b)(e \text{——} f) \\
&= ((ac + bd) \text{——} (ad + bc)) + ((ae + bf) \text{——} (af + be)) \\
&= ((ac + ae + bd + bf)) \text{——} ((ad + af + bc + be))
\end{aligned}
$$

9. $(y + z)x = yx + zx$. This latter identity is a direct consequence of commutativity of multiplication on the integers, and distributivity on the integers, both being already proved earlier in this exercise.

Exercise 4.1.5. — *Prove Proposition 4.1.8, i.e.: let x and y be integers such that $xy = 0$, then either $x = 0$ or $y = 0$ (or both).*

We will use here Lemma 4.1.5 (trichotomy of integers, which says that any integer is either zero, or equal to a positive natural number, or the negation of a positive natural number), and Lemma 2.3.3 (which provides an equivalent of Proposition 4.1.8 for natural numbers only). We will prove the proposition for (all) three possible cases: $x = 0$, x is a positive natural number, $-x$ is a positive natural number.

y will be considered as a fixed integer, $y = c \text{——} d$ with c, d natural numbers.

1. First let's take the case $x = 0$. There is nothing to prove here, the proposition is obviously true.

2. Then let's take the case where x is a positive natural number (and, consequently, is not equal to zero). In this case, as an integer, x can be written $n \text{——} 0$ with n a positive natural number.

   We have $xy = (n \text{——} 0) \times (c \text{——} d) = (nc + 0d) \text{——} (nd + 0c) = nc \text{——} nd$.

   Thus, $xy = 0$ iff $nc \text{——} nd = 0 \text{——} 0$, and by Definition 4.1.1, this means that $nc = nd$. But since all three n, c, d are natural numbers, we can use the cancellation law for natural numbers and conclude that $c = d$.

   This means that $y = c \text{——} c = 0 \text{——} 0 = 0$. Thus, in this case, if $xy = 0$ with x non-zero, we have showed that y is necessarily equal to 0.

3. Finally, let's take the case where x is the opposite of a positive natural number n, i.e. $x = 0 \text{——} n$. A very similar proof also leads to $c = d$, and to $y = 0$.

Exercise 4.1.6. — *Prove Corollary 4.1.9, i.e. if a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.*

If $ac = bc$, then $ac + (-bc) = bc + (-bc) = 0$. Thus, $ac - bc = 0$.

Let's use the property of distributivity (Proposition 4.1.6): we obtain $(a - b)c = 0$. According to Proposition 4.1.8 (see also the previous exercise), this implies either $c = 0$ or $a - b = 0$. The first option ($c = 0$) must be discarded since it does not fit the initial hypothesis. The only possibility is thus $a - b = 0$, and adding b to both sides finally leads to $a = b$.

Exercise 4.1.7. — *Prove Lemma 4.1.11.*

The statements to prove are the following:

1. Show that $a > b$ if and only if $a - b$ is a positive natural number.

   First suppose that $a > b$. This means (Definition 4.1.10) that there exists a natural number n such that $a = b + n$, and $a \neq b$. Then, we add to both sides the opposite of b, and we get $a + (-b) = b + n + (-b)$, i.e. $a - b = n$. In this latest equality, n cannot be zero, otherwise we would have $a = b$, which is excluded. The first implication is proved.

   Then suppose that $a - b = n$ with n a positive natural number. Adding b to both sides leads to $a = b + n$, i.e. $a \geqslant b$. We cannot have $a = b$, because this would be a contradiction with the fact that $n \neq 0$. Thus, $a > b$.

2. Show that addition preserves order, i.e. if $a > b$, then $a + c > b + c$.

   Suppose that $a > b$. According to the previous point, this means that $a - b = n$, with n a positive natural number. Then, we get successively:

   $$\begin{array}{rcll} a & = & b + n & \text{(by adding } b \text{ to both sides)} \\ a + c & = & b + c + n & \text{(by adding } c \text{ to both sides)} \\ a + c - b - c & = & n & \text{(by adding } (-b) + (-c) \text{ to both sides)} \\ a + c - (b + c) & = & n & \text{(by using the distributive law and Exercise 4.1.3)} \end{array}$$

   Using again the previous point, since $(a + c) - (b + c)$ is equal to a positive natural number, we can conclude that $a + c > b + c$.

3. Show that positive multiplication preserves order, i.e. if $a > b$ and c is positive, then $ac > bc$.

   Since $a > b$, according to the first point of this exercise, we have $a - b = n$, with n a positive natural number. According to the distributive law, $(a - b)c = ac - bc$. But we also have $(a - b)c = nc$, and nc is a positive natural number (as product of two positive numbers, see Lemma 2.3.3). Thus, $ac - bc$ is equal to a positive number, which means that $ac > bc$.

4. Show that negation reverses order, i.e. if $a > b$, then $-a < -b$.

   Here, we will need a small lemma, which says that for any natural number n, we have $n = -(-n)$. There are several ways to show this result: either by proving that $(-1) \times (-1) = 1$ and using Exercise 4.1.3, or simply by noting that $n + (n) = 0$ for all n, which means that n is the opposite of $-n$ (i.e., $n = -(-n)$).

Now this point is easy to prove. $a > b$ means that $a - b$ is a positive number, as shown earlier in this exercise. We want to prove that $-a < -b$, and proving this assertion requires to show that $-b - (-a)$ is a positive number. But $-b - (-a) = -b + a = a - b$, which is a positive natural number. Thus we are done.

5. Show that order is transitive, i.e. if $a > b$ and $b > c$, then $a > c$.

   Still using the first point of this exercise, we have $a - b = n$ and $b - c = m$, with n, m two positive natural numbers. We know that $n + m$ is positive as the sum of two positive numbers, thus $n + m = a - b + b - c = a - c$ is positive. This means that $a > c$.

6. Show order trichotomy, i.e.: exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

   - If $a = b$, then obviously (exactly) one of those statements is true.
   - Now consider the case $a \neq b$, and let's show that we have either $a > b$ or $a < b$ (and cannot have both).

     Let's consider the integer $a - b$. By trichotomy of integers (Lemma 4.1.5), we know that we have either $a - b = 0$ (which is excluded here), or $a - b = n$ with n positive, or $a - b = -n$ with n positive.

     If $a - b = n$, then $a > b$ according to the first point of this exercise. If $a - b = -n$, then $-n = -(a - b) = b - a$, thus $b > a$.

     Finally, we just have to show that we cannot have both $a > b$ and $b > a$ at the same time. If $a > b$, then the integer $a - b$ is positive. If $b > a$, then $b - a$ is positive, i.e. $-(b - a) = a - b$ is the opposite of a positive natural. Thus, $a - b$ is both positive and the opposite of a positive number, which is excluded by the trichotomy of integers.

Exercise 4.1.8. — *Show that the principle of induction (Axiom 2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n{+}{+})$ for all integers n, but that $P(n)$ is not true for all integers n.*

According to Lemma 4.1.5, an integer is either equal to 0, or equal to a positive natural number, or equal to the negation of a positive natural number.

Let's define $P(n)$ as the property "The integer n is a natural number, i.e. is either equal to 0 or equal to a positive natural number". Obviously, $P(0)$ is true. Furthermore, if n is a natural number, then $n{+}{+}$ is also a natural number (Axiom 2.2), so that if $P(n)$ is true, then $P(n{+}{+})$ is true. Thus, $P(n)$ matches the required condtions.

However, $P(-1)$ is obviously false.

Exercise 4.2.1. — *Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive.*

This exercise ressembles Exercise 4.1.1, and the same approach applies. Recall the Definition 4.2.1: two rational numbers $a /\!/ b$ and $c /\!/ d$ are equal iff $ad = bc$. This defines a binary relation on $\mathbb{Q}$, denoted "=". Let's show that this relation is reflexive, symmetric and transitive.

Hereafter, a, b, c, d, e, f are integers (and b, d, f are non-zero).

- Reflexivity: here we must prove that $a \mathbin{//} b = a \mathbin{//} b$. This is the case iff $ab = ba$, with is true because of (commutativity of multiplication on $\mathbb{Z}$ and) reflexivity of equality on $\mathbb{Z}$.
- Symmetry: here we must prove that if $a \mathbin{//} b = c \mathbin{//} d$, we also have $c \mathbin{//} d = a \mathbin{//} b$. We have successively:

$$\begin{array}{rrcl} & a \mathbin{//} b & = & c \mathbin{//} d \\ \iff & ad & = & bc \\ \iff & da & = & cb \text{ ($\times$ is commutative on $\mathbb{Z}$)} \\ \iff & cb & = & da \text{ ($=$ is symmetric on $\mathbb{Z}$)} \\ \iff & c \mathbin{//} d & = & a \mathbin{//} b \end{array}$$

- Transitivity: here we must prove that if $a \mathbin{//} b = c \mathbin{//} d$ and $c \mathbin{//} d = e \mathbin{//} f$, then $a \mathbin{//} b = e \mathbin{//} f$. I.e., we must prove that if $ad = bc$ and $cf = de$, then $af = be$.

  Let's multiply by e both sides of the equality $ad = bc$: we get $ade = bce$. Since $de = cf$, we also get $acf = bce$.

  In this latest equality, using the cancellation law (Corollary 4.1.9) for c would lead to $af = be$, which would close the proof. However, unlike b, d or f, the integer c may be equal to 0, and in this case we cannot use the cancellation law. There are thus two different cases:

  - If $c \neq 0$, we simply use the cancellation law: since $acf = bce$, then $af = be$, which means that $a \mathbin{//} b = e \mathbin{//} f$.
  - If $c = 0$, then $bc = 0$. But since $ad = bc$, we also have $ad = 0$, and we know that $d \neq 0$. According to Proposition 4.1.8, this leads to $a = 0$. A similar reasoning leads to $e = 0$. Thus, $a = c = e = 0$, and $0 = af = be$, which means $a \mathbin{//} b = e \mathbin{//} f$.

Exercise 4.2.2. — *Prove the remaining components of Lemma 4.2.3.*

Let $a \mathbin{//} b = a' \mathbin{//} b'$ be two rationals; let $c \mathbin{//} d = c' \mathbin{//} d'$ be two rationals. The remaining claims are the following:

- Prove that $-(a' \mathbin{//} b') = -(a \mathbin{//} b)$. This equality holds iff $(-a') \mathbin{//} b' = (-a) \mathbin{//} b$, i.e. iff $(-a')b = b'(-a)$. We have successively:

$$\begin{aligned} (-a')b &= (-1)a'b \text{ (see Exercise 4.1.3)} \\ &= (-1)ab' \text{ (because $a \mathbin{//} b = a' \mathbin{//} b'$)} \\ &= (-a)b' \text{ (using Exercise 4.1.3 one again)} \end{aligned}$$

  Thus we are done.

- Prove that $(a \mathbin{//} b) \times (c \mathbin{//} d) = (a' \mathbin{//} b') \times (c \mathbin{//} d)$. To prove this equality, we must show that $(ac) \mathbin{//} (bd) = (a'c) \mathbin{//} (b'd)$. By definition of equality between rationals, this holds iff $acb'd = bda'c$. Since $ab' = ba'$, the claim follows (using commutativity of multiplication on integers[10]).

[10] This kind of precision about very basic properties of naturals and integers will not be mentioned anymore.

- Prove that $(a \,/\!/\, b) \times (c \,/\!/\, d) = (a \,/\!/\, b) \times (c' \,/\!/\, d')$. To prove this equality, we must show that $(ac) \,/\!/\, (bd) = (ac') \,/\!/\, (bd')$. By definition of equality between rationals, this holds iff $acbd' = bdac'$. Since $cd' = dc'$, the claim follows.

Exercise 4.2.3. — *Prove the remaining components of Proposition 4.2.4.*

Let $x = a \,/\!/\, b$, $y = c \,/\!/\, d$, and $z = e \,/\!/\, f$ be rational numbers, with a, c, e integers, and b, d, f non-zero integers. The remaining claims are the following:

1. $x + y = y + x$, i.e. addition is commutative for the rationals.

   On the one hand, we have $x + y = a \,/\!/\, b + c \,/\!/\, d = (ad + bc) \,/\!/\, bd$.

   On the other hand, $y + x = c \,/\!/\, d + a \,/\!/\, b = (cb + da) \,/\!/\, db = (ad + bc) \,/\!/\, bd$ using the commutativity of addition and multiplication on the integers. Thus, the two expressions are equal.

2. $(x + y) + z = x + (y + z)$: already proved in the book.

3. $x + 0 = 0 + x = x$. By the first point of this exercise, we already know that $x + 0 = 0 + x$, so we have just to show that $x + 0 = x$. We have $x + 0 = a \,/\!/\, b + 0 \,/\!/\, 1 = (a1 + b0) \,/\!/\, (b1) = a \,/\!/\, b = x$, which is the required result.

4. $x + (-x) = (-x) + x = 0$. Once again, the part $x + (-x) = (-x) + x$ comes from the first point of this exercise, so we just have to prove that $x + (-x) = 0$. We have $x + (-x) = a \,/\!/\, b + -(a \,/\!/\, b) = a \,/\!/\, b + (-a) \,/\!/\, b = (ab - ba) \,/\!/\, b^2 = 0 \,/\!/\, b^2$. But we know that $0 \,/\!/\, m = 0 \,/\!/\, 1 = 0$ for all non-zero integer m, since $0 \times 1 = m \times 0$. Thus, $x + (-x) = 0 \,/\!/\, b^2 = 0$, as required.

5. $xy = yx$, i.e. multiplication is commutative on the rationals. Indeed, $xy = (a \,/\!/\, b) \times (c \,/\!/\, d) = (ac) \,/\!/\, (bd)$ by definition. On the other hand, $yx = (c \,/\!/\, d) \times (a \,/\!/\, b) = (ca) \,/\!/\, (db) = (ac) \,/\!/\, (bd)$ by commutativity of multiplication on the integers. Thus, $xy = yx$.

6. $(xy)z = x(yz)$, i.e. multiplication is associative on the rationals. We have $(xy)z = (ace) \,/\!/\, (bdf) = x(yz)$ by associativity of multiplication on the integers.

7. $x1 = 1x = x$. Once again, we already know that $x1 = 1x$, thanks to the fifth point of this exercise. So we just have to show that $x1 = x$. We have $x1 = (a \,/\!/\, b) \times (1 \,/\!/\, 1) = (a1) \,/\!/\, (b1) = a \,/\!/\, b = x$.

8. $x(y + z) = xy + xz$, i.e. distributivity of multplication for the rationals. On the one hand, we have:

$$\begin{aligned} x(y + z) &= (a \,/\!/\, b)(c \,/\!/\, d + e \,/\!/\, f) \\ &= (a \,/\!/\, b)((cf + de) \,/\!/\, (df)) \\ &= (acf + ade) \,/\!/\, (bdf) \end{aligned}$$

On the other hand[11]:

$$\begin{aligned} xy + xz &= (a \,//\, b)(c \,//\, d) + (a \,//\, b)(e \,//\, f) \\ &= (ac) \,//\, (bd) + (ae) \,//\, (bf) \\ &= (acbf + bdae) \,//\, (b^2 df) \\ &= (acf + ade) \,//\, (bdf) \end{aligned}$$

Thus we have indeed $x(y + z) = xy + xz$.

9. $(y + z)x = yx + zx$. This can be deduced immediately from commutativity of multiplication and the eighth point of this exercise.

10. For all $x \neq 0$, $xx^{-1} = x^{-1}x = 1$. Once again, the part $xx^{-1} = x^{-1}x$ comes from the fifth point of this exercise, so that we just have have to show that $xx^{-1} = 1$.

$$\begin{aligned} xx^{-1} &= (a \,//\, b) \times (b \,//\, a) \\ &= (ab) \,//\, (ba) \\ &= 1 \,//\, 1 = 1 \end{aligned}$$

Exercise 4.2.4. — *Prove Lemma 4.2.7. (trichotomy of rationals), i.e., if x is a rational number, then exactly one of the following three statements is true: (a) x is equal to 0, (b) x is a positive rational number, or (c) x is a negative rational number.*

Following the hint given by Terence Tao, we'll first prove that *at least* one of those statements is true, and then that *at most* one of them is true. Let be $x = a \,//\, b$, where a is an integer and b a non-zero integer.

1. Let's prove that at least one of those statements is true.

   First, an obvious case: if $a = 0$, then $x = a \,//\, b = 0$, thus one of the statements is true. Now consider the case where $a \neq 0$. By the trichotomy of integers, a can be either positive or negative. Similarly, b can also be either positive or negative (it cannot be null, by definition). Thus, there are four main cases:

   - $a > 0$ and $b > 0$. Here, by Definition 4.2.6, $x = a \,//\, b$ is positive.
   - $a > 0$ and $b < 0$. Here, $b = -m$, with m a positive natural number. Thus, $x = a \,//(-m)$. But $a \,//(-m) = (-a) \,// m$ (this is easy to verify: $am = (-a)(-m)$). This means that $x = (-a) \,//\, m$, with both a and m positive, i.e. x is negative.
   - $a < 0$ and $b > 0$. Here, $x = a \,//\, b$ is negative by Definition 4.2.6.
   - $a < 0$ and $b < 0$. Here, we can say that $a = -n$ and $b = -m$, with n, m positive natural numbers. Thus, $x = (-n) \,//(-m) = n \,// m$ (once again, this latest equality is easy to verify). Thus, x is positive.

   Conclusion: if all four cases, at least one of the three properties is true.

2. Now prove that at most one of those statements is true.

[11] We use implicitly here the fact that $(nm) \,//\, n = m \,//\, 1$ for all integers n, m with $m \neq 0$, which is straightforward to prove.

- Suppose, for the sake of contradiction, that we have both $x = 0$ and x positive. On the one hand, "$x = a \,//\, b = 0$" implies that $a = 0$ (see Terence Tao's remark, page 83). On the other hand, "x is positive" implies that $a > 0$. So we would have both $a = 0$ and a positive, which is not compatible with the trichotomy of integers.
- A similar argument holds if we suppose both $x = 0$ and x negative.
- Now suppose that we have both x positive and x negative, i.e. $x = c \,//\, d = (-n) \,//\, m$, with c, d, n, m positive natural numbers. Thus, we should have $cm = (-n)d$. On the one hand, cm is positive, as the multiplication of two positive natural numbers. On the other hand, $(-n)d = -(nd)$ is negative. The equality $cm = -nd$ is thus impossible.

Conclusion: all three statements are mutually exclusive.

Exercise 4.2.5. — *Prove Proposition 4.2.9.*

Let x, y, z be rational numbers. This proposition includes the following statements:

1. Prove that exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

   This statement is very close to Lemma 4.2.7, proved in the previous exercise. Let's consider the rational number $x - y$. According to the trichotomy of rationals, this number can be either zero, positive or negative (exactly one of these statements is true).

   If $x - y = 0$, then $x = y$. If $x - y$ is positive, then $x > y$. And if $x - y$ is negative, then $x < y$. Thus, the order trichotomy is a direct consequence of the ordering of rationals.

2. Prove that one has $x < y$ if and only if $y > x$.

   Since $x < y$, the rational number $x - y$ is negative, and can be written $(-a) \,//\, b$ for positive integers a, b. And since $x - y = -a \,//\, b$, we have $a \,//\, b = y - x$, i.e. $y - x$ is positive, i.e. $y > x$.

3. Prove that if $x < y$ and $y < z$, then $x < z$.

   Since $x < y$, $x - y$ is negative. Similarly, $y - z$ is negative. This means that $x - y$ can be written $(-a) \,//\, b$, and $y - z$ can be written $(-c) \,//\, d$, with a, b, c, d positive integers.

   On the one hand, their sum is $x - z$. On the other hand, their sum is $((-ad) + (-cb)) \,//\, (bd)$. This latest expression is a negative rational, thus we have $x < z$.

4. Prove that if $x < y$, then $x + z < y + z$.

   Suppose that $x < y$. Thus, $x - y$ is negative. But we have, for any rational z, $x - y = (x + z) - (y + z)$, and thus this latter expression is also negative. This means that $x + z < y + z$.

5. Prove that if $x < y$ and z is positive, then $xz < yz$.

   Since $x < y$, the rational number $x - y$ is negative. Furthermore, we know (by Proposition 4.2.4) that $xz - yz = (x - y)z$. In the expression $(x - y)z$, z is supposed to be positive and $x - y$ is negative, thus their product is negative[12]. This means that $xz < yz$.

Exercise 4.2.6. — *Show that if x, y, z are rational numbers such that $x < y$ and z is negative, then $xz > yz$.*

If $x < y$, then $x - y$ is negative. Thus, $(x - y)z$ is the product of two negative rationals: it is a positive rational[13].

But $(x - y)z = xz - yz$ by Proposition 4.2.4. And since we have showed that this number is positive, we have $xz > yz$.

Note: in particular, this exercise says that if $x > y$, then $-x < -y$ (with $z = -1$).

Exercise 4.3.1. — *Prove Proposition 4.3.3.*

Let x, y, z be rational numbers. The statements to prove are the following:

(a) Show that $|x| \geqslant 0$ for all x, and that $|x| = 0$ iff $x = 0$.

There are three cases:

- if $x = 0$, then $|x| := 0$, thus we have in particular $|x| \geqslant 0$.
- if $x > 0$, then $|x| := x$, thus $|x| > 0$. And in particular, this means that $|x| \geqslant 0$.
- if $x < 0$, then $|x| := -x$, thus $|x| > 0$. And in particular, $|x| \geqslant 0$.

We can note that the only case where $|x| = 0$ is when $x = 0$. Thus, by trichotomy of rationals, $|x| = 0$ iff $x = 0$.

(b) Show that $|x + y| \leqslant |x| + |y|$.

- If $x = 0$ or $y = 0$, this is immediate.
- If $x > 0$ and $y > 0$, $x + y$ is positive, thus $|x + y| = x + y = |x| + |y|$.
- If $x < 0$ and $y < 0$, $x + y$ is negative, thus $|x + y| = -(x + y) = -x - y$. On the other hand, $|x| + |y| = -x - y$.
- Finally, the case where x and y are of opposite signs. Say that x is positive and y negative, but they are exchangeable. On the one hand, $|x| + |y| = x - y > 0$. On the other hand, $|x + y|$ can be either equal to $x + y$ if $x + y > 0$, i.e. if $x > -y$; or equal to $-x - y$ if $x + y < 0$, i.e. if $x < -y$.
  In the first case, since $-y < 0 < y$ by hypothesis, we have $|x + y| = x + y < x - y = |x| + |y|$.
  In the second case, since $-x < 0 < x$ by hypothesis, we have $|x + y| = -x - y < x - y = |x| + |y|$.

Conclusion: in all cases, we have indeed $|x + y| \leqslant |x| + |y|$.

[12] This is never explicitly mentioned in the book. However, using Exercise 4.1.3, we know that for every integer a, we have $-a = (-1) \times a$. So let's consider the product $n(-m)$ where n, m are positive integers: this product is $(-1)nm = -(nm)$, and thus is negative.

[13] Similarly, if x and y are negative, then $-x$ and $-y$ are positive, and their product $(-x)(-y) = (-1)(-1)xy = xy$ is positive by definition. This can also be deduced from Proposition 4.2.9(e), by choosing $x = 0$.

(c) Show that $-y \leqslant x \leqslant y$ iff $y \geqslant |x|$. (Thus, in particular, $-|x| \leqslant x \leqslant |x|$.)

- First suppose that $y \geqslant |x|$. Note that, whatever could be the value of x, we have necessarily $y \geqslant 0$ according to the first point of this exercise. Now we can split into three cases.

  If $x = 0$ then $y \geqslant 0$ and the claim is immediate.

  If $x > 0$, then $|x| = x$, and the part $y \geqslant x$ is immediate. Furthermore, the other part $-y \leqslant x$ is also immediate since $-y$ is negative and x is positive.

  If $x < 0$, then $|x| = -x$, thus we have $y \geqslant -x$, i.e. $-y \leqslant x$ according to Exercise 4.2.6. Additionally, the part $x \leqslant y$ is immediate since x is negative and y is positive.
- Conversely, suppose that $-y \leqslant x \leqslant y$. If $x \geqslant 0$, then $|x| = x$, thus the rightmost inequality gives $x = |x| \leqslant y$. In the other case, if $x < 0$, then $|x| = -x$. The leftmost inequality $-y \leqslant x$ leads (according to Exercise 4.2.6) to $y \geqslant -x$, i.e. $y \geqslant |x|$.

(d) Show that $|xy| = |x| \times |y|$. (In particular, $|-x| = |x|$.)

Once again, we can split into several cases, as in the second point of this exercise.

- If $x = 0$ or $y = 0$, both sides of the equality are zero (cf. the first point of this exercise), thus the claim is immediate.
- If $x > 0$ and $y > 0$, the product xy is also positive. Thus, $|xy| = xy$, and $|x| \times |y| = xy$, and the claim follows.
- If $x < 0$ and $y < 0$, then the product xy is positive, and $|xy| = xy$. On the other hand, $|x| \times |y| = (-x)(-y) = xy$, and the claim follows.
- If x and y are of opposite signs (say x positive and y negative, but they are exchangeable), then xy is negative, and $|xy| = -xy$. On the other hand, $|x| \times |y| = -xy$, thus the claim follows.

(e) Show that $d(x, y) \leqslant 0$ for all x, y, and that $d(x, y) = 0$ iff $x = y$.

We have $d(x, y) = |x - y| \geqslant 0$ according to the first point of this exercise. Furthermore, still according to the first point, $d(x, y) = |x - y| = 0$ iff $x - y = 0$, i.e. $x = y$.

(f) Show that $d(x, y) = d(y, x)$.

We have $d(x, y) = |x - y|$ and $d(y, x) = |y - x|$ by definition. But $|y - x| = |-(x - y)| = |x - y|$ according to the fourth point of this exercise.

(g) Show that $d(x, z) = d(x, y) + d(y, z)$.

We have $d(x, z) = |x - z| = |(x - y) + (y - z)| \leqslant |x - y| + |y - z|$ according to the second point of this exercise. The claim follows.

Exercise 4.3.2. — *Prove the remaining claims in Proposition 4.3.7.*

Let x, y, z, w be rational numbers.

(a) If $x = y$, then $d(x,y) = 0$ according to Proposition 4.3.3(e). Thus, $d(x,y) \leqslant \varepsilon$ for any positive number ε.

Conversely, suppose that $d(x,y) \leqslant \varepsilon$ for any $\varepsilon > 0$, and let's prove that $x = y$. Suppose, for the the sake of contradiction, that $x \neq y$; and let be $\varepsilon = |x-y|/2$. Since $x \neq y$, we have $|x-y| > 0$, thus ε is a positive number. Furthermore, $d(x,y) = \varepsilon + \varepsilon$, thus $d(x,y) > \varepsilon$, which is a contradiction.

(b) This is a direct consequence from Proposition 4.3.3(f). Indeed, since $d(x,y) = d(y,x)$, we obviously have $d(y,x) \leqslant \varepsilon$ when $d(x,y) \leqslant \varepsilon$.

(c) Suppose that $d(x,y) \leqslant \varepsilon$ and $d(y,z) \leqslant \delta$. Thus, by triangle inequality, $d(x,z) \leqslant d(x,y) + d(y,z) \leqslant \varepsilon + \delta$.

(d) Suppose that $d(x,y) \leqslant \varepsilon$ and $d(z,w) \leqslant \delta$. Thus,

$$\begin{aligned} d(x+z, y+w) &= |(x+z) - (y+w)| \\ &= |(x-y) + (z-w)| \\ &\leqslant |x-y| + |z-w| \\ &\leqslant \varepsilon + \delta \end{aligned}$$

which means that $x+z$ and $y+w$ are $(\varepsilon+\delta)$-close.

Similarly, $d(x-z, y-w) = |(x-y) + (w-z)|$, and using just the symmetry of distance, we can conclude that $x-z$ and $y-w$ are $(\varepsilon+\delta)$-close according to the previous result.

(e) This is clear: we have $d(x,y) \leqslant \varepsilon < \varepsilon'$.

(f) Since $d(x,y) \leqslant \varepsilon$, we have $-\varepsilon \leqslant y - x \leqslant \varepsilon$. Similarly, we have $-\varepsilon \leqslant z - x \leqslant \varepsilon$.

y and z are exchangeable here, so we can suppose that $y \leqslant w \leqslant z$. From this inequality, we can get $y - x \leqslant w - x \leqslant z - x$. Extending this with the former inequalities, we have:

$$-\varepsilon \leqslant y - x \leqslant w - x \leqslant z - x \leqslant \varepsilon$$

and in particular $-\varepsilon \leqslant w - x \leqslant \varepsilon$, which means $d(w,x) \leqslant \varepsilon$.

(g) We have $d(x,y) = |x-y| \leqslant \varepsilon$. Since z is positive, we have $|z| > 0$, thus $|x-y|\,|z| \leqslant \varepsilon\,|z|$. But according to Proposition 4.3.3(d), $|x-y|\,z = |(x-y)z| = |xz - yz|$. Thus, $|xz - yz| \leqslant \varepsilon\,|z|$, i.e., xz and yz are $\varepsilon\,|z|$-close.

Exercise 4.3.3. — *Prove Proposition 4.3.10.*

Let x, y be rationals, and n, m be natural numbers. The claims to prove are the following (they are re-ordered and re-numbered here):

(a) Show that $x^n x^m = x^{n+m}$. We induct on n while keeping m fixed.

For the base case $n = 0$, we have on the one hand $x^n x^m = x^0 x^m = 1 \cdot x^m = x^m$. On the other hand, $x^{n+m} = x^{0+m} = x^m$. Thus, both sides are equal, and the base case is done.

Now suppose that $x^n x^m = x^{n+m}$, and let's show that $x^{n+1}x^m = x^{(n+1)m}$. We have:

$$\begin{array}{rcll} x^{n+1}x^m & = & (x^n x)x^m & \text{(by Definition 4.3.9)} \\ & = & x^n x^m x & \text{(by associativity and commutativity of multiplication)} \\ & = & x^{n+m}x & \text{(by induction hypothesis)} \\ & = & x^{n+m+1} & \text{(by Definition 4.3.9 once again)} \end{array}$$

This closes the induction.

(b) Show that $(xy)^n = x^n y^n$. Let's induct on n. The base case $n = 0$ is obvious, since both sides are equal to 1. Now suppose inductively that $(xy)^n = x^n y^n$. Thus we have:

$$\begin{array}{rcll} (xy)^{n+1} & = & (xy)^n(xy) & \text{(by Definition 4.3.9)} \\ & = & x^n y^n xy & \text{(by inductive hypothesis)} \\ & = & x^n x y^n y & \text{(by commutativity of multiplication)} \\ & = & x^{n+1}y^{n+1} & \text{(by Definition 4.3.9 once again)} \end{array}$$

(c) Show that $(x^n)^m = x^{nm}$. We induct on n while keeping m fixed.

For the base case $n = 0$, we have $(x^n)^m = 1^m = 1$, since $1^m = 1$ for all natural number m[14]. On the other hand, $x^{nm} = x^{0m} = 1$. Thus, both sides are equal, and the base case is done.

Now suppose inductively that $(x^n)^m = x^{nm}$. Then we have:

$$\begin{array}{rcll} (x^{n+1})^m & = & (x^n x)^m & \text{(by Definition 4.3.9)} \\ & = & (x^n)^m x^m & \text{(proved in 2. from this exercise)} \\ & = & x^{nm}x^m & \text{(by inductive hypothesis)} \\ & = & x^{nm+m} & \text{(proved in 1. from this exercise)} \\ & = & x^{(n+1)m} & \end{array}$$

This closes the induction.

(d) Show that if $n > 0$, then $x^n = 0$ iff $x = 0$. For that, let's induct on n. Here the base case starts with $n = 1$ since we suppose $n > 0$. For $n = 1$, $x^1 = x$, thus we obviously have $x^1 = 0 \Leftrightarrow x = 0$ since both objects are equal.

Now suppose inductively that $x^n = 0$ iff $x = 0$. We must show that $x^{n+1} = 0$ iff $x = 0$. Here we'll need the following lemma:

Lemma. *Let x, y be rational numbers. Then, if $xy = 0$, we have either $x = 0$ or $y = 0$.*

Proof. Let's denote $x = a /\!/ b$ and $y = c /\!/ d$. By Definition 4.2.2, $xy = (ac) /\!/ (bd)$. Thus, since $xy = 0$, we have $ac = 0$ (see Tao's remark p. 83). And, by Proposition 4.1.8, we have either $a = 0$ or $c = 0$. In the first case, this means that $x = 0$; in the second case this means that $y = 0$. $\square$

Now go back to the main proof. First, if $x = 0$, we have $x^{n+1} = x^n x = 0^n \times 0 = 0$. Conversely, if $x^{n+1} = 0$, then $x^n x = 0$. According to the previous lemma, this means that either $x^n = 0$ or $x = 0$. In the second case, we are done. In the first case, the induction hypothesis also allows to conclude that $x^n = 0$. This closes the induction.

[14] This can easily be proved by induction, which we'll not write formally here.

(e) Show that if $x \geqslant y \geqslant 0$, then $x^n \geqslant y^n \geqslant 0$. Let's induct on n.

For the base case $n = 0$, $x^0 = y^0 = 1$. Thus in particular we have indeed $x^0 \geqslant y^0 \geqslant 0$.

Now suppose inductively that $x^n \geqslant y^n \geqslant 0$, and show that $x^{n+1} \geqslant y^{n+1} \geqslant 0$. We start from $x^n \geqslant y^n \geqslant 0$ and multiply all terms by x (which preserves inequality since x is supposed to be positive): we get $x^{n+1} \geqslant xy^n \geqslant 0$. If we start from $x \geqslant y \geqslant 0$ and multiply all terms by y^n (which is also positive by induction hypothesis), we get $y^n x \geqslant y^{n+1} \geqslant 0$. Now combine all those inequalities:

$$x^{n+1} \geqslant xy^n \geqslant y^{n+1} \geqslant 0$$

This closes the induction.

(f) Show that $|x^n| = |x|^n$. Let's induct on n.

For the base case $n = 0$, we have $|x^n| = |x^0| = |1| = 1$; and $|x|^n = |x|^0 = 1$. Thus both sides are equal, and the base case is done.

Now suppose that $|x^n| = |x|^n$ and show that $|x^{n+1}| = |x|^{n+1}$. We have:

$$\begin{aligned} |x^{n+1}| &= |x^n x| && \text{(by Definition 4.3.9)} \\ &= |x^n| \cdot |x| && \text{(by Proposition 4.3.3d)} \\ &= |x|^n \cdot |x| && \text{(by inductive hypothesis)} \\ &= |x|^{n+1} \end{aligned}$$

This closes the induction.

Exercise 4.3.4. — *Prove Proposition 4.3.12.*

This is essentially the same exercise as 4.3.3, but dealing with integer exponents (instead of natural exponents). The claims to prove are the following (and once again, they are relabeled):

(a) Prove that $x^n x^m = x^{n+m}$. Let's distinguish three cases:

- If $n, m \geqslant 0$, then this is simply Proposition 4.3.10(a).
- If $n, m < 0$, then $n = -p$ and $m = -q$ with p, q positive natural numbers. Thus, $x^n x^m = (1/x^p) \cdot (1/x^q) = 1/(x^p x^q)$ by Definition 4.2.2. But since p, q are positive, this is also equal to $1/(x^{p+q})$ according to Proposition 4.3.10(a). This can also be written $x^{-(p+q)}$ by Definition 4.3.11, which is finally equal to x^{n+m}.
- If $n \geqslant 0$ and $m < 0$ (or inversely, since they are exchangeable), then $m = -q$ with q a positive natural number. Thus, $x^n x^m = x^n \times (1/x^q) = x^n/x^q$. We will (once again) split into two cases:
  - if $n \geqslant -m$, i.e. if $n - q \geqslant 0$, then we can note that $x^{n-q} \cdot x^q = x^n$ according to Proposition 4.3.10(a). Thus, let's multiply both sides of this equality by x^{-q} to get $x^{n-q} = x^n x^{-q}$; which can be rewritten $x^{n+m} = x^n x^m$ as required.
  - if $n < -m$, i.e. $q - n > 0$, then we can note that $x^{q-n} x^n = x^q$ according to Proposition 4.3.10(a). Also, since $n - q < 0$, according to Definition 4.3.11, we have $x^{n-q} = 1/x^{q-n}$. Let's multiply both sides by $1/x^n$, to get $x^{n-q}/x^n = 1/(x^{q-n}x^n) = 1/x^q = x^{-q}$. Finally, multiply both sides by x^n to get $x^{n+m} = x^n x^m$.

(b) Prove that $(x^n)^m = x^{nm}$.

(c) Show that $(xy)^n = x^n y^n$. If $n \geqslant 0$, this is simply Proposition 4.3.10(a). So let's consider the case $n < 0$. In this case, $n = -p$, with p a positive natural number. Thus we have successively:

$$\begin{array}{rcll} (xy)^n & = & (xy)^{-p} & \\ & = & 1/(xy)^p & \text{(by Definition 4.3.11)} \\ & = & 1/(x^p y^p) & \text{(Proposition 4.3.10(a))} \\ & = & 1/(x^p) \times 1/(y^p) & \text{(Definition 4.2.2)} \\ & = & x^{-p} \times y^{-p} & \text{(by Definition 4.3.11)} \\ & = & x^n \times y^n & \end{array}$$

(d) Show that if $x \geqslant y > 0$, then $x^n \geqslant y^n > 0$ if n is positive, and $0 < x^n \leqslant y^n$ if n is negative.

- If $n > 0$, according to Proposition 4.3.10(c), we already have $x^n \geqslant y^n \geqslant 0$, so that we just have to show that the rightmost inequality is strict, i.e. that $y^n > 0$. To show that, we only need to prove $y^n \neq 0$. For the sake of contradiction, let's suppose that $y^n = 0$. Our starting hypothesis was $x \geqslant y > 0$, thus we know that $y \neq 0$. According to Proposition 4.3.10(b), we can't have both $y \neq 0$ and $y^n = 0$, this is a contradiction. Thus, we indeed have $y^n \neq 0$, which shows the inequality $x^n \geqslant y^n > 0$ as required.
- If $n < 0$, this includes an important result, which is that taking the inverse reverses order. Indeed, let's begin by proving that if $x \geqslant y > 0$, then $1/x \leqslant 1/y$. Since both x and y are positive, their product xy is also positive, and $1/(xy)$ is also positive. Following Proposition 4.2.9(e), we can multiply both sides of $x \geqslant y$ by $1/(xy)$ to get $1/y \geqslant 1/x$. Then, we immediately get $(1/y)^p \geqslant (1/x)^p$ for any positive number p by Proposition 4.3.10(c), which can be rewritten $y^n \geqslant x^n$ with $n = -p$ negative. And since both numbers are positive (because x and y are positive), the claim follows.

(e) Prove that if $x, y > 0$ and $n \neq 0$, then $x^n = y^n \implies x = y$. Let's consider two cases: $n > 0$ and $n < 0$.

First, if $n > 0$, suppose for the sake of contradiction that we have both $x^n = y^n$ and $x \neq y$. According to the trichotomy of rationals (Lemma 4.2.7), this last claim means that we have either $x > y$ or $y > x$. Since x and y are exchangeable, we only prove the first case here, $x > y$. In this case, Proposition 4.3.10(c) leads to $x^n > y^n$, which is obviously not compatible with our initial hypothesis $x^n = y^n$. A similar contradiction follows in the case $y > x$. Thus, both $x > y$ and $y > x$ are impossible, and the only possibility is $x = y$.

Now, if $n < 0$, then $n = -p$, with p a positive natural number. Suppose that $x^n = y^n$, i.e. that $x^{-p} = y^{-p}$, or finally $1/x^p = 1/y^p$. From this last equality, by multiplying both sides by $x^p y^p$, we get $y^p = x^p$. We are thus back in the previous case, and obtain $x = y$.

(f) Prove that $|x^n| = |x|^n$. If $n \geqslant 0$, this is simply Proposition 4.3.10(d). So let's consider the case $n < 0$. We'll need a quick lemma:

Lemma. *For all rationals $x \neq 0$, we have $|1/x| = 1/|x|$.*

Proof. If $x > 0$, there is nothing to show. If $x < 0$, then $1/x$ is also negative[15]. Thus, $1/|x| = 1/(-x)$; and $|1/x| = -(1/x)$. And we have clearly $1/(-x) = -(1/x)$ because $1/(-x) + 1/x = 0$. □

In this case, $n = -p$, with p a positive natural number. We have successively:

$$\begin{array}{rcll} |x^n| & = & |x^{-p}| & \\ & = & |1/(x^p)| & \text{(by Definition 4.3.11)} \\ & = & |(1/x)^p| & \text{(Proposition 4.3.12(a))} \\ & = & |1/x|^p & \text{(Proposition 4.3.10(d))} \\ & = & (|1|/|x|)^p & \text{(lemma introduced just above)} \\ & = & 1/|x|^p & \text{(Proposition 4.3.12(a))} \\ & = & |x|^{-p} & \text{(Definition 4.3.11)} \\ & = & |x|^n & \end{array}$$

Exercise 4.3.5. — *Prove that $2^N \geqslant N$ for all positive integers N.*

Let's use induction on N. Since we only consider positive integers, we have here $N \geqslant 1$, and in particular, the base case starts at $N = 1$.

For the base case $N = 1$, the assertion is true, since we have indeed $2^1 \geqslant 1$.

Now suppose inductively that $2^N \geqslant N$, and show that $2^{N+1} \geqslant N + 1$. We have $2^{N+1} = 2^N \times 2 \geqslant N \times 2$ by induction hypothesis. But we know that $2N = N + N$ (recall Definition 2.3.1 for instance), thus we can rewrite this as $2^{N+1} \geqslant N + N$. And since $N \geqslant 1$, we finally get $2^{N+1} \geqslant N + 1$.

Exercise 4.4.1. — *Prove Proposition 4.4.1.*

We have to prove that, for any rational number x, there exists an integer n such that $n \leqslant x < n + 1$. Let's proceed through the following four steps:

- Suppose that $x \in \mathbb{Q}_+$. Thus, $x = a/b$, with a and b natural numbers. According to Proposition 2.3.9, there exists $n, r \in \mathbb{N}$ such that $a = bn + r$, with $0 \leqslant r < b$. By dividing all terms by b, this also means that $x = a/b = n + r/b$, with $0 \leqslant r/b < 1$.

  Since $0 \leqslant r/b < 1$, we have $n \leqslant n + r/b < n + 1$, i.e. $n \leqslant x < n + 1$, as required.

- Now suppose that $x \in \mathbb{Q}_-^\star$. Consequently, $-x \in \mathbb{Q}_+$, and we are back in the previous case: there exists a natural number n such that $n \leqslant -x < n + 1$, i.e. $-n - 1 < x \leqslant -n$. Now we have two possible cases:

  – if $x = -n$, then let be $m = -n$. Thus, $m - 1 < x \leqslant m$, and then $m \leqslant x < m + 1$, as required.

  – if $x \neq -n$, then let be $m = -n - 1$. Thus, $m < x \leqslant m + 1$, i.e. $m - 1 \leqslant x < m$. And by denoting $p = m - 1$, we have $p \leqslant x < p + 1$ as required.

[15] Formally, see Definition 4.2.6, and note that $a \,//\, (-b) = (-a) \,//\, b$ if a and b are positive integers.

- Let's prove that this integer n is unique. Suppose that we have two integers m, n such that:

$$n \leqslant x < n + 1 \tag{4.1}$$

$$m \leqslant x < m + 1 \tag{4.2}$$

From (4.2), we also have $-m - 1 < -x \leqslant -m$. And, by adding this inequality to (4.1), we get $n - m - 1 < 0 < n - m + 1$. The left-hand side says that $n < m + 1$, i.e. that $n \leqslant m$ (recall Proposition 2.2.12 (e)). Similarly, the right-hand side says that $n > m - 1$, i.e. that $n \geqslant m$. Thus, we have both $n \leqslant m$ and $n \geqslant m$, which means that $n = m$.

- Finally, this means in particular that there exists a natural number N such that $N > x$. Indeed, if x is negative, then $N = 0$ is suitable; and if x is positive, then N is directly given by $N = \lfloor x \rfloor + 1$.

Exercise 4.4.2. — *A sequence $a_0, a_1, a_2, \ldots$ of numbers (natural numbers, integers, rationals, or reals) is said to be in* infinite descent *if we have $a_n > a_{n+1}$ for all natural numbers n (i.e., $a_0 > a_1 > a_2 > \ldots$).*

1. *Prove the principle of infinite descent: that it is not possible to have a sequence of natural numbers which is in infinite descent.*

2. *Does the principle of infinite descent work if the sequence $a_1, a_2, a_3, \ldots$ is allowed to take integer values instead of natural number values? What about if it is allowed to take positive rational values instead of natural numbers? Explain.*

We follow the hints given by Terence Tao.

1. Assume for the sake of contradiction that we have a sequence of natural numbers (a_n) which is in infinite descent. Let k be a natural number, and P_k be the property "$a_n \geqslant k$ for all natural numbers n". Let's induct on k.

   For the base case, P_0 is true since a_n are natural numbers for all n, so that $a_n \geqslant 0$ for all n by definition.

   Now let's suppose inductively that P_k is true, i.e. that $a_0 > a_1 > a_2 > \ldots \geqslant k$. If we had $a_p = k$ for one given natural number p, then we would have $k = a_p > a_{p+1}$. But also, $a_{p+1} > a_{p+2} > \ldots > k$ by induction hypothesis. However, the inequality $k \geqslant a_{p+1} > k$ is a contradiction, so that $a_n \neq k$ for all n. Thus, P_{k+1} is also true: we have $a_n > k + 1$ for all n.

   However, having $a_n > k$ for all natural numbers k, n is a contradiction. Indeed, for $k = a_0$ and $n = 1$, we have $a_1 > a_0$, which contradicts the fact that (a_n) is in infinite descent.

   Thus, there are no such sequence of natural numbers.

2. A general note: to prove that the infinite descent principle does not work for integers or rationals, it is enough to find *one* sequence of such numbers which is actually in infinite descent. Instead of a formal proof as in the previous case, a simple counterexample will do the trick.

- If the sequence $a_0 > a_1 > \ldots$ can take integer values, lets define the sequence by $a_n = -n$. By definition, we have $a_n > a_{n+1}$ for all natural number n (since $-n > -n-1$, as a simple induction will show).
- If the sequence $a_0 > a_1 > \ldots$ can take rational values, lets define the sequence by $a_n = 1/n$. Thus, we have $a_n > a_{n+1}$ for all natural number n, since $1/n > 1/(n+1)$. (This can be shown as follows: $1/n - 1/(n+1) = 1/(n(n+1)) > 0$.)

5. The real numbers

Exercise 5.1.1. — *Prove Lemma 5.1.15, i.e. that every Cauchy sequence is bounded.*

Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence.

- By definition 5.1.8, for every rational $\varepsilon > 0$, there exists a natural number N such that if $j, k \geqslant N$, then $d(a_j, a_k) \leqslant \varepsilon$. In particular, let's rephrase this statement with the arbitrary value $\varepsilon = 1$ (valid, since 1 is a positive rational): there exists a natural number N such that if $j, k \geqslant N$, then $|a_j - a_k| \leqslant 1$.

  Since $N \geqslant N$, we can take in particular $k = N$ to get yet another particular formulation: if $j \geqslant N$, then $|a_j - a_N| \leqslant 1$.

  According to Proposition 4.3.3(b), we have $|x + y| \leqslant |x| + |y|$ for all rationals x, y. Let's consider $x = a_j - a_N$ and $y = a_N$: this leads to $|a_j| \leqslant |a_j + a_N| + |a_N|$, i.e. $|a_j| - |a_N| \leqslant |a_j - a_N|$.

  Thus, this means that $|a_j| - |a_N| \leqslant |a_j - a_N| \leqslant 1$ as soon as $j \geqslant N$, i.e. that $|a_j| \leqslant 1 + |a_N|$ for $j \geqslant N$. We have bounded part of the infinite sequence.

- The other part is simply the finite sequence $a_0, a_1, \cdots, a_{N-1}$. By Lemma 5.1.14, this finite sequence is necessarily bounded by a rational number M.

- Finally, let's consider the rational number $B = 1 + |a_N| + M$. Since we have both $B \geqslant M$ and $B \geqslant 1 + |a_N|$, both the infinite sequence $(a_n)_{n=N}^{\infty}$ and the finite sequence $a_0, \cdots, a_{N-1}$ are bounded by B. Thus, the whole Cauchy sequence $(a_n)_{n=1}^{\infty}$ is bounded by B.

Exercise 5.2.1. — *Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence if and only if $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.*

First note that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are exchangeable here, so that showing only one direction ("if $(a_n)_{n=1}^{\infty}$ is Cauchy, then $(b_n)_{n=1}^{\infty}$ is Cauchy") will be enough.

Let be $\varepsilon > 0$ a positive rational. Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, there exists a natural number N_1 such that $n \geqslant N_1 \Longrightarrow |a_n - b_n| \leqslant \frac{\varepsilon}{3}$. Furthermore, since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists a natural number N_2 such that $j, k \geqslant N_2 \Longrightarrow |a_j - a_k| \leqslant \frac{\varepsilon}{3}$.

Let be $N = \max(N_1, N_2)$. If $j, k \geqslant N$, then we have:

$$\begin{aligned}
|b_j - b_k| &= |b_j - a_j + a_j - a_k + a_k - b_k| \\
&\leqslant |b_j - a_j| + |a_j - a_k| + |a_k - b_k| \quad \text{(by triangle inequality)} \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&\leqslant \varepsilon
\end{aligned}$$

which means that $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Exercise 5.2.2. — *Let $\varepsilon > 0$. Show that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded.*

As in the previous exercise, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are exchangeable here, so that showing only one direction will be enough.

- Since $(a_n)_{n=1}^{\infty}$ is bounded, there exists a rational number M_1 such that $|a_n| \leqslant M_1$ for all natural n.
- Since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are eventually ε-close, there exists a positive natural number N such that $n \geqslant N \implies |a_n - b_n| \leqslant \varepsilon$.
- Let's decompose $(b_n)_{n=1}^{\infty}$ into a finite and an infinite part, and show that both parts are bounded.

  By Lemma 5.1.14, there exists a positive rational M_2 such that the finite sequence $b_0, \dots, b_{N-1}$ is bounded by M_2.

  Furthermore, we know by triangle inequality that, if $n \geqslant N$, we have $|b_n| - |a_n| \leqslant |b_n - a_n| \leqslant \varepsilon$. Consequently, $|b_n| \leqslant |a_n| + \varepsilon \leqslant M_1 + \varepsilon$.

  Finally, let be $M = M_1 + M_2 + \varepsilon$: we have indeed $|b_n| \leqslant M$ for all natural n.

Exercise 5.3.1. — *Prove Proposition 5.3.3.*

The three laws of equality must be verified.

1. Reflexivity: let's prove that $x = x$. By Definition 5.3.1, we have $x = x$ if and only if $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} a_n$, i.e. if and only if $(a_n)_{n=1}^{\infty}$ and $(a_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. Let be $\varepsilon > 0$ a positive rational, and let be $N = 1$. For all $n \geqslant N$, we have $|a_n - a_n| = 0 \leqslant \varepsilon$, QED.
2. Symmetry: let's suppose that $x = y$, and let's show that $y = x$. Let $\varepsilon > 0$ be a positive rational. We have:

$$\begin{aligned} x = y &\implies (a_n)_{n=1}^{\infty} \text{ and } (b_n)_{n=1}^{\infty} \text{ are equivalent Cauchy sequences} \\ &\implies \exists N \geqslant 1 \text{ such that } |a_n - b_n| \leqslant \varepsilon \text{ for } n \geqslant N \\ &\implies \exists N \geqslant 1 \text{ such that } |b_n - a_n| \leqslant \varepsilon \text{ for } n \geqslant N \\ &\implies (b_n)_{n=1}^{\infty} \text{ and } (a_n)_{n=1}^{\infty} \text{ are equivalent Cauchy sequences} \\ &\implies y = x \end{aligned}$$

3. Transitivity: if $x = y$ and $y = z$. Let be $\varepsilon > 0$ a positive rational. Since $x = y$, there exists $N_1 \geqslant 1$ such that $|a_n - b_n| \leqslant \varepsilon/2$ for $n \geqslant N_1$. Since $y = z$, there exists $N_2 \geqslant 1$ such that $|b_n - c_n| \leqslant \varepsilon/2$ for $n \geqslant N_2$. Thus, if $n \geqslant \max(N_1, N_2)$, a_n and b_n are $\varepsilon - 2$-close, and b_n and c_n are $\varepsilon - 2$-close. Thus, according to Proposition 4.3.7(c), a_n and c_n are ε-close, which means that $(a_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ are eventually ε-close for all ε, i.e. are equivalent Cauchy sequences. This closes the proof.

Exercise 5.3.2. — *Prove Proposition 5.3.10.*

To prove that xy is a real number, we must show that $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence. Let $\varepsilon > 0$ be a positive rational. We must show that there exists a natural number N such that $j, k \geqslant N \implies |a_j b_j - a_k b_k| \leqslant \varepsilon$.

We already know that:

- Since $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is bounded by a rational number M_a. Furthermore, there exists a natural number N_a such that $j, k \geqslant N_a \implies |a_j - a_k| \leqslant \frac{\varepsilon}{2M_a}$.

- In a similar fashion, since $(b_n)_{n=1}^{\infty}$ is a Cauchy sequence, it is bounded by a rational number M_b. Furthermore, there exists a natural number N_b such that $j, k \geqslant N_b \implies |b_j - b_k| \leqslant \frac{\varepsilon}{2M_b}$.

Now let's consider $N = \max(N_a, N_b)$. If $j, k \geqslant N$, we have:

$$\begin{aligned}|a_j b_j - a_k b_k| &= |a_j b_j - a_j b_k + a_j b_k - a_k b_k| \\ &\leqslant |a_j| \cdot |b_j - b_k| + |b_k| \cdot |a_j - a_k| \\ &\leqslant M_a \frac{\varepsilon}{2M_a} + M_b \frac{\varepsilon}{2M_b} \\ &\leqslant \varepsilon\end{aligned}$$

This proves that $(a_n b_n)_{n=1}^{\infty}$ is a Cauchy sequence, as required.

Exercise 5.3.3. — *Let a, b be rational numbers. Show that $a = b$ if and only if $LIM_{n\to\infty} a = LIM_{n\to\infty} b$ (i.e., the Cauchy sequences $a, a, a, a, \ldots$ and $b, b, b, b, \ldots$ are equivalent if and only if $a = b$).*

In what follows, we denote $(a_n)_{n=1}^{\infty}$ the constant sequence $a, a, a, \ldots$, and $(b_n)_{n=1}^{\infty}$ the constant sequence $b, b, b, \ldots$.

- If $a = b$, we have $|a - b| = 0$, i.e. $|a_n - b_n| = 0$ for all $n \geqslant 1$. Thus, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are ε-close for all $\varepsilon > 0$: they are equivalent Cauchy sequences. This means that $a = b$.
- If $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$, let's suppose (for the sake of contradiction) that $a \neq b$, and let's denote $\varepsilon = \frac{|a-b|}{2}$. Since $a \neq b$, we have $\varepsilon > 0$., and we also have $|a-b| > \varepsilon$. In other words, for all $n \geqslant 1$, we have found an ε such that $|a_n - b_n| > \varepsilon$: this is a contradiction with the fact that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent Cauchy sequences. Thus, $a = b$.

Exercise 5.3.4. — *Let $(a_n)_{n=1}^{\infty}$ be a sequence of rational numbers which is bounded. Let $(b_n)_{n=1}^{\infty}$ be another sequence of rational numbers which is equivalent to $(a_n)_{n=1}^{\infty}$. Show that $(b_n)_{n=1}^{\infty}$ is also bounded.*

This exercise is actually very close to Exercise 5.2.2, and the same proof could apply. But for short: in Exercise 5.2.2, we showed that if two sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are ε-close for a given positive rational ε, then $(a_n)_{n=1}^{\infty}$ is bounded if and only if $(b_n)_{n=1}^{\infty}$ is bounded. Here, since $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent, they are ε-close for any positive ε, and $(a_n)_{n=1}^{\infty}$ is bounded by hypothesis. Thus, $(b_n)_{n=1}^{\infty}$ is also bounded.

Exercise 5.3.5. — *Show that $LIM_{n\to\infty} 1/n = 0$.*

As a real number, by definition, 0 is the formal limit of the constant sequence $0, 0, 0, \ldots$. Thus, we must show that $\mathrm{LIM}_{n\to\infty} 1/n = \mathrm{LIM}_{n\to\infty} 0$. Still by definition, we must prove that the sequence $(1/n)_{n=1}^{\infty}$ is equivalent to the sequence $(0)_{n=1}^{\infty}$. This can be achieved by proving that for all $\varepsilon > 0$, there exists a natural $N \geqslant 1$ such that $n \geqslant N \implies |1/n - 0| \leqslant \varepsilon$, i.e. such that $n \geqslant 1/\varepsilon$.

By Proposition 4.4.1, there always exists a natural number N such that $N > 1/\varepsilon$. Then, if $n \geqslant N$, we have the desired proprty, which closes the proof.

Exercise 5.4.1. — *Prove Proposition 5.4.4.*

Let x be a real number.

1. First we show that at most one of the three statements above is true. To do this, we show that all those statements are "pairwise incompatible".

   - First suppose that we have both $x = 0$ and x positive. If $x = 0$, then x is the formal limit of the sequence $0, 0, \ldots$. If x is positive, then $x = \mathrm{LIM}_{n\to\infty} a_n$ with $a_n \geqslant c$ for all n and a certain rational $c > 0$. By Definition 5.3.1, this implies that both sequences $0, 0, \ldots$ and $(a_n)_{n=1}^{\infty}$ must be equivalent Cauchy sequences. I.e., for all $\varepsilon > 0$, there must exist $M \geqslant 1$ such that $|a_n - 0| \leqslant \varepsilon$ for all $n \geqslant M$. But taking $\varepsilon = c/2$ leads to an obvious contradiction: we cannot have both $a_n \geqslant c$ and $|a_n| \leqslant c/2$ for all $n \geqslant M$. Thus, x cannot be both zero and positive.
   - A similar argument shows that x cannot be both zero and negative.
   - Finally, we show that x cannot be both positive and negative. If x is positive, then $x = \mathrm{LIM}_{n\to\infty} a_n$, with $(a_n)_{n=1}^{\infty}$ positively bounded away from 0, i.e. $a_n \geqslant c$ for a certain rational $c > 0$. Similarly, if x is negative, then $x = \mathrm{LIM}_{n\to\infty} b_n$, with $(b_n)_{n=1}^{\infty}$ negatively bounded away from 0, i.e. $b_n \leqslant -d$, or $-b_n \geqslant d$, for a certain rational $d > 0$. But then, according to Definition 5.3.1, $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ should be equivalent Cauchy sequences, i.e. for any rational $\varepsilon > 0$, there should exist $M \geqslant 1$ such that $|a_n - b_n| \leqslant \varepsilon$ for $n \geqslant M$. But since $a_n - b_n$ is positive, this also can be written simply $a_n - b_n \leqslant \varepsilon$. But since we have both $a_n \geqslant c$ and $-b_n \geqslant d$, we know that $a_n - b_n \geqslant c + d$ for all natural numbers n, an obvious contradiction (take $\varepsilon = (c + d)/2$).
   - Thus, at most one of the three statements is true.

2. Next we show that at least one of the statements "x is zero", "x is positive", "x is negative" is true. Actually, if $x = 0$, we know that the statement "x is zero" is true, so we're done. We can thus suppose that $x \neq 0$, and we have just to show that x is either positive or negative.

   If $x \neq 0$, then by Lemma 5.3.14, $x = \mathrm{LIM}_{n\to\infty} a_n$ with $(a_n)_{n=1}^{\infty}$ bounded away from 0, i.e. there exists $c > 0$ such that $|a_n| \geqslant c$ for any natural n.

   It turns out that, for any Cauchy sequence $(a_n)_{n=1}^{\infty}$, if $(a_n)_{n=1}^{\infty}$ is bounded away from 0, it is either eventually positively bounded away from 0, or eventually negatively bounded away from 0.

   Indeed, suppose that we have at the same time $|a_n| \geqslant \varepsilon$ for all $n \geqslant N$, $a_k \geqslant \varepsilon$ for some $k \geqslant N$ and $a_j \leqslant -\varepsilon$ for some $j \geqslant N$. In such a case, we would have by triangular inequality:

   $$|a_k - a_j| \geqslant ||a_k| - |a_j|| \geqslant |\varepsilon + \varepsilon| \geqslant 2\varepsilon$$

   Thus, for all N, we could find two indexes $j, k \geqslant N$ such that $|a_k - a_j| > \varepsilon$, which contradicts that fact that $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. Consequently, the Cauchy sequence $(a_n)_{n=1}^{\infty}$ is either eventually positively bounded away from 0, or eventually negatively bounded away from 0.

   In the first case, $x = \mathrm{LIM}_{n\to\infty} a_n$ is a positive number[16], and in the second case it is a negative number.

[16] Actually, we also have to define an equivalent sequence $(b_n)_{n=1}^{\infty}$ defined by $a_n = b_n$ if $n \geqslant N$ and $b_n = 1$ otherwise to get what we want.

Exercise 5.4.2. — *Prove the remaining claims in Proposition 5.4.7.*

Let x, y, z be real numbers. Those claims are as follows:

(a) (Order trichotomy) Prove that exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

This is actually a simple rephrasing of the trichotomy of real numbers. Indeed, let's consider the real number $x - y$. According to Proposition 5.4.4, this real number is either null (in this case, $x = y$), positive (in this case, $x > y$) or negative (in this case, $x < y$).

(b) (Order is anti-symmetric) Prove that one has $x < y$ if and only if $y > x$.

The following statements are equivalent:

$$\begin{aligned} x < y &\iff x - y \text{ is positive} \\ &\iff -(x - y) \text{ is negative (Prop. 5.4.4)} \\ &\iff y - x \text{ is negative} \\ &\iff y > x \end{aligned}$$

(c) (Order is transitive) Prove that if $x < y$ and $y < z$, then $x < z$.

Note that $x - z = (x - y) + (y - z)$. In this sum, both terms are negative, so that their sum is also negative (this can easily be deduced from Proposition 5.4.4). Thus, $x - z$ is negative, which closes the proof.

(d) (Addition preserves order) Prove that if $x < y$, then $x + z < y + z$.

Note that if $x < y$, then $x - y$ is negative. Since $x - y = (x + z) - (y + z)$, we get the statement required.

(e) is already proven in the main text.

Exercise 5.4.4. — *Show that for any positive real number $x > 0$ there exists a positive integer N such that $x > 1/N > 0$.*

Let's use the archimedean property of the reals (Proposition 5.4.13), with $\varepsilon = 1$. Since x is a positive real number, there exists a positive integer N such that $Nx > 1$. Multiplying this inequality by the (positive[17]) rational $1/N$ leads to $x > 1/N$. Since we have $1/N > 0$, the claim follows.

Exercise 5.4.5. — *Prove Proposition 5.4.14: Given any two real numbers $x < y$, we can find a rational number q such that $x < q < y$.*

Following the hint given by Terence Tao, we can make use of Exercise 5.4.4. Actually, we do have a positive real number here: since $x < y$, then $y - x$ is positive. Thus, according to Exercise 5.4.4, we can find a positive integer N such that $y - x > 1/N > 0$.

Now let's multiply all terms by N. Since N is positive, we get the following inequality: $Ny - Nx > 1 > 0$. Intuitively: since $Ny - Nx > 1$, we should be able to find explicitly an integer lying between them. Then, dividing by N will provide the desired inequality.

[17] According to Proposition 5.4.8.

Nx is a real number. According to Exercise 5.4.3, there exists an integer n such that $n+1 > Nx \geqslant n$. In particular, since $Nx \geqslant n$, we also have $Nx + 1 \geqslant n+1$.

Thus, gathering all the inequalities we know:

$$Ny > Nx + 1 \geqslant n + 1 > Nx \tag{5.1}$$

Then let's divide by N: we finally get $y > \frac{n+1}{N} > x$, which is the required property with $q = \frac{n+1}{N}$.

Exercise 5.4.6. — *Let x, y be real numbers and let $\varepsilon > 0$ be a positive real. Show that $|x-y| < \varepsilon$ if and only if $y - \varepsilon < x < y + \varepsilon$; and that $|x-y| \leqslant \varepsilon$ iff $y - \varepsilon \leqslant x \leqslant y + \varepsilon$.*

We only give the proof for the strict version; the other one being totally similar.

- First suppose that $|x-y| < \varepsilon$, and let's consider two cases depending on the sign of $x - y$.
  - If $x - y \geqslant 0$, i.e. if $x \geqslant y$, then $|x-y| = x - y < \varepsilon$ by hypothesis. If we add y to both sides of this inequality, we get $x < y + \varepsilon$, which is the first part of the required result. Furthermore, we know that $y \leqslant x$. And since ε is positive, we have $y - \varepsilon < y \leqslant x$. Thus, by combining all those results, we finally get $y - \varepsilon < y \leqslant x \leqslant y + \varepsilon$, as required.
  - If $x - y < 0$, i.e. if $x < y$, then $|x-y| = y - x < \varepsilon$ by hypothesis. This leads to $y - \varepsilon < x$, which is the first part of the result. Also, since $x < y$, we have $x < y < y + \varepsilon$. Combining all those results, we finally have $y - \varepsilon < x < y < y + \varepsilon$ as required.
- Conversely, suppose that $y - \varepsilon < x < y + \varepsilon$. Adding $(-y)$ to each part leads to $-\varepsilon < x - y < \varepsilon$. There are now three cases depending on the sign of $x - y$:
  - If $x - y > 0$, then $|x-y| = x - y < \varepsilon$ as required.
  - If $x - y < 0$, then $|x-y| = y - x$. But we know that $y - \varepsilon < x$, i.e. $y - x < \varepsilon$, as required.
  - If $x - y = 0$, then by definition, $|x-y| = 0 < \varepsilon$, as required.

Exercise 5.4.7. — *Let x and y be real numbers. Show that $x \leqslant y + \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x \leqslant y$. Show that $|x-y| \leqslant \varepsilon$ for all real numbers $\varepsilon > 0$ if and only if $x = y$.*

1. Let's prove the first statement. It is obvious that if $x \leqslant y$, then $x \leqslant y + \varepsilon$ for all $\varepsilon > 0$. Let's thus prove the other direction. Let's use the contrapositive statement: suppose that $x > y$. Thus, $x - y > 0$. Now let δ be a positive real number defined by $\delta = \frac{x-y}{2} > 0$. We have:

$$\begin{aligned} y + \delta &= y + \frac{x}{2} + \frac{x}{2} \\ &= \frac{x}{2} + \frac{y}{2} \\ &< \frac{x}{2} + \frac{x}{2} = x \end{aligned}$$

Thus, if $x > y$, there exists $\delta > 0$ such that $x > y + \delta$. Using the contrapositive, we conclude that if $x \leqslant y + \varepsilon$ for all $\varepsilon > 0$, then $x \leqslant y$.

2. The second statement has a very similar proof. Once again, it is obvious that if $x = y$, then $|x - y| \leqslant \varepsilon$, since we have $|x - y| = 0$. For the other direction, let's use the contrapositive once again, and suppose that $x \neq y$. Thus, $|x-y| > 0$, and in particular, $|x - y| > \frac{|x-y|}{2} > 0$. Consequently, if $x \neq y$, there exists $\delta = \frac{|x-y|}{2} > 0$ such that $|x - y| > \delta$. The contrapositive means that if $|x - y| \leqslant \varepsilon$ for all $\varepsilon > 0$, then $x = y$, as required.

Exercise 5.4.8. — *Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and let x be a real number. Show that if $a_n \leqslant x$ for all $n \geqslant 1$, then $\mathrm{LIM}_{n\to\infty} a_n \leqslant x$. Similarly, show that if $a_n \geqslant x$ for all $n \geqslant 1$, then $\mathrm{LIM}_{n\to\infty} a_n \geqslant x$.*

Suppose, for the sake of contradiction, that we have both $a_n \leqslant x$ for all $n \geqslant 1$ and $\mathrm{LIM}_{n\to\infty} a_n > x$. Then, according to Proposition 5.4.14, there exists a rational q such that $x < q < \mathrm{LIM}_{n\to\infty} a_n$. We have thus $a_n \leqslant x < q$ for all $n \geqslant 1$. According to Corollary 5.4.10 with the constant sequence of rationals $(q_n)_{n=:}^{\infty} = q$, we should have $\mathrm{LIM}_{n\to\infty} a_n \leqslant q$.

Thus, we have both $\mathrm{LIM}_{n\to\infty} a_n \leqslant q$ and $x < q < \mathrm{LIM}_{n\to\infty} a_n$, which is a contradiction and closes the proof.

Exercise 5.5.1. — *Let E be a subset of the real numbers $\mathbb{R}$, and suppose that E has a least upper bound M which is a real number, i.e., $M = \sup(E)$. Let $-E$ be the set $-E := \{-x : x \in E\}$. Show that $-M$ is the greatest lower bound of $-E$, i.e., $-M = \inf(-E)$.*

According to the definition of the greatest lower bound, we have two separate things to show: first, that $-M$ is a lower bound for $-E$, and second, that any other lower bound for $-E$ is inferior to $-M$.

1. Let z be an element of $-E$. By definition, we have $z = -x$ for some $x \in E$. Since $x \in E$, we have $x \leqslant M$, i.e. $z = -x \geqslant -M$. This means that $-M$ is a lower bound for $-E$.

2. Let be $-A$ another lower bound for $-E$, and show that $-A \geqslant -M$. Let be $x \in E$. Thus, we have $-x \in -E$, and by definition, $-x \geqslant -A$. This can also be written $x \leqslant A$, meaning that A is an upper bound for E. But by definition of the least upper bound M, we have $A \leqslant M$, i.e. $-A \geqslant -M$ as required: this closes the proof.

Exercise 5.5.2. — *Let E be a non-empty subset of $\mathbb{R}$, let $n \geqslant 1$ be an integer, and let $L < K$ be integers. Suppose K/n is an upper bound for E, but that L/n is not an upper bound for E. Without using Theorem 5.5.9, show that there exists an integer $L < m \leqslant K$ such that m/n is an upper bound for E, but that $(m-1)/n$ is not an upper bound for E. (Hint: prove by contradiction, and use induction.)*

This is not an easy exercise, so that we begin by an informal draft of the proof. Tao's advice is to prove by contradiction, i.e. to suppose that the following statement holds:

$(\mathcal{H})$: there exists no integer m such that $L < m \leqslant K$, where m/n is an upper bound for E, but $\frac{m-1}{n}$ is not.

What would be the contradiction if we accept this fact? Let's proceed by "descending induction". Let's start with $m = K$: we already know that K/n is an upper bound, thus according to $(\mathcal{H})$, $(K-1)/n$ is necessarily also an upper bound. Now let's continue with $m = K-1$: we can see that, necessarily, $(K-2)/n$ should also be an upper bound. And so on, until we finally reach (after $K-L-1$ steps) $m = L+1$, which will be still supposed to be an upper bound at this stage; but according to $(\mathcal{H})$, L/n should also be an upper bound, which would be a contradiction with the fundamental assumption of this exercise.

Thus, we have to combine, in some way, an induction reasoning with this hypothesis $(\mathcal{H})$ that we would like to reject.

Actually, if we suppose that $(\mathcal{H})$ is true, we can show by induction that for any natural j, $(K-j)/n$ is an upper bound for E:

- The base case $j = 0$ is straightforward: we already know that K/n is an upper bound.
- Now let's suppose inductively that $(K-j)/n$ is an upper bound, and let's show that $(K-j-1)/n$ is also an upper bound. We know that L/n is not an upper bound, so there must exist some $x_0 \in E$ such that $x_0 > L/n$. But since $(K-j)/n$ is an upper bound, we have the following inequalities: $L/n < x_0 \leqslant (K-j)/n \leqslant K/n$. In particular, this means that $L < K-j$; and this also means that $L < K-j \leqslant K$. Thus, we also have necessarily $(K-j-1)/n$ as an upper bound, otherwise the integer $(K-j)$ would be a contradiction for $(\mathcal{H})$. This closes the induction.

Now the contradiction appears formally: take the (positive) natural number $K-L$, and apply the previous statement proved by induction. We should have $(K-(K-L))/n = L/n$ as an upper bound for E, which is a contradiction with the main assumption of this exercise.

Exercise 5.5.3. — *Let E be a non-empty subset of $\mathbb{R}$, let $n \geqslant 1$ be an integer, and let m, m' be integers with the properties that m/n and m'/n are upper bounds for E, but $(m-1)/n$ and $(m'-1)/n$ are not upper bounds for E. Show that $m = m'$. This shows that the integer m constructed in Exercise 5.5.2 is unique.*

We will show successively that $m' \leqslant m$ and $m \leqslant m'$, which will imply that $m = m'$.

- Since $(m-1)/n$ is not an upper bound for E, there exists $x_0 \in E$ such that:
$$x_0 > (m-1)/n \tag{5.2}$$
- But since $x_0 \in E$ and m'/n is an upper bound, we have actually:
$$(m-1)/n < x_0 \leqslant m'/n \tag{5.3}$$
- Similarly, since $(m'-1)/n$ is not an upper bound for E, there exists $x_1 \in E$ such that:
$$x_1 > (m'-1)/n \tag{5.4}$$
- But since $x_1 \in E$ and m/n is an upper bound, we have actually:
$$(m'-1)/n < x_1 \leqslant m/n \tag{5.5}$$
- Thus, combining (5.3) and (5.5), we have both $m'-1 < m$ and $m-1 < m'$. But m and m' are integers: recall that for integers, $a-1 < b$ and $a \leqslant b$ are equivalent (see for instance Proposition 2.2.12 for the naturals). Thus we have both $m' \leqslant m$ and $m \leqslant m'$, as required.

Exercise 5.5.4. — *Let $q_1, q_2, q_3, \dots$ be a sequence of rational numbers with the property that $|q_n - q_{n'}| \leqslant 1/M$ whenever $M \geqslant 1$ is an integer and $n, n' \geqslant M$. Show that $q_1, q_2, q_3, \dots$ is a Cauchy sequence. Furthermore, if $S := LIM_{n\to\infty} q_n$, show that $|q_M - S| \leqslant 1/M$ for every $M \geqslant 1$. (Hint: use Exercise 5.4.8.)*

1. Let $\varepsilon > 0$ be a positive rational number. To show that $(q_n)_{n=1}^{\infty}$ is a Cauchy sequence, we must prove that there exists a natural number $N \geqslant 1$ such that $n, n' \geqslant N \implies |q_n - q_{n'}| \leqslant \varepsilon$.

   Let's apply the archimedean property[18]: since 1 and ε are both positive real numbers, there exists a natural number M such that $M\varepsilon \geqslant 1$, i.e. such that $1/M \leqslant \varepsilon$.

   Thus, for a given value of ε, taking this natural number M provides the required result. Indeed, if $n, n' \geqslant M$, we have $|q_n - q_{n'}| \leqslant 1/M$ by initial hypothesis, and $1/M \leqslant \varepsilon$ by archimedean property. Thus $|q_n - q_{n'}| \leqslant \varepsilon$, as required.

2. Let be $S = \mathrm{LIM}_{n\to\infty} q_n$. We want to show that $|q_M - S| \leqslant 1/M$ for all $M \geqslant 1$.

   Recall that $|q_n - q_{n'}| \leqslant 1/M$ for any $n, n' \geqslant M$ by initial hypothesis. By fixing $n' := M$ to a given value, we get in particular:

   $$\forall n \geqslant M \geqslant 1, \quad |q_n - q_M| \leqslant \frac{1}{M} \tag{5.6}$$

   This is equivalent to:

   $$\forall n \geqslant M \geqslant 1, \quad -\frac{1}{M} + q_M \leqslant q_n \leqslant \frac{1}{M} + q_M \tag{5.7}$$

   At this point, we cannot immediately apply the insight from Exercise 5.4.8 because of this "$n \geqslant M \geqslant 1$" part (we would like to have this property for "$n \geqslant 1$" instead). We can overcome this difficulty by defining a new "instrumental" sequence $(a_n)_{n=1}^{\infty}$:

   $$a_n = \begin{cases} q_M & \text{if } n \leqslant M \\ q_n & \text{if } n > M \end{cases} \tag{5.8}$$

   The sequences $(a_n)_{n=1}^{\infty}$ and $(q_n)_{n=1}^{\infty}$ are equivalent since they are eventually equal. Thus, $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} q_n$ by definition of a real number.

   And this time, we can adapt the insight from (5.7) by removing the problematic "$n \geqslant M$" part: we have actually, for all $n \geqslant 1$,

   $$\forall n \geqslant M \geqslant 1, \quad -\frac{1}{M} + q_M \leqslant a_n \leqslant \frac{1}{M} + q_M \tag{5.9}$$

   so that, according to Exercise 5.4.8, we have $-1/M + q_M \leqslant S \leqslant 1/M + q_M$, i.e. $|q_M - S| \leqslant 1/M$.

   This is valid for any given $M \geqslant 1$, and thus closes the proof.

[18] This has actually previously been shown in Exercise 5.4.4.

Exercise 5.5.5. — *Establish an analogue of Proposition 5.4.14, in which "rational" is replaced by "irrational".*

We have to prove that there exists an irrational number s between any real numbers x, y. Actually, we only know one concrete example of irrational number so far in Tao's text: the number $\sqrt{2}$ (see Proposition 4.4.4). We thus should use it in this proof.

Let be x, y two real numbers, and consider the two real numbers $x + \sqrt{2}$ and $y + \sqrt{2}$. According to Proposition 5.4.14, there exists a rational number q such that $x + \sqrt{2} < q < y + \sqrt{2}$; i.e. $x < q - \sqrt{2} < y$.

But it is easy to show that $q - \sqrt{2}$ cannot be a rational number: if it was a rational, then we would have $q - \sqrt{2} = q'$, with q' a rational number. And then, we would have $\sqrt{2} = q - q' \in \mathbb{Q}$, which would contradict Proposition 4.4.4.

Thus we can indeed construct an irrational number between any pair of real numbers x, y.

Exercise 5.6.1. — *Prove Lemma 5.6.6. (Hints: review the proof of Proposition 5.5.12; proofs by contradiction will be useful.)*

The claims to prove are:

(a) If $y = x^{1/n}$, then $y^n = x$.

To prove this one, we can basically go back to Proposition 5.5.12 and adapt its proof, as suggested by Terence Tao. The general sketch is as follows: by Definition 5.6.4, we have $y := \sup\{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$. We should show that both the assertions $y^n > x$ and $y^n < x$ lead to contradictions with this definition. What could be such contradictions? For instance, it could be the fact that there exists a greater number than y in the set $\{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$, i.e. that y is *not* an upper bound. To show that, we must find a small number $\zeta > 0$ such that $(y + \zeta) \in \{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$, or in other words, $(y + \zeta)^n \leqslant x$. But another possible contradiction could be to show that y is an upper bound but not the *least* upper bound, i.e. that there exists a small number $0 < \varepsilon < 1$ such that $y - \varepsilon$ is also an upper bound of the set $\{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$.

Starting with each of the two hypotheses $y^n > x$ and $y^n < x$, we'll try to obtain one of these two contradictions.

- First suppose that we have $y^n < x$. Here we will refrain from using the binomial theorem in the proof, but we will use it as an intuition. Let be a small real number $0 < \varepsilon < 1$. First, we will show that if $y^n < x$, then we can find a real number $M > 0$ such that $(y + \varepsilon)^n \leqslant y^n + M\varepsilon$. This would be easiest with the binomial theorem, but it can also by shown by induction. Indeed, the base case $n = 1$ is obvious, since $M = 1$ is okay. Now, let's suppose inductively that there exists a natural number M such that we have $(y + \varepsilon)^n \leqslant y^n + M\varepsilon$. Thus:

$$\begin{aligned}(y + \varepsilon)^{n+1} &\leqslant (y + \varepsilon)(y^n + M\varepsilon) \\ &\leqslant y^{n+1} + \varepsilon(My + y^n + M\varepsilon) \\ &\leqslant y^{n+1} + \varepsilon\underbrace{(My + y^n + M)}_{:=M'} \text{ (because } \varepsilon < 1)\end{aligned}$$

  i.e. there exists also a natural number M' such that $(y + \varepsilon)^{n+1} \leqslant y^{n+1} + M'\varepsilon$, which closes the induction. And thus, we have: $(y + \varepsilon)^n \leqslant y^n + M\varepsilon < y^n < x$, for some $\varepsilon > 0$. This means that y is no longer the supremum of the set $\{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$, which contradicts our initial hypothesis. Thus, $y^n < x$ leads to a contradiction.

- Then suppose that we have $y^n > x$. A similar approach applies. Let $0 < \varepsilon < 1$ be a small number. Although not detailed here, it is now quite easy to show in a similar fashion that there exists a real number M such that $(y - \varepsilon)^n \geqslant y^n - M\varepsilon$. But since $y^n > x$ (strictly), we know that there exists a small number $\zeta > 0$ such that $y^n > y^n - \zeta > x$. If we choose ε so that we have $M\varepsilon = \zeta$, then we get $(y - \varepsilon)^n > x$. This means that there exists a smaller number than y which is also an upper bound for $\{z \in \mathbb{R} \mid z \geqslant 0, z^n \leqslant x\}$, which contradicts the definition of y as the supremum of this set.

Thus, both the statements $y^n > x$ and $y^n < x$ are impossible, which allows us to conclude that $y^n = x$, as required.

Note that, in other words, we've just showed that for any non-negative real x, we have:

$$\left(x^{1/n}\right)^n = x \tag{5.10}$$

and this is thus something we can use in the next steps.

(b) Conversely, if $y^n = x$, then $y = x^{1/n}$. (Additional hint: use the previous result, and Proposition 4.3.12.)

Suppose that we have $y^n = x$. Since the n-th root is well-defined, the n-th roots of two equal numbers are also equal, i.e. $(y^n)^{1/n} = x^{1/n}$. Now we can use the insight from equation (5.10). In one hand, we have $\left[(y^n)^{1/n}\right]^n = y^n$; and this is equal to $\left(x^{1/n}\right)^n$. But according to Proposition 4.3.12 (and its counterpart for the real numbers), the equality $y^n = \left(x^{1/n}\right)^n$ implies $y = x^{1/n}$ as required.

Note that, in other words, we've just showed that for any non-negative real x, we have:

$$(x^n)^{1/n} = x \tag{5.11}$$

(c) $x^{1/n}$ is a non-negative number, and is positive iff x is positive.

First, $x^{1/n}$ is obviously non-negative, since it is defined as the supremum of a (non-empty) set of non-negative numbers.

Now let's prove that it is positive iff $x > 0$.

- If $x^{1/n} > 0$, then we have $\left(x^{1/n}\right)^n > 0^n$ according to (the counterpart for real numbers of) Proposition 4.3.12(b). But $0^n = 0$, and $\left(x^{1/n}\right)^n = x$ according to equation (5.10). Thus we have indeed $x > 0$.
- If $x > 0$, let's suppose for the sake of contradiction that $x^{1/n} = 0$. Then, still according to equation (5.10), we should have $x = \left(x^{1/n}\right)^n = 0$, which is a contradiction.

(d) We have $x > y \iff x^{1/n} > y^{1/n}$.

- First suppose that $x^{1/n} > y^{1/n}$. According to Proposition 4.3.12(b), we have: $\left(x^{1/n}\right)^n > \left(y^{1/n}\right)^n$. Also, by equation (5.10), $\left(x^{1/n}\right)^n = x$ and $\left(y^{1/n}\right)^n = y$, so we have indeed $x > y$. as required.
- Now suppose that $x > y$. Let's suppose for the sake of contradiction that we have $x^{1/n} \leqslant y^{1/n}$. We would thus use Proposition 4.3.12(b) and equation (5.10) again and see that $\left(x^{1/n}\right)^n \leqslant \left(y^{1/n}\right)^n$, i.e. that $x \leqslant y$, which is a contradiction. Thus we have necessarily $x^{1/n} > y^{1/n}$.

(e) If $x > 1$, then $x^{1/k}$ is a decreasing (i.e. $x^{1/k} < x^{1/l}$ whenever $k > l$) function of k. If $0 < x < 1$, then $x^{1/k}$ is an increasing function of k. If $x = 1$, then $x^{1/k}$ for all k. Here k ranges over the positive integers.

- Let be $x = 1$. We know that $1^k = 1$ for any positive integer k. Now, applying equation (5.11) leads to $1 = 1^{1/k}$ for any positive integer k, as required.
- Let be $x > 1$, and two positive integers $k > l$. We have to show that $x^{1/k} < x^{1/l}$. First, note that if $k > l$, then we have $k = l + p$ with p a positive integer (recall Definition 4.1.10). Let's suppose, for the sake of contradiction, that we have $x^{1/k} \geqslant x^{1/l}$. Thus, we should have $(x^{1/k})^{kl} \geqslant (x^{1/l})^{kl}$ according to Proposition 4.3.10(c), i.e. $x^l \geqslant x^k$ (we use equation (5.10) for this latest claim). But this last inequality could be written $x^l \geqslant x^{l+p}$, i.e. $1 \geqslant x^p$ by cancellation law. But this is a contradiction: if $x > 1$, we cannot have $x^p \leqslant 1$ with p a positive integer[19]. Thus, $x^{1/k}$ is indeed a decreasing function of k in this case.
- Let be $0 < x < 1$, and two positive integers $k > l$ (thus, we still have $k = l + p$ for a certain positive integer p). We have to show that $x^{1/k} > x^{1/l}$. A very similar proof applies. Let's suppose, for the sake of contradiction, that we have $x^{1/k} \leqslant x^{1/l}$. We should thus have $(x^{1/k})^{kl} \leqslant (x^{1/l})^{kl}$, i.e. $x^l \leqslant x^k$. This means that $x^l \leqslant x^{l+p}$, i.e. $1 \leqslant x^p$, which is impossible if $0 < x < 1$. This shows the contradiction and closes the proof.

(f) We have $(xy)^{1/n} = x^{1/n}y^{1/n}$.

- On the one hand, we have $\left[(xy)^{1/n}\right]^n = xy$ according to equation (5.10).
- On the other hand, we have $\left[x^{1/n}y^{1/n}\right]^n = (x^{1/n})^n(y^{1/n})^n = xy$, where we used successively Proposition 4.3.12(a) and equation (5.10).

Thus, both expressions are equal.

(g) We have $(x^{1/n})^{1/m} = x^{1/nm}$.

- On the one hand, we have $(x^{1/nm})^{nm} = x$ according to equation (5.10).
- On the other hand, $((x^{1/n})^{1/m})^{nm} = ((x^{1/n})^{1/m})^{mn} = \left[((x^{1/n})^{1/m})^m\right]^n = \left[x^{1/n}\right]^n = x$; where we used several times equation (5.10), and Proposition 4.3.12.

Thus, both expressions are equal.

[19] Just perform a quick inductive proof if needed.

Exercise 5.6.2. — *Prove Lemma 5.6.9.* Let $x, y > 0$ be positive reals, and let q, r be rationals. In the whole exercise, let's say that $q = a/b$ and $r = a'/b'$, with $b, b' > 0$. The claims to prove are:

(a) x^q is a positive real.

By Definition 5.6.7, $x^q = (x^{1/b})^a$. Since $x > 0$, we know by Lemma 5.6.6(c) that $x^{1/b}$ is a positive real. Thus, $x^q = (x^{1/b})^a$ is also positive (by the counterpart for reals of Proposition 4.3.12(b)).

(b) $x^{q+r} = x^q x^r$ and $(x^q)^r = x^{qr}$.

- We have $q + r = \frac{ab'+a'b}{bb'}$ (recall Definition 4.2.2). By Definition 5.6.7, $x^{q+r} = (x^{1/bb'})^{ab'+a'b}$. But we also can write $q = ab'/bb'$ and $r = a'b/bb'$. We know by Lemma 5.6.8 that our choices of numerators and denominators for q and r do not matter as regards x^q and x^r. Thus we also have $x^q x^r = (x^{1/bb'})^{ab'}(x^{1/bb'})^{a'b}$, which is equal to $(x^{1/bb'})^{ab'+a'b}$ by Proposition 4.3.12(a). Thus, $x^{q+r} = x^q x^r$.
- We have:

$$\begin{aligned}
((x^q)^r)^{bb'} &= ((((x^{1/b})^a)^{1/b'})^{a'})^{bb'} \\
&= (((x^{1/b})^a)^{1/b'})^{a'bb'} \\
&= ((((x^{1/b})^a)^{1/b'})^{b'})^{a'b} \\
&= ((x^{1/b})^a)^{a'b} \\
&= ((x^{1/b})^b)^{aa'} \\
&= x^{aa'}
\end{aligned}$$

And, also:

$$\begin{aligned}
(x^{qr})^{bb'} &= ((x^{1/bb'})^{aa'})^{bb'} \\
&= (x^{1/bb'})^{aa'bb'} \\
&= ((x^{1/bb'})^{bb'})^{aa'} \\
&= x^{aa'}
\end{aligned}$$

Thus, both expressions are equal, and this implies $x^{qr} = (x^q)^r$ according to Proposition 4.3.12(c).

(c) $x^{-q} = 1/x^q$.

First, note that Definitions 4.3.11 and 5.6.2 only say that $x^{-n} = 1/x^n$ if n is a *positive* integer. But this is actually true for any integer n: if $n = 0$, then both x^{-n} and $1/x^n$ are equal to 1 and are thus equal; and if $n < 0$, then there exists a positive integer m such that $n = -m$, and we have $x^{-n} = x^m$, $1/x^n = 1/x^{-m} = 1/(1/x^m) = x^m$ by Definition 5.6.2. Thus in all cases,

$$x^{-n} = 1/x^n, \quad \forall n \in \mathbb{Z} \tag{5.12}$$

Now the claim is straightforward: if $q = a/b$ with $b > 0$, then $x^{-q} = (x^{1/b})^{-a} = 1/((x^{1/b})^a) = 1/x^q$ according to equation (5.12).

(d) If $q > 0$, then $x > y$ iff $x^q > y^q$.

First, note that if $q > 0$, then $q = a/b$ with both a, b as positive integers.

- If $x > y$, then $x^{1/b} > y^{1/b}$ according to Lemma 5.6.6(d). And then, $(x^{1/b})^a > (y^{1/b})^a$ according to (the counterpart for real numbers of) Proposition 4.3.12(b), i.e. $x^q > y^q$.
- If $x^q > y^q$, then by definition, $(x^{1/b})^a > (y^{1/b})^a$, and both terms are positive. Thus, by Lemma 5.6.6(d), we have $((x^{1/b})^a)^{1/a} > ((y^{1/b})^a)^{1/a}$, i.e. $x^{1/b} > y^{1/b}$. And, by Proposition 4.3.12(b), $x = (x^{1/b})^b > (y^{1/b})^b = y$, as required.

(e) If $x > 1$, then $x^q > x^r$ if and only if $q > r$. If $x < 1$, then $x^q > x^r$ if and only if $q < r$.

First, note that we have:

$$(x^{1/b})^a = (x^a)^{1/b} \tag{5.13}$$

because on the one hand, $((x^{1/b})^a)^b = (x^{1/b})^{ab} = (x^{1/b})^{ba} = ((x^{1/b})^b)^a = x^a$, where we used Proposition 4.3.12(a) twice. And on the other hand, $((x^a)^{1/b})^b = x^a$ by Lemma 5.6.6(a). Thus, we have $((x^{1/b})^a)^b = ((x^a)^{1/b})^b$, which implies $(x^{1/b})^a = (x^a)^{1/b}$ by Proposition 4.3.12(c). (This proof holds for $a \neq 0$, but equation (5.13) is obvious if $a = 0$.)

Now we go back to the main claims to prove.

- Let be $x > 1$. First suppose that $q > r$, and let's show that $x^q > x^r$. If $q > r$, then we have $a/b > a'/b'$, i.e. $ab'/bb' > a'b/bb'$, i.e. $ab' > a'b$. Since both ab' and $a'b$ are integers and $x > 1$, we thus have $x^{ab'} > x^{a'b}$. And, by Lemma 5.6.6(d), we have $(x^{ab'})^{1/bb'} > (x^{a'b})^{1/bb'}$, i.e. $x^{ab'/bb'} > x^{a'b/bb'}$, and finally $x^q > x^r$ by Lemma 5.6.8.

  Now suppose that $x^q > x^r$ (note that both of them are positive since $x > 1$), and let's show that $q > r$, i.e. that $ab' > a'b$. Since we can also write $q = ab'/bb'$ and $r = a'b/bb'$ (and this does not affect the result by Lemma 5.6.8), we have $x^{ab'} = (x^q)^{bb'} > (x^r)^{bb'} = x^{a'b}$. And, if we multiply both sides by $x^{-a'b}$, which is a positive number, we get $x^{ab'-a'b} > 1$. We see that, in this inequality, we obviously cannot have $ab' - ab' = 0$. We cannot have $ab' - a'b < 0$ either, because in this case, we would have $ab' - a'b = -n$ with n a positive integer. I.e., we would have $x^{-n} > 1$, i.e. $1/x^n > 1$, a fact which is incompatible with our initial hypothesis $x > 1$. Thus, we only have one possibility: $ab' - a'b > 0$, i.e. $q > r$.
- Let be $0 < x < 1$. Detailed proof is not given here, but is similar to the previous case.

(f) $(xy)^q = x^q y^q$.

We have:

$$\begin{aligned}
(xy)^q &= ((xy)^{1/b})^a \text{ (Definition 5.6.7)} \\
&= (x^{1/b}y^{1/b})^a \text{ (Lemma 5.6.6(f))} \\
&= (x^{1/b})^a(y^{1/b})^a \text{ (Proposition 4.3.12(a))} \\
&= x^q y^q
\end{aligned}$$

as required.

Exercise 5.6.3. — *If x is a real number, show that $|x| = (x^2)^{1/2}$.*

To show that $(x^2)^{1/2}$ is equal to $|x|$, we should consider three cases according to the definition of absolute values, and prove that $(x^2)^{1/2} = 0$ if $x = 0$, $(x^2)^{1/2} = x$ if $x > 0$, and $(x^2)^{1/2} = -x$ if $x < 0$.

- If $x = 0$, we have $x^2 = 0$, and thus $(x^2)^{1/2} = 0$ according to Lemma 5.6.6(c).
- If $x > 0$, we have $(x^2)^{1/2} = x$ according to equation (5.11), i.e. Lemma 5.6.6(b).
- If $x < 0$, then $-x > 0$. Also, we know that $x^2 = (-x)^2$ for every real x. Thus, $(x^2)^{1/2} = ((-x)^2)^{1/2} = -x$ as required, still by Lemma 5.6.6(b).

6. Limits of sequences

Exercise 6.1.1. — *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, such that $a_{n+1} > a_n$ for each natural number n. Prove that whenever n and m are natural numbers such that $m > n$, then we have $a_m > a_n$. (We refer to these sequences as increasing sequences.)*

Let be $k = m - n$, so that $k > 0$ represents the difference between the two indexes under comparison. Let's induct on k to prove that $a_{n+k} > a_n$ for all natural number n and any $k > 0$.

- The base case is $k = 1$, i.e. $m = n + 1$. We must show here that $a_{n+1} > a_n$, but this is true by our initial hypothesis.
- Now suppose inductively that this is true for a certain natural k, i.e. that we have $a_{n+k} > a_n$ for all n. We have to show that we have $a_{n+k+1} > a_n$ for all n. But we know by our initial hypothesis that $a_{n+k+1} > a_{n+k}$, thus we have $a_{n+k+1} > a_{n+k} > a_n$, i.e. by transitivity, $a_{n+k+1} > a_n$ as required. This closes the induction.

Exercise 6.1.2. — *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, and let L be a real number. Show that $(a_n)_{n=m}^{\infty}$ converges to L iff, given any real $\varepsilon > 0$, one can find an $N \geqslant m$ such that $|a_n - L| \leqslant \varepsilon$ for all $n \geqslant N$.*

- First suppose that $(a_n)_{n=m}^{\infty}$ converges to L. Let be ε a positive real number. By Definition 6.1.5, $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L, i.e., there exists a natural $N \geqslant m$ such that $|a_n - L| \leqslant \varepsilon$, as required.
- Now show to converse implication. Let be ε a positive real. We know that we can find an $N \geqslant m$ such that $|a_n - L| \leqslant \varepsilon$ for all $n \geqslant N$, and thus $(a_n)_{n=m}^{\infty}$ is eventually close to ε. This is true for all ε, so that $(a_n)_{n=m}^{\infty}$ converges to L.

Exercise 6.1.3. — *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $m' \geqslant m$ be an integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_n)_{n=m'}^{\infty}$ converges to c.*

We will prove each part of this equivalence separately.

- First, suppose that $(a_n)_{n=m'}^{\infty}$ converges to c. This means that, for every $\varepsilon > 0$, there exists $N' \geqslant m'$ such that $n \geqslant N' \implies |a_n - c| \leqslant \varepsilon$. And we want to show that, still for every $\varepsilon > 0$, there exists $N \geqslant m$ such that $n \geqslant N \implies |a_n - c| \leqslant \varepsilon$. We claim that taking $N := N'$ is convenient. Indeed, since $m' \geqslant m$, we have $n \geqslant m$ as soon as we have $n \geqslant m'$, so that N' satisfies the required condition.
- Now suppose that $(a_n)_{n=m}^{\infty}$ converges to c. This means that:

$$\forall \varepsilon > 0, \exists N \geqslant m \ : \ n \geqslant m \to |a_n - c| \leqslant \varepsilon \tag{6.1}$$

  So let be $\varepsilon > 0$, and let's take $N' := N + m'$. We have both $N' \geqslant N$ and $N' \geqslant m'$. Thus, if $n \geqslant N'$, we also have $n \geqslant N$, which implies $|a_n - c| \leqslant \varepsilon$ according to equation (6.1). Thus, we have indeed found a positive integer $N' \geqslant m'$ such that $n \geqslant N' \to |a_n - c| \leqslant \varepsilon$, i.e., we have proved that $(a_n)_{n=m'}^{\infty}$ converges to c.

EXERCISE 6.1.4. — *Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, let c be a real number, and let $k \geqslant 0$ be a non-negative integer. Show that $(a_n)_{n=m}^{\infty}$ converges to c if and only if $(a_{n+k})_{n=m}^{\infty}$ converges to c.*

This exercise is pretty similar to the previous one.[20]

- First, suppose that $(a_n)_{n=m}^{\infty}$ converges to c. Let $\varepsilon > 0$ be a real number. We must show that there exists $M \geqslant m$ such that $n + k \geqslant M \to |a_{n+k} - c| \leqslant \varepsilon$. We know that there exists $N \geqslant m$ such that $n \geqslant N \to |a_n - c| \leqslant \varepsilon$. But for any $k \leqslant 0$, we have $n + k \geqslant n$, so that as soon as $n \geqslant N$, we have $n + k \geqslant n \geqslant N$, and thus $|a_{n+k} - c| \leqslant \varepsilon$. So, choosing $M := N$ is suitable here, and we have showed the first part.
- Now prove the converse implication: suppose that $(a_{n+k})_{n=m}^{\infty}$ converges to c. We want to show that there exists $N \geqslant m$ such that $n \geqslant N \to |a_n - c| \leqslant \varepsilon$. We already know that there exists $M \geqslant m$ such that $n \geqslant M \to |a_{n+k} - c| \leqslant \varepsilon$. Let be $N := M + k$. We can see that if $n \geqslant N$, we have $n \geqslant M + k$, i.e. $n - k \geqslant M$, and thus $|a_{(n-k)+k} - c| \leqslant \varepsilon$ as required. So, $N := M + k$ is suitable, and $(a_n)_{n=m}^{\infty}$ converges to c.

EXERCISE 6.1.5. — *Prove Proposition 6.1.12.*

We must prove that convergent sequences are Cauchy sequences. Let $(a_n)_{n=m}^{\infty}$ be a convergent sequence to c, and let's prove that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence.

Let be $\varepsilon > 0$ a positive real number. By Definition 6.1.5, there exists a natural number $N \geqslant m$ such that $n \geqslant N \implies |a_n - c| \leqslant \varepsilon/2$.

Now let be $j, k \geqslant N$ two natural numbers. We have, by triangular inequality:

$$\begin{aligned} |a_j - a_k| &= |a_j - c + c - a_k| \\ &\leqslant |a_j - c| + |a_k - c| \\ &\leqslant \varepsilon/2 + \varepsilon/2 \\ &\leqslant \varepsilon \end{aligned}$$

Thus, we have showed that there exists $N \geqslant m$ such that, for any natural numbers $j, k \geqslant N$, we have $|a_j - a_k| \leqslant \varepsilon$. This means that $(a_n)_{n=m}^{\infty}$ is indeed a Cauchy sequence.

EXERCISE 6.1.6. — *Prove Proposition 6.1.15; i.e. that formal limits are genuine limits.*

Let $(a_n)_{n=m}^{\infty}$ be a Cauchy sequence of real numbers, and $L := \mathrm{LIM}_{n\to\infty} a_n$. We have to show that $(a_n)_{n=m}^{\infty}$ converges to L.

Now we assume, for the sake of contradiction, that $(a_n)_{n=m}^{\infty}$ is not eventually ε-close to L, i.e. that there exists $\varepsilon > 0$ such that:

$$\forall N \geqslant m, \exists n \geqslant N \ : \ |a_n - L| > \varepsilon \tag{6.2}$$

Precisely, let's consider this positive real ε in what follows. Also, recall that $(a_n)_{n=m}^{\infty}$ is a Cauchy sequence, so that:

$$\exists T \geqslant m \ : \ j, k \geqslant T \implies |a_j - a_k| \leqslant \varepsilon/2 \tag{6.3}$$

[20] Actually, it would be possible to use Exercise 6.1.3 here, but, quite unexpectedly, this would just make the proof a little harder, because it would require to use some notions (like composition of functions, or intervals) that were very rarely used until now. The present proof is easier to write at this point of our knowledge.

According to (6.2), there exists $t \geqslant T$ such that $|a_t - L| > \varepsilon$. And since $t \geqslant T$, let's also consider any $s \geqslant T$: according to (6.3), we have $|a_s - a_t| \leqslant \varepsilon/2$. In particular, let's now fix t so that we can rephrase this property as:

$$\forall s \geqslant T,\ |a_s - a_t| \leqslant \varepsilon/2 \quad ; \quad \text{i.e., } a_t - \varepsilon/2 \leqslant a_s \leqslant a_t + \varepsilon/2 \tag{6.4}$$

The inequality $|a_t - L| > \varepsilon$ opens two possible cases:

- if $a_t > L + \varepsilon$, then we have from (6.4):

  $$L + \varepsilon - \varepsilon/2 < a_t - \varepsilon/2 \leqslant a_s$$

  so that $L + \varepsilon/2 \leqslant a_s$ for all $s \geqslant T$. According to Exercise 5.4.8, we should conclude that $L + \varepsilon/2 \leqslant L$, which is clearly a contradiction.

- if $a_t < L - \varepsilon$, then then we have from (6.4):

  $$a_s \leqslant a_t + \varepsilon/2 \leqslant L - \varepsilon + \varepsilon/2$$

  so that $a_s \leqslant L - \varepsilon/2$ for all $s \geqslant T$. According to Exercise 5.4.8, we should conclude that $L \leqslant L - \varepsilon/2$, which is also a contradiction.

Thus, necessarily, $(a_n)_{n=m}^{\infty}$ is eventually ε-close to L for any $\varepsilon > 0$, i.e. $(a_n)_{n=m}^{\infty}$ converges to L.

Exercise 6.1.7. — *Show that Definition 6.1.16 is consistent with Definition 5.1.12 (i.e., prove an analogue of Proposition 6.1.4 for bounded sequences instead of Cauchy sequences).*

Let be $(a_n)_{n=m}^{\infty}$ a sequence of real[21] numbers.

- First suppose that $(a_n)_{n=m}^{\infty}$ is bounded according to Definition 5.1.12. It means that there exists a non-negative rational $M \geqslant 0$ such that $|a_n| \leqslant M$ for all $n \geqslant m$. We have to show that there exists a positive real $M' > 0$ such that $|a_n| \leqslant M'$ for all $n \geqslant m$. If we just take $M' := M + 1$, we have $M' > M$ and $M' \in \mathbb{R}$ because $M' \in \mathbb{Q}$. Thus, we have $|a_n| \leqslant M < M'$ for all $n \geqslant m$, as required. Thus, $(a_n)_{n=m}^{\infty}$ is bounded according to Definition 6.1.16.

- Now suppose that $(a_n)_{n=m}^{\infty}$ is bounded according to Definition 6.1.16. It means that there exists a positive real number $M' > 0$ such that $|a_n| \leqslant M'$ for all $n \geqslant m$. We have to show that there exists a non-negative rational number $M \geqslant 0$ such that $|a_n| \leqslant M$ for all $n \geqslant m$. According to Proposition 5.4.12, there exists a positive integer N such that $M' \leqslant N$. This implies that $|a_n| \leqslant M' \leqslant N$ for all $n \geqslant m$. And since N is a positive integer, it's also a non-negative rational. Thus, $(a_n)_{n=m}^{\infty}$ is bounded according to Definition 5.1.12.

Exercise 6.1.8. — *Prove Theorem 6.1.19 about limit laws.*

We'll prove each statement successively.

[21] If $(a_n)_{n=m}^{\infty}$ is a sequence of rational numbers, there is literally nothing to prove, so that we can skip this case.

(a) Let be $\varepsilon > 0$. We have to prove that there exists a natural number $N \geqslant m$ such that for all $n \geqslant N$, we have $|(a_n + b_n) - (x + y)| \leqslant \varepsilon$.

Since $(a_n)_{n=m}^{\infty}$ converges to x, there exists $N_1 \geqslant m$ such that $n \geqslant N_1 \to |a_n - x| \leqslant \varepsilon/2$.

Similarly, there exists $N_2 \geqslant m$ such that $n \geqslant N_2 \to |b_n - y| \leqslant \varepsilon/2$.

Let be $N := \max(N_1, N_2)$. Thus, by triangular inequality, we have for all $n \geqslant N$:

$$\begin{aligned} |(a_n + b_n) - (x + y)| &\leqslant |a_n + x| + |b_n + y| \\ &\leqslant \varepsilon/2 + \varepsilon/2 \\ &\leqslant \varepsilon \end{aligned}$$

as required. This closes the proof.

(b) Let be $\varepsilon > 0$. We have to prove that there exists a natural number $N \geqslant m$ such that for all $n \geqslant N$, we have $|a_n b_n - xy| \leqslant \varepsilon$.

Let's start by some algebraic manipulations:

$$\begin{aligned} |a_n b_n - xy| &= |a_n b_n - a_n y + a_n y - xy| \\ &= |a_n(b_n - y) + y(a_n - x)| \\ &\leqslant |a_n| \times |b_n - y| + |y| \times |a_n - x| \end{aligned}$$

Hopefully, there exists an upper bound for each term of this last expression, at least eventually. Indeed:

- Since $(a_n)_{n=m}^{\infty}$ is convergent, it is bounded according to Corollary 6.1.17. Thus, there exists $M \geqslant 0$ such that $|a_n| \leqslant M$ for all $n \geqslant m$.
- Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists an integer $N_b \geqslant m$ such that $n \geqslant N_b \to |b_n - y| \leqslant \frac{\varepsilon}{2M}$.
- Since $(a_n)_{n=m}^{\infty}$ converges to x, there exists an integer $N_a \geqslant m$ such that $n \geqslant N_b \to |a_n - x| \leqslant \frac{\varepsilon}{2(|y|+1)}$. (Note that we don't choose $\frac{\varepsilon}{2|y|}$, because we don't know whether $|y| \neq 0$ or not, so we need an additional precaution.)

Let be $N := \max(N_a, N_b)$. For $n \geqslant N$, we thus have:

$$\begin{aligned} |a_n b_n - xy| &\leqslant |a_n| \times |b_n - y| + |y| \times |a_n - x| \\ &\leqslant M \times \frac{\varepsilon}{2M} + |y| \times \frac{\varepsilon}{2(|y| + 1)} \\ &\leqslant \varepsilon/2 + \varepsilon/2 \\ &\leqslant \varepsilon \end{aligned}$$

as required. This closes the proof.

(c) Here, a direct proof would be possible (and short), for instance by using Proposition 4.3.3(d). But following Tao's hint, let's use the previous results of this exercise instead. Let's consider the constant sequence $(c_n)_{n=m}^{\infty} = c, c, c, \ldots$; so that the sequence $(ca_n)_{n=m}^{\infty}$ is actually the product of the two sequences $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$. Obviously, $(c_n)_{n=m}^{\infty}$ converges to c, so that according to statement *(b)*, we have $\lim_{n\to\infty} ca_n = \lim_{n\to\infty} c_n a_n = \lim_{n\to\infty} c_n \lim_{n\to\infty} a_n = cx$, which closes the proof.

(d) According to *(c)* taking $c = -1$, we have $\lim_{n\to\infty}(-b_n) = -\lim_{n\to\infty} b_n = -y$. Furthermore, according to *(a)*, we have $\lim_{n\to\infty}(a_n + (-b_n)) = \lim_{n\to\infty} a_n + \lim_{n\to\infty}(-b_n) = x - y$, as requested.

(e) Following Tao's hint, we begin by proving an intermediate result, which is the following: if $(b_n)_{n=m}^{\infty}$ is a sequence of non-zero real numbers, and converges to $y \neq 0$, then $(b_n)_{n=m}^{\infty}$ is bounded away from zero. Note that, by the "second formula" of triangular inequality, we have:

$$|y| - |b_n| \leqslant ||y| - |b_n|| \leqslant |y - b_n| \tag{6.5}$$

Furthermore, since $(b_n)_{n=m}^{\infty}$ converges to y, there exists a positive integer $M \geqslant m$ such that, for all $n \geqslant M$, we have $|b_n - y| \leqslant |y|/2$. This result, along with equation (6.5), implies that $|b_n| \geqslant |y|/2$ for all $n \geqslant M$, so that $(b_n)_{n=m}^{\infty}$ is eventually bounded away from zero, for $n \geqslant M$. But since $b_n \neq 0$ for all n, just consider $c = \min(|b_m|, |b_{m+1}|, \ldots, |b_{M-1}|, |y|/2)$: we thus have $|b_n| \geqslant c$ for all $n \geqslant m$, and c is positive. Thus, $(b_n)_{n=m}^{\infty}$ is indeed bounded away from zero.

Note that saying "$(b_n)_{n=m}^{\infty}$ is bounded away from zero", or $|b_n| \geqslant c$ for all $n \geqslant m$, means that $|1/b_n| \leqslant 1/c$, i.e. that $(1/b_n)_{n=m}^{\infty}$ is bounded.

Let be $\varepsilon > 0$ a positive real number. Since $(b_n)_{n=m}^{\infty}$ converges to y, there exists a positive integer N such that for all $n \geqslant N$, we have $|y - b_n| \leqslant \varepsilon |y| c$ (which is a positive real number, because $|y| \neq 0$. Now let's consider:

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{y}\right| &= \left|\frac{y - b_n}{y b_n}\right| \\
&= \left|\frac{1}{y}\right| \times \left|\frac{1}{b_n}\right| \times |y - b_n| \\
&\leqslant \frac{1}{|y|} \times \frac{1}{c} \times |y - b_n| \\
&\leqslant \frac{1}{|y|} \times \frac{1}{c} \times \varepsilon |y| c \quad \text{for all } n \geqslant N \\
&\leqslant \varepsilon
\end{aligned}$$

This means that, for any positive real ε, one can find a positive integer $N \geqslant m$ such that $\left|\frac{1}{b_n} - \frac{1}{y}\right| \leqslant \varepsilon$. Thus, $(1/b_n)_{n=m}^{\infty}$ converges to $1/y$.

(f) This is a direct consequence of the parts *(b)* and *(e)*. Indeed, by part *(b)*, we have: $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty}(a_n \times 1/b_n) = \lim_{n\to\infty} a_n \times \lim_{n\to\infty} 1/b_n$. And by part *(e)*, we have $\lim_{n\to\infty} 1/b_n = 1/\lim_{n\to\infty} b_n$, which gives the property we wanted to show.

(g) We must show that $\lim_{n\to\infty} \max(a_n, b_n) = \max(x, y)$. We immediately see that we have two different cases: if $x \geqslant y$, then $\max(x, y) = x$ and then we must show that $\lim_{n\to\infty} \max(a_n, b_n) = x$; else if $x < y$, then $\max(x, y) = y$ and we must show that $\lim_{n\to\infty} \max(a_n, b_n) = y$. Let's consider those two cases separately. In what follows, let be $\varepsilon > 0$ a positive real.

- If $x \geqslant y$, we actually have to prove that there exists a positive integer $N \geqslant m$ such that for all $n \geqslant N$, we have $|\max(a_n, b_n) - x| \leqslant \varepsilon$. We already know that $(a_n)_{n=m}^{\infty}$ converges to x, thus:

$$\exists N_a \geqslant m \ : \ n \geqslant N_a \Longrightarrow |a_n - x| \leqslant \varepsilon \tag{6.6}$$

Similarly, since $(b_n)_{n=m}^{\infty}$ converges to y:

$$\exists N_b \geqslant m \ : \ n \geqslant N_b \Longrightarrow |b_n - y| \leqslant \varepsilon \tag{6.7}$$

Now let's consider $N := \max(N_a, N_b)$, which is a positive integer. For all $n \geqslant N$, we have both $x - \varepsilon \leqslant a_n \leqslant \varepsilon + x$ by (6.6), and $y - \varepsilon \leqslant b_n \leqslant \varepsilon + y \leqslant \varepsilon + x$ by (6.7). Combining those two relationships[22] leads to:

$$x - \varepsilon \leqslant a_n \leqslant \max(a_n, b_n) \leqslant \varepsilon + x$$

whose the most important part is $x - \varepsilon \leqslant \max(a_n, b_n) \leqslant x + \varepsilon$, which is equivalent to $|\max(a_n, b_n) - x| \leqslant \varepsilon$, as requested initially. This closes the proof for this first case.

- If $x < y$, the proof is very similar. We have to prove that there exists a positive integer $N \geqslant m$ such that for all $n \geqslant N$, we have $|\max(a_n, b_n) - y| \leqslant \varepsilon$. Once again, we can combine the previous results to get:

$$y - \varepsilon \leqslant b_n \leqslant \max(a_n, b_n) \leqslant \varepsilon + y$$

i.e. $|\max(a_n, b_n) - y| \leqslant \varepsilon$ as requested.

(h) Taking the inspiration in the way we defined the greatest lower bound from the least upper bound (Exercise 5.5.1), we could first note that $\min(a, b) = -\max(-a, -b)$. Then, this would be a direct consequence from part *(c)* and *(g)*.

Exercise 6.1.9. — *Explain why Theorem 6.1.19(f) fails when the limit of the denominator is* 0.

Theroem 6.1.19(f) says that if $(a_n)_{n=m}^{\infty}$ converges to x and $(b_n)_{n=m}^{\infty}$ converges to $y \neq 0$ (with $b_n \neq 0$ for all $n \geqslant m$), then $(a_n/b_n)_{n=m}^{\infty}$ converges to x/y. If $y = 0$, the number x/y simply does not exist, so that the statement is pointless.

Note however that the sequence $(a_n/b_n)_{n=m}^{\infty}$ *may* converge even if we have $\lim b_n = 0$: think of the situation $a_n = b_n = 1/n$, for instance.

Exercise 6.1.10. — *Show that the concept of equivalent Cauchy sequence, as defined in Definition 5.2.6, does not change if ε is required to be positive real instead of positive rational. More precisely, if $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are sequences of reals, show that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close for every rational $\varepsilon > 0$ if and only if they are eventually ε-close for every real $\varepsilon > 0$.*

Following Tao's hint, we just adapt the proof of Proposition 6.1.4.

- First suppose that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close for every real $\varepsilon > 0$. In particular, this means that they are ε-close for any *rational* $\varepsilon > 0$, so there is nothing to prove.

- Then suppose that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close for every rational $\varepsilon > 0$. Now let $\varepsilon > 0$ be a *real* number. According to Proposition 5.4.12, there exists a positive

[22] We also use the property that if $x \leqslant a$ and $y \leqslant a$, then $\max(x, y) \leqslant a$.

rational q such that $q \leqslant \varepsilon$. Since q is a rational, $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are q-close, according to our initial hypothesis, i.e., $|a_n - b_n| \leqslant q \leqslant \varepsilon$ for all n greater than some positive integer N. In particular, this means that $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are eventually ε-close, as requested.

EXERCISE 6.2.1. — *Prove Proposition 6.2.5.*

Hereafter, x, y, z are extended real numbers. Thus, they can be either real numbers, or $\pm\infty$.

(a) We have to show that $x \leqslant x$. We have three cases here. If x is a real number, this is an obvious statement. If $x = +\infty$, then by Definition 6.2.3(b), we have $x \leqslant +\infty$ for all extended real x, so that the claim $x \leqslant x$ is still true. Similarly, if $x = -\infty$, we have $-\infty \leqslant x$ for all extended real x according to Definition 6.2.3(c).

(b) We have to show the trichotomy of extended real numbers, i.e. that we always have exactly one of the statements $x < y$, $x = y$ or $x > y$. Both x and y can be either real numbers, $+\infty$ or $-\infty$, so that we have nine cases to study exhaustively.

(i) If both x, y are real numbers, then this is simply Proposition 5.4.7(a)[23].

(ii) If x is a real number and $y = +\infty$, then we have $x \leqslant y$ by Definition 6.2.3(b), and we also have $x \neq y$ by Definition 6.2.1. Thus, $x < y$ is true, and the two other statements $x = y$ and $x > y$ are false. (It is easy to see that $x > y$ is false, because it corresponds to none of the cases listed in Definition 6.2.3.)

(iii) If x is a real number and $y = -\infty$, then we have $y \leqslant x$ according to Definition 6.2.3(c), and $x \neq y$ according to Definition 6.2.1. Thus, $x > y$ is true; whereas $x = y$ is false, and $x < y$ is also false because it corresponds to none of the cases listed in Definition 6.2.3.

(iv) If $x = +\infty$ and y is a real number, then we have $x \geqslant y$ by Definition 6.2.3(b), and $x \neq y$ by Definition 6.2.1. Thus, $x > y$ is true; whereas $x = y$ is false, and $x < y$ is also false because it corresponds to none of the cases listed in Definition 6.2.3.

(v) If $x = +\infty$ and $y = +\infty$, then we have $x = y$. Thus, both statements $x < y$ and $x > y$ are incompatible with this one. It means that exactly one statement ($x = y$) is true in this case.

(vi) If $x = +\infty$ and $y = -\infty$, then we have $y \leqslant x$ by Definition 6.2.3(b-c). According to Definition 6.2.1, we also have $x \neq y$. Thus, $y > x$ is true; whereas $x = y$ is false, and $x > y$ is also false because it corresponds to none of the cases listed in Definition 6.2.3.

(vii) The three last cases where $x = -\infty$ and y is either $-\infty$, a real, or $+\infty$, can be shown in a similar fashion, and are "symmetrical" with all previous proofs.

(c) We have to show that if $x \leqslant y$ and $y \leqslant z$, then $x \leqslant z$. We'll choose wisely below some of the cases that really make sense in such a siuation.

- If $z = +\infty$, then according to Definition 6.2.3(b), $x \leqslant z$ is always true, regardless of the value of y.

[23] Actually, the statement (a) from Proposition 4.2.9, and applied to reals by Proposition 5.4.7, but I'll shorten this for the rest of the exercise.

- If $z = -\infty$, only possible situation makes sense: the situation where $x = y = z = -\infty$. Thus, the transitivity ($x \leqslant z$) is still true in this case.
- If z is a real number, then x, y cannot be $+\infty$. If $y = -\infty$, then necessarily $x = -\infty$, and we thus have $x \leqslant z$. If x, y are real numbers, then it is simply Proposition 5.4.7(c). And if y is a real number and $x = -\infty$, then $x \leqslant z$ by Definition 6.2.3(c).

(d) Finally we have to show that if $x \leqslant y$, then $-y \leqslant -x$. If $y = +\infty$, then $-y = -\infty$ and thus $-y \leqslant -x$ for all extended real $-x$, thus the statement is true in this case. If $y = -\infty$, then we have necessarily $x = -\infty$, and thus $-x = -y$, so that the statement still holds. Finally, if y is a real number, then the statement is simply Proposition 5.4.7 if x is real, and is obvious if $x = -\infty$.

Exercise 6.2.2. — *Prove Theorem 6.2.11.*

We have to prove the following statements:

(a) For every $x \in E$ we have $x \leqslant \sup(E)$ and $x \geqslant \inf(E)$.

- First, suppose that $+\infty \in E$. We thus have $\sup(E) = +\infty$ by Definition 6.2.6(b); and by Definition 6.2.3(b), $x \leqslant +\infty = \sup(E)$ for all $x \in E$, as required.
- Now suppose that $+\infty \notin E$, but $-\infty \in E$. There are two possible cases here. If $x = -\infty$, then $-\infty = x \leqslant y$ for all extended real$y$ by Definition 6.2.3(c); and in particular, with $y = \sup(E)$, we have $x \leqslant \sup(E)$ as required. On the other hand, if x is a real number, then $x \in E \backslash \{-\infty\}$, so that $x \leqslant \sup(E \backslash \{-\infty\}) := \sup(E)$ by Definition 6.2.6(c), as required.
- Finally, if E consists only of real numbers, we have $x \leqslant \sup(E)$ by Definition 5.5.10 as long as E has an upper bound; otherwise we have $\sup(E) = +\infty$ by Definition 6.2.6(b), and thus $x \leqslant \sup(E)$ bu Definition 6.2.3(b). Thus, in all cases, we have indeed $x \leqslant \sup(E)$.
- The other statement, $x \geqslant \inf(E)$, is a direct consequence. Indeed, if $x \in E$, then by definition, $-x \in -E$, and thus $-x \leqslant \sup(-E)$ according to our previous conclusions. And finally, $-x \leqslant \sup(-E)$ is equivalent to $x \geqslant -\sup(-E)$, i.e. $x \geqslant \inf(E)$, according to Proposition 6.2.5(d) and the definition of the greatest lower bound.

(b) Suppose that $M \in \overline{\mathbb{R}}$ is an upper bound for E, i.e., $x \leqslant M$ for all $x \in E$. Then we have $\sup(E) \leqslant M$.

- If $+\infty \in E$, then $\sup(E) = +\infty$ by Definition 6.2.6(b). But the only possible upper bound for E is $+\infty$, so that we have necessarily $\sup(E) = M = +\infty$. The statement is thus true is this case.
- If E consists only of real numbers, there are three sub-cases. If E is empty, the $\sup(E) = -\infty$, so that $\sup(E) \leqslant M$ regardless of the value of M. If E is non-empty and not bounded above, then $\sup(E) = +\infty$ and M can only be equal to $+\infty$, so that $\sup(E) = M$. And finally, if E is non-empty but bounded above, then the results comes from Definition 5.5.10.

- If $+\infty \notin E$ but $-\infty \in E$, then $\sup(E) := \sup(E \backslash \{-\infty\})$. But the set $E \backslash \{-\infty\}$ consists only of real numbers, so that we can basically go back to the previous case, to get $\sup(E) = \sup(E \backslash \{-\infty\}) \leqslant M$.

Exercise 6.3.1. — *Verify the claim in Example 6.3.4: Let $a_n := 1/n$; thus $(a_n)_{n=1}^{\infty}$ is the sequence $1, 1/2, 1/3, \ldots$. Then $\sup(a_n)_{n=1}^{\infty} = 1$ and $\inf(a_n)_{n=1}^{\infty} = 0$.*

Let's prove those two statements separately.

- First, note that 1 is an upper bound for $(a_n)_{n=1}^{\infty}$: since $n \geqslant 1$, we always have $1/n \leqslant 1$. Furthermore, let M be an upper bound for $(a_n)_{n=1}^{\infty}$. By definition, we must have $M \geqslant a_n$ for all $n \geqslant 1$; and in particular $M \geqslant a_1 = 1$. Thus, 1 is indeed the least upper bound of $(a_n)_{n=1}^{\infty}$.
- Second, 0 is a lower bound for $(a_n)_{n=1}^{\infty}$: we obviously have $1/n \geqslant 0$ for all $n \geqslant 1$. Let's suppose for the sake of contradiction that there exists a greater lower bound m of $(a_n)_{n=1}^{\infty}$. By definition, this means that $a_n \geqslant 0$ for all $n \geqslant 1$, and that $m > 0$. But according to the archimedean property with $\varepsilon = 1$, there exists a natural number N such that $mN \geqslant 1$, i.e. $m \geqslant 1/N$, that is to say $a_N \leqslant m$. This is a contradiction, since m is not a lower bound anymore. Thus, 0 is indeed the greatest lower bound of $(a_n)_{n=1}^{\infty}$.

Exercise 6.3.2. — *Prove Proposition 6.3.6.*

Let $E = \{a_n : n \geqslant m\}$, and $x = \sup(a_n)_{n=m}^{\infty}$. By Definition 6.3.1, we thus have $x = \sup(E)$. Let's prove all statements from the Proposition.

- Obviously, we have $a_n \leqslant x$ for all $n \geqslant m$, according to Theorem 6.2.11(a).
- Let be $M \in \overline{\mathbb{R}}$ an upper bound for $(a_n)_{n=m}^{\infty}$. This is equivalent to say that M is an upper bound for E. Thus, according to Theorem 6.2.11(b), we have $M \geqslant x$.
- Now let be $y \in \overline{\mathbb{R}}$ such that $y < x$. Suppose, for the sake of contradiction, that $y \geqslant a_n$ for all $n \geqslant m$. This means that y is an upper bound for $(a_n)_{n=m}^{\infty}$, and that $y < x$. This contradicts the conclusion from the previous bullet point of this exercise. Thus, there exists an $n \geqslant m$ such that $y < a_n$. And since $a_n \in E$, we also have $y < a_n \leqslant x$, as required.

Exercise 6.3.3. — *Prove Proposition 6.3.8. (increasing bounded sequences converge).*

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers. The hypotheses are, first, that $a_{n+1} \geqslant a_n$ for all $n \geqslant m$; and second, that there exists a positive integer M such that $|a_n| \leqslant M$ for all $n \geqslant m$.

Let be $\ell = \sup(a_n)_{n=m}^{\infty}$. Since $(a_n)_{n=m}^{\infty}$ is bounded, ℓ is a real number. We will show that $(a_n)_{n=m}^{\infty}$ converges to ℓ.

Let be $\varepsilon > 0$ a real number. First, we already know that:

$$a_n \leqslant \ell \tag{6.8}$$

for all $n \geqslant m$. Thus, in particular, we have $a_n \leqslant \ell + \varepsilon$ for all $n \geqslant m$.

Also, according to the third statement of Proposition 6.3.6, if y is a real number such that $y < \ell$, there exists at least one element of $(a_n)_{n=m}^{\infty}$, say a_{n_0}, such that $y < a_{n_0} < \ell$. In particular, with $y = \ell - \varepsilon$, we thus have:

$$\ell - \varepsilon < a_{n_0} \leqslant a_n \tag{6.9}$$

But recall that the sequence $(a_n)_{n=m}^{\infty}$ is an increasing sequence: we have (as a quick induction shows) $a_n \geqslant a_{n_0}$ for all $n \geqslant n_0$. Combining the two equations (6.8) and (6.9), we get $\ell - \varepsilon < a_{n_0} \leqslant a_n \leqslant \ell \leqslant \ell + \varepsilon$ for all $n \geqslant n_0$.

Let's summarise:

$$\forall \varepsilon > 0, \exists n_0 \geqslant m \quad : \quad n \geqslant n_0 \Longrightarrow \ell - \varepsilon \leqslant a_n \leqslant \ell + \varepsilon \tag{6.10}$$

which means precisely that $(a_n)_{n=m}^{\infty}$ converges to ℓ, as required.

Exercise 6.3.4. — *Explain why Proposition 6.3.10 fails when $x > 1$. In fact, show that the sequence $(x^n)_{n=1}^{\infty}$ diverges when $x > 1$. Compare this with the argument in Example 1.2.3; can you now explain the flaws in the reasoning in that example?*

First, the proof of Proposition 6.3.10 supposes that the sequence $(x^n)_{n=1}^{\infty}$ is decreasing and has a lower bound of 0, which is not the case here: the sequence $(x^n)_{n=1}^{\infty}$ is increasing and has no real upper bound (or, let's say that its upper bound is $+\infty$). So the situation is not identical, and even not "symmetrical".

Actually, when $x > 1$, the sequence $(x^n)_{n=1}^{\infty}$ diverges to $+\infty$. Let's suppose, for the sake of contradiction, that it converges to a real number ℓ instead. Since $x > 1$, we have $0 < 1/x < 1$, so that by Proposition 6.3.10, $(1/x^n)_{n=1}^{\infty}$ is a sequence that converges to 0. Thus, if we denote $a_n = x^n \times 1/x^n$, the sequence $(a_n)_{n=1}^{\infty}$ is the product of two convergent sequences, so that if we apply the limit laws (Theorem 6.1.19(b)), we would have $\lim_{n\to\infty} a_n = \ell \times 0 = 0$. But this is impossible, since $a_n = 1$ for all $n \geqslant 1$. It is thus impossible for $(x^n)_{n=1}^{\infty}$ to be a convergent sequence if $x > 1$.

Finally, the issue in Example 1.2.3 is that there is a "hidden" assumption in it: it is not explicitly stated, but it is a strong one. In this example, we begin with the statement: "let L be the limit $L = \lim_{n\to\infty} x^n$", and then we apply the limit laws (Theorem 6.1.19) as if we were sure that $(x^n)_{n=1}^{\infty}$ is a convergent sequence, i.e. that L is a real number. But in the next steps of this example (where we take $x = -1$ or $x = 2$), this is absolutely not the case: instead this sequence diverges. Thus, it was wrong to apply the limit laws, and the whole reasoning is flawed. It is only correct in the case where $x \leqslant 1$, i.e. where $L = 0$ or $x = 1$, which is also a remark we make during this example.

Exercise 6.4.1. — *Prove Proposition 6.4.5.*

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, which converges to c. There are two statements to prove.

- First, let's prove that c is a limit point for $(a_n)_{n=m}^{\infty}$. Intuitively this is obvious since being a limit point is somewhat weaker than being a limit, but show it rigorously this requires some manipulations on quantifiers.

  Let be $\varepsilon > 0$ a real number and $N \geqslant m$ a positive integer. We must show that there exists $k \geqslant N$ such that $|a_k - c| \leqslant \varepsilon$.

Actually, we know that $(a_n)_{n=m}^{\infty}$ converges to c. It means that there exists $N' \geqslant m$ such that, for *all* $n \geqslant N'$, we have $|a_n - c| \leqslant \varepsilon$.

Thus, any positive integer $n \geqslant \max(N, N')$ will do the trick, and in particular $n = \max(N, N')$: we will have $n \geqslant N$ and $|a_n - c|$ as required.

- Now let's prove that c is the *only* limit point of $(a_n)_{n=m}^{\infty}$. Here, drawing a picture will help a lot. Let's suppose, for the sake of contradiction, that there exists another limit point $c' \neq c$ for $(a_n)_{n=m}^{\infty}$. Let's consider the positive real number $\varepsilon = |c - c'|/3$ (we know it is positive, because $c - c' \neq 0$ by hypothesis).

  Since $(a_n)_{n=m}^{\infty}$ converges to c, there exists a positive integer $N \geqslant m$ such that $|a_n - c| \leqslant \varepsilon$ for all $n \geqslant N$. Also, since c' is a limit point, there exists $n_0 \geqslant N$ such that $|a_{n_0} - c'| \leqslant \varepsilon$. Thus:

$$\begin{aligned}
|c - c'| &\leqslant |c - a_{n_0} + a_{n_0} - c'| \\
&\leqslant |a_{n_0} - c| + |a_{n_0} - c'| \\
&\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3} = \frac{2}{3}|c - c'|
\end{aligned}$$

  This is a clear contradiction. Thus, $(a_n)_{n=m}^{\infty}$ has no other limit point than c.

Exercise 6.4.2. — *State and prove analogues of Exercises 6.1.3 and 6.1.4 for limit points, limit superior and limit inferior.*

As stated before, Exercises 6.1.3 and 6.1.4 provided very similar statements, so that we'll just prove here the analogues for Exercise 6.1.3. They can be stated as follows.

Let $m' \geqslant m$ two positive integers, and c a real number.

1. Statement for limit points: c is a limit point for $(a_n)_{n=m}^{\infty}$ iff c is a limit point for $(a_n)_{n=m'}^{\infty}$.

   Actually, c is a limit point for $(a_n)_{n=m}^{\infty}$ iff:

$$\forall \varepsilon > 0, \forall N \geqslant m, \exists n \geqslant N \ : \ |a_n - c| \leqslant \varepsilon \tag{6.11}$$

   Similarly, c is a limit point for $(a_n)_{n=m'}^{\infty}$ iff:

$$\forall \varepsilon > 0, \forall N' \geqslant m', \exists n \geqslant N' \ : \ |a_n - c| \leqslant \varepsilon \tag{6.12}$$

   - First suppose that c is a limit point for $(a_n)_{n=m}^{\infty}$, and let's show that it is a limit point for $(a_n)_{n=m'}^{\infty}$. Let be $\varepsilon > 0$ a positive real and $N' \geqslant m'$ a positive integer. Since $m' \geqslant m$, we also have $N' \geqslant m$. Thus, by (6.11), we know that there exists $n \geqslant N'$ such that $|a_n - c|$, as required. Thus, c is a limit point for $(a_n)_{n=m'}^{\infty}$.
   - Now suppose that c is a limit point for $(a_n)_{n=m'}^{\infty}$, and let's show that it is a limit point for $(a_n)_{n=m}^{\infty}$. Let be $\varepsilon > 0$ a positive real and $N \geqslant m$ a positive integer. We can distinguish two sub-cases here. If $N \geqslant m'$, then (6.12) indeed provides a $n \geqslant N$ such that $|a_n - c| \leqslant \varepsilon$, as required. Else, if $m \leqslant N < m'$, then according to (6.12), there exists an $n \geqslant m'$ such that $|a_n - c| \leqslant \varepsilon$. We thus have $m \leqslant N < m' \leqslant n$, i.e. in particular $N < n'$, and $|a_n - c| \leqslant \varepsilon$, as required. Thus, in both sub-cases, c is a limit point for $(a_n)_{n=m}^{\infty}$.

2. Statement for limit superior: ℓ is the limit superior of $(a_n)_{n=m}^{\infty}$ iff ℓ is the limit superior of $(a_n)_{n=m'}^{\infty}$.

   Actually, ℓ is the limit superior of $(a_n)_{n=m}^{\infty}$ iff:

$$\ell = \inf(a_N^+)_{N=m}^{\infty} \tag{6.13}$$

   Similarly, ℓ' is the limit superior of $(a_n)_{n=m'}^{\infty}$ iff:

$$\ell' = \inf(a_N^+)_{N=m'}^{\infty} \tag{6.14}$$

   First, note that the sequence $(a_N^+)_{N=m}^{\infty}$ is decreasing. In particular, thanks to Proposition 6.3.8, it means that its limit is equal to its greatest lower bound, i.e., $\lim_{N\to+\infty}(a_N^+) = \inf(a_N^+)_{N=m}^{\infty}$. But we already know, thanks to Exercises 6.1.3 and 6.1.4, that this limit does not depend on the starting index of the sequence. Thus, we indeed have $\ell = \ell'$.

3. The statement and proof for limit inferior are similar to the previous one, since the sequence $(a_N^-)_{N=m}^{\infty}$ is increasing.

Exercise 6.4.3. — *Prove parts (c), (d) and (e) of Proposition 6.4.12.*

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers, L^+ its limit superior and L^- its limit superior.

(c) Prove that $\inf(a_n)_{n=m}^{\infty} \leqslant L^- \leqslant L^+ \leqslant \sup(a_n)_{n=m}^{\infty}$.

We will prove each inequality separately.

- First let's prove that $\inf(a_n)_{n=m}^{\infty} \leqslant L^-$. By Definition 6.4.6, $L^- = \sup(a_N^-)_{N=m}^{\infty}$. Thus, we have $L^- \geqslant a_N^-$ for all $N \geqslant m$. In particular, for $N = m$, we have $L^- \geqslant a_m^- = \inf(a_n)_{n=m}^{\infty}$, as required.
- The proof for $L^+ \leqslant \sup(a_n)_{n=m}^{\infty}$ is similar. By Definition 6.4.6, $L^+ = \inf(a_N^+)_{N=m}^{\infty}$. Thus, we have $L^+ \leqslant a_N^+$ for all $N \geqslant m$. In particular, for $N = m$, we have $L^+ \leqslant a_m^+ = \sup(a_n)_{n=m}^{\infty}$, as required.
- Finally, let's prove that $L^- \leqslant L^+$. Let be $\varepsilon > 0$ a positive real. First, according to Proposition 6.3.6, there exists a positive integer N_1 such that $a_{N_1}^- + \varepsilon > L^- \geqslant a_{N_1}^-$. Similarly, there exists a positive integer N_2 such that $a_{N_2}^+ - \varepsilon < L^+ \leqslant a_{N_2}^+$. Suppose that $N_1 \leqslant N_2$. Given that we have $a_N^- \leqslant a_N^+$ for all N, and that $(a_N^-)_{N=m}^{\infty}$ is an increasing sequence[24], we get:

$$L^- - \varepsilon \leqslant a_{N_1}^- \leqslant a_{N_2}^- \leqslant a_{N_2}^+ < L^+ + \varepsilon$$

  which leads to $L^- - L^+ \leqslant 2\varepsilon$ for all $\varepsilon > 0$. This implies that $L^- - L^+ \leqslant 0$ (otherwise we would have an obvious contradiction).

  The case $N_2 \leqslant N_1$ can be written in a similar fashion.

  Thus, the whole inequality is proven.

[24]Both results are easy to prove, although not stated explicitly in the main text.

(d) Prove that if c is any limit point of $(a_n)_{n=m}^{\infty}$, then we have $L^- \leqslant c \leqslant L^+$.

We only show here the inequality $c \leqslant L^+$; its counterpart $L^- \leqslant c$ can be proved the same way.

Let's suppose, for the sake of contradiction, that we have $c > L^+$. Let be $\varepsilon = \frac{c-L^+}{3}$, which is by hypothesis a positive real number. We have $L^+ + \varepsilon > L^+$. Thus, by Proposition 6.4.2(a), there exists $N \geqslant m$ such that:

$$a_n < L^+ + \varepsilon = \frac{2L^+ + c}{3} \text{ for all } n \geqslant N \tag{6.15}$$

But this is a contradiction with the fact that c is a limit point for $(a_n)_{n=m}^{\infty}$. Indeed, if c is a limit point, then there exists $n \geqslant N$ such that $|a_n - c| \leqslant \varepsilon$, i.e.: $c - \varepsilon < a_n < c + \varepsilon$, and in particular

$$a_n > c - \varepsilon = \frac{3c - c + L^+}{3} = \frac{2c + L^+}{3} \tag{6.16}$$

But equations (6.15) and (6.16) are incompatible, since they would lead to $a_n < \frac{2L^+ + c}{3} < \frac{2c + L^+}{3} < a_n$, i.e. $a_n < a_n$. Thus, our starting hypothesis $c \leqslant L^+$ is false, and we have necessarily $c > L^+$ as required.

(e) Prove that if L^+ is finite, then it is a limit point of $(a_n)_{n=m}^{\infty}$ (and similarly for L^-).

Once again, we only give the proof for L^+, the other one being very similar.

Let be $\varepsilon > 0$ a positive real, and $N \geqslant m$ a positive integer. Since $L^+ + \varepsilon > L^+$, according to Proposition 6.4.2(a), there exists $N' \geqslant m$ such that $a_n < L^+ + \varepsilon$ for all $n \geqslant N'$. In particular, if $n \geqslant M := \max(N, N')$, then we have $a_n < L^+ + \varepsilon$.

Also, we have $L^+ - \varepsilon < L^+$. Thus, according to Proposition 6.4.2(b), there exists $n \geqslant M$ such that $a_n > L^+ - \varepsilon$.

Thus, for any given $\varepsilon > 0$ and any given $N \geqslant m$, there exists $n \geqslant M \geqslant N$ such that $L^+ - \varepsilon < a_n < L^+ + \varepsilon$, i.e. $|a_n - L^+| \leqslant \varepsilon$. L^+ is thus a limit point, as required.

(f) Prove that $(a_n)_{n=m}^{\infty}$ converges to c iff $L^+ = L^- = c$.

- First, let's suppose that $L^+ = L^- = c$. According to Proposition 6.4.5, if $(a_n)_{n=m}^{\infty}$ is a convergent sequence, then its limit is a limit point. But according to Proposition 6.4.2(d), if $L^- = L^+ = c$, the only possible limit point for $(a_n)_{n=m}^{\infty}$ is c. However, we have only shown that *if* $(a_n)_{n=m}^{\infty}$ *is a convergent sequence, then it converges to* c, but we do not know whether or not $(a_n)_{n=m}^{\infty}$ is convergent.
  Let be $\varepsilon > 0$. According to Proposition 6.4.2(a), if $x = L^+ + \varepsilon = c + \varepsilon > L^+$, then there exists $N \geqslant m$ such that $a_n < c + \varepsilon$ for all $n \geqslant N$. Similarly, since $y = L^- - \varepsilon = c - \varepsilon < L^-$, then there exists $N' \geqslant m$ such that $a_n > c - \varepsilon$ for all $n \geqslant N'$.
  Thus, for all $n \geqslant \max(N, N')$, we have $c - \varepsilon < a_n < c + \varepsilon$, i.e. $|a_n - c| \leqslant \varepsilon$. This means that $(a_n)_{n=m}^{\infty}$ converges to c.
- Conversely, let's suppose that $(a_n)_{n=m}^{\infty}$ converges to c.
  Let be $\varepsilon > 0$. First, since $(an)_{n=m}^{\infty}$ converges to c, there exists $N \geqslant m$ such that $|a_n - c| \leqslant \varepsilon/2$ for all $n \geqslant N$. Also, since L^- is a limit point, there exists $k \geqslant N$ such that $|a_k - L^-| \leqslant \varepsilon/2$. Thus, we have both $|a_k - c| \leqslant \varepsilon/2$ and $|a_k - L^-| \leqslant \varepsilon/2$.

We then derive, by triangular inequality: $|L^- - c| \leqslant |L^- - a_n| + |a_n - c| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$. This means that $|L^- - c| \leqslant \varepsilon$ for all $\varepsilon > 0$, which is equivalent to $L^- = c$.

Similarly, we could prove that $c = L^+$.

Finally, we indeed have $L^- = c = L^+$ as required.

Exercise 6.4.4. — *Prove Lemma 6.4.13.*

Let be $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ two sequences of real numbers, such that $a_n \leqslant b_n$ for all $n \geqslant m$. We will prove the four statements of this lemma.

1. Prove that $\sup(a_n)_{n=m}^{\infty} \leqslant \sup(b_n)_{n=m}^{\infty}$.

   Let be $B = \sup(b_n)_{n=m}^{\infty}$. By definition, B is an upper bound for $(b_n)_{n=m}^{\infty}$, so that we have $B \geqslant b_n \geqslant a_n$ for all $n \geqslant m$. In particular, B is also an upper bound for $(a_n)_{n=m}^{\infty}$. And, by Definition 5.5.5 of a least upper bound, we thus have $B \geqslant \sup(a_n)_{n=m}^{\infty}$ as required.

2. Prove that $\inf(a_n)_{n=m}^{\infty} \leqslant \inf(b_n)_{n=m}^{\infty}$.

   A very similar argument applies. Let be $A = \sup(a_n)_{n=m}^{\infty}$. By definition, A is an lower bound for $(a_n)_{n=m}^{\infty}$, so that we have $A \leqslant a_n \leqslant b_n$ for all $n \geqslant m$. In particular, A is also a lower bound for $(b_n)_{n=m}^{\infty}$. We thus have $A \leqslant \inf(b_n)_{n=m}^{\infty}$ as required.

3. Prove that $\limsup a_n \leqslant \limsup b_n$.

   Let be, for any $N \geqslant m$, $a_N^+ = \sup(a_n)_{n=N}^{\infty}$, and $b_N^+ = \sup(b_n)_{n=N}^{\infty}$. According to the first point of this exercise, we have $a_N^+ \leqslant b_N^+$ for all $N \geqslant m$. According to the second point, we have $\inf(a_N^+)_{N=m}^{\infty} \leqslant \inf(b_N^+)_{N=m}^{\infty}$, i.e. $\limsup a_n \leqslant \limsup b_n$ as required.

4. Prove that $\liminf a_n \leqslant \liminf b_n$.

   Once again, the proof is similar to the previous point.

Exercise 6.4.5. — *Use Lemma 6.4.13 to prove Corollary 6.4.14.*

Let $(a_n)_{n=m}^{\infty}, (b_n)_{n=m}^{\infty}, (c_n)_{n=m}^{\infty}$ be sequences of real numbers such that $a_n \leqslant b_n \leqslant c_n$ for all $n \geqslant m$; and the sequences $(a_n)_{n=m}^{\infty}$ and $(c_n)_{n=m}^{\infty}$ converge to L.

According to Corollary 6.4.14, we have $\limsup a_n \leqslant \limsup b_n \leqslant \limsup c_n$. But according to Proposition 6.4.12(f), when a sequence converges to a real number L, its limit superior is simply equal to its limit L. Thus, we have $L = \limsup a_n \leqslant \limsup b_n \leqslant \limsup c_n = L$, i.e. $\limsup b_n = L$.

The same statement apply for $\liminf b_n$, which is also equal to L.

Thus, still according to Proposition 6.4.12(f), $(b_n)_{n=m}^{\infty}$ converges to L, as required.

Exercise 6.4.6. — *Give an example of two bounded sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ such that $a_n < b_n$ for all $n \geqslant 1$, but that $\sup(a_n)_{n=1}^{\infty} \not< (b_n)_{n=1}^{\infty}$. Explain why this does not contradict Lemma 6.4.13.*

The sequences defined by $a_n = -1/n$ and $b_n = 0$ for all $n \geqslant 1$ are suitable, since $-1/n < 0$ for all $n \geqslant 1$, but both least upper bounds are equal to 0.

This does not contradict Lemma 6.4.13 which deals with large inequalities; instead, it is even perfectly in accordance with Remark 5.4.11 in the previous chapter.

Exercise 6.4.7. — *Prove Corollary 6.4.17.*

This corollary says that $(a_n)_{n=m}^{\infty}$ converges to 0 iff $(|a_n|)_{n=m}^{\infty}$ converges to 0. Let's denote $b_n = |a_n|$ in what follows.

- First suppose that $\lim_{n\to\infty}|a_n| = 0$. We know that we have, for all $n \geqslant m$, $-|a_n| \leqslant a_n \leqslant |a_n|$. According to the squeeze test, we have indeed $\lim_{n\to\infty}a_n = 0$.

- Now suppose that $\lim_{n\to\infty}a_n = 0$. Let $\varepsilon > 0$ be a positive real number. We have $|b_n - 0| = |b_n| = ||a_n|| = |a_n| = |a_n - 0|$. We know, by definition, that there exists $N \geqslant m$ such that, for all $n \geqslant N$, we have $|a_n - 0| \leqslant \varepsilon$, and thus $|b_n - 0| \leqslant \varepsilon$. This means that $(b_n)_{n=m}^{\infty}$ converges to 0, as required.

Exercise 6.5.1. — *Show that $\lim_{n\to\infty}1/n^q = 0$ for any rational $q > 0$.*

Since q is a positive rational, let's suppose that $q := a/b$, with $a, b > 0$ two positive integers. With a little algebra, we can rewrite $1/n^q = (1/n)^q = (1/n)^{a/b} = ((1/n)^{1/b})^a$, using in particular Definition 5.6.7, and in a somewhat hidden manner, equation (5.13) from this document.

Using Corollary 6.5.1, we know that $\lim_{n\to\infty}1/n^{1/k} = 0$ for every integer $k \geqslant 1$. In particular, we have $\lim_{n\to\infty}(1/n)^{1/b} = \lim_{n\to\infty}(1/n^{1/b}) = 0$. Thus, using the limit laws (Theorem 6.1.19(b), iterated a times), we conclude that $\lim_{n\to\infty}1/n^q = \lim_{n\to\infty}(1/n^{1/b})^a = 0^a = 0$.

Exercise 6.5.2. — *Prove Lemma 6.5.2.*

Here, there are a lot of cases to consider.

- First, if $x = 1$, then $x^n = 1$ for all $n \geqslant 1$, so that $(x^n)_n$ is a constant sequence. Thus, $\lim_{n\to\infty}x^n = 1$.

- Similarly, if $x = 0$, then $\lim_{n\to\infty}x^n = 0$.

- If $-1 < x < 1$ and $x \neq 0$, we have $0 < |x| < 1$. This means, according to Proposition 6.3.10, that $\lim_{n\to\infty}|x|^n = 0$. But, we can note that we have, for all $n \geqslant 1$, the inequality $-|x^n| \leqslant x^n \leqslant |x|^n$. Since $\lim_{n\to\infty}|x|^n = \lim_{n\to\infty} - |x|^n = 0$, then according to the squeeze test, we also have $\lim_{n\to\infty}x^n = 0$.

- If $x = -1$, then the sequence $(x^n)_n$ is alternatively equal to 1 or -1. Consequently, it has two limits points, which are 1 and -1; it is even possible to show that $L^- = -1$ and $L^+ = 1$. Thus, it cannot converge, since a convergent sequence has only one limit point, and cannot have $L^- \neq L^+$.

- If $x > 1$, then Exercise 6.3.4 says that the sequence $(x^n)_n$ is divergent.

- Finally, if $x < -1$, then the exact same reasoning performed in Exercise 6.3.4 would also prove that $(x^n)_n$ is divergent.

Thus, as required, we have shown that $\lim_{n\to\infty}x^n$ exists and is equal to zero when $|x| < 1$, exists and is equal to 1 when $x = 1$, and diverges when $x = -1$ or when $|x| > 1$.

Exercise 6.5.3. — *Prove Lemma 6.5.3. (You may need to treat the cases $x \geqslant 1$ and $x < 1$ separately. You might wish to first use Lemma 6.5.2 to prove the preliminary result that for every $\varepsilon > 0$ and every real number $M > 0$, there exists an n such that $M^{1/n} \leqslant 1 + \varepsilon$.)*

We first prove the preliminary result suggested by Terence Tao. Let be $\varepsilon > 0$ a positive real number, so that $1 + \varepsilon > 1$. By Lemma 6.5.2, this implies that the sequence $(a_n)_{n=1}^{\infty}$ defined by $a_n = (1+\varepsilon)^n$ diverges. Furthermore, by Lemma 5.6.9(e), $(1+\varepsilon)^n$ is a strictly increasing function of n. It cannot be bounded because, by Proposition 6.3.8, it would imply that $(a_n)_{n=1}^{\infty}$ is a convergent sequence. Thus, $(a_n)_{n=1}^{\infty}$ is strictly increasing and not bounded above. In other words, for all $M > 0$, there exists a natural number N such that $(1+\varepsilon)^N \geqslant M$; and since we have an increasing function of n, we even get $(1+\varepsilon)^n \geqslant M$ for all $n \geqslant N$. This is equivalent to $(1+\varepsilon) \geqslant M^{1/n}$ by Lemma 5.6.6(d), which proves this preliminary result.

Let's go back to the main claim. We'll distinguish two cases. Let be $\varepsilon > 0$.

1. If $x \geqslant 1$, we have $x^{1/n} > 1$ for all $n \geqslant 1$ (because if we suppose, for the sake of contradiction, that $x^{1/n} < 1$, we would have $(x^{1/n})^n < 1^n$, i.e. $x < 1$). Thus, we have $0 < x^{1/n} - 1$ for all $n \geqslant 1$. Furthermore, according to the preliminary result, there exists an N such that $x^{1/n} - 1 \leqslant \varepsilon$ for all $n \geqslant N$. We thus have $-\varepsilon \leqslant 0 \leqslant x^{1/n} - 1 \leqslant \varepsilon$ for all $n \geqslant N$, i.e. $|x^{1/n} - 1| \leqslant \varepsilon$ for all $n \geqslant N$. This means that $\lim_{n\to\infty} x^{1/n} = 1$, as required.

2. If $x < 1$, we have $1/x > 1$. Thus, when considering $1/x$, we are back to the previous case, i.e. we have $\lim_{n\to\infty}(1/x)^{1/n} = 1$. But, by Lemma 5.6.9(b), we have $(1/x)^{1/n} = (x^{-1})^{1/n} = x^{-1/n} = 1/x^{1/n}$. Thus, we have $\lim_{n\to\infty} 1/x^{1/n} = 1$. Finally, by the limit laws (Theorem 6.1.19(e)), we get $\lim_{n\to\infty} x^{1/n} = 1/1 = 1$, as required.

Exercise 6.6.1. — *Prove Lemma 6.6.4.*

This lemma states that being a subsequence is reflexive and transitive.

- Let's show that $(a_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$. This is obvious: let's take the strictly increasing function $f(n) = n$ for all $n \in \mathbb{N}^*$: this will indeed give $a_n = a_{f(n)}$, i.e. $(a_n)_{n=1}^{\infty}$ as a subsequence of $(a_n)_{n=1}^{\infty}$.

- Now consider that $(b_n)_{n=1}^{\infty}$ is a subsequence of $(a_n)_{n=1}^{\infty}$, and that $(c_n)_{n=1}^{\infty}$ is a subsequence of $(b_n)_{n=1}^{\infty}$. It means that we have, for all $n \in \mathbb{N}^*$, $b_n = a_{f(n)}$ and $c_n = b_{g(n)}$, with f, g two strictly increasing functions from $\mathbb{N}^*$ to $\mathbb{N}^*$. The function $f \circ g$ is also strictly increasing, as the composition of two strictly increasing functions. Thus, we have $c_n = b_{g(n)} = a_{f\circ g(n)}$ for all $n \in \mathbb{N}^*$, i.e. $(c_n)_{n=1}^{\infty}$ as a subsequence of $(a_n)_{n=1}^{\infty}$.

Exercise 6.6.2. — *Can you find two sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ which are not the same sequence, but such that each is a subsequence of the other?*

It might appear counterintuitive, but the answer is yes! One can think of the sequences defined by $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. We have, for any natural number n, $a_n = b_{n+1}$ and $b_n = a_{n+1}$. I.e., we have found a strictly increasing function $f : n \mapsto n+1$ such that $a_n = b_{f(n)}$ and $b_n = a_{f(n)}$ for all $n \in \mathbb{N}$; and these sequences are not equal.

Note that any example where $(b_n)_{n=0}^{\infty}$ is simply a "shift" of a periodic sequence $(a_n)_{n=0}^{\infty}$ would do the trick. For instance, the sequence $a_n = 0, 0, 1, 1, 0, 0, \ldots$ and the sequence $b_n = 1, 1, 0, 0, 1, 1, \ldots$, with the function $f : n \mapsto n + 2$.

Exercise 6.6.3. — *Let $(a_n)_{n=0}^{\infty}$ be a sequence which is not bounded. Show that there exists a subsequence $(b_n)_{n=0}^{\infty}$ of $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} 1/b_n$ exists and is equal to zero. (Hint: for each natural number j, recursively introduce the quantity $n_j := \min\{n \in \mathbb{N} : |a_n| \geqslant j, n > n_j\}$,* *omitting the condition $n > n_{j-1}$ when $j = 0$. First explain why the set $\{n \in \mathbb{N} : |a_n| \geqslant j, n > n_j\}$ is non-empty; then set $b_j := a_{n_j}$. To ensure the existence and uniqueness of the minimum, one either needs to invoke the well ordering principle (which we have placed in Proposition 8.1.4, but whose proof does not rely on any material not already presented), or the least upper bound principle (Theorem 5.5.19).)*

For any natural number j, let's denote $A_j := \{n \in \mathbb{N} : |a_n| \geqslant j, n > n_{j-1}\}$, where n_j is recursively defined by:

$$\begin{cases} n_0 := \min \{n \in \mathbb{N} : |a_n| \geqslant 0\} \\ n_j := \min \{n \in \mathbb{N} : |a_n| \geqslant j; n > n_j\} \end{cases}$$

- For all natural number j, the set A_j is non-empty because $(a_n)_{n=0}^{\infty}$ is not bounded. Indeed, let's suppose for the sake of contradiction that there exists a natural number j such that $A_j = \varnothing$. It means that, for all natural numbers $n > n_j$, we have $|a_n| < j$. In such a case, $(a_n)_{n=j}^{\infty}$ is bounded by j, and thus $(a_n)_{n=0}^{\infty}$ is bounded by $M := \max(j, |a_0|, \dots, |a_{j-1}|)$. This contradicts our initial hypothesis that $(a_n)_{n=0}^{\infty}$ is not bounded. Thus, A_j is non-empty for all $j \in \mathbb{N}$.

- A_j also has a lower bound for all $j \in \mathbb{N}$. Indeed, by definition, n_{j-1} is a lower bound for A_j. So, A_j is always non-empty and has a lower bound. Thus, by Theorem 5.5.19, A_j has a greatest lower bound, that we will denote $\inf(A_j)$. (Furthermore, it is unique, by Theorem 5.5.18.)

- Let's show that $\inf(A_j) \in A_j$ for all $j \in \mathbb{N}$. Since $\inf(A_j)$ is the greatest lower bound, $\inf(A_j) + 1/2$ is not a lower bound for A_j (otherwise we would have an obvious contradiction). Thus, there exists $n \in A_j$ such that $\inf(A_j) \leqslant n < \inf(A_j) + 1/2$. Let's suppose, for the sake of contradiction, that there exists an $m \in A_j$ such that $m < n$. We would have $\inf(A_j) \leqslant m < n < \inf(A_j) + 1/2$, so that we could place two distinct integers within a range of width 1/2, which is impossible[25]. Thus, we have $n \leqslant m$ for all $m \in A_j$. It means that n is also a lower bound for A_j, and thus $n \leqslant \inf(A_j)$. And since we have both $n \leqslant \inf(A_j)$ and $n \geqslant \inf(A_j)$, we finally get $\inf(A_j) = n \in A_j$, as required[26]. Thus, the numbers n_j are well-defined.

- Finally, let be $\varepsilon > 0$ a positive real number, and a function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(j) = n_j$. It is a strictly increasing function by definition of the numbers n_j. Let's consider the sequence $(b_n)_{n=0}^{\infty}$ defined by $b_j := a_{n_j} = a_{f(j)}$ for all $j \in \mathbb{N}$.

  By definition, we also have $|a_{n_j}| \geqslant j$ for all $j \in \mathbb{N}$, which is equivalent to $1/|a_{n_j}| \leqslant 1/j$.

  By the archimedean property (Corollary 5.4.13, and Exercise 5.4.4), there exists a natural number j such that $\varepsilon > 1/j > 0$.

  Finally, let be a natural number $k > j$: we thus have $|a_{n_k}| \geqslant k > j$, i.e. $1/|a_{n_k}| < 1/j$.

  Thus, unwrapping all these findings: for any $\varepsilon > 0$, there exists a natural number j such that, for any $k > j$, we have $1/|a_{n_k}| = |b_k - 0| \leqslant 1/j \leqslant \varepsilon$, i.e., $\lim_{n\to\infty} b_n = 0$, as required.

[25] One may also give a proof by contradiction by writing $x + 1/2 - x = \underbrace{x + 1/2 - m}_{>0} + \underbrace{m - n}_{>1} + \underbrace{n - x}_{>0}$, to prove this fact in this particular situation.

[26] We actually have proven a generalizable lemma: for any non-empty set $X \subset \mathbb{N}$, the greatest lower bound $\inf(X)$ is in X. This will be useful in Exercise 6.6.5 as well; and will be useful in Chapter 8.

Exercise 6.6.4. — *Prove Proposition 6.6.5. (Note that one of the two implications has a very short proof.)*

First let's prove a preliminary result: if $f : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, then $f(n) \geqslant n$ for all $n \in \mathbb{N}$. We can use induction on n:

- for the base case $n = 0$, we indeed have $f(0) \geqslant 0$ because $f(0) \in \mathbb{N}$;
- now suppose inductively that $f(n) \geqslant n$. Since f is strictly increasing, we have $f(n+1) > f(n) \geqslant n$, and thus, $f(n+1) > n$, i.e. $f(n+1) \geqslant n+1$. This closes the induction for this preliminary result.

For the exercise itself, we must prove that the two statements "The sequence $(a_n)_{n=0}^{\infty}$ converges to L" and "Every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L" are equivalent.

- First suppose that every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L. According to Lemma 6.6.4, $(a_n)_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$, thus in particular, $(a_n)_{n=0}^{\infty}$ converges to L.
- Now suppose that the sequence $(a_n)_{n=0}^{\infty}$ converges to L. Let be $\varepsilon > 0$, and $f : \mathbb{N} \to \mathbb{N}$ a strictly increasing function such that $(a_{f(n)})_{n=0}^{\infty}$ is a subsequence of $(a_n)_{n=0}^{\infty}$.

  Since $(a_n)_{n=0}^{\infty}$ converges to L, there exists a natural number N such that $|a_n - L| \leqslant \varepsilon$ for all $n \geqslant N$. But, according to our preliminary result, we have $f(n) \geqslant n$, and thus by transitivity, $f(n) \geqslant N$. This means that $|a_{f(n)} - L| \leqslant \varepsilon$. Unfolding those findings: there exists an N such that $|a_{f(n)} - L| \leqslant \varepsilon$ for all $n \geqslant N$, i.e., $(a_{f(n)})_{n=0}^{\infty}$ converges to L.

Thus, both statements are indeed equivalent.

Exercise 6.6.5. — *Prove Proposition 6.6.6. (Hint: to show that (a) implies (b), define the numbers n_j for each natural number j by the formula $n_j := \min\{n > n_{j-1} : |a_n - L| \leqslant 1/j\}$, with the convention $n_0 := 0$, explaining why the set $\{n > n_{j-1} : |a_n - L| \leqslant 1/j\}$ is non-empty. Then consider the sequence a_{n_j}. To ensure the existence and uniqueness of the minimum, one either needs to invoke the well ordering principle (which we have placed in Proposition 8.1.4, but whose proof does not rely on any material not already presented), or the least upper bound principle (Theorem 5.5.19).)*

This exercise is pretty similar to Exercise 6.6.3, and we will re-use here a preliminary result shown there—see also note 26.

1. First we show that (b) implies (a). Let be $f : \mathbb{N} \to \mathbb{N}$ a strictly increasing function such that $(a_{f(n)})_{n=0}^{\infty}$ converges to L. We must show that L is a limit point of $(a_n)_{n=0}^{\infty}$, i.e. that:
$$\forall \varepsilon > 0, \forall N \geqslant 0, \exists n \geqslant N \ : \ |a_n - L| \leqslant \varepsilon \tag{6.17}$$
   So let be $\varepsilon > 0$ and $N \geqslant 0$. If $(a_{f(n)})_{n=0}^{\infty}$ converges to L, there exists $M \geqslant 0$ such that $n \geqslant M \to |a_{f(n)} - L| < \varepsilon$. We know that f is a strictly increasing function, so that $f(n) \geqslant n$ for all $n \in \mathbb{N}$, as previously shown in Exercise 6.6.4. Thus, if one considers $p := \max(M, N)$, we have:
   - $f(p) \geqslant p \geqslant N$

- $f(p) \geqslant p \geqslant M$, so that $|a_{f(p)} - L| \leqslant \varepsilon$

  Choosing $n := f(p) \geqslant N$, the condition in formula (6.17) is thus verified.

2. Now we prove that (a) implies (b).

   - First, we note that the set $\{n > n_{j-1} : |a_n - L| \leqslant 1/j\}$ is non-empty for all $j \geqslant 1$. For instance, with $j = 1$, this set is equal to $\{n > 0 : |a_n - L| \leqslant 1\}$: this set is non-empty since L is a limit point of $(a_n)_{n=0}^{\infty}$. This gives an intuition for the general case. Generally speaking, let's suppose for the sake of contradiction that the set $A_j := \{n > n_{j-1} : |a_n - L| \leqslant 1/j\}$ is non empty for one given $j \geqslant 1$. It means that we can define $\varepsilon := 1/j$ and $N := n_{j-1}$ such that, for all $n \geqslant N$, we have $|a_n - L| > \varepsilon$. This is precisely the negation of the fact that L is a limit point of $(a_n)_{n=0}^{\infty}$ (see formula (6.17)): we have a clear contradiction. Thus, this set A_j is non-empty for every natural number $j \geqslant 1$.
   - Thus, for all $j \geqslant 1$, this set A_j is non-empty, and it has a lower bound (for instance, 0 is obviously a lower bound, since A_j is a subset of $\mathbb{N}$). By Theorem 5.5.9, we thus know that this set has a (unique) greatest lower bound, which we can write $\inf(A_j)$. By note 26, we have $\inf(A_j) \in A_j$, i.e., $\inf(A_j) = \min(A_j) = n_j$ is well-defined.
   - Now let be $f : \mathbb{N} \to \mathbb{N}$ the strictly increasing function defined by $f(j) := n_j$; and let's consider the subsequence of $(a_n)_{n=0}^{\infty}$ defined by $a_{f(j)} = a_{n_j}$. Let be $\varepsilon > 0$. By the archimedean property (or Exercise 5.4.4), there exists a natural number $j \geqslant 1$ such that $\varepsilon > 1/j > 0$. We thus have a natural number n_j such that $|a_{n_j} - L| \leqslant 1/j < \varepsilon$. Furthermore, if $k > j$, we have $n_k > n_j$, and thus we have: $|a_{n_k} - L| \leqslant 1/k \leqslant 1/j < \varepsilon$. Unfolding all these findings:

   $$\forall \varepsilon > 0, \exists j \geqslant 1 \ : \ n \geqslant j \to |a_{f(j)} - L| \leqslant \varepsilon \tag{6.18}$$

   which means that the subsequence $(a_{f(n)})_{n=0}^{\infty}$ converges to 0, as required.

7. Series

Exercise 7.1.1. — *Prove Lemma 7.1.4. (Hint: you will need to use induction, but the base case might not necessarily be at 0.)*

Recall that, by Definition 7.1.1, a series $\sum_{i=m}^{n} a_i$ is recursively defined by $\sum_{i=m}^{n} a_i = 0$ for $n < m$, and $\sum_{i=m}^{n+1} a_i = \sum_{i=m}^{n} a_i + a_{n+1}$ for $m \geqslant n$.

The claims to prove are the following.

(a) For all integers $m \leqslant n < p$, we have $\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i$.

Let's use induction on p, while keeping m, n fixed. Since we have $n < p$, the base case is $p = n + 1$. For $p = n + 1$, we have:

$$\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{n} a_i + \sum_{i=n+1}^{n+1} a_i = \sum_{i=m}^{n} a_i + a_{n+1} = \sum_{i=m}^{n+1} a_i = \sum_{i=m}^{p} a_i$$

as expected. The base case is done. Now let's suppose inductively that $\sum_{i=m}^{n} a_i + \sum_{i=n+1}^{p} a_i = \sum_{i=m}^{p} a_i$, and let's prove that this equality still holds for $p + 1$. We have:

$$\begin{aligned}
\sum_{i=m}^{n} a_i + \sum_{i=m}^{p+1} &= \sum_{i=m}^{n} a_i + \sum_{i=m}^{p} a_i + a_{p+1} \text{ (Definition 7.1.1)} \\
&= \sum_{i=m}^{p} a_i + a_{p+1} \text{ (induction hypothesis)} \\
&= \sum_{i=m}^{p+1} a_i \text{ (Definition 7.1.1)}
\end{aligned}$$

which closes the induction and proves the equality for all integers $m \leqslant n < p$.

(b) For k an integer, and $m \leqslant n$ two integers, we have $\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}$.

Let's use induction on n, the "offset" k being fixed. For the base case $n = m$, we have $\sum_{j=m+k}^{n+k} a_{j-k} = \sum_{j=m+k}^{m+k} a_{j-k} = a_{m+k-k} = a_m$. On the other hand, we also have $\sum_{i=m}^{n} a_i = \sum_{i=m}^{m} a_i = a_m$, which proves the base case.

Now let's suppose inductively that $\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}$. We thus have:

$$\begin{aligned}
\sum_{j=m+k}^{n+k+1} a_{j-k} &= \sum_{j=m+k}^{n+k} a_{j-k} + a_{n+1} \\
&= \sum_{i=m}^{n} a_i + a_{n+1} \text{ (induction hypothesis)} \\
&= \sum_{i=m}^{n+1} a_i
\end{aligned}$$

which closes the induction, the property being true for all $n \geqslant m$, with any arbitrary offset k.

(c) For $m \leqslant n$ two integers, we have $\sum_{i=m}^{n}(a_i + b_i) = (\sum_{i=m}^{n} a_i) + (\sum_{i=m}^{n} b_i)$.

Let's use induction on n. The base case is $m = n$ and is obvious, since we have on the one hand $\sum_{i=m}^{m}(a_i + b_i) = a_m + b_m$, and on the other hand $(\sum_{i=m}^{m} a_i) + (\sum_{i=m}^{m} b_i) = a_m + b_m$.

Now let's suppose inductively that $\sum_{i=m}^{n}(a_i + b_i) = (\sum_{i=m}^{n} a_i) + (\sum_{i=m}^{n} b_i)$. Then we have:

$$\begin{aligned}
\sum_{i=m}^{n+1}(a_i + b_i) &= \sum_{i=m}^{n}(a_i + b_i) + (a_{n+1} + b_{n+1}) \\
&= \left(\sum_{i=m}^{n} a_i + \sum_{i=m}^{n} b_i\right) + a_{n+1} + b_{n+1} \text{ (induction hypothesis)} \\
&= \sum_{i=m}^{n} a_i + a_{n+1} + \sum_{i=m}^{n} b_i + b_{n+1} \\
&= \sum_{i=m}^{n+1} a_i + \sum_{i=m}^{n+1} b_i
\end{aligned}$$

which closes the induction. The property is thus true for all $n \geqslant m$.

(d) For $m \leqslant n$ integers, and c a real number, we have $\sum_{i=m}^{n} ca_i = c\,(\sum_{i=m}^{n} a_i)$.

Let's induct on n, first considering the base case $n = m$. We have $\sum_{i=m}^{m} ca_i = ca_m = c\,(\sum_{i=m}^{m} a_i)$ as required.

Now let's suppose inductively that $\sum_{i=m}^{n} ca_i = c\,(\sum_{i=m}^{n} a_i)$. Then we have:

$$\begin{aligned}
\sum_{i=m}^{n+1} ca_i &= \sum_{i=m}^{n} ca_i + ca_{n+1} \\
&= c\left(\sum_{i=m}^{n} a_i\right) + ca_{n+1} \text{ (induction hypothesis)} \\
&= c\left(\sum_{i=m}^{n} a_i + a_{n+1}\right) \\
&= c\sum_{i=m}^{n+1} a_i
\end{aligned}$$

which closes the induction. The property is thus true for all $n \geqslant m$.

(e) For $m \leqslant n$ integers, prove that $|\sum_{i=m}^{n} a_i| \leqslant \sum_{i=m}^{n} |a_i|$.

Let's use induction on n, first considering the base case $n = m$. We have $|\sum_{i=m}^{m} a_i| = |a_m| = \sum_{im}^{m} |a_i|$, as required.

Now let's suppose inductively that we have $|\sum_{i=m}^{n} a_i| \leqslant \sum_{i=m}^{n} |a_i|$. Then we have:

$$\begin{aligned}\left|\sum_{i=m}^{n+1} a_i\right| &:= \left|\sum_{i=m}^{n} a_i + a_{n+1}\right| \\ &\leqslant \left|\sum_{i=m}^{n} a_i\right| + |a_{n+1}| \text{ (Proposition 4.3.3(b))} \\ &\leqslant \sum_{i=m}^{n} |a_i| + |a_{n+1}| \text{ (induction hypothesis)} \\ &\leqslant \sum_{i=m}^{n+1} |a_i|\end{aligned}$$

which closes the induction. The property is thus true for all $n \geqslant m$.

(f) For $m \leqslant n$ integers, prove that, if $a_i \leqslant b_i$ for all $m \leqslant i \leqslant n$, we have $\sum_{i=m}^{n} \leqslant \sum_{i=m}^{n} b_i$.

Once again, let's use induction on n, starting with the base case $n = m$. In this case, we have $\sum_{i=m}^{n} a_i := a_m \leqslant b_m := \sum_{i=m}^{n} b_i$, as required.

Now let's suppose inductively that we have $\sum_{i=m}^{n} \leqslant \sum_{i=m}^{n} b_i$. Then, if $a_{n+1} \leqslant b_{n+1}$, we have:

$$\begin{aligned}\sum_{i=m}^{n+1} a_i &:= \sum_{i=m}^{n} a_i + a_{n+1} \\ &\leqslant \sum_{i=m}^{n} a_i + b_{n+1} \\ &\leqslant \sum_{i=m}^{n} b_i + b_{n+1} \text{ (induction hypothesis)} \\ &\leqslant \sum_{i=m}^{n+1} b_i\end{aligned}$$

which closes the induction. The property is thus true for all $n \geqslant m$.

EXERCISE 7.1.2. — *Prove Proposition 7.1.11.*

First recall the main definition: if X is a finite set with n elements, $f : X \to \mathbb{R}$ a function, and g a bijection from $[\![1, n]\!]$ to X, then we have $\sum_{x \in X} f(x) := \sum_{i=1}^{n} f(g(i))$.

The statements to prove are:

(a) If $X = \varnothing$ and f is the empty function, we have $\sum_{x \in X} f(x) = 0$.

In this case, X has $n = 0$ elements, and the empty function g is a bijection between X and $[\![1, 0]\!]$, so that by definition, we have $\sum_{x \in X} f(x) = \sum_{i=1}^{0} f(g(i)) = 0$.

(b) If $X = \{x_0\}$, we have $\sum_{x \in X} f(x) = f(x_0)$.

In this case, X has $n = 1$ element, and $g : \{1\} \to X$ such that $g(1) = x_0$ is a bijection. Thus, by definition, we have $\sum_{x \in X} f(x) = \sum_{i=1}^{1} f(g(i)) = f(g(1)) = f(x_0)$.

(c) If X is a finite set and $g : Y \to X$ a bijection, then $\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y))$.

First note that since g is a bijection, Y also has n elements.

On the one hand, since X has n elements, there exists a bijection $h : [\![1, n]\!] \to X$ such that $\sum_{x \in X} f(x) = \sum_{i=1}^{n} f(h(i))$.

On the other hand, let be $k : [\![1, n]\!] \to Y$ a function defined by $k = g^{-1} \circ h$. Since k is the composition of two bijections, it is itself a bijection (cf. Exercise 3.3.2). We thus have : $\sum_{y \in Y} f(g(y)) = \sum_{i=1}^{n} f \circ g \circ g^{-1} \circ h(i) = \sum_{i=1}^{n} f(h(i))$.

Thus, we have indeed $\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y))$.

(d) For $n \leqslant m$ two integers and $X = \{i \in \mathbb{Z} : n \leqslant i \leqslant m\}$, we have $\sum_{i=n}^{m} a_i = \sum_{i \in X} a_i$.

Let be $f : [\![n, m]\!] \to X$ defined by $f(i) = a_i$ a function; so that we have $\sum_{i \in X} a_i = \sum_{i \in X} f(i)$. First of all, note that X has $m - n + 1$ elements. Furthermore, let be $g : [\![1, m-n+1]\!] \to X$ the bijection defined by $g(i) = i+n-1$ (we could show rigorously that this function is bijective, but this is straightforward). We thus have, by definition: $\sum_{i \in X} a_i = \sum_{i \in X} f(i) = \sum_{i=1}^{m-n+1} f(g(i)) = \sum_{i=1}^{m-n+1} f(i + n - 1) = \sum_{i=1}^{m-n+1} a_{i+n-1}$.

Now we can use Lemma 7.1.4(b), to get $\sum_{i=1}^{m-n+1} a_{i+n-1} = \sum_{i=n}^{m} a_i$, so that we have finally showed that $\sum_{i \in X} a_i = \sum_{i=n}^{m} a_i$.

(e) If X, Y are sets such that $X \cap Y = \varnothing$, and $f : X \cup Y \to \mathbb{R}$, we have $\sum_{z \in X \cup Y} f(z) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$.

Suppose that X has n elements and Y has m elements. Thus, there exist two bijections $g : [\![1, n]\!] \to X$ and $h : [\![1, m]\!] \to Y$. Thus, by definition, we have $\sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g(i))$ and $\sum_{y \in Y} f(y) = \sum_{i=1}^{m} f(h(i))$.

By Lemma 7.1.4(b), we have $\sum_{i=1}^{m} f(h(i)) = \sum_{i=n+1}^{n+m} f(h(i - n))$.

Now let's construct a new function $k : [\![1, n + m]\!] \to X \cup Y$, defined by: $k(i) = g(i) \in X$ if $1 \leqslant i \leqslant n$ and $k(i) = h(i) \in Y$ if $n + 1 \leqslant i \leqslant n + m$. In particular, $k(i) \in X \cup Y$ for all $i \in [\![1, n + m]\!]$. Let's prove that k is a bijection.

- Let's suppose that there exists $z \in X \cup Y$ such that $k(i) \neq z$ for all $i \in [\![1, n + m]\!]$. If $z \in X$, this means in particular that there exists one $z \in X$ such that $g(i) \neq z$ for all $i \in [\![1, n]\!]$, a contradiction. A similar contradiction follows if $z \in Y$. Thus, k is surjective.
- Now suppose that there exist two $i, j \in [\![1, n + m]\!]$ such that $k(i) = k(j)$. If both $i, j \in [\![1, n]\!]$, it is a contradiction with the fact that g is injective; a similar contradiction follows for h if both $i, j \in [\![n + 1, m]\!]$. Finally, if $i \in [\![1, n]\!]$ and $j \in [\![n+1, m]\!]$ (or the converse), it is a contradiction with the fact that $X \cap Y = \varnothing$. Thus, k is injective.

Thus, we have:

$$\begin{aligned}
\sum_{x \in X \cup Y} f(x) &= \sum_{i=1}^{n+m} f(k(i)) \\
&= \sum_{i=1}^{n} f(k(i)) + \sum_{i=n+1}^{n+m} f(k(i)) \text{ (by Lemma 7.1.4(a))} \\
&= \sum_{i=1}^{n} f(g(i)) + \sum_{i=n+1}^{n+m} f(h(i)) \\
&= \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)
\end{aligned}$$

as required, which closes the proof.

(f) If X is a finite set, then we have $\sum_{x \in X}(f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x)$.

Suppose that X has n elements, so that there exists a bijection $h : [\![1, n]\!] \to X$. Then we have:

$$\begin{aligned}
\sum_{x \in X} f(x) + g(x) &:= \sum_{i=1}^{n} f(h(i)) + g(h(i)) \\
&= \sum_{i=1}^{n} f(h(i)) + \sum_{i=1}^{n} g(h(i)) \text{ (Lemma 7.1.4(e))} \\
&:= \sum_{x \in X} f(x) + \sum_{x \in X} g(x)
\end{aligned}$$

(g) If c is a real number, the we have $\sum_{x \in X} cf(x) = c \sum_{x \in X} f(x)$.

If we define a function g as $g(x) := cf(x)$, and h a bijection $[\![1, n]\!] \to X$, we have:

$$\begin{aligned}
\sum_{x \in X} cf(x) = \sum_{x \in X} g(x) = \sum_{i=1}^{n} g(h(i)) &= \sum_{i=1}^{n} cf(h(i)) \\
&= c \times \sum_{i=1}^{n} f(h(i)) \text{ (Lemma 7.1.4(d))} \\
&= c \times \sum_{x \in X} f(x)
\end{aligned}$$

(h) If f, g are functions such that $f(x) \leqslant g(x)$ for all $i \in \mathbb{R}$, then we have $\sum_{x \in X} f(x) \leqslant \sum_{x \in X} g(x)$.

If X has n elements, let $h : [\![1, n]\!] \to X$ be a bijection. Then we have:

$$\begin{aligned}
\sum_{x \in X} f(x) &= \sum_{i=1}^{n} f(h(i)) \\
&\leqslant \sum_{i=1}^{n} g(h(i)) \text{ (Lemma 7.1.4(f))} \\
&= \sum_{x \in X} g(x)
\end{aligned}$$

(i) Triangle inequality: we have $|\sum_{x\in X} f(x)| \leqslant \sum_{x\in X} |f(x)|$.

If we consider the function $k : X \to \mathbb{R}$ such that $k(x) = |f(x)|$ for all $x \in X$; and $h : [\![1, n]\!] \to X$ a bijection, we have:

$$\begin{aligned}\left|\sum_{x\in X} f(x)\right| &= \left|\sum_{i=1}^{n} f(h(i))\right| \\ &\leqslant \sum_{i=1}^{n} |f(h(i))| \text{ (Lemma 7.1.4(e))} \\ &= \sum_{i=1}^{n} k(h(i)) = \sum_{x\in X} k(x) \\ &= \sum_{x\in X} |f(x)|\end{aligned}$$

Exercise 7.1.4. — *Define the factorial function $n!$ for natural numbers n by the recursive definition $0! := 1$ and $(n+1)! := n! \times (n+1)$. If x and y are real numbers, prove the binomial formula $(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$ for all natural numbers n. (Hint: induct on n.)*

First we begin with a small preliminary result, which is that for all n, j natural numbers, we have $\frac{(n+1)!}{j!(n+1-j)!} = \frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!}$. Indeed:

$$\begin{aligned}\frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!} &= \frac{n! \times j}{j!(n-j+1)!} + \frac{n! \times (n-j+1)}{j!(n-j+1)!} \\ &= \frac{n! \times (n+1)}{j!(n-j+1)!} \\ &= \frac{(n+1)!}{j!(n+1-j)!}\end{aligned}$$

Now we can go back to the main proof, for which we will induct on n.

- Let's start with the base case $n = 0$. On the one hand, by Definition 5.6.1, we have $(x+y)^0 = 1$. On the other hand, we have $\sum_{j=0}^{0} \frac{n!}{j!(n-j)!} x^j y^{n-j} = \frac{0!}{0! \times 0!} x^0 y^0 = 1$, so that the base case is done.
- Now suppose inductively that we have $(x+y)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j y^{n-j}$, and let's prove that this property is still true for $n+1$. We have:

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)\times(x+y)^n \\
&= x\left(\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}x^j y^{n-j}\right) + y\left(\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}x^j y^{n-j}\right) \text{ (by induction hypothesis)} \\
&= \left(\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}x^{j+1} y^{n-j}\right) + \left(\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}x^j y^{n-j+1}\right) \\
&= \left(x^{n+1} + \sum_{j=0}^{n-1}\frac{n!}{j!(n-j)!}x^{j+1} y^{n-j}\right) + \left(y^{n+1} + \sum_{j=1}^{n}\frac{n!}{j!(n-j)!}x^j y^{n-j+1}\right) \\
&= \left(x^{n+1} + \sum_{j=1}^{n}\frac{n!}{(j-1)!(n-j+1)!}x^{j} y^{n-j+1}\right) + \left(y^{n+1} + \sum_{j=1}^{n}\frac{n!}{j!(n-j)!}x^j y^{n-j+1}\right) \\
&= x^{n+1} + \sum_{j=1}^{n}\left(\frac{n!}{(j-1)!(n-j+1)!} + \frac{n!}{j!(n-j)!}\right)x^j y^{n-j+1} + y^{n+1} \\
&= x^{n+1} + \sum_{j=1}^{n}\frac{(n+1)!}{j!(n+1-j)!}x^j y^{n-j+1} + y^{n+1} \\
&= \sum_{j=0}^{n+1}\frac{(n+1)!}{j!(n+1-j)!}x^j y^{n-j+1}
\end{aligned}
$$

so that the property is true for all natural number n.

Exercise 7.1.5. — *Let X be a finite set, let m be an integer, and for each $x \in X$ let $(a_n(x))_{n=m}^{\infty}$ be a convergent sequence of real numbers. Show that the sequence $(\sum_{x\in X} a_n(x))_{n=m}^{\infty}$ is convergent, and that $lim_{n\to\infty} \sum_{x\in X} a_n(x) = \sum_{x\in X} lim_{n\to\infty} a_n(x)$.*

Let's suppose that X has r elements. Note that, in particular, the property is equivalent to $\lim_{n\to\infty}\sum_{i=1}^{r} a_n(h(i)) = \sum_{i=1}^{r}\lim_{n\to\infty} a_n(h(i))$ for any bijection $h : [\![1,r]\!] \to X$.

Let's induct on r, the cardinality of X.

- We start with the base case $r = 0$, i.e. $X = \varnothing$. In this case, by Proposition 7.1.11(a), we have immediately $\sum_{x\in X} a_n(x) = 0$, so that $\lim_{n\to\infty}\sum_{x\in X} a_n(x) = 0$ as the limit of a constant sequence. Similarly, we have $\sum_{x\in X}\lim_{n\to\infty} a_n(x) = 0$, as a sum over an empty set X. Thus, the base case is done.
- Now suppose inductively that $\lim_{n\to\infty}\sum_{x\in X} a_n(x) = \sum_{x\in X}\lim_{n\to\infty} a_n(x)$ when X has r elements, and let's show that this property is still true when X has $r+1$ elements.

  Let be $h : [\![1, r+1]\!] \to X$ a bijection. By Definition 7.1.6, we have:

$$
\begin{aligned}
\sum_{x\in X} a_n(x) &= \sum_{i=1}^{r+1} a_n(h(i)) \\
&= \sum_{i=1}^{r} a_n(h(i)) + a_n(h(r+1))
\end{aligned}
$$

By the induction hypothesis, $\sum_{i=1}^{r} a_n(h(i))$ is a convergent sequence; and $a_n(h(r+1))$ is a convergent sequence by the initial hypothesis; so that we can apply Theorem 6.1.19(a):

$$\begin{aligned}
\lim_{n\to\infty} \sum_{x\in X} a_n(x) &= \lim_{n\to\infty} \left(\sum_{i=1}^{r} a_n(h(i)) + a_n(h(r+1)) \right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{r} a_n(h(i)) + \lim_{n\to\infty} a_n(h(r+1)) \\
&= \sum_{i=1}^{r} \lim_{n\to\infty} a_n(h(i)) + \lim_{n\to\infty} a_n(h(r+1)) \text{ (by induction hypothesis)} \\
&= \sum_{i=1}^{r+1} \lim_{n\to\infty} a_n(h(i)) \text{ (Definition 7.1.1)} \\
&= \sum_{x\in X} \lim_{n\to\infty} a_n(x) \text{ (by Proposition 7.1.11(c))}
\end{aligned}$$

so that the property is also true when X has $n+1$ elements. This closes the induction.

Exercise 7.2.2. — *Prove Proposition 7.2.5.*

We have to prove that the formal series $\sum_{n=m}^{\infty} a_n$ converges iff, for any $\varepsilon > 0$, there exists an integer $N \geqslant m$ such that $\left|\sum_{n=p}^{q} a_n\right| \leqslant \varepsilon$ for all $p, q \geqslant N$.

First, note that the second statement is just another way to say that the partial sum $S_n = \sum_{i=m}^{n} a_n$ is a Cauchy sequence. Indeed, for $q \geqslant p \geqslant N$, we have $|S_q - S_p| = |\sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n| = \left|\sum_{n=p}^{q} a_n\right|$ (by Lemma 7.1.4(a)). Thus, the equivalence to prove is simply the fact that $\sum_{n=m}^{\infty} a_n$ converges iff the sequence of its partial sums $(S_N)_{N=m}^{\infty}$ is a Cauchy sequence. Proposition 6.1.12 and Theorem 6.4.18 immediately provide this equivalence.

Exercise 7.2.3. — *Prove Corollary 7.2.6.*

Let's suppose that $\sum_{n=m}^{\infty} a_n$ is a convergent sequence. Thus, according to Proposition 7.2.5 (see also previous exercise), for all $\varepsilon > 0$, there exists $N \geqslant 0$ such as $p, q \geqslant N \implies \left|\sum_{n=p}^{q} a_n\right| \leqslant \varepsilon$.

In particular, let's take $p = q$, and we have $|a_p| \leqslant \varepsilon$ for all $p \geqslant N$, which says precisely that $(a_n)_{n=m}^{\infty}$ converges to 0.

Exercise 7.2.5. — *Prove Proposition 7.2.14.*

We have to prove several statements here:

(a) Let's consider the partial sums $S_N := \sum_{n=m}^{N} a_n$ and $T_N := \sum_{n=m}^{N} b_n$. Saying that the formal series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ are convergent means that the sequences of their partials sums, $(S_N)_{N=m}^{\infty}$ and $(T_N)_{N=m}^{\infty}$ are convergent, towards x and y respectively.

By Theorem 6.1.19(a), the sequence $(S_N+T_N)_{N=m}^{\infty}$ converges to $x+y$. But by definition, we have $S_N + T_N = \sum_{n=m}^{N} a_n + \sum_{n=m}^{N} b_n = \sum_{n=m}^{N} a_n + b_n$ by Lemma 7.1.4(c). It means that the partial sum of the formal series $\sum_{n=m}^{\infty} a_n + b_n$ converges to $x + y$, so that we have proved the result.

(b) The proof would be very similar to the previous one; just use Lemma 7.1.4(d) and Theorem 6.1.19(c) instead.

(c) For all integers $N \geqslant m$ and $k \geqslant 0$, we know by Lemma 7.1.4(a) that we have $\sum_{n=m}^{N} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{N} a_n$. If we set $S_N := \sum_{n=m}^{N} a_n$; $x := \sum_{n=m}^{m+k-1} a_n$; and $T_N := \sum_{n=m+k}^{N} a_n$, we have for all $N \geqslant m$, the equality $S_N = x + T_N$. As a very general fact (not shown in the book, but easy to prove), the sequence $(S_N)_{N=m}^{\infty}$ converges to L iff the sequence $(T_N)_{N=m}^{\infty}$ converges to $L - x$. The statement follows.

(d) Let be $\varepsilon > 0$ a positive real number. Since the formal series $\sum_{n=m}^{\infty} a_n$ converges to x, there exists a positive integer M such that $|\sum_{n=m}^{N} a_n - x| \leqslant \varepsilon$ for all $N \geqslant M$. By Lemma 7.1.4(b), this is equivalent to $|\sum_{n=m+k}^{N+k} a_{n-k} - x| \leqslant \varepsilon$, for k a positive integer and for all $N \geqslant M$.

Thus, there exists a positive integer $M' := M + k$ such that, for all $N \geqslant M'$, we have $|\sum_{n=m+k}^{N} a_{n-k} - x| \leqslant \varepsilon$, which means that $\sum_{n=m+k}^{\infty} a_{n-k}$ also converges to x.

Exercise 7.2.6. — *Prove Lemma 7.2.15.*

Let be $(a_n)_{n=0}^{\infty}$ a sequence that converges to 0, and let's consider the formal series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$. Its partials sums are, for any integer $N \geqslant 0$, $S_N := \sum_{n=0}^{N}(a_n - a_{n+1}) = a_0 - a_{N+1}$. This can be shown by an induction on $N \geqslant 0$:

- Let's start with the base case $N = 0$. In this case, we have $S_0 = \sum_{n=0}^{0}(a_n - a_{n+1}) = a_0 - a1$, as expected.

- Now suppose inductively that the property for a given positive integer N, and let's show that it is still true for $N + 1$. We have:

$$\begin{aligned} S_{N+1} &:= \sum_{n=0}^{N+1}(a_n - a_{n+1}) \\ &= \sum_{n=0}^{N}(a_n - a_{n+1}) + (a_{N+1} - a_{N+2}) \text{ (Definition 7.1.1)} \\ &= a_0 - a_{N+1} + a_{N+1} - a_{N+2} \text{ (induction hypothesis)} \\ &= a_0 - a_{N+2} \end{aligned}$$

which closes the induction, so that the property is true for all $N \geqslant 0$.

Now, let be $\varepsilon > 0$ a positive real number. Recall that $(a_n)_{n=0}^{\infty}$ converges to 0, so that there exists a positive integer $M \geqslant 0$ such that $|a_n| \leqslant \varepsilon$ for all $n \geqslant M$. On the other hand, we have, for all $N \geqslant 0$, $|S_N - a_0| = |a_0 - a_{N+1} - a_0| = |a_{N+1}|$. Thus, there exists a positive integer $M' := M - 1$ such that, for all $N \geqslant M'$, we have $|S_N - a_0| \leqslant \varepsilon$. This means that $(S_N)_{N=0}^{\infty}$ converges to a_0, i.e. that the formal series $\sum_{n=0}^{\infty}(a_n - a_{n+1})$ converges to a_0, as required.

Exercise 7.3.1. — *Use Proposition 7.3.1 to prove Corollary 7.3.2.*

First, an important remark: if we have $|a_n| \leqslant b_n$ for all $n \geqslant m$, we have in particular $b_n \geqslant 0$ for all $n \geqslant m$, so that the formal series $\sum_{n=m}^{\infty} b_n$ is a series of non-negative numbers. Since it is convergent by hypothesis, it is thus possible to apply Proposition 7.3.1 to this series: there exists a real number M such that $T_N := \sum_{n=m}^{N} b_n \leqslant M$ for all $N \geqslant m$.

- First, let's show the first part of Corollary 7.3.2, that is to say that $\sum_{n=m}^{\infty} a_n$ is absolutely convergent as soon as $\sum_{n=m}^{\infty} b_n$ is convergent.

  Since we have $|a_n| \leqslant b_n$ for all $n \geqslant m$ by hypothesis, we have also for all $N \geqslant m$ the inequality $\sum_{n=m}^{N} |a_n| \leqslant \sum_{n=m}^{N} b_n \leqslant M$ by Lemma 7.1.4(f), so that the formal series $\sum_{n=m}^{\infty} |a_n|$ is itself convergent by Proposition 7.3.1 again. This proves the first statement.

- Now let's prove that $|\sum_{n=m}^{\infty} a_n| \leqslant \sum_{n=m}^{\infty} |a_n|$. This is actually simply Proposition 7.2.9, so there is nothing to add here.

- Finally, let's prove that $\sum_{n=m}^{\infty} |a_n| \leqslant \sum_{n=m}^{\infty} b_n$. By Proposition 7.2.14(a)-(b), we know that, since $\sum_{n=m}^{\infty} |a_n|$ and $\sum_{n=m}^{\infty} b_n$ are convergent, then $\sum_{n=m}^{\infty} b_n - |a_n|$ is convergent, so that its partial sum $V_N := \sum_{n=m}^{N} b_n - |a_n|$ converges to some limit L. Also, we already know that $b_n - |a_n| \geqslant 0$ for all $n \geqslant m$, so that $V_N \geqslant 0$ for all $N \geqslant m$, and thus $L \geqslant 0$ by Proposition 5.4.9.

  To summarize: $\sum_{n=m}^{\infty}(b_n - |a_n|) = \sum_{n=m}^{\infty} b_n - \sum_{n=m}^{\infty} |a_n| = L \geqslant 0$, and thus in particular, $\sum_{n=m}^{\infty} b_n \geqslant \sum_{n=m}^{\infty} |a_n|$, as required.

  This closes the proof.

Exercise 7.3.2. — *Prove Lemma 7.3.3.*

Let be x a real number and let's consider the formal series $\sum_{n=0}^{\infty} x^n$.

- If $|x| \geqslant 1$, then $(x^n)_{n=0}^{\infty}$ does not converge to 0, by Lemma 6.5.2. Thus, by the zero test (Corollary 7.2.6), the formal series $\sum_{n=0}^{\infty} x^n$ is not convergent.

- If $|x| < 1$, then let's prove that for all $N \geqslant 0$, the partial sum S_N is equal to $\sum_{n=0}^{N} x^n$. Let's induct on N, starting with the base case $N = 0$. In this case, we have $S_0 = \sum_{n=0}^{0} x^n = 1$; and on the other side, we have $\frac{1-x^{N+1}}{1-x} = \frac{1-x}{1-x} = 1$, so the base case is done.

  Now let's suppose inductively that $S_N = \frac{1-x^{N+1}}{1-x}$ and let's prove that this property still holds for S_{N+1}. On the one hand we have:

$$\begin{aligned} S_{N+1} &:= \sum_{n=0}^{N} x^n + x^{N+1} \\ &= \frac{1-x^{N+1}}{1-x} + x^{N+1} \text{ (induction hypothesis)} \\ &= \frac{1-x^{N+1} + (1-x)x^{N+1}}{1-x} \\ &= \frac{1-x^{N+2}}{1-x} \end{aligned}$$

  as required, so that the property holds for all $N \geqslant 0$: we always have $S_N = \frac{1-x^{N+1}}{1-x}$.

  But, by Lemma 6.5.2 (combined with Exercise 6.1.4), we know that $(x^{n+1})_{n=0}^{\infty}$ converges to 0; so that by the convergence laws, $(S_N)_{N=0}^{\infty}$ converges to $\frac{1}{1-x}$.

  However, this only shows that $(S_N)_{N=0}^{\infty}$ is conditionally convergent. Actually, it is absolutely convergent. Indeed, according to Proposition 4.3.10(d), we have $|x^n| = |x|^n$, and the series $\sum_{n=0}^{\infty} |x|^n$ is convergent, by the previous result. This closes the proof.

8. Infinite sets

Exercise 8.1.2. — *Prove Proposition 8.1.4. (Hint: you can either use induction, or use the principle of infinite descent, Exercise 4.4.2, or use the least upper bound (or greatest lower bound) principle, Theorem 5.5.9.) Does the well- ordering principle work if we replace the natural numbers by the integers? What if we replace the natural numbers by the positive rationals? Explain.*

Let be $X \subset \mathbb{N}$ such that $\#X = p$; let's show that there exists $n \in X$ such that $n \leqslant m$ for all $m \in X$.

We could indeed show this property by using the greatest lower bound principle: we have showed in Exercise 6.6.3 that, for every subset $X \subset \mathbb{N}$, the greatest lower bound $\inf(X)$ belongs to X. But we will give here another proof, using induction on p.

- Let's start with the base case $n = 1$ (note that the base case is not $n = 0$, since X is supposed to be non-empty). In such a case, X is a singleton, i.e. there exists $n \in \mathbb{N}$ such that $X = \{n\}$. By reflexivity, we have of course $n \leqslant n$, i.e. we have $n \leqslant m$ for all $m \in X$, as required.

- Now suppose inductively that the property is true for a cardinality of p, and let's show that it is still true if $\#X = p + 1$. By Lemma 3.6.9, there exists $x \in \mathbb{N}$ and a subset $Y \subset \mathbb{N}$ with $\#Y = p$, such that $X = Y \cup \{x\}$. By the induction hypothesis, there exists $n \in Y$ such that $n \leqslant m$ for all $m \in Y$.

  Now, let be $n' := \min(n, x)$. We thus have $n' \leqslant n \leqslant m$ for all $m \in Y$ (and thus, in particular, $n' \leqslant m$); and $n' \leqslant x$ by definition. Thus, we have $n' \leqslant m$ for all $m \in Y \cup \{x\}$, that is to say $n' \leqslant m$ for all $m \in X$, as required. This closes the induction.

Note that the well ordering principle does not apply for the integers, because a non-empty set of integers can have arbitrarily small negative numbers: consider for instance the set $A = \{\dots, -3, -2, -1, 0\}$. This is not applicable to the positive rationals either: one can think of the set $B = \{1/n : n \in \mathbb{N}^\star\}$, which has an infimum of 0, but has no smallest element in B.

Exercise 8.1.3. — *Fill in the gaps marked (?) in Proposition 8.1.5.*

Those gaps only state a few results that we will better show below.

1. Show that the set $\{x \in X : x \neq a_m \text{ for all } m < n\}$ is infinite.

   By definition, we have $X = \{a_0, \cdots, a_{n-1}\} \cup \{x \in X : x \neq a_m \text{ for all } m < n\}$. Obviously, the set $\{a_0, \cdots, a_{n-1}\}$ is finite. Let's suppose for the sake of contradiction that $\{x \in X : x \neq a_m \text{ for all } m < n\}$ is also finite. Then, by Proposition 3.6.14(b), X would also be finite as the union of two finite sets, which is a contradiction.

2. Show that $(a_n)_{n=0}^{\infty}$ is strictly increasing.

   By definition, we have:

$$a_n := \min\{x \in X : x \neq a_m \text{ for all } m < n\}$$
$$a_{n+1} := \min\{x \in X : x \neq a_m \text{ for all } m < n + 1\}$$

Let's write $A_n := \{x \in X : x \neq a_m \text{ for all } m < n\}$; we thus have $A_{n+1} \subset A_n$ for all natural number n. Consequently, $\min A_{n+1} \geqslant \min A_n$ for all n; i.e. $a_{n+1} \geqslant a_n$ for all n. Furthermore, by definition, $a_{n+1} \neq a_n$, so that we finally have $a_{n+1} > a_n$ for all n, i.e. $(a_n)_{n=0}^{\infty}$ is a strictly increasing sequence.

3. Show that $a_n \neq a_m$ for all $n \neq m$.

   Suppose, for the sake of contradiction, that there exists two distinct natural numbers n, m such that $a_n = a_m$. This would be a contradiction with the fact that $(a_n)_{n=0}^{\infty}$ is a strictly increasing sequence.

4. Show that $a_n \in X$ for all $n \in \mathbb{N}$.

   By Proposition 8.1.4 (well-ordering principle), the minimum of a subset of natural numbers is well-defined: $\min X = \inf X \in X$ if $X \subset \mathbb{N}$.

5. Obvious statement, simple rephrasing of the definition.

6. Show that $a_n \geqslant n$ for all n.

   This can be shown by a simple induction: $(a_n)_{n=0}^{\infty}$ is a strictly increasing sequence of natural numbers, so that $a_0 \geqslant 0$ (base case), and if we suppose inductively that $a_n \geqslant n$, we thus have $a_{n+1} > a_n \geqslant n$, i.e. $a_{n+1} \geqslant n + 1$. This closes the induction.

Exercise 8.1.4. — *Prove Proposition 8.1.8.*

Let Y be a set, and $f : \mathbb{N} \to Y$ a function (non necessarily bijective). We have to show that $f(\mathbb{N})$ is at most countable.

Let's define the set $A := \{n \in \mathbb{N} : f(m) \neq f(n) \text{ for all } 0 \leqslant m < n\}$, intuitively the set of natural numbers n such that $f(n)$ does not appear in the sequence $f(0), \cdots, f(n-1)$. We write $f_{|A}$ the restriction of f to A, and let's show that $f_{|A}$ is a bijection from A to $f(\mathbb{N})$.

- First, $f_{|A}$ is injective: let's suppose that we have two natural numbers $a, b \in A$ such that $f(a) = f(b)$. By definition of A, we must have $f(m) \neq f(a)$ for all $m < a$. Since $f(a) = f(b)$, we necessarily have $b \geqslant a$. Similarly, we must have $f(m) \neq f(b)$ for all $m < b$, which implies $a \geqslant b$. The two inequalities $a \geqslant b$ and $b \geqslant a$ imply $a = b$, which shows that $f_{|A}$ is injective.

- Now, let's prove that $f_{|A} : A \to f(\mathbb{N})$ is surjective. Let be $y \in f(\mathbb{N})$, and let's suppose, for the sake of contradiction, that $f(a) \neq y$ for all $a \in A$.

  By definition of $f(\mathbb{N})$, there exists $a_1 \in \mathbb{N}$ such that $f(a_1) = y$. Since we suppose $f(a) \neq y$ for all $a \in A$, we have $a_1 \notin A$. By definition of the set A, it means that there exists a natural number $a_2 < a_1$ such that $f(a_2) = f(a_1) = y$, and still $a_2 \notin A$. Similarly, there exists a natural number $a_3 < a_2$ such that $a_3 \notin A$ and $f(a_3) = f(a_2) = f(a_1) = y$. Actually, we are constructing like this a sequence $(a_n)_{n=1}^{\infty}$ of natural numbers which is in infinite descent. But this is impossible (see Exercise 4.4.2). Thus, for every $y \in f(\mathbb{N})$, we necessarily have a $a \in A$ such that $f(a) = y$; i.e., $f_{|A} : A \to f(\mathbb{N})$ is surjective.

Thus, $f_{|A} : A \to f(\mathbb{N})$ is bijective. By Corollary 8.1.6, every subset of $\mathbb{N}$ is at most countable, so that A (which is clearly a subset of $\mathbb{N}$) is at most countable. And since $f(\mathbb{N})$ has the same cardinality as A, this shows that $f(\mathbb{N})$ is at most countable and closes the proof.

EXERCISE 8.1.5. — *Use Proposition 8.1.8 to prove Corollary 8.1.9.*

We have to show the following claim: if X is a countable set, and $f : X \to Y$ a function, then $f(X)$ is at most countable (i.e., any image of a countable set is itself countable).

By definition, if X is a countable set, there exists a bijective function $g : \mathbb{N} \to X$. Let's consider the function $h = f \circ g$, which is a function from $\mathbb{N}$ to Y. We will show that $f(X) = h(\mathbb{N})$.

- First, $f(X) \subseteq h(\mathbb{N})$. Indeed, let be $y \in f(X)$. By definition, there exists $x \in X$ such that $y = f(x)$. But since g is bijective, there exists $n \in \mathbb{N}$ such that $y = f(g(n))$, i.e., there exists $n \in \mathbb{N}$ such that $y = h(n)$. Thus, $y \in h(\mathbb{N})$.
- Furthermore, $h(\mathbb{N}) \subseteq f(X)$. Let be $y \in h(\mathbb{N})$. By definition, there exists $n \in \mathbb{N}$ such that $y = h(n) = f \circ g(n) = f(g(n))$. But since $g(n) \in X$, we have $y = f(g(n)) \in f(X)$ as required.

Thus, we have indeed $f(X) = h(\mathbb{N})$, and since $h(\mathbb{N})$ is a countable set by Proposition 8.1.8, $f(X)$ is indeed countable.

EXERCISE 8.1.6. — *Let A be a set. Show that A is at most countable if and only if there exists an injective map $f : A \to \mathbb{N}$.*

First suppose that there exists an injective function $f : A \to \mathbb{N}$. By definition, f is thus a bijection between A and $f(A)$, and those two sets thus have the same cardinality. But since we have $f(A) \subset \mathbb{N}$, the set $f(A)$ is at most countable by Corollary 8.1.6. Thus, A is at most countable, as required.

Now suppose that A is at most countable. We have two options here:

1. If A is countably infinite, then there exists a bijection $f : A \to \mathbb{N}$, and in particular, f is thus injective.
2. If A is finite, then A has a finite number of elements—say n elements. Consequently, there exists a bijective function $g : A \to [\![1, n]\!]$. Now let be $f : A \to \mathbb{N}$ such that $f(a) = g(a)$ for all $a \in A$. In particular, f is injective—but not necessarily surjective.

In both cases, we have found an injective map $f : A \to \mathbb{N}$ as required. This closes the proof.

EXERCISE 8.1.7. — *Prove Proposition 8.1.10.*

We have to prove that if X and Y are countable, then $X \cup Y$ is countable. We follow the hint given by Terence Tao for this proof.

Since X and Y are countable, there exist two bijections $f : \mathbb{N} \to X$ and $g : \mathbb{N} \to Y$. Let be $h : \mathbb{N} \to X \cup Y$ a function defined by $h(2n) = f(n)$ and $h(2n + 1) = g(n)$ for all natural numbers n. Let's show that we have $h(\mathbb{N}) = X \cup Y$.

- First, the fact that $h(\mathbb{N}) \subset X \cup Y$ is obvious: for all $m \in \mathbb{N}$, $h(m)$ belongs either to X or Y depending on whether m is odd or even, but in both cases, $h(m) \in X \cup Y$.
- Now let's prove that $X \cup Y \subset h(\mathbb{N})$. Let be $z \in X \cup Y$. If $z \in X$, then there exists $n \in \mathbb{N}$ such that $f(n) = z$, and thus $h(2n) = z$. This means that $z \in h(\mathbb{N})$. A similar argument applies if $z \in Y$. Thus, we have indeed $X \cup Y \subset h(\mathbb{N})$.

Those two properties show that $X \cup Y = h(\mathbb{N})$. According to Proposition 8.1.8 or Corollary 8.1.9, $h(\mathbb{N})$ is at most countable, as the image of the countable set $\mathbb{N}$. Thus, $h(\mathbb{N})$ can be either finite or countable. Let's suppose that it is a finite set. In this case, $X \cup Y$ is a finite set. By Proposition 3.6.14(c), X is also finite, as a subset of a finite set. This is a clear contradiction with our initial hypothesis that X is countably infinite. Thus, $h(\mathbb{N}) = X \cup Y$ is countable, as required.

Exercise 8.1.8. — *Use Corollary 8.1.13 to prove Corollary 8.1.14.*

We must show that any cartesian product of two countable sets is itself countable, i.e., if X and Y are countable, then $X \times Y$ is countable.

By definition, there exist two bijections $f : X \to \mathbb{N}$ and $g : Y \to \mathbb{N}$. Now let be the function $h : X \times Y \to \mathbb{N} \times \mathbb{N}$ defined by $h(x, y) = (f(x), g(y))$ for all $x \in X$ and $y \in Y$. We will show that h is also bijective.

- h is injective: if we suppose that $h(x, y) = h(x', y')$, then we have $(f(x), g(y)) = (f(x'), g(y'))$, i.e. $f(x) = f(x')$ and $g(y) = g(y')$. Since f and g are bijective, this implies $x = x'$ and $y = y'$, so that h is injective.
- h is also surjective because f and g are surjective: for all $n \in \mathbb{N}$, there exists $x \in X$ such that $n = f(x)$; and similarly, for all $m \in \mathbb{N}$ there exists $y \in Y$ such that $m = g(y)$. Thus, for all $(n, m) \in \mathbb{N} \times \mathbb{N}$ there exists $(x, y) \in X \times Y$ such that $h(x, y) = (n, m)$, i.e., h is surjective.

Thus, $X \times Y$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality. By Corollary 8.1.13, this means that $X \times Y$ is countable.

Exercise 8.1.9. — *Suppose that I is an at most countable set, and for each $\alpha \in I$, let A_α be an at most countable set. Show that the set $\bigcup_{\alpha \in I} A_\alpha$ is also at most countable. In particular, countable unions of countable sets are countable. (This exercise requires the axiom of choice, see Section 8.4.)*

This statement, although quite intuitive, is actually tricky to prove rigorously. There are a bunch of things that make the proof even trickier; for instance:

- the sets I and A_α are said to be *at most* countable, i.e. either finite or countably infinite: do we have to handle those cases separately?
- intuitively and informally, we have actually a denumerable sequence of sets, and in each set, we can also count the elements. For instance, we have a set $A_{\alpha_1} = \{a_{11}, a_{12}, \ldots\}$, a set $A_{\alpha_2} = \{a_{21}, a_{22}, \ldots\}$, and so on. Each element of $\bigcup_\alpha A_\alpha$ thus has two indices (the index of its set, and the index of its place in this set), so that we can think of a map between $\mathbb{N} \times \mathbb{N}$ and $\bigcup_\alpha A_\alpha$. But the sets A_α are not supposed to be disjoint, so that we do not really see how this map could be necessarily bijective (a same object x can belong to several A_α, and thus, several pairs (n, k) can provide an $x = a_{nk}$).

We thus need to think to a way to overcome these problems. A first remark can address (at least part of) them: if there exists an injection $f : A \to C$ between a set A and a countable set C, then A is at most countable (see also Exercise 3.6.7). Indeed, f is a bijection between A and $f(A)$, so that they have the same cardinality; and $f(A)$ is at most countable because it is a subset of C (Corollary 8.1.7). Alternatively, if A is at most countable, there exists a bijection between A and a subset of $\mathbb{N}$.

Using the notion of injection instead of bijection seems to be the way to go. So, let's begin the main proof!

- Since I is at most countable, there exists a bijection $g : N \to I$, where N is a subset of the natural numbers. Thus, $(A_\alpha)_{\alpha \in I} = (A_{g(m)})_{m \in N}$.
- Also, since each set $A_{g(m)}$ is at most countable, there exists an injective function $f_m : A_{g(m)} \to \mathbb{N}$ for each given $m \in N$. So, for each $m \in N$, let be $\mathcal{F}_m$ the set of all possible injections from $A_{g(m)}$ to $\mathbb{N}$. By the axiom of choice, we can choose simultaneously an injection in each of these sets (although we do not know which one exactly), so that we end up with an at most countable set of injections $\{f_m\}_{m \in N}$.
- Now let's consider $x \in \bigcup_{\alpha \in I} A_\alpha = \bigcup_{m \in N} A_{g(m)}$. As we said previously, x can belong to one or several sets in this union. Thus, let's consider the set $\{m \in N \,:\, x \in A_{g(m)}\}$. It's a subset of $\mathbb{N}$, so that by the well-ordering principle (Proposition 8.1.4), there exists exactly one minimal element n in this set.
- Now let's define $\theta : \bigcup_{m \in N} A_{g(m)} \to \mathbb{N} \times \mathbb{N}$ the function such that, for all $x \in \bigcup_{m \in N} A_{g(m)}$, we have $\theta(x) = (n, f_n(x))$ with n the minimal element defined above. θ is injective since n is uniquely defined and f_n is injective. Thus, $\bigcup_{m \in N} A_{g(m)}$ is at most countable, as required.

EXERCISE 8.2.1. — *Prove Lemma 8.2.3.*

We have to prove that, if X is a countable set, then $\sum_{x \in X} f(x)$ is absolutely convergent iff $\sup \left\{\sum_{x \in A} |f(x)| \,:\, A \subseteq X,\ A \text{ finite}\right\}$.

- First suppose that $\sup \left\{\sum_{x \in A} |f(x)| \,:\, A \subseteq X,\ A \text{ finite}\right\} < \infty$, and let's show that the series $\sum_{x \in X} f(x)$ is absolutely convergent.

  A preliminary remark: since X is a countable set, there exists a bijection $g : \mathbb{N} \to X$. Let be N a natural number. Since g is a bijection from $\mathbb{N}$ to X, the set $A_N := g(\llbracket 0, N \rrbracket)$ is a (finite) subset of X for any value of N. Furthermore, since the restriction $g : \llbracket 0, N \rrbracket \to A_N$ is bijective, we have $\sum_{x \in A_N} |f(x)| = \sum_{n=0}^{N} |f(g(n))|$. Thus, by our initial hypothesis,

$$\sup \left\{ \sum_{n=0}^{N} |f(g(n))| \,:\, N \in \mathbb{N} \right\} = \sup \left\{ \sum_{x \in A_N} |f(x)| \,:\, N \in \mathbb{N} \right\} \tag{8.1}$$

$$\leqslant \sup \left\{ \sum_{x \in A} |f(x)| \,:\, A \subseteq X,\ A \text{ finite} \right\} < \infty \tag{8.2}$$

  Let's denote $S_N := \sum_{n=0}^{N} |f(g(n))|$. The sequence $(S_N)_{N=0}^{\infty}$ is a sequence of partial sums of non-negative numbers, so that it is increasing, and thus converges iff it is bounded (Proposition 6.3.8). By equations (8.1)–(8.2), we have $\sup(S_N)_{N=0}^{\infty} < \infty$, which means that $(S_N)_{N=0}^{\infty}$ converges. It means that $\sum_{n=0}^{\infty} |f(g(n))|$ converges for some bijection $g : \mathbb{N} \to X$, i.e. that $\sum_{x \in X} f(x)$ is absolutely convergent by Definition 8.2.1.

- Now suppose that $\sum_{x\in X} f(x)$ is absolutely convergent and let's show that we have $\sup\{\sum_{x\in A}|f(x)| : A\subseteq X,\ A \text{ finite}\} < \infty$.

  Since $\sum_{x\in X} f(x)$ is absolutely convergent, then by Definition 8.2.1, there exists some bijection $g:\mathbb{N}\to X$ such that $\sum_{n=0}^{\infty} f(g(n))$ is absolutely convergent. But by Proposition 7.4.1, we know that if one such bijection g exists, then the series $\sum_{n=0}^{\infty} f(h(n))$ is also absolutely convergent for any other bijection $h:\mathbb{N}\to X$.

  So, let's choose a bijection h that suits us. Let A be a finite (non-empty) subset of X having N elements; we define $h:\mathbb{N}\to X$ a bijection such that $h([\![0,N-1]\!]) = A$. We thus have (by Proposition 7.2.14(c)):

$$\sum_{n=0}^{\infty}|f(h(n))| = \sum_{n=0}^{N-1}|f(h(n))| + \sum_{n=N}^{\infty}|f(h(n))| \tag{8.3}$$

  which is equivalent to

$$\sum_{x\in X}|f(x)| = \sum_{x\in A}|f(x)| + \sum_{n=N}^{\infty}|f(h(n))| \tag{8.4}$$

  And since $\sum_{n=N}^{\infty}|f(h(n))|$ converges (Proposition 7.2.14(d)) to a positive real number (Proposition 5.4.9), we get: $\sum_{x\in A}|f(x)| \leqslant \sum_{x\in X}|f(x)|$ for any subset A of X.

  Finally, our initial hypothesis was that $\sum_{x\in X} f(x)$ is absolutely convergent, i.e. that $\sum_{x\in X}|f(x)|$ is a (positive) real number M. Thus, there exists $M\in\mathbb{R}$ such that $\sum_{x\in A}|f(x)| \leqslant M$, which is equivalent to $\sup\{\sum_{x\in A}|f(x)| : A\subseteq X,\ A \text{ finite}\} < \infty$, as required. This closes the proof.

Exercise 8.2.2. — *Prove Lemma 8.2.5.*

Let's follow the hint, and first consider the number:

$$M := \sup\left\{\sum_{x\in A}|f(x)| \,:\, A\subseteq X, A \text{ finite}\right\} \tag{8.5}$$

which is (by Definition 8.2.4) a finite real number since $\sum_{x\in X} f(x)$ is supposed to be absolutely convergent; and for each positive integer n, the set:

$$A_n = \{x\in X \,:\, |f(x)| > 1/n\}$$

- Let's show that all the sets A_n are finite. Suppose, for the sake of contradiction, that there exists a natural number n such that A_n is infinite. Since it is an infinite set, in particular there exists a subset finite $A\subset A_n$ such that $\#A = 2Mn$. Thus we have $\sum_{x\in A}|f(x)| > \sum_{x\in A} 1/n = 2M$. I.e., we have found a finite subset $A\subset A_n\subseteq X$ such that:

$$\sum_{x\in A}|f(x)| > \sup\left\{\sum_{x\in A}|f(x)| \,:\, A\subseteq X, A \text{ finite}\right\},$$

  an obvious contradiction. Thus, A_n is finite for all $n>0$.

- Since A_n is a finite subset of X, then by equation (8.5), we have $\sum_{x\in A_n} |f(x)| \leqslant M$. But also, we have by definition of A_n (and Proposition 7.1.11(h)) that $\sum_{x\in A_n} 1/n < \sum_{x\in A_n} |f(x)|$, so that by transitivity:

$$M > \sum_{x\in A_n} 1/n = \#A_n \times (1/n)$$

and thus $\#A_n < Mn$ for all natural number $n > 0$.

- Now, we show that

$$A := \{x \in X \ : \ f(x) \neq 0\} = \bigcup_{n=1}^{\infty} A_n \tag{8.6}$$

On the one hand, if $x \in A$, then $f(x) \neq 0$, and in particular $|f(x)| > 0$. By Exercise 5.4.4, there exists a positive integer m such that $|f(x)| > 1/m > 0$, i.e. such that $x \in A_m$.

Conversely, if $x \in \bigcup_{n=1}^{\infty} A_n$, then by definition there exists a positive integer m such that $x \in A_m$, so that $|f(x)| > /m$, and thus $x \in A$.

Both sets are thus equal.

- Finally, by Exercise 8.1.9, a countable union of countable set is countable. Thus, $\bigcup_{n=1}^{\infty} A_n$ is countable, i.e. $\{x \in X \ : \ f(x) \neq 0\}$ is countable. This closes the proof.

Exercise 8.2.3. — *Prove Proposition 8.2.6.*

All statements can be deduced in a similar fashion from the usual series laws given in Chapter 7, so that we won't prove all of them in painful details below; we just give an example of proof for one of them. Remember that since X is possibly uncountable here, only Definition 8.2.4 and Lemma 8.2.5 can be used to prove the statements. However, Lemma 8.2.5 allows to reduce them into a case where the sum is computed on an at most countable set, so that Propositions 7.1.11 and 7.2.14 apply (more or less) immediately.

For what follows, let's define $F := \{x \in X \ : \ f(x) \neq 0\}$ and $G := \{x \in X \ : \ g(x) \neq 0\}$. By Lemma 8.2.5, both F and G are at most countable.

Also, let's define $M_f := \sup\{\sum_{x\in A} |f(x)| \ : \ A \subseteq X, A \text{ finite}\}$, by Definition 8.2.4 we know that M_f is a finite real number; similarly we define $M_g := \sup\{\sum_{x\in A} |g(x)| \ : \ A \subseteq X, A \text{ finite}\}$ which is also a finite real number.

(b) Let be c a real number. First a general remark: if $\sum_{x\in X} f(x)$ is absolutely convergent, then $\sum_{x\in X} cf(x)$ is also absolutely convergent. Indeed, we know that for all finite subset $A \subseteq X$, we have $\sum_{x\in A} |f(x)| \leqslant M_f$, so that by Proposition 7.1.11(g) we have $\sum_{x\in A} |cf(x)| \leqslant |c| \sum_{x\in A} |f(x)| \leqslant |c| M_f$. It means that the set $\{\sum_{x\in A} |cf(x)| \ : \ A \subseteq X, A \text{ finite}\}$ is bounded, i.e. that $\sum_{x\in X} cf(x)$ is absolutely convergent. Now consider several cases.

If $c = 0$ there is almost nothing to prove. On the one hand, we have $c \times \sum_{x\in X} f(x) = 0$ (because $\sum_{x\in X} f(x)$ is a finite real number); on the other hand we have $\sum_{x\in X} cf(x) := \sum_{\{x\in X \,:\, cf(x)=0\}} f(x) = \sum_{\varnothing} f(x) = 0$ by Proposition 7.1.11(a). Thus, we have $\sum_{x\in X} cf(x) = 0 = c\sum_{x\in X} f(x)$, and the claim follows.

Now, suppose instead that $c \neq 0$. Note that, in this case, $f(x) = 0 \iff cf(x) = 0$, so that:

$$\{x \in X : f(x) \neq 0\} = \{x \in X : cf(x) \neq 0\} \tag{8.7}$$

Also, we know that $c \times \sum_{x \in X} f(x) := c \times \sum_{x \in F} f(x)$, the set F being at most countable.

- If F is finite, then by Proposition 7.1.11(g), we have $c \times \sum_{x \in F} f(x) = \sum_{x \in F} cf(x)$. Thus, we have $c \times \sum_{x \in X} f(x) := c \times \sum_{x \in F} f(x) = \sum_{x \in F} cf(x) =: \sum_{x \in X} cf(x)$ (using equation (8.7) under the hood for the last equality), as expected.
- If F is countably infinite, since we have already shown that $\sum_{x \in X} cf(x) := \sum_{x \in F} cf(x)$ is absolutely convergent, we can apply Definition 8.2.1: there exists a bijection $g : \mathbb{N} \to F$ such that $\sum_{n=0}^{\infty} cf(g(n))$ is absolutely convergent. By Proposition 7.2.14(b), we finally have:

$$\sum_{x \in F} cf(x) := \sum_{n=0}^{\infty} cf(g(n)) = c \sum_{n=0}^{\infty} f(g(n)) =: c \sum_{x \in F} f(x)$$

  as expected.

Exercise 8.2.4. — *Prove Lemma 8.2.7.*

Let be $\sum_{n=0}^{\infty} a_n$ a series which is conditionally but not absolutely convergent; and let's define the two sets $A_+ := \{n \in \mathbb{N} : a_n \geqslant 0\}$ and $A_- := \{n \in \mathbb{N} : a_n < 0\}$. We have to prove that neither $\sum_{n \in A_+} a_n$ nor $\sum_{n \in A_-} a_n$ are conditionally convergent.

We'll use a proof by contradiction.

- First, we are going to suppose something which is a little too strong: let's suppose that *both* $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$ are conditionally convergent[27].

  In this case, $\sum_{n \in A_+} a_n$ is absolutely convergent (because for the positive series, conditional and absolute convergence are the same thing). Also, the series $\sum_{n \in A_-} (-a_n)$ is convergent by Proposition 7.2.14(b); and as it is a positive series, it is absolutely convergent; so that $\sum_{n \in A_-} a_n$ itself is absolutely convergent by Proposition 8.2.6(b). Thus, for both $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$, conditional convergence implies absolute convergence.

  Since $\mathbb{N} = A_+ \sqcup A_-$, we have by Proposition 8.2.6(c) that $\sum_{n \in A_+} a_n + \sum_{n \in A_+} a_n = \sum_{n \in A} a_n$ is absolutely convergent; a contradiction.

  Thus, $\sum_{n \in A_+} a_n$ and $\sum_{n \in A_-} a_n$ cannot be both conditionally convergent.

- However, we have not really proved Lemma 8.2.7 when doing this. Instead, we need to prove that *even only one of them* cannot be conditionally convergent.

  Actually, it turns out that when one of them is convergent, the other one is also convergent, which closes the proof.

  Indeed, let's suppose that $\sum_{n \in A_+} a_n$ is (conditionally, and thus absolutely) convergent. Let's define the (positive) series $\sum_{n=0}^{\infty} b_n$, where $b_n = 0$ whenever $a_n < 0$, and $b_n = a_n$ whenever $a_n \geqslant 0$.

[27] This hypothesis is too strong because the proper negation which should be the starting point for our proof by contradiction would be: "one of $\sum_{n \in A_+} a_n$ or $\sum_{n \in A_-} a_n$ is conditionally convergent". We'll fix that later.

We obviously have $\sum_{n\in A_+} b_n = \sum_{n\in A_+} a_n$ (so that $\sum_{n\in A_+} b_n$ is absolutely convergent); and $\sum_{n\in A_-} b_n = 0$ (so that it is also absolutely convergent). Thus, by Proposition 8.2.6(c), $\sum_{n=0}^{\infty} b_n$ is absolutely convergent, and is equal to $\sum_{n\in A_+} a_n$.

Finally, since $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are both (at least conditionally) convergent, then $\sum_{n=0}^{\infty}(a_n - b_n)$ is also convergent. Since we always have $a_n - b_n \leqslant 0$ for all natural number n, conditional convergence implies absolute convergence, so that by Proposition 8.2.6(c), both $\sum_{n\in A_+}(a_n - b_n) = 0$ and $\sum_{n\in A_-}(a_n - b_n) = \sum_{n\in A_-} a_n$ are absolutely convergent.

In particular, we have shown that whenever $\sum_{n\in A_+} a_n$ is convergent, the other series $\sum_{n\in A_-} a_n$ is also convergent, as expected. This closes the proof.

Exercise 8.5.2. — *Give examples of a set X and a relation $\leqslant_X$ such that the relation $\leqslant_X$ is...*

We have the cases below:

(a) Reflexive and anti-symmetric but not transitive. Consider $X = \mathbb{N}$, and $\leqslant_X$ defined by $n \leqslant_X m$ iff $0 \leqslant n - m \leqslant 1$. This relation is obviously reflexive. It is also anti-symmetric, since $n \leqslant_X m$ and $m \leqslant_X n$ imply that we have both "$n = m$ or $n = 1 + m$" and "$n = m$ or $m = 1 + n$", which lets $n = m$ for the only possibility. But it is not transitive since $1 \leqslant_X 2$ and $2 \leqslant_X 3$ but we do not have $1 \leqslant_X 3$.

(b) Reflexive and transitive but not anti-symmetric. Consider $X = \mathbb{R}^2$ and $\leqslant_X$ defined by $(x, y) \leqslant_X (x', y')$ iff $x \leqslant x'$. It is obviously reflexive and transitive, but not anti-symmetric since $(2, 3) \leqslant_X (2, 4)$ and $(2, 4) \leqslant_X (2, 3)$ but we do not have $(2, 3) = (2, 4)$.

(c) Anti-symmetric and transitive but not reflexive. Consider $X = \mathbb{R}$ and $\leqslant_X$ defined by the usual strict inequality $<$. Obviously, it is transitive. It might not immediately clear why it is also anti-symmetric, however: how could it happen that we have $x = y$ after both statements $x < y$ and $y < x$, since each of them implies in particular that $x \neq y$? By trichotomy of order, we can never have both of them at the same time, so that the implication $(x < y)$ and $(y < x) \implies x = y$ is *vacuously* true. And thus, $<$ is indeed anti-symmetric.

Exercise 8.5.3. — *Given two positive integers $n, m \in \mathbb{N} - \{0\}$, we say that n divides m, and write $n|m$, if there exists a positive integer a such that $m = na$. Show that the set $\mathbb{N} - \{0\}$ with the ordering relation $|$ is a partially ordered set but not a totally ordered one.*

To show that this defines a partially order set, we just have to prove that the relation $|$ is reflexive, anti-symmetric and transitive.

- It is obviously reflexive because we have $n = n \times 1$, and 1 is a positive integer.
- It is anti-symmetric because if we suppose that $n|m$, we must have $m = na$ for some positive integer a. Similarly, if we have $m|n$, we must have $n = mb$ for some positive integer b. Gathering these two statements, we get $n = n(ab)$, i.e. $ab = 1$ by cancellation law (Corollary 2.3.7). The only possibility for $ab = 1$ on the natural numbers is that both $a = 1$ and $b = 1$, as a quick proof by contradiction would show. Thus, we have $n = mb = m$, as required.

- It is also transitive because if $n|m$ and $m|p$, we have $m = na$ and $p = mb$ for some natural numbers a, b. Thus, $p = n(ab)$, with ab a natural number, so that $n|p$ as required.

However, this does not define a totally ordered set, since we have neither $2|3$ nor $3|2$.

Exercise 8.5.4. — *Show that the set of positive real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$, has no minimal element.*

This is actually a direct consequence of Exercise 5.4.4, and thus of the Archimedean property of $\mathbb{R}$. Suppose that there exists $m \in \mathbb{R}_+$ such that $m = \min(\mathbb{R}_+)$. Obviously, we must have $m < 1$, since there exists elements such as $1/2 \in \mathbb{R}_+$. Let's apply the Archimedean property with $x = 1$: there exists a positive integer M such that $Mm > 1$, i.e. such that $1/M < m$. But $1/M$ is a positive rational number, thus a positive real number, which is a contradiction.

Exercise 8.5.5. — *Let $f : X \to Y$ be a function from one set X to another set Y. Suppose that Y is partially ordered with some ordering relation $\leqslant_Y$. Define a relation $\leqslant_X$ on X by defining $x \leqslant_X x'$ if and only if $f(x) <_Y f(x')$ or $x = x'$. Show that this relation $\leqslant_X$ turns X into a partially ordered set. If we know in addition that the relation $\leqslant_Y$ makes Y totally ordered, does this mean that the relation $\leqslant_X$ makes X totally ordered also? If not, what additional assumption needs to be made on f in order to ensure that $\leqslant_X$ makes X totally ordered?*

First we prove that $\leqslant_X$ makes X a partially ordered set, by showing that the three properties of ordering relations hold for $\leqslant_X$:

- Reflexivity is obviously okay: $x \leqslant_X x$ is true iff $f(x) <_Y f(x)$ or $x = x$; one of these two statements is clearly true.

- $\leqslant_X$ is anti-symmetric: suppose that we have both $x \leqslant_X y$ and $y \leqslant_X x$ for $x, y \in X$. Since $x \leqslant_X y$, we have either $x = y$ (and in this case we are done) or $f(x) <_Y f(y)$. Let's suppose that $f(x) <_Y f(y)$. By the second hypothesis, we have $y \leqslant_X x$, so that we have either $y = x$ (and in this case we are done) or $f(y) <_Y f(x)$. But since we already have supposed that $f(x) <_Y f(y)$, we cannot have $f(y) <_Y f(x)$. Thus, the only possibility is $x = y$, as expected.

- $\leqslant_X$ is also transitive: if we suppose that we have both $x \leqslant_X y$ and $y \leqslant_X z$, we have actually $[x = y$ or $f(x) <_Y f(y)]$ and $[y = z$ or $f(y) <_Y f(z)]$; and we must show that this implies $[x = z$ or $f(x) <_Y f(z)]$. A cumbersome but easy distinction into four pairs of cases would show that $\leqslant_X$ is transitive.

However, the hypothesis that Y is totally ordered does not make X a totally ordered set. For instance, let's take $X = Y = \mathbb{R}$ and f the constant function $f(x) = 0\,\forall x \in \mathbb{R}$. In this case, we have neither $2 \leqslant_X 3$ (because $2 = 3$ is false, and $f(2) < f(3)$ is false), nor $3 \leqslant_X 2$ (the same remark applies). So, at least, f must not be constant. But it's still not enough! Indeed, let's take $f(x) = x^2$. We have neither $-1 \leqslant_X 1$ (because $-1 = 1$ is false, and $f(-1) < f(1)$ is false) nor $1 \leqslant_X -1$ (same remark). Thus, f must be at least injective.

Exercise 8.5.7. — *Let X be a partially ordered set, and let Y be a totally ordered subset of X. Show that Y can have at most one maximum and at most one minimum.*

Let's suppose that there exist two distinct elements $m \neq m'$ in Y such that both m and m' are minimal elements. Since Y is totally ordered, we have either $m \leqslant m'$, or $m' \leqslant m$. Suppose that $m \leqslant m'$. By Definition 8.5.5, since m' is a minimal element, it is impossible that $m < m'$; thus $m = m'$ is the only possibility. The same conclusion applies if we suppose that $m' \leqslant m$, so we are done.

A similar argument shows that Y can have at most one maximal element.

Exercise 8.5.8. — *Show that every finite non-empty subset of a totally ordered set has a minimum and a maximum. (Hint: use induction.) Conclude in particular that every finite totally ordered set is well-ordered.*

Let be X a totally ordered set, and Y a finite non-empty subset of X. Since Y is finite, we may say that it has n elements. Let's induct on n.

- For the base case, let's suppose that $n = 1$ (we don't start from 0 since Y is supposed to be non-empty). It means that Y is a singleton set, i.e., $Y = \{x\}$ for some $x \in X$. We can say that x is a minimal element, since there is no $y \in Y$ such that $y < x$ (because there is simply no other element in Y). Similarly, x is a maximal element, so the base case is done.

- Now suppose inductively that the property holds for any subset Y which has n elements, and let's prove that it is still true when Y is supposed to have $n+1$ elements. By Lemma 3.6.9, if we suppose $\#Y = n + 1$, we can write $Y = A \cup \{x\}$ with $\#A = n$. By the induction hypothesis, A has a minimum m and a maximum M. If we have $x < m$, then x is a minimum for Y; else (i.e., if $m \leqslant x$) m is still a minimum for Y. Similarly, if we have $x > M$, then x is a maximum for Y; else (i.e., if $M \geqslant x$) M is still a maximum for Y. In all cases, Y has a minimum and a maximum, as expected. This closes the induction, and proves that every finite non-empty subset of a totally ordered set has a minimum and a maximum.

If X is a finite totally ordered set, all non-empty subsets $Y \subseteq X$ are finite. Thus, we have even proved that every finite totally ordered set is well-ordered, in the sense of Definition 8.5.8.

Exercise 8.5.9. — *Let X be a totally ordered set such that every non-empty subset of X has both a minimum and a maximum. Show that X is finite.*

Suppose for sake of contradiction that X is infinite. By the initial hypothesis, there exists a minimal element $x_0 \in X$. Since X is infinite, the set $X - \{x_0\}$ is non-empty; let be x_1 its minimum. We have $x_0 \leqslant x_1$ because $x_1 \in X$ and $x_0 = \min(X)$, but we also have $x_0 \neq x_1$ because $x_1 \in X - \{x_0\}$ by definition. Thus, we have $x_0 < x_1$. Now let's consider the subset of X defined by $X - \{x_0, x_1\}$: it it still a non-empty subset (otherwise X could not be infinite), and the same argument as above would show that its minimum x_2 (wich certainly exists) is such that $x_0 < x_1 < x_2$. We can thus repeat these steps indefinitely to construct an increasing sequence $x_0 < x_1 < x_2 < \cdots$.

More formally, we can define recursively $x_0 := \min(X)$ and, for each natural number n, $x_{n+1} := \min(X - \cup_{i=0}^{n}\{x_i\})$; so that $(x_n)_{n=0}^{\infty}$ is an increasing sequence.

By definition, the set $\{x_0, x_1, \cdots\}$ is an infinite set, is thus a non-empty subset of X, but has no maximal element. This contradicts our initial hypothesis. Thus, X cannot be infinite: it is a finite set.

Exercise 8.5.10. — *Prove Proposition 8.5.10.*

Let be X a totally ordered set, and P a property such that, for any $n \in X$, we have the following implication: $[P(m)$ is true for all $m \in X$ with $m < n] \Rightarrow [P(n)$ is true$]$. We have to prove that $P(n)$ is true for all $n \in X$.

First, note that, if we denote $0 := \min(X)$ (and this minium necessarily exists since X is totally ordered), $P(0)$ must be true. Indeed, the statement "$P(m)$ is true for all $m \in X$ with $m < 0$" is vacuously true, so that $P(0)$ must be true.

Now suppose, for the sake of contradiction, that the set Y defined by

$$Y := \{n \in X \,:\, P(m) \text{ is false for some } m \in X \text{ with } m \leqslant n\}$$

is non-empty. Since X is totally ordered, there exists a minimal element M in Y, i.e. there exists some $M := \min(Y)$. In particular, M is the lowest element of X for which $P(M)$ is false.

It is impossible that $M = 0$, because it would imply that $P(0)$ is false, which has been excluded. Thus, we have $M \neq 0$, or more precisely, $M > 0$.

Since $M > 0$, for all elements $m \in X$ such that $0 \leqslant m < M$ (there is at least one such element, which is 0), $P(m)$ is true. By our initial hypothesis, it would thus imply that $P(M)$ is true; a contradiction.

Thus, Y is empty, and $P(n)$ is true for all $n \in X$.