

# Propositions of solutions for *Analysis II* by Terence Tao

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**Remarks.** The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

## 12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence  $(a_n)_{n=m}^{\infty}$  defined by  $a_n := d(x_n, x) = |x_n - x|$  for all  $n \geq m$ . We have to prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if and only if  $\lim_{n \rightarrow \infty} x_n = x$ .

- Let be  $\varepsilon > 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then there exists an  $N \geq m$  such that  $|a_n| < \varepsilon$  whenever  $n \geq N$ . Thus, there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ , which means that  $\lim_{n \rightarrow \infty} x_n = x$ .
- Let be  $\varepsilon > 0$ . Conversely, if  $\lim_{n \rightarrow \infty} x_n = x$ , then there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . But since  $|a_n| := |x_n - x|$ , it means that  $\lim_{n \rightarrow \infty} a_n = 0$ , as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric  $d(x, y) := |x - y|$  is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function. With respect to Definition 12.1.2, give an example of a pair  $(X, d)$  which...*

- (a) obeys the axioms (bcd) but not (a).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, x) = 1$  and  $d(x, y) = 5$  for all  $x \neq y \in \mathbb{R}$ .

- (b) obeys the axioms (acd) but not (b).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = 0$  for all  $x, y \in \mathbb{R}$ .

- (c) obeys the axioms (abd) but not (c).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = \max(x - y, 0)$  for all  $x, y \in \mathbb{R}$ .

- (d) obeys the axioms (abc) but not (d).

Consider the finite set  $X := \{1, 2, 3\}$  and the application  $d$  defined by  $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$ , and  $d(1, 3) = d(3, 1) := 5$ , and  $d(x, x) = 0$  for all  $x \in X$ .