Propositions of solutions for *Analysis II* by Terence Tao

Frédéric Santos

August 3, 2022

Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

Exercise 12.1.4. — *Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y\times Y}(x, y) := d(x, y)$, then the application $d|_{Y\times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y\times Y})$ is indeed a metric space.

EXERCISE 12.1.5. — *Let $n \geqslant 1$, and let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Verify the identity $\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.*

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n.

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geqslant 1$, and let's prove that it is still true for $n+1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i\right)^2}_{:=A} + \underbrace{\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right)}_{:=C} \tag{12.1}$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A, we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i\right)^2 \\ &= \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2 \\ &= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \end{aligned}$$

- for B, we have

$$\begin{aligned} B &:= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n+1}(a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2\times S} + \underbrace{\frac{1}{2}\sum_{j=1}^{n}(a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2\times S} \\ &\quad + \underbrace{\frac{1}{2}(a_{n+1}b_{n+1} - b_{n+1}a_{n+1})^2}_{=0} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for $A+B$, we now use the induction hypothesis (IH) to get:

$$
\begin{aligned}
A+B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^{n} a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2\sum_{i=1}^{n} a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^{n}(a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right) + \sum_{i=1}^{n}(a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right) \\
&= C
\end{aligned}
$$

so that the identity is indeed true for all natural number n.

(ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$
\left|\sum_{i=1}^{n} a_i b_i\right| \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2}. \tag{12.2}
$$

Indeed, since $B \geqslant 0$ in the identity (12.1), we have

$$
\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{j=1}^{n} b_j^2\right)
$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$
\begin{aligned}
\sum_{i=1}^{n}(a_i^2 + b_i^2) &= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \\
&\leqslant \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(by eq. (12.2))} \\
&\leqslant \left(\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\right)^2
\end{aligned}
$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

Exercise 12.1.6. — *Show that $(\mathbb{R}^n, d_{l^2})$ in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $(x_i - y_i)^2 > 0$, and $d_{l^2}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(y,x) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_{l^2}(x,y)$$

as expected.

(d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$
\begin{aligned}
d_{l^2}(x,z) :&= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} && \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(Exercise 12.1.5(iii))} \\
&\leqslant \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2} \\
&\leqslant d_{l^2}(x,y) + d_{l^2}(y,z)
\end{aligned}
$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^2})$ is indeed a metric space.

Exercise 12.1.7. — *Show that $(\mathbb{R}^n, d_{l^1})$ in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x,x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x,y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y,x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x,y)$$

as expected.

(d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a-c| \leqslant |a-b| + |b-c|$ for all $a, b, c \in \mathbb{R}$. Thus, for all $x, y, z \in \mathbb{R}^n$, we have

$$d_{l^1}(x,z) := \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x,y) + d_{l^1}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x,y)^2 \leqslant d_{l^1}(x,y)^2$, and we will prove it in this form.

  Indeed, using a trivial product expansion, we have

$$\begin{aligned}
d_{l_1}(x,y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\
&= \left(\sum_{i=1}^n |x_i - y_i|\right) \times \left(\sum_{i=1}^n |x_i - y_i|\right) \\
&= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leqslant i,j \leqslant n;\, i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geqslant 0} \\
&\geqslant \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x,y)^2
\end{aligned}$$

  as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x,y) &:= \sum_{i=1}^{n} |x_i - y_i| \\
&= \left| \sum_{i=1}^{n} |x_i - y_i| \times 1 \right| \\
&\leqslant \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} \\
&\leqslant d_{l^2}(x,y) \times \sqrt{n}
\end{aligned}$$

  as expected.

Exercise 12.1.9. — *Show that the pair $(\mathbb{R}^n, d_{l^\infty})$ in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we clearly have $d_{l^\infty}(x,x) = \sup\{|x_i - x_i| : 1 \leqslant i \leqslant n\} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant j \leqslant n$ such that $x_j \neq y_j$. Thus $|x_j - y_j| > 0$, and $d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} \geqslant |x_j - y_j| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} = \sup\{|y_i - x_i| : 1 \leqslant i \leqslant n\} = d_{l^\infty}(y,x)$$

as expected.

(d) Triangle inequality. Let be $x, y, z \in \mathbb{R}^n$. We have $|x_i - z_i| \leqslant |x_i - y_i| + |y_i - z_i|$ for all $1 \leqslant i \leqslant n$, by Proposition 4.3.3(g). But, by definition of the supremum, we have $|x_i - y_i| \leqslant d_{l^\infty}(x,y)$ and $|y_i - z_i| \leqslant d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$. Thus, we have $|x_i - z_i| \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$; i.e., $d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ is an upper bound of the set $\{|x_i - z_i| : 1 \leqslant i \leqslant n\}$. By definition of the supremum, it implies that

$$d_{l^\infty}(x,z) := \sup\{|x_i - z_i| : 1 \leqslant i \leqslant n\} \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y).$$

First, a preliminary remark. By definition, we have $d_{l^\infty}(x,y) := \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\}$ for all $x, y \in \mathbb{R}^n$. Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a $1 \leqslant m \leqslant n$ such that $d_{l^\infty}(x,y) = |x_m - y_m|$ (the supremum belongs to the set). The index "m" will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}
\frac{1}{\sqrt{n}} d_{l^2}(x, y) &:= \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_i - y_i)^2} \\
&\leqslant \sqrt{\frac{1}{n} \sum_{i=1}^{n} (x_m - y_m)^2} \\
&\leqslant \sqrt{\frac{n}{n} (x_m - y_m)^2} \\
&= |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}
d_{l^2}(x, y) &:= \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \\
&= \sqrt{(x_m - y_m)^2 + \sum_{1 \leqslant i \leqslant n;\, i \neq m} (x_i - y_i)^2} \\
&\geqslant \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

Exercise 12.1.11. — *Show that the discrete metric (X, d_{disc}) in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in X$, we have $d_{\text{disc}}(x, x) := 0$ by definition, so that there is nothing to prove here.

(b) Positivity: for all $x \neq y \in X$, we have $d_{\text{disc}}(x, y) := 1 > 0$ by definition, so that there's still nothing to prove.

(c) Symmetry: for all $x, y \in X$, we have $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x) = 1$, so that d_{disc} obeys the symmetry property.

(d) Triangle inequality. Let be $x, y, z \in X$, and let's consider $d_{\text{disc}}(x, z)$.

  - If $x = z$, then $d_{\text{disc}}(x, z) = 0$. And since d_{disc} is a non-negative application, we clearly have $0 =: d_{\text{disc}}(x, z) \leqslant d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$ for all $y \in X$.
  - If $x \neq z$, then we cannot have both $x = y$ and $y = z$ (it would be a clear contradiction with $x \neq z$). Thus, at least one of the propositions "$x \neq y$", "$y \neq z$" is true. Another way to say that is $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geqslant 1$. But since $d_{\text{disc}}(x, z) := 1$, we have actually $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geqslant d_{\text{disc}}(x, z)$, as expected.

Exercise 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all $x, y \in \mathbb{R}^n$, we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}}\, d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y).$$

Note that n is a real constant here.

- Let's prove that $(a) \implies (b)$. If $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, then by the limit laws, the sequence $t_k := \sqrt{n}\, d_{l^2}(x^{(k)}, x)$ also converges to 0 as $k \to \infty$, since $\sqrt{n}$ is a constant real number. Thus, we have

  $$d_{l^2}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x) \leqslant \sqrt{n}\, d_{l^2}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x)$ as expected.

- Let's prove that $(b) \implies (c)$. If $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x) = 0$, then we have

  $$0 \leqslant d_{l^\infty}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x)$ as expected.

- Let's prove that $(c) \implies (d)$. Suppose that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x) = 0$. Then, for all $1 \leqslant j \leqslant n$, we have $0 \leqslant |x_j^k - x_j| \leqslant d_{l^\infty}(x^{(k)}, x)$. Still by the squeeze test, this implies that $\lim_{k\to\infty} |x_j^k - x_j| = 0$, i.e. that $(x_j^k)_{k=m}^\infty$ converges to x_j as $k \to \infty$ (by Lemma 12.1.1), as expected.

- Finally, let's prove that $(d) \implies (a)$. Using the definition of convergence is more appropriate here. Let be $\varepsilon > 0$ a positive real number, and let be $1 \leqslant j \leqslant n$. By definition, there exists a natural number $N \geqslant m$ such that $|x_j^{(k)} - x_j| \leqslant \varepsilon/\sqrt{n}$ whenever $k \geqslant N$. Thus, if $k \geqslant N$, we have

  $$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^n (x_j^{(k)} - x_j)^2} \leqslant \sqrt{\sum_{j=1}^n \frac{\varepsilon^2}{n}} \leqslant \varepsilon$$

  so that $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, i.e., $(x^k)_{k=m}^\infty$ converges to x as $k \to \infty$ in the l^2 metric (by Lemma 12.1.1), as expected.

Exercise 12.1.13. — *Prove Proposition 12.1.19.*

Let be $(x^{(n)})_{n=m}^\infty$ a sequence of elements of a set X.

- First suppose that $(x^{(n)})_{n=m}^\infty$ is eventually constant. Thus, by definition, there exists an $N \geqslant m$ and an element $x \in X$ such that $(x^{(n)})_{n=m}^\infty = x$ for all $n \geqslant N$. This implies that we have $d_{\text{disc}}(x^{(n)}, x) = 0$ for all $n \geqslant N$. In particular, for all $\varepsilon > 0$, we have $d_{\text{disc}}(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N$, so that $(x^{(n)})_{n=m}^\infty$ indeed converges to x with respect to d_{disc}.

- Conversely, suppose that $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d_{disc}. Let be $\varepsilon = 1/2$. By definition, there exists an $N \geqslant m$ such that $d_{\text{disc}}(x^{(n)}, x) \leqslant 1/2$ whenever $n \geqslant N$. Since $d_{\text{disc}}(x^{(n)}, x)$ cannot be 1, it is necessarily equal to 0, so that $x^{(n)} = x$ whenever $n \geqslant N$. Thus, the sequence $x^{(n)}$ is indeed eventually constant.

Exercise 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have $\lim_{n\to\infty} d(x^{(n)}, x) = 0$ and $\lim_{n\to\infty} d(x^{(n)}, x') = 0$. Suppose, for the sake of contradiction, that we have $x \neq x'$. Thus, the real number $\varepsilon := \frac{d(x,x')}{3}$ is positive.

Since $x^{(n)}$ converges to x, there exists a $N_1 \geqslant m$ such that $d(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N_1$.

Similarly, since $x^{(n)}$ converges to x', there exists a $N_2 \geqslant m$ such that $d(x^{(n)}, x') \leqslant \varepsilon$ whenever $n \geqslant N_2$.

By the triangle inequality, we thus have, for all $n \geqslant \max(N_1, N_2)$,

$$d(x, x') \leqslant d(x, x^{(n)}) + d(x^{(n)}, x') \leqslant \varepsilon + \varepsilon = \frac{2}{3} d(x, x')$$

which is a contradiction (since $d(x, x') > 0$ by hypothesis).

Thus, the limit is unique, and we must have $x = x'$.

Exercise 12.1.15. — *Let be $X := \{(a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty\}$. We define on this space the metrics $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sum_{n=0}^\infty |a_n - b_n|$, and $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sup_{n\in\mathbb{N}} |a_n - b_n|$. Then...*

We have to prove the following statements.

1. d_{l^1} is a metric on X.

   We have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^1}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^1}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |b_n - a_n| = \sum_{n=0}^\infty |a_n - b_n| = d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately

   $$\begin{aligned} d_{l^1}((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty) &:= \sum_{n=0}^\infty |a_n - c_n| \\ &\leqslant \sum_{n=0}^\infty (|a_n - b_n| + |b_n - c_n|) \text{ (consequence of Prop. 7.1.11(h))} \\ &\leqslant \sum_{n=0}^\infty |a_n - b_n| + \sum_{n=0}^\infty |b_n - c_n| \text{ (by Proposition 7.2.14(a))} \\ &\leqslant d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) + d_{l^1}((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty). \end{aligned}$$

   Thus, d_{l^1} is indeed a metric on X.

2. d_{l^∞} is a metric on X.

   Once again, we have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |b_n - a_n| = \sup_{n\in\mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately $|a_m - c_m| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$ for all $m \in \mathbb{N}$, by definition of the supremum. In other words, $(\sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|)$ is an upper bound for the set $\{|a_m - c_m| : m \in \mathbb{N}\}$. Thus we have, still by definition of the supremum, $\sup_{n\in\mathbb{N}} |a_n - c_n| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$, as expected.

   Thus, d_{l^∞} is indeed a metric on X.

3. There exist sequences $x^{(1)}$, $x^{(2)}$, ..., of elements of X (i.e., sequences of sequences) which are convergent with respect to d_{l^∞}, but are not convergent with respect to d_{l^1}.

   Here we are dealing with sequences of sequences: we have a sequence $(x^{(k)})_{k=1}^\infty$ where each $x^{(k)}$ is itself a sequence of real numbers. Thus, let's define $(x^{(k)})_{k=1}^\infty$ as follows:

   $$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

   Just to make things clearer, we have for instance

   $$\begin{aligned} x^{(1)} &:= \frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ 0,\ \ldots \\ x^{(2)} &:= \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0,\ 0,\ \ldots \\ x^{(3)} &:= \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 0,\ \ldots \end{aligned}$$

   Also, let be the null sequence $(a_n)_{n=0}^\infty$ defined by $a_n := 0$ for all $n \in \mathbb{N}$. Thus:

   - $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞}. Indeed, if we consider a given positive integer k (fixed), we have

     $$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

     so that $d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) := \sup_{n\in\mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$.

     Thus, $\lim_{k\to\infty} d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = 0$, or in other words, $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} in X.

- But $(x^{(k)})_{k=1}^{\infty}$ does not converges to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}. Indeed, we have, for each given (fixed) k,

$$d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{k} \frac{1}{k+1} = 1$$

  Thus, we clearly do not have $\lim_{k\to\infty} d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right) = 0$, i.e., $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

4. Conversely, any sequence which converges with respect to d_{l^1} also converges with respect to d_{l^∞}.

   Suppose, for the sake of contradiction, that $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^∞}, but does converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

   In this case, there exists a $\varepsilon > 0$ such that, for all $k \geqslant 1$, we have $(\sup_{n\geqslant 0} |x_n^{(k)} - a_n|) > \varepsilon$. In particulier, for all $k \geqslant 1$ and all $n \geqslant 0$, we have $|x_n^{(k)} - a_n| > \varepsilon$. Thus, $\sum_{n=0}^{\infty} |x_n^{(k)} - a_n|$ is not even a convergent series, and we cannot have $\lim_{k\to\infty}\left(\sum_{n=0}^{\infty} |x_n^{(k)} - a_n|\right) = 0$.

Note that this exercise actually shows that in this space X, the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

Exercise 12.1.16. — *Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be two sequences in a metric space (X, d). Suppose that $(x_n)_{n=1}^{\infty}$ converges to a point $x \in X$, and $(y_n)_{n=1}^{\infty}$ converges to a point $y \in X$. Show that* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

On the one hand, the triangle inequality applied two times to d gives us

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leqslant d(x_n, y_n) - d(x, y) \leqslant d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y).$$

Let be $\varepsilon > 0$. By hypothesis, there exists a $N_1 \geqslant 1$ such that $d(x_n, x) \leqslant \varepsilon/2$ whenever $n \geqslant N_1$. Similarly, there exists a $N_2 \geqslant 1$ such that $d(y_n, y) \leqslant \varepsilon/2$ whenever $n \geqslant N_2$. Thus, if we set $N := \max(N_1, N_2)$, then for all $n \geqslant N$ we have

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y) \leqslant 2\varepsilon/2\ leq \varepsilon$$

which shows that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$, as expected.

Exercise 12.2.1. — *Verify the claims in Example 12.2.8*

Let be (X, d_{disc}) a metric space, and E a subset of X.

- Let be $x \in E$. Then x is an interior point of E. Indeed, we have $B(x, 1/2) = \{x\} \subseteq E$.
- Let be $y \notin E$. Then y is an exterior point of E. Indeed, we have $B(y, 1/2) \cap E = \{y\} \cap E = \varnothing$.
- Finally, there are no boundary points of E in (X, d_{disc}). Indeed, let be $r > 0$ and any $x \in X$. We will always have $B(x, r) = \{x\}$ by definition of the discrete metric d_{disc}. Thus, we have either $x \in E$ and then $x \in \text{int}(E)$, or $x \notin E$ and then $x \in \text{ext}(E)$. Thus, E has no boundary points.

Exercise 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that $(a) \implies (b)$. We will use the contrapositive, assuming that x_0 is neither an interior point of E, nor a boundary point of E. By definition, it means that x_0 is an exterior point of E, i.e. that there exists $r > 0$ such that $B(x_0, r) \cap E = \varnothing$. This is precisely the negation of x_0 being an adherent point of E. Thus, we have showed that if x_0 is an adherent of of E, it is either an interior point of a boundary point.
- Let's show that $(b) \implies (c)$. Let be a positive integer $n > 0$, and suppose that x_0 is either an interior point of E, or a boundary point of E. In either case, the set $A_n := B(x_0, 1/n) \cap E$ is non empty, i.e., there exists $a_n \in X$ such that $d(a_n, x_0) < 1/n$. By the (countable) axiom of choice, we can define a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A_n$ for all $n \geqslant 1$.

  Let be $\varepsilon > 0$. There exists $N > 0$ such that $1/N < \varepsilon$ (Exercise 5.4.4). Thus, for all $n \geqslant N$, we have
  $$d(a_n, x_0) < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$
  i.e., the sequence $(a_n)_{n=1}^{\infty}$ converges to x_0 with respect to the metric d, as expected.
- Finally, let's show that $(c) \implies (a)$. Let be $r > 0$. If $(a_n)_{n=1}^{\infty}$ in E converges to x_0 with respect to d, then there exists a n such that $d(x_0, a_n) < r$. But since $a_n \in E$, it means that $B(x_0, r) \cap E$ is non empty, i.e. that x_0 is an adherent point of E.

Exercise 12.2.3. — *Prove Proposition 12.2.5.*

Let be (X, d) a metric space.

(a) Let be $E \subseteq X$. First suppose that E is open; this means that $E \cap \partial E = \varnothing$. Let be $x \in E$, then we have $x \notin \partial E$. But since $x \in E$, we have $x \in \overline{E}$, and thus $x \in \text{int}(E)$ by Proposition 12.2.10(b). We have shown that $x \in E \implies x \in \text{int}(E)$, and since the converse implication is trivial (Remark 12.2.6), we have $E = \text{int}(E)$ as expected.

Now suppose that $E = \text{int}(E)$. Let be $x \in E$. We thus have $x \in \text{int}(E)$. By definition, x is thus not a boundary point of E, i.e. x $\notin \partial E$. This means that $E \cap \partial E = \varnothing$, i.e. that E is open, as expected.

(b) Let be $E \subseteq X$. First suppose that E is closed; i.e. that $\partial E \subseteq E$. Let be $x \in \overline{E}$. By Proposition 12.2.10, we have $\overline{E} = \text{int}(E) \cup \partial E$; such that $\overline{E}$ is the union of two subsets of E, and thus is itself a subset of E, as expected.

Conversely, suppose that $\overline{E} \subseteq E$. It means that $\text{int}(E) \cup \partial E \subseteq E$, and in particular that $\partial E \subseteq E$, i.e. that E is closed, as expected.

(c) Let be $x_0 \in X$, $r > 0$ and $E := B(x_0, r)$. To show that E is open, we must show that $E = \text{int}(E)$ (by Proposition 1.2.15(a)), and in particular that $E \subseteq \text{int}(E)$ (the converse inclusion being trivial). Let be $x \in E$, and let's show that $x \in \text{int}(E)$. By definition, we have $d(x, x_0) < r$, so that $\varepsilon := r - d(x, x_0)$ is a positive real number. Thus, let be $y \in B(x, \varepsilon)$. By the triangle inequality, we have

$$\begin{aligned} d(x_0, y) &< d(x, x_0) + d(x, y) \\ &< d(x, x_0) + \varepsilon \\ &< d(x, x_0) + r - d(x, x_0) = r \end{aligned}$$

so that $y \in E$. Thus, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq E$, i.e., x is an interior point of E. This shows that $E \subseteq \text{int}(E)$, as expected.

Now let be $F := \{x \in X : d(x, x_0) \leqslant r\}$, and let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in F. To show that F is closed, we have to show that $\ell := \lim_{n\to\infty} a_n$ lies in F (Proposition 12.2.15(b)). Suppose, for the sake of contradiction, that $\ell \notin F$. We thus have $d(\ell, x_0) > r$, so that $\varepsilon := d(\ell, x_0) - r$ is a positive real number. Since $(a_n)_{n=1}^{\infty}$ converges to ℓ, there exists a $N > 0$ such that $d(a_n, \ell) < \varepsilon$ whenever $n \geqslant N$. By the triangle inequality, for $n \geqslant N$, we have

$$\begin{aligned} d(x_0, \ell) &\leqslant d(x_0, a_n) + d(a_n, \ell) \\ d(x_0, a_n) &\geqslant d(x_0, \ell) - d(a_n, \ell) \\ &\geqslant d(x_0, \ell) - \varepsilon \\ &\geqslant d(x_0, \ell) + r - d(\ell, x_0) \\ &\geqslant r \end{aligned}$$

and thus, $a_n \notin B(x_0, r)$, a contradiction. Thus, we must have $\ell \in F$, so that F is indeed a closed set.

(d) Let be $\{x_0\}$ a singleton with $x_0 \in X$. To show that E is closed, we may use Proposition 12.2.15(b), and show that $\{x_0\}$ contains all its adherent points. Let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in $\{x_0\}$; it can only be the constant sequence $x_0, x_0, \ldots$. Since it is a constant sequence, its limit can only be x_0 itself, and this limit belongs to $\{x_0\}$. Thus, $\{x_0\}$ is a closed set.

(e) First we can form a lemma: for any subset $E \subseteq X$, we have $\text{int}(E) = \text{ext}(X \backslash E)$. This is a direct consequence of Definition 12.2.5. Indeed, $x \in \text{int}(E)$ iff there exists a $r > 0$ such that $B(x, r) \subseteq E$, which is equivalent to "$\exists r > 0 : B(x, r) \cap (X \backslash E) = \varnothing$", which is equivalent to $x \in \text{ext}(X \backslash E)$.

This implies that the interior points of E are the exterior points of $X \backslash E$, and conversely, that the exterior points of E are the interior points of $X \backslash E$. Thus, in particular, we have this useful fact:

$$\partial E = \partial(X \backslash E). \tag{12.4}$$

Now we go back to the main proof. First suppose that E is open. Thus, by Definition 12.2.12, we have $E \cap \partial E = \varnothing$, so that $\partial E \subseteq X \backslash E$, which means that $X \backslash E$ is a closed set. The converse also applies: if we suppose that $X \backslash E$ is closed, then $\partial(X \backslash E) \subseteq X \backslash E$. By equation (12.4) above, this is equivalent to $\partial E \subseteq X \backslash E$, and thus we have $\partial E \cap E = \varnothing$. This means that E is open, as expected.[1]

(f) Let $E_1, \ldots, E_n$ be open sets. Thus, for all $1 \leqslant i \leqslant n$, if $x \in E_i$, there exists a $r_i > 0$ such that $B(x, r_i) \subseteq E_i$. Let's define $r := \min(r_1, \ldots, r_n)$. We have $B(x,r) \subseteq B(x, r_i) \subseteq E_i$ for all $1 \leqslant i \leqslant n$, i.e. $B(x,r) \subseteq E_1 \cap \ldots \cap E_n$. Thus, $E_1 \cap \ldots \cap E_n$ is an open set.

Also, let $F_1, \ldots, F_n$ be closed sets. By the previous result (e), the complementary sets $X \backslash F_1, \ldots X \backslash F_n$ are open sets. Thus, we have just proved that $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n)$ is an open set. But we have $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n) = X \backslash (F_1 \cup \ldots \cup F_n)$, and this set is open. Thus, by (e), its complementary set, $F_1 \cup \ldots \cup F_n$, is closed, as expected.

(g) Let $(E_\alpha)_{\alpha \in I}$ be open sets. Suppose that we have $x \in \bigcup_{\alpha \in I} E_\alpha$. By definition, there exists a $i \in I$ such that $x \in E_i$. Since E_i is an open set, there exists $r_i > 0$ such that $B(x, r_i) \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$. Thus, by (a), $\bigcup_{\alpha \in I} E_\alpha$ is an open set, as expected.

Now let be $(F_\alpha)_{\alpha \in I}$ be closed sets. Suppose that we have a convergent sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \in \bigcap_{\alpha \in I} F_\alpha$ for all $n \geqslant 1$. Thus, for all $\alpha \in I$, the sequence $(x_n)_{n=1}^{\infty}$ entirely belongs to the closed set F_α, so that its limit ℓ also lies in F_α according to (b). Thus, $\ell \in \bigcup_{\alpha \in I} F_\alpha$, so that $\bigcap_{\alpha \in I} F_\alpha$ is a closed set, as expected.

(h) Let be $E \subseteq X$.

- Let's show that $\operatorname{int}(E)$ is the largest open set included in E. It has not clearly be proved in the main text that $\operatorname{int}(E)$ is an open set, so we begin by proving it. Let be $x \in \operatorname{int}(E)$. By definition, there exists $r > 0$ so that $B(x,r) \subseteq E$. But by (c), we know that $B(x,r)$ is an open set, so that any point y of $B(x,r)$ is an interior point of this open ball, and thus an interior point of E. Thus, $\operatorname{int}(E)$ is open.

  Now consider another open set $V \subseteq E$, and let's show that $V \subseteq \operatorname{int}(E)$. If $x \in \operatorname{int}(V)$, then there exists $r > 0$ such that $B(x,r) \subseteq V \subseteq E$, so that $x \in \operatorname{int}(E)$. This shows that $V \subseteq \operatorname{int}(E)$, as expected.

- Similarly, let's show that $\overline{E}$ is the smallest closed set that contains E. First we show that $\overline{E}$ is closed, i.e. that $\overline{\overline{E}} \subseteq \overline{E}$. (Hint: see Exercise 9.1.6 for an intuition.) Let be $x \in \overline{\overline{E}}$. By definition, for all $r > 0$, $B(x,r) \cap \overline{E} \neq \varnothing$. Thus, there exists $y \in B(x,r)$ such that $y \in \overline{E}$. Thus, because $B(x,r)$ is an open set and y is adherent to E, there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq B(x,r)$ and $B(y,\varepsilon) \cap E \neq \varnothing$; i.e., there exists $z \in B(y, \varepsilon) \subseteq B(x,r)$ such that $z \in E$. We have showed that whenever $x \in \overline{\overline{E}}$, we have $B(x,r) \cap E \neq \varnothing$ for all $r > 0$, i.e. that x is an adherent point of E, as expected. Thus, $\overline{E}$ is closed.

  Now we consider a closed set K such that $E \subseteq K$, and we have to show that $\overline{E} \subseteq K$. Let be $x \in \overline{E}$. By definition, for all $r > 0$, we have $B(x,r) \cap E \neq \varnothing$. But since $E \subseteq K$, we also have $B(x,r) \cap K \neq \varnothing$ for all $r > 0$. Thus, x is an adherent point of K, i.e., $x \in \overline{K}$. But since K is closed, we have $K = \overline{K}$, and thus $x \in K$. This shows that $\overline{E} \subseteq K$, as expected.

[1] This important result will be used in future proofs to turn any statement on closed sets into a statement on open sets.

Exercise 12.2.4. — *Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leqslant r\}$.*

Let's prove the following claims:

(a) Show that $\overline{B} \subseteq C$.

Let be $x \in \overline{B}$. By definition, since x is an adherent point of B, for all $\varepsilon > 0$ we have $B(x, \varepsilon) \cap B \neq \varnothing$. In other words, there exists y such that we have both $d(x, y) < \varepsilon$ and $d(x_0, y) < r$. Thus, by the triangle inequality, we have

$$\begin{aligned} d(x, x_0) &\leqslant d(x, y) + d(y, x_0) \\ &\leqslant \varepsilon + r \quad \text{for all } \varepsilon > 0 \end{aligned}$$

which is equivalent (as a quick proof by contradiction would show) to $d(x, x_0) \leqslant r$. Thus, $x \in C$.

We have indeed proved that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

Let's take $X = \mathbb{R}$, $d = d_{\text{disc}}$, $x = 0$ and $r = 1$. One the one hand, we have $B := \{0\}$ and $C := \mathbb{R}$. Now let's work out $\overline{B}$. By Proposition 12.2.15(bd), B is closed, so that we have $\overline{B} = B$. Thus, we clearly do not have $\overline{B} \neq C$ here. (Note however that any $x_0 \in \mathbb{R}$ would be convenient here; there is nothing special about 0.)

Exercise 12.3.1. — *Prove Proposition 12.3.4(b).*

Let's show each direction of the equivalence.

- First suppose that E is relatively closed w.r.t. Y, and let's show that there exists a closed subset $K \subseteq X$ such that $E = K \cap Y$.

  Since E is closed w.r.t. Y, the set $Y \backslash E$ is open w.r.t. Y (by Proposition 12.2.15(e)). Thus, by (a), there exists an open subset $V \subseteq X$ such that $Y \backslash E = V \cap Y$.

  Let be $K := X \backslash V$; this subset $K \subseteq X$ is closed w.r.t. X by Proposition 12.2.15(e) since it is the complementary set of an open set. We have to show that $E = K \cap Y$.

  - Let be $x \in E$. Thus, we have $x \in Y$, since $E \subseteq Y$. And since $x \in E$, by definition, we have $x \notin Y \backslash E$. Thus, $x \notin V \cap Y$, which implies that $x \notin V$ (since $x \in Y$). Thus, by definition, $x \in K$, and thus, $x \in K \cap Y$.
  - Conversely, let be $x \in K \cap Y$. By definition, $x \in Y$ and $x \notin V$. Thus, $x \notin V \cap Y$, or, in other words, $x \notin Y \backslash E$. We finally get $x \in E$, as expected.

  Thus, we have indeed $E = K \cap Y$, for some closed subset $K \subseteq X$, as expected.

- Now let's prove the converse implication: suppose that $E = K \cap Y$ for some closed subset $K \subseteq X$, and let's prove that E is relatively closed w.r.t. Y.

  Still by Proposition 12.2.15(e), we know that the subset $V := X \backslash K$ is open w.r.t. X. Thus, by the previous result from this exercise, $V \cap Y$ is relatively open w.r.t. Y. Thus, its complementary set $Y \backslash (V \cap Y) = Y \backslash V$ is relatively closed w.r.t. Y. Now we want to show that $E = Y \backslash V$ to close the proof.

– First suppose that $x \in E$. Since $E = K \cap Y$, we thus have $x \in Y$ and $x \in K$, i.e. $x \notin V$. Thus, $x \in Y \backslash V$.

– Now suppose that $x \in Y \backslash V$. We thus have $x \in X$ (since $Y \subseteq X$) and $x \notin V$, so that we necessarily have $x \in K$. Thus $x \in Y \cap K$, i.e. $x \in E$.

Thus $E = Y \backslash V$ is relatively closed w.r.t. Y, as expected.

Exercise 12.4.1. — *Prove Lemma 12.4.3.*

We have to prove that any subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of a convergent sequence $(x^{(n)})_{n=m}^{\infty}$ converges to the same limit as the whole sequence itself.

Suppose that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to x_0. Let be $\varepsilon > 0$. By definition, we have a positive integer $N \geqslant m$ such that $n \geqslant N \implies d(x^{(n)}, x_0) \leqslant \varepsilon$. Our aim here is to show that there exists a positive integer $J \geqslant 1$ such that $j \geqslant J \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon$.

By Definition 12.4.1, we know that we have $m \leqslant n_1 < n_2 < n_3 < \ldots$. Thus, as a quick induction would show, we have $n_j \geqslant m + j - 1$ for all $j \geqslant 1$. Let's take $J := N$. In this case, if $j \geqslant J$, i.e. if $j \geqslant N$, we have $n_j \geqslant m + N - 1 \geqslant N$. Thus:

$$j \geqslant J \implies n_j \geqslant N \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it means that $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.2. — *Prove Proposition 12.4.5.*

Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space. We have to prove that the following two statements are equivalent:

(a) L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.

(b) There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence which converges to L.

We will prove the two implications, but first, note that (with the notations from Definition 12.4.1) if we have $1 \leqslant m \leqslant n_1 < n_2 < n_3 < \ldots$, then a quick induction shows that we have $n_j \geqslant j$ for all $j \geqslant 1$.

- First we prove that (b) implies (a). If some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to L, then we have by definition:

$$\forall \varepsilon > 0, \exists J \geqslant 1 : j \geqslant J \implies d(x^{(n_j)}, L) \leqslant \varepsilon \tag{12.5}$$

Now, consider any $\varepsilon > 0$ and any $N \geqslant m$. For this particular choice of ε, consider the corresponding real number J given by equation (12.5), and let's define $p := \max(N, J)$. Thus, we have $n_p \geqslant p \geqslant J$, and by equation (12.5), we thus have $d(x^{(n_p)}, L) \leqslant \varepsilon$. If we set $n := n_p$, we have indeed found an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, as required.

- Now we prove that (a) implies (b). Suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, there exists a natural number $n_1 \geqslant m$ such that $d(x^{(n_1)}, L) \leqslant 1$. Now, for $j > 1$, let's define inductively $n_j := \min\{n > n_{j-1} : d(x^{(n)}, L) \leqslant 1/j\}$. This set is non-empty (by definition of a limit point), so that the well-ordering principle

(Proposition 8.1.4) ensures that it has a (unique) minimal element, i.e. that n_j indeed exists. Let's define the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ obtained following this process. We thus have $d(x^{(n_j)}, L) \leqslant 1/j$ for all $j \geqslant 1$, by construction.

Thus, let be $\varepsilon > 0$. There exists a $j \geqslant 1$ such that $0 < 1/j < \varepsilon$ (Exercise 5.4.4). Thus, for this positive integer j, we have $d(x^{(n_j)}, L) \leqslant 1/j < \varepsilon$. By construction, for all other natural numbers $k \geqslant j+1$, we have $d(x^{(n_k)}, L) \leqslant 1/k \leqslant 1/j \leqslant \varepsilon$.

In summary, for our arbitrary choice of ε, we have showed that there exists $j \geqslant 1$ such that, for all $k \geqslant j$, we have $d(x^{(n_k)}, L) \leqslant \varepsilon$. Thus, the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ constructed in this way converges to L, as expected.

Exercise 12.4.3. — *Prove Lemma 12.4.7.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a convergent sequence of points in a metric space (X, d), and that its limit is x_0. Let's show that it is a Cauchy sequence.

By the triangle inequality, we know that for all $j, k \geqslant m$, we have:

$$d(x^{(j)}, x^{(k)}) \leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0).$$

Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, there exists an $N \geqslant m$ such that we have $d(x^{(n)}, x_0) \leqslant \varepsilon/3$ for all $n \geqslant N$. Thus, if we take $j, k \geqslant N$, we have:

$$\begin{aligned} d(x^{(j)}, x^{(k)}) &\leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \\ &\leqslant \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, as expected.

Exercise 12.4.4. — *Prove Lemma 12.4.9.*

Let be an arbitrary $\varepsilon > 0$. Since the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, there exists a $J \geqslant 1$ such that $d(x^{(n_j)}, x_0) \leqslant \varepsilon/3$ whenever $j \geqslant J$.

But the whole sequence $(x^{(n)})_{n=m}^{\infty}$ is supposed to be a Cauchy sequence. Thus, there also exists a $N \geqslant m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon/3$ whenever $j, k \geqslant N$.

Now, let be $K := \max(J, N)$. If $k \geqslant K$, we have

$$\begin{aligned} d(x^{(k)}, x_0) &\leqslant d(x^{(k)}, x^{(n_k)}) + d(x^{(n_k)}, x_0) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.5. — *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d) and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\{x^{(n)} : n \geqslant m\}$. Is the converse true?*

First suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, it means that

$$\forall \varepsilon > 0,\ \forall N \geqslant m,\ \exists n \geqslant N\ :\ d(x^{(n)}, L) \leqslant \varepsilon \tag{12.6}$$

Let be an arbitrary $\varepsilon > 0$, and let's take $N = m$. By formula (12.6) above, there exists an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, this $x^{(n)}$ belongs to both sets $\{x^{(n)} : n \geqslant m\}$ and $B(L, \varepsilon)$. We have just proved that for all $\varepsilon > 0$, the intersection $B(L, \varepsilon) \cap \{x^{(n)} : n \geqslant m\}$ is always non-empty. In other words, L is thus an adherent point of $\{x^{(n)} : n \geqslant m\}$.

However, the converse is not true. Indeed, consider the sequence $(x^{(n)})_{n=1}^{\infty}$ defined in $(\mathbb{R}, d)$ by $x^{(1)} = 1$ and $x^{(n)} = 0$ for all $n \geqslant 2$, i.e. the sequence $1, 0, 0, 0, \ldots$. It is clear that $L := 1$ is an adherent point of $\{x^{(n)} : n \geqslant 1\}$ (which is just the set $\{0, 1\}$). But 1 is not a limit point of $(x^{(n)})_{n=1}^{\infty}$, since we have $d(x^{(n)}, 1) > 1/2$ for all $n \geqslant 2$.

Exercise 12.4.6. — *Show that every Cauchy sequence can have at most one limit point.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in a metric space (X, d), such that L, L' are limit points. Then we have $L = L'$. We will give two different proofs for this fact.

- **Proof 1** (short proof using previous results). By Proposition 12.4.5, since L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, there exists a subsequence that converges to L. But by Lemma 12.4.9, it means that the whole original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to L. The same argument can be used to show that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to L'. But by uniqueness of limits (Proposition 12.1.20), we must have $L = L'$, as expected.

- **Proof 2** (a more "manual" proof). Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, there exists $N \geqslant m$ such that $d(x^{(p)}, x^{(q)}) \leqslant \varepsilon/3$ for all $p, q \geqslant N$.

  If L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, then for this $N \geqslant m$, there exists $p \geqslant N$ such that $d(x^{(p)}, L) \leqslant \varepsilon/3$. Similarly, there exists $q \geqslant N$ such that $d(x^{(q)}, L') \leqslant \varepsilon/3$.

  We thus have, by triangle inequality:

  $$\begin{aligned} d(L, L') &\leqslant d(L, x^{(p)}) + d(x^{(p)}, x^{(q)}) + d(x^{(q)}, L') \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leqslant \varepsilon \end{aligned}$$

  Thus, $d(L, L') \leqslant \varepsilon$ for all $\varepsilon > 0$, which implies $L = L'$.

Exercise 12.4.7. — *Prove Proposition 12.4.12.*

For statement (a), consider a convergent sequence $(y^{(n)})_{n=m}^{\infty}$ of elements of $Y \subseteq X$. Since it is convergent, it is a Cauchy sequence (Lemma 12.4.7). Saying that $(Y, d_{Y \times Y})$ is complete means that $(y^{(n)})_{n=m}^{\infty}$ converges in $(Y, d_{Y \times Y})$. Thus, every convergent sequence in Y has its limit in Y: this is exactly the characterization of closed sets given by Proposition 12.2.15(b).

For statement (b), consider a Cauchy sequence $(y^{(n)})_{n=m}^{\infty}$ of elements of a given closed subset $Y \subseteq X$. Since (X, d) is complete, $(y^{(n)})_{n=m}^{\infty}$ must converge to some value $L \in X$. But since Y is closed, we have $L \in Y$ by Proposition 12.2.15(b). Thus, every Cauchy sequence in Y converges in Y. This means that $(Y, d_{Y \times Y})$ is complete, as expected.

Exercise 12.4.8. — *The following construction generalizes the construction of the reals from the rationals in Chapter 5. In what follows, we let (X, d) be a metric space.*

We have to prove the following statements. Note that this is a generalization of the process of construction of the real numbers, so that we can use all results relative to the real numbers below.

(a) Given any Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ in X, we denote $\mathrm{LIM}_{n\to\infty} x_n$ its formal limit. We say that two formal limits $\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n$ are equal iff $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then, this equality relation obeys the reflexive, symmetry and transitive axioms.

- This relation is reflexive: for every Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$, we have $d(x_n, x_n) = 0$ for all $n \geqslant 1$, by definition of a metric. Thus, $d(x_n, x_n)$ is constant and equal to zero, so that $\lim_{n\to\infty} d(x_n, x_n) = 0$.
- By the property of symmetry of the metric d, we have $d(x_n, y_n) = d(y_n, x_n)$ for all $n \geqslant 1$ and all Cauchy sequence $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$. Thus, $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} y_n$ iff $\lim_{n\to\infty} d(x_n, y_n) = 0$, iff $\lim_{n\to\infty} d(y_n, x_n) = 0$, which is equivalent to $\mathrm{LIM}_{n\to\infty} y_n = \mathrm{LIM}_{n\to\infty} x_n$.
- For transitivity, suppose that $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ and $(z^{(n)})_{n=1}^{\infty}$ are Cauchy sequences in X. If $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} y_n$ and $\mathrm{LIM}_{n\to\infty} y_n = \mathrm{LIM}_{n\to\infty} z_n$, then by definition we have $\lim_{n\to\infty} d(x_n, y_n) = 0$ and $\lim_{n\to\infty} d(y_n, z_n) = 0$. Let be $\varepsilon > 0$. By definition, there exists $N_1 \geqslant 1$ such that $d(x_n, y_n) \leqslant \varepsilon/2$ whenever $n \geqslant N_1$. Similarly, there exists $N_2 \geqslant 1$ such that $d(y_n, z_n) \leqslant \varepsilon/2$ whenever $n \geqslant N_2$. Thus, if $n \geqslant N := \max(N_1, N_2)$, we have by the triangle inequality $d(x_n, z_n) \leqslant d(x_n, y_n) + d(y_n, z_n) \leqslant \varepsilon$. It means that $\lim_{n\to\infty} d(x_n, z_n)$, i.e. that $\mathrm{LIM}_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} z_n$, as expected.

(b) Let $\overline{X}$ be the space of all formal limits of Cauchy sequences in X, with the above equality relation. Define a metric $d_{\overline{X}} : \overline{X} \times \overline{X} \to \mathbb{R}^+$ by setting

$$d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} y_n) := \lim_{n\to\infty} d(x_n, y_n).$$

Then this function is well-defined and gives $\overline{X}$ the structure of a metric space.

- First we have to show that the limit $\lim_{n\to\infty} d(x_n, y_n)$ exists (in $\mathbb{R}^+$) for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$. We already know that $\mathbb{R}$ is complete, thus $\mathbb{R}^+$ is complete as a closed subset of the complete space $\mathbb{R}$ (Proposition 12.4.12(b)).

  Let be the sequence defined by $u_n := d(x_n, y_n)$ for all $n \geqslant 1$. Obviously, this sequence is in $\mathbb{R}^+$, which is a complete space. Thus, to show that it converges, we just have to show that it is a Cauchy sequence.

  Consider the usual metric on $\mathbb{R}^+$. We have, for all $p, q \geqslant 1$,

$$\begin{aligned} |u_p - u_q| &= |d(x_p, y_p) - d(x_q, y_q)| \\ &\leqslant |d(x_p, x_q) + d(x_q, y_q) + d(y_q, y_p) - d(x_q, y_q)| \\ &\leqslant |d(x_p, x_q)| + |d(y_p, y_q)|. \end{aligned}$$

  Now let be $\varepsilon > 0$. Since $(x^{(n)})_{n=1}^{\infty}$ and $(y^{(n)})_{n=1}^{\infty}$ are Cauchy sequences, there exists $N_1, N_2 \geqslant 1$ such that $d(x_p, x_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_1$, and $d(y_p, y_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_2$. Thus, if $p, q \geqslant N := \max(N_1, N_2)$, we have

$$|u_p - u_q| \leqslant |d(x_p, x_q)| + |d(y_p, y_q)| \leqslant \varepsilon.$$

  This shows that $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence, and thus, that $\lim_{n\to\infty} d(x_n, y_n)$ exists.

- Now we must show that the axiom of substitution is obeyed. In other words, consider a Cauchy sequence $(z^{(n)})_{n=1}^{\infty}$ in (X,d) such that $\mathrm{LIM}_{n\to\infty}z_n = \mathrm{LIM}_{n\to\infty}x_n$. We must show that $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}z_n, \mathrm{LIM}_{n\to\infty}y_n) = d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n)$, i.e. that
$$\lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_n, y_n) \tag{12.7}$$
By the previous bullet point, we know that both limits in (12.7) do exist. Thus, the limit laws apply. We have:
$$d(z_n, y_n) \leqslant d(z_n, x_n) + d(x_n, y_n)$$
but since $\lim_{n\to\infty} d(z_n, x_n) = 0$ by definition, we obtain
$$\lim_{n\to\infty} d(z_n, y_n) \leqslant \lim_{n\to\infty} d(x_n, y_n)$$
if we take the limits of both sides in the previous inequality.
But similarly, we have $d(x_n, y_n) \leqslant d(x_n, z_n) + d(z_n, y_n)$, so that a similar argument gives
$$\lim_{n\to\infty} d(x_n, y_n) \leqslant \lim_{n\to\infty} d(z_n, y_n).$$
Thus, we have indeed $\lim_{n\to\infty} d(z_n, y_n) = \lim_{n\to\infty} d(x_n, y_n)$, as expected.
- Finally, we must show that $d_{\overline{X}}$ is a metric on $\overline{X}$. To prove this statement, we must show that $d_{\overline{X}}$ obeys all four axioms that define a metric.
  - First, it is clear that $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}x_n) = \lim_{n\to\infty} d(x_n, x_n) = 0$ for all Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ in (X,d).
  - Now let be two Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ in X, such that $\mathrm{LIM}_{n\to\infty}x_n \neq \mathrm{LIM}_{n\to\infty}y_n$. This latest property implies that $\lim_{n\to\infty} d(x_n, y_n) > 0$, by definition. Thus, $d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) > 0$.
  - Symmetry: we have
$$\begin{aligned} d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) &= \lim_{n\to\infty} d(x_n, y_n) \\ &= \lim_{n\to\infty} d(y_n, x_n) \text{ (symmetry of } d \text{ on } \mathbb{R}^+) \\ &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}y_n, \mathrm{LIM}_{n\to\infty}x_n) \end{aligned}$$
for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$.
  - Triangle inequality: by the limit laws, we have
$$\begin{aligned} &d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}z_n) = \lim_{n\to\infty} d(x_n, z_n) \\ &\leqslant \lim_{n\to\infty} (d(x_n, y_n) + d(y_n, z_n)) \\ &\leqslant \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n) \\ &\leqslant d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}x_n, \mathrm{LIM}_{n\to\infty}y_n) + d_{\overline{X}}(\mathrm{LIM}_{n\to\infty}y_n, \mathrm{LIM}_{n\to\infty}z_n) \end{aligned}$$
for all Cauchy sequences $(x^{(n)})_{n=1}^{\infty}$, $(y^{(n)})_{n=1}^{\infty}$ and $(z^{(n)})_{n=1}^{\infty}$.

Thus, $d_{\overline{X}}$ is indeed a metric on $\overline{X}$.

(c) The metric space $(\overline{X}, d_{\overline{X}})$ is complete.

To prove this statement, consider a Cauchy sequence $(u_n)_{n=1}^{\infty}$ in $\overline{X}$: we have to prove that this sequence converges in $(\overline{X}, d_{\overline{X}})$.

By definition, $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence of formal limits of Cauchy sequences that take their values in X; i.e., for all $k \geqslant 1$, there exists a Cauchy sequence $(x_n^{(k)})_{n=1}^{\infty}$ of elements of X such that $u_k := \mathrm{LIM}_{n\to\infty} x_n^{(k)}$.

Since all $(x_n^{(k)})_{n=1}^{\infty}$ are Cauchy sequences, then for all $k \geqslant 1$, there exists a threshold N_k such that $d(x_n^{(k)}, x_{N_k}^{(k)}) < 1/k$ whenever $n \geqslant N_k$. Thus, (using the countable axiom of choice) we can build a sequence $(z_k)_{k=1}^{\infty}$ defined by

$$z_k := \left(x_{N_k}^{(k)}\right)$$

for all $k \geqslant 1$. Now:

- We claim that $(z_k)_{k=1}^{\infty}$ is itself a Cauchy sequence. Indeed, consider an arbitrary positive real number $\varepsilon > 0$. We must prove that $d(z_p, z_q) := d(x_{N_p}^{(p)}, x_{N_q}^{(q)})$ is eventually lesser than ε.

  Since $(u_n)_{n=1}^{\infty}$ is a Cauchy sequence in $\overline{X}$, there exists a $N \geqslant 1$ such that, if $p, q \geqslant N$, we have $d_{\overline{X}}(u_p, u_q) < \varepsilon/3$, i.e.:

$$\begin{aligned}\varepsilon/3 &> d_{\overline{X}}(u_p, u_q)\\ &\geqslant d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n^{(p)}, \mathrm{LIM}_{n\to\infty} x_n^{(q)})\\ &\geqslant \lim_{n\to\infty} d(x_n^{(p)}, x_n^{(q)})\end{aligned}$$

  Thus, there exists a $N' \geqslant 1$ such that, if $n \geqslant N'$, we have $d(x_n^{(p)}, x_n^{(q)}) \leqslant \varepsilon/3$[2]. Also, by Exercise 5.4.4, there exists a $k > 0$ such that $1/k \leqslant \varepsilon/3$. Thus, if $n, p, q \geqslant \max(k, N', N_p, N_q)$, we have

$$\begin{aligned}d(z_p, z_q) &= d(x_{N_p}^{(p)}, x_{N_q}^{(q)})\\ &\leqslant \underbrace{d(x_{N_p}^{(p)}, x_n^{(p)})}_{\leqslant 1/p \leqslant \varepsilon/3} + \underbrace{d(x_n^{(p)}, x_n^{(q)})}_{\leqslant \varepsilon/3} + \underbrace{d(x_n^{(q)}, x_{N_q}^{(q)})}_{\leqslant 1/q \leqslant \varepsilon/3}\\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3\\ &\leqslant \varepsilon\end{aligned}$$

  Thus, $(z_k)_{k=1}^{\infty}$ is indeed a Cauchy sequence in X.

- Consequently, we can take the formal limit $L := \mathrm{LIM}_{n\to\infty} z_n$, and this formal limit L lies in $\overline{X}$ by definition. We claim that $\lim_{n\to\infty} u_n = L \in \overline{X}$; proving this claim will close the proof of (c).

  Let be $\varepsilon > 0$. Since $(z_n)_{n=1}^{\infty}$ is a Cauchy sequence in X, there exists a $N_1 \geqslant 1$ such that $d(z_p, z_q) \leqslant \varepsilon/2$ whenever $p, q \geqslant N_1$.

[2]Indeed, for any sequence $(v_n)_{n=1}^{\infty}$ that converges to ℓ, if we have $0 \leqslant \ell < \varepsilon$, then there exists an $N \geqslant 1$ such that $v_n \leqslant \varepsilon$ whenever $n \geqslant N$ (why? use a proof by contradiction.).

Once again, by Exercise 5.4.4, there exists a $K' \geqslant 1$ such that $1/K' < \varepsilon/2$. Thus, if $k \geqslant K$ and $n > N_k$, we have

$$d(x_n^{(k)}, z_k) := d(x_n^{(k)}, x_{N_k}^{(k)}) < \frac{1}{k} \leqslant \frac{1}{K} < \frac{\varepsilon}{2}.$$

Thus, by the triangle inequality, we have, for all $n > \max(N_k, N_1)$,

$$d(x_n^{(k)}, z_n) \leqslant d(x_n^{(k)}, z_k) + d(z_k, z_n) \leqslant \varepsilon/2 + \varepsilon/2 \leqslant \varepsilon.$$

Consequently, we have, for all $k > K'$,

$$d_{\overline{X}}(u_k, L) := \lim_{n\to\infty} d(x_n^{(k)}, b_n) < \varepsilon.$$

This shows that $(u_n)_{n=1}^{\infty} \to L$ in $(\overline{X}, d_{\overline{X}})$, which closes the proof.

(d) We identify an element $x \in X$ with the corresponding formal limit $\mathrm{LIM}_{n\to\infty} x$ in $\overline{X}$.

- This is legitimate since we have $x = y \iff \mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$.
  Indeed, it is clear that if $x = y$, then we have $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$ by definition. Conversely, if $\mathrm{LIM}_{n\to\infty} x = \mathrm{LIM}_{n\to\infty} y$, then we have $\lim_{n\to\infty} d(x, y) = 0$, i.e. $d(x, y) = 0$, i.e. $x = y$. Thus, this identification is legitimate.
- With this identification, we have $d(x, y) = d_{\overline{X}}(x, y)$. Indeed:

$$\begin{aligned} d_{\overline{X}}(x, y) &= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x, \mathrm{LIM}_{n\to\infty} y) \\ &= \lim_{n\to\infty} d(x, y) \\ &= d(x, y). \end{aligned}$$

Thus, (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$.

(e) The closure of X in $\overline{X}$ is $\overline{X}$.

Indeed, let be C the closure of X in $\overline{X}$. We clearly have $C \subseteq \overline{X}$, by definition. Thus we just have to show that $\overline{X} \subseteq C$.

Let be $x \in \overline{X}$, and let's show that $x \in C$. By definition, $x \in C$ means that x is an adherent point of X in $\overline{X}$, i.e. that for all $\varepsilon > 0$, $B_{(\overline{X}, d_{\overline{X}})}(x, \varepsilon) \cap X \neq \varnothing$. In other words, for all $\varepsilon > 0$, we must show that there exists a $y \in X$ such that $d_{\overline{X}}(x, y) < \varepsilon$.

Thus, let be $\varepsilon > 0$. By definition, x is the formal limit of a Cauchy sequence $(x_n)_{n=1}^{\infty}$ of elements of X, so that $x := \mathrm{LIM}_{n\to\infty} x_n$. Since $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence, there exists an $N \geqslant 1$ such that $d(x_n, x_N) < \varepsilon/2$ whenever $n \geqslant N$. Thus:

$$\begin{aligned} d_{\overline{X}}(x, x_N) &:= d_{\overline{X}}(\mathrm{LIM}_{n\to\infty} x_n, \mathrm{LIM}_{n\to\infty} x_N) \\ &= \lim_{n\to\infty} d(x_n, x_N) \\ &\leqslant \varepsilon/2 < \varepsilon \end{aligned}$$

so that $y := x_N$ is a convenient choice. This shows that x is an adherent point of X in $\overline{X}$, as expected.

(f) Finally, the formal limit agrees with the actual limit, i.e., $\lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} x_n \in \overline{X}$ for all Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X.

Indeed, let be $(x_n)_{n=1}^{\infty}$ a Cauchy sequence of elements of X. We know that (X, d) can be thought of as a subspace of $(\overline{X}, d_{\overline{X}})$, so that $(x_n)_{n=1}^{\infty}$ can be thought of as a sequence of elements of $\overline{X}$. But we have showed that $(\overline{X}, d_{\overline{X}})$ is complete. Thus, the sequence $(x_n)_{n=1}^{\infty}$ converges in $\overline{X}$ to a certain limit $L \in \overline{X}$; i.e., we have $\lim_{n\to\infty} x_n = L$ for some $L \in \overline{X}$.

Consider this limit L. By definition of $\overline{X}$, there exists a Cauchy sequence $(a_n)_{n=1}^{\infty}$ of elements of X such that $L := \mathrm{LIM}_{n\to\infty} a_n$. What we need to prove is that we have

$$L = \lim_{n\to\infty} x_n = \mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n \tag{12.8}$$

and thus, it is sufficient to show that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n$, since we already have the other equalities. And, by definition of the equality relation established in (a), in order to prove that $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} x_n$, we just have to show that $\lim_{n\to\infty} d(x_n, a_n) = 0$. Or, in yet another equivalent way, we have to show that for all $\varepsilon > 0$, there exists an $N \geqslant 1$ such that $d(x_n, a_n) \leqslant \varepsilon$ whenever $n \geqslant N$.

Thus, let be an arbitrary $\varepsilon > 0$. Let's unfold our hypotheses.

- We know that the sequence $(x_n)_{n=1}^{\infty}$ converges to L in $\overline{X}$. Thus, by definition, there exists a $N_1 \geqslant 1$ such that $d_{\overline{X}}(x_k, L) \leqslant \varepsilon/2$ whenever $k \geqslant N_1$. In other words, $\lim_{n\to\infty} d(x_k, a_n) \leqslant \varepsilon/3 < \varepsilon/2$ whenever $k \geqslant N_1$.

  Thus, there exists a N_2 such that $d(x_k, a_n) \leqslant \varepsilon/2$ whenever $k \geqslant N_1$ and $n \geqslant N_2$ (see footnote 2 p. 22 from the present document).
- We also know that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence. It means that there exists a $N_3 \geqslant 1$ such that $d(x_p, x_q) \leqslant \varepsilon/2$ for all $p, q \geqslant N_3$.

Let be $N := \max(N_1, N_2, N_3)$. Using the triangle inequality, we finally get, for all $n \geqslant N$,

$$\begin{aligned} d(x_n, a_n) &\leqslant d(x_n, x_N) + d(x_N, a_n) \\ &\leqslant \varepsilon/2 + \varepsilon/2 \\ &\leqslant \varepsilon \end{aligned}$$

This closes the proof.

Exercise 12.5.1. — *Show that Definitions 9.1.22 and 12.5.3 match when talking about subsets of the real line with the standard metric.*

Consider $Y \subseteq \mathbb{R}$ and the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. We have to show that both definitions of boundedness are equivalent in this case.

- First, suppose that Y is bounded in the sense of Definition 12.5.3. Thus, there exists a real number x and a positive real number $r > 0$ such that $Y \subseteq B(x, r)$. In other words, we have $Y \subseteq]x - r, x + r[\subseteq [x - r, x + r]$. Let be $M := |x| + |r|$. We clearly have $x + r \leqslant M$, and $-M \leqslant x - r$. Thus, we have $Y \subseteq [-M, M]$, and Y is bounded in the sense of Definition 9.1.22.

- Conversely, suppose that Y is bounded in the sense of Definition 9.1.22. Thus, there exists a positive real $M > 0$ such that $Y \subseteq [-M, M] \subset]-2M, 2M[$. But this later interval is simply $B(0, 2M)$, so that Y is bounded in the sense of Definition 12.5.1, taking $x := 0$ and $r := 2M$.

EXERCISE 12.5.2. — *Prove Proposition 12.5.5.*

We must prove that any compact space (X, d) is both complete and bounded. In both cases, we will use a proof by contradiction.

- First, let's prove completeness. Suppose, for the sake of contradiction, that the compact space (X, d) is not complete. Since it is not complete, there exists a Cauchy sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X which does not converge in (X, d). But since it is compact, there exists a subsequence $(x^{(n_k)})_{k=1}^{\infty}$ of this Cauchy sequence, which converges in (X, d). But, by Lemma 12.4.9, if a Cauchy sequence has a convergent subsequence, then it is convergent itself; thus $(x^{(n)})_{n=1}^{\infty}$ converges. It is a clear contradiction. Thus, (X, d) must be complete.

- Now we show boundedness. Similarly, suppose for the sake of contradiction that (X, d) is not bounded. It means that, for all positive real $r > 0$ and all $x \in X$, we have $X \nsubseteq B(x, r)$. In particular, for any positive natural number $n \geqslant 1$ and an arbitrary $x \in X$, the set $X \backslash B(x, n)$ is not empty. Thus, using the (countable) axiom of choice, we can build a sequence $(x^{(n)})_{n=1}^{\infty}$ such that $x^{(n)} \in X \backslash B(x, n)$ for all positive integer $n \geqslant 1$. Or, in other words, we have $d(x, x^{(n)}) \geqslant n$ for all $n \geqslant 1$.

  But recall that (X, d) is compact. Thus, there must exist a convergent subsequence $(x^{(n_k)})_{k=1}^{\infty}$ of the original sequence. Say that this subsequence converges to some value L. Thus, by definition,

  $$\forall \varepsilon > 0, \exists K \geqslant 1 \,:\, k \geqslant K \implies d(x^{(n_k)}, L) \leqslant \varepsilon.$$

  Let's take $\varepsilon := 1$ (there is nothing special about this value; this is just any arbitrary ε to obtain a contradiction). There must exist a $K_1 \geqslant 1$ such that $d(x^{(n_k)}, L) \leqslant 1$ whenever $k \geqslant K_1$. But, at the same time, we have by the triangle inequality

  $$\begin{aligned} & d(x^{(n_k)}, x) \leqslant d(x^{(n_k)}, L) + d(L, x) \\ \implies & d(x^{(n_k)}, L) \geqslant d(x^{(n_k)}, x) - d(L, x) \end{aligned}$$

  For instance by the Archimedean principle, there exists an $N \in \mathbb{N}$ such that $N \geqslant d(L, x) + 3$. Let be $K_2 := \min\{k \in \mathbb{N} : n_k \geqslant N\}$ (this natural number exists simply because $n_N \geqslant N$, so that the set is not empty). We thus have

  $$d(x, x^{(n_k)}) \geqslant n_k \geqslant N \geqslant d(L, x) + 3$$

  for all $k \geqslant K_2$.

  Thus, for all $k \geqslant \max(K_1, K_2)$, we have both $d(x^{(n_k)}, x) \leqslant 1$ (because $k \geqslant K_1$), and $d(x^{(n_k)}, L) \geqslant d(x^{(n_k)}, x) - d(L, x) \geqslant d(L, x) + 3 - d(L, x) \geqslant 3$ (because $k \geqslant K_2$). This is a contradiction. Thus, (X, d) is bounded.

Exercise 12.5.3. — *Prove Theorem 12.5.7.*

Let be $(\mathbb{R}^n, d)$ an Euclidean space, where d is either the Euclidean, taxicab or sup norm metric. Also, let be $E \subseteq \mathbb{R}^n$. We have to prove that E is compact iff E is closed and bounded. By Corollary 12.5.6, we already know that if E is compact, then it is closed and bounded. We thus have to prove the converse implication.

Suppose that E is both closed and bounded. Since E is a subset of $\mathbb{R}^n$, we can write $E := E_1 \times \ldots \times E_n$, where $E_j \subseteq \mathbb{R}$ for all $1 \leqslant j \leqslant n$.

We have to prove that any sequence $(x^{(k)})_{k=1}^{\infty}$ in E has a convergent subsequence in (E, d). This sequence can be written as a sequence of vectors of length n, i.e., we have $x^{(k)} = (x_1^{(k)}, \ldots, x_n^{(k)})$, where $x_j^{(k)} \in E_j$ for all $k \geqslant 1$ and all $1 \leqslant j \leqslant n$.

We will first need a lemma:

Lemma. If E is bounded, then each $E_j \subseteq \mathbb{R}$ is also bounded.

Sketch of proof. Suppose that d is the sup norm metric. If E is bounded, we have $E \subseteq B(x, r)$ for some $x \in \mathbb{R}^n$ and some $r > 0$ (Definition 12.5.3). In other words, we have $d(x, y) < r$ for all $y \in E$. Since d is the sup norm metric, this implies that

$$\forall j \in [\![1, n]\!],\ |x_j - y_j| \leqslant \max_{j=1,\ldots,n} |x_j - y_j| := d(x, y) < r.$$

Thus, $E_j \subseteq B(x_j, r)$, i.e. E_j is bounded for all $1 \leqslant j \leqslant n$.

The proof is similar if d is the Euclidean metric, or the taxicab metric. □

Now we go back to the main proof. Since each sequence $(x_j^{(k)})_{k=1}^{\infty}$ is a sequence of real numbers in the bounded subset $E_j \subseteq \mathbb{R}$, then by Theorem 9.1.24 this sequence has a convergent subsequence $(x_j^{(k_l)})_{l=1}^{\infty}$, which converges to $L_j \in \mathbb{R}_j$. But by Proposition 12.1.18, this implies that the whole subsequence $(x^{(k_l)})_{l=1}^{\infty}$ converges to $(L_1, \ldots, L_n)$ (since it converges component-wise).

Thus, $(x_j^{(k)})_{k=1}^{\infty}$ indeed has a convergent subsequence, as expected; and E is compact.

Exercise 12.5.4. — *Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and an open set $V \subseteq \mathbb{R}$, such that the image $f(V) := \{f(x) : x \in V\}$ of V is not open.*

As a simple example, consider the constant function $f(x) = 0$ defined on $V :=]-1, 1[$. The interval V is clearly open, but we have $f(V) = \{0\}$. This singleton (or more generally, any singleton) is not open in $(\mathbb{R}, d)$, since for all $r > 0$, there always exists a real number x such that $x \in B(0, r) \backslash \{0\}$.

Exercise 12.5.5. — *Let $(\mathbb{R}, d)$ be the real line with the standard metric. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$, and closed set $F \subseteq \mathbb{R}$, such that $f(F)$ is not closed.*

One can give the example of the function $\tan^{-1}(x)$ defined on the closed set $F := \mathbb{R}$, but this function has not really been defined so far in the book. So, let's use a simpler example.

Consider the closed set $F := [1, +\infty[$ and the function $f(x) = 1/x$. We have $f(F) =]0, 1]$, which is not a closed set.

Exercise 12.5.6. — *Prove Corollary 12.5.9.*

Consider a sequence $K_1 \supset K_2 \supset K_3 \supset \ldots$ of non-empty compact sets in a metric space (X, d). We have to show that $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Let's work in the space $(K_1, d_{K_1 \times K_1})$. We define the sets $V_n := K_1 \backslash K_n$ for all $n \geqslant 1$, i.e.,

$$\begin{aligned} V_1 &:= K_1 \backslash K_1 = \varnothing \\ V_2 &:= K_1 \backslash K_2 \\ V_3 &:= K_1 \backslash K_3 \\ &\ldots \end{aligned}$$

so that the V_n clearly constitute an increasing sequence:

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \ldots,$$

so that $\bigcup_{k=1}^{n} V_k = V_n$ for all $n \geqslant 1$.

Furthermore, each set V_n is open in $(K_1, d_{K_1 \times K_1})$, since it is the complementary set of a compact (and then closed) set (Proposition 12.2.15 (e)).

Suppose, for the sake of contradiction, that we have $\bigcap_{n=1}^{\infty} K_n = \varnothing$. We would thus have:

$$\begin{aligned} \bigcup_{n=1}^{\infty} V_n &= \bigcup_{n=1}^{\infty} (K_1 \backslash K_n) \\ &= K_1 \backslash \left(\bigcap_{n=1}^{\infty} K_n \right) \text{ (Exercise 3.4.11)} \\ &= K_1 \backslash \varnothing \text{ (by hypothesis)} \\ &= K_1. \end{aligned}$$

But since K_1 is compact, then by Theorem 12.5.8, there exists a finite open cover of K_1, i.e., there exists a finite number k of indices $n_1 < \ldots < n_k\}$ such that

$$\bigcup_{n \in \{n_1, \ldots, n_k\}} V_n = K_1.$$

But since the V_n form an increasing sequence, this implies $V_{n_k} = K_1$, i.e., $K_1 \backslash K_{n_k} = K_1$, so that we finally get $K_{n_k} = \varnothing$.

But all the sets K_n were supposed to be non empty: this is thus a contradiction, and we must have $\bigcap_{n=1}^{\infty} K_n \neq \varnothing$.

Exercise 12.5.7. — *Prove Theorem 12.5.10.*

Let be (X, d) a metric space.

(a) Let be $Z \subseteq Y \subseteq X$, with Y compact. We have to show that Z is closed iff it is compact. We already know that if Z is compact, then it is closed (Corollary 12.5.6); so that we just have to show the converse implication.

Suppose that Z is closed, and let be $(z^{(n)})_{n=1}^{\infty}$ a sequence of elements of Z. Since $Z \subseteq Y$, $(z^{(n)})_{n=1}^{\infty}$ is also a sequence of elements of Y; and since Y is compact, there exists a subsequence $(z^{(n_k)})_{k=1}^{\infty}$ that converges to some $z \in Y$. But since Z is closed, we must have $z \in Z$ (by Proposition 12.2.15(b)). Thus, any sequence of elements of Z has a subsequence that converges in Z, i.e., Z is indeed compact.

(b) Let be $Y_1, \ldots, Y_n$ be n compact subsets of X; we have to show that the finite union $Y_1 \cup \ldots \cup Y_n$ is compact. Let's use the topological characterization of compact sets: suppose that we have an open cover $\bigcup_{\alpha \in I} V_\alpha$ (possibly uncountable), i.e. that

$$Y_1 \cup \ldots \cup Y_n \subseteq \bigcup_{\alpha \in I} V_\alpha.$$

Clearly, we have $Y_1 \subseteq \bigcup_{\alpha \in I} V_\alpha$, and since V_1 is compact, there exists a finite open cover, i.e. $Y_1 \subseteq \bigcup_{i=1}^{s_1} V_{a_i}$. Similarly, there exist finite open covers for each other subset Y_i, i.e.,

$$Y_2 \subseteq \bigcup_{i=1}^{s_2} V_{b_i}$$
$$\ldots$$
$$Y_n \subseteq \bigcup_{i=1}^{s_n} V_{n_i}.$$

Thus, there exists a finite open cover

$$Y_1 \cup \ldots \cup Y_n \subseteq \bigcup_{\alpha \in \{a_1, \ldots, a_{s_1}, b_1, \ldots, b_{s_2}, \ldots, n_1, \ldots, n_{s_n}\}} V_\alpha$$

so that $Y_1 \cup \ldots \cup Y_n$ is indeed compact.

(c) Let be Y a finite subset of X; we have to show that Y is compact.

First, suppose that Y is a singleton $\{a\}$. By definition, any sequence of elements of Y can only be the constant sequence $a, a, a, \ldots$. Thus, any subsequence of this sequence is still the constant sequence $a, a, \ldots$, and still converges to a. Thus, any sequence of elements of Y has a subsequence that converges in Y, i.e., Y is compact.

Now suppose that Y is a finite subset of cardinality n. Let's write $Y := \{y_1, \ldots, y_n\}$. This can also be written $Y := \{y_1\} \cup \ldots \cup \{y_n\}$, so that we are back in the previous case (b): Y is the finite union of compact subsets of X. Thus, Y is itself compact.

Note that for the limit case $Y = \varnothing$, we can say that the empty set is just a closed[3] subset of the compact set $\{a\}$, so that by the previous case (a), $Y = \varnothing$ is compact.

Exercise 12.5.8. — *Let (X, d_{l^1}) be the metric space from Exercise 12.1.15. For each natural number n, let $e^{(n)} = (e_j^{(n)})_{j=0}^\infty$ be the sequence in X such that $e_j^{(n)} := 1$ when $n = j$ and $e_j^{(n)} := 0$ when $n \neq j$. Show that the set $\{e^{(n)} : n \in \mathbb{N}\}$ is a closed and bounded subset of X, but is not compact.*

Recall that (X, d_{l^1}) is the metric space of absolutely convergent sequences, with the metric defined by $d_{l^1}((a^{(n)}), (b^{(n)})) := \sum_{n=0}^\infty |a_n - b_n|$. Hereafter, we denote $E := \{e^{(n)} : n \in \mathbb{N}\}$, with

$$e^{(0)} := 1, 0, 0, 0, \ldots$$
$$e^{(1)} := 0, 1, 0, 0, \ldots$$
$$e^{(2)} := 0, 0, 1, 0, \ldots$$
$$\ldots$$

[3] See Remark 12.2.14.

- First, we show that E is not compact. To prove this statement, we just have to find one sequence of elements of E that has no convergent subsequence in E.

  Consider the "canonical" sequence of elements of E defined by $e^{(0)}, e^{(1)}, e^{(2)}, \ldots$. The distance between any two distinct elements of this sequence is

  $$d_{l^{(1)}}(e^{(j)}, e^{(k)}) := \sum_{i=0}^{\infty} |e_i^{(j)} - e_i^{(k)}| = 2 > 1.$$

  Thus, this sequence is not a Cauchy sequence itself, and it is clear that no subsequence can be a Cauchy sequence either. Thus, no subsequence of this sequence can converge in E, i.e., E is not compact.

- However, E is a closed subset of X. To prove this property, consider a convergent sequence of elements of E; we have to prove that its limit lies in E. We've just shown that the distance between any two distinct terms $e^{(j)}, e^{(k)}$ for $j \neq k$ is equal to 2. Thus, if a sequence of elements of E converges, it must be eventually 0.5-stable, and the only possibility for that is to be eventually constant. In other words, it must be eventually equal to $e^{(n_0)}$ for $n_0 \in \mathbb{N}$, so that it necessarily converges to $e^{(n_0)}$, which is an element of E. This shows that E is closed.

- Furthermore, E is bounded. To show the boundedness of E, we have to show that $E \subseteq B_{(X,d_{l^1})}((x_j)_{j=0}^{\infty}, r)$ for some $r > 0$ and some sequence $(x_j)_{j=0}^{\infty} \in X$. Consider the zero sequence $(z_j)_{j=0}^{\infty} := 0, 0, 0, \ldots$. This is clearly a sequence in X (since it converges to 0), and we have

  $$d_{l^1}\left((z_j)_{j=0}^{\infty}, (e_j^{(n)})_{j=0}^{\infty}\right) = \sum_{j=0}^{\infty} |z_j - e_j^{(n)}| = 1 < 2$$

  for all $n \in \mathbb{N}$. Thus, we have $E \subseteq B_{(X,d_{l^1})}((z_j)_{j=0}^{\infty}, 2)$, which shows that E is bounded.

Thus, the case of the subset E of the metric space (X, d_{l^1}) shows that the Heine-Borel theorem (stated for the metric space $(\mathbb{R}^n, d)$) is not valid in more general metric spaces.

Exercise 12.5.9. — *Show that a metric space (X, d) is compact if and only if every sequence in X has at least one limit point.*

A metric space (X, d) is compact iff any sequence of elements of X has a subsequence that converges in (X, d). Thus, the statement is a direct consequence of Proposition 12.4.5, which says basically that "having a convergent subsequence" and "having a limit point" are synonymous.

Exercise 12.5.13. — *Let E and F be two compact subsets of $\mathbb{R}$ (with the standard metric $d(x, y) = |x - y|$). Show that the Cartesian product $E \times F := \{(x, y) : x \in E, y \in F\}$ is a compact subset of $\mathbb{R}^2$ (with the Euclidean metric d_{l^2}).*

To prove that $E \times F$ is compact, we will show that it is both closed and bounded (by Heine-Borel theorem).

- First we show that $E \times F$ is bounded.

  Since E and F are compact, they are themselves bounded (by Heine-Borel theorem). Thus, there exist $a \in E$, $b \in F$ and $r_1, r_2 > 0$ such that $E \subseteq B_d(a, r_1)$ and $F \subseteq B_d(b, r_2)$, by Definition 12.5.3. In other words, we have:

$$\begin{aligned} \forall x \in E, &|x - a| < r_1 \\ \forall y \in F, &|y - b| < r_2. \end{aligned}$$

  Thus, let be $(x, y) \in E \times F$. We have:

$$\begin{aligned} d_{l^2}\left((x, y), (a, b)\right) &= \sqrt{(x-a)^2 + (y-b)^2} \\ &< \sqrt{r_1^2 + r_2^2}. \end{aligned}$$

  This means that each $(x, y) \in E \times F$ lies in $B_{d_{l^2}}\left((a, b), \sqrt{r_1^2 + r_2^2}\right)$. Thus, $E \times F$ is indeed bounded.

- Now let's show that $E \times F$ is closed.

  Since E and F are compact, they are themselves closed (by Heine-Borel theorem). Consider a sequence $((x^{(n)}, y^{(n)}))_{n=1}^{\infty}$ of elements of $E \times F$ which converges to (x_0, y_0) with respect to d_{l^2}. By Proposition 12.1.18, this means that this sequence converges component-wise, i.e. that $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, and $(y^{(n)})_{n=1}^{\infty}$ converges to y_0. By definition, we have $x_0 \in E$ and $y_0 \in F$, since E and F are closed. Thus, $(x_0, y_0) \in E \times F$. This shows that $E \times F$ is indeed bounded.

Thus, $E \times F$ is compact, as expected.

13. Continuous functions on metric spaces

Exercise 13.1.1. — *Prove Theorem 13.1.4.*

Since the implication $(b) \implies (c)$ may be slightly more difficult to write, we will prove the implications $(a) \implies (c)$, $(c) \implies (b)$ and $(b) \implies (a)$ in this order.

Let be $f : (X, d_X) \to (Y, d_Y)$, and $x_0 \in X$.

- First let's prove $(a) \implies (c)$. Suppose that f is continuous at x_0, and let be $V \subseteq Y$ an open set that contains $f(x_0)$. By Proposition 12.2.15(a), there exists a $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq V$. But since f is continuous at x_0, we know that there exists a $\delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$. Thus, if we set $U := B_X(x_0, \delta)$, we have found an open set $U \subseteq X$ such that $f(U) \subseteq B_Y(f(x_0), \varepsilon) \subseteq V$, as required.

- Now we prove $(c) \implies (b)$. Consider a sequence $(x^{(n)})_{n=1}^{\infty}$ in X which converges to x_0 with respect to d_X. Let be an arbitrary $\varepsilon > 0$; we set $V_\varepsilon := B_Y(f(x_0), \varepsilon)$. By (c), we know that there exists an open set $U \subseteq X$ containing x_0 and such that $f(U) \subseteq V_\varepsilon$. But since U is open set, by Proposition 12.2.15(a), there exists a $\delta > 0$ such that $B_X(x_0, \delta) \subseteq U$.

  Since $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, there exists a natural number $N \geqslant 1$ such that $d_X(x^{(n)}, x_0) < \delta$ whenever $n \geqslant N$. Or, in other words, we have $x^{(n)} \in B_X(x_0, \delta) \subseteq U$ whenever $n \geqslant N$.

  But since $f(U) \subseteq V$ by hypothesis, we thus have $f(x^{(n)}) \in V_\varepsilon$ whenever $n \geqslant N$. Since this is true for any arbitrary $\varepsilon > 0$, this shows that the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ with respect to d_Y, as expected.

- Finally, we prove $(b) \implies (a)$. Suppose that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ whenever $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, and let's show that f is continuous at x_0.

  Suppose, for the sake of contradiction, that f is *not* continuous at x_0. Thus, there exists an $\varepsilon > 0$ such that for all $\delta > 0$, there exists an $x \in X$ such that $d_Y(f(x), f(x_0)) \geqslant \varepsilon$ although $d_X(x, x_0) < \delta$.

  Thus, using the (countable) axiom of choice, we build a sequence $(x^{(n)})_{n=1}^{\infty}$ such that, for all $n \geqslant 1$, we have $d_Y(f(x^{(n)}), f(x_0)) \geqslant \varepsilon$ although $d_X(x^{(n)}, x_0) < \frac{1}{n}$. It is thus clear that $(x^{(n)})_{n=1}^{\infty}$ converges to x_0, but that $(f(x^{(n)}))_{n=1}^{\infty}$ does not converge to $f(x_0)$, since $f(x^{(n)})$ and $f(x_0)$ are never $\varepsilon/2$-close. This is a contradiction with (c). Thus, f must be continuous at x_0, as expected.

Exercise 13.1.2. — *Prove Theorem 13.1.5.*

We already know from Theorem 13.1.4 that (a) and (b) are equivalent. Let's prove the other implications.

- First we prove that $(a) \implies (c)$. Let be V an open set in Y. We must show that $f^{-1}(V)$ is an open set in X. Thus, if we take an arbitrary $x_0 \in f^{-1}(V)$, we must show that there exists an $r_0 > 0$ such that $B_X(x_0, r_0) \subseteq f^{-1}(V)$ (cf. Theorem 12.2.15(a)).

  Consider this arbitrary $x_0 \in f^{-1}(V)$. By definition, we have $f(x_0) \in V$. But since V is an open set, there exists an $\varepsilon > 0$ such that $B_Y(f(x_0), \varepsilon) \subseteq V$.

But f is continuous: for this $\varepsilon > 0$, there exists a $\delta > 0$ such that, for $x \in X$, we have $d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon$. In other words, we have $x \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon) \subseteq V$.

Thus, if we set $r_0 := \delta$, we are done: for all $x \in B_X(x_0, r_0)$, we have $f(x) \in V$, i.e. $x \in f^{-1}(V)$. This shows that $B_X(x_0, \delta) \subseteq f^{-1}(V)$, and thus that $f^{-1}(V)$ is an open set, as expected.

- Now we show that $(c) \implies (d)$. By Theorem 12.2.15(e), we know that $F \subseteq X$ is closed iff $X \backslash F$ is open. Thus, consider $F \subseteq Y$ a closed set in Y. Let be $V := Y \backslash F$ its complementary set, which is thus an open set. By (c), the set $f^{-1}(V)$ is an open set in X. But we have :

$$\begin{aligned} f^{-1}(F) &= \{x \in X : f(x) \in F\} \\ &= \{x \in X : f(x) \in Y \backslash V\} \\ &= \{x \in X : f(x) \notin V\} \end{aligned}$$

  so that $f^{-1}(F) = X \backslash f^{-1}(V)$. Since $f^{-1}(F)$ is the complementary set of the open set $f^{-1}(V)$, it is closed in X, as expected.

- The implication $(d) \implies (c)$ can be shown in exactly the same way as above.

- Finally, let's show that $(c) \implies (a)$. Let be $\varepsilon > 0$, let be $x_0 \in X$. Consider $V := B_Y(f(x_0), \varepsilon)$, which is an open set in Y. By (c), the set $f^{-1}(V)$ is open in X. Thus, by Theorem 12.2.15(a), there exists a $\delta > 0$ such that $B_X(x_0, \delta) \subseteq f^{-1}(V)$. Thus, if $x \in B_X(x_0, \delta)$, we have $f(x) \in V$.

  In other words, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x0)) < \varepsilon$. This shows that f is continuous at x_0, for any arbitrary $x_0 \in X$, as expected.

Exercise 13.1.3. — *Use Theorem 13.1.4 and Theorem 13.1.5 to prove Corollary 13.1.7.*

To show (a), consider $(x^{(n)})_{n=1}^{\infty}$ a sequence of elements of X that converges to $x_0 \in X$. Since f is continuous at x_0, then by Theorem 13.1.4(b), we know that $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0) \in Y$. But $(f(x^{(n)}))_{n=1}^{\infty}$ is a sequence of elements of Y. Since g is continuous at $f(x_0)$, then still by Theorem 13.1.4(b), we know that $(g(f(x^{(n)})))_{n=1}^{\infty}$ converges to $g(f(x_0)) \in Z$.

Thus, we have proved that for any sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X that converges to $x_0 \in X$, the sequence $(g \circ f(x^{(n)}))_{n=1}^{\infty}$ converges to $g \circ f(x_0)$. This shows that $g \circ f$ is continuous at x_0, as expected.

Once (a) is proved, the result (b) is clear, since it is just (a) at any arbitrary $x_0 \in X$.

Exercise 13.1.4. — *Give an example of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that (a) f is not continuous, but g and $g \circ f$ are continuous; (b) g is not continuous, but f and $g \circ f$ are continuous; (c) f and g are not continuous, but $g \circ f$ is continuous. Explain briefly why these examples do not contradict Corollary 13.1.7.*

Here, the simplest way is to use piecewise constant functions, at least for one of the functions f, g.

(a) Let be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

and the constant function $g(x) := 3$. We thus have $g \circ f(x) = 3$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

(b) Let be, for instance,

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0 \end{cases}$$

and $f(x) := x^2 + 1$. We thus have $g \circ f(x) = 1$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

(c) Let be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 3 & \text{if } x \geqslant 0 \end{cases}$$

and

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geqslant 0. \end{cases}$$

We thus have $g \circ f(x) = 1$ for all $x \in \mathbb{R}$, so that $g \circ f$ is continuous.

This does not contradict Corollary 13.1.7, since the initial hypothesis of this corollary is that both functions f, g are continuous, and it says nothing about non discontinuous functions.

Exercise 13.1.5. — *Let (X, d) be a metric space, and let $(E, d|_{E\times E})$ be a subspace of (X, d). Let $\iota_{E\to X} : E \to X$ be the inclusion map, defined by setting $\iota_{E\to X}(x) := x$ for all $x \in E$. Show that $\iota_{E\to X}$ is continuous.*

Let be $x_0 \in E$ an arbitrary point in E, and let be $\varepsilon > 0$ a positive real number. Note that we have, for all $x \in E$,

$$d(\iota_{E\to X}(x_0), \iota_{E\to X}(x)) = d_{E\times E}(x_0, x).$$

Thus, if we take $\delta := \varepsilon$ in Definition 13.1.1 of continuity, we are done: if $d_{E\times E}(x_0, x) < \varepsilon$, we automatically have $d(\iota_{E\to X}(x_0), \iota_{E\to X}(x)) < \varepsilon$, so that $\iota_{E\to X}$ is continuous at any arbitrary $x_0 \in E$, as expected.

Exercise 13.1.6. — *Let $f : X \to Y$ be a function from one metric space (X, d_X) to another (Y, d_Y). Let E be a subset of X (which we give the induced metric $d_X|_{E\times E}$), and let $f|_E : E \to Y$ be the restriction of f to E, thus $f|_E(x) := f(x)$ when $x \in E$. If $x_0 \in E$ and f is continuous at x_0, show that $f|_E$ is also continuous at x_0. (Is the converse of this statement true? Explain.) Conclude that if f is continuous, then $f|_E$ is continuous.*

Let's use Exercise 13.1.5. First we note that we have $f|_E = f \circ \iota_{E\to X}$. Indeed, $f \circ \iota_{E\to X}$ is a function from E to Y just like f, and for all $x \in E$, we clearly have $f \circ \iota E \to X(x) = f(x) = f|_E(x)$.

We have shown in Exercise 13.1.5 that $\iota_{E\to X}$ is continuous at any $x_0 \in E$, and f is supposed to be continuous at $x_0 \in E$. Thus, by Corollary 13.1.7, $f|_E$ is continuous at x_0 since it is the composition of two continuous functions. Since this is true for any arbitrary $x_0 \in E$, the function $f|_E$ is continuous on E.

The converse statement is not true: consider the piecewise constant function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geqslant 0$ for all $x \in \mathbb{R}$; and let be $E := [0, +\infty[$. The restriction $f|_E$ is clearly continuous (as a constant function) at 0, but the function f itself is clearly not continuous at 0.

Exercise 13.2.1. — *Prove Lemma 13.2.1.*

Here we just have to prove the statement (a), since the statement (b) is essentially (a) applied to any arbitrary $x_0 \in X$.

- First suppose that f and g are both continuous at $x_0 \in X$, and let be $(x^{(n)})_{n=1}^{\infty}$ a sequence of elements of X. Then, by Theorem 13.1.4(b), the sequence $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ in $\mathbb{R}$, and the sequence $(g(x^{(n)}))_{n=1}^{\infty}$ converges to $g(x_0)$ in $\mathbb{R}$.

  Thus, by Theorem 12.1.18, the sequence $((f(x^{(n)}), g(x^{(n)})))_{n=1}^{\infty}$ of elements of $\mathbb{R}^2$ converges to $(f(x_0), g(x_0))$ in $\mathbb{R}^2$ with respect to the metric d_{l^2} (since it converges componentwise). In other words, for any arbitrary sequence $(x^{(n)})_{n=1}^{\infty}$ that converges to x_0, the sequence $(f \oplus g(x^{(n)}))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0)) = f \oplus g(x_0)$. Thus, $f \oplus g$ is continuous at x_0, by Theorem 13.1.4.

- Conversely, if $f \oplus g$ is continuous at x_0, then for any sequence $(x^{(n)})_{n=1}^{\infty}$ of elements of X, the sequence $(f \oplus g(x^{(n)}))_{n=1}^{\infty}$ converges to $(f(x_0), g(x_0))$, by Theorem 13.1.4. Thus, by Theorem 12.1.18, $(f(x^{(n)}))_{n=1}^{\infty}$ converges to $f(x_0)$ in $\mathbb{R}$, and $(g(x^{(n)}))_{n=1}^{\infty}$ converges to $g(x_0)$ in $\mathbb{R}$. Thus, f and g are both continuous at x_0.