Propositions of solutions for *Analysis I* by Terence Tao

Frédéric Santos

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1. Introduction

No exercises in this chapter.

2. The natural numbers

Exercise 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a + b) + 0 = a + (b + 0)$. The left hand side is equal to $(a + b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b + 0)$ part, we get $a + (b + 0) = a + b$. Both sides are equal to $a + b$, and the base case is thus done.

- Now let's suppose inductively that $(a + b) + c = a + (b + c)$: we have to prove that $(a + b) + c{++} = a + (b + c{++})$. Using Lemma 2.2.3 on the right hand side leads to $a + (b + c){++}$. Now consider the left hand side. Using still the same lemma, we get $(a + b) + c{++} = ((a + b) + c){++}$. By the inductive hypothesis, this is also equal to $(a + (b + c)){++}$. And, using the lemma 2.2.3 again, this also leads to $a + b + c{++}$. Therefore, both sides are equal to $a + b + c{++}$, and we have closed the induction.

Exercise 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b{++} = a$.*

Let's use induction on a.

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0{++} = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b{++} = 1$. Then, we would have $b{++} = 0{++}$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.

- Let's suppose inductively that there is exactly one natural number b such that $b{++} = a$. We have to prove that there is exactly one natural number b' such that $b'{++} = a{++}$. By the induction hypothesis, and taking $b' = b{++}$, we have $b'{++} = (b{++}){++} = a{++}$. So there exists a solution, with $b' = b{++} = a$. Uniqueness is given by Axiom 2.4.: if $b'{++} = a{++}$, then we necessarily have $b' = a$.

EXERCISE 2.2.3. — *Let a, b, c be natural numbers. Prove the following properties of order for natural numbers:*

(a) Reflexivity: $a \geqslant a$. This is true since $a = 0 + a$ by Definition 2.2.1. By commutativity of addition, we can also write $a = a + 0$. So there is indeed a natural number n (with $n = 0$) such that $a = a + n$, i.e. $a \geqslant a$.

(b) Transitivity: if $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$ according to Definition 2.2.11. A similar consideration for the part $b \geqslant c$ leads to $b = c + m$, m being a natural number. Combining together those two equalities, we can write $a = b + n = (c + m) + n = c + (m + n)$ by associativity (see Exercise 2.2.1). Then, $n + m$ being a natural number[1], the transitivity is demonstrated.

(c) Anti-symmetry: if $a \geqslant b$ and $b \geqslant a$, then $a = b$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$. Similarly, there exists a natural number m such that $b = a + m$. Combining those two equalities leads to $a = b + n = (a + m) + n = a + (m + n)$. By cancellation law (Proposition 2.2.6), we can conclude that $0 = m + n$. According to Corollary 2.2.9, this leads to $m = n = 0$. Therefore, both m and n are null, meaning that $a = b + 0 = b$.

(d) Preservation of order: $a \geqslant b$ iff $a+c \geqslant b+c$. First, let's prove that $a+c \geqslant b+c \Longrightarrow a \geqslant b$. If $a+c \geqslant b+c$, there exists a natural number n such that $a+c = b+c+n$. By cancellation law (Proposition 2.2.6)[2], we conclude that $a = b + n$, i.e. $a \geqslant b$, thus demonstrating the first implication. Conversely, let's suppose that $a \geqslant b$. There exists a natural number m such that $a = b + m$. Therefore, $a + c = b + m + c$ for any natural number c. Still by associativity and commutativity, we can rewrite this as $a + c = (b + c) + m$, i.e. $a + c \geqslant b + c$.

(e) $a < b$ iff $a{+}{+} \leqslant b$. First, let's prove that $a{+}{+} \leqslant b \Longrightarrow a < b$. By definition of ordering, there exists a natural number n such that $b = (a{+}{+}) + n$. By definition of addition, we can re-write: $b = (a{+}{+} + n){+}{+}$. Then, by commutativity and yet again by definition of addition, $b = (n + a{+}{+}){+}{+} = (n{+}{+}) + (a{+}{+})$. Thus, there exists a natural number $n{+}{+}$ such that $b = n{+}{+} + a$, which means that $b \geqslant a$. But we still have to prove that $a \neq b$. Let's suppose that $a = b$: in this case, by cancellation law, we would have $n{+}{+} = 0$, which is impossible according to Axiom 2.3 (0 is not the successor of any natural number). Thus, $a \neq b$ et $b \geqslant a$: we have showed that $a < b$.

Conversely, let's prove that $a < b \Longrightarrow a{+}{+} \leqslant b$. Starting from that strict inequality, there exists a *positive*[3] natural number n such that $b = a + n$. By Lemma 2.2.10, since n is positive, it has one unique antecessor m, so that n can be written $m{+}{+}$. Thus, $b = a + (m{+}{+}) = (a + m){+}{+} = (m + a){+}{+} = m + (a{+}{+}) = (a{+}{+}) + m$. And, m being a natural number, this corresponds to the statement $b \geqslant a$.

(f) $a < b$ iff $b = a + d$ for some positive number d. First, let's prove the first implication, $a < b \Longrightarrow b = a + d$ with $d \neq 0$. Since $a < b$, we have in particular $a \leqslant b$, and

[1] This is a trivial induction from the definition of addition.

[2] And also associativity and commutativity that we do not detail explicitly here.

[3] We make use here of the statement *(f)* demonstrated below. There is no circularity here, since proving *(f)* will not make use of *(e)*.

there exists a natural number d such that $b = a + d$. For the sake of contradiction, let's suppose that $d = 0$. We would have $b = a$, which would contradict the condition $a \neq b$ of the strict inequality. Thus, d is a positive number, which demonstrates the left-to-right implication.

Conversely, let's suppose that $b = a + d$, with $d \neq 0$. This expression gives immediately $a \leqslant b$. But if $a = b$, by cancellation law, this would lead to $0 = d$, a contradiction with the fact that d is a positive number. Thus, $a \neq b$ and $a \leqslant b$, which demonstrates $a < b$.

Exercise 2.2.4. — *Demonstrate three lemmas used to prove the trichotomy of order for natural numbers.*

(a) Show that we have $0 \leqslant b$ for any natural number b. This is obvious since, by definition of addition, $0 + b = b$ for any natural number b. This is precisely the definition of $0 \leqslant b$.

(b) Show that if $a > b$, then $a{+\!+} > b$. If $a > b$, then $a = b + d$, d being a positive natural number. Let's recall that $a{+\!+} = a + 1$. Thus, $a{+\!+} = a + 1 = b + d + 1 = b + (d + 1)$ by associativity of addition. Furthermore, $d+1$ is a positive natural number (by Proposition 2.2.8). Thus, $a{+\!+} > b$.

(c) Show that if $a = b$, then $a{+\!+} > b$. Once again, let's use the fact that $a{+\!+} = a + 1$. Thus, $a{+\!+} = a + 1 = b + 1$, and 1 is a positive natural number. This is the definition of $a{+\!+} > b$.

Exercise 2.2.5. — *Prove the strong principle of induction, formulated as follows: Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geqslant m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leqslant m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geqslant m_0$.*

First let's introduce a small lemma (similar to Proposition 2.2.12*(e)*).

Lemma. *For any natural number a and b, $a < b{+\!+}$ iff $a \leqslant b$.*

Proof. If $a < b{+\!+}$, then $b{+\!+} = a + n$ for a given positive natural n. By Lemma 2.2.10, there exists one natural number m such as $n = m{+\!+}$. Thus $b{+\!+} = a + m{+\!+}$, which can be rewritten $b{+\!+} = (a + m){+\!+}$ by Lemma 2.2.3[4]. By Axiom 2.4., this is equivalent to $b = a + n$, which can also be written $a \leqslant b$.

Conversely, if $a \leqslant b$, there exists a natural number m such as $b = a + m$. Thus, $b{+\!+} = (a + m){+\!+} = a + (m{+\!+})$ by Definition of addition (2.2.1). And, $m{+\!+}$ being a positive number, this means that $b > a$ according to Proposition 2.2.12*(f)*. □

Now we can prove the main proposition. Let $Q(n)$ be the property "$P(m)$ is true for all m such that $m_0 \leqslant m < n$". Let's induct on n.

- (Although this is not necessary,) we could consider two types of base cases. If $n < m_0$, $Q(n)$ is the proposition "$P(m)$ is true for all m such that $m_0 \leqslant m < n$", but there is no such natural number m. Thus, $Q(n)$ is vacuously true. If $n = m_0$, $P(m_0)$ is true by hypothesis, thus $Q(m_0)$ is also true.

[4] We could also rewrite $b + 1 = a + m + 1$ and then use the cancellation law.

- Now let's suppose inductively that $Q(n)$ is true, and show that $Q(n{+\!+})$ is also true. If $Q(n)$ is true, $P(m)$ is true for all m such that $m_0 \leqslant m < n$. By hypothesis, this implies that $P(n)$ is true. Thus, $P(m)$ is true for any natural number m such that $m_0 \leqslant m \leqslant n$, i.e. such that $m_0 \leqslant m < n{+\!+}$ according to the lemma introduced above. This is precisely $Q(n{+\!+})$, and this closes the induction.

Thus, $Q(n)$ is true for all natural numbers n, which means in particular that $P(m)$ is true for any natural number $m \geqslant m_0$. This demonstrates the principle of strong induction.

Exercise 2.2.6. — *Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m{+\!+})$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leqslant n$; this is known as the* principle of backwards induction.

Terence Tao suggests to use induction on n. So let $Q(n)$ be the following property: "if $P(n)$ is true, then $P(m)$ is true for all $m \leqslant n$. The goal is to prove $Q(n)$ for all natural numbers n.

- Base case $n = 0$: here, $Q(n)$ means that if $P(0)$ is true, then $P(m)$ is true for any $m \leqslant 0$. By Definition 2.2.11, if $m \leqslant 0$, there exists a natural number d such that $0 = m + d$. But, by Corollary 2.2.9, this implies that both $m = 0$ and $d = 0$. Thus, the only number m such that $m \leqslant 0$ is 0 itself. Therefore, $Q(0)$ is simply the tautology "if $P(0)$ is true, then $P(0)$ is true"— a statement that we can safely accept. The base case is the, demonstrated.

- Let's suppose inductively that $Q(n)$ is true: we must show that $Q(n{+\!+})$ is also true. If $P(n{+\!+})$ is true, then by definition of P, $P(n)$ is also true. Then, by induction hypothesis, $P(m)$ is true for all $m \leqslant n$. We have showed that $P(n{+\!+})$ implies $P(m)$ for all $m \leqslant n{+\!+}$[5], which is precisely $Q(n{+\!+})$. This closes the induction.

Exercise 2.3.1. — *Show that multiplication is commutative, i.e., if n and m are natural numbers, show that $n \times m = m \times n$.*

We will use an induction of n while keeping m fixed. However, this is not a trivial result, and even the base case is not straightforward. We will first introduce some lemmas.

Lemma. *For any natural number n, $n \times 0 = 0$.*

Proof. Let's induct on n. For the base case $n = 0$, we know by Definition 2.3.1 of multiplication that $0 \times 0 = 0$, since $0 \times m = 0$ for any natural number m.

Now let's suppose that $n \times 0 = 0$. Thus, $n{+\!+} \times 0 = (n \times 0) + 0$ by Definition 2.3.1. But by induction hypothesis, $n \times 0 = 0$, so that $n{+\!+} \times 0 = 0 + 0 = 0$. This closes the induction. ☐

Lemma. *For all natural numbers m and n, we have $m \times n{+\!+} = (m \times n) + m$.*

[5] Actually, we use here yet another lemma, similar to the one introduced for the previous exercise. We use the fact that $m \leqslant n{+\!+}$ is equivalent to $m = n{+\!+}$ or $m \leqslant n$, which is easy to prove, but is not part of the "standard" results presented in the textbook.

Proof. Let's induct on m. The base case $m = 0$ is easy to prove: $0 \times n{+}{+} = 0$ by Definition 2.3.1 of multiplication, and $(0 \times n) + 0 = 0$.

Now suppose inductively that $m \times n{+}{+} = (m \times n) + m$, and we must show that

$$m{+}{+} \times n{+}{+} = (m{+}{+} \times n) + m{+}{+} \tag{1}$$

We begin by the left hand side: by Definition 2.3.1, $m{+}{+} \times n{+}{+} = (m \times n{+}{+}) + n{+}{+}$. By induction hypothesis, this is equal to $(m \times n) + m + n{+}{+}$.

Then, apply the definition of multiplication to the right hand side: $(m{+}{+} \times n) + m{+}{+} = (m \times n) + n + m{+}{+}$. The Lemma 2.2.3 and the commutativity of addition leads to $(m \times n) + n + m{+}{+} = (m \times n) + (n + m){+}{+} = (m \times n) + (m + n){+}{+} = (m \times n) + m + n{+}{+}$, which is equal to the left hand side.

Thus, both sides of equation (1) are equal, and we can close the induction. □

Now it is easier to prove the main result ($n \times m = m \times n$), by an induction on n.

- Base case $n = 0$: we already know by Definition 2.3.1 that $0 \times m = 0$. The first lemma introduced in this exercise also provides $m \times 0 = 0$. Thus, the base case is proven, since $0 \times m = m \times 0$ $(= 0)$.

- Now we suppose inductively that $n \times m = m \times n$, and we must prove that:

  $$n{+}{+} \times m = m \times n{+}{+} \tag{2}$$

  By Definition 2.3.1 of multiplication, the left hand side is equal to $(n \times m) + m$.

  Using the lemma introduced above, the right hand side is equal to $(m \times n) + m$. By induction hypothesis, this is also equal to $(n \times m) + m$, which closes the induction.