Propositions of solutions for *Analysis II* by Terence Tao

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Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

Exercise 12.1.4. — *Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y\times Y}(x, y) := d(x, y)$, then the application $d|_{Y\times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y\times Y})$ is indeed a metric space.

Exercise 12.1.5. — *Let $n \geqslant 1$, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Verify the identity $\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.*

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n.

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geqslant 1$, and let's prove that it is still true for $n + 1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i\right)^2}_{:=A} + \underbrace{\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right)}_{:=C} \tag{12.1}$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A, we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i\right)^2 \\ &= \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2 \\ &= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \end{aligned}$$

- for B, we have

$$\begin{aligned} B &:= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n+1}(a_{n+1}b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2\times S} + \underbrace{\frac{1}{2}\sum_{j=1}^{n}(a_{n+1}b_j - a_j b_{n+1})^2}_{:=1/2\times S} \\ &\quad + \underbrace{\frac{1}{2}(a_{n+1}b_{n+1} - b_{n+1}a_{n+1})^2}_{=0} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for $A+B$, we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A+B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^n a_ib_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_ib_j - a_jb_i)^2 + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^n a_ib_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_ib_j - a_jb_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2\sum_{i=1}^n a_ia_{n+1}b_ib_{n+1} + \sum_{i=1}^n (a_i^2b_{n+1}^2 - 2a_ib_{n+1}a_{n+1}b_i + a_{n+1}^2b_i^2) \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + \sum_{i=1}^n (a_i^2b_{n+1}^2 + a_{n+1}^2b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number n.

(ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left|\sum_{i=1}^n a_ib_i\right| \leqslant \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}. \tag{12.2}$$

Indeed, since $B \geqslant 0$ in the identity (12.1), we have

$$\left(\sum_{i=1}^n a_ib_i\right)^2 \leqslant \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality. We have

$$\begin{aligned}
\sum_{i=1}^{n}(a_i^2+b_i^2) &= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \\
&\leqslant \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(by eq. (12.2))} \\
&\leqslant \left(\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

Exercise 12.1.6. — *Show that $(\mathbb{R}^n, d_{l^2})$ in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $(x_i - y_i)^2 > 0$, and $d_{l^2}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(y,x) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_{l^2}(x,y)$$

as expected.

(d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned}
d_{l^2}(x,z) &:= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} && \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(Exercise 12.1.5(iii))} \\
&\leqslant \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2} \\
&\leqslant d_{l^2}(x,y) + d_{l^2}(y,z)
\end{aligned}$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^2})$ is indeed a metric space.

Exercise 12.1.7. — *Show that $(\mathbb{R}^n, d_{l^1})$ in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x,x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x,y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y,x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x,y)$$

as expected.

(d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a-c| \leqslant |a-b| + |b-c|$ for all $a, b, c \in \mathbb{R}$. Thus, for all $x, y, z \in \mathbb{R}^n$, we have

$$d_{l^1}(x,z) := \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x,y) + d_{l^1}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x,y)^2 \leqslant d_{l^1}(x,y)^2$, and we will prove it in this form.

  Indeed, using a trivial product expansion, we have

$$\begin{aligned}
d_{l_1}(x,y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\
&= \left(\sum_{i=1}^n |x_i - y_i|\right) \times \left(\sum_{i=1}^n |x_i - y_i|\right) \\
&= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leqslant i,j \leqslant n;\, i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geqslant 0} \\
&\geqslant \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x,y)^2
\end{aligned}$$

  as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned} d_{l^1}(x,y) &:= \sum_{i=1}^{n} |x_i - y_i| \\ &= \left| \sum_{i=1}^{n} |x_i - y_i| \times 1 \right| \\ &\leqslant \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} \\ &\leqslant d_{l^2}(x,y) \times \sqrt{n} \end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair* $(\mathbb{R}^n, d_{l^\infty})$ *in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we clearly have $d_{l^\infty}(x,x) = \sup\{|x_i - x_i| : 1 \leqslant i \leqslant n\} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant j \leqslant n$ such that $x_j \neq y_j$. Thus $|x_j - y_j| > 0$, and $d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} \geqslant |x_j - y_j| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} = \sup\{|y_i - x_i| : 1 \leqslant i \leqslant n\} = d_{l^\infty}(y,x)$$

as expected.

(d) Triangle inequality. Let be $x, y, z \in \mathbb{R}^n$. We have $|x_i - z_i| \leqslant |x_i - y_i| + |y_i - z_i|$ for all $1 \leqslant i \leqslant n$, by Proposition 4.3.3(g). But, by definition of the supremum, we have $|x_i - y_i| \leqslant d_{l^\infty}(x,y)$ and $|y_i - z_i| \leqslant d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$. Thus, we have $|x_i - z_i| \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$; i.e., $d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ is an upper bound of the set $\{|x_i - z_i| : 1 \leqslant i \leqslant n\}$. By definition of the supremum, it implies that

$$d_{l^\infty}(x,z) := \sup\{|x_i - z_i| : 1 \leqslant i \leqslant n\} \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y).$$

First, a preliminary remark. By definition, we have $d_{l^\infty}(x,y) := \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\}$ for all $x, y \in \mathbb{R}^n$. Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a $1 \leqslant m \leqslant n$ such that $d_{l^\infty}(x,y) = |x_m - y_m|$ (the supremum belongs to the set). The index "m" will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}\frac{1}{\sqrt{n}} d_{l^2}(x,y) &:= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2} \\ &\leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_m - y_m)^2} \\ &\leqslant \sqrt{\frac{n}{n}(x_m - y_m)^2} \\ &= |x_m - y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}d_{l^2}(x,y) &:= \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} \\ &= \sqrt{(x_m - y_m)^2 + \sum_{1 \leqslant i \leqslant n;\, i \neq m}(x_i - y_i)^2} \\ &\geqslant \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

Exercise 12.1.11. — *Show that the discrete metric (X, d_{disc}) in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in X$, we have $d_{\text{disc}}(x,x) := 0$ by definition, so that there is nothing to prove here.

(b) Positivity: for all $x \neq y \in X$, we have $d_{\text{disc}}(x,y) := 1 > 0$ by definition, so that there's still nothing to prove.

(c) Symmetry: for all $x, y \in X$, we have $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x) = 1$, so that d_{disc} obeys the symmetry property.

(d) Triangle inequality. Let be $x, y, z \in X$, and let's consider $d_{\text{disc}}(x,z)$.

  - If $x = z$, then $d_{\text{disc}}(x,z) = 0$. And since d_{disc} is a non-negative application, we clearly have $0 =: d_{\text{disc}}(x,z) \leqslant d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$ for all $y \in X$.
  - If $x \neq z$, then we cannot have both $x = y$ and $y = z$ (it would be a clear contradiction with $x \neq z$). Thus, at least one of the propositions "$x \neq y$", "$y \neq z$" is true. Another way to say that is $d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z) \geqslant 1$. But since $d_{\text{disc}}(x,z) := 1$, we have actually $d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z) \geqslant d_{\text{disc}}(x,z)$, as expected.

Exercise 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all $x, y \in \mathbb{R}^n$, we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y).$$

Note that n is a real constant here.

- Let's prove that $(a) \implies (b)$. If $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, then by the limit laws, the sequence $t_k := \sqrt{n}\, d_{l^2}(x^{(k)}, x)$ also converges to 0 as $k \to \infty$, since $\sqrt{n}$ is a constant real number. Thus, we have

  $$d_{l^2}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x) \leqslant \sqrt{n}\, d_{l^2}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x)$ as expected.

- Let's prove that $(b) \implies (c)$. If $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x) = 0$, then we have

  $$0 \leqslant d_{l^\infty}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x)$ as expected.

- Let's prove that $(c) \implies (d)$. Suppose that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x) = 0$. Then, for all $1 \leqslant j \leqslant n$, we have $0 \leqslant |x_j^k - x_j| \leqslant d_{l^\infty}(x^{(k)}, x)$. Still by the squeeze test, this implies that $\lim_{k\to\infty} |x_j^k - x_j| = 0$, i.e. that $(x_j^k)_{k=m}^\infty$ converges to x_j as $k \to \infty$ (by Lemma 12.1.1), as expected.

- Finally, let's prove that $(d) \implies (a)$. Using the definition of convergence is more appropriate here. Let be $\varepsilon > 0$ a positive real number, and let be $1 \leqslant j \leqslant n$. By definition, there exists a natural number $N \geqslant m$ such that $|x_j^{(k)} - x_j| \leqslant \varepsilon/\sqrt{n}$ whenever $k \geqslant N$. Thus, if $k \geqslant N$, we have

  $$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^{n} (x_j^{(k)} - x_j)^2} \leqslant \sqrt{\sum_{j=1}^{n} \frac{\varepsilon^2}{n}} \leqslant \varepsilon$$

  so that $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, i.e., $(x^k)_{k=m}^\infty$ converges to x as $k \to \infty$ in the l^2 metric (by Lemma 12.1.1), as expected.

Exercise 12.1.13. — *Prove Proposition 12.1.19.*

Let be $(x^{(n)})_{n=m}^\infty$ a sequence of elements of a set X.

- First suppose that $(x^{(n)})_{n=m}^\infty$ is eventually constant. Thus, by definition, there exists an $N \geqslant m$ and an element $x \in X$ such that $(x^{(n)})_{n=m}^\infty = x$ for all $n \geqslant N$. This implies that we have $d_{\text{disc}}(x^{(n)}, x) = 0$ for all $n \geqslant N$. In particular, for all $\varepsilon > 0$, we have $d_{\text{disc}}(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N$, so that $(x^{(n)})_{n=m}^\infty$ indeed converges to x with respect to d_{disc}.

- Conversely, suppose that $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d_{disc}. Let be $\varepsilon = 1/2$. By definition, there exists an $N \geqslant m$ such that $d_{\text{disc}}(x^{(n)}, x) \leqslant 1/2$ whenever $n \geqslant N$. Since $d_{\text{disc}}(x^{(n)}, x)$ cannot be 1, it is necessarily equal to 0, so that $x^{(n)} = x$ whenever $n \geqslant N$. Thus, the sequence $x^{(n)}$ is indeed eventually constant.

Exercise 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have $\lim_{n\to\infty} d(x^{(n)}, x) = 0$ and $\lim_{n\to\infty} d(x^{(n)}, x') = 0$. Suppose, for the sake of contradiction, that we have $x \neq x'$. Thus, the real number $\varepsilon := \frac{d(x,x')}{3}$ is positive.

Since $x^{(n)}$ converges to x, there exists a $N_1 \geqslant m$ such that $d(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N_1$.

Similarly, since $x^{(n)}$ converges to x', there exists a $N_2 \geqslant m$ such that $d(x^{(n)}, x') \leqslant \varepsilon$ whenever $n \geqslant N_2$.

By the triangle inequality, we thus have, for all $n \geqslant \max(N_1, N_2)$,

$$d(x, x') \leqslant d(x, x^{(n)}) + d(x^{(n)}, x') \leqslant \varepsilon + \varepsilon = \frac{2}{3} d(x, x')$$

which is a contradiction (since $d(x, x') > 0$ by hypothesis).

Thus, the limit is unique, and we must have $x = x'$.

Exercise 12.1.15. — *Let be $X := \{(a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty\}$. We define on this space the metrics $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|$, and $d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n\in\mathbb{N}} |a_n - b_n|$. Then...*

We have to prove the following statements.

1. d_{l^1} is a metric on X.

   We have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^{\infty} \in X$. We have $d_{l^1}((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^1}((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |b_n - a_n| = \sum_{n=0}^{\infty} |a_n - b_n| = d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately

   $$\begin{aligned} d_{l^1}((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}) &:= \sum_{n=0}^{\infty} |a_n - c_n| \\ &\leqslant \sum_{n=0}^{\infty} (|a_n - b_n| + |b_n - c_n|) \ \ \text{(consequence of Prop. 7.1.11(h))} \\ &\leqslant \sum_{n=0}^{\infty} |a_n - b_n| + \sum_{n=0}^{\infty} |b_n - c_n| \ \ \text{(by Proposition 7.2.14(a))} \\ &\leqslant d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) + d_{l^1}((b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}). \end{aligned}$$

   Thus, d_{l^1} is indeed a metric on X.

2. d_{l^∞} is a metric on X.

   Once again, we have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |b_n - a_n| = \sup_{n\in\mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately $|a_m - c_m| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$ for all $m \in \mathbb{N}$, by definition of the supremum. In other words, $(\sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|)$ is an upper bound for the set $\{|a_m - c_m| : m \in \mathbb{N}\}$. Thus we have, still by definition of the supremum, $\sup_{n\in\mathbb{N}} |a_n - c_n| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$, as expected.

   Thus, d_{l^∞} is indeed a metric on X.

3. There exist sequences $x^{(1)}$, $x^{(2)}$, ..., of elements of X (i.e., sequences of sequences) which are convergent with respect to d_{l^∞}, but are not convergent with respect to d_{l^1}.

   Here we are dealing with sequences of sequences: we have a sequence $(x^{(k)})_{k=1}^\infty$ where each $x^{(k)}$ is itself a sequence of real numbers. Thus, let's define $(x^{(k)})_{k=1}^\infty$ as follows:

   $$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

   Just to make things clearer, we have for instance

   $$\begin{aligned} x^{(1)} &:= \frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ 0,\ \ldots \\ x^{(2)} &:= \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0,\ 0,\ \ldots \\ x^{(3)} &:= \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 0,\ \ldots \end{aligned}$$

   Also, let be the null sequence $(a_n)_{n=0}^\infty$ defined by $a_n := 0$ for all $n \in \mathbb{N}$. Thus:

   - $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞}. Indeed, if we consider a given positive integer k (fixed), we have

     $$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

     so that $d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) := \sup_{n\in\mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$.

     Thus, $\lim_{k\to\infty} d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = 0$, or in other words, $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} in X.

- But $(x^{(k)})_{k=1}^{\infty}$ does not converges to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}. Indeed, we have, for each given (fixed) k,

$$d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right) = \sum_{n=0}^{k} \frac{1}{k+1} = 1$$

  Thus, we clearly do not have $\lim_{k\to\infty} d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}\right) = 0$, i.e., $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

4. Conversely, any sequence which converges with respect to d_{l^1} also converges with respect to d_{l^∞}.

   Suppose, for the sake of contradiction, that $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^∞}, but does converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

   In this case, there exists a $\varepsilon > 0$ such that, for all $k \geqslant 1$, we have $(\sup_{n\geqslant 0} |x_n^{(k)} - a_n|) > \varepsilon$. In particulier, for all $k \geqslant 1$ and all $n \geqslant 0$, we have $|x_n^{(k)} - a_n| > \varepsilon$. Thus, $\sum_{n=0}^{\infty} |x_n^{(k)} - a_n|$ is not even a convergent series, and we cannot have $\lim_{k\to\infty}\left(\sum_{n=0}^{\infty} |x_n^{(k)} - a_n|\right) = 0$.

Note that this exercise actually shows that in this space X, the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

Exercise 12.1.16. — *Let* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *be two sequences in a metric space* (X, d). *Suppose that* $(x_n)_{n=1}^{\infty}$ *converges to a point* $x \in X$, *and* $(y_n)_{n=1}^{\infty}$ *converges to a point* $y \in X$. *Show that* $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.

On the one hand, the triangle inequality applied two times to d gives us

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leqslant d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leqslant d(x_n, y_n) - d(x, y) \leqslant d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y).$$

Let be $\varepsilon > 0$. By hypothesis, there exists a $N_1 \geqslant 1$ such that $d(x_n, x) \leqslant \varepsilon/2$ whenever $n \geqslant N_1$. Similarly, there exists a $N_2 \geqslant 1$ such that $d(y_n, y) \leqslant \varepsilon/2$ whenever nN_2. Thus, if we set $N := \max(N_1, N_2)$, then for all $n \geqslant N$ we have

$$|d(x_n, y_n) - d(x, y)| \leqslant d(x_n, x) + d(y_n, y) \leqslant 2\varepsilon/2 \; leq\varepsilon$$

which shows that $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$, as expected.