Propositions of solutions for *Analysis I* by Terence Tao

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1. Introduction

No exercises in this chapter.

2. The natural numbers

Exercise 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c, we have $(a+b)+c = a+(b+c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a+b)+0 = a+(b+0)$. The left hand side is equal to $(a+b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b+0)$ part, we get $a+(b+0) = a+b$. Both sides are equal to $a+b$, and the base case is thus done.

- Now let's suppose inductively that $(a+b)+c = a+(b+c)$: we have to prove that $(a+b)+c\mathrel{++} = a+(b+c\mathrel{++})$. Using Lemma 2.2.3 on the right hand side leads to $a+(b+c)\mathrel{++}$. Now consider the left hand side. Using still the same lemma, we get $(a+b)+c\mathrel{++} = ((a+b)+c)\mathrel{++}$. By the inductive hypothesis, this is also equal to $(a+(b+c))\mathrel{++}$. And, using the lemma 2.2.3 again, this also leads to $a+b+c\mathrel{++}$. Therefore, both sides are equal to $a+b+c\mathrel{++}$, and we have closed the induction.

Exercise 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b\mathrel{++} = a$.*

Let's use induction on a.

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0\mathrel{++} = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b\mathrel{++} = 1$. Then, we would have $b\mathrel{++} = 0\mathrel{++}$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.

- Let's suppose inductively that there is exactly one natural number b such that $b\mathrel{++} = a$. We have to prove that there is exactly one natural number b' such that $b'\mathrel{++} = a\mathrel{++}$. By the induction hypothesis, and taking $b' = b\mathrel{++}$, we have $b'\mathrel{++} = (b\mathrel{++})\mathrel{++} = a\mathrel{++}$. So there exists a solution, with $b' = b\mathrel{++} = a$. Uniqueness is given by Axiom 2.4.: if $b'\mathrel{++} = a\mathrel{++}$, then we necessarily have $b' = a$.

Exercise 2.2.3. — *Let a, b, c be natural numbers. Prove the following properties of order for natural numbers:*

(a) Reflexivity: $a \geqslant a$. This is true since $a = 0 + a$ by Definition 2.2.1. By commutativity of addition, we can also write $a = a + 0$. So there is indeed a natural number n (with $n = 0$) such that $a = a + n$, i.e. $a \geqslant a$.

(b) Transitivity: if $a \geqslant b$ and $b \geqslant c$, then $a \geqslant c$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$ according to Definition 2.2.11. A similar consideration for the part $b \geqslant c$ leads to $b = c + m$, m being a natural number. Combining together those two equalities, we can write $a = b + n = (c + m) + n = c + (m + n)$ by associativity (see Exercise 2.2.1). Then, $n + m$ being a natural number[1], the transitivity is demonstrated.

(c) Anti-symmetry: if $a \geqslant b$ and $b \geqslant a$, then $a = b$. From the part $a \geqslant b$, there exists a natural number n such that $a = b + n$. Similarly, there exists a natural number m such that $b = a + m$. Combining those two equalities leads to $a = b + n = (a + m) + n = a + (m + n)$. By cancellation law (Proposition 2.2.6), we can conclude that $0 = m + n$. According to Corollary 2.2.9, this leads to $m = n = 0$. Therefore, both m and n are null, meaning that $a = b + 0 = b$.

(d) Preservation of order: $a \geqslant b$ iff $a+c \geqslant b+c$. First, let's prove that $a+c \geqslant b+c \Longrightarrow a \geqslant b$. If $a+c \geqslant b+c$, there exists a natural number n such that $a+c = b+c+n$. By cancellation law (Proposition 2.2.6)[2], we conclude that $a = b + n$, i.e. $a \geqslant b$, thus demonstrating the first implication. Conversely, let's suppose that $a \geqslant b$. There exists a natural number m such that $a = b + m$. Therefore, $a + c = b + m + c$ for any natural number c[3]. Still by associativity and commutativity, we can rewrite this as $a + c = (b + c) + m$, i.e. $a + c \geqslant b + c$.

(e) $a < b$ iff $a{+\!+} \leqslant b$. First, let's prove that $a{+\!+} \leqslant b \Longrightarrow a < b$. By definition of ordering, there exists a natural number n such that $b = (a{+\!+}) + n$. By definition of addition, we can re-write: $b = (a{+\!+} + n){+\!+}$. Then, by commutativity and yet again by definition of addition, $b = (n + a{+\!+}){+\!+} = (n{+\!+}) + (a{+\!+})$. Thus, there exists a natural number $n{+\!+}$ such that $b = n{+\!+} + a$, which means that $b \geqslant a$. But we still have to prove that $a \neq b$. Let's suppose that $a = b$: in this case, by cancellation law, we would have $n{+\!+} = 0$, which is impossible according to Axiom 2.3 (0 is not the successor of any natural number). Thus, $a \neq b$ et $b \geqslant a$: we have showed that $a < b$.

Conversely, let's prove that $a < b \Longrightarrow a{+\!+} \leqslant b$. Starting from that strict inequality, there exists a *positive*[4] natural number n such that $b = a + n$. By Lemma 2.2.10, since n is positive, it has one unique antecessor m, so that n can be written $m{+\!+}$. Thus, $b = a + (m{+\!+}) = (a + m){+\!+} = (m + a){+\!+} = m + (a{+\!+}) = (a{+\!+}) + m$. And, m being a natural number, this corresponds to the statement $b \geqslant a$.

[1] This is a trivial induction from the definition of addition.

[2] And also associativity and commutativity that we do not detail explicitly here.

[3] It is easy to demonstrate that, if $a = b$, then $a + n = b + n$ for any natural numbers a, b, n. This is a trivial induction on n, but it seems to me that it should be proved. I am not totally sure that we can use that starting only from Peano's axioms.

[4] We make use here of the statement *(f)* demonstrated below. There is no circularity here, since proving *(f)* will not make use of *(e)*.

(f) $a < b$ iff $b = a + d$ for some positive number d. First, let's prove the first implication, $a < b \implies b = a + d$ with $d \neq 0$. Since $a < b$, we have in particular $a \leqslant b$, and there exists a natural number d such that $b = a + d$. For the sake of contradiction, let's suppose that $d = 0$. We would have $b = a$, which would contradict the condition $a \neq b$ of the strict inequality. Thus, d is a positive number, which demonstrates the left-to-right implication.

Conversely, let's suppose that $b = a + d$, with $d \neq 0$. This expression gives immediately $a \leqslant b$. But if $a = b$, by cancellation law, this would lead to $0 = d$, a contradiction with the fact that d is a positive number. Thus, $a \neq b$ and $a \leqslant b$, which demonstrates $a < b$.