

# Propositions of solutions for *Analysis II* by Terence Tao

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March 22, 2022

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**Remarks.** The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

## 12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence  $(a_n)_{n=m}^{\infty}$  defined by  $a_n := d(x_n, x) = |x_n - x|$  for all  $n \geq m$ . We have to prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if and only if  $\lim_{n \rightarrow \infty} x_n = x$ .

- Let be  $\varepsilon > 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then there exists an  $N \geq m$  such that  $|a_n| < \varepsilon$  whenever  $n \geq N$ . Thus, there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ , which means that  $\lim_{n \rightarrow \infty} x_n = x$ .
- Let be  $\varepsilon > 0$ . Conversely, if  $\lim_{n \rightarrow \infty} x_n = x$ , then there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . But since  $|a_n| := |x_n - x|$ , it means that  $\lim_{n \rightarrow \infty} a_n = 0$ , as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric  $d(x, y) := |x - y|$  is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function. With respect to Definition 12.1.2, give an example of a pair  $(X, d)$  which...*

- (a) obeys the axioms (bcd) but not (a).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, x) = 1$  and  $d(x, y) = 5$  for all  $x \neq y \in \mathbb{R}$ .

- (b) obeys the axioms (acd) but not (b).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = 0$  for all  $x, y \in \mathbb{R}$ .

- (c) obeys the axioms (abd) but not (c).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = \max(x - y, 0)$  for all  $x, y \in \mathbb{R}$ .

- (d) obeys the axioms (abc) but not (d).

Consider the finite set  $X := \{1, 2, 3\}$  and the application  $d$  defined by  $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$ , and  $d(1, 3) = d(3, 1) := 5$ , and  $d(x, x) = 0$  for all  $x \in X$ .

EXERCISE 12.1.4. — *Show that the pair  $(Y, d|_{Y \times Y})$  defined in Example 12.1.5 is indeed a metric space.*

By definition, since  $Y \subseteq X$ , we have  $x, y \in X$  whenever  $x, y \in Y$ . And furthermore, since  $d|_{Y \times Y}(x, y) := d(x, y)$ , then the application  $d|_{Y \times Y}$  obeys all four statements (a)–(d) of Definition 12.1.2. Thus,  $(Y, d|_{Y \times Y})$  is indeed a metric space.

EXERCISE 12.1.5. — Let  $n \geq 1$ , and let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Verify the identity  $(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$ , and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.

Let's prove these three statements.

(i) To prove the first identity, let's use induction on  $n$ .

The base case  $n = 1$  is obvious: on the left-hand side, we just get  $(a_1 b_1)^2$ , and on the right-hand side, we get  $a_1^2 b_1^2$ , hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer  $n \geq 1$ , and let's prove that it is still true for  $n + 1$ . We have to prove that

$$\underbrace{\left( \sum_{i=1}^{n+1} a_i b_i \right)^2}_{:=A} + \underbrace{\frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right)}_{:=C} \quad (12.1)$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for  $A$ , we have

$$\begin{aligned} A &:= \left( \sum_{i=1}^{n+1} a_i b_i \right)^2 \\ &= \left( a_{n+1} b_{n+1} + \sum_{i=1}^n a_i b_i \right)^2 \\ &= (a_{n+1} b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1} b_{n+1}) \sum_{i=1}^n a_i b_i \end{aligned}$$

- for  $B$ , we have

$$\begin{aligned} B &:= \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{j=1}^{n+1} (a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2} \sum_{i=1}^n (a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2 \times S} + \underbrace{\frac{1}{2} \sum_{j=1}^n (a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2 \times S} \\ &\quad + \underbrace{\frac{1}{2} (a_{n+1} b_{n+1} - b_{n+1} a_{n+1})^2}_{=0} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for  $A + B$ , we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A + B &:= (a_{n+1}b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2 \sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number  $n$ .

- (ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2}. \quad (12.2)$$

Indeed, since  $B \geq 0$  in the identity (12.1), we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$\begin{aligned}
\sum_{i=1}^n (a_i^2 + b_i^2) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \\
&\leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{by eq. (12.2)}) \\
&\leq \left( \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

EXERCISE 12.1.6. — *Show that  $(\mathbb{R}^n, d_{l^2})$  in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^2}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2} = 0$ , as expected.
- (b) Positivity: for all  $x, y \in \mathbb{R}^n$ , we have  $d_{l^2}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \geq 0$ , as the square root of a sum of  $n$  non-negative terms, as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(y, x) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_{l^2}(x, y)$$

as expected.

- (d) Triangle inequality: for all  $x, y, z \in \mathbb{R}^n$ , we have

$$\begin{aligned}
d_{l^2}(x, z) &:= \left( \sum_{i=1}^n (x_i - z_i)^2 \right)^{1/2} \\
&= \left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} && \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} && (\text{Exercise 12.1.5(iii)}) \\
&\leq \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} + \left( \sum_{i=1}^n (y_i - z_i)^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) + d_{l^2}(y, z)
\end{aligned}$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^2})$  is indeed a metric space.