

# Propositions of solutions for *Analysis II* by Terence Tao

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**Remarks.** The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

## 12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence  $(a_n)_{n=m}^{\infty}$  defined by  $a_n := d(x_n, x) = |x_n - x|$  for all  $n \geq m$ . We have to prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if and only if  $\lim_{n \rightarrow \infty} x_n = x$ .

- Let be  $\varepsilon > 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then there exists an  $N \geq m$  such that  $|a_n| < \varepsilon$  whenever  $n \geq N$ . Thus, there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ , which means that  $\lim_{n \rightarrow \infty} x_n = x$ .
- Let be  $\varepsilon > 0$ . Conversely, if  $\lim_{n \rightarrow \infty} x_n = x$ , then there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . But since  $|a_n| := |x_n - x|$ , it means that  $\lim_{n \rightarrow \infty} a_n = 0$ , as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric  $d(x, y) := |x - y|$  is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function. With respect to Definition 12.1.2, give an example of a pair  $(X, d)$  which...*

- (a) obeys the axioms (bcd) but not (a).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, x) = 1$  and  $d(x, y) = 5$  for all  $x \neq y \in \mathbb{R}$ .

- (b) obeys the axioms (acd) but not (b).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = 0$  for all  $x, y \in \mathbb{R}$ .

- (c) obeys the axioms (abd) but not (c).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = \max(x - y, 0)$  for all  $x, y \in \mathbb{R}$ .

- (d) obeys the axioms (abc) but not (d).

Consider the finite set  $X := \{1, 2, 3\}$  and the application  $d$  defined by  $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$ , and  $d(1, 3) = d(3, 1) := 5$ , and  $d(x, x) = 0$  for all  $x \in X$ .

EXERCISE 12.1.4. — *Show that the pair  $(Y, d|_{Y \times Y})$  defined in Example 12.1.5 is indeed a metric space.*

By definition, since  $Y \subseteq X$ , we have  $x, y \in X$  whenever  $x, y \in Y$ . And furthermore, since  $d|_{Y \times Y}(x, y) := d(x, y)$ , then the application  $d|_{Y \times Y}$  obeys all four statements (a)–(d) of Definition 12.1.2. Thus,  $(Y, d|_{Y \times Y})$  is indeed a metric space.

EXERCISE 12.1.5. — Let  $n \geq 1$ , and let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Verify the identity  $(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$ , and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.

Let's prove these three statements.

(i) To prove the first identity, let's use induction on  $n$ .

The base case  $n = 1$  is obvious: on the left-hand side, we just get  $(a_1 b_1)^2$ , and on the right-hand side, we get  $a_1^2 b_1^2$ , hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer  $n \geq 1$ , and let's prove that it is still true for  $n + 1$ . We have to prove that

$$\underbrace{\left( \sum_{i=1}^{n+1} a_i b_i \right)^2}_{:=A} + \underbrace{\frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right)}_{:=C} \quad (12.1)$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for  $A$ , we have

$$\begin{aligned} A &:= \left( \sum_{i=1}^{n+1} a_i b_i \right)^2 \\ &= \left( a_{n+1} b_{n+1} + \sum_{i=1}^n a_i b_i \right)^2 \\ &= (a_{n+1} b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1} b_{n+1}) \sum_{i=1}^n a_i b_i \end{aligned}$$

- for  $B$ , we have

$$\begin{aligned} B &:= \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{j=1}^{n+1} (a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2} \sum_{i=1}^n (a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2 \times S} + \underbrace{\frac{1}{2} \sum_{j=1}^n (a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2 \times S} \\ &\quad + \underbrace{\frac{1}{2} (a_{n+1} b_{n+1} - b_{n+1} a_{n+1})^2}_{=0} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for  $A + B$ , we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A + B &:= (a_{n+1}b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2 \sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number  $n$ .

- (ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2}. \quad (12.2)$$

Indeed, since  $B \geq 0$  in the identity (12.1), we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$\begin{aligned}
\sum_{i=1}^n (a_i^2 + b_i^2) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \\
&\leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{by eq. (12.2)}) \\
&\leq \left( \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

EXERCISE 12.1.6. — *Show that  $(\mathbb{R}^n, d_{l^2})$  in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^2}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $(x_i - y_i)^2 > 0$ , and  $d_{l^2}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(y, x) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_{l^2}(x, y)$$

as expected.

- (d) Triangle inequality: for all  $x, y, z \in \mathbb{R}^n$ , we have

$$\begin{aligned}
d_{l^2}(x, z) &:= \left( \sum_{i=1}^n (x_i - z_i)^2 \right)^{1/2} \\
&= \left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \quad \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{Exercise 12.1.5(iii)}) \\
&\leq \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} + \left( \sum_{i=1}^n (y_i - z_i)^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) + d_{l^2}(y, z)
\end{aligned}$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^2})$  is indeed a metric space.

EXERCISE 12.1.7. — *Show that  $(\mathbb{R}^n, d_{l^1})$  in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^1}(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $|x_i - y_i| > 0$ , and  $d_{l^1}(x, y) = \sum_{i=1}^n |x_i - y_i| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^1}(y, x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x, y)$$

as expected.

- (d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality  $|a - c| \leq |a - b| + |b - c|$  for all  $a, b, c \in \mathbb{R}$ . Thus, for all  $x, y, z \in \mathbb{R}^n$ , we have

$$d_{l^1}(x, z) := \sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x, y) + d_{l^1}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to  $d_{l^2}(x, y)^2 \leq d_{l^1}(x, y)^2$ , and we will prove it in this form.

Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l^1}(x, y)^2 &:= \left( \sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \left( \sum_{i=1}^n |x_i - y_i| \right) \times \left( \sum_{i=1}^n |x_i - y_i| \right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leq i, j \leq n; i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geq 0} \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x, y)^2 \end{aligned}$$

as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x, y) &:= \sum_{i=1}^n |x_i - y_i| \\
&= \left| \sum_{i=1}^n |x_i - y_i| \times 1 \right| \\
&\leq \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left( \sum_{i=1}^n 1^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) \times \sqrt{n}
\end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair  $(\mathbb{R}^n, d_{l^\infty})$  in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we clearly have  $d_{l^\infty}(x, x) = \sup\{|x_i - x_i| : 1 \leq i \leq n\} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq j \leq n$  such that  $x_j \neq y_j$ . Thus  $|x_j - y_j| > 0$ , and  $d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_j - y_j| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} = \sup\{|y_i - x_i| : 1 \leq i \leq n\} = d_{l^\infty}(y, x)$$

as expected.

- (d) Triangle inequality. Let be  $x, y, z \in \mathbb{R}^n$ . We have  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$  for all  $1 \leq i \leq n$ , by Proposition 4.3.3(g). But, by definition of the supremum, we have  $|x_i - y_i| \leq d_{l^\infty}(x, y)$  and  $|y_i - z_i| \leq d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ . Thus, we have  $|x_i - z_i| \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ ; i.e.,  $d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  is an upper bound of the set  $\{|x_i - z_i| : 1 \leq i \leq n\}$ . By definition of the supremum, it implies that

$$d_{l^\infty}(x, z) := \sup\{|x_i - z_i| : 1 \leq i \leq n\} \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ ,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y).$$

First, a preliminary remark. By definition, we have  $d_{l^\infty}(x, y) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}$  for all  $x, y \in \mathbb{R}^n$ . Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a  $1 \leq m \leq n$  such that  $d_{l^\infty}(x, y) = |x_m - y_m|$  (the supremum belongs to the set). The index “ $m$ ” will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}
\frac{1}{\sqrt{n}}d_{l^2}(x, y) &:= \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - y_i)^2} \\
&\leq \sqrt{\frac{1}{n} \sum_{i=1}^n (x_m - y_m)^2} \\
&\leq \sqrt{\frac{n}{n} (x_m - y_m)^2} \\
&= |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}
d_{l^2}(x, y) &:= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \\
&= \sqrt{(x_m - y_m)^2 + \sum_{1 \leq i \leq n; i \neq m} (x_i - y_i)^2} \\
&\geq \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

EXERCISE 12.1.11. — *Show that the discrete metric  $(X, d_{\text{disc}})$  in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in X$ , we have  $d_{\text{disc}}(x, x) := 0$  by definition, so that there is nothing to prove here.
- (b) Positivity: for all  $x \neq y \in X$ , we have  $d_{\text{disc}}(x, y) := 1 > 0$  by definition, so that there's still nothing to prove.
- (c) Symmetry: for all  $x, y \in X$ , we have  $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x) = 1$ , so that  $d_{\text{disc}}$  obeys the symmetry property.
- (d) Triangle inequality. Let be  $x, y, z \in X$ , and let's consider  $d_{\text{disc}}(x, z)$ .
  - If  $x = z$ , then  $d_{\text{disc}}(x, z) = 0$ . And since  $d_{\text{disc}}$  is a non-negative application, we clearly have  $0 =: d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$  for all  $y \in X$ .
  - If  $x \neq z$ , then we cannot have both  $x = y$  and  $y = z$  (it would be a clear contradiction with  $x \neq z$ ). Thus, at least one of the propositions " $x \neq y$ ", " $y \neq z$ " is true. Another way to say that is  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq 1$ . But since  $d_{\text{disc}}(x, z) := 1$ , we have actually  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq d_{\text{disc}}(x, z)$ , as expected.



EXERCISE 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all  $x, y \in \mathbb{R}^n$ , we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y).$$

Note that  $n$  is a real constant here.

- Let's prove that (a)  $\implies$  (b). If  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , then by the limit laws, the sequence  $t_k := \sqrt{n} d_{l^2}(x^{(k)}, x)$  also converges to 0 as  $k \rightarrow \infty$ , since  $\sqrt{n}$  is a constant real number. Thus, we have

$$d_{l^2}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x) \leq \sqrt{n} d_{l^2}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x)$  as expected.

- Let's prove that (b)  $\implies$  (c). If  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x) = 0$ , then we have

$$0 \leq d_{l^\infty}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x)$  as expected.

- Let's prove that (c)  $\implies$  (d). Suppose that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x) = 0$ . Then, for all  $1 \leq j \leq n$ , we have  $0 \leq |x_j^{(k)} - x_j| \leq d_{l^\infty}(x^{(k)}, x)$ . Still by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} |x_j^{(k)} - x_j| = 0$ , i.e. that  $(x_j^{(k)})_{k=m}^\infty$  converges to  $x_j$  as  $k \rightarrow \infty$  (by Lemma 12.1.1), as expected.
- Finally, let's prove that (d)  $\implies$  (a). Using the definition of convergence is more appropriate here. Let be  $\varepsilon > 0$  a positive real number, and let be  $1 \leq j \leq n$ . By definition, there exists a natural number  $N \geq m$  such that  $|x_j^{(k)} - x_j| \leq \varepsilon/\sqrt{n}$  whenever  $k \geq N$ . Thus, if  $k \geq N$ , we have

$$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^n (x_j^{(k)} - x_j)^2} \leq \sqrt{\sum_{j=1}^n \frac{\varepsilon^2}{n}} \leq \varepsilon$$

so that  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , i.e.,  $(x^{(k)})_{k=m}^\infty$  converges to  $x$  as  $k \rightarrow \infty$  in the  $l^2$  metric (by Lemma 12.1.1), as expected.

EXERCISE 12.1.13. — *Prove Proposition 12.1.19.*

Let be  $(x^{(n)})_{n=m}^\infty$  a sequence of elements of a set  $X$ .

- First suppose that  $(x^{(n)})_{n=m}^\infty$  is eventually constant. Thus, by definition, there exists an  $N \geq m$  and an element  $x \in X$  such that  $(x^{(n)})_{n=m}^\infty = x$  for all  $n \geq N$ . This implies that we have  $d_{\text{disc}}(x^{(n)}, x) = 0$  for all  $n \geq N$ . In particular, for all  $\varepsilon > 0$ , we have  $d_{\text{disc}}(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N$ , so that  $(x^{(n)})_{n=m}^\infty$  indeed converges to  $x$  with respect to  $d_{\text{disc}}$ .
- Conversely, suppose that  $(x^{(n)})_{n=m}^\infty$  converges to  $x$  with respect to  $d_{\text{disc}}$ . Let be  $\varepsilon = 1/2$ . By definition, there exists an  $N \geq m$  such that  $d_{\text{disc}}(x^{(n)}, x) \leq 1/2$  whenever  $n \geq N$ . Since  $d_{\text{disc}}(x^{(n)}, x)$  cannot be 1, it is necessarily equal to 0, so that  $x^{(n)} = x$  whenever  $n \geq N$ . Thus, the sequence  $x^{(n)}$  is indeed eventually constant.

EXERCISE 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have  $\lim_{n \rightarrow \infty} d(x^{(n)}, x) = 0$  and  $\lim_{n \rightarrow \infty} d(x^{(n)}, x') = 0$ . Suppose, for the sake of contradiction, that we have  $x \neq x'$ . Thus, the real number  $\varepsilon := \frac{d(x, x')}{3}$  is positive.

Since  $x^{(n)}$  converges to  $x$ , there exists a  $N_1 \geq m$  such that  $d(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N_1$ .

Similarly, since  $x^{(n)}$  converges to  $x'$ , there exists a  $N_2 \geq m$  such that  $d(x^{(n)}, x') \leq \varepsilon$  whenever  $n \geq N_2$ .

By the triangle inequality, we thus have, for all  $n \geq \max(N_1, N_2)$ ,

$$d(x, x') \leq d(x, x^{(n)}) + d(x^{(n)}, x') \leq \varepsilon + \varepsilon = \frac{2}{3}d(x, x')$$

which is a contradiction (since  $d(x, x') > 0$  by hypothesis).

Thus, the limit is unique, and we must have  $x = x'$ .

EXERCISE 12.1.15. — *Let be  $X := \{(a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty\}$ . We define on this space the metrics  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sum_{n=0}^\infty |a_n - b_n|$ , and  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |a_n - b_n|$ . Then...*

We have to prove the following statements.

1.  $d_{l^1}$  is a metric on  $X$ .

We have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^1}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^1}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |b_n - a_n| = \sum_{n=0}^\infty |a_n - b_n| = d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately

$$\begin{aligned} d_{l^1}((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty) &:= \sum_{n=0}^\infty |a_n - c_n| \\ &\leq \sum_{n=0}^\infty (|a_n - b_n| + |b_n - c_n|) \quad (\text{consequence of Prop. 7.1.11(h)}) \\ &\leq \sum_{n=0}^\infty |a_n - b_n| + \sum_{n=0}^\infty |b_n - c_n| \quad (\text{by Proposition 7.2.14(a)}) \\ &\leq d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) + d_{l^1}((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty). \end{aligned}$$

Thus,  $d_{l^1}$  is indeed a metric on  $X$ .

2.  $d_{l^\infty}$  is a metric on  $X$ .

Once again, we have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |b_n - a_n| = \sup_{n \in \mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately  $|a_m - c_m| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$  for all  $m \in \mathbb{N}$ , by definition of the supremum. In other words,  $(\sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|)$  is an upper bound for the set  $\{|a_m - c_m| : m \in \mathbb{N}\}$ . Thus we have, still by definition of the supremum,  $\sup_{n \in \mathbb{N}} |a_n - c_n| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$ , as expected.

Thus,  $d_{l^\infty}$  is indeed a metric on  $X$ .

3. There exist sequences  $x^{(1)}, x^{(2)}, \dots$ , of elements of  $X$  (i.e., sequences of sequences) which are convergent with respect to  $d_{l^\infty}$ , but are not convergent with respect to  $d_{l^1}$ .

Here we are dealing with sequences of sequences: we have a sequence  $(x^{(k)})_{k=1}^\infty$  where each  $x^{(k)}$  is itself a sequence of real numbers. Thus, let's define  $(x^{(k)})_{k=1}^\infty$  as follows:

$$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

Just to make things clearer, we have for instance

$$\begin{aligned} x^{(1)} &:= \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \\ x^{(2)} &:= \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots \\ x^{(3)} &:= \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots \end{aligned}$$

Also, let be the null sequence  $(a_n)_{n=0}^\infty$  defined by  $a_n := 0$  for all  $n \in \mathbb{N}$ . Thus:

- $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ . Indeed, if we consider a given positive integer  $k$  (fixed), we have

$$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

so that  $d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$ .

Thus,  $\lim_{k \rightarrow \infty} d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) = 0$ , or in other words,  $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$  in  $X$ .

- But  $(x_n^{(k)})_{n=0}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ . Indeed, we have, for each given (fixed)  $k$ ,

$$d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = \sum_{n=0}^k \frac{1}{k+1} = 1$$

Thus, we clearly do not have  $\lim_{k \rightarrow \infty} d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = 0$ , i.e.,  $(x_n^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

4. Conversely, any sequence which converges with respect to  $d_{l^1}$  also converges with respect to  $d_{l^\infty}$ .

Suppose, for the sake of contradiction, that  $(x_n^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ , but does converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

In this case, there exists a  $\varepsilon > 0$  such that, for all  $k \geq 1$ , we have  $(\sup_{n \geq 0} |x_n^{(k)} - a_n|) > \varepsilon$ . In particular, for all  $k \geq 1$  and all  $n \geq 0$ , we have  $|x_n^{(k)} - a_n| > \varepsilon$ . Thus,  $\sum_{n=0}^\infty |x_n^{(k)} - a_n|$  is not even a convergent series, and we cannot have  $\lim_{k \rightarrow \infty} \left( \sum_{n=0}^\infty |x_n^{(k)} - a_n| \right) = 0$ .

Note that this exercise actually shows that in this space  $X$ , the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

**EXERCISE 12.1.16.** — Let  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  be two sequences in a metric space  $(X, d)$ . Suppose that  $(x_n)_{n=1}^\infty$  converges to a point  $x \in X$ , and  $(y_n)_{n=1}^\infty$  converges to a point  $y \in X$ . Show that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ .

On the one hand, the triangle inequality applied two times to  $d$  gives us

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leq d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

Let be  $\varepsilon > 0$ . By hypothesis, there exists a  $N_1 \geq 1$  such that  $d(x_n, x) \leq \varepsilon/2$  whenever  $n \geq N_1$ . Similarly, there exists a  $N_2 \geq 1$  such that  $d(y_n, y) \leq \varepsilon/2$  whenever  $n \geq N_2$ . Thus, if we set  $N := \max(N_1, N_2)$ , then for all  $n \geq N$  we have

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \leq 2\varepsilon/2 = \varepsilon$$

which shows that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ , as expected.

EXERCISE 12.2.1. — *Verify the claims in Example 12.2.8*

Let be  $(X, d_{\text{disc}})$  a metric space, and  $E$  a subset of  $X$ .

- Let be  $x \in E$ . Then  $x$  is an interior point of  $E$ . Indeed, we have  $B(x, 1/2) = \{x\} \subseteq E$ .
- Let be  $y \notin E$ . Then  $y$  is an exterior point of  $E$ . Indeed, we have  $B(y, 1/2) \cap E = \{y\} \cap E = \emptyset$ .
- Finally, there are no boundary points of  $E$  in  $(X, d_{\text{disc}})$ . Indeed, let be  $r > 0$  and any  $x \in X$ . We will always have  $B(x, r) = \{x\}$  by definition of the discrete metric  $d_{\text{disc}}$ . Thus, we have either  $x \in E$  and then  $x \in \text{int}(E)$ , or  $x \notin E$  and then  $x \in \text{ext}(E)$ . Thus,  $E$  has no boundary points.

EXERCISE 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that  $(a) \implies (b)$ . We will use the contrapositive, assuming that  $x_0$  is neither an interior point of  $E$ , nor a boundary point of  $E$ . By definition, it means that  $x_0$  is an exterior point of  $E$ , i.e. that there exists  $r > 0$  such that  $B(x_0, r) \cap E = \emptyset$ . This is precisely the negation of  $x_0$  being an adherent point of  $E$ . Thus, we have showed that if  $x_0$  is an adherent of  $E$ , it is either an interior point or a boundary point.
- Let's show that  $(b) \implies (c)$ . Let be a positive integer  $n > 0$ , and suppose that  $x_0$  is either an interior point of  $E$ , or a boundary point of  $E$ . In either case, the set  $A_n := B(x_0, 1/n) \cap E$  is non empty, i.e., there exists  $a_n \in X$  such that  $d(a_n, x_0) < 1/n$ . By the (countable) axiom of choice, we can define a sequence  $(a_n)_{n=1}^\infty$  such that  $a_n \in A_n$  for all  $n \geq 1$ .

Let be  $\varepsilon > 0$ . There exists  $N > 0$  such that  $1/N < \varepsilon$  (Exercise 5.4.4). Thus, for all  $n \geq N$ , we have

$$d(a_n, x_0) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

i.e., the sequence  $(a_n)_{n=1}^\infty$  converges to  $x_0$  with respect to the metric  $d$ , as expected.

- Finally, let's show that  $(c) \implies (a)$ . Let be  $r > 0$ . If  $(a_n)_{n=1}^\infty$  in  $E$  converges to  $x_0$  with respect to  $d$ , then there exists a  $n$  such that  $d(x_0, a_n) < r$ . But since  $a_n \in E$ , it means that  $B(x_0, r) \cap E$  is non empty, i.e. that  $x_0$  is an adherent point of  $E$ .

EXERCISE 12.2.3. — *Prove Proposition 12.2.5.*

Let be  $(X, d)$  a metric space.

- (a) Let be  $E \subseteq X$ . First suppose that  $E$  is open; this means that  $E \cap \partial E = \emptyset$ . Let be  $x \in E$ , then we have  $x \notin \partial E$ . But since  $x \in E$ , we have  $x \in \overline{E}$ , and thus  $x \in \text{int}(E)$  by Proposition 12.2.10(b). We have shown that  $x \in E \implies x \in \text{int}(E)$ , and since the converse implication is trivial (Remark 12.2.6), we have  $E = \text{int}(E)$  as expected.

Now suppose that  $E = \text{int}(E)$ . Let be  $x \in E$ . We thus have  $x \in \text{int}(E)$ . By definition,  $x$  is thus not a boundary point of  $E$ , i.e.  $x \notin \partial E$ . This means that  $E \cap \partial E = \emptyset$ , i.e. that  $E$  is open, as expected.

- (b) Let be  $E \subseteq X$ . First suppose that  $E$  is closed; i.e. that  $\partial E \subseteq E$ . Let be  $x \in \overline{E}$ . By Proposition 12.2.10, we have  $\overline{E} = \text{int}(E) \cup \partial E$ ; such that  $\overline{E}$  is the union of two subsets of  $E$ , and thus is itself a subset of  $E$ , as expected.

Conversely, suppose that  $\overline{E} \subseteq E$ . It means that  $\text{int}(E) \cup \partial E \subseteq E$ , and in particular that  $\partial E \subseteq E$ , i.e. that  $E$  is closed, as expected.

- (c) Let be  $x_0 \in X$ ,  $r > 0$  and  $E := B(x_0, r)$ . To show that  $E$  is open, we must show that  $E = \text{int}(E)$  (by Proposition 1.2.15(a)), and in particular that  $E \subseteq \text{int}(E)$  (the converse inclusion being trivial). Let be  $x \in E$ , and let's show that  $x \in \text{int}(E)$ . By definition, we have  $d(x, x_0) < r$ , so that  $\varepsilon := r - d(x, x_0)$  is a positive real number. Thus, let be  $y \in B(x, \varepsilon)$ . By the triangle inequality, we have

$$\begin{aligned} d(x_0, y) &< d(x, x_0) + d(x, y) \\ &< d(x, x_0) + \varepsilon \\ &< d(x, x_0) + r - d(x, x_0) = r \end{aligned}$$

so that  $y \in E$ . Thus, there exists  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq E$ , i.e.,  $x$  is an interior point of  $E$ . This shows that  $E \subseteq \text{int}(E)$ , as expected.

Now let be  $F := \{x \in X : d(x, x_0) \leq r\}$ , and let be  $(a_n)_{n=1}^\infty$  a convergent sequence in  $F$ . To show that  $F$  is closed, we have to show that  $\ell := \lim_{n \rightarrow \infty} a_n$  lies in  $F$  (Proposition 12.2.15(b)). Suppose, for the sake of contradiction, that  $\ell \notin F$ . We thus have  $d(\ell, x_0) > r$ , so that  $\varepsilon := d(\ell, x_0) - r$  is a positive real number. Since  $(a_n)_{n=1}^\infty$  converges to  $\ell$ , there exists a  $N > 0$  such that  $d(a_n, \ell) < \varepsilon$  whenever  $n \geq N$ . By the triangle inequality, for  $n \geq N$ , we have

$$\begin{aligned} d(x_0, \ell) &\leq d(x_0, a_n) + d(a_n, \ell) \\ d(x_0, a_n) &\geq d(x_0, \ell) - d(a_n, \ell) \\ &\geq d(x_0, \ell) - \varepsilon \\ &\geq d(x_0, \ell) + r - d(\ell, x_0) \\ &\geq r \end{aligned}$$

and thus,  $a_n \notin B(x_0, r)$ , a contradiction. Thus, we must have  $\ell \in F$ , so that  $F$  is indeed a closed set.

- (d) Let be  $\{x_0\}$  a singleton with  $x_0 \in X$ . To show that  $E$  is closed, we may use Proposition 12.2.15(b), and show that  $\{x_0\}$  contains all its adherent points. Let be  $(a_n)_{n=1}^\infty$  a convergent sequence in  $\{x_0\}$ ; it can only be the constant sequence  $x_0, x_0, \dots$ . Since it is a constant sequence, its limit can only be  $x_0$  itself, and this limit belongs to  $\{x_0\}$ . Thus,  $\{x_0\}$  is a closed set.

- (e) First we can form a lemma: for any subset  $E \subseteq X$ , we have  $\text{int}(E) = \text{ext}(X \setminus E)$ . This is a direct consequence of Definition 12.2.5. Indeed,  $x \in \text{int}(E)$  iff there exists a  $r > 0$  such that  $B(x, r) \subseteq E$ , which is equivalent to “ $\exists r > 0 : B(x, r) \cap (X \setminus E) = \emptyset$ ”, which is equivalent to  $x \in \text{ext}(X \setminus E)$ .

This implies that the interior points of  $E$  are the exterior points of  $X \setminus E$ , and conversely, that the exterior points of  $E$  are the interior points of  $X \setminus E$ . Thus, in particular, we have this useful fact:

$$\partial E = \partial(X \setminus E). \quad (12.4)$$

Now we go back to the main proof. First suppose that  $E$  is open. Thus, by Definition 12.2.12, we have  $E \cap \partial E = \emptyset$ , so that  $\partial E \subseteq X \setminus E$ , which means that  $X \setminus E$  is a closed set. The converse also applies: if we suppose that  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ . By equation (12.4) above, this is equivalent to  $\partial E \subseteq X \setminus E$ , and thus we have  $\partial E \cap E = \emptyset$ . This means that  $E$  is open, as expected.<sup>1</sup>

- (f) Let  $E_1, \dots, E_n$  be open sets. Thus, for all  $1 \leq i \leq n$ , if  $x \in E_i$ , there exists a  $r_i > 0$  such that  $B(x, r_i) \subseteq E_i$ . Let's define  $r := \min(r_1, \dots, r_n)$ . We have  $B(x, r) \subseteq B(x, r_i) \subseteq E_i$  for all  $1 \leq i \leq n$ , i.e.  $B(x, r) \subseteq E_1 \cap \dots \cap E_n$ . Thus,  $E_1 \cap \dots \cap E_n$  is an open set.

Also, let  $F_1, \dots, F_n$  be closed sets. By the previous result (e), the complementary sets  $X \setminus F_1, \dots, X \setminus F_n$  are open sets. Thus, we have just proved that  $(X \setminus F_1) \cap \dots \cap (X \setminus F_n)$  is an open set. But we have  $(X \setminus F_1) \cap \dots \cap (X \setminus F_n) = X \setminus (F_1 \cup \dots \cup F_n)$ , and this set is open. Thus, by (e), its complementary set,  $F_1 \cup \dots \cup F_n$ , is closed, as expected.

- (g) Let  $(E_\alpha)_{\alpha \in I}$  be open sets. Suppose that we have  $x \in \bigcup_{\alpha \in I} E_\alpha$ . By definition, there exists a  $i \in I$  such that  $x \in E_i$ . Since  $E_i$  is an open set, there exists  $r_i > 0$  such that  $B(x, r_i) \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$ . Thus, by (a),  $\bigcup_{\alpha \in I} E_\alpha$  is an open set, as expected.

Now let be  $(F_\alpha)_{\alpha \in I}$  be closed sets. Suppose that we have a convergent sequence  $(x_n)_{n=1}^\infty$  such that  $x_n \in \bigcap_{\alpha \in I} F_\alpha$  for all  $n \geq 1$ . Thus, for all  $\alpha \in I$ , the sequence  $(x_n)_{n=1}^\infty$  entirely belongs to the closed set  $F_\alpha$ , so that its limit  $\ell$  also lies in  $F_\alpha$  according to (b). Thus,  $\ell \in \bigcap_{\alpha \in I} F_\alpha$ , so that  $\bigcap_{\alpha \in I} F_\alpha$  is a closed set, as expected.

- (h) Let be  $E \subseteq X$ .

- Let's show that  $\text{int}(E)$  is the largest open set included in  $E$ . It has not clearly been proved in the main text that  $\text{int}(E)$  is an open set, so we begin by proving it. Let be  $x \in \text{int}(E)$ . By definition, there exists  $r > 0$  so that  $B(x, r) \subseteq E$ . But by (c), we know that  $B(x, r)$  is an open set, so that any point  $y$  of  $B(x, r)$  is an interior point of this open ball, and thus an interior point of  $E$ . Thus,  $\text{int}(E)$  is open.

Now consider another open set  $V \subseteq E$ , and let's show that  $V \subseteq \text{int}(E)$ . If  $x \in \text{int}(V)$ , then there exists  $r > 0$  such that  $B(x, r) \subseteq V \subseteq E$ , so that  $x \in \text{int}(E)$ . This shows that  $V \subseteq \text{int}(E)$ , as expected.

- Similarly, let's show that  $\overline{E}$  is the smallest closed set that contains  $E$ . First we show that  $\overline{E}$  is closed, i.e. that  $\overline{\overline{E}} \subseteq \overline{E}$ . (Hint: see Exercise 9.1.6 for an intuition.) Let be  $x \in \overline{\overline{E}}$ . By definition, for all  $r > 0$ ,  $B(x, r) \cap \overline{E} \neq \emptyset$ . Thus, there exists  $y \in B(x, r)$  such that  $y \in \overline{E}$ . Thus, because  $B(x, r)$  is an open set and  $y$  is adherent to  $E$ , there exists  $\varepsilon > 0$  such that  $B(y, \varepsilon) \subseteq B(x, r)$  and  $B(y, \varepsilon) \cap E \neq \emptyset$ ; i.e., there exists  $z \in B(y, \varepsilon) \subseteq B(x, r)$  such that  $z \in E$ . We have showed that whenever  $x \in \overline{\overline{E}}$ , we have  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ , i.e. that  $x$  is an adherent point of  $E$ , as expected. Thus,  $\overline{E}$  is closed.

Now we consider a closed set  $K$  such that  $E \subseteq K$ , and we have to show that  $\overline{E} \subseteq K$ . Let be  $x \in \overline{E}$ . By definition, for all  $r > 0$ , we have  $B(x, r) \cap E \neq \emptyset$ . But since  $E \subseteq K$ , we also have  $B(x, r) \cap K \neq \emptyset$  for all  $r > 0$ . Thus,  $x$  is an adherent point of  $K$ , i.e.,  $x \in \overline{K}$ . But since  $K$  is closed, we have  $K = \overline{K}$ , and thus  $x \in K$ . This shows that  $\overline{E} \subseteq K$ , as expected.

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<sup>1</sup>This important result will be used in future proofs to turn any statement on closed sets into a statement on open sets.

EXERCISE 12.2.4. — Let  $(X, d)$  be a metric space,  $x_0$  be a point in  $X$ , and  $r > 0$ . Let  $B$  be the open ball  $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ , and let  $C$  be the closed ball  $C := \{x \in X : d(x, x_0) \leq r\}$ .

Let's prove the following claims:

(a) Show that  $\overline{B} \subseteq C$ .

Let be  $x \in \overline{B}$ . By definition, since  $x$  is an adherent point of  $B$ , for all  $\varepsilon > 0$  we have  $B(x, \varepsilon) \cap B \neq \emptyset$ . In other words, there exists  $y$  such that we have both  $d(x, y) < \varepsilon$  and  $d(x_0, y) < r$ . Thus, by the triangle inequality, we have

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ &\leq \varepsilon + r \quad \text{for all } \varepsilon > 0 \end{aligned}$$

which is equivalent (as a quick proof by contradiction would show) to  $d(x, x_0) \leq r$ . Thus,  $x \in C$ .

We have indeed proved that  $\overline{B} \subseteq C$ .

(b) Give an example of a metric space  $(X, d)$ , a point  $x_0$ , and a radius  $r > 0$  such that  $\overline{B}$  is *not* equal to  $C$ .

Let's take  $X = \mathbb{R}$ ,  $d = d_{\text{disc}}$ ,  $x = 0$  and  $r = 1$ . On the one hand, we have  $B := \{0\}$  and  $C := \mathbb{R}$ . Now let's work out  $\overline{B}$ . By Proposition 12.2.15(bd),  $B$  is closed, so that we have  $\overline{B} = B$ . Thus, we clearly do not have  $\overline{B} = C$  here. (Note however that any  $x_0 \in \mathbb{R}$  would be convenient here; there is nothing special about 0.)

EXERCISE 12.3.1. — Prove Proposition 12.3.4(b).

Let's show each direction of the equivalence.

- First suppose that  $E$  is relatively closed w.r.t.  $Y$ , and let's show that there exists a closed subset  $K \subseteq X$  such that  $E = K \cap Y$ .

Since  $E$  is closed w.r.t.  $Y$ , the set  $Y \setminus E$  is open w.r.t.  $Y$  (by Proposition 12.2.15(e)). Thus, by (a), there exists an open subset  $V \subseteq X$  such that  $Y \setminus E = V \cap Y$ .

Let be  $K := X \setminus V$ ; this subset  $K \subseteq X$  is closed w.r.t.  $X$  by Proposition 12.2.15(e) since it is the complementary set of an open set. We have to show that  $E = K \cap Y$ .

- Let be  $x \in E$ . Thus, we have  $x \in Y$ , since  $E \subseteq Y$ . And since  $x \in E$ , by definition, we have  $x \notin Y \setminus E$ . Thus,  $x \notin V \cap Y$ , which implies that  $x \notin V$  (since  $x \in Y$ ). Thus, by definition,  $x \in K$ , and thus,  $x \in K \cap Y$ .
- Conversely, let be  $x \in K \cap Y$ . By definition,  $x \in Y$  and  $x \notin V$ . Thus,  $x \notin V \cap Y$ , or, in other words,  $x \notin Y \setminus E$ . We finally get  $x \in E$ , as expected.

Thus, we have indeed  $E = K \cap Y$ , for some closed subset  $K \subseteq X$ , as expected.

- Now let's prove the converse implication: suppose that  $E = K \cap Y$  for some closed subset  $K \subseteq X$ , and let's prove that  $E$  is relatively closed w.r.t.  $Y$ .

Still by Proposition 12.2.15(e), we know that the subset  $V := X \setminus K$  is open w.r.t.  $X$ . Thus, by the previous result from this exercise,  $V \cap Y$  is relatively open w.r.t.  $Y$ . Thus, its complementary set  $Y \setminus (V \cap Y) = Y \setminus V$  is relatively closed w.r.t.  $Y$ . Now we want to show that  $E = Y \setminus V$  to close the proof.



- First suppose that  $x \in E$ . Since  $E = K \cap Y$ , we thus have  $x \in Y$  and  $x \in K$ , i.e.  $x \notin V$ . Thus,  $x \in Y \setminus V$ .
- Now suppose that  $x \in Y \setminus V$ . We thus have  $x \in X$  (since  $Y \subseteq X$ ) and  $x \notin V$ , so that we necessarily have  $x \in K$ . Thus  $x \in Y \cap K$ , i.e.  $x \in E$ .

Thus  $E = Y \setminus V$  is relatively closed w.r.t.  $Y$ , as expected.

EXERCISE 12.4.1. — *Prove Lemma 12.4.3.*

We have to prove that any subsequence  $(x^{(n_j)})_{j=1}^\infty$  of a convergent sequence  $(x^{(n)})_{n=m}^\infty$  converges to the same limit as the whole sequence itself.

Suppose that the whole sequence  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ . Let be  $\varepsilon > 0$ . By definition, we have a positive integer  $N \geq m$  such that  $n \geq N \implies d(x^{(n)}, x_0) \leq \varepsilon$ . Our aim here is to show that there exists a positive integer  $J \geq 1$  such that  $j \geq J \implies d(x^{(n_j)}, x_0) \leq \varepsilon$ .

By Definition 12.4.1, we know that we have  $m \leq n_1 < n_2 < n_3 < \dots$ . Thus, as a quick induction would show, we have  $n_j \geq m + j - 1$  for all  $j \geq 1$ . Let's take  $J := N$ . In this case, if  $j \geq J$ , i.e. if  $j \geq N$ , we have  $n_j \geq m + N - 1 \geq N$ . Thus:

$$j \geq J \implies n_j \geq N \implies d(x^{(n_j)}, x_0) \leq \varepsilon.$$

Since this is true for all  $\varepsilon > 0$ , it means that  $(x^{(n_j)})_{j=1}^\infty$  converges to  $x_0$ , as expected.

EXERCISE 12.4.2. — *Prove Proposition 12.4.5.*

Let  $(x^{(n)})_{n=m}^\infty$  be a sequence of points in a metric space. We have to prove that the following two statements are equivalent:

- (a)  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ .
- (b) There exists a subsequence  $(x^{(n_j)})_{j=1}^\infty$  of the original sequence which converges to  $L$ .

We will prove the two implications, but first, note that (with the notations from Definition 12.4.1) if we have  $1 \leq m \leq n_1 < n_2 < n_3 < \dots$ , then a quick induction shows that we have  $n_j \geq j$  for all  $j \geq 1$ .

- First we prove that (b) implies (a). If some subsequence  $(x^{(n_j)})_{j=1}^\infty$  converges to  $L$ , then we have by definition:

$$\forall \varepsilon > 0, \exists J \geq 1 : j \geq J \implies d(x^{(n_j)}, L) \leq \varepsilon \quad (12.5)$$

Now, consider any  $\varepsilon > 0$  and any  $N \geq m$ . For this particular choice of  $\varepsilon$ , consider the corresponding real number  $J$  given by equation (12.5), and let's define  $p := \max(N, J)$ . Thus, we have  $n_p \geq p \geq J$ , and by equation (12.5), we thus have  $d(x^{(n_p)}, L) \leq \varepsilon$ . If we set  $n := n_p$ , we have indeed found an  $n \geq N$  such that  $d(x^{(n)}, L) \leq \varepsilon$ . Thus,  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ , as required.

- Now we prove that (a) implies (b). Suppose that  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ . By definition, there exists a natural number  $n_1 \geq m$  such that  $d(x^{(n_1)}, L) \leq 1$ . Now, for  $j > 1$ , let's define inductively  $n_j := \min\{n > n_{j-1} : d(x^{(n)}, L) \leq 1/j\}$ . This set is non-empty (by definition of a limit point), so that the well-ordering principle

(Proposition 8.1.4) ensures that it has a (unique) minimal element, i.e. that  $n_j$  indeed exists. Let's define the subsequence  $(x^{(n_j)})_{j=1}^\infty$  obtained following this process. We thus have  $d(x^{(n_j)}, L) \leq 1/j$  for all  $j \geq 1$ , by construction.

Thus, let be  $\varepsilon > 0$ . There exists a  $j \geq 1$  such that  $0 < 1/j < \varepsilon$  (Exercise 5.4.4). Thus, for this positive integer  $j$ , we have  $d(x^{(n_j)}, L) \leq 1/j < \varepsilon$ . By construction, for all other natural numbers  $k \geq j + 1$ , we have  $d(x^{(n_k)}, L) \leq 1/k \leq 1/j \leq \varepsilon$ .

In summary, for our arbitrary choice of  $\varepsilon$ , we have showed that there exists  $j \geq 1$  such that, for all  $k \geq j$ , we have  $d(x^{(n_k)}, L) \leq \varepsilon$ . Thus, the subsequence  $(x^{(n_j)})_{j=1}^\infty$  constructed in this way converges to  $L$ , as expected.

EXERCISE 12.4.3. — *Prove Lemma 12.4.7.*

Suppose that  $(x^{(n)})_{n=m}^\infty$  is a convergent sequence of points in a metric space  $(X, d)$ , and that its limit is  $x_0$ . Let's show that it is a Cauchy sequence.

By the triangle inequality, we know that for all  $j, k \geq m$ , we have:

$$d(x^{(j)}, x^{(k)}) \leq d(x^{(j)}, x_0) + d(x^{(k)}, x_0).$$

Let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ , there exists an  $N \geq m$  such that we have  $d(x^{(n)}, x_0) \leq \varepsilon/3$  for all  $n \geq N$ . Thus, if we take  $j, k \geq N$ , we have:

$$\begin{aligned} d(x^{(j)}, x^{(k)}) &\leq d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \\ &\leq \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that  $(x^{(n)})_{n=m}^\infty$  is a Cauchy sequence, as expected.

EXERCISE 12.4.4. — *Prove Lemma 12.4.9.*

Let be an arbitrary  $\varepsilon > 0$ . Since the subsequence  $(x^{(n_j)})_{j=1}^\infty$  converges to  $x_0$ , there exists a  $J \geq 1$  such that  $d(x^{(n_j)}, x_0) \leq \varepsilon/3$  whenever  $j \geq J$ .

But the whole sequence  $(x^{(n)})_{n=m}^\infty$  is supposed to be a Cauchy sequence. Thus, there also exists a  $N \geq m$  such that  $d(x^{(j)}, x^{(k)}) < \varepsilon/3$  whenever  $j, k \geq N$ .

Now, let be  $K := \max(J, N)$ . If  $k \geq K$ , we have

$$\begin{aligned} d(x^{(k)}, x_0) &\leq d(x^{(k)}, x^{(n_k)}) + d(x^{(n_k)}, x_0) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ , as expected.

EXERCISE 12.4.5. — *Let  $(x^{(n)})_{n=m}^\infty$  be a sequence of points in a metric space  $(X, d)$  and let  $L \in X$ . Show that if  $L$  is a limit point of the sequence  $(x^{(n)})_{n=m}^\infty$ , then  $L$  is an adherent point of the set  $\{x^{(n)} : n \geq m\}$ . Is the converse true?*

First suppose that  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ . By definition, it means that

$$\forall \varepsilon > 0, \forall N \geq m, \exists n \geq N : d(x^{(n)}, L) \leq \varepsilon \quad (12.6)$$

Let be an arbitrary  $\varepsilon > 0$ , and let's take  $N = m$ . By formula (12.6) above, there exists an  $n \geq N$  such that  $d(x^{(n)}, L) \leq \varepsilon$ . Thus, this  $x^{(n)}$  belongs to both sets  $\{x^{(n)} : n \geq m\}$  and  $B(L, \varepsilon)$ . We have just proved that for all  $\varepsilon > 0$ , the intersection  $B(L, \varepsilon) \cap \{x^{(n)} : n \geq m\}$  is always non-empty. In other words,  $L$  is thus an adherent point of  $\{x^{(n)} : n \geq m\}$ .

However, the converse is not true. Indeed, consider the sequence  $(x^{(n)})_{n=1}^{\infty}$  defined in  $(\mathbb{R}, d)$  by  $x^{(1)} = 1$  and  $x^{(n)} = 0$  for all  $n \geq 2$ , i.e. the sequence  $1, 0, 0, 0, \dots$ . It is clear that  $L := 1$  is an adherent point of  $\{x^{(n)} : n \geq 1\}$  (which is just the set  $\{0, 1\}$ ). But 1 is not a limit point of  $(x^{(n)})_{n=1}^{\infty}$ , since we have  $d(x^{(n)}, 1) > 1/2$  for all  $n \geq 2$ .

EXERCISE 12.4.6. — *Show that every Cauchy sequence can have at most one limit point.*

Suppose that  $(x^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence in a metric space  $(X, d)$ , such that  $L, L'$  are limit points. Then we have  $L = L'$ . We will give two different proofs for this fact.

- **Proof 1** (short proof using previous results). By Proposition 12.4.5, since  $L$  is a limit point of  $(x^{(n)})_{n=m}^{\infty}$ , there exists a subsequence that converges to  $L$ . But by Lemma 12.4.9, it means that the whole original sequence  $(x^{(n)})_{n=m}^{\infty}$  also converges to  $L$ . The same argument can be used to show that the whole sequence  $(x^{(n)})_{n=m}^{\infty}$  converges to  $L'$ . But by uniqueness of limits (Proposition 12.1.20), we must have  $L = L'$ , as expected.
- **Proof 2** (a more “manual” proof). Let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence, there exists  $N \geq m$  such that  $d(x^{(p)}, x^{(q)}) \leq \varepsilon/3$  for all  $p, q \geq N$ .

If  $L$  is a limit point of  $(x^{(n)})_{n=m}^{\infty}$ , then for this  $N \geq m$ , there exists  $p \geq N$  such that  $d(x^{(p)}, L) \leq \varepsilon/3$ . Similarly, there exists  $q \geq N$  such that  $d(x^{(q)}, L') \leq \varepsilon/3$ .

We thus have, by triangle inequality:

$$\begin{aligned} d(L, L') &\leq d(L, x^{(p)}) + d(x^{(p)}, x^{(q)}) + d(x^{(q)}, L') \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leq \varepsilon \end{aligned}$$

Thus,  $d(L, L') \leq \varepsilon$  for all  $\varepsilon > 0$ , which implies  $L = L'$ .

EXERCISE 12.4.7. — *Prove Proposition 12.4.12.*

For statement (a), consider a convergent sequence  $(y^{(n)})_{n=m}^{\infty}$  of elements of  $Y \subseteq X$ . Since it is convergent, it is a Cauchy sequence (Lemma 12.4.7). Saying that  $(Y, d_{Y \times Y})$  is complete means that  $(y^{(n)})_{n=m}^{\infty}$  converges in  $(Y, d_{Y \times Y})$ . Thus, every convergent sequence in  $Y$  has its limit in  $Y$ : this is exactly the characterization of closed sets given by Proposition 12.2.15(b).

For statement (b), consider a Cauchy sequence  $(y^{(n)})_{n=m}^{\infty}$  of elements of a given closed subset  $Y \subseteq X$ . Since  $(X, d)$  is complete,  $(y^{(n)})_{n=m}^{\infty}$  must converge to some value  $L \in X$ . But since  $Y$  is closed, we have  $L \in Y$  by Proposition 12.2.15(b). Thus, every Cauchy sequence in  $Y$  converges in  $Y$ . This means that  $(Y, d_{Y \times Y})$  is complete, as expected.

EXERCISE 12.4.8. — *The following construction generalizes the construction of the reals from the rationals in Chapter 5. In what follows, we let  $(X, d)$  be a metric space.*

We have to prove the following statements. Note that this is a generalization of the process of construction of the real numbers, so that we can use all results relative to the real numbers below.

- (a) Given any Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $X$ , we denote  $\text{LIM}_{n \rightarrow \infty} x_n$  its formal limit. We say that two formal limits  $\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n$  are equal iff  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ . Then, this equality relation obeys the reflexive, symmetry and transitive axioms.
- This relation is reflexive: for every Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$ , we have  $d(x_n, x_n) = 0$  for all  $n \geq 1$ , by definition of a metric. Thus,  $d(x_n, x_n)$  is constant and equal to zero, so that  $\lim_{n \rightarrow \infty} d(x_n, x_n) = 0$ .
  - By the property of symmetry of the metric  $d$ , we have  $d(x_n, y_n) = d(y_n, x_n)$  for all  $n \geq 1$  and all Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ . Thus,  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} y_n$  iff  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ , iff  $\lim_{n \rightarrow \infty} d(y_n, x_n) = 0$ , which is equivalent to  $\text{LIM}_{n \rightarrow \infty} y_n = \text{LIM}_{n \rightarrow \infty} x_n$ .
  - For transitivity, suppose that  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  and  $(z^{(n)})_{n=1}^{\infty}$  are Cauchy sequences in  $X$ . If  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} y_n$  and  $\text{LIM}_{n \rightarrow \infty} y_n = \text{LIM}_{n \rightarrow \infty} z_n$ , then by definition we have  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$  and  $\lim_{n \rightarrow \infty} d(y_n, z_n) = 0$ . Let be  $\varepsilon > 0$ . By definition, there exists  $N_1 \geq 1$  such that  $d(x_n, y_n) \leq \varepsilon/2$  whenever  $n \geq N_1$ . Similarly, there exists  $N_2 \geq 1$  such that  $d(y_n, z_n) \leq \varepsilon/2$  whenever  $n \geq N_2$ . Thus, if  $n \geq N := \max(N_1, N_2)$ , we have by the triangle inequality  $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \leq \varepsilon$ . It means that  $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$ , i.e. that  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} z_n$ , as expected.
- (b) Let  $\bar{X}$  be the space of all formal limits of Cauchy sequences in  $X$ , with the above equality relation. Define a metric  $d_{\bar{X}} : \bar{X} \times \bar{X} \rightarrow \mathbb{R}^+$  by setting

$$d_{\bar{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) := \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Then this function is well-defined and gives  $\bar{X}$  the structure of a metric space.

- First we have to show that the limit  $\lim_{n \rightarrow \infty} d(x_n, y_n)$  exists (in  $\mathbb{R}^+$ ) for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ . We already know that  $\mathbb{R}$  is complete, thus  $\mathbb{R}^+$  is complete as a closed subset of the complete space  $\mathbb{R}$  (Proposition 12.4.12(b)).

Let be the sequence defined by  $u_n := d(x_n, y_n)$  for all  $n \geq 1$ . Obviously, this sequence is in  $\mathbb{R}^+$ , which is a complete space. Thus, to show that it converges, we just have to show that it is a Cauchy sequence.

Consider the usual metric on  $\mathbb{R}^+$ . We have, for all  $p, q \geq 1$ ,

$$\begin{aligned} |u_p - u_q| &= |d(x_p, y_p) - d(x_q, y_q)| \\ &\leq |d(x_p, x_q) + d(x_q, y_q) + d(y_q, y_p) - d(x_q, y_q)| \\ &\leq |d(x_p, x_q)| + |d(y_p, y_q)|. \end{aligned}$$

Now let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=1}^{\infty}$  and  $(y^{(n)})_{n=1}^{\infty}$  are Cauchy sequences, there exists  $N_1, N_2 \geq 1$  such that  $d(x_p, x_q) \leq \varepsilon/2$  whenever  $p, q \geq N_1$ , and  $d(y_p, y_q) \leq \varepsilon/2$  whenever  $p, q \geq N_2$ . Thus, if  $p, q \geq N := \max(N_1, N_2)$ , we have

$$|u_p - u_q| \leq |d(x_p, x_q)| + |d(y_p, y_q)| \leq \varepsilon.$$

This shows that  $(u_n)_{n=1}^{\infty}$  is a Cauchy sequence, and thus, that  $\lim_{n \rightarrow \infty} d(x_n, y_n)$  exists.

- Now we must show that the axiom of substitution is obeyed. In other words, consider a Cauchy sequence  $(z^{(n)})_{n=1}^{\infty}$  in  $(X, d)$  such that  $\text{LIM}_{n \rightarrow \infty} z_n = \text{LIM}_{n \rightarrow \infty} x_n$ . We must show that  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} z_n, \text{LIM}_{n \rightarrow \infty} y_n) = d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n)$ , i.e. that

$$\lim_{n \rightarrow \infty} d(z_n, y_n) = \lim_{n \rightarrow \infty} d(x_n, y_n) \quad (12.7)$$

By the previous bullet point, we know that both limits in (12.7) do exist. Thus, the limit laws apply. We have:

$$d(z_n, y_n) \leq d(z_n, x_n) + d(x_n, y_n)$$

but since  $\lim_{n \rightarrow \infty} d(z_n, x_n) = 0$  by definition, we obtain

$$\lim_{n \rightarrow \infty} d(z_n, y_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n)$$

if we take the limits of both sides in the previous inequality.

But similarly, we have  $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ , so that a similar argument gives

$$\lim_{n \rightarrow \infty} d(x_n, y_n) \leq \lim_{n \rightarrow \infty} d(z_n, y_n).$$

Thus, we have indeed  $\lim_{n \rightarrow \infty} d(z_n, y_n) = \lim_{n \rightarrow \infty} d(x_n, y_n)$ , as expected.

- Finally, we must show that  $d_{\overline{X}}$  is a metric on  $\overline{X}$ . To prove this statement, we must show that  $d_{\overline{X}}$  obeys all four axioms that define a metric.
  - First, it is clear that  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0$  for all Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $(X, d)$ .
  - Now let be two Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  in  $X$ , such that  $\text{LIM}_{n \rightarrow \infty} x_n \neq \text{LIM}_{n \rightarrow \infty} y_n$ . This latest property implies that  $\lim_{n \rightarrow \infty} d(x_n, y_n) > 0$ , by definition. Thus,  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) > 0$ .
  - Symmetry: we have

$$\begin{aligned} d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} d(y_n, x_n) \text{ (symmetry of } d \text{ on } \mathbb{R}^+) \\ &= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} y_n, \text{LIM}_{n \rightarrow \infty} x_n) \end{aligned}$$

for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ .

- Triangle inequality: by the limit laws, we have

$$\begin{aligned} d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} z_n) &= \lim_{n \rightarrow \infty} d(x_n, z_n) \\ &\leq \lim_{n \rightarrow \infty} (d(x_n, y_n) + d(y_n, z_n)) \\ &\leq \lim_{n \rightarrow \infty} d(x_n, y_n) + \lim_{n \rightarrow \infty} d(y_n, z_n) \\ &\leq d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) + d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} y_n, \text{LIM}_{n \rightarrow \infty} z_n) \end{aligned}$$

for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  and  $(z^{(n)})_{n=1}^{\infty}$ .

Thus,  $d_{\overline{X}}$  is indeed a metric on  $\overline{X}$ .

(c) The metric space  $(\overline{X}, d_{\overline{X}})$  is complete.

To prove this statement, consider a Cauchy sequence  $(u_n)_{n=1}^\infty$  in  $\overline{X}$ : we have to prove that this sequence converges in  $(\overline{X}, d_{\overline{X}})$ .

By definition,  $(u_n)_{n=1}^\infty$  is a Cauchy sequence of formal limits of Cauchy sequences that take their values in  $X$ ; i.e., for all  $k \geq 1$ , there exists a Cauchy sequence  $(x_n^{(k)})_{n=1}^\infty$  of elements of  $X$  such that  $u_k := \text{LIM}_{n \rightarrow \infty} x_n^{(k)}$ .

Since all  $(x_n^{(k)})_{n=1}^\infty$  are Cauchy sequences, then for all  $k \geq 1$ , there exists a threshold  $N_k$  such that  $d(x_n^{(k)}, x_{N_k}^{(k)}) < 1/k$  whenever  $n \geq N_k$ . Thus, (using the countable axiom of choice) we can build a sequence  $(z_k)_{k=1}^\infty$  defined by

$$z_k := \left( x_{N_k}^{(k)} \right)$$

for all  $k \geq 1$ . Now:

- We claim that  $(z_k)_{k=1}^\infty$  is itself a Cauchy sequence. Indeed, consider an arbitrary positive real number  $\varepsilon > 0$ . We must prove that  $d(z_p, z_q) := d(x_{N_p}^{(p)}, x_{N_q}^{(q)})$  is eventually lesser than  $\varepsilon$ .

Since  $(u_n)_{n=1}^\infty$  is a Cauchy sequence in  $\overline{X}$ , there exists a  $N \geq 1$  such that, if  $p, q \geq N$ , we have  $d_{\overline{X}}(u_p, u_q) < \varepsilon/3$ , i.e.:

$$\begin{aligned} \varepsilon/3 &> d_{\overline{X}}(u_p, u_q) \\ &\geq d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n^{(p)}, \text{LIM}_{n \rightarrow \infty} x_n^{(q)}) \\ &\geq \lim_{n \rightarrow \infty} d(x_n^{(p)}, x_n^{(q)}) \end{aligned}$$

Thus, there exists a  $N' \geq 1$  such that, if  $n \geq N'$ , we have  $d(x_n^{(p)}, x_n^{(q)}) \leq \varepsilon/3^2$ . Also, by Exercise 5.4.4, there exists a  $k > 0$  such that  $1/k \leq \varepsilon/3$ . Thus, if  $n, p, q \geq \max(k, N', N_p, N_q)$ , we have

$$\begin{aligned} d(z_p, z_q) &= d(x_{N_p}^{(p)}, x_{N_q}^{(q)}) \\ &\leq \underbrace{d(x_{N_p}^{(p)}, x_n^{(p)})}_{\leq 1/p \leq \varepsilon/3} + \underbrace{d(x_n^{(p)}, x_n^{(q)})}_{\leq \varepsilon/3} + \underbrace{d(x_n^{(q)}, x_{N_q}^{(q)})}_{\leq 1/q \leq \varepsilon/3} \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leq \varepsilon \end{aligned}$$

Thus,  $(z_k)_{k=1}^\infty$  is indeed a Cauchy sequence in  $X$ .

- Consequently, we can take the formal limit  $L := \text{LIM}_{n \rightarrow \infty} z_n$ , and this formal limit  $L$  lies in  $\overline{X}$  by definition. We claim that  $\lim_{n \rightarrow \infty} u_n = L \in \overline{X}$ ; proving this claim will close the proof of (c).

Let be  $\varepsilon > 0$ . Since  $(z_n)_{n=1}^\infty$  is a Cauchy sequence in  $X$ , there exists a  $N_1 \geq 1$  such that  $d(z_p, z_q) \leq \varepsilon/2$  whenever  $p, q \geq N_1$ .

---

<sup>2</sup>Indeed, for any sequence  $(v_n)_{n=1}^\infty$  that converges to  $\ell$ , if we have  $0 \leq \ell < \varepsilon$ , then there exists an  $N \geq 1$  such that  $v_n \leq \varepsilon$  whenever  $n \geq N$  (why? use a proof by contradiction.).

Once again, by Exercise 5.4.4, there exists a  $K' \geq 1$  such that  $1/K' < \varepsilon/2$ . Thus, if  $k \geq K$  and  $n > N_k$ , we have

$$d(x_n^{(k)}, z_k) := d(x_n^{(k)}, x_{N_k}^{(k)}) < \frac{1}{k} \leq \frac{1}{K} < \frac{\varepsilon}{2}.$$

Thus, by the triangle inequality, we have, for all  $n > \max(N_k, N_1)$ ,

$$d(x_n^{(k)}, z_n) \leq d(x_n^{(k)}, z_k) + d(z_k, z_n) \leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon.$$

Consequently, we have, for all  $k > K'$ ,

$$d_{\overline{X}}(u_k, L) := \lim_{n \rightarrow \infty} d(x_n^{(k)}, b_n) < \varepsilon.$$

This shows that  $(u_n)_{n=1}^\infty \rightarrow L$  in  $(\overline{X}, d_{\overline{X}})$ , which closes the proof.

(d) We identify an element  $x \in X$  with the corresponding formal limit  $\text{LIM}_{n \rightarrow \infty} x$  in  $\overline{X}$ .

- This is legitimate since we have  $x = y \iff \text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$ .  
Indeed, it is clear that if  $x = y$ , then we have  $\text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$  by definition. Conversely, if  $\text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$ , then we have  $\lim_{n \rightarrow \infty} d(x, y) = 0$ , i.e.  $d(x, y) = 0$ , i.e.  $x = y$ . Thus, this identification is legitimate.
- With this identification, we have  $d(x, y) = d_{\overline{X}}(x, y)$ . Indeed:

$$\begin{aligned} d_{\overline{X}}(x, y) &= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x, \text{LIM}_{n \rightarrow \infty} y) \\ &= \lim_{n \rightarrow \infty} d(x, y) \\ &= d(x, y). \end{aligned}$$

Thus,  $(X, d)$  can be thought of as a subspace of  $(\overline{X}, d_{\overline{X}})$ .

(e) The closure of  $X$  in  $\overline{X}$  is  $\overline{X}$ .

Indeed, let be  $C$  the closure of  $X$  in  $\overline{X}$ . We clearly have  $C \subseteq \overline{X}$ , by definition. Thus we just have to show that  $\overline{X} \subseteq C$ .

Let be  $x \in \overline{X}$ , and let's show that  $x \in C$ . By definition,  $x \in C$  means that  $x$  is an adherent point of  $X$  in  $\overline{X}$ , i.e. that for all  $\varepsilon > 0$ ,  $B_{(\overline{X}, d_{\overline{X}})}(x, \varepsilon) \cap X \neq \emptyset$ . In other words, for all  $\varepsilon > 0$ , we must show that there exists a  $y \in X$  such that  $d_{\overline{X}}(x, y) < \varepsilon$ .

Thus, let be  $\varepsilon > 0$ . By definition,  $x$  is the formal limit of a Cauchy sequence  $(x_n)_{n=1}^\infty$  of elements of  $X$ , so that  $x := \text{LIM}_{n \rightarrow \infty} x_n$ . Since  $(x_n)_{n=1}^\infty$  is a Cauchy sequence, there exists an  $N \geq 1$  such that  $d(x_n, x_N) < \varepsilon/2$  whenever  $n \geq N$ . Thus:

$$\begin{aligned} d_{\overline{X}}(x, x_N) &:= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} x_N) \\ &= \lim_{n \rightarrow \infty} d(x_n, x_N) \\ &\leq \varepsilon/2 < \varepsilon \end{aligned}$$

so that  $y := x_N$  is a convenient choice. This shows that  $x$  is an adherent point of  $X$  in  $\overline{X}$ , as expected.

- (f) Finally, the formal limit agrees with the actual limit, i.e.,  $\lim_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} x_n \in \overline{X}$  for all Cauchy sequence  $(x_n)_{n=1}^\infty$  in  $X$ .

Indeed, let be  $(x_n)_{n=1}^\infty$  a Cauchy sequence of elements of  $X$ . We know that  $(X, d)$  can be thought of as a subspace of  $(\overline{X}, d_{\overline{X}})$ , so that  $(x_n)_{n=1}^\infty$  can be thought of as a sequence of elements of  $\overline{X}$ . But we have showed that  $(\overline{X}, d_{\overline{X}})$  is complete. Thus, the sequence  $(x_n)_{n=1}^\infty$  converges in  $\overline{X}$  to a certain limit  $L \in \overline{X}$ ; i.e., we have  $\lim_{n \rightarrow \infty} x_n = L$  for some  $L \in \overline{X}$ .

Consider this limit  $L$ . By definition of  $\overline{X}$ , there exists a Cauchy sequence  $(a_n)_{n=1}^\infty$  of elements of  $X$  such that  $L := \text{LIM}_{n \rightarrow \infty} a_n$ . What we need to prove is that we have

$$L = \lim_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n \quad (12.8)$$

and thus, it is sufficient to show that  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n$ , since we already have the other equalities. And, by definition of the equality relation established in (a), in order to prove that  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n$ , we just have to show that  $\lim_{n \rightarrow \infty} d(x_n, a_n) = 0$ . Or, in yet another equivalent way, we have to show that for all  $\varepsilon > 0$ , there exists an  $N \geq 1$  such that  $d(x_n, a_n) \leq \varepsilon$  whenever  $n \geq N$ .

Thus, let be an arbitrary  $\varepsilon > 0$ . Let's unfold our hypotheses.

- We know that the sequence  $(x_n)_{n=1}^\infty$  converges to  $L$  in  $\overline{X}$ . Thus, by definition, there exists a  $N_1 \geq 1$  such that  $d_{\overline{X}}(x_k, L) \leq \varepsilon/2$  whenever  $k \geq N_1$ . In other words,  $\lim_{n \rightarrow \infty} d(x_k, a_n) \leq \varepsilon/3 < \varepsilon/2$  whenever  $k \geq N_1$ .  
Thus, there exists a  $N_2$  such that  $d(x_k, a_n) \leq \varepsilon/2$  whenever  $k \geq N_1$  and  $n \geq N_2$  (see footnote 2 p. 22 from the present document).
- We also know that  $(x_n)_{n=1}^\infty$  is a Cauchy sequence. It means that there exists a  $N_3 \geq 1$  such that  $d(x_p, x_q) \leq \varepsilon/2$  for all  $p, q \geq N_3$ .

Let be  $N := \max(N_1, N_2, N_3)$ . Using the triangle inequality, we finally get, for all  $n \geq N$ ,

$$\begin{aligned} d(x_n, a_n) &\leq d(x_n, x_N) + d(x_N, a_n) \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &\leq \varepsilon \end{aligned}$$

This closes the proof.

**EXERCISE 12.5.1.** — *Show that Definitions 9.1.22 and 12.5.3 match when talking about subsets of the real line with the standard metric.*

Consider  $Y \subseteq \mathbb{R}$  and the standard metric  $d(x, y) = |x - y|$  for all  $x, y \in \mathbb{R}$ . We have to show that both definitions of boundedness are equivalent in this case.

- First, suppose that  $Y$  is bounded in the sense of Definition 12.5.3. Thus, there exists a real number  $x$  and a positive real number  $r > 0$  such that  $Y \subseteq B(x, r)$ . In other words, we have  $Y \subseteq ]x - r, x + r[ \subseteq [x - r, x + r]$ . Let be  $M := |x| + |r|$ . We clearly have  $x + r \leq M$ , and  $-M \leq x - r$ . Thus, we have  $Y \subseteq [-M, M]$ , and  $Y$  is bounded in the sense of Definition 9.1.22.



- Conversely, suppose that  $Y$  is bounded in the sense of Definition 9.1.22. Thus, there exists a positive real  $M > 0$  such that  $Y \subseteq [-M, M] \subset ]-2M, 2M[$ . But this later interval is simply  $B(0, 2M)$ , so that  $Y$  is bounded in the sense of Definition 12.5.1, taking  $x := 0$  and  $r := 2M$ .

EXERCISE 12.5.2. — *Prove Proposition 12.5.5.*

We must prove that any compact space  $(X, d)$  is both complete and bounded. In both cases, we will use a proof by contradiction.

- First, let's prove completeness. Suppose, for the sake of contradiction, that the compact space  $(X, d)$  is not complete. Since it is not complete, there exists a Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  of elements of  $X$  which does not converge in  $(X, d)$ . But since it is compact, there exists a subsequence  $(x^{(n_k)})_{k=1}^{\infty}$  of this Cauchy sequence, which converges in  $(X, d)$ . But, by Lemma 12.4.9, if a Cauchy sequence has a convergent subsequence, then it is convergent itself; thus  $(x^{(n)})_{n=1}^{\infty}$  converges. It is a clear contradiction. Thus,  $(X, d)$  must be complete.
- Now we show boundedness. Similarly, suppose for the sake of contradiction that  $(X, d)$  is not bounded. It means that, for all positive real  $r > 0$  and all  $x \in X$ , we have  $X \not\subseteq B(x, r)$ . In particular, for any positive natural number  $n \geq 1$  and an arbitrary  $x \in X$ , the set  $X \setminus B(x, n)$  is not empty. Thus, using the (countable) axiom of choice, we can build a sequence  $(x^{(n)})_{n=1}^{\infty}$  such that  $x^{(n)} \in X \setminus B(x, n)$  for all positive integer  $n \geq 1$ . Or, in other words, we have  $d(x, x^{(n)}) \geq n$  for all  $n \geq 1$ .

But recall that  $(X, d)$  is compact. Thus, there must exist a convergent subsequence  $(x^{(n_k)})_{k=1}^{\infty}$  of the original sequence. Say that this subsequence converges to some value  $L$ . Thus, by definition,

$$\forall \varepsilon > 0, \exists K \geq 1 : k \geq K \implies d(x^{(n_k)}, L) \leq \varepsilon.$$

Let's take  $\varepsilon := 1$  (there is nothing special about this value; this is just any arbitrary  $\varepsilon$  to obtain a contradiction). There must exist a  $K_1 \geq 1$  such that  $d(x^{(n_k)}, L) \leq 1$  whenever  $k \geq K_1$ . But, at the same time, we have by the triangle inequality

$$\begin{aligned} d(x^{(n_k)}, x) &\leq d(x^{(n_k)}, L) + d(L, x) \\ \implies d(x^{(n_k)}, L) &\geq d(x^{(n_k)}, x) - d(L, x) \end{aligned}$$

For instance by the Archimedean principle, there exists an  $N \in \mathbb{N}$  such that  $N \geq d(L, x) + 3$ . Let be  $K_2 := \min\{k \in \mathbb{N} : n_k \geq N\}$  (this natural number exists simply because  $n_N \geq N$ , so that the set is not empty). We thus have

$$d(x, x^{(n_k)}) \geq n_k \geq N \geq d(L, x) + 3$$

for all  $k \geq K_2$ .

Thus, for all  $k \geq \max(K_1, K_2)$ , we have both  $d(x^{(n_k)}, x) \leq 1$  (because  $k \geq K_1$ ), and  $d(x^{(n_k)}, L) \geq d(x^{(n_k)}, x) - d(L, x) \geq d(L, x) + 3 - d(L, x) \geq 3$  (because  $k \geq K_2$ ). This is a contradiction. Thus,  $(X, d)$  is bounded.

EXERCISE 12.5.3. — *Prove Theorem 12.5.7.*

Let be  $(\mathbb{R}^n, d)$  an Euclidean space, where  $d$  is either the Euclidean, taxicab or sup norm metric. Also, let be  $E \subseteq \mathbb{R}^n$ . We have to prove that  $E$  is compact iff  $E$  is closed and bounded. By Corollary 12.5.6, we already know that if  $E$  is compact, then it is closed and bounded. We thus have to prove the converse implication.

Suppose that  $E$  is both closed and bounded. Since  $E$  is a subset of  $\mathbb{R}^n$ , we can write  $E := E_1 \times \dots \times E_n$ , where  $E_j \subseteq \mathbb{R}$  for all  $1 \leq j \leq n$ .

We have to prove that any sequence  $(x^{(k)})_{k=1}^\infty$  in  $E$  has a convergent subsequence in  $(E, d)$ . This sequence can be written as a sequence of vectors of length  $n$ , i.e., we have  $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$ , where  $x_j^{(k)} \in E_j$  for all  $k \geq 1$  and all  $1 \leq j \leq n$ .

We will first need a lemma:

**Lemma.** If  $E$  is bounded, then each  $E_j \subseteq \mathbb{R}$  is also bounded.

*Sketch of proof.* Suppose that  $d$  is the sup norm metric. If  $E$  is bounded, we have  $E \subseteq B(x, r)$  for some  $x \in \mathbb{R}^n$  and some  $r > 0$  (Definition 12.5.3). In other words, we have  $d(x, y) < r$  for all  $y \in E$ . Since  $d$  is the sup norm metric, this implies that

$$\forall j \in \llbracket 1, n \rrbracket, |x_j - y_j| \leq \max_{j=1, \dots, n} |x_j - y_j| := d(x, y) < r.$$

Thus,  $E_j \subseteq B(x_j, r)$ , i.e.  $E_j$  is bounded for all  $1 \leq j \leq n$ .

The proof is similar if  $d$  is the Euclidean metric, or the taxicab metric.  $\square$

Now we go back to the main proof. Since each sequence  $(x_j^{(k)})_{k=1}^\infty$  is a sequence of real numbers in the bounded subset  $E_j \subseteq \mathbb{R}$ , then by Theorem 9.1.24 this sequence has a convergent subsequence  $(x_j^{(k_l)})_{l=1}^\infty$ , which converges to  $L_j \in \mathbb{R}_j$ . But by Proposition 12.1.18, this implies that the whole subsequence  $(x^{(k_l)})_{l=1}^\infty$  converges to  $(L_1, \dots, L_n)$  (since it converges component-wise).

Thus,  $(x_j^{(k)})_{k=1}^\infty$  indeed has a convergent subsequence, as expected; and  $E$  is compact.

EXERCISE 12.5.4. — *Let  $(\mathbb{R}, d)$  be the real line with the standard metric. Give an example of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and an open set  $V \subseteq \mathbb{R}$ , such that the image  $f(V) := \{f(x) : x \in V\}$  of  $V$  is not open.*

As a simple example, consider the constant function  $f(x) = 0$  defined on  $V := ]-1, 1[$ . The interval  $V$  is clearly open, but we have  $f(V) = \{0\}$ . This singleton (or more generally, any singleton) is not open in  $(\mathbb{R}, d)$ , since for all  $r > 0$ , there always exists a real number  $x$  such that  $x \in B(0, r) \setminus \{0\}$ .

EXERCISE 12.5.5. — *Let  $(\mathbb{R}, d)$  be the real line with the standard metric. Give an example of a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and closed set  $F \subseteq \mathbb{R}$ , such that  $f(F)$  is not closed.*

One can give the example of the function  $\tan^{-1}(x)$  defined on the closed set  $F := \mathbb{R}$ , but this function has not really been defined so far in the book. So, let's use a simpler example.

Consider the closed set  $F := [1, +\infty[$  and the function  $f(x) = 1/x$ . We have  $f(F) = ]0, 1]$ , which is not a closed set.

EXERCISE 12.5.6. — *Prove Corollary 12.5.9.*

Consider a sequence  $K_1 \supset K_2 \supset K_3 \supset \dots$  of non-empty compact sets in a metric space  $(X, d)$ . We have to show that  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ .

Let's work in the space  $(K_1, d_{K_1 \times K_1})$ . We define the sets  $V_n := K_1 \setminus K_n$  for all  $n \geq 1$ , i.e.,

$$V_1 := K_1 \setminus K_1 = \emptyset$$

$$V_2 := K_1 \setminus K_2$$

$$V_3 := K_1 \setminus K_3$$

...

so that the  $V_n$  clearly constitute an increasing sequence:

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots,$$

so that  $\bigcup_{k=1}^n V_k = V_n$  for all  $n \geq 1$ .

Furthermore, each set  $V_n$  is open in  $(K_1, d_{K_1 \times K_1})$ , since it is the complementary set of a compact (and then closed) set (Proposition 12.2.15 (e)).

Suppose, for the sake of contradiction, that we have  $\bigcap_{n=1}^{\infty} K_n = \emptyset$ . We would thus have:

$$\begin{aligned} \bigcup_{n=1}^{\infty} V_n &= \bigcup_{n=1}^{\infty} (K_1 \setminus K_n) \\ &= K_1 \setminus \left( \bigcap_{n=1}^{\infty} K_n \right) \quad (\text{Exercise 3.4.11}) \\ &= K_1 \setminus \emptyset \quad (\text{by hypothesis}) \\ &= K_1. \end{aligned}$$

But since  $K_1$  is compact, then by Theorem 12.5.8, there exists a finite open cover of  $K_1$ , i.e., there exists a finite number  $k$  of indices  $n_1 < \dots < n_k$  such that

$$\bigcup_{n \in \{n_1, \dots, n_k\}} V_n = K_1.$$

But since the  $V_n$  form an increasing sequence, this implies  $V_{n_k} = K_1$ , i.e.,  $K_1 \setminus K_{n_k} = K_1$ , so that we finally get  $K_{n_k} = \emptyset$ .

But all the sets  $K_n$  were supposed to be non empty: this is thus a contradiction, and we must have  $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$ .

EXERCISE 12.5.7. — *Prove Theorem 12.5.10.*

Let be  $(X, d)$  a metric space.

- (a) Let be  $Z \subseteq Y \subseteq X$ , with  $Y$  compact. We have to show that  $Z$  is closed iff it is compact. We already know that if  $Z$  is compact, then it is closed (Corollary 12.5.6); so that we just have to show the converse implication.

Suppose that  $Z$  is closed, and let be  $(z^{(n)})_{n=1}^{\infty}$  a sequence of elements of  $Z$ . Since  $Z \subseteq Y$ ,  $(z^{(n)})_{n=1}^{\infty}$  is also a sequence of elements of  $Y$ ; and since  $Y$  is compact, there exists a subsequence  $(z^{(n_k)})_{k=1}^{\infty}$  that converges to some  $z \in Y$ . But since  $Z$  is closed, we must have  $z \in Z$  (by Proposition 12.2.15(b)). Thus, any sequence of elements of  $Z$  has a subsequence that converges in  $Z$ , i.e.,  $Z$  is indeed compact.

- (b) Let be  $Y_1, \dots, Y_n$  be  $n$  compact subsets of  $X$ ; we have to show that the finite union  $Y_1 \cup \dots \cup Y_n$  is compact. Let's use the topological characterization of compact sets: suppose that we have an open cover  $\bigcup_{\alpha \in I} V_\alpha$  (possibly uncountable), i.e. that

$$Y_1 \cup \dots \cup Y_n \subseteq \bigcup_{\alpha \in I} V_\alpha.$$

Clearly, we have  $Y_1 \subseteq \bigcup_{\alpha \in I} V_\alpha$ , and since  $Y_1$  is compact, there exists a finite open cover, i.e.  $Y_1 \subseteq \bigcup_{i=1}^{s_1} V_{a_i}$ . Similarly, there exist finite open covers for each other subset  $Y_i$ , i.e.,

$$\begin{aligned} Y_2 &\subseteq \bigcup_{i=1}^{s_2} V_{b_i} \\ &\dots \\ Y_n &\subseteq \bigcup_{i=1}^{s_n} V_{n_i}. \end{aligned}$$

Thus, there exists a finite open cover

$$Y_1 \cup \dots \cup Y_n \subseteq \bigcup_{\alpha \in \{a_1, \dots, a_{s_1}, b_1, \dots, b_{s_2}, \dots, n_1, \dots, n_{s_n}\}} V_\alpha$$

so that  $Y_1 \cup \dots \cup Y_n$  is indeed compact.

- (c) Let be  $Y$  a finite subset of  $X$ ; we have to show that  $Y$  is compact.

First, suppose that  $Y$  is a singleton  $\{a\}$ . By definition, any sequence of elements of  $Y$  can only be the constant sequence  $a, a, a, \dots$ . Thus, any subsequence of this sequence is still the constant sequence  $a, a, \dots$ , and still converges to  $a$ . Thus, any sequence of elements of  $Y$  has a subsequence that converges in  $Y$ , i.e.,  $Y$  is compact.

Now suppose that  $Y$  is a finite subset of cardinality  $n$ . Let's write  $Y := \{y_1, \dots, y_n\}$ . This can also be written  $Y := \{y_1\} \cup \dots \cup \{y_n\}$ , so that we are back in the previous case (b):  $Y$  is the finite union of compact subsets of  $X$ . Thus,  $Y$  is itself compact.

Note that for the limit case  $Y = \emptyset$ , we can say that the empty set is just a closed<sup>3</sup> subset of the compact set  $\{a\}$ , so that by the previous case (a),  $Y = \emptyset$  is compact.

EXERCISE 12.5.8. — Let  $(X, d_{l^1})$  be the metric space from Exercise 12.1.15. For each natural number  $n$ , let  $e^{(n)} = (e_j^{(n)})_{j=0}^\infty$  be the sequence in  $X$  such that  $e_j^{(n)} := 1$  when  $n = j$  and  $e_j^{(n)} := 0$  when  $n \neq j$ . Show that the set  $\{e^{(n)} : n \in \mathbb{N}\}$  is a closed and bounded subset of  $X$ , but is not compact.

Recall that  $(X, d_{l^1})$  is the metric space of absolutely convergent sequences, with the metric defined by  $d_{l^1}((a^{(n)}), (b^{(n)})) := \sum_{n=0}^\infty |a_n - b_n|$ . Hereafter, we denote  $E := \{e^{(n)} : n \in \mathbb{N}\}$ , with

$$e^{(0)} := 1, 0, 0, 0, \dots$$

$$e^{(1)} := 0, 1, 0, 0, \dots$$

$$e^{(2)} := 0, 0, 1, 0, \dots$$

...

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<sup>3</sup>See Remark 12.2.14.

- First, we show that  $E$  is not compact. To prove this statement, we just have to find one sequence of elements of  $E$  that has no convergent subsequence in  $E$ .

Consider the “canonical” sequence of elements of  $E$  defined by  $e^{(0)}, e^{(1)}, e^{(2)}, \dots$ . The distance between any two distinct elements of this sequence is

$$d_{l^1}(e^{(j)}, e^{(k)}) := \sum_{i=0}^{\infty} |e_i^{(j)} - e_i^{(k)}| = 2 > 1.$$

Thus, this sequence is not a Cauchy sequence itself, and it is clear that no subsequence can be a Cauchy sequence either. Thus, no subsequence of this sequence can converge in  $E$ , i.e.,  $E$  is not compact.

- However,  $E$  is a closed subset of  $X$ . To prove this property, consider a convergent sequence of elements of  $E$ ; we have to prove that its limit lies in  $E$ . We’ve just shown that the distance between any two distinct terms  $e^{(j)}, e^{(k)}$  for  $j \neq k$  is equal to 2. Thus, if a sequence of elements of  $E$  converges, it must be eventually 0.5-stable, and the only possibility for that is to be eventually constant. In other words, it must be eventually equal to  $e^{(n_0)}$  for  $n_0 \in \mathbb{N}$ , so that it necessarily converges to  $e^{(n_0)}$ , which is an element of  $E$ . This shows that  $E$  is closed.
- Furthermore,  $E$  is bounded. To show the boundedness of  $E$ , we have to show that  $E \subseteq B_{(X, d_{l^1})}((x_j)_{j=0}^{\infty}, r)$  for some  $r > 0$  and some sequence  $(x_j)_{j=0}^{\infty} \in X$ . Consider the zero sequence  $(z_j)_{j=0}^{\infty} := 0, 0, 0, \dots$ . This is clearly a sequence in  $X$  (since it converges to 0), and we have

$$d_{l^1}\left((z_j)_{j=0}^{\infty}, (e_j^{(n)})_{j=0}^{\infty}\right) = \sum_{j=0}^{\infty} |z_j - e_j^{(n)}| = 1 < 2$$

for all  $n \in \mathbb{N}$ . Thus, we have  $E \subseteq B_{(X, d_{l^1})}((z_j)_{j=0}^{\infty}, 2)$ , which shows that  $E$  is bounded.

Thus, the case of the subset  $E$  of the metric space  $(X, d_{l^1})$  shows that the Heine-Borel theorem (stated for the metric space  $(\mathbb{R}^n, d)$ ) is not valid in more general metric spaces.

**EXERCISE 12.5.9.** — *Show that a metric space  $(X, d)$  is compact if and only if every sequence in  $X$  has at least one limit point.*

A metric space  $(X, d)$  is compact iff any sequence of elements of  $X$  has a subsequence that converges in  $(X, d)$ . Thus, the statement is a direct consequence of Proposition 12.4.5, which says basically that “having a convergent subsequence” and “having a limit point” are synonymous.

**EXERCISE 12.5.13.** — *Let  $E$  and  $F$  be two compact subsets of  $\mathbb{R}$  (with the standard metric  $d(x, y) = |x - y|$ ). Show that the Cartesian product  $E \times F := \{(x, y) : x \in E, y \in F\}$  is a compact subset of  $\mathbb{R}^2$  (with the Euclidean metric  $d_2$ ).*

To prove that  $E \times F$  is compact, we will show that it is both closed and bounded (by Heine-Borel theorem).

- First we show that  $E \times F$  is bounded.

Since  $E$  and  $F$  are compact, they are themselves bounded (by Heine-Borel theorem). Thus, there exist  $a \in E$ ,  $b \in F$  and  $r_1, r_2 > 0$  such that  $E \subseteq B_d(a, r_1)$  and  $F \subseteq B_d(b, r_2)$ , by Definition 12.5.3. In other words, we have:

$$\begin{aligned}\forall x \in E, |x - a| &< r_1 \\ \forall y \in F, |y - b| &< r_2.\end{aligned}$$

Thus, let be  $(x, y) \in E \times F$ . We have:

$$\begin{aligned}d_{l^2}((x, y), (a, b)) &= \sqrt{(x - a)^2 + (y - b)^2} \\ &< \sqrt{r_1^2 + r_2^2}.\end{aligned}$$

This means that each  $(x, y) \in E \times F$  lies in  $B_{d_{l^2}}\left((a, b), \sqrt{r_1^2 + r_2^2}\right)$ . Thus,  $E \times F$  is indeed bounded.

- Now let's show that  $E \times F$  is closed.

Since  $E$  and  $F$  are compact, they are themselves closed (by Heine-Borel theorem). Consider a sequence  $((x^{(n)}, y^{(n)}))_{n=1}^{\infty}$  of elements of  $E \times F$  which converges to  $(x_0, y_0)$  with respect to  $d_{l^2}$ . By Proposition 12.1.18, this means that this sequence converges component-wise, i.e. that  $(x^{(n)})_{n=1}^{\infty}$  converges to  $x_0$ , and  $(y^{(n)})_{n=1}^{\infty}$  converges to  $y_0$ . By definition, we have  $x_0 \in E$  and  $y_0 \in F$ , since  $E$  and  $F$  are closed. Thus,  $(x_0, y_0) \in E \times F$ . This shows that  $E \times F$  is indeed bounded.

Thus,  $E \times F$  is compact, as expected.

### 13. Continuous functions on metric spaces

EXERCISE 13.1.1. — *Prove Theorem 13.1.4.*

Since the implication  $(b) \implies (c)$  may be slightly more difficult to write, we will prove the implications  $(a) \implies (c)$ ,  $(c) \implies (b)$  and  $(b) \implies (a)$  in this order.

Let be  $f : (X, d_X) \rightarrow (Y, d_Y)$ , and  $x_0 \in X$ .

- First let's prove  $(a) \implies (c)$ . Suppose that  $f$  is continuous at  $x_0$ , and let be  $V \subseteq Y$  an open set that contains  $f(x_0)$ . By Proposition 12.2.15(a), there exists a  $\varepsilon > 0$  such that  $B_Y(f(x_0), \varepsilon) \subseteq V$ . But since  $f$  is continuous at  $x_0$ , we know that there exists a  $\delta > 0$  such that  $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$ . Thus, if we set  $U := B_X(x_0, \delta)$ , we have found an open set  $U \subseteq X$  such that  $f(U) \subseteq B_Y(f(x_0), \varepsilon) \subseteq V$ , as required.
- Now we prove  $(c) \implies (b)$ . Consider a sequence  $(x^{(n)})_{n=1}^\infty$  in  $X$  which converges to  $x_0$  with respect to  $d_X$ . Let be an arbitrary  $\varepsilon > 0$ ; we set  $V_\varepsilon := B_Y(f(x_0), \varepsilon)$ . By (c), we know that there exists an open set  $U \subseteq X$  containing  $x_0$  and such that  $f(U) \subseteq V_\varepsilon$ . But since  $U$  is open set, by Proposition 12.2.15(a), there exists a  $\delta > 0$  such that  $B_X(x_0, \delta) \subseteq U$ .

Since  $(x^{(n)})_{n=1}^\infty$  converges to  $x_0$ , there exists a natural number  $N \geq 1$  such that  $d_X(x^{(n)}, x_0) < \delta$  whenever  $n \geq N$ . Or, in other words, we have  $x^{(n)} \in B_X(x_0, \delta) \subseteq U$  whenever  $n \geq N$ .

But since  $f(U) \subseteq V$  by hypothesis, we thus have  $f(x^{(n)}) \in V_\varepsilon$  whenever  $n \geq N$ . Since this is true for any arbitrary  $\varepsilon > 0$ , this shows that the sequence  $(f(x^{(n)}))_{n=1}^\infty$  converges to  $f(x_0)$  with respect to  $d_Y$ , as expected.

- Finally, we prove  $(b) \implies (a)$ . Suppose that  $(f(x^{(n)}))_{n=1}^\infty$  converges to  $f(x_0)$  whenever  $(x^{(n)})_{n=1}^\infty$  converges to  $x_0$ , and let's show that  $f$  is continuous at  $x_0$ .

Suppose, for the sake of contradiction, that  $f$  is *not* continuous at  $x_0$ . Thus, there exists an  $\varepsilon > 0$  such that for all  $\delta > 0$ , there exists an  $x \in X$  such that  $d_Y(f(x), f(x_0)) \geq \varepsilon$  although  $d_X(x, x_0) < \delta$ .

Thus, using the (countable) axiom of choice, we build a sequence  $(x^{(n)})_{n=1}^\infty$  such that, for all  $n \geq 1$ , we have  $d_Y(f(x^{(n)}), f(x_0)) \geq \varepsilon$  although  $d_X(x^{(n)}, x_0) < \frac{1}{n}$ . It is thus clear that  $(x^{(n)})_{n=1}^\infty$  converges to  $x_0$ , but that  $(f(x^{(n)}))_{n=1}^\infty$  does not converge to  $f(x_0)$ , since  $f(x^{(n)})$  and  $f(x_0)$  are never  $\varepsilon/2$ -close. This is a contradiction with (c). Thus,  $f$  must be continuous at  $x_0$ , as expected.

EXERCISE 13.1.2. — *Prove Theorem 13.1.5.*

We already know from Theorem 13.1.4 that  $(a)$  and  $(b)$  are equivalent. Let's prove the other implications.

- First we prove that  $(a) \implies (c)$ . Let be  $V$  an open set in  $Y$ . We must show that  $f^{-1}(V)$  is an open set in  $X$ . Thus, if we take an arbitrary  $x_0 \in f^{-1}(V)$ , we must show that there exists an  $r_0 > 0$  such that  $B_X(x_0, r_0) \subseteq f^{-1}(V)$  (cf. Theorem 12.2.15(a)).

Consider this arbitrary  $x_0 \in f^{-1}(V)$ . By definition, we have  $f(x_0) \in V$ . But since  $V$  is an open set, there exists an  $\varepsilon > 0$  such that  $B_Y(f(x_0), \varepsilon) \subseteq V$ .

But  $f$  is continuous: for this  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that, for  $x \in X$ , we have  $d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x)) < \varepsilon$ . In other words, we have  $x \in B_X(x_0, \delta) \implies f(x) \in B_Y(f(x_0), \varepsilon) \subseteq V$ .

Thus, if we set  $r_0 := \delta$ , we are done: for all  $x \in B_X(x_0, r_0)$ , we have  $f(x) \in V$ , i.e.  $x \in f^{-1}(V)$ . This shows that  $B_X(x_0, \delta) \subseteq f^{-1}(V)$ , and thus that  $f^{-1}(V)$  is an open set, as expected.

- Now we show that (c)  $\implies$  (d). By Theorem 12.2.15(e), we know that  $F \subseteq X$  is closed iff  $X \setminus F$  is open. Thus, consider  $F \subseteq Y$  a closed set in  $Y$ . Let be  $V := Y \setminus F$  its complementary set, which is thus an open set. By (c), the set  $f^{-1}(V)$  is an open set in  $X$ . But we have :

$$\begin{aligned} f^{-1}(F) &= \{x \in X : f(x) \in F\} \\ &= \{x \in X : f(x) \in Y \setminus V\} \\ &= \{x \in X : f(x) \notin V\} \end{aligned}$$

so that  $f^{-1}(F) = X \setminus f^{-1}(V)$ . Since  $f^{-1}(V)$  is the complementary set of the open set  $f^{-1}(V)$ , it is closed in  $X$ , as expected.

- The implication (d)  $\implies$  (c) can be shown in exactly the same way as above.
- Finally, let's show that (c)  $\implies$  (a). Let be  $\varepsilon > 0$ , let be  $x_0 \in X$ . Consider  $V := B_Y(f(x_0), \varepsilon)$ , which is an open set in  $Y$ . By (c), the set  $f^{-1}(V)$  is open in  $X$ . Thus, by Theorem 12.2.15(a), there exists a  $\delta > 0$  such that  $B_X(x_0, \delta) \subseteq f^{-1}(V)$ . Thus, if  $x \in B_X(x_0, \delta)$ , we have  $f(x) \in V$ .

In other words, for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$ . This shows that  $f$  is continuous at  $x_0$ , for any arbitrary  $x_0 \in X$ , as expected.

EXERCISE 13.1.3. — Use Theorem 13.1.4 and Theorem 13.1.5 to prove Corollary 13.1.7.

To show (a), consider  $(x^{(n)})_{n=1}^{\infty}$  a sequence of elements of  $X$  that converges to  $x_0 \in X$ . Since  $f$  is continuous at  $x_0$ , then by Theorem 13.1.4(b), we know that  $(f(x^{(n)}))_{n=1}^{\infty}$  converges to  $f(x_0) \in Y$ . But  $(f(x^{(n)}))_{n=1}^{\infty}$  is a sequence of elements of  $Y$ . Since  $g$  is continuous at  $f(x_0)$ , then still by Theorem 13.1.4(b), we know that  $(g(f(x^{(n)})))_{n=1}^{\infty}$  converges to  $g(f(x_0)) \in Z$ .

Thus, we have proved that for any sequence  $(x^{(n)})_{n=1}^{\infty}$  of elements of  $X$  that converges to  $x_0 \in X$ , the sequence  $(g \circ f(x^{(n)}))_{n=1}^{\infty}$  converges to  $g \circ f(x_0)$ . This shows that  $g \circ f$  is continuous at  $x_0$ , as expected.

Once (a) is proved, the result (b) is clear, since it is just (a) at any arbitrary  $x_0 \in X$ .

EXERCISE 13.1.4. — Give an example of functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that (a)  $f$  is not continuous, but  $g$  and  $g \circ f$  are continuous; (b)  $g$  is not continuous, but  $f$  and  $g \circ f$  are continuous; (c)  $f$  and  $g$  are not continuous, but  $g \circ f$  is continuous. Explain briefly why these examples do not contradict Corollary 13.1.7.

Here, the simplest way is to use piecewise constant functions, at least for one of the functions  $f, g$ .



(a) Let  $f$  be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

and the constant function  $g(x) := 3$ . We thus have  $g \circ f(x) = 3$  for all  $x \in \mathbb{R}$ , so that  $g \circ f$  is continuous.

(b) Let  $f$  be, for instance,

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

and  $f(x) := x^2 + 1$ . We thus have  $g \circ f(x) = 1$  for all  $x \in \mathbb{R}$ , so that  $g \circ f$  is continuous.

(c) Let  $f$  be, for instance,

$$f(x) := \begin{cases} 0 & \text{if } x < 0 \\ 3 & \text{if } x \geq 0 \end{cases}$$

and

$$g(x) := \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

We thus have  $g \circ f(x) = 1$  for all  $x \in \mathbb{R}$ , so that  $g \circ f$  is continuous.

This does not contradict Corollary 13.1.7, since the initial hypothesis of this corollary is that both functions  $f, g$  are continuous, and it says nothing about non discontinuous functions.

**EXERCISE 13.1.5.** — *Let  $(X, d)$  be a metric space, and let  $(E, d|_{E \times E})$  be a subspace of  $(X, d)$ . Let  $\iota_{E \rightarrow X} : E \rightarrow X$  be the inclusion map, defined by setting  $\iota_{E \rightarrow X}(x) := x$  for all  $x \in E$ . Show that  $\iota_{E \rightarrow X}$  is continuous.*

Let  $x_0 \in E$  be an arbitrary point in  $E$ , and let  $\varepsilon > 0$  be a positive real number. Note that we have, for all  $x \in E$ ,

$$d(\iota_{E \rightarrow X}(x_0), \iota_{E \rightarrow X}(x)) = d_{E \times E}(x_0, x).$$

Thus, if we take  $\delta := \varepsilon$  in Definition 13.1.1 of continuity, we are done: if  $d_{E \times E}(x_0, x) < \varepsilon$ , we automatically have  $d(\iota_{E \rightarrow X}(x_0), \iota_{E \rightarrow X}(x)) < \varepsilon$ , so that  $\iota_{E \rightarrow X}$  is continuous at any arbitrary  $x_0 \in E$ , as expected.

**EXERCISE 13.1.6.** — *Let  $f : X \rightarrow Y$  be a function from one metric space  $(X, d_X)$  to another  $(Y, d_Y)$ . Let  $E$  be a subset of  $X$  (which we give the induced metric  $d_X|_{E \times E}$ ), and let  $f|_E : E \rightarrow Y$  be the restriction of  $f$  to  $E$ , thus  $f|_E(x) := f(x)$  when  $x \in E$ . If  $x_0 \in E$  and  $f$  is continuous at  $x_0$ , show that  $f|_E$  is also continuous at  $x_0$ . (Is the converse of this statement true? Explain.) Conclude that if  $f$  is continuous, then  $f|_E$  is continuous.*

Let's use Exercise 13.1.5. First we note that we have  $f|_E = f \circ \iota_{E \rightarrow X}$ . Indeed,  $f \circ \iota_{E \rightarrow X}$  is a function from  $E$  to  $Y$  just like  $f$ , and for all  $x \in E$ , we clearly have  $f \circ \iota_{E \rightarrow X}(x) = f(x) = f|_E(x)$ .

We have shown in Exercise 13.1.5 that  $\iota_{E \rightarrow X}$  is continuous at any  $x_0 \in E$ , and  $f$  is supposed to be continuous at  $x_0 \in E$ . Thus, by Corollary 13.1.7,  $f|_E$  is continuous at  $x_0$  since

it is the composition of two continuous functions. Since this is true for any arbitrary  $x_0 \in E$ , the function  $f|_E$  is continuous on  $E$ .

The converse statement is not true: consider the piecewise constant function  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = -1$  if  $x < 0$  and  $f(x) = 1$  if  $x \geq 0$  for all  $x \in \mathbb{R}$ ; and let be  $E := [0, +\infty[$ . The restriction  $f|_E$  is clearly continuous (as a constant function) at 0, but the function  $f$  itself is clearly not continuous at 0.

EXERCISE 13.2.1. — *Prove Lemma 13.2.1.*

Here we just have to prove the statement (a), since the statement (b) is essentially (a) applied to any arbitrary  $x_0 \in X$ .

- First suppose that  $f$  and  $g$  are both continuous at  $x_0 \in X$ , and let be  $(x^{(n)})_{n=1}^\infty$  a sequence of elements of  $X$ . Then, by Theorem 13.1.4(b), the sequence  $(f(x^{(n)}))_{n=1}^\infty$  converges to  $f(x_0)$  in  $\mathbb{R}$ , and the sequence  $(g(x^{(n)}))_{n=1}^\infty$  converges to  $g(x_0)$  in  $\mathbb{R}$ .

Thus, by Theorem 12.1.18, the sequence  $((f(x^{(n)}), g(x^{(n)})))_{n=1}^\infty$  of elements of  $\mathbb{R}^2$  converges to  $(f(x_0), g(x_0))$  in  $\mathbb{R}^2$  with respect to the metric  $d_{l^2}$  (since it converges component-wise). In other words, for any arbitrary sequence  $(x^{(n)})_{n=1}^\infty$  that converges to  $x_0$ , the sequence  $(f \oplus g(x^{(n)}))_{n=1}^\infty$  converges to  $(f(x_0), g(x_0)) = f \oplus g(x_0)$ . Thus,  $f \oplus g$  is continuous at  $x_0$ , by Theorem 13.1.4.

- Conversely, if  $f \oplus g$  is continuous at  $x_0$ , then for any sequence  $(x^{(n)})_{n=1}^\infty$  of elements of  $X$ , the sequence  $(f \oplus g(x^{(n)}))_{n=1}^\infty$  converges to  $(f(x_0), g(x_0))$ , by Theorem 13.1.4. Thus, by Theorem 12.1.18,  $(f(x^{(n)}))_{n=1}^\infty$  converges to  $f(x_0)$  in  $\mathbb{R}$ , and  $(g(x^{(n)}))_{n=1}^\infty$  converges to  $g(x_0)$  in  $\mathbb{R}$ . Thus,  $f$  and  $g$  are both continuous at  $x_0$ .

EXERCISE 13.2.2. — *Prove Lemma 13.2.2.*

First we prove that the addition function,  $f : (x, y) \mapsto x + y$ , defined from  $\mathbb{R}^2$  to  $\mathbb{R}$ , is continuous. Consider a sequence  $(x_n, y_n)_{n=1}^\infty$  of elements of  $\mathbb{R}^2$ , that converges to  $(x_0, y_0)$  with respect to the metric  $d_{l^2}$ . In particular,  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  are both sequences of real numbers, and we know from Proposition 12.1.8 that  $(x_n)_{n=1}^\infty$  converges to  $x_0$ , and  $(y_n)_{n=1}^\infty$  converges to  $y_0$ . But we have  $f(x_n, y_n) = x_n + y_n$ , and we know by the limit laws (Proposition 6.1.19(a)) that  $\lim_{n \rightarrow \infty} (x_n + y_n) = x_0 + y_0 =: f(x_0, y_0)$ . Thus, for any sequence  $((x_n, y_n))_{n=1}^\infty$  of elements of  $\mathbb{R}^2$  which converges to  $(x_0, y_0)$ , we have proved that  $(f(x_n, y_n))_{n=1}^\infty$  converges to  $f(x_0, y_0)$ . Thus, the addition function  $f$  is continuous at any  $(x_0, y_0) \in \mathbb{R}^2$ , and thus is continuous on  $\mathbb{R}^2$ .

A similar proof applies for the subtraction function, the multiplication function, and the maximum and minimum functions.

An additional precaution is required for the division function,  $f : (x, y) \mapsto x/y$  defined from  $\mathbb{R} \times \mathbb{R} \setminus \{0\}$  to  $\mathbb{R}$ . Let be  $(x_0, y_0) \in \mathbb{R} \times \mathbb{R} \setminus \{0\}$ , and let be  $(x_n, y_n)_{n=1}^\infty$  a sequence of elements in  $\mathbb{R} \times \mathbb{R} \setminus \{0\}$  which converges to  $(x_0, y_0)$ . In the proof above, we can indeed apply Proposition 6.1.19(f) since  $y_n \neq 0$  for all  $n \geq 1$ .

EXERCISE 13.2.3. — *Show that if  $f : X \rightarrow \mathbb{R}$  is a continuous function, so is the function  $|f| : X \rightarrow \mathbb{R}$  defined by  $|f|(x) := |f(x)|$ .*

The function  $|f|$  is the composition of the functions  $f$  and  $x \mapsto |x|$ . We already know (from Proposition 9.4.12) that the function  $x \mapsto |x|$  is continuous on  $\mathbb{R}$  (basically because

$|x| := \max(x, -x)$ , and thus is the composition of two continuous functions). Thus, by Corollary 13.1.7,  $|f|$  is continuous on  $X$ .

EXERCISE 13.2.4. — Let  $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\pi_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the functions  $\pi_1(x, y) := x$  and  $\pi_2(x, y) := y$ . Show that  $\pi_1$  and  $\pi_2$  are continuous. Conclude that if  $f : \mathbb{R} \rightarrow X$  is any continuous function into a metric space  $(X, d)$ , then the functions  $g_1 : \mathbb{R}^2 \rightarrow X$  and  $g_2 : \mathbb{R}^2 \rightarrow X$  defined by  $g_1(x, y) := f(x)$  and  $g_2(x, y) := f(y)$  are also continuous.

Let be a sequence of elements of  $\mathbb{R}^2$ , that we will note as  $((x_n, y_n))_{n=1}^\infty$ , and suppose that this sequence converges to  $(x_0, y_0) \in \mathbb{R}^2$ . By Proposition 12.1.18, we know that this sequence converges component-wise, i.e. that the sequence of real numbers  $(x_n)_{n=1}^\infty$  converges to  $x_0$ , and the sequence of real numbers  $(y_n)_{n=1}^\infty$  converges to  $y_0$ . But we have  $x_n = \pi_1(x_n, y_n)$ ,  $x_0 = \pi_1(x_0, y_0)$ ,  $y_n = \pi_2(x_n, y_n)$ , and  $y_0 = \pi_2(x_0, y_0)$ . To summarise, for any sequence  $((x_n, y_n))_{n=1}^\infty$  of elements of  $\mathbb{R}^2$  that converges to  $(x_0, y_0)$ , the sequence  $\pi_1(x_n, y_n)$  converges to  $\pi_1(x_0, y_0)$ , so that  $\pi_1$  is continuous. A similar argument shows that  $\pi_2$  is continuous.

Furthermore, the function  $g_1$  defined above can be expressed as  $g_1 := f \circ \pi_1$ , and is thus continuous as the composition of two continuous functions (Corollary 13.1.7). Similarly,  $g_2 = f \circ \pi_2$  is continuous.

EXERCISE 13.2.5. — Let  $n, m \geq 0$  be integers. Suppose that for every  $0 \leq i \leq n$  and  $0 \leq j \leq m$  we have a real number  $c_{ij}$ . Form the function  $P : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$P(x, y) := \sum_{i=0}^n \sum_{j=0}^m c_{ij} x^i y^j.$$

Show that  $P$  is continuous. Conclude that if  $f : X \rightarrow \mathbb{R}$  and  $g : X \rightarrow \mathbb{R}$  are continuous functions, then the function  $P(f, g) : X \rightarrow \mathbb{R}$  defined by  $P(f, g)(x) := P(f(x), g(x))$  is also continuous.

By Exercise 13.2.4, we know that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous, then the function  $f \circ \pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  is also continuous. It is clear that the function  $f : x \mapsto x^i$  is continuous for all integer  $i \geq 0$  (as a product of  $i$  continuous functions: apply Corollary 13.2.3  $i$  times to the identity function  $x \mapsto x$ ). Thus, the function  $f_1 = f \circ \pi_1$  defined by  $f_1(x, y) = x^i$  is continuous on  $\mathbb{R}^2$ . Similarly, the function  $f_2(x, y) = y^j$  is continuous on  $\mathbb{R}^2$  for all  $j \geq 0$ .

By Corollary 13.2.3 another time, the function  $(x, y) \mapsto x^i y^j$  is thus continuous (since it is the product of two continuous functions), and still by Corollary 13.2.3, the function  $(x, y) \mapsto c_{ij} x^i y^j$  is also continuous, for all real constant  $c_{ij}$ . And thus,  $P(x, y)$  is also continuous as a sum of continuous functions.

Now consider the function  $H := P(f, g)$ , such that  $H(x) := P(f(x), g(x))$ . We have  $H = P \circ (f \oplus g)$ . But we have just showed that  $P$  is continuous, and we already know that  $f \oplus g$  is continuous whenever  $f$  and  $g$  are continuous (Lemma 2.2.1). Thus,  $H$  continuous (as the composition of two continuous functions), as expected.

EXERCISE 13.2.6. — Let  $\mathbb{R}^m$  and  $\mathbb{R}^n$  be Euclidean spaces. If  $f : X \rightarrow \mathbb{R}^m$  and  $g : X \rightarrow \mathbb{R}^n$  are continuous functions, show that  $f \oplus g : X \rightarrow \mathbb{R}^{m+n}$  is also continuous, where we have identified  $\mathbb{R}^m \times \mathbb{R}^n$  with  $\mathbb{R}^{m+n}$  in the obvious manner. Is the converse statement true?

This exercise generalizes the result of Lemma 13.2.1. The proof will thus be very close to the approach adopted in Exercise 13.2.1.

- Let be an arbitrary  $x_0 \in X$ , and let's show that  $f \oplus g$  is continuous at  $x_0$ .

Let be  $(x^{(n)})_{n=1}^\infty$  a sequence of elements of  $X$  that converges to  $x_0$ . We will use below the following notations:

$$\begin{aligned} f(x_0) &:= (x_1, \dots, x_m) \\ g(x_0) &:= (y_1, \dots, y_m) \\ f(x^{(k)}) &:= (x_1^{(k)}, \dots, x_m^{(k)}) \text{ for all } k \geq 1 \\ g(x^{(k)}) &:= (y_1^{(k)}, \dots, y_n^{(k)}) \text{ for all } k \geq 1 \end{aligned}$$

First, note that by Proposition 12.1.18, the sequence  $(x^{(n)})_{n=1}^\infty$  converges component-wise, i.e., we have  $\lim_{k \rightarrow \infty} x_p^{(k)} = x_p$  for all  $1 \leq p \leq m$ .

Since  $f$  is continuous, the sequence  $(f(x^{(n)}))_{n=1}^\infty$  converges to  $f(x_0) \in \mathbb{R}^m$ . Similarly for  $g$ , the sequence  $(g(x^{(n)}))_{n=1}^\infty$  converges to  $g(x_0) \in \mathbb{R}^n$ .

By definition, we have  $f \oplus g(x_0) = (x_1, \dots, x_m, y_1, \dots, y_n) \in \mathbb{R}^{m+n}$ .

But we also have, for all  $k \geq 1$ ,

$$\begin{aligned} f \oplus g(x^{(k)}) &:= (f(x^{(k)}), g(x^{(k)})) \\ &= (x_1^{(k)}, \dots, x_m^{(k)}, y_1^{(k)}, \dots, y_n^{(k)}) \end{aligned}$$

And since we already know that this sequence converges component-wise, we have by Proposition 12.1.18

$$\lim_{k \rightarrow \infty} f \oplus g(x^{(k)}) = (x_1, \dots, x_m, y_1, \dots, y_n) =: f \oplus g(x_0).$$

Thus,  $f \oplus g$  is continuous at  $x_0$ . And since this is true for any arbitrary  $x_0 \in X$ ,  $f \oplus g$  is continuous on  $X$ .

- The converse statement is also true; this can also be proved in a very similar fashion as Exercise 13.2.1.

EXERCISE 13.2.7. — Let  $k \geq 1$ , let  $I$  be a finite subset of  $\mathbb{N}^k$ , and let  $c : I \rightarrow \mathbb{R}$  be a function. Form the function  $P : \mathbb{R}^k \rightarrow \mathbb{R}$  defined by

$$P(x_1, \dots, x_k) := \sum_{(i_1, \dots, i_k) \in I} c(i_1, \dots, i_k) x_1^{i_1} \dots x_k^{i_k}.$$

Show that  $P$  is continuous.

Let's use induction on  $k$ .

- First, if  $k = 1$ , we have  $I \subset \mathbb{N}$ , and  $P$  is simply of the form  $P(x) := \sum_{i \in I} c_i x^i$ . Such a function is clearly continuous by Lemma 13.2.2, as the product and sum of continuous functions.
- We can also note that, for  $k = 2$ , the result corresponds exactly to Exercise 13.2.5.

- Now suppose that the property is true for a given positive integer  $k$ , and let's show that it is still true for  $k + 1$ .

Let be  $I$  a finite subset of  $\mathbb{N}^{k+1}$ , and a function  $P(x_1, \dots, x_k, x_{k+1})$  as defined above. Note that (for example by Corollary 3.6.14(e)), since  $I$  is supposed to be finite, we have  $I = I_1 \times \dots \times I_k \times I_{k+1}$  where each  $I_j$  is also a finite subset of  $\mathbb{N}$ . In particular,  $I_{k+1}$  is finite.

We thus have:

$$\begin{aligned} P(x_1, \dots, x_k, x_{k+1}) &:= \sum_{(i_1, \dots, i_k, i_{k+1}) \in I} c(i_1, \dots, i_k, i_{k+1}) x_1^{i_1} \dots x_k^{i_k} x_{k+1}^{i_{k+1}} \\ &= \sum_{i_{k+1} \in I_{k+1}} \left( \sum_{(i_1, \dots, i_k) \in I_1 \times \dots \times I_k} c(i_1, \dots, i_k, i_{k+1}) x_1^{i_1} \dots x_k^{i_k} \right) x_{k+1}^{i_{k+1}} \end{aligned}$$

By the induction hypothesis, the expression enclosed in the parentheses is a continuous function. Thus,  $P(x_1, \dots, x_{k+1})$  is also continuous, as a (finite) sum and product of continuous functions.

EXERCISE 13.2.8. — Let  $(X, d_X)$  and  $(Y, d_Y)$  be metric spaces. Define the metric  $d_{X \times Y} : (X \times Y) \times (X \times Y) \rightarrow [0, \infty[$  by the formula

$$d_{X \times Y}((x, y), (x', y')) := d_X(x, x') + d_Y(y, y').$$

Show that  $(X \times Y, d_{X \times Y})$  is a metric space, and deduce an analogue of Proposition 12.1.18 and Lemma 13.2.1.

1. First we prove that  $d_{X \times Y}$  is indeed a metric. In all the proofs below, we simply use the fact that  $d_X$  and  $d_Y$  are themselves metrics.

- For all  $(x, y) \in X \times Y$ , we have:

$$d_{X \times Y}((x, y), (x, y)) = \underbrace{d_X(x, x)}_{=0} + \underbrace{d_Y(y, y)}_{=0} = 0.$$

- For all distinct points  $(x, y), (x', y') \in X \times Y$ , we have

$$d_{X \times Y}((x, y), (x', y')) = \underbrace{d_X(x, x')}_{>0} + \underbrace{d_Y(y, y')}_{>0} > 0.$$

- Symmetry: for all  $(x, y), (x', y') \in X \times Y$ , we have

$$\begin{aligned} d_{X \times Y}((x, y), (x', y')) &= d_X(x, x') + d_Y(y, y') \\ &= d_X(x', x) + d_Y(y', y) \\ &= d_{X \times Y}((x', y'), (x, y)). \end{aligned}$$

- Triangle inequality: for all  $(x, y), (x', y'), (x'', y'') \in X \times Y$ , we have

$$\begin{aligned} &d_{X \times Y}((x, y), (x', y')) + d_{X \times Y}((x', y'), (x'', y'')) \\ &= d_X(x, x') + d_X(x', x'') + d_Y(y, y') + d_Y(y', y'') \\ &\leq d_X(x, x'') + d_Y(y, y'') \\ &\leq d_{X \times Y}((x, x''), (y, y'')). \end{aligned}$$

Thus,  $d_{X \times Y}$  is indeed a metric on  $X \times Y$ .

2. Now we give an analogue of Proposition 12.1.18. Note that  $X$  and  $Y$  are “abstract” metric spaces here, and  $d_X, d_Y$  also are “abstract” distances, so that we cannot really give an analogue to the notion of the equivalence of metrics  $d_{l^1}, d_{l^2}, d_{l^\infty}$ . However, we can give an analogue of the notion of “component-wise” convergence. Indeed, let be  $(a^{(n)})_{n=1}^\infty$  a sequence of elements of  $X \times Y$  (so that we can write  $a^{(n)} := (x^{(n)}, y^{(n)})$  for all  $n \geq 1$  with obvious notations), and let be  $(x_0, y_0)$  a point in  $X \times Y$ . The following two statements are equivalent:

- (i)  $(a^{(n)})_{n=1}^\infty := (x^{(n)}, y^{(n)})_{n=1}^\infty$  converges to  $(x_0, y_0)$  with respect to the metric  $d_{X \times Y}$
- (ii)  $(x^{(n)})_{n=1}^\infty$  converges to  $x_0$  with respect to  $d_X$  and  $(y^{(n)})_{n=1}^\infty$  converges to  $y_0$  with respect to  $d_Y$ .

*Proof.* First we show that (i)  $\implies$  (ii). If  $(x^{(n)}, y^{(n)})_{n=1}^\infty$  converges to  $(x_0, y_0)$  with respect to  $d_{X \times Y}$ , then by definition we have  $\lim_{n \rightarrow \infty} d_{X \times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) = 0$ . Thus, for any arbitrary  $\varepsilon > 0$ , there exists  $N \geq 1$  such that for all  $n \geq N$  we have

$$\begin{aligned} d_{X \times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) &< \varepsilon \\ d_X(x^{(n)}, x_0) + d_Y(y^{(n)}, y_0) &< \varepsilon. \end{aligned}$$

In particular, we have both  $d_X(x^{(n)}, x_0) < \varepsilon$  and  $d_Y(y^{(n)}, y_0) < \varepsilon$  whenever  $n \geq N$ . Thus, we have  $\lim_{n \rightarrow \infty} d_X(x^{(n)}, x_0) = 0$  and  $\lim_{n \rightarrow \infty} d_Y(y^{(n)}, y_0) = 0$ , which is precisely the statement (ii).

Now we show that (ii)  $\implies$  (i). Let be  $\varepsilon > 0$ . Since we have both  $\lim_{n \rightarrow \infty} d_X(x^{(n)}, x_0) = 0$  and  $\lim_{n \rightarrow \infty} d_Y(y^{(n)}, y_0) = 0$ , there exists  $N_1, N_2 \geq 1$  such that  $d_X(x^{(n)}, x_0) < \varepsilon/2$  whenever  $n \geq N_1$ , and  $d_Y(y^{(n)}, y_0) < \varepsilon/2$  whenever  $n \geq N_2$ . Thus, for all  $n \geq \max(N_1, N_2)$ , we have

$$\begin{aligned} d_{X \times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) &= d_X(x^{(n)}, x_0) + d_Y(y^{(n)}, y_0) \\ &< \varepsilon/2 + \varepsilon/2 \\ &< \varepsilon \end{aligned}$$

which means that  $\lim_{n \rightarrow \infty} d_{X \times Y}((x^{(n)}, y^{(n)}), (x_0, y_0)) = 0$ . This is the statement (i).  $\square$

3. Similarly, we can give an analogue for Lemma 13.2.1:

Let be  $S$  an arbitrary domain for the functions  $f : S \rightarrow X$  and  $g : S \rightarrow Y$ . Consider the metric spaces  $(X, d_X)$ ,  $(Y, d_Y)$  and  $(X \times Y, d_{X \times Y})$ . Then the function  $f \oplus g : S \rightarrow X \times Y$  is continuous at  $s_0 \in S$  iff both  $f$  and  $g$  are continuous at  $s_0$ .

*Proof.* Suppose that both  $f$  and  $g$  are continuous at  $s_0 \in S$ . Let be  $\varepsilon > 0$ , and let be  $(s^{(n)})_{n=1}^\infty$  a sequence of elements of  $S$  that converges to  $s_0$ . By definition, there exists  $N_1 \geq 1$  such that  $d_X(f(s^{(n)}), f(s_0)) < \varepsilon/2$  whenever  $n \geq N_1$ ; and there exists  $N_2 \geq 1$  such that  $d_Y(g(s^{(n)}), g(s_0)) < \varepsilon/2$  whenever  $n \geq N_2$ . Thus, for  $n \geq N := \max(N_1, N_2)$ ,

$$d_{X \times Y}(f \oplus g(s^{(n)}), f \oplus g(s_0)) := \underbrace{d_X(f(s^{(n)}), f(s_0))}_{< \varepsilon/2} + \underbrace{d_Y(g(s^{(n)}), g(s_0))}_{< \varepsilon/2} < \varepsilon$$

which means that  $f \oplus g$  is continuous at  $s_0$ , as expected.

The converse statement can be proved similarly (see also the proof of 2. above).  $\square$

EXERCISE 13.2.10. — *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a continuous function. Show that for each  $x \in \mathbb{R}$ , the function  $y \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ , and for each  $y \in \mathbb{R}$ , the function  $x \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ . Thus a function  $f(x, y)$  which is jointly continuous in  $(x, y)$  is also continuous in each variable  $x, y$  separately.*

Let be  $x_0 \in \mathbb{R}$  a real number. We will just prove the result for the function  $g : y \mapsto f(x_0, y)$ , since the result for the other function can be shown in the same way. We will work below with the metric space  $(\mathbb{R}^2, d_{l^2})$  but the proof can easily be adapted for the metrics  $d_{l^1}$  or  $d_{l^\infty}$ .

Let be  $y_0 \in \mathbb{R}$  an arbitrary real number: we have to prove that  $g$  is continuous at  $y_0$ . Let be  $\varepsilon > 0$  an arbitrary positive real number. Since  $f$  is continuous on  $\mathbb{R}^2$ , it is continuous in particular at  $(x_0, y_0) \in \mathbb{R}^2$ . Thus, there exists a  $\delta > 0$  such that  $|f(x, y) - f(x_0, y_0)| < \varepsilon$  whenever  $d((x, y), (x_0, y_0)) < \delta$ .

If we set  $x := x_0$  and choose any  $y \in \mathbb{R}$  such that  $|y - y_0| < \delta$ , we will have  $d((x_0, y), (x_0, y_0)) = |y - y_0| < \delta$ . Thus, we will have  $\varepsilon > |f(x_0, y) - f(x_0, y_0)| = |g(y) - g(y_0)|$ . This means that  $g$  is continuous at  $y_0$ , as expected.

EXERCISE 13.2.11. — *Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be the function defined by  $f(x, y) := \frac{xy}{x^2 + y^2}$  when  $(x, y) \neq (0, 0)$ , and  $f(x, y) = 0$  otherwise. Show that for each fixed  $x \in \mathbb{R}$ , the function  $y \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ , and that for each fixed  $y \in \mathbb{R}$ , the function  $x \mapsto f(x, y)$  is continuous on  $\mathbb{R}$ , but that the function  $f : \mathbb{R} \mapsto \mathbb{R}$  is not continuous on  $\mathbb{R}$ . This shows that the converse to Exercise 13.2.10 fails; it is possible to be continuous in each variable separately without being jointly continuous.*

First, let be  $x_0 \in \mathbb{R}$  an arbitrary real number, and let's consider the function  $g : \mathbb{R} \rightarrow \mathbb{R}$  such that  $g(y) = f(x_0, y)$  for all  $y \in \mathbb{R}$ . We have two cases:

- If  $x_0 = 0$ , then  $g$  is the constant null function, i.e.  $g(y) = 0$  for all  $y \in \mathbb{R}$ . Thus (for example by Exercise 9.4.2),  $g$  is continuous on  $\mathbb{R}$ .
- If  $x_0 \neq 0$ , then  $x_0^2 + y^2 \neq 0$  for all  $y \in \mathbb{R}$ . Thus,  $g(y) = \frac{x_0 y}{x_0^2 + y^2}$  is the ratio between the continuous function  $y \mapsto x_0 y$  and the continuous, positive function  $y \mapsto x_0^2 + y^2$  (these functions can be shown to be continuous by applying Lemma 13.2.2 several times). Thus, by Corollary 13.2.3,  $g$  is continuous on  $\mathbb{R}$ .

Of course, the same demonstration shows that the function  $h : x \mapsto f(x, y)$  is also continuous on  $\mathbb{R}$ .

Now let's show that  $f$  is not jointly continuous on  $\mathbb{R}$ , and more precisely that it is not continuous at the point  $(0, 0)$ . By Theorem 13.1.5, to show that a function  $f$  is not continuous, it is enough to find a sequence  $(x^{(n)})_{n=1}^\infty$  that converges to  $x_0$ , but such that  $(f(x^{(n)}))_{n=1}^\infty$  does not converge to  $f(x_0)$ . Consider the sequence  $(1/n, 1/n)_{n=1}^\infty$  of elements of  $\mathbb{R}^2$ . It is clear that this sequence converges to  $(0, 0)$ . However, we have  $f(1/n, 1/n) = \frac{1/n^2}{2/n^2} = \frac{1}{2}$ . Thus,  $(f(1/n, 1/n))_{n=1}^\infty$  does not converge to  $f(0, 0) = 0$ , which means that  $f$  is not continuous at  $(0, 0)$ .