Propositions of solutions for *Analysis II* by Terence Tao

Frédéric Santos

March 23, 2022

Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

Exercise 12.1.4. — *Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y\times Y}(x, y) := d(x, y)$, then the application $d|_{Y\times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y\times Y})$ is indeed a metric space.

EXERCISE 12.1.5. — *Let $n \geqslant 1$, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Verify the identity $\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.*

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n.

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geqslant 1$, and let's prove that it is still true for $n + 1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i\right)^2}_{:=A} + \underbrace{\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right)}_{:=C} \tag{12.1}$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A, we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i\right)^2 \\ &= \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2 \\ &= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \end{aligned}$$

- for B, we have

$$\begin{aligned} B &:= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n+1}(a_{n+1}b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1}b_i)^2}_{:=1/2\times S} + \underbrace{\frac{1}{2}\sum_{j=1}^{n}(a_{n+1}b_j - a_j b_{n+1})^2}_{:=1/2\times S} \\ &\quad + \underbrace{\frac{1}{2}(a_{n+1}b_{n+1} - b_{n+1}a_{n+1})^2}_{=0} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \end{aligned}$$

- and thus, for $A+B$, we now use the induction hypothesis (IH) to get:

$$
\begin{aligned}
A+B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^n a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2\sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right) \\
&= C
\end{aligned}
$$

  so that the identity is indeed true for all natural number n.

(ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left|\sum_{i=1}^n a_i b_i\right| \leqslant \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}. \tag{12.2}$$

Indeed, since $B \geqslant 0$ in the identity (12.1), we have

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \leqslant \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality. We have

$$\begin{aligned}
\sum_{i=1}^{n}(a_i^2+b_i^2) &= \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\sum_{i=1}^{n} a_i b_i \\
&\leqslant \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} b_i^2 + 2\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}\left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(by eq. (12.2))} \\
&\leqslant \left(\left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}\right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

Exercise 12.1.6. — *Show that $(\mathbb{R}^n, d_{l^2})$ in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i - x_i)^2} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $(x_i - y_i)^2 > 0$, and $d_{l^2}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(y,x) = \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2} = d_{l^2}(x,y)$$

as expected.

(d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned}
d_{l^2}(x,z) &:= \left(\sum_{i=1}^{n}(x_i - z_i)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} && \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2} && \text{(Exercise 12.1.5(iii))} \\
&\leqslant \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2} + \left(\sum_{i=1}^{n}(y_i - z_i)^2\right)^{1/2} \\
&\leqslant d_{l^2}(x,y) + d_{l^2}(y,z)
\end{aligned}$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^2})$ is indeed a metric space.

Exercise 12.1.7. — *Show that $(\mathbb{R}^n, d_{l^1})$ in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x,x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x,y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y,x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x,y)$$

as expected.

(d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a-c| \leqslant |a-b| + |b-c|$ for all $a,b,c \in \mathbb{R}$. Thus, for all $x,y,z \in \mathbb{R}^n$, we have

$$d_{l^1}(x,z) := \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x,y) + d_{l^1}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x,y)^2 \leqslant d_{l^1}(x,y)^2$, and we will prove it in this form.

  Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l_1}(x,y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \left(\sum_{i=1}^n |x_i - y_i|\right) \times \left(\sum_{i=1}^n |x_i - y_i|\right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leqslant i,j \leqslant n;\, i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geqslant 0} \\ &\geqslant \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x,y)^2 \end{aligned}$$

  as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x,y) &:= \sum_{i=1}^{n} |x_i - y_i| \\
&= \left| \sum_{i=1}^{n} |x_i - y_i| \times 1 \right| \\
&\leqslant \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^{n} 1^2 \right)^{1/2} \\
&\leqslant d_{l^2}(x,y) \times \sqrt{n}
\end{aligned}$$

  as expected.