Propositions of solutions for *Analysis II* by Terence Tao

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Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X \to [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

Exercise 12.1.4. — *Show that the pair $(Y, d|_{Y\times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y\times Y}(x, y) := d(x, y)$, then the application $d|_{Y\times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y\times Y})$ is indeed a metric space.

EXERCISE 12.1.5. — *Let $n \geqslant 1$, and let $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ be real numbers. Verify the identity $\left(\sum_{i=1}^n a_i b_i\right)^2 + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.*

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n.

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geqslant 1$, and let's prove that it is still true for $n + 1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i\right)^2}_{:=A} + \underbrace{\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right)}_{:=C} \tag{12.1}$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A, we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i\right)^2 \\ &= \left(a_{n+1}b_{n+1} + \sum_{i=1}^{n} a_i b_i\right)^2 \\ &= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^{n} a_i b_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^{n} a_i b_i \end{aligned}$$

- for B, we have

$$\begin{aligned} B &:= \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n+1}(a_i b_j - a_j b_i)^2 + \frac{1}{2}\sum_{j=1}^{n+1}(a_{n+1}b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2}\sum_{i=1}^{n}(a_i b_{n+1} - a_{n+1}b_i)^2}_{:=1/2\times S} + \underbrace{\frac{1}{2}\sum_{j=1}^{n}(a_{n+1}b_j - a_j b_{n+1})^2}_{:=1/2\times S} \\ &\quad + \underbrace{\frac{1}{2}(a_{n+1}b_{n+1} - b_{n+1}a_{n+1})^2}_{=0} \\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_j - a_j b_i)^2 + \sum_{k=1}^{n}(a_k b_{n+1} - a_{n+1}b_k)^2 \end{aligned}$$

- and thus, for $A+B$, we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A+B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^n a_ib_i\right)^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i \\
&\quad + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_ib_j - a_jb_i)^2 + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^n a_ib_i\right)^2 + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n (a_ib_j - a_jb_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1})\sum_{i=1}^n a_ib_i + \sum_{k=1}^n (a_kb_{n+1} - a_{n+1}b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2\sum_{i=1}^n a_ia_{n+1}b_ib_{n+1} + \sum_{i=1}^n (a_i^2b_{n+1}^2 - 2a_ib_{n+1}a_{n+1}b_i + a_{n+1}^2b_i^2) \\
&= \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right) + \sum_{i=1}^n (a_i^2b_{n+1}^2 + a_{n+1}^2b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2\right)\left(\sum_{j=1}^{n+1} b_j^2\right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number n.

(ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left|\sum_{i=1}^n a_ib_i\right| \leqslant \left(\sum_{i=1}^n a_i^2\right)^{1/2}\left(\sum_{i=1}^n b_i^2\right)^{1/2}. \tag{12.2}$$

Indeed, since $B \geqslant 0$ in the identity (12.1), we have

$$\left(\sum_{i=1}^n a_ib_i\right)^2 \leqslant \left(\sum_{i=1}^n a_i^2\right)\left(\sum_{j=1}^n b_j^2\right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality. We have

$$\begin{aligned}
\sum_{i=1}^{n}(a_i^2+b_i^2) &= \sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2+2\sum_{i=1}^{n}a_ib_i \\
&\leqslant \sum_{i=1}^{n}a_i^2+\sum_{i=1}^{n}b_i^2+2\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}\left(\sum_{i=1}^{n}b_i^2\right)^{1/2} && \text{(by eq. (12.2))} \\
&\leqslant \left(\left(\sum_{i=1}^{n}a_i^2\right)^{1/2}+\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}\right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

Exercise 12.1.6. — *Show that $(\mathbb{R}^n, d_{l^2})$ in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x,x) = \sqrt{\sum_{i=1}^{n}(x_i-x_i)^2} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $(x_i-y_i)^2 > 0$, and $d_{l^2}(x,y) = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(y,x) = \sqrt{\sum_{i=1}^{n}(y_i-x_i)^2} = \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} = d_{l^2}(x,y)$$

as expected.

(d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned}
d_{l^2}(x,z) &:= \left(\sum_{i=1}^{n}(x_i-z_i)^2\right)^{1/2} \\
&= \left(\sum_{i=1}^{n}(a_i+b_i)^2\right)^{1/2} && \text{with } a_i := x_i-y_i \text{ and } b_i := y_i-z_i \\
&\leqslant \left(\sum_{i=1}^{n}a_i^2\right)^{1/2}+\left(\sum_{i=1}^{n}b_i^2\right)^{1/2} && \text{(Exercise 12.1.5(iii))} \\
&\leqslant \left(\sum_{i=1}^{n}(x_i-y_i)^2\right)^{1/2}+\left(\sum_{i=1}^{n}(y_i-z_i)^2\right)^{1/2} \\
&\leqslant d_{l^2}(x,y)+d_{l^2}(y,z)
\end{aligned}$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^2})$ is indeed a metric space.

Exercise 12.1.7. — *Show that* $(\mathbb{R}^n, d_{l^1})$ *in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x,x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant i \leqslant n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x,y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y,x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x,y)$$

as expected.

(d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a-c| \leqslant |a-b| + |b-c|$ for all $a, b, c \in \mathbb{R}$. Thus, for all $x, y, z \in \mathbb{R}^n$, we have

$$d_{l^1}(x,z) := \sum_{i=1}^n |x_i - z_i| \leqslant \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x,y) + d_{l^1}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

Exercise 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x,y)^2 \leqslant d_{l^1}(x,y)^2$, and we will prove it in this form.

  Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l_1}(x,y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i|\right)^2 \\ &= \left(\sum_{i=1}^n |x_i - y_i|\right) \times \left(\sum_{i=1}^n |x_i - y_i|\right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leqslant i,j \leqslant n;\, i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geqslant 0} \\ &\geqslant \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x,y)^2 \end{aligned}$$

  as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned} d_{l^1}(x,y) &:= \sum_{i=1}^n |x_i - y_i| \\ &= \left| \sum_{i=1}^n |x_i - y_i| \times 1 \right| \\ &\leqslant \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^n 1^2 \right)^{1/2} \\ &\leqslant d_{l^2}(x,y) \times \sqrt{n} \end{aligned}$$

  as expected.

EXERCISE 12.1.9. — *Show that the pair* $(\mathbb{R}^n, d_{l^\infty})$ *in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in \mathbb{R}^n$, we clearly have $d_{l^\infty}(x,x) = \sup\{|x_i - x_i| : 1 \leqslant i \leqslant n\} = 0$, as expected.

(b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leqslant j \leqslant n$ such that $x_j \neq y_j$. Thus $|x_j - y_j| > 0$, and $d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} \geqslant |x_j - y_j| > 0$, as expected.

(c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^\infty}(x,y) = \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\} = \sup\{|y_i - x_i| : 1 \leqslant i \leqslant n\} = d_{l^\infty}(y,x)$$

as expected.

(d) Triangle inequality. Let be $x, y, z \in \mathbb{R}^n$. We have $|x_i - z_i| \leqslant |x_i - y_i| + |y_i - z_i|$ for all $1 \leqslant i \leqslant n$, by Proposition 4.3.3(g). But, by definition of the supremum, we have $|x_i - y_i| \leqslant d_{l^\infty}(x,y)$ and $|y_i - z_i| \leqslant d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$. Thus, we have $|x_i - z_i| \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ for all $1 \leqslant i \leqslant n$; i.e., $d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$ is an upper bound of the set $\{|x_i - z_i| : 1 \leqslant i \leqslant n\}$. By definition of the supremum, it implies that

$$d_{l^\infty}(x,z) := \sup\{|x_i - z_i| : 1 \leqslant i \leqslant n\} \leqslant d_{l^\infty}(x,y) + d_{l^\infty}(y,z)$$

as expected.

Thus, $(\mathbb{R}^n, d_{l^1})$ is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y).$$

First, a preliminary remark. By definition, we have $d_{l^\infty}(x,y) := \sup\{|x_i - y_i| : 1 \leqslant i \leqslant n\}$ for all $x, y \in \mathbb{R}^n$. Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a $1 \leqslant m \leqslant n$ such that $d_{l^\infty}(x,y) = |x_m - y_m|$ (the supremum belongs to the set). The index "m" will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}\frac{1}{\sqrt{n}} d_{l^2}(x,y) &:= \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-y_i)^2} \\ &\leqslant \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_m-y_m)^2} \\ &\leqslant \sqrt{\frac{n}{n}(x_m-y_m)^2} \\ &= |x_m-y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}d_{l^2}(x,y) &:= \sqrt{\sum_{i=1}^{n}(x_i-y_i)^2} \\ &= \sqrt{(x_m-y_m)^2 + \sum_{1\leqslant i\leqslant n;\, i\neq m}(x_i-y_i)^2} \\ &\geqslant \sqrt{(x_m-y_m)^2} = |x_m-y_m| =: d_{l^\infty}(x,y)\end{aligned}$$

as expected.

EXERCISE 12.1.11. — *Show that the discrete metric (X, d_{disc}) in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

(a) For all $x \in X$, we have $d_{\text{disc}}(x,x) := 0$ by definition, so that there is nothing to prove here.

(b) Positivity: for all $x \neq y \in X$, we have $d_{\text{disc}}(x,y) := 1 > 0$ by definition, so that there's still nothing to prove.

(c) Symmetry: for all $x, y \in X$, we have $d_{\text{disc}}(x,y) = d_{\text{disc}}(y,x) = 1$, so that d_{disc} obeys the symmetry property.

(d) Triangle inequality. Let be $x, y, z \in X$, and let's consider $d_{\text{disc}}(x,z)$.

  - If $x = z$, then $d_{\text{disc}}(x,z) = 0$. And since d_{disc} is a non-negative application, we clearly have $0 =: d_{\text{disc}}(x,z) \leqslant d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z)$ for all $y \in X$.
  - If $x \neq z$, then we cannot have both $x = y$ and $y = z$ (it would be a clear contradiction with $x \neq z$). Thus, at least one of the propositions "$x \neq y$", "$y \neq z$" is true. Another way to say that is $d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z) \geqslant 1$. But since $d_{\text{disc}}(x,z) := 1$, we have actually $d_{\text{disc}}(x,y) + d_{\text{disc}}(y,z) \geqslant d_{\text{disc}}(x,z)$, as expected.

Exercise 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all $x, y \in \mathbb{R}^n$, we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}}\, d_{l^2}(x,y) \leqslant d_{l^\infty}(x,y) \leqslant d_{l^2}(x,y) \leqslant d_{l^1}(x,y) \leqslant \sqrt{n}\, d_{l^2}(x,y).$$

Note that n is a real constant here.

- Let's prove that $(a) \implies (b)$. If $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, then by the limit laws, the sequence $t_k := \sqrt{n}\, d_{l^2}(x^{(k)}, x)$ also converges to 0 as $k \to \infty$, since $\sqrt{n}$ is a constant real number. Thus, we have

$$d_{l^2}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x) \leqslant \sqrt{n}\, d_{l^2}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x)$ as expected.

- Let's prove that $(b) \implies (c)$. If $\lim_{k\to\infty} d_{l^1}(x^{(k)}, x) = 0$, then we have

$$0 \leqslant d_{l^\infty}(x^{(k)}, x) \leqslant d_{l^1}(x^{(k)}, x)$$

  and, by the squeeze test, this implies that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x)$ as expected.

- Let's prove that $(c) \implies (d)$. Suppose that $\lim_{k\to\infty} d_{l^\infty}(x^{(k)}, x) = 0$. Then, for all $1 \leqslant j \leqslant n$, we have $0 \leqslant |x_j^k - x_j| \leqslant d_{l^\infty}(x^{(k)}, x)$. Still by the squeeze test, this implies that $\lim_{k\to\infty} |x_j^k - x_j| = 0$, i.e. that $(x_j^k)_{k=m}^\infty$ converges to x_j as $k \to \infty$ (by Lemma 12.1.1), as expected.

- Finally, let's prove that $(d) \implies (a)$. Using the definition of convergence is more appropriate here. Let be $\varepsilon > 0$ a positive real number, and let be $1 \leqslant j \leqslant n$. By definition, there exists a natural number $N \geqslant m$ such that $|x_j^{(k)} - x_j| \leqslant \varepsilon/\sqrt{n}$ whenever $k \geqslant N$. Thus, if $k \geqslant N$, we have

$$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^{n} (x_j^{(k)} - x_j)^2} \leqslant \sqrt{\sum_{j=1}^{n} \frac{\varepsilon^2}{n}} \leqslant \varepsilon$$

  so that $\lim_{k\to\infty} d_{l^2}(x^{(k)}, x) = 0$, i.e., $(x^k)_{k=m}^\infty$ converges to x as $k \to \infty$ in the l^2 metric (by Lemma 12.1.1), as expected.

Exercise 12.1.13. — *Prove Proposition 12.1.19.*

Let be $(x^{(n)})_{n=m}^\infty$ a sequence of elements of a set X.

- First suppose that $(x^{(n)})_{n=m}^\infty$ is eventually constant. Thus, by definition, there exists an $N \geqslant m$ and an element $x \in X$ such that $(x^{(n)})_{n=m}^\infty = x$ for all $n \geqslant N$. This implies that we have $d_{\text{disc}}(x^{(n)}, x) = 0$ for all $n \geqslant N$. In particular, for all $\varepsilon > 0$, we have $d_{\text{disc}}(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N$, so that $(x^{(n)})_{n=m}^\infty$ indeed converges to x with respect to d_{disc}.

- Conversely, suppose that $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d_{disc}. Let be $\varepsilon = 1/2$. By definition, there exists an $N \geqslant m$ such that $d_{\text{disc}}(x^{(n)}, x) \leqslant 1/2$ whenever $n \geqslant N$. Since $d_{\text{disc}}(x^{(n)}, x)$ cannot be 1, it is necessarily equal to 0, so that $x^{(n)} = x$ whenever $n \geqslant N$. Thus, the sequence $x^{(n)}$ is indeed eventually constant.

Exercise 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have $\lim_{n\to\infty} d(x^{(n)}, x) = 0$ and $\lim_{n\to\infty} d(x^{(n)}, x') = 0$. Suppose, for the sake of contradiction, that we have $x \neq x'$. Thus, the real number $\varepsilon := \frac{d(x,x')}{3}$ is positive.

Since $x^{(n)}$ converges to x, there exists a $N_1 \geqslant m$ such that $d(x^{(n)}, x) \leqslant \varepsilon$ whenever $n \geqslant N_1$.

Similarly, since $x^{(n)}$ converges to x', there exists a $N_2 \geqslant m$ such that $d(x^{(n)}, x') \leqslant \varepsilon$ whenever $n \geqslant N_2$.

By the triangle inequality, we thus have, for all $n \geqslant \max(N_1, N_2)$,

$$d(x, x') \leqslant d(x, x^{(n)}) + d(x^{(n)}, x') \leqslant \varepsilon + \varepsilon = \frac{2}{3} d(x, x')$$

which is a contradiction (since $d(x, x') > 0$ by hypothesis).

Thus, the limit is unique, and we must have $x = x'$.

Exercise 12.1.15. — *Let be $X := \{(a_n)_{n=0}^{\infty} : \sum_{n=0}^{\infty} |a_n| < \infty\}$. We define on this space the metrics $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sum_{n=0}^{\infty} |a_n - b_n|$, and $d_{l^\infty}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) := \sup_{n\in\mathbb{N}} |a_n - b_n|$. Then...*

We have to prove the following statements.

1. d_{l^1} is a metric on X.

   We have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^{\infty} \in X$. We have $d_{l^1}((a_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^1}((b_n)_{n=0}^{\infty}, (a_n)_{n=0}^{\infty}) = \sum_{n=0}^{\infty} |b_n - a_n| = \sum_{n=0}^{\infty} |a_n - b_n| = d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty} \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately

   $$\begin{aligned} d_{l^1}((a_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}) &:= \sum_{n=0}^{\infty} |a_n - c_n| \\ &\leqslant \sum_{n=0}^{\infty} (|a_n - b_n| + |b_n - c_n|) \text{ (consequence of Prop. 7.1.11(h))} \\ &\leqslant \sum_{n=0}^{\infty} |a_n - b_n| + \sum_{n=0}^{\infty} |b_n - c_n| \text{ (by Proposition 7.2.14(a))} \\ &\leqslant d_{l^1}((a_n)_{n=0}^{\infty}, (b_n)_{n=0}^{\infty}) + d_{l^1}((b_n)_{n=0}^{\infty}, (c_n)_{n=0}^{\infty}). \end{aligned}$$

   Thus, d_{l^1} is indeed a metric on X.

2. d_{l^∞} is a metric on X.

   Once again, we have to prove the four axioms of Definition 12.1.2.

   (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - a_n| = 0$, as expected.

   (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X. Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |a_n - b_n| \geqslant |a_m - b_m| > 0$, as expected.

   (c) Symmetry: we clearly have

   $$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n\in\mathbb{N}} |b_n - a_n| = \sup_{n\in\mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

   (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on $\mathbb{R}$ (i.e., we have $|a_n - c_n| \leqslant |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately $|a_m - c_m| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$ for all $m \in \mathbb{N}$, by definition of the supremum. In other words, $(\sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|)$ is an upper bound for the set $\{|a_m - c_m| : m \in \mathbb{N}\}$. Thus we have, still by definition of the supremum, $\sup_{n\in\mathbb{N}} |a_n - c_n| \leqslant \sup_{n\in\mathbb{N}} |a_n - b_n| + \sup_{n\in\mathbb{N}} |b_n - c_n|$, as expected.

   Thus, d_{l^∞} is indeed a metric on X.

3. There exist sequences $x^{(1)}$, $x^{(2)}$, ..., of elements of X (i.e., sequences of sequences) which are convergent with respect to d_{l^∞}, but are not convergent with respect to d_{l^1}.

   Here we are dealing with sequences of sequences: we have a sequence $(x^{(k)})_{k=1}^\infty$ where each $x^{(k)}$ is itself a sequence of real numbers. Thus, let's define $(x^{(k)})_{k=1}^\infty$ as follows:

   $$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

   Just to make things clearer, we have for instance

   $$\begin{aligned} x^{(1)} &:= \frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ 0,\ \ldots \\ x^{(2)} &:= \frac{1}{3},\ \frac{1}{3},\ \frac{1}{3},\ 0,\ 0,\ \ldots \\ x^{(3)} &:= \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ \frac{1}{4},\ 0,\ \ldots \end{aligned}$$

   Also, let be the null sequence $(a_n)_{n=0}^\infty$ defined by $a_n := 0$ for all $n \in \mathbb{N}$. Thus:

   - $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞}. Indeed, if we consider a given positive integer k (fixed), we have

     $$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leqslant n \leqslant k \\ 0 & \text{if } n > k. \end{cases}$$

     so that $d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) := \sup_{n\in\mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$.

     Thus, $\lim_{k\to\infty} d_{l^\infty}\left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty\right) = 0$, or in other words, $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} in X.

- But $(x^{(k)})_{k=1}^{\infty}$ does not converges to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}. Indeed, we have, for each given (fixed) k,

$$d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty},(a_n)_{n=0}^{\infty}\right)=\sum_{n=0}^{k}\frac{1}{k+1}=1$$

  Thus, we clearly do not have $\lim_{k\to\infty} d_{l^1}\left((x_n^{(k)})_{n=0}^{\infty},(a_n)_{n=0}^{\infty}\right)=0$, i.e., $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

4. Conversely, any sequence which converges with respect to d_{l^1} also converges with respect to d_{l^∞}.

   Suppose, for the sake of contradiction, that $(x^{(k)})_{k=1}^{\infty}$ does not converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^∞}, but does converge to $(a_n)_{n=0}^{\infty}$ w.r.t. the metric d_{l^1}.

   In this case, there exists a $\varepsilon>0$ such that, for all $k\geqslant 1$, we have $(\sup_{n\geqslant 0}|x_n^{(k)}-a_n|)>\varepsilon$. In particulier, for all $k\geqslant 1$ and all $n\geqslant 0$, we have $|x_n^{(k)}-a_n|>\varepsilon$. Thus, $\sum_{n=0}^{\infty}|x_n^{(k)}-a_n|$ is not even a convergent series, and we cannot have $\lim_{k\to\infty}\left(\sum_{n=0}^{\infty}|x_n^{(k)}-a_n|\right)=0$.

Note that this exercise actually shows that in this space X, the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

Exercise 12.1.16. — *Let* $(x_n)_{n=1}^{\infty}$ *and* $(y_n)_{n=1}^{\infty}$ *be two sequences in a metric space* (X,d). *Suppose that* $(x_n)_{n=1}^{\infty}$ *converges to a point* $x\in X$, *and* $(y_n)_{n=1}^{\infty}$ *converges to a point* $y\in X$. *Show that* $\lim_{n\to\infty} d(x_n,y_n)=d(x,y)$.

On the one hand, the triangle inequality applied two times to d gives us

$$d(x_n,y_n)\leqslant d(x_n,x)+d(x,y)+d(y,y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x,y)\leqslant d(x,x_n)+d(x_n,y_n)+d(y_n,y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n,x)-d(y_n,y)\leqslant d(x_n,y_n)-d(x,y)\leqslant d(x_n,x)+d(y_n,y)$$

i.e.,

$$|d(x_n,y_n)-d(x,y)|\leqslant d(x_n,x)+d(y_n,y).$$

Let be $\varepsilon>0$. By hypothesis, there exists a $N_1\geqslant 1$ such that $d(x_n,x)\leqslant\varepsilon/2$ whenever $n\geqslant N_1$. Similarly, there exists a $N_2\geqslant 1$ such that $d(y_n,y)\leqslant\varepsilon/2$ whenever $n\geqslant N_2$. Thus, if we set $N:=\max(N_1,N_2)$, then for all $n\geqslant N$ we have

$$|d(x_n,y_n)-d(x,y)|\leqslant d(x_n,x)+d(y_n,y)\leqslant 2\varepsilon/2\ leq\varepsilon$$

which shows that $\lim_{n\to\infty} d(x_n,y_n)=d(x,y)$, as expected.

Exercise 12.2.1. — *Verify the claims in Example 12.2.8*

Let be (X, d_{disc}) a metric space, and E a subset of X.

- Let be $x \in E$. Then x is an interior point of E. Indeed, we have $B(x, 1/2) = \{x\} \subseteq E$.
- Let be $y \notin E$. Then y is an exterior point of E. Indeed, we have $B(y, 1/2) \cap E = \{y\} \cap E = \varnothing$.
- Finally, there are no boundary points of E in (X, d_{disc}). Indeed, let be $r > 0$ and any $x \in X$. We will always have $B(x, r) = \{x\}$ by definition of the discrete metric d_{disc}. Thus, we have either $x \in E$ and then $x \in \text{int}(E)$, or $x \notin E$ and then $x \in \text{ext}(E)$. Thus, E has no boundary points.

Exercise 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that $(a) \implies (b)$. We will use the contrapositive, assuming that x_0 is neither an interior point of E, nor a boundary point of E. By definition, it means that x_0 is an exterior point of E, i.e. that there exists $r > 0$ such that $B(x_0, r) \cap E = \varnothing$. This is precisely the negation of x_0 being an adherent point of E. Thus, we have showed that if x_0 is an adherent of of E, it is either an interior point of a boundary point.
- Let's show that $(b) \implies (c)$. Let be a positive integer $n > 0$, and suppose that x_0 is either an interior point of E, or a boundary point of E. In either case, the set $A_n := B(x_0, 1/n) \cap E$ is non empty, i.e., there exists $a_n \in X$ such that $d(a_n, x_0) < 1/n$. By the (countable) axiom of choice, we can define a sequence $(a_n)_{n=1}^{\infty}$ such that $a_n \in A_n$ for all $n \geqslant 1$.

  Let be $\varepsilon > 0$. There exists $N > 0$ such that $1/N < \varepsilon$ (Exercise 5.4.4). Thus, for all $n \geqslant N$, we have
  $$d(a_n, x_0) < \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$$
  i.e., the sequence $(a_n)_{n=1}^{\infty}$ converges to x_0 with respect to the metric d, as expected.
- Finally, let's show that $(c) \implies (a)$. Let be $r > 0$. If $(a_n)_{n=1}^{\infty}$ in E converges to x_0 with respect to d, then there exists a n such that $d(x_0, a_n) < r$. But since $a_n \in E$, it means that $B(x_0, r) \cap E$ is non empty, i.e. that x_0 is an adherent point of E.

Exercise 12.2.3. — *Prove Proposition 12.2.5.*

Let be (X, d) a metric space.

(a) Let be $E \subseteq X$. First suppose that E is open; this means that $E \cap \partial E = \varnothing$. Let be $x \in E$, then we have $x \notin \partial E$. But since $x \in E$, we have $x \in \overline{E}$, and thus $x \in \text{int}(E)$ by Proposition 12.2.10(b). We have shown that $x \in E \implies x \in \text{int}(E)$, and since the converse implication is trivial (Remark 12.2.6), we have $E = \text{int}(E)$ as expected.

Now suppose that $E = \text{int}(E)$. Let be $x \in E$. We thus have $x \in \text{int}(E)$. By definition, x is thus not a boundary point of E, i.e. x $\notin \partial E$. This means that $E \cap \partial E = \varnothing$, i.e. that E is open, as expected.

(b) Let be $E \subseteq X$. First suppose that E is closed; i.e. that $\partial E \subseteq E$. Let be $x \in \overline{E}$. By Proposition 12.2.10, we have $\overline{E} = \text{int}(E) \cup \partial E$; such that $\overline{E}$ is the union of two subsets of E, and thus is itself a subset of E, as expected.

Conversely, suppose that $\overline{E} \subseteq E$. It means that $\text{int}(E) \cup \partial E \subseteq E$, and in particular that $\partial E \subseteq E$, i.e. that E is closed, as expected.

(c) Let be $x_0 \in X$, $r > 0$ and $E := B(x_0, r)$. To show that E is open, we must show that $E = \text{int}(E)$ (by Proposition 1.2.15(a)), and in particular that $E \subseteq \text{int}(E)$ (the converse inclusion being trivial). Let be $x \in E$, and let's show that $x \in \text{int}(E)$. By definition, we have $d(x, x_0) < r$, so that $\varepsilon := r - d(x, x_0)$ is a positive real number. Thus, let be $y \in B(x, \varepsilon)$. By the triangle inequality, we have

$$\begin{aligned} d(x_0, y) &< d(x, x_0) + d(x, y) \\ &< d(x, x_0) + \varepsilon \\ &< d(x, x_0) + r - d(x, x_0) = r \end{aligned}$$

so that $y \in E$. Thus, there exists $\varepsilon > 0$ such that $B(x, \varepsilon) \subseteq E$, i.e., x is an interior point of E. This shows that $E \subseteq \text{int}(E)$, as expected.

Now let be $F := \{x \in X : d(x, x_0) \leqslant r\}$, and let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in F. To show that F is closed, we have to show that $\ell := \lim_{n\to\infty} a_n$ lies in F (Proposition 12.2.15(b)). Suppose, for the sake of contradiction, that $\ell \notin F$. We thus have $d(\ell, x_0) > r$, so that $\varepsilon := d(\ell, x_0) - r$ is a positive real number. Since $(a_n)_{n=1}^{\infty}$ converges to ℓ, there exists a $N > 0$ such that $d(a_n, \ell) < \varepsilon$ whenever $n \geqslant N$. By the triangle inequality, for $n \geqslant N$, we have

$$\begin{aligned} d(x_0, \ell) &\leqslant d(x_0, a_n) + d(a_n, \ell) \\ d(x_0, a_n) &\geqslant d(x_0, \ell) - d(a_n, \ell) \\ &\geqslant d(x_0, \ell) - \varepsilon \\ &\geqslant d(x_0, \ell) + r - d(\ell, x_0) \\ &\geqslant r \end{aligned}$$

and thus, $a_n \notin B(x_0, r)$, a contradiction. Thus, we must have $\ell \in F$, so that F is indeed a closed set.

(d) Let be $\{x_0\}$ a singleton with $x_0 \in X$. To show that E is closed, we may use Proposition 12.2.15(b), and show that $\{x_0\}$ contains all its adherent points. Let be $(a_n)_{n=1}^{\infty}$ a convergent sequence in $\{x_0\}$; it can only be the constant sequence $x_0, x_0, \ldots$. Since it is a constant sequence, its limit can only be x_0 itself, and this limit belongs to $\{x_0\}$. Thus, $\{x_0\}$ is a closed set.

(e) First we can form a lemma: for any subset $E \subseteq X$, we have $\text{int}(E) = \text{ext}(X \backslash E)$. This is a direct consequence of Definition 12.2.5. Indeed, $x \in \text{int}(E)$ iff there exists a $r > 0$ such that $B(x, r) \subseteq E$, which is equivalent to "$\exists r > 0 : B(x, r) \cap (X \backslash E) = \varnothing$", which is equivalent to $x \in \text{ext}(X \backslash E)$.

This implies that the interior points of E are the exterior points of $X \backslash E$, and conversely, that the exterior points of E are the interior points of $X \backslash E$. Thus, in particular, we have this useful fact:

$$\partial E = \partial(X \backslash E). \tag{12.4}$$

Now we go back to the main proof. First suppose that E is open. Thus, by Definition 12.2.12, we have $E \cap \partial E = \varnothing$, so that $\partial E \subseteq X \backslash E$, which means that $X \backslash E$ is a closed set. The converse also applies: if we suppose that $X \backslash E$ is closed, then $\partial(X \backslash E) \subseteq X \backslash E$. By equation (12.4) above, this is equivalent to $\partial E \subseteq X \backslash E$, and thus we have $\partial E \cap E = \varnothing$. This means that E is open, as expected.[1]

(f) Let $E_1, \ldots, E_n$ be open sets. Thus, for all $1 \leqslant i \leqslant n$, if $x \in E_i$, there exists a $r_i > 0$ such that $B(x, r_i) \subseteq E_i$. Let's define $r := \min(r_1, \ldots, r_n)$. We have $B(x, r) \subseteq B(x, r_i) \subseteq E_i$ for all $1 \leqslant i \leqslant n$, i.e. $B(x, r) \subseteq E_1 \cap \ldots \cap E_n$. Thus, $E_1 \cap \ldots \cap E_n$ is an open set.

Also, let $F_1, \ldots, F_n$ be closed sets. By the previous result (e), the complementary sets $X \backslash F_1, \ldots X \backslash F_n$ are open sets. Thus, we have just proved that $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n)$ is an open set. But we have $(X \backslash F_1) \cap \ldots \cap (X \backslash F_n) = X \backslash (F_1 \cup \ldots \cup F_n)$, and this set is open. Thus, by (e), its complementary set, $F_1 \cup \ldots \cup F_n$, is closed, as expected.

(g) Let $(E_\alpha)_{\alpha \in I}$ be open sets. Suppose that we have $x \in \bigcup_{\alpha \in I} E_\alpha$. By definition, there exists a $i \in I$ such that $x \in E_i$. Since E_i is an open set, there exists $r_i > 0$ such that $B(x, r_i) \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$. Thus, by (a), $\bigcup_{\alpha \in I} E_\alpha$ is an open set, as expected.

Now let be $(F_\alpha)_{\alpha \in I}$ be closed sets. Suppose that we have a convergent sequence $(x_n)_{n=1}^{\infty}$ such that $x_n \in \bigcap_{\alpha \in I} F_\alpha$ for all $n \geqslant 1$. Thus, for all $\alpha \in I$, the sequence $(x_n)_{n=1}^{\infty}$ entirely belongs to the closed set F_α, so that its limit ℓ also lies in F_α according to (b). Thus, $\ell \in \bigcup_{\alpha \in I} F_\alpha$, so that $\bigcap_{\alpha \in I} F_\alpha$ is a closed set, as expected.

(h) Let be $E \subseteq X$.

- Let's show that $\mathrm{int}(E)$ is the largest open set included in E. It has not clearly be proved in the main text that $\mathrm{int}(E)$ is an open set, so we begin by proving it. Let be $x \in \mathrm{int}(E)$. By definition, there exists $r > 0$ so that $B(x, r) \subseteq E$. But by (c), we know that $B(x, r)$ is an open set, so that any point y of $B(x, r)$ is an interior point of this open ball, and thus an interior point of E. Thus, $\mathrm{int}(E)$ is open.

  Now consider another open set $V \subseteq E$, and let's show that $V \subseteq \mathrm{int}(E)$. If $x \in \mathrm{int}(V)$, then there exists $r > 0$ such that $B(x, r) \subseteq V \subseteq E$, so that $x \in \mathrm{int}(E)$. This shows that $V \subseteq \mathrm{int}(E)$, as expected.

- Similarly, let's show that $\overline{E}$ is the smallest closed set that contains E. First we show that $\overline{E}$ is closed, i.e. that $\overline{\overline{E}} \subseteq \overline{E}$. (Hint: see Exercise 9.1.6 for an intuition.) Let be $x \in \overline{\overline{E}}$. By definition, for all $r > 0$, $B(x, r) \cap \overline{E} \neq \varnothing$. Thus, there exists $y \in B(x, r)$ such that $y \in \overline{E}$. Thus, because $B(x, r)$ is an open set and y is adherent to E, there exists $\varepsilon > 0$ such that $B(y, \varepsilon) \subseteq B(x, r)$ and $B(y, \varepsilon) \cap E \neq \varnothing$; i.e., there exists $z \in B(y, \varepsilon) \subseteq B(x, r)$ such that $z \in E$. We have showed that whenever $x \in \overline{\overline{E}}$, we have $B(x, r) \cap E \neq \varnothing$ for all $r > 0$, i.e. that x is an adherent point of E, as expected. Thus, $\overline{E}$ is closed.

  Now we consider a closed set K such that $E \subseteq K$, and we have to show that $\overline{E} \subseteq K$. Let be $x \in \overline{E}$. By definition, for all $r > 0$, we have $B(x, r) \cap E \neq \varnothing$. But since $E \subseteq K$, we also have $B(x, r) \cap K \neq \varnothing$ for all $r > 0$. Thus, x is an adherent point of K, i.e., $x \in \overline{K}$. But since K is closed, we have $K = \overline{K}$, and thus $x \in K$. This shows that $\overline{E} \subseteq K$, as expected.

[1] This important result will be used in future proofs to turn any statement on closed sets into a statement on open sets.

Exercise 12.2.4. — *Let (X, d) be a metric space, x_0 be a point in X, and $r > 0$. Let B be the open ball $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$, and let C be the closed ball $C := \{x \in X : d(x, x_0) \leqslant r\}$.*

Let's prove the following claims:

(a) Show that $\overline{B} \subseteq C$.

Let be $x \in \overline{B}$. By definition, since x is an adherent point of B, for all $\varepsilon > 0$ we have $B(x, \varepsilon) \cap B \neq \varnothing$. In other words, there exists y such that we have both $d(x, y) < \varepsilon$ and $d(x_0, y) < r$. Thus, by the triangle inequality, we have

$$\begin{aligned} d(x, x_0) &\leqslant d(x, y) + d(y, x_0) \\ &\leqslant \varepsilon + r \quad \text{for all } \varepsilon > 0 \end{aligned}$$

which is equivalent (as a quick proof by contradiction would show) to $d(x, x_0) \leqslant r$. Thus, $x \in C$.

We have indeed proved that $\overline{B} \subseteq C$.

(b) Give an example of a metric space (X, d), a point x_0, and a radius $r > 0$ such that $\overline{B}$ is *not* equal to C.

Let's take $X = \mathbb{R}$, $d = d_{\text{disc}}$, $x = 0$ and $r = 1$. One the one hand, we have $B := \{0\}$ and $C := \mathbb{R}$. Now let's work out $\overline{B}$. By Proposition 12.2.15(bd), B is closed, so that we have $\overline{B} = B$. Thus, we clearly do not have $\overline{B} \neq C$ here. (Note however that any $x_0 \in \mathbb{R}$ would be convenient here; there is nothing special about 0.)

Exercise 12.3.1. — *Prove Proposition 12.3.4(b).*

Let's show each direction of the equivalence.

- First suppose that E is relatively closed w.r.t. Y, and let's show that there exists a closed subset $K \subseteq X$ such that $E = K \cap Y$.

  Since E is closed w.r.t. Y, the set $Y \backslash E$ is open w.r.t. Y (by Proposition 12.2.15(e)). Thus, by (a), there exists an open subset $V \subseteq X$ such that $Y \backslash E = V \cap Y$.

  Let be $K := X \backslash V$; this subset $K \subseteq X$ is closed w.r.t. X by Proposition 12.2.15(e) since it is the complementary set of an open set. We have to show that $E = K \cap Y$.

  - Let be $x \in E$. Thus, we have $x \in Y$, since $E \subseteq Y$. And since $x \in E$, by definition, we have $x \notin Y \backslash E$. Thus, $x \notin V \cap Y$, which implies that $x \notin V$ (since $x \in Y$). Thus, by definition, $x \in K$, and thus, $x \in K \cap Y$.
  - Conversely, let be $x \in K \cap Y$. By definition, $x \in Y$ and $x \notin V$. Thus, $x \notin V \cap Y$, or, in other words, $x \notin Y \backslash E$. We finally get $x \in E$, as expected.

  Thus, we have indeed $E = K \cap Y$, for some closed subset $K \subseteq X$, as expected.

- Now let's prove the converse implication: suppose that $E = K \cap Y$ for some closed subset $K \subseteq X$, and let's prove that E is relatively closed w.r.t. Y.

  Still by Proposition 12.2.15(e), we know that the subset $V := X \backslash K$ is open w.r.t. X. Thus, by the previous result from this exercise, $V \cap Y$ is relatively open w.r.t. Y. Thus, its complementary set $Y \backslash (V \cap Y) = Y \backslash V$ is relatively closed w.r.t. Y. Now we want to show that $E = Y \backslash V$ to close the proof.

- First suppose that $x \in E$. Since $E = K \cap Y$, we thus have $x \in Y$ and $x \in K$, i.e. $x \notin V$. Thus, $x \in Y \backslash V$.
- Now suppose that $x \in Y \backslash V$. We thus have $x \in X$ (since $Y \subseteq X$) and $x \notin V$, so that we necessarily have $x \in K$. Thus $x \in Y \cap K$, i.e. $x \in E$.

Thus $E = Y \backslash V$ is relatively closed w.r.t. Y, as expected.

Exercise 12.4.1. — *Prove Lemma 12.4.3.*

We have to prove that any subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of a convergent sequence $(x^{(n)})_{n=m}^{\infty}$ converges to the same limit as the whole sequence itself.

Suppose that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to x_0. Let be $\varepsilon > 0$. By definition, we have a positive integer $N \geqslant m$ such that $n \geqslant N \implies d(x^{(n)}, x_0) \leqslant \varepsilon$. Our aim here is to show that there exists a positive integer $J \geqslant 1$ such that $j \geqslant J \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon$.

By Definition 12.4.1, we know that we have $m \leqslant n_1 < n_2 < n_3 < \ldots$. Thus, as a quick induction would show, we have $n_j \geqslant m + j - 1$ for all $j \geqslant 1$. Let's take $J := N$. In this case, if $j \geqslant J$, i.e. if $j \geqslant N$, we have $n_j \geqslant m + N - 1 \geqslant N$. Thus:

$$j \geqslant J \implies n_j \geqslant N \implies d(x^{(n_j)}, x_0) \leqslant \varepsilon.$$

Since this is true for all $\varepsilon > 0$, it means that $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.2. — *Prove Proposition 12.4.5.*

Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space. We have to prove that the following two statements are equivalent:

(a) L is a limit point of $(x^{(n)})_{n=m}^{\infty}$.

(b) There exists a subsequence $(x^{(n_j)})_{j=1}^{\infty}$ of the original sequence which converges to L.

We will prove the two implications, but first, note that (with the notations from Definition 12.4.1) if we have $1 \leqslant m \leqslant n_1 < n_2 < n_3 < \ldots$, then a quick induction shows that we have $n_j \geqslant j$ for all $j \geqslant 1$.

- First we prove that (b) implies (a). If some subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to L, then we have by definition:

$$\forall \varepsilon > 0, \exists J \geqslant 1 : j \geqslant J \implies d(x^{(n_j)}, L) \leqslant \varepsilon \tag{12.5}$$

  Now, consider any $\varepsilon > 0$ and any $N \geqslant m$. For this particular choice of ε, consider the corresponding real number J given by equation (12.5), and let's define $p := \max(N, J)$. Thus, we have $n_p \geqslant p \geqslant J$, and by equation (12.5), we thus have $d(x^{(n_p)}, L) \leqslant \varepsilon$. If we set $n := n_p$, we have indeed found an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, as required.

- Now we prove that (a) implies (b). Suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, there exists a natural number $n_1 \geqslant m$ such that $d(x^{(n_1)}, L) \leqslant 1$. Now, for $j > 1$, let's define inductively $n_j := \min\{n > n_{j-1} : d(x^{(n)}, L) \leqslant 1/j\}$. This set is non-empty (by definition of a limit point), so that the well-ordering principle

(Proposition 8.1.4) ensures that it has a (unique) minimal element, i.e. that n_j indeed exists. Let's define the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ obtained following this process. We thus have $d(x^{(n_j)}, L) \leqslant 1/j$ for all $j \geqslant 1$, by construction.

Thus, let be $\varepsilon > 0$. There exists a $j \geqslant 1$ such that $0 < 1/j < \varepsilon$ (Exercise 5.4.4). Thus, for this positive integer j, we have $d(x^{(n_j)}, L) \leqslant 1/j < \varepsilon$. By construction, for all other natural numbers $k \geqslant j+1$, we have $d(x^{(n_k)}, L) \leqslant 1/k \leqslant 1/j \leqslant \varepsilon$.

In summary, for our arbitrary choice of ε, we have showed that there exists $j \geqslant 1$ such that, for all $k \geqslant j$, we have $d(x^{(n_k)}, L) \leqslant \varepsilon$. Thus, the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ constructed in this way converges to L, as expected.

Exercise 12.4.3. — *Prove Lemma 12.4.7.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a convergent sequence of points in a metric space (X, d), and that its limit is x_0. Let's show that it is a Cauchy sequence.

By the triangle inequality, we know that for all $j, k \geqslant m$, we have:

$$d(x^{(j)}, x^{(k)}) \leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0).$$

Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, there exists an $N \geqslant m$ such that we have $d(x^{(n)}, x_0) \leqslant \varepsilon/3$ for all $n \geqslant N$. Thus, if we take $j, k \geqslant N$, we have:

$$\begin{aligned} d(x^{(j)}, x^{(k)}) &\leqslant d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \\ &\leqslant \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, as expected.

Exercise 12.4.4. — *Prove Lemma 12.4.9.*

Let be an arbitrary $\varepsilon > 0$. Since the subsequence $(x^{(n_j)})_{j=1}^{\infty}$ converges to x_0, there exists a $J \geqslant 1$ such that $d(x^{(n_j)}, x_0) \leqslant \varepsilon/3$ whenever $j \geqslant J$.

But the whole sequence $(x^{(n)})_{n=m}^{\infty}$ is supposed to be a Cauchy sequence. Thus, there also exists a $N \geqslant m$ such that $d(x^{(j)}, x^{(k)}) < \varepsilon/3$ whenever $j, k \geqslant N$.

Now, let be $K := \max(J, N)$. If $k \geqslant K$, we have

$$\begin{aligned} d(x^{(k)}, x_0) &\leqslant d(x^{(k)}, x^{(n_k)}) + d(x^{(n_k)}, x_0) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that $(x^{(n)})_{n=m}^{\infty}$ converges to x_0, as expected.

Exercise 12.4.5. — *Let $(x^{(n)})_{n=m}^{\infty}$ be a sequence of points in a metric space (X, d) and let $L \in X$. Show that if L is a limit point of the sequence $(x^{(n)})_{n=m}^{\infty}$, then L is an adherent point of the set $\{x^{(n)} : n \geqslant m\}$. Is the converse true?*

First suppose that L is a limit point of $(x^{(n)})_{n=m}^{\infty}$. By definition, it means that

$$\forall \varepsilon > 0,\ \forall N \geqslant m,\ \exists n \geqslant N \ :\ d(x^{(n)}, L) \leqslant \varepsilon \tag{12.6}$$

Let be an arbitrary $\varepsilon > 0$, and let's take $N = m$. By formula (12.6) above, there exists an $n \geqslant N$ such that $d(x^{(n)}, L) \leqslant \varepsilon$. Thus, this $x^{(n)}$ belongs to both sets $\{x^{(n)} : n \geqslant m\}$ and $B(L, \varepsilon)$. We have just proved that for all $\varepsilon > 0$, the intersection $B(L, \varepsilon) \cap \{x^{(n)} : n \geqslant m\}$ is always non-empty. In other words, L is thus an adherent point of $\{x^{(n)} : n \geqslant m\}$.

However, the converse is not true. Indeed, consider the sequence $(x^{(n)})_{n=1}^{\infty}$ defined in $(\mathbb{R}, d)$ by $x^{(1)} = 1$ and $x^{(n)} = 0$ for all $n \geqslant 2$, i.e. the sequence $1, 0, 0, 0, \ldots$. It is clear that $L := 1$ is an adherent point of $\{x^{(n)} : n \geqslant 1\}$ (which is just the set $\{0, 1\}$). But 1 is not a limit point of $(x^{(n)})_{n=1}^{\infty}$, since we have $d(x^{(n)}, 1) > 1/2$ for all $n \geqslant 2$.

Exercise 12.4.6. — *Show that every Cauchy sequence can have at most one limit point.*

Suppose that $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence in a metric space (X, d), such that L, L' are limit points. Then we have $L = L'$. We will give two different proofs for this fact.

- **Proof 1** (short proof using previous results). By Proposition 12.4.5, since L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, there exists a subsequence that converges to L. But by Lemma 12.4.9, it means that the whole original sequence $(x^{(n)})_{n=m}^{\infty}$ also converges to L. The same argument can be used to show that the whole sequence $(x^{(n)})_{n=m}^{\infty}$ converges to L'. But by uniqueness of limits (Proposition 12.1.20), we must have $L = L'$, as expected.

- **Proof 2** (a more "manual" proof). Let be $\varepsilon > 0$. Since $(x^{(n)})_{n=m}^{\infty}$ is a Cauchy sequence, there exists $N \geqslant m$ such that $d(x^{(p)}, x^{(q)}) \leqslant \varepsilon/3$ for all $p, q \geqslant N$.

  If L is a limit point of $(x^{(n)})_{n=m}^{\infty}$, then for this $N \geqslant m$, there exists $p \geqslant N$ such that $d(x^{(p)}, L) \leqslant \varepsilon/3$. Similarly, there exists $q \geqslant N$ such that $d(x^{(q)}, L') \leqslant \varepsilon/3$.

  We thus have, by triangle inequality:

  $$\begin{aligned} d(L, L') &\leqslant d(L, x^{(p)}) + d(x^{(p)}, x^{(q)}) + d(x^{(q)}, L') \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leqslant \varepsilon \end{aligned}$$

  Thus, $d(L, L') \leqslant \varepsilon$ for all $\varepsilon > 0$, which implies $L = L'$.