

Propositions of solutions for *Analysis I* by Terence Tao

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1. Introduction

No exercises in this chapter.

2. The natural numbers

EXERCISE 2.2.1. — *Prove that the addition is associative, i.e. that for any natural numbers a, b, c , we have $(a + b) + c = a + (b + c)$.*

Let's use induction on c while keeping a and b fixed.

- Base case for $c = 0$: let's prove that $(a + b) + 0 = a + (b + 0)$. The left hand side is equal to $(a + b)$ according to Lemma 2.2.3. For the right hand side, if we apply the same lemma to the $(b + 0)$ part, we get $a + (b + 0) = a + b$. Both sides are equal to $a + b$, and the base case is thus done.
- Now let's suppose inductively that $(a + b) + c = a + (b + c)$: we have to prove that $(a + b) + c++ = a + (b + c++)$. Using Lemma 2.2.3 on the right hand side leads to $a + (b + c)++$. Now consider the left hand side. Using still the same lemma, we get $(a + b) + c++ = ((a + b) + c)++$. By the inductive hypothesis, this is also equal to $(a + (b + c))++$. And, using the lemma 2.2.3 again, this also leads to $a + b + c++$. Therefore, both sides are equal to $a + b + c++$, and we have closed the induction.

EXERCISE 2.2.2. — *Let a be a positive number. Prove that there exists exactly one natural number b such that $b++ = a$.*

Let's use induction on a .

- Base case for $a = 1$: we know that $b = 0$ matches this property, since $0++ = 1$ by Definition 2.1.3. Furthermore, there is only one solution. Suppose that is another natural number b such that $b++ = 1$. Then, we would have $b++ = 0++$, which would imply $b = 0$ by Axiom 2.4. The base case is demonstrated.
- Let's suppose inductively that there is exactly one natural number b such that $b++ = a$. We have to prove that there is exactly one natural number b' such that $b'++ = a++$. By the induction hypothesis, and taking $b' = b++$, we have $b'++ = (b++)++ = a++$. So there exists a solution, with $b' = b++ = a$. Uniqueness is given by Axiom 2.4.: if $b'++ = a++$, then we necessarily have $b' = a$.

EXERCISE 2.2.3. — Let a, b, c be natural numbers. Prove the following properties of order for natural numbers:

- (a) Reflexivity: $a \geq a$. This is true since $a = 0 + a$ by Definition 2.2.1. By commutativity of addition, we can also write $a = a + 0$. So there is indeed a natural number n (with $n = 0$) such that $a = a + n$, i.e. $a \geq a$.
- (b) Transitivity: if $a \geq b$ and $b \geq c$, then $a \geq c$. From the part $a \geq b$, there exists a natural number n such that $a = b + n$ according to Definition 2.2.11. A similar consideration for the part $b \geq c$ leads to $b = c + m$, m being a natural number. Combining together those two equalities, we can write $a = b + n = (c + m) + n = c + (m + n)$ by associativity (see Exercise 2.2.1). Then, $n + m$ being a natural number¹, the transitivity is demonstrated.
- (c) Anti-symmetry: if $a \geq b$ and $b \geq a$, then $a = b$. From the part $a \geq b$, there exists a natural number n such that $a = b + n$. Similarly, there exists a natural number m such that $b = a + m$. Combining those two equalities leads to $a = b + n = (a + m) + n = a + (m + n)$. By cancellation law (Proposition 2.2.6), we can conclude that $0 = m + n$. According to Corollary 2.2.9, this leads to $m = n = 0$. Therefore, both m and n are null, meaning that $a = b + 0 = b$.
- (d) Preservation of order: $a \geq b$ iff $a + c \geq b + c$. First, let's prove that $a + c \geq b + c \implies a \geq b$. If $a + c \geq b + c$, there exists a natural number n such that $a + c = b + c + n$. By cancellation law (Proposition 2.2.6)², we conclude that $a = b + n$, i.e. $a \geq b$, thus demonstrating the first implication. Conversely, let's suppose that $a \geq b$. There exists a natural number m such that $a = b + m$. Therefore, $a + c = b + m + c$ for any natural number c . Still by associativity and commutativity, we can rewrite this as $a + c = (b + c) + m$, i.e. $a + c \geq b + c$.
- (e) $a < b$ iff $a++ \leq b$. First, let's prove that $a++ \leq b \implies a < b$. By definition of ordering, there exists a natural number n such that $b = (a++) + n$. By definition of addition, we can re-write: $b = (a++ + n)++$. Then, by commutativity and yet again by definition of addition, $b = (n + a++)++ = (n++) + (a++)$. Thus, there exists a natural number $n++$ such that $b = n++ + a$, which means that $b \geq a$. But we still have to prove that $a \neq b$. Let's suppose that $a = b$: in this case, by cancellation law, we would have $n++ = 0$, which is impossible according to Axiom 2.3 (0 is not the successor of any natural number). Thus, $a \neq b$ et $b \geq a$: we have showed that $a < b$.
Conversely, let's prove that $a < b \implies a++ \leq b$. Starting from that strict inequality, there exists a *positive*³ natural number n such that $b = a + n$. By Lemma 2.2.10, since n is positive, it has one unique antecessor m , so that n can be written $m++$. Thus, $b = a + (m++) = (a + m)++ = (m + a)++ = m + (a++) = (a++) + m$. And, m being a natural number, this corresponds to the statement $b \geq a$.
- (f) $a < b$ iff $b = a + d$ for some positive number d . First, let's prove the first implication, $a < b \implies b = a + d$ with $d \neq 0$. Since $a < b$, we have in particular $a \leq b$, and

¹This is a trivial induction from the definition of addition.

²And also associativity and commutativity that we do not detail explicitly here.

³We make use here of the statement (f) demonstrated below. There is no circularity here, since proving (f) will not make use of (e).

there exists a natural number d such that $b = a + d$. For the sake of contradiction, let's suppose that $d = 0$. We would have $b = a$, which would contradict the condition $a \neq b$ of the strict inequality. Thus, d is a positive number, which demonstrates the left-to-right implication.

Conversely, let's suppose that $b = a + d$, with $d \neq 0$. This expression gives immediately $a \leq b$. But if $a = b$, by cancellation law, this would lead to $0 = d$, a contradiction with the fact that d is a positive number. Thus, $a \neq b$ and $a \leq b$, which demonstrates $a < b$.

EXERCISE 2.2.4. — *Demonstrate three lemmas used to prove the trichotomy of order for natural numbers.*

- (a) Show that we have $0 \leq b$ for any natural number b . This is obvious since, by definition of addition, $0 + b = b$ for any natural number b . This is precisely the definition of $0 \leq b$.
- (b) Show that if $a > b$, then $a++ > b$. If $a > b$, then $a = b + d$, d being a positive natural number. Let's recall that $a++ = a + 1$. Thus, $a++ = a + 1 = b + d + 1 = b + (d + 1)$ by associativity of addition. Furthermore, $d + 1$ is a positive natural number (by Proposition 2.2.8). Thus, $a++ > b$.
- (c) Show that if $a = b$, then $a++ > b$. Once again, let's use the fact that $a++ = a + 1$. Thus, $a++ = a + 1 = b + 1$, and 1 is a positive natural number. This is the definition of $a++ > b$.

EXERCISE 2.2.5. — *Prove the strong principle of induction, formulated as follows: Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m . Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.*

First let's introduce a small lemma (similar to Proposition 2.2.12(e)).

Lemma. *For any natural number a and b , $a < b++$ iff $a \leq b$.*

Proof. If $a < b++$, then $b++ = a + n$ for a given positive natural n . By Lemma 2.2.10, there exists one natural number m such as $n = m++$. Thus $b++ = a + m++$, which can be rewritten $b++ = (a + m)++$ by Lemma 2.2.3⁴. By Axiom 2.4., this is equivalent to $b = a + n$, which can also be written $a \leq b$.

Conversely, if $a \leq b$, there exists a natural number m such as $b = a + m$. Thus, $b++ = (a + m)++ = a + (m++)$ by Definition of addition (2.2.1). And, $m++$ being a positive number, this means that $b > a$ according to Proposition 2.2.12(f). \square

Now we can prove the main proposition. Let $Q(n)$ be the property " $P(m)$ is true for all m such that $m_0 \leq m < n$ ". Let's induct on n .

- (Although this is not necessary,) we could consider two types of base cases. If $n < m_0$, $Q(n)$ is the proposition " $P(m)$ is true for all m such that $m_0 \leq m < n$ ", but there is no such natural number m . Thus, $Q(n)$ is vacuously true. If $n = m_0$, $P(m_0)$ is true by hypothesis, thus $Q(m_0)$ is also true.

⁴We could also rewrite $b + 1 = a + m + 1$ and then use the cancellation law.

- Now let's suppose inductively that $Q(n)$ is true, and show that $Q(n++)$ is also true. If $Q(n)$ is true, $P(m)$ is true for all m such that $m_0 \leq m < n$. By hypothesis, this implies that $P(n)$ is true. Thus, $P(m)$ is true for any natural number m such that $m_0 \leq m \leq n$, i.e. such that $m_0 \leq m < n++$ according to the lemma introduced above. This is precisely $Q(n++)$, and this closes the induction.

Thus, $Q(n)$ is true for all natural numbers n , which means in particular that $P(m)$ is true for any natural number $m \geq m_0$. This demonstrates the principle of strong induction.

EXERCISE 2.2.6. — *Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m++)$ is true, then $P(m)$ is true. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$; this is known as the principle of backwards induction.*

Terence Tao suggests to use induction on n . So let $Q(n)$ be the following property: “if $P(n)$ is true, then $P(m)$ is true for all $m \leq n$ ”. The goal is to prove $Q(n)$ for all natural numbers n .

- Base case $n = 0$: here, $Q(n)$ means that if $P(0)$ is true, then $P(m)$ is true for any $m \leq 0$. By Definition 2.2.11, if $m \leq 0$, there exists a natural number d such that $0 = m + d$. But, by Corollary 2.2.9, this implies that both $m = 0$ and $d = 0$. Thus, the only number m such that $m \leq 0$ is 0 itself. Therefore, $Q(0)$ is simply the tautology “if $P(0)$ is true, then $P(0)$ is true”—a statement that we can safely accept. The base case is the, demonstrated.
- Let's suppose inductively that $Q(n)$ is true: we must show that $Q(n++)$ is also true. If $P(n++)$ is true, then by definition of P , $P(n)$ is also true. Then, by induction hypothesis, $P(m)$ is true for all $m \leq n$. We have showed that $P(n++)$ implies $P(m)$ for all $m \leq n++$ ⁵, which is precisely $Q(n++)$. This closes the induction.

EXERCISE 2.3.1. — *Show that multiplication is commutative, i.e., if n and m are natural numbers, show that $n \times m = m \times n$.*

We will use an induction of n while keeping m fixed. However, this is not a trivial result, and even the base case is not straightforward. We will first introduce some lemmas.

Lemma. *For any natural number n , $n \times 0 = 0$.*

Proof. Let's induct on n . For the base case $n = 0$, we know by Definition 2.3.1 of multiplication that $0 \times 0 = 0$, since $0 \times m = 0$ for any natural number m .

Now let's suppose that $n \times 0 = 0$. Thus, $n++ \times 0 = (n \times 0) + 0$ by Definition 2.3.1. But by induction hypothesis, $n \times 0 = 0$, so that $n++ \times 0 = 0 + 0 = 0$. This closes the induction. \square

Lemma. *For all natural numbers m and n , we have $m \times n++ = (m \times n) + m$.*

⁵Actually, we use here yet another lemma, similar to the one introduced for the previous exercise. We use the fact that $m \leq n++$ is equivalent to $m = n++$ or $m \leq n$, which is easy to prove, but is not part of the “standard” results presented in the textbook.

Proof. Let's induct on m . The base case $m = 0$ is easy to prove: $0 \times n++ = 0$ by Definition 2.3.1 of multiplication, and $(0 \times n) + 0 = 0$.

Now suppose inductively that $m \times n++ = (m \times n) + m$, and we must show that

$$m++ \times n++ = (m++ \times n) + m++ \quad (1)$$

We begin by the left hand side: by Definition 2.3.1, $m++ \times n++ = (m \times n++) + n++$. By induction hypothesis, this is equal to $(m \times n) + m + n++$.

Then, apply the definition of multiplication to the right hand side: $(m++ \times n) + m++ = (m \times n) + n + m++$. The Lemma 2.2.3 and the commutativity of addition leads to $(m \times n) + n + m++ = (m \times n) + (n + m)++ = (m \times n) + (m + n)++ = (m \times n) + m + n++$, which is equal to the left hand side.

Thus, both sides of equation (1) are equal, and we can close the induction. \square

Now it is easier to prove the main result ($n \times m = m \times n$), by an induction on n .

- Base case $n = 0$: we already know by Definition 2.3.1 that $0 \times m = 0$. The first lemma introduced in this exercise also provides $m \times 0 = 0$. Thus, the base case is proven, since $0 \times m = m \times 0 (= 0)$.
- Now we suppose inductively that $n \times m = m \times n$, and we must prove that:

$$n++ \times m = m \times n++ \quad (2)$$

By Definition 2.3.1 of multiplication, the left hand side is equal to $(n \times m) + m$.

Using the lemma introduced above, the right hand side is equal to $(m \times n) + m$. By induction hypothesis, this is also equal to $(n \times m) + m$, which closes the induction.

EXERCISE 2.3.2. — *Show that positive natural numbers have no zero divisors, i.e. that $nm = 0$ iff $n = 0$ or $m = 0$. In particular, if n and m are both positive, then nm is also positive.*

We will prove the second statement first. Suppose, for the sake of contradiction, that $nm = 0$ and that both n and m are positive numbers. Since they are positive, by Lemma 2.2.10, there exists two (unique) natural numbers a and b such that $n = a++$ and $m = b++$. Thus, the hypothesis $nm = 0$ can also be written $(a++) \times (b++) = 0$. But, by Definition 2.3.1 of multiplication, $(a++) \times (b++) = (a \times b++) + b++$. Thus, we should have $(a \times b++) + b++ = 0$. By Corollary 2.2.9, this implies that both $(a \times b++) = 0$ and $b++ = 0$, which is impossible since zero is the successor of no natural number (Axiom 2.3).

Thus, we have proved that if n and m are both positive, then nm is also positive. The main statement can now be proved more easily.

- The right-to-left implication is straightforward: if $n = 0$, then by Definition of multiplication, $n \times m = 0 \times m = 0$. Since multiplication is commutative, we have the same result if $m = 0$.
- The left-to-right implication is exactly the contrapositive of the statement we have just proved above.

EXERCISE 2.3.3. — *Show that multiplication is associative, i.e., for any natural numbers a, b, c , we have $(a \times b) \times c = a \times (b \times c)$.*

We will induct on c while keeping a and b fixed.

- Base case: for $c = 0$, we must prove that $(a \times b) \times 0 = a \times (b \times 0)$. The left hand side is equal to 0 by definition (and commutativity) of multiplication⁶. The right hand side is equal to $a0$, which is also 0. Both sides are null, and the base case is proved.
- Suppose inductively that $(a \times b) \times c = a \times (b \times c)$, and let's prove that $(a \times b) \times c++ = a \times (b \times c++)$. By definition (and commutativity) of multiplication, the left hand side is equal to $(a \times b) \times c + (a \times b)$. The right hand side is equal to $a \times (b \times c + b)$, and by distributive law (i.e., Proposition 2.3.4), this is also $a \times (b \times c) + a \times b$. But then, by inductive hypothesis, this can be rewritten $(a \times b) \times c + a \times b$, which is equal to the left hand side. The induction is closed.

EXERCISE 2.3.4. — *Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b .*

By distribution law (i.e., Proposition 2.3.4) and commutativity of multiplication, we have:

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) = (a + b)a + (a + b)b \\ &= a \times a + b \times a + a \times b + b \times b \\ &= a^2 + a \times b + a \times b + b^2 \\ &= a^2 + 2ab + b^2\end{aligned}$$

(For the last step, we recall that, by Definition 2.3.1, $2 \times m = m + m$ for any natural number m .)

EXERCISE 2.3.5. — *Euclidean algorithm. Let n be a natural number, and let q be a positive number. Prove that there exists natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

We will induct on n while remaining q fixed.

- Base case: if $n = 0$, there exists an obvious solution, namely $m = 0$ and $r = 0$.
- Suppose inductively that there exists m, r such that $n = mq + r$ with $0 \leq r < q$, and let's prove that there exists m', r' such that $n + 1 = m'q + r'$, with $0 \leq r' < q$.

By the induction hypothesis, we have $n + 1 = mq + r + 1$. Since $r < q$, we have $r + 1 \leq q$ (this is Proposition 2.2.12). Thus, we have two cases here:

1. If $r + 1 < q$, then $n + 1 = mq + (r + 1)$, with $0 \leq r + 1 < q$, so that choosing $m' = m$ and $r' = r + 1$ is convenient.
2. If $r + 1 = q$, then $n + 1 = mq + q = (m + 1)q$ according to the distributive law (Proposition 2.3.4). Thus, choosing $m' = m + 1$ and $r' = 0$ is convenient.

This closes the induction.

⁶Actually, we use the second lemma introduced for the resolution of Exercise 2.3.1.

3. Set theory

EXERCISE 3.1.2. — *Using only Definition 3.1.4, Axiom 3.1, Axiom 3.2, and Axiom 3.3, prove that the sets \emptyset , $\{\emptyset\}$, $\{\{\emptyset\}\}$, and $\{\emptyset, \{\emptyset\}\}$ are all distinct (i.e., no two of them are equal to each other).*

As a general reminder, we recall that sets are objects (Axiom 3.1) and the empty set \emptyset is such that no object is an element of \emptyset , thus $\emptyset \notin \emptyset$.

1. First let's show that \emptyset is different from all other sets. \emptyset is an element of $\{\emptyset\}$ and $\{\emptyset, \{\emptyset\}\}$, and $\{\emptyset\}$ is an element of $\{\{\emptyset\}\}$. But none of those two objects are elements of \emptyset (by Axiom 3.2), thus \emptyset is different from all three other sets.
2. Then let's show that $\{\emptyset\} \neq \{\{\emptyset\}\}$. By Axiom 3.3, the singleton $\{\emptyset\}$ is such that $x \in \{\emptyset\} \iff x = \emptyset$. Similarly, the singleton $\{\{\emptyset\}\}$ is such that $x \in \{\{\emptyset\}\} \iff x = \{\emptyset\}$. But we already know that $\emptyset \neq \{\emptyset\}$ so there exists an object, \emptyset , which is a element of $\{\emptyset\}$ but not an element of $\{\{\emptyset\}\}$. Those sets are not equal.
3. Now let's show that $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$. By Axiom 3.3, the pair $\{\emptyset, \{\emptyset\}\}$ is such that x is an element of this set iff $x = \emptyset$ or $x = \{\emptyset\}$. Thus, $\{\emptyset\}$ is an element of $\{\emptyset, \{\emptyset\}\}$, but is not an element of $\{\emptyset\}$ (if it was, we should have $\emptyset = \{\emptyset\}$, which would be a contradiction with the first point of this proof). Those two sets are thus different.
4. Finally, we also have $\{\{\emptyset\}\} \neq \{\emptyset, \{\emptyset\}\}$. Indeed, we have $\emptyset \in \{\emptyset, \{\emptyset\}\}$ by Axiom 3.3. However, $\emptyset \in \{\{\emptyset\}\} \iff \emptyset = \{\emptyset\}$ by definition of a singleton, and we know this latest statement is false by the first point of this proof. Those two sets are also different.