

Propositions of solutions for *Analysis II* by Terence Tao

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Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geq m$. We have to prove that $\lim_{n \rightarrow \infty} a_n = 0$ if and only if $\lim_{n \rightarrow \infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n \rightarrow \infty} a_n = 0$, then there exists an $N \geq m$ such that $|a_n| < \varepsilon$ whenever $n \geq N$. Thus, there exists an $N \geq m$ such that $|x_n - x| < \varepsilon$ whenever $n \geq N$, which means that $\lim_{n \rightarrow \infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n \rightarrow \infty} x_n = x$, then there exists an $N \geq m$ such that $|x_n - x| < \varepsilon$ whenever $n \geq N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n \rightarrow \infty} a_n = 0$, as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let X be a set, and let $d : X \times X \rightarrow [0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which...*

- (a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

- (b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

- (c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

- (d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.

EXERCISE 12.1.4. — *Show that the pair $(Y, d|_{Y \times Y})$ defined in Example 12.1.5 is indeed a metric space.*

By definition, since $Y \subseteq X$, we have $x, y \in X$ whenever $x, y \in Y$. And furthermore, since $d|_{Y \times Y}(x, y) := d(x, y)$, then the application $d|_{Y \times Y}$ obeys all four statements (a)–(d) of Definition 12.1.2. Thus, $(Y, d|_{Y \times Y})$ is indeed a metric space.

EXERCISE 12.1.5. — Let $n \geq 1$, and let a_1, a_2, \dots, a_n and b_1, b_2, \dots, b_n be real numbers. Verify the identity $(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$, and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.

Let's prove these three statements.

(i) To prove the first identity, let's use induction on n .

The base case $n = 1$ is obvious: on the left-hand side, we just get $(a_1 b_1)^2$, and on the right-hand side, we get $a_1^2 b_1^2$, hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer $n \geq 1$, and let's prove that it is still true for $n + 1$. We have to prove that

$$\underbrace{\left(\sum_{i=1}^{n+1} a_i b_i \right)^2}_{:=A} + \underbrace{\frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left(\sum_{i=1}^{n+1} a_i^2 \right) \left(\sum_{j=1}^{n+1} b_j^2 \right)}_{:=C} \quad (12.1)$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for A , we have

$$\begin{aligned} A &:= \left(\sum_{i=1}^{n+1} a_i b_i \right)^2 \\ &= \left(a_{n+1} b_{n+1} + \sum_{i=1}^n a_i b_i \right)^2 \\ &= (a_{n+1} b_{n+1})^2 + \left(\sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1} b_{n+1}) \sum_{i=1}^n a_i b_i \end{aligned}$$

- for B , we have

$$\begin{aligned} B &:= \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{j=1}^{n+1} (a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2} \sum_{i=1}^n (a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2 \times S} + \underbrace{\frac{1}{2} \sum_{j=1}^n (a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2 \times S} \\ &\quad + \underbrace{\frac{1}{2} (a_{n+1} b_{n+1} - b_{n+1} a_{n+1})^2}_{=0} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for $A + B$, we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A + B &:= (a_{n+1}b_{n+1})^2 + \left(\sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left(\sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2 \sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left(\sum_{i=1}^{n+1} a_i^2 \right) \left(\sum_{j=1}^{n+1} b_j^2 \right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number n .

- (ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{i=1}^n b_i^2 \right)^{1/2}. \quad (12.2)$$

Indeed, since $B \geq 0$ in the identity (12.1), we have

$$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \left(\sum_{i=1}^n a_i^2 \right) \left(\sum_{j=1}^n b_j^2 \right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$\begin{aligned}
\sum_{i=1}^n (a_i^2 + b_i^2) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \\
&\leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \left(\sum_{i=1}^n a_i^2 \right)^{1/2} \left(\sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{by eq. (12.2)}) \\
&\leq \left(\left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

EXERCISE 12.1.6. — *Show that (\mathbb{R}^n, d_{l^2}) in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

- (a) For all $x \in \mathbb{R}^n$, we have $d_{l^2}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2} = 0$, as expected.
- (b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leq i \leq n$ such that $x_i \neq y_i$, so that $(x_i - y_i)^2 > 0$, and $d_{l^2}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} > 0$, as expected.
- (c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(y, x) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_{l^2}(x, y)$$

as expected.

- (d) Triangle inequality: for all $x, y, z \in \mathbb{R}^n$, we have

$$\begin{aligned}
d_{l^2}(x, z) &:= \left(\sum_{i=1}^n (x_i - z_i)^2 \right)^{1/2} \\
&= \left(\sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \quad \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leq \left(\sum_{i=1}^n a_i^2 \right)^{1/2} + \left(\sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{Exercise 12.1.5(iii)}) \\
&\leq \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} + \left(\sum_{i=1}^n (y_i - z_i)^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) + d_{l^2}(y, z)
\end{aligned}$$

as expected.

Thus, (\mathbb{R}^n, d_{l^2}) is indeed a metric space.

EXERCISE 12.1.7. — *Show that (\mathbb{R}^n, d_{l^1}) in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all $x \in \mathbb{R}^n$, we have $d_{l^1}(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$, as expected.
- (b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leq i \leq n$ such that $x_i \neq y_i$, so that $|x_i - y_i| > 0$, and $d_{l^1}(x, y) = \sum_{i=1}^n |x_i - y_i| > 0$, as expected.
- (c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^1}(y, x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x, y)$$

as expected.

- (d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality $|a - c| \leq |a - b| + |b - c|$ for all $a, b, c \in \mathbb{R}$. Thus, for all $x, y, z \in \mathbb{R}^n$, we have

$$d_{l^1}(x, z) := \sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x, y) + d_{l^1}(y, z)$$

as expected.

Thus, (\mathbb{R}^n, d_{l^1}) is indeed a metric space.

EXERCISE 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y) \tag{12.3}$$

- The first inequality, since everything is non-negative, is equivalent to $d_{l^2}(x, y)^2 \leq d_{l^1}(x, y)^2$, and we will prove it in this form.

Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l^1}(x, y)^2 &:= \left(\sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \left(\sum_{i=1}^n |x_i - y_i| \right) \times \left(\sum_{i=1}^n |x_i - y_i| \right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leq i, j \leq n; i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geq 0} \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x, y)^2 \end{aligned}$$

as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x, y) &:= \sum_{i=1}^n |x_i - y_i| \\
&= \left| \sum_{i=1}^n |x_i - y_i| \times 1 \right| \\
&\leq \left(\sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left(\sum_{i=1}^n 1^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) \times \sqrt{n}
\end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair $(\mathbb{R}^n, d_{l^\infty})$ in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all $x \in \mathbb{R}^n$, we clearly have $d_{l^\infty}(x, x) = \sup\{|x_i - x_i| : 1 \leq i \leq n\} = 0$, as expected.
- (b) Positivity: for all $x \neq y \in \mathbb{R}^n$, there exists at least one $1 \leq j \leq n$ such that $x_j \neq y_j$. Thus $|x_j - y_j| > 0$, and $d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_j - y_j| > 0$, as expected.
- (c) Symmetry: for all $x, y \in \mathbb{R}^n$, we have

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} = \sup\{|y_i - x_i| : 1 \leq i \leq n\} = d_{l^\infty}(y, x)$$

as expected.

- (d) Triangle inequality. Let be $x, y, z \in \mathbb{R}^n$. We have $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$ for all $1 \leq i \leq n$, by Proposition 4.3.3(g). But, by definition of the supremum, we have $|x_i - y_i| \leq d_{l^\infty}(x, y)$ and $|y_i - z_i| \leq d_{l^\infty}(y, z)$ for all $1 \leq i \leq n$. Thus, we have $|x_i - z_i| \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$ for all $1 \leq i \leq n$; i.e., $d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$ is an upper bound of the set $\{|x_i - z_i| : 1 \leq i \leq n\}$. By definition of the supremum, it implies that

$$d_{l^\infty}(x, z) := \sup\{|x_i - z_i| : 1 \leq i \leq n\} \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$$

as expected.

Thus, (\mathbb{R}^n, d_{l^1}) is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y).$$

First, a preliminary remark. By definition, we have $d_{l^\infty}(x, y) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}$ for all $x, y \in \mathbb{R}^n$. Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a $1 \leq m \leq n$ such that $d_{l^\infty}(x, y) = |x_m - y_m|$ (the supremum belongs to the set). The index “ m ” will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}
\frac{1}{\sqrt{n}}d_{l^2}(x, y) &:= \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - y_i)^2} \\
&\leq \sqrt{\frac{1}{n} \sum_{i=1}^n (x_m - y_m)^2} \\
&\leq \sqrt{\frac{n}{n} (x_m - y_m)^2} \\
&= |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}
d_{l^2}(x, y) &:= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \\
&= \sqrt{(x_m - y_m)^2 + \sum_{1 \leq i \leq n; i \neq m} (x_i - y_i)^2} \\
&\geq \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

EXERCISE 12.1.11. — *Show that the discrete metric (X, d_{disc}) in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all $x \in X$, we have $d_{\text{disc}}(x, x) := 0$ by definition, so that there is nothing to prove here.
- (b) Positivity: for all $x \neq y \in X$, we have $d_{\text{disc}}(x, y) := 1 > 0$ by definition, so that there's still nothing to prove.
- (c) Symmetry: for all $x, y \in X$, we have $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x) = 1$, so that d_{disc} obeys the symmetry property.
- (d) Triangle inequality. Let be $x, y, z \in X$, and let's consider $d_{\text{disc}}(x, z)$.
 - If $x = z$, then $d_{\text{disc}}(x, z) = 0$. And since d_{disc} is a non-negative application, we clearly have $0 =: d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$ for all $y \in X$.
 - If $x \neq z$, then we cannot have both $x = y$ and $y = z$ (it would be a clear contradiction with $x \neq z$). Thus, at least one of the propositions " $x \neq y$ ", " $y \neq z$ " is true. Another way to say that is $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq 1$. But since $d_{\text{disc}}(x, z) := 1$, we have actually $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq d_{\text{disc}}(x, z)$, as expected.

EXERCISE 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all $x, y \in \mathbb{R}^n$, we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y).$$

Note that n is a real constant here.

- Let's prove that (a) \implies (b). If $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$, then by the limit laws, the sequence $t_k := \sqrt{n} d_{l^2}(x^{(k)}, x)$ also converges to 0 as $k \rightarrow \infty$, since \sqrt{n} is a constant real number. Thus, we have

$$d_{l^2}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x) \leq \sqrt{n} d_{l^2}(x^{(k)}, x)$$

and, by the squeeze test, this implies that $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x)$ as expected.

- Let's prove that (b) \implies (c). If $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x) = 0$, then we have

$$0 \leq d_{l^\infty}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x)$$

and, by the squeeze test, this implies that $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x)$ as expected.

- Let's prove that (c) \implies (d). Suppose that $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x) = 0$. Then, for all $1 \leq j \leq n$, we have $0 \leq |x_j^{(k)} - x_j| \leq d_{l^\infty}(x^{(k)}, x)$. Still by the squeeze test, this implies that $\lim_{k \rightarrow \infty} |x_j^{(k)} - x_j| = 0$, i.e. that $(x_j^{(k)})_{k=m}^\infty$ converges to x_j as $k \rightarrow \infty$ (by Lemma 12.1.1), as expected.
- Finally, let's prove that (d) \implies (a). Using the definition of convergence is more appropriate here. Let be $\varepsilon > 0$ a positive real number, and let be $1 \leq j \leq n$. By definition, there exists a natural number $N \geq m$ such that $|x_j^{(k)} - x_j| \leq \varepsilon/\sqrt{n}$ whenever $k \geq N$. Thus, if $k \geq N$, we have

$$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^n (x_j^{(k)} - x_j)^2} \leq \sqrt{\sum_{j=1}^n \frac{\varepsilon^2}{n}} \leq \varepsilon$$

so that $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$, i.e., $(x^{(k)})_{k=m}^\infty$ converges to x as $k \rightarrow \infty$ in the l^2 metric (by Lemma 12.1.1), as expected.

EXERCISE 12.1.13. — *Prove Proposition 12.1.19.*

Let be $(x^{(n)})_{n=m}^\infty$ a sequence of elements of a set X .

- First suppose that $(x^{(n)})_{n=m}^\infty$ is eventually constant. Thus, by definition, there exists an $N \geq m$ and an element $x \in X$ such that $(x^{(n)})_{n=m}^\infty = x$ for all $n \geq N$. This implies that we have $d_{\text{disc}}(x^{(n)}, x) = 0$ for all $n \geq N$. In particular, for all $\varepsilon > 0$, we have $d_{\text{disc}}(x^{(n)}, x) \leq \varepsilon$ whenever $n \geq N$, so that $(x^{(n)})_{n=m}^\infty$ indeed converges to x with respect to d_{disc} .
- Conversely, suppose that $(x^{(n)})_{n=m}^\infty$ converges to x with respect to d_{disc} . Let be $\varepsilon = 1/2$. By definition, there exists an $N \geq m$ such that $d_{\text{disc}}(x^{(n)}, x) \leq 1/2$ whenever $n \geq N$. Since $d_{\text{disc}}(x^{(n)}, x)$ cannot be 1, it is necessarily equal to 0, so that $x^{(n)} = x$ whenever $n \geq N$. Thus, the sequence $x^{(n)}$ is indeed eventually constant.

EXERCISE 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have $\lim_{n \rightarrow \infty} d(x^{(n)}, x) = 0$ and $\lim_{n \rightarrow \infty} d(x^{(n)}, x') = 0$. Suppose, for the sake of contradiction, that we have $x \neq x'$. Thus, the real number $\varepsilon := \frac{d(x, x')}{3}$ is positive.

Since $x^{(n)}$ converges to x , there exists a $N_1 \geq m$ such that $d(x^{(n)}, x) \leq \varepsilon$ whenever $n \geq N_1$.

Similarly, since $x^{(n)}$ converges to x' , there exists a $N_2 \geq m$ such that $d(x^{(n)}, x') \leq \varepsilon$ whenever $n \geq N_2$.

By the triangle inequality, we thus have, for all $n \geq \max(N_1, N_2)$,

$$d(x, x') \leq d(x, x^{(n)}) + d(x^{(n)}, x') \leq \varepsilon + \varepsilon = \frac{2}{3}d(x, x')$$

which is a contradiction (since $d(x, x') > 0$ by hypothesis).

Thus, the limit is unique, and we must have $x = x'$.

EXERCISE 12.1.15. — *Let be $X := \{(a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty\}$. We define on this space the metrics $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sum_{n=0}^\infty |a_n - b_n|$, and $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |a_n - b_n|$. Then...*

We have to prove the following statements.

1. d_{l^1} is a metric on X .

We have to prove the four axioms of Definition 12.1.2.

- (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^1}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - a_n| = 0$, as expected.
- (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X . Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - b_n| \geq |a_m - b_m| > 0$, as expected.
- (c) Symmetry: we clearly have

$$d_{l^1}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |b_n - a_n| = \sum_{n=0}^\infty |a_n - b_n| = d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on \mathbb{R} (i.e., we have $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately

$$\begin{aligned} d_{l^1}((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty) &:= \sum_{n=0}^\infty |a_n - c_n| \\ &\leq \sum_{n=0}^\infty (|a_n - b_n| + |b_n - c_n|) \quad (\text{consequence of Prop. 7.1.11(h)}) \\ &\leq \sum_{n=0}^\infty |a_n - b_n| + \sum_{n=0}^\infty |b_n - c_n| \quad (\text{by Proposition 7.2.14(a)}) \\ &\leq d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) + d_{l^1}((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty). \end{aligned}$$

Thus, d_{l^1} is indeed a metric on X .

2. d_{l^∞} is a metric on X .

Once again, we have to prove the four axioms of Definition 12.1.2.

- (a) Let be $(a_n)_{n=0}^\infty \in X$. We have $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - a_n| = 0$, as expected.
- (b) Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$ two distinct elements of X . Since they are distinct, there exists at least one $m \in \mathbb{N}$ such as $|a_m - b_m| > 0$. Thus, $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - b_n| \geq |a_m - b_m| > 0$, as expected.
- (c) Symmetry: we clearly have

$$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |b_n - a_n| = \sup_{n \in \mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$. Since we have the triangle inequality for the usual distance d on \mathbb{R} (i.e., we have $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$ for all $n \in \mathbb{N}$), we have immediately $|a_m - c_m| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$ for all $m \in \mathbb{N}$, by definition of the supremum. In other words, $(\sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|)$ is an upper bound for the set $\{|a_m - c_m| : m \in \mathbb{N}\}$. Thus we have, still by definition of the supremum, $\sup_{n \in \mathbb{N}} |a_n - c_n| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$, as expected.

Thus, d_{l^∞} is indeed a metric on X .

3. There exist sequences $x^{(1)}, x^{(2)}, \dots$, of elements of X (i.e., sequences of sequences) which are convergent with respect to d_{l^∞} , but are not convergent with respect to d_{l^1} .

Here we are dealing with sequences of sequences: we have a sequence $(x^{(k)})_{k=1}^\infty$ where each $x^{(k)}$ is itself a sequence of real numbers. Thus, let's define $(x^{(k)})_{k=1}^\infty$ as follows:

$$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

Just to make things clearer, we have for instance

$$\begin{aligned} x^{(1)} &:= \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \\ x^{(2)} &:= \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots \\ x^{(3)} &:= \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots \end{aligned}$$

Also, let be the null sequence $(a_n)_{n=0}^\infty$ defined by $a_n := 0$ for all $n \in \mathbb{N}$. Thus:

- $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} . Indeed, if we consider a given positive integer k (fixed), we have

$$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

so that $d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$.

Thus, $\lim_{k \rightarrow \infty} d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) = 0$, or in other words, $(x^{(k)})_{k=1}^\infty$ converges to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} in X .

- But $(x_n^{(k)})_{n=0}^\infty$ does not converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1} . Indeed, we have, for each given (fixed) k ,

$$d_{l^1} \left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = \sum_{n=0}^k \frac{1}{k+1} = 1$$

Thus, we clearly do not have $\lim_{k \rightarrow \infty} d_{l^1} \left((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = 0$, i.e., $(x_n^{(k)})_{k=1}^\infty$ does not converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1} .

4. Conversely, any sequence which converges with respect to d_{l^1} also converges with respect to d_{l^∞} .

Suppose, for the sake of contradiction, that $(x_n^{(k)})_{k=1}^\infty$ does not converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^∞} , but does converge to $(a_n)_{n=0}^\infty$ w.r.t. the metric d_{l^1} .

In this case, there exists a $\varepsilon > 0$ such that, for all $k \geq 1$, we have $(\sup_{n \geq 0} |x_n^{(k)} - a_n|) > \varepsilon$. In particular, for all $k \geq 1$ and all $n \geq 0$, we have $|x_n^{(k)} - a_n| > \varepsilon$. Thus, $\sum_{n=0}^\infty |x_n^{(k)} - a_n|$ is not even a convergent series, and we cannot have $\lim_{k \rightarrow \infty} \left(\sum_{n=0}^\infty |x_n^{(k)} - a_n| \right) = 0$.

Note that this exercise actually shows that in this space X , the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

EXERCISE 12.1.16. — Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be two sequences in a metric space (X, d) . Suppose that $(x_n)_{n=1}^\infty$ converges to a point $x \in X$, and $(y_n)_{n=1}^\infty$ converges to a point $y \in X$. Show that $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$.

On the one hand, the triangle inequality applied two times to d gives us

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leq d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

Let be $\varepsilon > 0$. By hypothesis, there exists a $N_1 \geq 1$ such that $d(x_n, x) \leq \varepsilon/2$ whenever $n \geq N_1$. Similarly, there exists a $N_2 \geq 1$ such that $d(y_n, y) \leq \varepsilon/2$ whenever $n \geq N_2$. Thus, if we set $N := \max(N_1, N_2)$, then for all $n \geq N$ we have

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \leq 2\varepsilon/2 = \varepsilon$$

which shows that $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$, as expected.

EXERCISE 12.2.1. — *Verify the claims in Example 12.2.8*

Let be (X, d_{disc}) a metric space, and E a subset of X .

- Let be $x \in E$. Then x is an interior point of E . Indeed, we have $B(x, 1/2) = \{x\} \subseteq E$.
- Let be $y \notin E$. Then y is an exterior point of E . Indeed, we have $B(y, 1/2) \cap E = \{y\} \cap E = \emptyset$.
- Finally, there are no boundary points of E in (X, d_{disc}) . Indeed, let be $r > 0$ and any $x \in X$. We will always have $B(x, r) = \{x\}$ by definition of the discrete metric d_{disc} . Thus, we have either $x \in E$ and then $x \in \text{int}(E)$, or $x \notin E$ and then $x \in \text{ext}(E)$. Thus, E has no boundary points.