

# Propositions of solutions for *Analysis II* by Terence Tao

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**Remarks.** The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to “Exercise 4.3.3” (for instance) will always mean “Exercise 4.3.3 from *Analysis I*”.

## 12. Metric spaces

EXERCISE 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence  $(a_n)_{n=m}^{\infty}$  defined by  $a_n := d(x_n, x) = |x_n - x|$  for all  $n \geq m$ . We have to prove that  $\lim_{n \rightarrow \infty} a_n = 0$  if and only if  $\lim_{n \rightarrow \infty} x_n = x$ .

- Let be  $\varepsilon > 0$ . If  $\lim_{n \rightarrow \infty} a_n = 0$ , then there exists an  $N \geq m$  such that  $|a_n| < \varepsilon$  whenever  $n \geq N$ . Thus, there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ , which means that  $\lim_{n \rightarrow \infty} x_n = x$ .
- Let be  $\varepsilon > 0$ . Conversely, if  $\lim_{n \rightarrow \infty} x_n = x$ , then there exists an  $N \geq m$  such that  $|x_n - x| < \varepsilon$  whenever  $n \geq N$ . But since  $|a_n| := |x_n - x|$ , it means that  $\lim_{n \rightarrow \infty} a_n = 0$ , as expected.

EXERCISE 12.1.2. — *Show that the real line with the metric  $d(x, y) := |x - y|$  is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

- (a) comes from Proposition 4.3.3(e)
- (b) also comes from Proposition 4.3.3(e)
- (c) comes from Proposition 4.3.3(f)
- (d) comes from Proposition 4.3.3(g).

EXERCISE 12.1.3. — *Let  $X$  be a set, and let  $d : X \times X \rightarrow [0, \infty)$  be a function. With respect to Definition 12.1.2, give an example of a pair  $(X, d)$  which...*

- (a) obeys the axioms (bcd) but not (a).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, x) = 1$  and  $d(x, y) = 5$  for all  $x \neq y \in \mathbb{R}$ .

- (b) obeys the axioms (acd) but not (b).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = 0$  for all  $x, y \in \mathbb{R}$ .

- (c) obeys the axioms (abd) but not (c).

Consider  $X = \mathbb{R}$ , and  $d$  defined by  $d(x, y) = \max(x - y, 0)$  for all  $x, y \in \mathbb{R}$ .

- (d) obeys the axioms (abc) but not (d).

Consider the finite set  $X := \{1, 2, 3\}$  and the application  $d$  defined by  $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$ , and  $d(1, 3) = d(3, 1) := 5$ , and  $d(x, x) = 0$  for all  $x \in X$ .

EXERCISE 12.1.4. — *Show that the pair  $(Y, d|_{Y \times Y})$  defined in Example 12.1.5 is indeed a metric space.*

By definition, since  $Y \subseteq X$ , we have  $x, y \in X$  whenever  $x, y \in Y$ . And furthermore, since  $d|_{Y \times Y}(x, y) := d(x, y)$ , then the application  $d|_{Y \times Y}$  obeys all four statements (a)–(d) of Definition 12.1.2. Thus,  $(Y, d|_{Y \times Y})$  is indeed a metric space.

EXERCISE 12.1.5. — Let  $n \geq 1$ , and let  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  be real numbers. Verify the identity  $(\sum_{i=1}^n a_i b_i)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 = \sum_{i=1}^n a_i^2 \sum_{j=1}^n b_j^2$ , and conclude the Cauchy-Schwarz inequality. Then use the Cauchy-Schwarz inequality to prove the triangle inequality.

Let's prove these three statements.

(i) To prove the first identity, let's use induction on  $n$ .

The base case  $n = 1$  is obvious: on the left-hand side, we just get  $(a_1 b_1)^2$ , and on the right-hand side, we get  $a_1^2 b_1^2$ , hence the statement.

Now let's suppose inductively that this identity is true for a given positive integer  $n \geq 1$ , and let's prove that it is still true for  $n + 1$ . We have to prove that

$$\underbrace{\left( \sum_{i=1}^{n+1} a_i b_i \right)^2}_{:=A} + \underbrace{\frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2}_{:=B} = \underbrace{\left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right)}_{:=C} \quad (12.1)$$

where we gave a name to each part of the identity for an easier computation below. Indeed,

- for  $A$ , we have

$$\begin{aligned} A &:= \left( \sum_{i=1}^{n+1} a_i b_i \right)^2 \\ &= \left( a_{n+1} b_{n+1} + \sum_{i=1}^n a_i b_i \right)^2 \\ &= (a_{n+1} b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1} b_{n+1}) \sum_{i=1}^n a_i b_i \end{aligned}$$

- for  $B$ , we have

$$\begin{aligned} B &:= \frac{1}{2} \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^{n+1} (a_i b_j - a_j b_i)^2 + \frac{1}{2} \sum_{j=1}^{n+1} (a_{n+1} b_j - a_j b_{n+1})^2 \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \underbrace{\frac{1}{2} \sum_{i=1}^n (a_i b_{n+1} - a_{n+1} b_i)^2}_{:=1/2 \times S} + \underbrace{\frac{1}{2} \sum_{j=1}^n (a_{n+1} b_j - a_j b_{n+1})^2}_{:=1/2 \times S} \\ &\quad + \underbrace{\frac{1}{2} (a_{n+1} b_{n+1} - b_{n+1} a_{n+1})^2}_{=0} \\ &= \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \end{aligned}$$

- and thus, for  $A + B$ , we now use the induction hypothesis (IH) to get:

$$\begin{aligned}
A + B &:= (a_{n+1}b_{n+1})^2 + \left( \sum_{i=1}^n a_i b_i \right)^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i \\
&\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2 + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \underbrace{\left( \sum_{i=1}^n a_i b_i \right)^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n (a_i b_j - a_j b_i)^2}_{\text{apply (IH) here}} \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) \\
&\quad + (a_{n+1}b_{n+1})^2 + 2(a_{n+1}b_{n+1}) \sum_{i=1}^n a_i b_i + \sum_{k=1}^n (a_k b_{n+1} - a_{n+1} b_k)^2 \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + (a_{n+1}b_{n+1})^2 \\
&\quad + 2 \sum_{i=1}^n a_i a_{n+1} b_i b_{n+1} + \sum_{i=1}^n (a_i^2 b_{n+1}^2 - 2a_i b_{n+1} a_{n+1} b_i + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right) + \sum_{i=1}^n (a_i^2 b_{n+1}^2 + a_{n+1}^2 b_i^2) \\
&= \left( \sum_{i=1}^{n+1} a_i^2 \right) \left( \sum_{j=1}^{n+1} b_j^2 \right) \\
&= C
\end{aligned}$$

so that the identity is indeed true for all natural number  $n$ .

- (ii) We can use this identity to prove the Cauchy-Schwarz identity,

$$\left| \sum_{i=1}^n a_i b_i \right| \leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2}. \quad (12.2)$$

Indeed, since  $B \geq 0$  in the identity (12.1), we have

$$\left( \sum_{i=1}^n a_i b_i \right)^2 \leq \left( \sum_{i=1}^n a_i^2 \right) \left( \sum_{j=1}^n b_j^2 \right)$$

and thus, taking the square root on both sides, we get (12.2), as expected.

(iii) Finally, we can use the Cauchy-Schwarz inequality to prove the triangle inequality.

We have

$$\begin{aligned}
\sum_{i=1}^n (a_i^2 + b_i^2) &= \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \sum_{i=1}^n a_i b_i \\
&\leq \sum_{i=1}^n a_i^2 + \sum_{i=1}^n b_i^2 + 2 \left( \sum_{i=1}^n a_i^2 \right)^{1/2} \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{by eq. (12.2)}) \\
&\leq \left( \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \right)^2
\end{aligned}$$

and, since everything is positive, we get the triangle inequality by taking square roots on both sides.

EXERCISE 12.1.6. — *Show that  $(\mathbb{R}^n, d_{l^2})$  in Example 12.1.6 is indeed a metric space.*

We have to show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^2}(x, x) = \sqrt{\sum_{i=1}^n (x_i - x_i)^2} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $(x_i - y_i)^2 > 0$ , and  $d_{l^2}(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(y, x) = \sqrt{\sum_{i=1}^n (y_i - x_i)^2} = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} = d_{l^2}(x, y)$$

as expected.

- (d) Triangle inequality: for all  $x, y, z \in \mathbb{R}^n$ , we have

$$\begin{aligned}
d_{l^2}(x, z) &:= \left( \sum_{i=1}^n (x_i - z_i)^2 \right)^{1/2} \\
&= \left( \sum_{i=1}^n (a_i + b_i)^2 \right)^{1/2} \quad \text{with } a_i := x_i - y_i \text{ and } b_i := y_i - z_i \\
&\leq \left( \sum_{i=1}^n a_i^2 \right)^{1/2} + \left( \sum_{i=1}^n b_i^2 \right)^{1/2} \quad (\text{Exercise 12.1.5(iii)}) \\
&\leq \left( \sum_{i=1}^n (x_i - y_i)^2 \right)^{1/2} + \left( \sum_{i=1}^n (y_i - z_i)^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) + d_{l^2}(y, z)
\end{aligned}$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^2})$  is indeed a metric space.

EXERCISE 12.1.7. — *Show that  $(\mathbb{R}^n, d_{l^1})$  in Example 12.1.7 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we have  $d_{l^1}(x, x) = \sum_{i=1}^n |x_i - x_i| = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq i \leq n$  such that  $x_i \neq y_i$ , so that  $|x_i - y_i| > 0$ , and  $d_{l^1}(x, y) = \sum_{i=1}^n |x_i - y_i| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^1}(y, x) = \sum_{i=1}^n |y_i - x_i| = \sum_{i=1}^n |x_i - y_i| = d_{l^1}(x, y)$$

as expected.

- (d) Triangle inequality: we already know from Proposition 4.3.3(g) (generalized to real numbers) that we have the triangle inequality  $|a - c| \leq |a - b| + |b - c|$  for all  $a, b, c \in \mathbb{R}$ . Thus, for all  $x, y, z \in \mathbb{R}^n$ , we have

$$d_{l^1}(x, z) := \sum_{i=1}^n |x_i - z_i| \leq \sum_{i=1}^n (|x_i - y_i| + |y_i - z_i|) =: d_{l^1}(x, y) + d_{l^1}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.8. — *Prove the two inequalities in equation (12.1).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y) \quad (12.3)$$

- The first inequality, since everything is non-negative, is equivalent to  $d_{l^2}(x, y)^2 \leq d_{l^1}(x, y)^2$ , and we will prove it in this form.

Indeed, using a trivial product expansion, we have

$$\begin{aligned} d_{l^1}(x, y)^2 &:= \left( \sum_{i=1}^n |x_i - y_i| \right)^2 \\ &= \left( \sum_{i=1}^n |x_i - y_i| \right) \times \left( \sum_{i=1}^n |x_i - y_i| \right) \\ &= \sum_{i=1}^n |x_i - y_i|^2 + \overbrace{\sum_{1 \leq i, j \leq n; i \neq j} |x_i - y_i| \times |x_j - y_j|}^{\geq 0} \\ &\geq \sum_{i=1}^n |x_i - y_i|^2 =: d_{l^2}(x, y)^2 \end{aligned}$$

as expected.

- For the second inequality, we use the Cauchy-Schwarz inequality, which says that

$$\begin{aligned}
d_{l^1}(x, y) &:= \sum_{i=1}^n |x_i - y_i| \\
&= \left| \sum_{i=1}^n |x_i - y_i| \times 1 \right| \\
&\leq \left( \sum_{i=1}^n |x_i - y_i|^2 \right)^{1/2} \left( \sum_{i=1}^n 1^2 \right)^{1/2} \\
&\leq d_{l^2}(x, y) \times \sqrt{n}
\end{aligned}$$

as expected.

EXERCISE 12.1.9. — *Show that the pair  $(\mathbb{R}^n, d_{l^\infty})$  in Example 12.1.9 is a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in \mathbb{R}^n$ , we clearly have  $d_{l^\infty}(x, x) = \sup\{|x_i - x_i| : 1 \leq i \leq n\} = 0$ , as expected.
- (b) Positivity: for all  $x \neq y \in \mathbb{R}^n$ , there exists at least one  $1 \leq j \leq n$  such that  $x_j \neq y_j$ . Thus  $|x_j - y_j| > 0$ , and  $d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} \geq |x_j - y_j| > 0$ , as expected.
- (c) Symmetry: for all  $x, y \in \mathbb{R}^n$ , we have

$$d_{l^\infty}(x, y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\} = \sup\{|y_i - x_i| : 1 \leq i \leq n\} = d_{l^\infty}(y, x)$$

as expected.

- (d) Triangle inequality. Let be  $x, y, z \in \mathbb{R}^n$ . We have  $|x_i - z_i| \leq |x_i - y_i| + |y_i - z_i|$  for all  $1 \leq i \leq n$ , by Proposition 4.3.3(g). But, by definition of the supremum, we have  $|x_i - y_i| \leq d_{l^\infty}(x, y)$  and  $|y_i - z_i| \leq d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ . Thus, we have  $|x_i - z_i| \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  for all  $1 \leq i \leq n$ ; i.e.,  $d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$  is an upper bound of the set  $\{|x_i - z_i| : 1 \leq i \leq n\}$ . By definition of the supremum, it implies that

$$d_{l^\infty}(x, z) := \sup\{|x_i - z_i| : 1 \leq i \leq n\} \leq d_{l^\infty}(x, y) + d_{l^\infty}(y, z)$$

as expected.

Thus,  $(\mathbb{R}^n, d_{l^1})$  is indeed a metric space.

EXERCISE 12.1.10. — *Prove the two inequalities in equation (12.2).*

We have to prove that for all  $x, y \in \mathbb{R}^n$ ,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y).$$

First, a preliminary remark. By definition, we have  $d_{l^\infty}(x, y) := \sup\{|x_i - y_i| : 1 \leq i \leq n\}$  for all  $x, y \in \mathbb{R}^n$ . Since this distance is defined as the supremum of a finite set, we know (see Chapter 8 of *Analysis I*) that there exists a  $1 \leq m \leq n$  such that  $d_{l^\infty}(x, y) = |x_m - y_m|$  (the supremum belongs to the set). The index “ $m$ ” will have this meaning below.

- Let's prove the first inequality.

$$\begin{aligned}
\frac{1}{\sqrt{n}}d_{l^2}(x, y) &:= \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - y_i)^2} \\
&\leq \sqrt{\frac{1}{n} \sum_{i=1}^n (x_m - y_m)^2} \\
&\leq \sqrt{\frac{n}{n} (x_m - y_m)^2} \\
&= |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

- Now we prove the second one. We have

$$\begin{aligned}
d_{l^2}(x, y) &:= \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \\
&= \sqrt{(x_m - y_m)^2 + \sum_{1 \leq i \leq n; i \neq m} (x_i - y_i)^2} \\
&\geq \sqrt{(x_m - y_m)^2} = |x_m - y_m| =: d_{l^\infty}(x, y)
\end{aligned}$$

as expected.

EXERCISE 12.1.11. — *Show that the discrete metric  $(X, d_{\text{disc}})$  in Example 12.1.11 is indeed a metric space.*

Once again, let's show the four axioms of Definition 12.1.2.

- (a) For all  $x \in X$ , we have  $d_{\text{disc}}(x, x) := 0$  by definition, so that there is nothing to prove here.
- (b) Positivity: for all  $x \neq y \in X$ , we have  $d_{\text{disc}}(x, y) := 1 > 0$  by definition, so that there's still nothing to prove.
- (c) Symmetry: for all  $x, y \in X$ , we have  $d_{\text{disc}}(x, y) = d_{\text{disc}}(y, x) = 1$ , so that  $d_{\text{disc}}$  obeys the symmetry property.
- (d) Triangle inequality. Let be  $x, y, z \in X$ , and let's consider  $d_{\text{disc}}(x, z)$ .
  - If  $x = z$ , then  $d_{\text{disc}}(x, z) = 0$ . And since  $d_{\text{disc}}$  is a non-negative application, we clearly have  $0 =: d_{\text{disc}}(x, z) \leq d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z)$  for all  $y \in X$ .
  - If  $x \neq z$ , then we cannot have both  $x = y$  and  $y = z$  (it would be a clear contradiction with  $x \neq z$ ). Thus, at least one of the propositions " $x \neq y$ ", " $y \neq z$ " is true. Another way to say that is  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq 1$ . But since  $d_{\text{disc}}(x, z) := 1$ , we have actually  $d_{\text{disc}}(x, y) + d_{\text{disc}}(y, z) \geq d_{\text{disc}}(x, z)$ , as expected.



EXERCISE 12.1.12. — *Prove Proposition 12.1.18.*

First, recall that for all  $x, y \in \mathbb{R}^n$ , we have, from Examples 12.1.7 and 12.1.9,

$$\frac{1}{\sqrt{n}} d_{l^2}(x, y) \leq d_{l^\infty}(x, y) \leq d_{l^2}(x, y) \leq d_{l^1}(x, y) \leq \sqrt{n} d_{l^2}(x, y).$$

Note that  $n$  is a real constant here.

- Let's prove that (a)  $\implies$  (b). If  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , then by the limit laws, the sequence  $t_k := \sqrt{n} d_{l^2}(x^{(k)}, x)$  also converges to 0 as  $k \rightarrow \infty$ , since  $\sqrt{n}$  is a constant real number. Thus, we have

$$d_{l^2}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x) \leq \sqrt{n} d_{l^2}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x)$  as expected.

- Let's prove that (b)  $\implies$  (c). If  $\lim_{k \rightarrow \infty} d_{l^1}(x^{(k)}, x) = 0$ , then we have

$$0 \leq d_{l^\infty}(x^{(k)}, x) \leq d_{l^1}(x^{(k)}, x)$$

and, by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x)$  as expected.

- Let's prove that (c)  $\implies$  (d). Suppose that  $\lim_{k \rightarrow \infty} d_{l^\infty}(x^{(k)}, x) = 0$ . Then, for all  $1 \leq j \leq n$ , we have  $0 \leq |x_j^{(k)} - x_j| \leq d_{l^\infty}(x^{(k)}, x)$ . Still by the squeeze test, this implies that  $\lim_{k \rightarrow \infty} |x_j^{(k)} - x_j| = 0$ , i.e. that  $(x_j^{(k)})_{k=m}^\infty$  converges to  $x_j$  as  $k \rightarrow \infty$  (by Lemma 12.1.1), as expected.
- Finally, let's prove that (d)  $\implies$  (a). Using the definition of convergence is more appropriate here. Let be  $\varepsilon > 0$  a positive real number, and let be  $1 \leq j \leq n$ . By definition, there exists a natural number  $N \geq m$  such that  $|x_j^{(k)} - x_j| \leq \varepsilon/\sqrt{n}$  whenever  $k \geq N$ . Thus, if  $k \geq N$ , we have

$$d_{l^2}(x^{(k)}, x) := \sqrt{\sum_{j=1}^n (x_j^{(k)} - x_j)^2} \leq \sqrt{\sum_{j=1}^n \frac{\varepsilon^2}{n}} \leq \varepsilon$$

so that  $\lim_{k \rightarrow \infty} d_{l^2}(x^{(k)}, x) = 0$ , i.e.,  $(x^{(k)})_{k=m}^\infty$  converges to  $x$  as  $k \rightarrow \infty$  in the  $l^2$  metric (by Lemma 12.1.1), as expected.

EXERCISE 12.1.13. — *Prove Proposition 12.1.19.*

Let be  $(x^{(n)})_{n=m}^\infty$  a sequence of elements of a set  $X$ .

- First suppose that  $(x^{(n)})_{n=m}^\infty$  is eventually constant. Thus, by definition, there exists an  $N \geq m$  and an element  $x \in X$  such that  $(x^{(n)})_{n=m}^\infty = x$  for all  $n \geq N$ . This implies that we have  $d_{\text{disc}}(x^{(n)}, x) = 0$  for all  $n \geq N$ . In particular, for all  $\varepsilon > 0$ , we have  $d_{\text{disc}}(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N$ , so that  $(x^{(n)})_{n=m}^\infty$  indeed converges to  $x$  with respect to  $d_{\text{disc}}$ .
- Conversely, suppose that  $(x^{(n)})_{n=m}^\infty$  converges to  $x$  with respect to  $d_{\text{disc}}$ . Let be  $\varepsilon = 1/2$ . By definition, there exists an  $N \geq m$  such that  $d_{\text{disc}}(x^{(n)}, x) \leq 1/2$  whenever  $n \geq N$ . Since  $d_{\text{disc}}(x^{(n)}, x)$  cannot be 1, it is necessarily equal to 0, so that  $x^{(n)} = x$  whenever  $n \geq N$ . Thus, the sequence  $x^{(n)}$  is indeed eventually constant.

EXERCISE 12.1.14. — *Prove Proposition 12.1.20.*

Suppose that we have  $\lim_{n \rightarrow \infty} d(x^{(n)}, x) = 0$  and  $\lim_{n \rightarrow \infty} d(x^{(n)}, x') = 0$ . Suppose, for the sake of contradiction, that we have  $x \neq x'$ . Thus, the real number  $\varepsilon := \frac{d(x, x')}{3}$  is positive.

Since  $x^{(n)}$  converges to  $x$ , there exists a  $N_1 \geq m$  such that  $d(x^{(n)}, x) \leq \varepsilon$  whenever  $n \geq N_1$ .

Similarly, since  $x^{(n)}$  converges to  $x'$ , there exists a  $N_2 \geq m$  such that  $d(x^{(n)}, x') \leq \varepsilon$  whenever  $n \geq N_2$ .

By the triangle inequality, we thus have, for all  $n \geq \max(N_1, N_2)$ ,

$$d(x, x') \leq d(x, x^{(n)}) + d(x^{(n)}, x') \leq \varepsilon + \varepsilon = \frac{2}{3}d(x, x')$$

which is a contradiction (since  $d(x, x') > 0$  by hypothesis).

Thus, the limit is unique, and we must have  $x = x'$ .

EXERCISE 12.1.15. — *Let be  $X := \{(a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty\}$ . We define on this space the metrics  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sum_{n=0}^\infty |a_n - b_n|$ , and  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |a_n - b_n|$ . Then...*

We have to prove the following statements.

1.  $d_{l^1}$  is a metric on  $X$ .

We have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^1}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sum_{n=0}^\infty |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^1}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sum_{n=0}^\infty |b_n - a_n| = \sum_{n=0}^\infty |a_n - b_n| = d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately

$$\begin{aligned} d_{l^1}((a_n)_{n=0}^\infty, (c_n)_{n=0}^\infty) &:= \sum_{n=0}^\infty |a_n - c_n| \\ &\leq \sum_{n=0}^\infty (|a_n - b_n| + |b_n - c_n|) \quad (\text{consequence of Prop. 7.1.11(h)}) \\ &\leq \sum_{n=0}^\infty |a_n - b_n| + \sum_{n=0}^\infty |b_n - c_n| \quad (\text{by Proposition 7.2.14(a)}) \\ &\leq d_{l^1}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) + d_{l^1}((b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty). \end{aligned}$$

Thus,  $d_{l^1}$  is indeed a metric on  $X$ .

2.  $d_{l^\infty}$  is a metric on  $X$ .

Once again, we have to prove the four axioms of Definition 12.1.2.

- (a) Let be  $(a_n)_{n=0}^\infty \in X$ . We have  $d_{l^\infty}((a_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - a_n| = 0$ , as expected.
- (b) Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty$  two distinct elements of  $X$ . Since they are distinct, there exists at least one  $m \in \mathbb{N}$  such as  $|a_m - b_m| > 0$ . Thus,  $d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |a_n - b_n| \geq |a_m - b_m| > 0$ , as expected.
- (c) Symmetry: we clearly have

$$d_{l^\infty}((b_n)_{n=0}^\infty, (a_n)_{n=0}^\infty) = \sup_{n \in \mathbb{N}} |b_n - a_n| = \sup_{n \in \mathbb{N}} |a_n - b_n| = d_{l^\infty}((a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty).$$

- (d) Finally, let's prove the triangle inequality. Let be  $(a_n)_{n=0}^\infty, (b_n)_{n=0}^\infty, (c_n)_{n=0}^\infty \in X$ . Since we have the triangle inequality for the usual distance  $d$  on  $\mathbb{R}$  (i.e., we have  $|a_n - c_n| \leq |a_n - b_n| + |b_n - c_n|$  for all  $n \in \mathbb{N}$ ), we have immediately  $|a_m - c_m| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$  for all  $m \in \mathbb{N}$ , by definition of the supremum. In other words,  $(\sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|)$  is an upper bound for the set  $\{|a_m - c_m| : m \in \mathbb{N}\}$ . Thus we have, still by definition of the supremum,  $\sup_{n \in \mathbb{N}} |a_n - c_n| \leq \sup_{n \in \mathbb{N}} |a_n - b_n| + \sup_{n \in \mathbb{N}} |b_n - c_n|$ , as expected.

Thus,  $d_{l^\infty}$  is indeed a metric on  $X$ .

3. There exist sequences  $x^{(1)}, x^{(2)}, \dots$ , of elements of  $X$  (i.e., sequences of sequences) which are convergent with respect to  $d_{l^\infty}$ , but are not convergent with respect to  $d_{l^1}$ .

Here we are dealing with sequences of sequences: we have a sequence  $(x^{(k)})_{k=1}^\infty$  where each  $x^{(k)}$  is itself a sequence of real numbers. Thus, let's define  $(x^{(k)})_{k=1}^\infty$  as follows:

$$x_n^{(k)} := \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

Just to make things clearer, we have for instance

$$\begin{aligned} x^{(1)} &:= \frac{1}{2}, \frac{1}{2}, 0, 0, 0, \dots \\ x^{(2)} &:= \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots \\ x^{(3)} &:= \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, \dots \end{aligned}$$

Also, let be the null sequence  $(a_n)_{n=0}^\infty$  defined by  $a_n := 0$  for all  $n \in \mathbb{N}$ . Thus:

- $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ . Indeed, if we consider a given positive integer  $k$  (fixed), we have

$$|x^{(k)} - a_n| = |x^{(k)}| = \begin{cases} 1/(k+1) & \text{if } 0 \leq n \leq k \\ 0 & \text{if } n > k. \end{cases}$$

so that  $d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) := \sup_{n \in \mathbb{N}} |x^{(k)} - a_n| = \frac{1}{k+1}$ .

Thus,  $\lim_{k \rightarrow \infty} d_{l^\infty}((x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty) = 0$ , or in other words,  $(x^{(k)})_{k=1}^\infty$  converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$  in  $X$ .

- But  $(x^{(k)})_{k=1}^\infty$  does not converges to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ . Indeed, we have, for each given (fixed)  $k$ ,

$$d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = \sum_{n=0}^k \frac{1}{k+1} = 1$$

Thus, we clearly do not have  $\lim_{k \rightarrow \infty} d_{l^1} \left( (x_n^{(k)})_{n=0}^\infty, (a_n)_{n=0}^\infty \right) = 0$ , i.e.,  $(x^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

4. Conversely, any sequence which converges with respect to  $d_{l^1}$  also converges with respect to  $d_{l^\infty}$ .

Suppose, for the sake of contradiction, that  $(x^{(k)})_{k=1}^\infty$  does not converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^\infty}$ , but does converge to  $(a_n)_{n=0}^\infty$  w.r.t. the metric  $d_{l^1}$ .

In this case, there exists a  $\varepsilon > 0$  such that, for all  $k \geq 1$ , we have  $(\sup_{n \geq 0} |x_n^{(k)} - a_n|) > \varepsilon$ . In particular, for all  $k \geq 1$  and all  $n \geq 0$ , we have  $|x_n^{(k)} - a_n| > \varepsilon$ . Thus,  $\sum_{n=0}^\infty |x_n^{(k)} - a_n|$  is not even a convergent series, and we cannot have  $\lim_{k \rightarrow \infty} \left( \sum_{n=0}^\infty |x_n^{(k)} - a_n| \right) = 0$ .

Note that this exercise actually shows that in this space  $X$ , the metrics are not equivalent; instead, the convergence in the taxi cab metric is stronger than the convergence in the sup norm metric. Thus, Proposition 12.1.18 is not true for *any* metric space.

EXERCISE 12.1.16. — Let  $(x_n)_{n=1}^\infty$  and  $(y_n)_{n=1}^\infty$  be two sequences in a metric space  $(X, d)$ . Suppose that  $(x_n)_{n=1}^\infty$  converges to a point  $x \in X$ , and  $(y_n)_{n=1}^\infty$  converges to a point  $y \in X$ . Show that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ .

On the one hand, the triangle inequality applied two times to  $d$  gives us

$$d(x_n, y_n) \leq d(x_n, x) + d(x, y) + d(y, y_n)$$

but this is only half of what we need to prove the result.

Similarly, we have

$$d(x, y) \leq d(x, x_n) + d(x_n, y_n) + d(y_n, y)$$

so that we can combine the previous two inequalities to get

$$-d(x_n, x) - d(y_n, y) \leq d(x_n, y_n) - d(x, y) \leq d(x_n, x) + d(y_n, y)$$

i.e.,

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y).$$

Let be  $\varepsilon > 0$ . By hypothesis, there exists a  $N_1 \geq 1$  such that  $d(x_n, x) \leq \varepsilon/2$  whenever  $n \geq N_1$ . Similarly, there exists a  $N_2 \geq 1$  such that  $d(y_n, y) \leq \varepsilon/2$  whenever  $n \geq N_2$ . Thus, if we set  $N := \max(N_1, N_2)$ , then for all  $n \geq N$  we have

$$|d(x_n, y_n) - d(x, y)| \leq d(x_n, x) + d(y_n, y) \leq 2\varepsilon/2 = \varepsilon$$

which shows that  $\lim_{n \rightarrow \infty} d(x_n, y_n) = d(x, y)$ , as expected.

EXERCISE 12.2.1. — *Verify the claims in Example 12.2.8*

Let be  $(X, d_{\text{disc}})$  a metric space, and  $E$  a subset of  $X$ .

- Let be  $x \in E$ . Then  $x$  is an interior point of  $E$ . Indeed, we have  $B(x, 1/2) = \{x\} \subseteq E$ .
- Let be  $y \notin E$ . Then  $y$  is an exterior point of  $E$ . Indeed, we have  $B(y, 1/2) \cap E = \{y\} \cap E = \emptyset$ .
- Finally, there are no boundary points of  $E$  in  $(X, d_{\text{disc}})$ . Indeed, let be  $r > 0$  and any  $x \in X$ . We will always have  $B(x, r) = \{x\}$  by definition of the discrete metric  $d_{\text{disc}}$ . Thus, we have either  $x \in E$  and then  $x \in \text{int}(E)$ , or  $x \notin E$  and then  $x \in \text{ext}(E)$ . Thus,  $E$  has no boundary points.

EXERCISE 12.2.2. — *Prove Proposition 12.2.10.*

We have to prove the following implications:

- Let's show that  $(a) \implies (b)$ . We will use the contrapositive, assuming that  $x_0$  is neither an interior point of  $E$ , nor a boundary point of  $E$ . By definition, it means that  $x_0$  is an exterior point of  $E$ , i.e. that there exists  $r > 0$  such that  $B(x_0, r) \cap E = \emptyset$ . This is precisely the negation of  $x_0$  being an adherent point of  $E$ . Thus, we have showed that if  $x_0$  is an adherent of  $E$ , it is either an interior point or a boundary point.
- Let's show that  $(b) \implies (c)$ . Let be a positive integer  $n > 0$ , and suppose that  $x_0$  is either an interior point of  $E$ , or a boundary point of  $E$ . In either case, the set  $A_n := B(x_0, 1/n) \cap E$  is non empty, i.e., there exists  $a_n \in X$  such that  $d(a_n, x_0) < 1/n$ . By the (countable) axiom of choice, we can define a sequence  $(a_n)_{n=1}^\infty$  such that  $a_n \in A_n$  for all  $n \geq 1$ .

Let be  $\varepsilon > 0$ . There exists  $N > 0$  such that  $1/N < \varepsilon$  (Exercise 5.4.4). Thus, for all  $n \geq N$ , we have

$$d(a_n, x_0) < \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

i.e., the sequence  $(a_n)_{n=1}^\infty$  converges to  $x_0$  with respect to the metric  $d$ , as expected.

- Finally, let's show that  $(c) \implies (a)$ . Let be  $r > 0$ . If  $(a_n)_{n=1}^\infty$  in  $E$  converges to  $x_0$  with respect to  $d$ , then there exists a  $n$  such that  $d(x_0, a_n) < r$ . But since  $a_n \in E$ , it means that  $B(x_0, r) \cap E$  is non empty, i.e. that  $x_0$  is an adherent point of  $E$ .

EXERCISE 12.2.3. — *Prove Proposition 12.2.5.*

Let be  $(X, d)$  a metric space.

- (a) Let be  $E \subseteq X$ . First suppose that  $E$  is open; this means that  $E \cap \partial E = \emptyset$ . Let be  $x \in E$ , then we have  $x \notin \partial E$ . But since  $x \in E$ , we have  $x \in \overline{E}$ , and thus  $x \in \text{int}(E)$  by Proposition 12.2.10(b). We have shown that  $x \in E \implies x \in \text{int}(E)$ , and since the converse implication is trivial (Remark 12.2.6), we have  $E = \text{int}(E)$  as expected.

Now suppose that  $E = \text{int}(E)$ . Let be  $x \in E$ . We thus have  $x \in \text{int}(E)$ . By definition,  $x$  is thus not a boundary point of  $E$ , i.e.  $x \notin \partial E$ . This means that  $E \cap \partial E = \emptyset$ , i.e. that  $E$  is open, as expected.

- (b) Let be  $E \subseteq X$ . First suppose that  $E$  is closed; i.e. that  $\partial E \subseteq E$ . Let be  $x \in \overline{E}$ . By Proposition 12.2.10, we have  $\overline{E} = \text{int}(E) \cup \partial E$ ; such that  $\overline{E}$  is the union of two subsets of  $E$ , and thus is itself a subset of  $E$ , as expected.

Conversely, suppose that  $\overline{E} \subseteq E$ . It means that  $\text{int}(E) \cup \partial E \subseteq E$ , and in particular that  $\partial E \subseteq E$ , i.e. that  $E$  is closed, as expected.

- (c) Let be  $x_0 \in X$ ,  $r > 0$  and  $E := B(x_0, r)$ . To show that  $E$  is open, we must show that  $E = \text{int}(E)$  (by Proposition 1.2.15(a)), and in particular that  $E \subseteq \text{int}(E)$  (the converse inclusion being trivial). Let be  $x \in E$ , and let's show that  $x \in \text{int}(E)$ . By definition, we have  $d(x, x_0) < r$ , so that  $\varepsilon := r - d(x, x_0)$  is a positive real number. Thus, let be  $y \in B(x, \varepsilon)$ . By the triangle inequality, we have

$$\begin{aligned} d(x_0, y) &< d(x, x_0) + d(x, y) \\ &< d(x, x_0) + \varepsilon \\ &< d(x, x_0) + r - d(x, x_0) = r \end{aligned}$$

so that  $y \in E$ . Thus, there exists  $\varepsilon > 0$  such that  $B(x, \varepsilon) \subseteq E$ , i.e.,  $x$  is an interior point of  $E$ . This shows that  $E \subseteq \text{int}(E)$ , as expected.

Now let be  $F := \{x \in X : d(x, x_0) \leq r\}$ , and let be  $(a_n)_{n=1}^\infty$  a convergent sequence in  $F$ . To show that  $F$  is closed, we have to show that  $\ell := \lim_{n \rightarrow \infty} a_n$  lies in  $F$  (Proposition 12.2.15(b)). Suppose, for the sake of contradiction, that  $\ell \notin F$ . We thus have  $d(\ell, x_0) > r$ , so that  $\varepsilon := d(\ell, x_0) - r$  is a positive real number. Since  $(a_n)_{n=1}^\infty$  converges to  $\ell$ , there exists a  $N > 0$  such that  $d(a_n, \ell) < \varepsilon$  whenever  $n \geq N$ . By the triangle inequality, for  $n \geq N$ , we have

$$\begin{aligned} d(x_0, \ell) &\leq d(x_0, a_n) + d(a_n, \ell) \\ d(x_0, a_n) &\geq d(x_0, \ell) - d(a_n, \ell) \\ &\geq d(x_0, \ell) - \varepsilon \\ &\geq d(x_0, \ell) + r - d(\ell, x_0) \\ &\geq r \end{aligned}$$

and thus,  $a_n \notin B(x_0, r)$ , a contradiction. Thus, we must have  $\ell \in F$ , so that  $F$  is indeed a closed set.

- (d) Let be  $\{x_0\}$  a singleton with  $x_0 \in X$ . To show that  $E$  is closed, we may use Proposition 12.2.15(b), and show that  $\{x_0\}$  contains all its adherent points. Let be  $(a_n)_{n=1}^\infty$  a convergent sequence in  $\{x_0\}$ ; it can only be the constant sequence  $x_0, x_0, \dots$ . Since it is a constant sequence, its limit can only be  $x_0$  itself, and this limit belongs to  $\{x_0\}$ . Thus,  $\{x_0\}$  is a closed set.

- (e) First we can form a lemma: for any subset  $E \subseteq X$ , we have  $\text{int}(E) = \text{ext}(X \setminus E)$ . This is a direct consequence of Definition 12.2.5. Indeed,  $x \in \text{int}(E)$  iff there exists a  $r > 0$  such that  $B(x, r) \subseteq E$ , which is equivalent to “ $\exists r > 0 : B(x, r) \cap (X \setminus E) = \emptyset$ ”, which is equivalent to  $x \in \text{ext}(X \setminus E)$ .

This implies that the interior points of  $E$  are the exterior points of  $X \setminus E$ , and conversely, that the exterior points of  $E$  are the interior points of  $X \setminus E$ . Thus, in particular, we have this useful fact:

$$\partial E = \partial(X \setminus E). \quad (12.4)$$

Now we go back to the main proof. First suppose that  $E$  is open. Thus, by Definition 12.2.12, we have  $E \cap \partial E = \emptyset$ , so that  $\partial E \subseteq X \setminus E$ , which means that  $X \setminus E$  is a closed set. The converse also applies: if we suppose that  $X \setminus E$  is closed, then  $\partial(X \setminus E) \subseteq X \setminus E$ . By equation (12.4) above, this is equivalent to  $\partial E \subseteq X \setminus E$ , and thus we have  $\partial E \cap E = \emptyset$ . This means that  $E$  is open, as expected.<sup>1</sup>

- (f) Let  $E_1, \dots, E_n$  be open sets. Thus, for all  $1 \leq i \leq n$ , if  $x \in E_i$ , there exists a  $r_i > 0$  such that  $B(x, r_i) \subseteq E_i$ . Let's define  $r := \min(r_1, \dots, r_n)$ . We have  $B(x, r) \subseteq B(x, r_i) \subseteq E_i$  for all  $1 \leq i \leq n$ , i.e.  $B(x, r) \subseteq E_1 \cap \dots \cap E_n$ . Thus,  $E_1 \cap \dots \cap E_n$  is an open set.

Also, let  $F_1, \dots, F_n$  be closed sets. By the previous result (e), the complementary sets  $X \setminus F_1, \dots, X \setminus F_n$  are open sets. Thus, we have just proved that  $(X \setminus F_1) \cap \dots \cap (X \setminus F_n)$  is an open set. But we have  $(X \setminus F_1) \cap \dots \cap (X \setminus F_n) = X \setminus (F_1 \cup \dots \cup F_n)$ , and this set is open. Thus, by (e), its complementary set,  $F_1 \cup \dots \cup F_n$ , is closed, as expected.

- (g) Let  $(E_\alpha)_{\alpha \in I}$  be open sets. Suppose that we have  $x \in \bigcup_{\alpha \in I} E_\alpha$ . By definition, there exists a  $i \in I$  such that  $x \in E_i$ . Since  $E_i$  is an open set, there exists  $r_i > 0$  such that  $B(x, r_i) \subseteq E_i \subseteq \bigcup_{\alpha \in I} E_\alpha$ . Thus, by (a),  $\bigcup_{\alpha \in I} E_\alpha$  is an open set, as expected.

Now let be  $(F_\alpha)_{\alpha \in I}$  be closed sets. Suppose that we have a convergent sequence  $(x_n)_{n=1}^\infty$  such that  $x_n \in \bigcap_{\alpha \in I} F_\alpha$  for all  $n \geq 1$ . Thus, for all  $\alpha \in I$ , the sequence  $(x_n)_{n=1}^\infty$  entirely belongs to the closed set  $F_\alpha$ , so that its limit  $\ell$  also lies in  $F_\alpha$  according to (b). Thus,  $\ell \in \bigcap_{\alpha \in I} F_\alpha$ , so that  $\bigcap_{\alpha \in I} F_\alpha$  is a closed set, as expected.

- (h) Let be  $E \subseteq X$ .

- Let's show that  $\text{int}(E)$  is the largest open set included in  $E$ . It has not clearly been proved in the main text that  $\text{int}(E)$  is an open set, so we begin by proving it. Let be  $x \in \text{int}(E)$ . By definition, there exists  $r > 0$  so that  $B(x, r) \subseteq E$ . But by (c), we know that  $B(x, r)$  is an open set, so that any point  $y$  of  $B(x, r)$  is an interior point of this open ball, and thus an interior point of  $E$ . Thus,  $\text{int}(E)$  is open.

Now consider another open set  $V \subseteq E$ , and let's show that  $V \subseteq \text{int}(E)$ . If  $x \in \text{int}(V)$ , then there exists  $r > 0$  such that  $B(x, r) \subseteq V \subseteq E$ , so that  $x \in \text{int}(E)$ . This shows that  $V \subseteq \text{int}(E)$ , as expected.

- Similarly, let's show that  $\overline{E}$  is the smallest closed set that contains  $E$ . First we show that  $\overline{E}$  is closed, i.e. that  $\overline{\overline{E}} \subseteq \overline{E}$ . (Hint: see Exercise 9.1.6 for an intuition.) Let be  $x \in \overline{\overline{E}}$ . By definition, for all  $r > 0$ ,  $B(x, r) \cap \overline{E} \neq \emptyset$ . Thus, there exists  $y \in B(x, r)$  such that  $y \in \overline{E}$ . Thus, because  $B(x, r)$  is an open set and  $y$  is adherent to  $E$ , there exists  $\varepsilon > 0$  such that  $B(y, \varepsilon) \subseteq B(x, r)$  and  $B(y, \varepsilon) \cap E \neq \emptyset$ ; i.e., there exists  $z \in B(y, \varepsilon) \subseteq B(x, r)$  such that  $z \in E$ . We have showed that whenever  $x \in \overline{\overline{E}}$ , we have  $B(x, r) \cap E \neq \emptyset$  for all  $r > 0$ , i.e. that  $x$  is an adherent point of  $E$ , as expected. Thus,  $\overline{E}$  is closed.

Now we consider a closed set  $K$  such that  $E \subseteq K$ , and we have to show that  $\overline{E} \subseteq K$ . Let be  $x \in \overline{E}$ . By definition, for all  $r > 0$ , we have  $B(x, r) \cap E \neq \emptyset$ . But since  $E \subseteq K$ , we also have  $B(x, r) \cap K \neq \emptyset$  for all  $r > 0$ . Thus,  $x$  is an adherent point of  $K$ , i.e.,  $x \in \overline{K}$ . But since  $K$  is closed, we have  $K = \overline{K}$ , and thus  $x \in K$ . This shows that  $\overline{E} \subseteq K$ , as expected.

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<sup>1</sup>This important result will be used in future proofs to turn any statement on closed sets into a statement on open sets.

EXERCISE 12.2.4. — Let  $(X, d)$  be a metric space,  $x_0$  be a point in  $X$ , and  $r > 0$ . Let  $B$  be the open ball  $B := B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ , and let  $C$  be the closed ball  $C := \{x \in X : d(x, x_0) \leq r\}$ .

Let's prove the following claims:

(a) Show that  $\overline{B} \subseteq C$ .

Let be  $x \in \overline{B}$ . By definition, since  $x$  is an adherent point of  $B$ , for all  $\varepsilon > 0$  we have  $B(x, \varepsilon) \cap B \neq \emptyset$ . In other words, there exists  $y$  such that we have both  $d(x, y) < \varepsilon$  and  $d(x_0, y) < r$ . Thus, by the triangle inequality, we have

$$\begin{aligned} d(x, x_0) &\leq d(x, y) + d(y, x_0) \\ &\leq \varepsilon + r \quad \text{for all } \varepsilon > 0 \end{aligned}$$

which is equivalent (as a quick proof by contradiction would show) to  $d(x, x_0) \leq r$ . Thus,  $x \in C$ .

We have indeed proved that  $\overline{B} \subseteq C$ .

(b) Give an example of a metric space  $(X, d)$ , a point  $x_0$ , and a radius  $r > 0$  such that  $\overline{B}$  is *not* equal to  $C$ .

Let's take  $X = \mathbb{R}$ ,  $d = d_{\text{disc}}$ ,  $x = 0$  and  $r = 1$ . On the one hand, we have  $B := \{0\}$  and  $C := \mathbb{R}$ . Now let's work out  $\overline{B}$ . By Proposition 12.2.15(bd),  $B$  is closed, so that we have  $\overline{B} = B$ . Thus, we clearly do not have  $\overline{B} = C$  here. (Note however that any  $x_0 \in \mathbb{R}$  would be convenient here; there is nothing special about 0.)

EXERCISE 12.3.1. — Prove Proposition 12.3.4(b).

Let's show each direction of the equivalence.

- First suppose that  $E$  is relatively closed w.r.t.  $Y$ , and let's show that there exists a closed subset  $K \subseteq X$  such that  $E = K \cap Y$ .

Since  $E$  is closed w.r.t.  $Y$ , the set  $Y \setminus E$  is open w.r.t.  $Y$  (by Proposition 12.2.15(e)). Thus, by (a), there exists an open subset  $V \subseteq X$  such that  $Y \setminus E = V \cap Y$ .

Let be  $K := X \setminus V$ ; this subset  $K \subseteq X$  is closed w.r.t.  $X$  by Proposition 12.2.15(e) since it is the complementary set of an open set. We have to show that  $E = K \cap Y$ .

- Let be  $x \in E$ . Thus, we have  $x \in Y$ , since  $E \subseteq Y$ . And since  $x \in E$ , by definition, we have  $x \notin Y \setminus E$ . Thus,  $x \notin V \cap Y$ , which implies that  $x \notin V$  (since  $x \in Y$ ). Thus, by definition,  $x \in K$ , and thus,  $x \in K \cap Y$ .
- Conversely, let be  $x \in K \cap Y$ . By definition,  $x \in Y$  and  $x \notin V$ . Thus,  $x \notin V \cap Y$ , or, in other words,  $x \notin Y \setminus E$ . We finally get  $x \in E$ , as expected.

Thus, we have indeed  $E = K \cap Y$ , for some closed subset  $K \subseteq X$ , as expected.

- Now let's prove the converse implication: suppose that  $E = K \cap Y$  for some closed subset  $K \subseteq X$ , and let's prove that  $E$  is relatively closed w.r.t.  $Y$ .

Still by Proposition 12.2.15(e), we know that the subset  $V := X \setminus K$  is open w.r.t.  $X$ . Thus, by the previous result from this exercise,  $V \cap Y$  is relatively open w.r.t.  $Y$ . Thus, its complementary set  $Y \setminus (V \cap Y) = Y \setminus V$  is relatively closed w.r.t.  $Y$ . Now we want to show that  $E = Y \setminus V$  to close the proof.



- First suppose that  $x \in E$ . Since  $E = K \cap Y$ , we thus have  $x \in Y$  and  $x \in K$ , i.e.  $x \notin V$ . Thus,  $x \in Y \setminus V$ .
- Now suppose that  $x \in Y \setminus V$ . We thus have  $x \in X$  (since  $Y \subseteq X$ ) and  $x \notin V$ , so that we necessarily have  $x \in K$ . Thus  $x \in Y \cap K$ , i.e.  $x \in E$ .

Thus  $E = Y \setminus V$  is relatively closed w.r.t.  $Y$ , as expected.

EXERCISE 12.4.1. — *Prove Lemma 12.4.3.*

We have to prove that any subsequence  $(x^{(n_j)})_{j=1}^\infty$  of a convergent sequence  $(x^{(n)})_{n=m}^\infty$  converges to the same limit as the whole sequence itself.

Suppose that the whole sequence  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ . Let be  $\varepsilon > 0$ . By definition, we have a positive integer  $N \geq m$  such that  $n \geq N \implies d(x^{(n)}, x_0) \leq \varepsilon$ . Our aim here is to show that there exists a positive integer  $J \geq 1$  such that  $j \geq J \implies d(x^{(n_j)}, x_0) \leq \varepsilon$ .

By Definition 12.4.1, we know that we have  $m \leq n_1 < n_2 < n_3 < \dots$ . Thus, as a quick induction would show, we have  $n_j \geq m + j - 1$  for all  $j \geq 1$ . Let's take  $J := N$ . In this case, if  $j \geq J$ , i.e. if  $j \geq N$ , we have  $n_j \geq m + N - 1 \geq N$ . Thus:

$$j \geq J \implies n_j \geq N \implies d(x^{(n_j)}, x_0) \leq \varepsilon.$$

Since this is true for all  $\varepsilon > 0$ , it means that  $(x^{(n_j)})_{j=1}^\infty$  converges to  $x_0$ , as expected.

EXERCISE 12.4.2. — *Prove Proposition 12.4.5.*

Let  $(x^{(n)})_{n=m}^\infty$  be a sequence of points in a metric space. We have to prove that the following two statements are equivalent:

- (a)  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ .
- (b) There exists a subsequence  $(x^{(n_j)})_{j=1}^\infty$  of the original sequence which converges to  $L$ .

We will prove the two implications, but first, note that (with the notations from Definition 12.4.1) if we have  $1 \leq m \leq n_1 < n_2 < n_3 < \dots$ , then a quick induction shows that we have  $n_j \geq j$  for all  $j \geq 1$ .

- First we prove that (b) implies (a). If some subsequence  $(x^{(n_j)})_{j=1}^\infty$  converges to  $L$ , then we have by definition:

$$\forall \varepsilon > 0, \exists J \geq 1 : j \geq J \implies d(x^{(n_j)}, L) \leq \varepsilon \quad (12.5)$$

Now, consider any  $\varepsilon > 0$  and any  $N \geq m$ . For this particular choice of  $\varepsilon$ , consider the corresponding real number  $J$  given by equation (12.5), and let's define  $p := \max(N, J)$ . Thus, we have  $n_p \geq p \geq J$ , and by equation (12.5), we thus have  $d(x^{(n_p)}, L) \leq \varepsilon$ . If we set  $n := n_p$ , we have indeed found an  $n \geq N$  such that  $d(x^{(n)}, L) \leq \varepsilon$ . Thus,  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ , as required.

- Now we prove that (a) implies (b). Suppose that  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ . By definition, there exists a natural number  $n_1 \geq m$  such that  $d(x^{(n_1)}, L) \leq 1$ . Now, for  $j > 1$ , let's define inductively  $n_j := \min\{n > n_{j-1} : d(x^{(n)}, L) \leq 1/j\}$ . This set is non-empty (by definition of a limit point), so that the well-ordering principle

(Proposition 8.1.4) ensures that it has a (unique) minimal element, i.e. that  $n_j$  indeed exists. Let's define the subsequence  $(x^{(n_j)})_{j=1}^\infty$  obtained following this process. We thus have  $d(x^{(n_j)}, L) \leq 1/j$  for all  $j \geq 1$ , by construction.

Thus, let be  $\varepsilon > 0$ . There exists a  $j \geq 1$  such that  $0 < 1/j < \varepsilon$  (Exercise 5.4.4). Thus, for this positive integer  $j$ , we have  $d(x^{(n_j)}, L) \leq 1/j < \varepsilon$ . By construction, for all other natural numbers  $k \geq j + 1$ , we have  $d(x^{(n_k)}, L) \leq 1/k \leq 1/j \leq \varepsilon$ .

In summary, for our arbitrary choice of  $\varepsilon$ , we have showed that there exists  $j \geq 1$  such that, for all  $k \geq j$ , we have  $d(x^{(n_k)}, L) \leq \varepsilon$ . Thus, the subsequence  $(x^{(n_j)})_{j=1}^\infty$  constructed in this way converges to  $L$ , as expected.

EXERCISE 12.4.3. — *Prove Lemma 12.4.7.*

Suppose that  $(x^{(n)})_{n=m}^\infty$  is a convergent sequence of points in a metric space  $(X, d)$ , and that its limit is  $x_0$ . Let's show that it is a Cauchy sequence.

By the triangle inequality, we know that for all  $j, k \geq m$ , we have:

$$d(x^{(j)}, x^{(k)}) \leq d(x^{(j)}, x_0) + d(x^{(k)}, x_0).$$

Let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ , there exists an  $N \geq m$  such that we have  $d(x^{(n)}, x_0) \leq \varepsilon/3$  for all  $n \geq N$ . Thus, if we take  $j, k \geq N$ , we have:

$$\begin{aligned} d(x^{(j)}, x^{(k)}) &\leq d(x^{(j)}, x_0) + d(x^{(k)}, x_0) \\ &\leq \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that  $(x^{(n)})_{n=m}^\infty$  is a Cauchy sequence, as expected.

EXERCISE 12.4.4. — *Prove Lemma 12.4.9.*

Let be an arbitrary  $\varepsilon > 0$ . Since the subsequence  $(x^{(n_j)})_{j=1}^\infty$  converges to  $x_0$ , there exists a  $J \geq 1$  such that  $d(x^{(n_j)}, x_0) \leq \varepsilon/3$  whenever  $j \geq J$ .

But the whole sequence  $(x^{(n)})_{n=m}^\infty$  is supposed to be a Cauchy sequence. Thus, there also exists a  $N \geq m$  such that  $d(x^{(j)}, x^{(k)}) < \varepsilon/3$  whenever  $j, k \geq N$ .

Now, let be  $K := \max(J, N)$ . If  $k \geq K$ , we have

$$\begin{aligned} d(x^{(k)}, x_0) &\leq d(x^{(k)}, x^{(n_k)}) + d(x^{(n_k)}, x_0) \\ &< \varepsilon/3 + \varepsilon/3 \\ &< \varepsilon \end{aligned}$$

which means that  $(x^{(n)})_{n=m}^\infty$  converges to  $x_0$ , as expected.

EXERCISE 12.4.5. — *Let  $(x^{(n)})_{n=m}^\infty$  be a sequence of points in a metric space  $(X, d)$  and let  $L \in X$ . Show that if  $L$  is a limit point of the sequence  $(x^{(n)})_{n=m}^\infty$ , then  $L$  is an adherent point of the set  $\{x^{(n)} : n \geq m\}$ . Is the converse true?*

First suppose that  $L$  is a limit point of  $(x^{(n)})_{n=m}^\infty$ . By definition, it means that

$$\forall \varepsilon > 0, \forall N \geq m, \exists n \geq N : d(x^{(n)}, L) \leq \varepsilon \quad (12.6)$$

Let be an arbitrary  $\varepsilon > 0$ , and let's take  $N = m$ . By formula (12.6) above, there exists an  $n \geq N$  such that  $d(x^{(n)}, L) \leq \varepsilon$ . Thus, this  $x^{(n)}$  belongs to both sets  $\{x^{(n)} : n \geq m\}$  and  $B(L, \varepsilon)$ . We have just proved that for all  $\varepsilon > 0$ , the intersection  $B(L, \varepsilon) \cap \{x^{(n)} : n \geq m\}$  is always non-empty. In other words,  $L$  is thus an adherent point of  $\{x^{(n)} : n \geq m\}$ .

However, the converse is not true. Indeed, consider the sequence  $(x^{(n)})_{n=1}^{\infty}$  defined in  $(\mathbb{R}, d)$  by  $x^{(1)} = 1$  and  $x^{(n)} = 0$  for all  $n \geq 2$ , i.e. the sequence  $1, 0, 0, 0, \dots$ . It is clear that  $L := 1$  is an adherent point of  $\{x^{(n)} : n \geq 1\}$  (which is just the set  $\{0, 1\}$ ). But 1 is not a limit point of  $(x^{(n)})_{n=1}^{\infty}$ , since we have  $d(x^{(n)}, 1) > 1/2$  for all  $n \geq 2$ .

EXERCISE 12.4.6. — *Show that every Cauchy sequence can have at most one limit point.*

Suppose that  $(x^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence in a metric space  $(X, d)$ , such that  $L, L'$  are limit points. Then we have  $L = L'$ . We will give two different proofs for this fact.

- **Proof 1** (short proof using previous results). By Proposition 12.4.5, since  $L$  is a limit point of  $(x^{(n)})_{n=m}^{\infty}$ , there exists a subsequence that converges to  $L$ . But by Lemma 12.4.9, it means that the whole original sequence  $(x^{(n)})_{n=m}^{\infty}$  also converges to  $L$ . The same argument can be used to show that the whole sequence  $(x^{(n)})_{n=m}^{\infty}$  converges to  $L'$ . But by uniqueness of limits (Proposition 12.1.20), we must have  $L = L'$ , as expected.
- **Proof 2** (a more “manual” proof). Let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=m}^{\infty}$  is a Cauchy sequence, there exists  $N \geq m$  such that  $d(x^{(p)}, x^{(q)}) \leq \varepsilon/3$  for all  $p, q \geq N$ .

If  $L$  is a limit point of  $(x^{(n)})_{n=m}^{\infty}$ , then for this  $N \geq m$ , there exists  $p \geq N$  such that  $d(x^{(p)}, L) \leq \varepsilon/3$ . Similarly, there exists  $q \geq N$  such that  $d(x^{(q)}, L') \leq \varepsilon/3$ .

We thus have, by triangle inequality:

$$\begin{aligned} d(L, L') &\leq d(L, x^{(p)}) + d(x^{(p)}, x^{(q)}) + d(x^{(q)}, L') \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leq \varepsilon \end{aligned}$$

Thus,  $d(L, L') \leq \varepsilon$  for all  $\varepsilon > 0$ , which implies  $L = L'$ .

EXERCISE 12.4.7. — *Prove Proposition 12.4.12.*

For statement (a), consider a convergent sequence  $(y^{(n)})_{n=m}^{\infty}$  of elements of  $Y \subseteq X$ . Since it is convergent, it is a Cauchy sequence (Lemma 12.4.7). Saying that  $(Y, d_{Y \times Y})$  is complete means that  $(y^{(n)})_{n=m}^{\infty}$  converges in  $(Y, d_{Y \times Y})$ . Thus, every convergent sequence in  $Y$  has its limit in  $Y$ : this is exactly the characterization of closed sets given by Proposition 12.2.15(b).

For statement (b), consider a Cauchy sequence  $(y^{(n)})_{n=m}^{\infty}$  of elements of a given closed subset  $Y \subseteq X$ . Since  $(X, d)$  is complete,  $(y^{(n)})_{n=m}^{\infty}$  must converge to some value  $L \in X$ . But since  $Y$  is closed, we have  $L \in Y$  by Proposition 12.2.15(b). Thus, every Cauchy sequence in  $Y$  converges in  $Y$ . This means that  $(Y, d_{Y \times Y})$  is complete, as expected.

EXERCISE 12.4.8. — *The following construction generalizes the construction of the reals from the rationals in Chapter 5. In what follows, we let  $(X, d)$  be a metric space.*

We have to prove the following statements. Note that this is a generalization of the process of construction of the real numbers, so that we can use all results relative to the real numbers below.

- (a) Given any Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $X$ , we denote  $\text{LIM}_{n \rightarrow \infty} x_n$  its formal limit. We say that two formal limits  $\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n$  are equal iff  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ . Then, this equality relation obeys the reflexive, symmetry and transitive axioms.
- This relation is reflexive: for every Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$ , we have  $d(x_n, x_n) = 0$  for all  $n \geq 1$ , by definition of a metric. Thus,  $d(x_n, x_n)$  is constant and equal to zero, so that  $\lim_{n \rightarrow \infty} d(x_n, x_n) = 0$ .
  - By the property of symmetry of the metric  $d$ , we have  $d(x_n, y_n) = d(y_n, x_n)$  for all  $n \geq 1$  and all Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ . Thus,  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} y_n$  iff  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$ , iff  $\lim_{n \rightarrow \infty} d(y_n, x_n) = 0$ , which is equivalent to  $\text{LIM}_{n \rightarrow \infty} y_n = \text{LIM}_{n \rightarrow \infty} x_n$ .
  - For transitivity, suppose that  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  and  $(z^{(n)})_{n=1}^{\infty}$  are Cauchy sequences in  $X$ . If  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} y_n$  and  $\text{LIM}_{n \rightarrow \infty} y_n = \text{LIM}_{n \rightarrow \infty} z_n$ , then by definition we have  $\lim_{n \rightarrow \infty} d(x_n, y_n) = 0$  and  $\lim_{n \rightarrow \infty} d(y_n, z_n) = 0$ . Let be  $\varepsilon > 0$ . By definition, there exists  $N_1 \geq 1$  such that  $d(x_n, y_n) \leq \varepsilon/2$  whenever  $n \geq N_1$ . Similarly, there exists  $N_2 \geq 1$  such that  $d(y_n, z_n) \leq \varepsilon/2$  whenever  $n \geq N_2$ . Thus, if  $n \geq N := \max(N_1, N_2)$ , we have by the triangle inequality  $d(x_n, z_n) \leq d(x_n, y_n) + d(y_n, z_n) \leq \varepsilon$ . It means that  $\lim_{n \rightarrow \infty} d(x_n, z_n) = 0$ , i.e. that  $\text{LIM}_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} z_n$ , as expected.
- (b) Let  $\bar{X}$  be the space of all formal limits of Cauchy sequences in  $X$ , with the above equality relation. Define a metric  $d_{\bar{X}} : \bar{X} \times \bar{X} \rightarrow \mathbb{R}^+$  by setting

$$d_{\bar{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) := \lim_{n \rightarrow \infty} d(x_n, y_n).$$

Then this function is well-defined and gives  $\bar{X}$  the structure of a metric space.

- First we have to show that the limit  $\lim_{n \rightarrow \infty} d(x_n, y_n)$  exists (in  $\mathbb{R}^+$ ) for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ . We already know that  $\mathbb{R}$  is complete, thus  $\mathbb{R}^+$  is complete as a closed subset of the complete space  $\mathbb{R}$  (Proposition 12.4.12(b)).

Let be the sequence defined by  $u_n := d(x_n, y_n)$  for all  $n \geq 1$ . Obviously, this sequence is in  $\mathbb{R}^+$ , which is a complete space. Thus, to show that it converges, we just have to show that it is a Cauchy sequence.

Consider the usual metric on  $\mathbb{R}^+$ . We have, for all  $p, q \geq 1$ ,

$$\begin{aligned} |u_p - u_q| &= |d(x_p, y_p) - d(x_q, y_q)| \\ &\leq |d(x_p, x_q) + d(x_q, y_q) + d(y_q, y_p) - d(x_q, y_q)| \\ &\leq |d(x_p, x_q)| + |d(y_p, y_q)|. \end{aligned}$$

Now let be  $\varepsilon > 0$ . Since  $(x^{(n)})_{n=1}^{\infty}$  and  $(y^{(n)})_{n=1}^{\infty}$  are Cauchy sequences, there exists  $N_1, N_2 \geq 1$  such that  $d(x_p, x_q) \leq \varepsilon/2$  whenever  $p, q \geq N_1$ , and  $d(y_p, y_q) \leq \varepsilon/2$  whenever  $p, q \geq N_2$ . Thus, if  $p, q \geq N := \max(N_1, N_2)$ , we have

$$|u_p - u_q| \leq |d(x_p, x_q)| + |d(y_p, y_q)| \leq \varepsilon.$$

This shows that  $(u_n)_{n=1}^{\infty}$  is a Cauchy sequence, and thus, that  $\lim_{n \rightarrow \infty} d(x_n, y_n)$  exists.

- Now we must show that the axiom of substitution is obeyed. In other words, consider a Cauchy sequence  $(z^{(n)})_{n=1}^{\infty}$  in  $(X, d)$  such that  $\text{LIM}_{n \rightarrow \infty} z_n = \text{LIM}_{n \rightarrow \infty} x_n$ . We must show that  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} z_n, \text{LIM}_{n \rightarrow \infty} y_n) = d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n)$ , i.e. that

$$\lim_{n \rightarrow \infty} d(z_n, y_n) = \lim_{n \rightarrow \infty} d(x_n, y_n) \quad (12.7)$$

By the previous bullet point, we know that both limits in (12.7) do exist. Thus, the limit laws apply. We have:

$$d(z_n, y_n) \leq d(z_n, x_n) + d(x_n, y_n)$$

but since  $\lim_{n \rightarrow \infty} d(z_n, x_n) = 0$  by definition, we obtain

$$\lim_{n \rightarrow \infty} d(z_n, y_n) \leq \lim_{n \rightarrow \infty} d(x_n, y_n)$$

if we take the limits of both sides in the previous inequality.

But similarly, we have  $d(x_n, y_n) \leq d(x_n, z_n) + d(z_n, y_n)$ , so that a similar argument gives

$$\lim_{n \rightarrow \infty} d(x_n, y_n) \leq \lim_{n \rightarrow \infty} d(z_n, y_n).$$

Thus, we have indeed  $\lim_{n \rightarrow \infty} d(z_n, y_n) = \lim_{n \rightarrow \infty} d(x_n, y_n)$ , as expected.

- Finally, we must show that  $d_{\overline{X}}$  is a metric on  $\overline{X}$ . To prove this statement, we must show that  $d_{\overline{X}}$  obeys all four axioms that define a metric.
  - First, it is clear that  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} d(x_n, x_n) = 0$  for all Cauchy sequence  $(x^{(n)})_{n=1}^{\infty}$  in  $(X, d)$ .
  - Now let be two Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  in  $X$ , such that  $\text{LIM}_{n \rightarrow \infty} x_n \neq \text{LIM}_{n \rightarrow \infty} y_n$ . This latest property implies that  $\lim_{n \rightarrow \infty} d(x_n, y_n) > 0$ , by definition. Thus,  $d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) > 0$ .
  - Symmetry: we have

$$\begin{aligned} d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) &= \lim_{n \rightarrow \infty} d(x_n, y_n) \\ &= \lim_{n \rightarrow \infty} d(y_n, x_n) \text{ (symmetry of } d \text{ on } \mathbb{R}^+) \\ &= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} y_n, \text{LIM}_{n \rightarrow \infty} x_n) \end{aligned}$$

for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$ .

- Triangle inequality: by the limit laws, we have

$$\begin{aligned} d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} z_n) &= \lim_{n \rightarrow \infty} d(x_n, z_n) \\ &\leq \lim_{n \rightarrow \infty} (d(x_n, y_n) + d(y_n, z_n)) \\ &\leq \lim_{n \rightarrow \infty} d(x_n, y_n) + \lim_{n \rightarrow \infty} d(y_n, z_n) \\ &\leq d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} y_n) + d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} y_n, \text{LIM}_{n \rightarrow \infty} z_n) \end{aligned}$$

for all Cauchy sequences  $(x^{(n)})_{n=1}^{\infty}, (y^{(n)})_{n=1}^{\infty}$  and  $(z^{(n)})_{n=1}^{\infty}$ .

Thus,  $d_{\overline{X}}$  is indeed a metric on  $\overline{X}$ .

(c) The metric space  $(\overline{X}, d_{\overline{X}})$  is complete.

To prove this statement, consider a Cauchy sequence  $(u_n)_{n=1}^\infty$  in  $\overline{X}$ : we have to prove that this sequence converges in  $(\overline{X}, d_{\overline{X}})$ .

By definition,  $(u_n)_{n=1}^\infty$  is a Cauchy sequence of formal limits of Cauchy sequences that take their values in  $X$ ; i.e., for all  $k \geq 1$ , there exists a Cauchy sequence  $(x_n^{(k)})_{n=1}^\infty$  of elements of  $X$  such that  $u_k := \text{LIM}_{n \rightarrow \infty} x_n^{(k)}$ .

Since all  $(x_n^{(k)})_{n=1}^\infty$  are Cauchy sequences, then for all  $k \geq 1$ , there exists a threshold  $N_k$  such that  $d(x_n^{(k)}, x_{N_k}^{(k)}) < 1/k$  whenever  $n \geq N_k$ . Thus, (using the countable axiom of choice) we can build a sequence  $(z_k)_{k=1}^\infty$  defined by

$$z_k := \left( x_{N_k}^{(k)} \right)$$

for all  $k \geq 1$ . Now:

- We claim that  $(z_k)_{k=1}^\infty$  is itself a Cauchy sequence. Indeed, consider an arbitrary positive real number  $\varepsilon > 0$ . We must prove that  $d(z_p, z_q) := d(x_{N_p}^{(p)}, x_{N_q}^{(q)})$  is eventually lesser than  $\varepsilon$ .

Since  $(u_n)_{n=1}^\infty$  is a Cauchy sequence in  $\overline{X}$ , there exists a  $N \geq 1$  such that, if  $p, q \geq N$ , we have  $d_{\overline{X}}(u_p, u_q) < \varepsilon/3$ , i.e.:

$$\begin{aligned} \varepsilon/3 &> d_{\overline{X}}(u_p, u_q) \\ &\geq d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n^{(p)}, \text{LIM}_{n \rightarrow \infty} x_n^{(q)}) \\ &\geq \lim_{n \rightarrow \infty} d(x_n^{(p)}, x_n^{(q)}) \end{aligned}$$

Thus, there exists a  $N' \geq 1$  such that, if  $n \geq N'$ , we have  $d(x_n^{(p)}, x_n^{(q)}) \leq \varepsilon/3^2$ . Also, by Exercise 5.4.4, there exists a  $k > 0$  such that  $1/k \leq \varepsilon/3$ . Thus, if  $n, p, q \geq \max(k, N', N_p, N_q)$ , we have

$$\begin{aligned} d(z_p, z_q) &= d(x_{N_p}^{(p)}, x_{N_q}^{(q)}) \\ &\leq \underbrace{d(x_{N_p}^{(p)}, x_n^{(p)})}_{\leq 1/p \leq \varepsilon/3} + \underbrace{d(x_n^{(p)}, x_n^{(q)})}_{\leq \varepsilon/3} + \underbrace{d(x_n^{(q)}, x_{N_q}^{(q)})}_{\leq 1/q \leq \varepsilon/3} \\ &\leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\ &\leq \varepsilon \end{aligned}$$

Thus,  $(z_k)_{k=1}^\infty$  is indeed a Cauchy sequence in  $X$ .

- Consequently, we can take the formal limit  $L := \text{LIM}_{n \rightarrow \infty} z_n$ , and this formal limit  $L$  lies in  $\overline{X}$  by definition. We claim that  $\lim_{n \rightarrow \infty} u_n = L \in \overline{X}$ ; proving this claim will close the proof of (c).

Let be  $\varepsilon > 0$ . Since  $(z_n)_{n=1}^\infty$  is a Cauchy sequence in  $X$ , there exists a  $N_1 \geq 1$  such that  $d(z_p, z_q) \leq \varepsilon/2$  whenever  $p, q \geq N_1$ .

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<sup>2</sup>Indeed, for any sequence  $(v_n)_{n=1}^\infty$  that converges to  $\ell$ , if we have  $0 \leq \ell < \varepsilon$ , then there exists an  $N \geq 1$  such that  $v_n \leq \varepsilon$  whenever  $n \geq N$  (why? use a proof by contradiction.).

Once again, by Exercise 5.4.4, there exists a  $K' \geq 1$  such that  $1/K' < \varepsilon/2$ . Thus, if  $k \geq K$  and  $n > N_k$ , we have

$$d(x_n^{(k)}, z_k) := d(x_n^{(k)}, x_{N_k}^{(k)}) < \frac{1}{k} \leq \frac{1}{K} < \frac{\varepsilon}{2}.$$

Thus, by the triangle inequality, we have, for all  $n > \max(N_k, N_1)$ ,

$$d(x_n^{(k)}, z_n) \leq d(x_n^{(k)}, z_k) + d(z_k, z_n) \leq \varepsilon/2 + \varepsilon/2 \leq \varepsilon.$$

Consequently, we have, for all  $k > K'$ ,

$$d_{\overline{X}}(u_k, L) := \lim_{n \rightarrow \infty} d(x_n^{(k)}, b_n) < \varepsilon.$$

This shows that  $(u_n)_{n=1}^\infty \rightarrow L$  in  $(\overline{X}, d_{\overline{X}})$ , which closes the proof.

(d) We identify an element  $x \in X$  with the corresponding formal limit  $\text{LIM}_{n \rightarrow \infty} x$  in  $\overline{X}$ .

- This is legitimate since we have  $x = y \iff \text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$ .  
Indeed, it is clear that if  $x = y$ , then we have  $\text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$  by definition. Conversely, if  $\text{LIM}_{n \rightarrow \infty} x = \text{LIM}_{n \rightarrow \infty} y$ , then we have  $\lim_{n \rightarrow \infty} d(x, y) = 0$ , i.e.  $d(x, y) = 0$ , i.e.  $x = y$ . Thus, this identification is legitimate.
- With this identification, we have  $d(x, y) = d_{\overline{X}}(x, y)$ . Indeed:

$$\begin{aligned} d_{\overline{X}}(x, y) &= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x, \text{LIM}_{n \rightarrow \infty} y) \\ &= \lim_{n \rightarrow \infty} d(x, y) \\ &= d(x, y). \end{aligned}$$

Thus,  $(X, d)$  can be thought of as a subspace of  $(\overline{X}, d_{\overline{X}})$ .

(e) The closure of  $X$  in  $\overline{X}$  is  $\overline{X}$ .

Indeed, let be  $C$  the closure of  $X$  in  $\overline{X}$ . We clearly have  $C \subseteq \overline{X}$ , by definition. Thus we just have to show that  $\overline{X} \subseteq C$ .

Let be  $x \in \overline{X}$ , and let's show that  $x \in C$ . By definition,  $x \in C$  means that  $x$  is an adherent point of  $X$  in  $\overline{X}$ , i.e. that for all  $\varepsilon > 0$ ,  $B_{(\overline{X}, d_{\overline{X}})}(x, \varepsilon) \cap X \neq \emptyset$ . In other words, for all  $\varepsilon > 0$ , we must show that there exists a  $y \in X$  such that  $d_{\overline{X}}(x, y) < \varepsilon$ .

Thus, let be  $\varepsilon > 0$ . By definition,  $x$  is the formal limit of a Cauchy sequence  $(x_n)_{n=1}^\infty$  of elements of  $X$ , so that  $x := \text{LIM}_{n \rightarrow \infty} x_n$ . Since  $(x_n)_{n=1}^\infty$  is a Cauchy sequence, there exists an  $N \geq 1$  such that  $d(x_n, x_N) < \varepsilon/2$  whenever  $n \geq N$ . Thus:

$$\begin{aligned} d_{\overline{X}}(x, x_N) &:= d_{\overline{X}}(\text{LIM}_{n \rightarrow \infty} x_n, \text{LIM}_{n \rightarrow \infty} x_N) \\ &= \lim_{n \rightarrow \infty} d(x_n, x_N) \\ &\leq \varepsilon/2 < \varepsilon \end{aligned}$$

so that  $y := x_N$  is a convenient choice. This shows that  $x$  is an adherent point of  $X$  in  $\overline{X}$ , as expected.

- (f) Finally, the formal limit agrees with the actual limit, i.e.,  $\lim_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} x_n \in \overline{X}$  for all Cauchy sequence  $(x_n)_{n=1}^\infty$  in  $X$ .

Indeed, let be  $(x_n)_{n=1}^\infty$  a Cauchy sequence of elements of  $X$ . We know that  $(X, d)$  can be thought of as a subspace of  $(\overline{X}, d_{\overline{X}})$ , so that  $(x_n)_{n=1}^\infty$  can be thought of as a sequence of elements of  $\overline{X}$ . But we have showed that  $(\overline{X}, d_{\overline{X}})$  is complete. Thus, the sequence  $(x_n)_{n=1}^\infty$  converges in  $\overline{X}$  to a certain limit  $L \in \overline{X}$ ; i.e., we have  $\lim_{n \rightarrow \infty} x_n = L$  for some  $L \in \overline{X}$ .

Consider this limit  $L$ . By definition of  $\overline{X}$ , there exists a Cauchy sequence  $(a_n)_{n=1}^\infty$  of elements of  $X$  such that  $L := \text{LIM}_{n \rightarrow \infty} a_n$ . What we need to prove is that we have

$$L = \lim_{n \rightarrow \infty} x_n = \text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n \quad (12.8)$$

and thus, it is sufficient to show that  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n$ , since we already have the other equalities. And, by definition of the equality relation established in (a), in order to prove that  $\text{LIM}_{n \rightarrow \infty} a_n = \text{LIM}_{n \rightarrow \infty} x_n$ , we just have to show that  $\lim_{n \rightarrow \infty} d(x_n, a_n) = 0$ . Or, in yet another equivalent way, we have to show that for all  $\varepsilon > 0$ , there exists an  $N \geq 1$  such that  $d(x_n, a_n) \leq \varepsilon$  whenever  $n \geq N$ .

Thus, let be an arbitrary  $\varepsilon > 0$ . Let's unfold our hypotheses.

- We know that the sequence  $(x_n)_{n=1}^\infty$  converges to  $L$  in  $\overline{X}$ . Thus, by definition, there exists a  $N_1 \geq 1$  such that  $d_{\overline{X}}(x_k, L) \leq \varepsilon/2$  whenever  $k \geq N_1$ . In other words,  $\lim_{n \rightarrow \infty} d(x_k, a_n) \leq \varepsilon/3 < \varepsilon/2$  whenever  $k \geq N_1$ .

Thus, there exists a  $N_2$  such that  $d(x_k, a_n) \leq \varepsilon/2$  whenever  $k \geq N_1$  and  $n \geq N_2$  (see footnote 2 p. 22 from the present document).

- We also know that  $(x_n)_{n=1}^\infty$  is a Cauchy sequence. It means that there exists a  $N_3 \geq 1$  such that  $d(x_p, x_q) \leq \varepsilon/2$  for all  $p, q \geq N_3$ .

Let be  $N := \max(N_1, N_2, N_3)$ . Using the triangle inequality, we finally get, for all  $n \geq N$ ,

$$\begin{aligned} d(x_n, a_n) &\leq d(x_n, x_N) + d(x_N, a_n) \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &\leq \varepsilon \end{aligned}$$

This closes the proof.

**EXERCISE 12.5.1.** — *Show that Definitions 9.1.22 and 12.5.3 match when talking about subsets of the real line with the standard metric.*

Consider  $Y \subseteq \mathbb{R}$  and the standard metric  $d(x, y) = |x - y|$  for all  $x, y \in \mathbb{R}$ . We have to show that both definitions of boundedness are equivalent in this case.

- First, suppose that  $Y$  is bounded in the sense of Definition 12.5.3. Thus, there exists a real number  $x$  and a positive real number  $r > 0$  such that  $Y \subseteq B(x, r)$ . In other words, we have  $Y \subseteq ]x - r, x + r[ \subseteq [x - r, x + r]$ . Let be  $M := |x| + |r|$ . We clearly have  $x + r \leq M$ , and  $-M \leq x - r$ . Thus, we have  $Y \subseteq [-M, M]$ , and  $Y$  is bounded in the sense of Definition 9.1.22.



- Conversely, suppose that  $Y$  is bounded in the sense of Definition 9.1.22. Thus, there exists a positive real  $M > 0$  such that  $Y \subseteq [-M, M] \subset ]-2M, 2M[$ . But this later interval is simply  $B(0, 2M)$ , so that  $Y$  is bounded in the sense of Definition 12.5.1, taking  $x := 0$  and  $r := 2M$ .