Propositions of solutions for *Analysis II* by Terence Tao

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March 16, 2022

Contents

Remarks. The numbering of the Exercises follows the fourth edition of *Analysis II*. In order to make the references to *Analysis I* easier, we consider that we begin with Chapter 12 here, as in earlier editions of the textbook. Thus, in particular, a reference to "Exercise 4.3.3" (for instance) will always mean "Exercise 4.3.3 from *Analysis I*".

12. Metric spaces

Exercise 12.1.1. — *Prove Lemma 12.1.1*

Consider the sequence $(a_n)_{n=m}^{\infty}$ defined by $a_n := d(x_n, x) = |x_n - x|$ for all $n \geqslant m$. We have to prove that $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} x_n = x$.

- Let be $\varepsilon > 0$. If $\lim_{n\to\infty} a_n = 0$, then there exists an $N \geqslant m$ such that $|a_n| < \varepsilon$ whenever $n \geqslant N$. Thus, there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$, which means that $\lim_{n\to\infty} x_n = x$.
- Let be $\varepsilon > 0$. Conversely, if $\lim_{n\to\infty} x_n = x$, then there exists an $N \geqslant m$ such that $|x_n - x| < \varepsilon$ whenever $n \geqslant N$. But since $|a_n| := |x_n - x|$, it means that $\lim_{n\to\infty} a_n = 0$, as expected.

Exercise 12.1.2. — *Show that the real line with the metric $d(x, y) := |x - y|$ is indeed a metric space.*

Using Proposition 4.3.3, this claim is obvious. All claims (a)–(d) of Definition 12.1.2 are satisfied because:

(a) comes from Proposition 4.3.3(e)

(b) also comes from Proposition 4.3.3(e)

(c) comes from Proposition 4.3.3(f)

(d) comes from Proposition 4.3.3(g).

Exercise 12.1.3. — *Let X be a set, and let $d : X \times X$ß$[0, \infty)$ be a function. With respect to Definition 12.1.2, give an example of a pair (X, d) which obeys to the following properties...*

We have the following cases for the pair (X, d):

(a) obeys the axioms (bcd) but not (a).

Consider $X = \mathbb{R}$, and d defined by $d(x, x) = 1$ and $d(x, y) = 5$ for all $x \neq y \in \mathbb{R}$.

(b) obeys the axioms (acd) but not (b).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = 0$ for all $x, y \in \mathbb{R}$.

(c) obeys the axioms (abd) but not (c).

Consider $X = \mathbb{R}$, and d defined by $d(x, y) = \max(x - y, 0)$ for all $x, y \in \mathbb{R}$.

(d) obeys the axioms (abc) but not (d).

Consider the finite set $X := \{1, 2, 3\}$ and the application d defined by $d(1, 2) = d(2, 1) = d(2, 3) = d(3, 2) := 1$, and $d(1, 3) = d(3, 1) := 5$, and $d(x, x) = 0$ for all $x \in X$.