Tree level correlation functions (Medium)

Exercises for the lecture of Nikolay Gromov

Theory

For this exsercise it is proposed to write a program for computing tree level Feynman diagrams. The goal is to compute two and tree point functions

$$C^{(2)} = \langle O_1(x)\bar{O}_1(y)\rangle \quad , \quad C^{(3)} = \langle O_1(x)O_2(y)O_3(z)\rangle$$

where the operators are

$$O_1 = \mathrm{tr}(ZXZ\dots X) \ , \ O_2 = \mathrm{tr}(\bar{Z}\bar{Z}\dots\bar{X}) \ , \ O_3 = \mathrm{tr}(\bar{X}ZZ\dots\bar{X}) \, .$$

For the tree level contructions we only need propagators. Note that our fields $X, Z, \bar{X}, \bar{Z}$ are $N\times N$ matrices. The only nonzero propagators are:

$$\langle X_{ij}(x)\bar{X}_{kl}(y)\rangle = \frac{\delta_{jk}\delta_{il}}{4\pi^2(x-y)^2} \ , \ \langle Z_{ij}(x)\bar{Z}_{kl}(y)\rangle = \frac{\delta_{jk}\delta_{il}}{4\pi^2(x-y)^2}$$

Implementation

- Define in mathematica a function `Prop` for the propagator. (up to a constant)

```
Prop[X[x][i, j], Xb[y][k, l]]
```

$$\frac{\delta(i,l)\,\delta(j,k)}{(x-y)^2}$$

```
Prop[X[x][i, j], Z[y][k, l]]
```

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- Create a function `ToInd` which rewrites the expressions for the single trace operators O_1 explicitly with all indexes

```
O1 = ToInd[tr[Z[x], X[x], Z[x], X[x]]]
```

$X(x)(i(2), i(3))\, X(x)(i(4), i(1))\, Z(x)(i(1), i(2))\, Z(x)(i(3), i(4))$

- Use the `Notation` (see Help) package to make it more readable.

```
Notation[ i_a_ ⟺ i[a_] ]
```

```
O1
```

$X^x_{i_2,i_3}\, X^x_{i_4,i_1}\, Z^x_{i_1,i_2}\, Z^x_{i_3,i_4}$

- Create a function `Contr` which do the Wick‘s contractions . The fastest way is to define a recursion function.

```
O2 = ToInd[tr[Zb[y], Xb[y], Zb[y], Xb[y]]] /. i → j;
Contr[O1 O2] // Simplify
```

$$\frac{(\delta(i_1, j_4)\,\delta(i_2, j_3)\,\delta(i_3, j_2)\,\delta(i_4, j_1) + \delta(i_1, j_2)\,\delta(i_2, j_1)\,\delta(i_3, j_4)\,\delta(i_4, j_3))^2}{(x-y)^8}$$

- Again Notation package could be used

```
Contr[O1 O2] // Simplify
```

$$\frac{\left(\delta_{i_1,j_4}\,\delta_{i_2,j_3}\,\delta_{i_3,j_2}\,\delta_{i_4,j_1} + \delta_{i_1,j_2}\,\delta_{i_2,j_1}\,\delta_{i_3,j_4}\,\delta_{i_4,j_3}\right)^2}{(x-y)^8}$$

- Define a function to Simp simplify the tensor structure. Use that $\delta_{ij}\delta_{jk} = \delta_{ik}$, $\delta_{ii} = N_c$ ets. After simplifying the above expression you should get

$$\frac{2\,\mathrm{Nc}^4}{(x-y)^8} + \frac{2\,\mathrm{Nc}^2}{(x-y)^8}$$

- Check your program for the following three operators

```
o1[y_] = ToInd[tr[Z[y], X[y], Z[y], Z[y], X[y]]];
o2[y_] = ToInd[tr[Zb[y], Xb[y], Zb[y], Xb[y], Xb[y]]] /. i → j;
o3[y_] = ToInd[tr[Zb[y], X[y]]] /. i → k;
```

$$\frac{2\,\mathrm{Nc}^5}{(x-y)^8\,(x-z)^2\,(z-y)^2} + \frac{23\,\mathrm{Nc}^3}{(x-y)^8\,(x-z)^2\,(z-y)^2} + \frac{11\,\mathrm{Nc}}{(x-y)^8\,(x-z)^2\,(z-y)^2}$$

Notice that you always get the same spacial structure $\frac{1}{(x-y)^8(x-z)^2(z-y)^2}$. What are these numbers?

- Notice that already for four operators the result contains several different spacial structure. For o4[w]

```
o4[w_] = ToInd[tr[Zb[w], Z[w]]] /. i → l;
```

You get

$$\frac{2\,\mathrm{Nc}^5}{(x-w)^2\,(w-z)^2\,(x-y)^8\,(z-y)^2} + \frac{4\,\mathrm{Nc}^5}{(x-w)^2\,(w-y)^2\,(x-y)^6\,(x-z)^2\,(z-y)^2} + \frac{23\,\mathrm{Nc}^3}{(x-w)^2\,(w-z)^2\,(x-y)^8\,(z-y)^2} + \frac{46\,\mathrm{Nc}^3}{(x-w)^2\,(w-y)^2\,(x-y)^6\,(x-z)^2\,(z-y)^2} + \frac{11\,\mathrm{Nc}}{(x-w)^2\,(w-z)^2\,(x-y)^8\,(z-y)^2} + \frac{22\,\mathrm{Nc}}{(x-w)^2\,(w-y)^2\,(x-y)^6\,(x-z)^2\,(z-y)^2}$$

BPS operators

The operators of real interest are the operators which diagonalize the dilatation. The simplest example of such operators are the so-called BPS operators. They are constructed as follows:

$$O_1 = \sum_{\text{all permutations}} \operatorname{tr}(XZZZX\ldots)\,.$$

The simplest example is

$$O_1 = \operatorname{tr}(XZXZ) + 2\operatorname{tr}(ZZXX)\,.$$

The operators are characterized by L - length (total number of fields) and M - number of X's.

• Build a function `BPS[L_,M_]`

```
BPS[4, 2]
```

$\operatorname{tr}(X, X, Z, Z) + \operatorname{tr}(X, Z, X, Z) + \operatorname{tr}(X, Z, Z, X) + \operatorname{tr}(Z, X, X, Z) + \operatorname{tr}(Z, X, Z, X) + \operatorname{tr}(Z, Z, X, X)$

• Define a function `Sm` which simplifies the operators using the rotational symmetry

```
Sm[BPS[4, 2]]
```

$4\operatorname{tr}(X, X, Z, Z) + 2\operatorname{tr}(X, Z, X, Z)$

• Compute the three point function of 3 BPS operators. You will need to improve `ToInd` and `Contr` functions to work with a linear combinations of single traces.
Compute the contructions for `O1=BPS[5,3]` , `O2=BPS[5,2]` and `O3=BPS[2,1]`

the result you should get:

$$\frac{300\,\mathrm{Nc}^5}{(x-y)^8(x-z)^2(z-y)^2} + \frac{4500\,\mathrm{Nc}^3}{(x-y)^8(x-z)^2(z-y)^2} + \frac{2400\,\mathrm{Nc}}{(x-y)^8(x-z)^2(z-y)^2}$$

• Normalize the result in the following way:

$$C_{123} \equiv \frac{\langle O_1(x)O_2(y)O_3(z)\rangle}{\sqrt{\frac{\langle O_1(x)\bar{O}_1(y)\rangle\langle O_2(x)\bar{O}_2(y)\rangle}{\langle O_3(x)\bar{O}_3(y)\rangle}\frac{\langle O_1(x)\bar{O}_1(z)\rangle\langle O_3(x)\bar{O}_3(z)\rangle}{\langle O_2(x)\bar{O}_2(z)\rangle}\frac{\langle O_2(y)\bar{O}_2(z)\rangle\langle O_3(y)\bar{O}_3(z)\rangle}{\langle O_1(y)\bar{O}_1(z)\rangle}}}$$

And compare it with the exact analytic result:

$$C_{123} = \frac{\binom{L_1-N_1+N_2}{N_2}}{N_c\sqrt{\binom{L1}{N1}/L1}\sqrt{\binom{L2}{N2}/L2}\sqrt{\binom{L3}{N3}/L3}}$$

• For the large N_c limit think how the program can be improved to work much faster