Answer ALL SIX questions in Section A and TWO questions from Section B

The numbers in square brackets at the right-hand side of the text indicate the provisional allocation of maximum marks per question or sub-section of a question.

You may find the following constants and theorems useful.

For any vector $\mathbf{F}$, $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$
In spherical polar coordinates, $\boldsymbol{\nabla} \times \mathbf{F}$ is given by:

$$\boldsymbol{\nabla} \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \mathbf{i}_r & r\mathbf{i}_\theta & r\sin\theta \mathbf{i}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

For any vector function which can be written $\mathbf{C}(\mathbf{r}, t) = \mathbf{D} \exp i (\mathbf{k} \cdot \mathbf{r} - \omega t)$ where $\mathbf{D}$ is a constant, then:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{C} &= i\mathbf{k} \cdot \mathbf{C} \\ \boldsymbol{\nabla} \times \mathbf{C} &= i\mathbf{k} \times \mathbf{C} \\ \partial \mathbf{C}/\partial t &= -i\omega \mathbf{C} \end{aligned}$$

Useful four-vectors and operators:

$$\begin{aligned} \partial_\mu &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t} \right) \\ x^\mu &= (x, y, z, ct) \\ a^\mu &= (A_x, A_y, A_z, \frac{\phi}{c}) \\ j^\mu &= (J_x, J_y, J_z, c\rho) \\ \mathrm{D'Alembertian}\ \Box &= \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \end{aligned}$$

The Lorentz transform from a reference frame S to a reference frame S$'$ with relative velocity $(0, 0, v)$ for a general contravariant four-vector $f^\mu = (f^1, f^2, f^3, f^4)$ can be written as:

$$\begin{aligned} f'^1 &= f^1 \\ f'^2 &= f^2 \\ f'^3 &= \gamma \left(f^3 - \beta f^4 \right) \\ f'^4 &= \gamma \left(f^4 - \beta f^3 \right), \end{aligned}$$

where $\beta = v/c, \gamma = 1/\sqrt{1 - \beta^2}$.

SECTION A

1. (a) Describe *briefly* how domains form in a ferromagnetic material as it is cooled through the Curie temperature with no external field, and indicate the types of forces involved in the process. [4]

   (b) Describe the difference between a hard and a soft ferromagnetic material, using sketches to illustrate. [3]

2. We can write the Maxwell equation $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$ as $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$.

   (a) Define ρ and ρ_f and explain the difference between them. [4]

   (b) How is the polarisation charge density (both bulk and surface) related to the polarisation ? Give appropriate equations, defining the terms where necessary. [3]

3. (a) Define the Poynting vector N in terms of electric and magnetic fields. What are its dimensions (use SI units if you wish) ? [3]

   (b) What is its physical meaning for electromagnetic fields ? Be sure to make clear the conditions under which your answer applies. [3]

4. (a) Derive the boundary conditions on $\mathbf{D}$ and $\mathbf{B}$ at an interface between two materials, in the absence of free charges. [5]

   (b) How will the boundary conditions change in the presence of free charges ? [2]

5. (a) Give the definition of the magnetic field $\mathbf{B}$ in terms of the vector potential $\mathbf{A}$. [2]

   (b) Show that this is consistent with the Maxwell Equation $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and explain the physical significance of the magnetic field having zero divergence. [2]

   (c) Starting from Ampere's Law, $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$, show that $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$. You should use the Coulomb gauge ($\boldsymbol{\nabla} \cdot \mathbf{A} = 0$). [3]

6. (a) Explain what is meant by the *polarisation* of the electric field given below, defining any new terms you may need [3]

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

   N.B. $\mathbf{E}_0$ is, in general, a complex constant.

   (b) Give an equation for a plane wave electric field which has:

      i. Circular polarisation [1]

      ii. Elliptical polarisation [1]

      iii. Linear polarisation [1]

SECTION B

7. (a) When considering radiation emitted by a moving charge, the concept of *retarded time* is often used. For an observer at a time t and point $\mathbf{r}$ from an emitting body at the origin, define the retarded time for electromagnetic waves, and explain its physical significance. [4]

(b) Show that the differentials of a function of *retarded* time with respect to time and position are:

$$\frac{\partial f(t')}{\partial t} = \frac{\partial f(t')}{\partial t'} \quad [2]$$

$$\frac{\partial f(t')}{\partial r} = -\frac{1}{c}\frac{\partial f(t')}{\partial t'} \quad [4]$$

(c) A small current loop carrying an alternating current of frequency ω can serve as a model for a magnetic dipole with dipole moment $\mathbf{m}(t) = m_0 \cos \omega t \mathbf{i}_z$. It has potentials which, at large distances, can be written in spherical polar coordinates as:

$$\begin{aligned} V(\mathbf{r}, t') &= 0 \\ \mathbf{A}(\mathbf{r}, t') &= -\frac{\mu_0 m_0 \omega}{4\pi c}\frac{\sin\theta}{r}\sin \omega t' \mathbf{i}_\phi \end{aligned}$$

with t' the *retarted* time. Using the form for curl in spherical polar coordinates given in the rubric, show that at large distances ($r \gg c/\omega$) the fields can be written:

$$\begin{aligned} \mathbf{E} &= \frac{\mu_0 m_0 \omega^2}{4\pi c}\frac{\sin\theta}{r}\cos \omega t' \mathbf{i}_\phi \\ \mathbf{B} &= -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\frac{\sin\theta}{r}\cos \omega t' \mathbf{i}_\theta \end{aligned} \quad [6]$$

(d) Using the above results, find the Poynting vector for the dipole in this limit. Is there radiation from the system ? [4]

(e) The fields for a Hertzian dipole oriented along z with length d and time-varying charge $q = q_0 \cos \omega t$ are:

$$\begin{aligned} \mathbf{E} &= \frac{\mu_0 I_0 d\omega}{4\pi}\frac{\sin\theta}{r}\cos \omega t' \mathbf{i}_\theta \\ \mathbf{B} &= \frac{\mu_0 I_0 d\omega}{4\pi c}\frac{\sin\theta}{r}\cos \omega t' \mathbf{i}_\phi \end{aligned}$$

i. Show that the magnetic dipole and the Hertzian dipole radiate energy in the same direction at any point in space [2]

ii. Consider the time-averaged Poynting vectors P_{Hertzian} and P_{loop}. Show that the ratio of these quantities $P_{\text{loop}}/P_{\text{Hertzian}} = m_0^2/(p_0^2 c^2)$, where p_0 is the magnitude of the Hertzian dipole (choose an appropriate value for I_0). [4]

iii. If the systems are arranged so that $d = \pi a$ (with a the radius of the magnetic dipole) and the currents are the same in the two dipoles, what form does the ratio take ? What does this tell us about how the relative magnitudes vary with frequency ? [4]

8. (a) The electromagnetic potential four-vector can be written as:

$$a^\alpha = (A_1, A_2, A_3, \phi/c)$$

Define the terms A_1, A_2, A_3, ϕ in the equation. [2]

(b) Show that it can be used with the differential four-vector, which can be written as:

$$\partial_\alpha = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\right)$$

to impose the Lorenz condition. Consider a set of electromagnetic potentials in an inertial reference frame S which obey the Lorenz condition. What does this imply about these potentials in another inertial reference frame S'? [6]

(c) Similarly, explain what the equation $\Box a^\mu = -\mu_0 j^\mu$ tells us about the laws of electromagnetism, and why. [4]

(d) Consider an infinite line of charge with density λ (units of charge per unit length) lying along the z-axis. Using Gauss' Law, show that the electric field in cylindrical polar coordinates can be written:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r}\mathbf{i}_r$$

[2]

(e) Using $I = \lambda v$ show by applying Ampère's Law that the magnetic field for a moving line of charge with density λ and velocity v can be written:

$$\mathbf{B} = \frac{\mu_0 \lambda v}{2\pi r}\mathbf{i}_\phi$$

[2]

(f) If the line of charge in (d) is stationary in frame S, use the Lorentz transforms to find the current and charge densities in a frame S' which is moving with velocity v parallel to the z-axis of S. Use cartesian coordinates. [6]

(g) The Lorentz transforms for the electric and magnetic fields from a frame S to another frame S' moving with velocity $\mathbf{v}$ relative to S are given by:

$$\begin{aligned}
E'_\parallel &= E_\parallel \\
\mathbf{E}'_\perp &= \gamma \mathbf{E}_\perp + \gamma \mathbf{v} \times \mathbf{B} \\
B'_\parallel &= B_\parallel \\
\mathbf{B}'_\perp &= \gamma \mathbf{B}_\perp - \frac{1}{c^2}\gamma \mathbf{v} \times \mathbf{E},
\end{aligned}$$

where the subscripts $\parallel$ and $\perp$ refer to components of the field parallel and perpendicular to the velocity. Find the fields $\mathbf{E}'$ and $\mathbf{B}'$ for the line of charge *stationary* in S, as observed from S$'$. How do these compare to the fields found non-relativistically for the *moving* line of charge in part (e)? At what velocities will the differences be significant ? [8]

9. (a) The following identity can be written for an electromagnetic field in a fixed volume V bounded by a surface S,

$$-\oint_S \mathbf{E} \times \mathbf{H} \cdot \mathbf{n} da = \frac{\partial}{\partial t} \int_V \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H})\, dv + \int_V \mathbf{J} \cdot \mathbf{E} dv$$

State the physical significance of the two integrals on the right hand side, and hence give a physical interpretation of the Poynting vector. [6]

(b) For a plane electromagnetic wave in a vacuum, we can relate the magnitudes of the wavevector and angular frequency by: $k = \omega/c$. Write the equivalent relationship for a dielectric medium. [2]

(c) Consider an electromagnetic plane wave of the form $\mathbf{E} = \mathbf{E}_0 \exp i\, (\mathbf{k} \cdot \mathbf{r} - \omega t)$, in a medium with refractive index n and relative permeability $\mu_r = 1$. Using an appropriate Maxwell equation, show that the magnetic induction field can be written as:

$$\mathbf{H} = \frac{n}{c\mu_0} \hat{\mathbf{k}} \times \mathbf{E}$$ [4]

(d) Hence or otherwise, show that the amplitude of the Poynting vector can be written as:

$$N = \frac{1}{\mu_0} \frac{n}{c} E_0^2$$ [2]

(e) Derive, using Ampère's Law, an expression for the magnetic field inside a long solenoid with n turns per unit length carrying a constant current I, stating any approximations you need to make. [2]

(f) How will this expression change if the interior of the solenoid is filled by an *ideal* soft ferromagnetic material (take μ_r to be significantly larger than one, but let the material be linear) ? [2]

(g) The current is now given a time variation, $I = I_0 \cos \omega t$ (N.B. this is a slow enough variation that changes in the magnetic field induced by the electric field can be neglected). Show that the electric field inside the solenoid can be written:

$$\mathbf{E} = \frac{\omega \mu n I_0 r}{2} \sin \omega t \mathbf{i}_\phi$$

What will the field be outside the solenoid (assume that the solenoid has a radius a) ? [6]

(h) We can approximate (very roughly !) the behaviour of a *hard* ferromagnet by keeping a fixed μ_r and introducing a phase delay between $\mathbf{H}$ and $\mathbf{B}$, so that if $\mathbf{H} = \mathbf{H}_0 \cos \omega t$, the time variation of $\mathbf{B}$ is given by $\mathbf{B}_0 \sin(\omega t + \pi/4)$. Find the electric field outside the solenoid if the soft ferromagnet is replaced by the hard ferromagnet. [2]

(i) Assume that the phase delay, ϕ, between $\mathbf{B}$ and $\mathbf{H}$ can be tuned by varying the material in the solenoid. The energy density in the *magnetic* field is given by $U_M = \int \mathbf{B} \cdot \mathbf{dH}$. Using the forms for $\mathbf{H}$ and $\mathbf{B}$ in part (h), write the energy density as an integral over time (you do not need to evaluate the integral) and describe how the energy density in the solenoid over one period of the $\mathbf{H}$ field varies as the phase delay varies. [4]

10. (a) The electric field in a plane wave is written as:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

i. Define $\mathbf{k}$ and ω in terms of physical properties of the wave [2]

ii. What is the relationship between the directions of $\mathbf{k}$ and $\mathbf{E}$ in free space ? Justify your answer with reference to one of Maxwell's equations. [2]

iii. Use Faraday's Law to show that the associated magnetic field can be written as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

What assumption must you make about the form of the $\mathbf{B}$ field ? [4]

(b) i. Write down the dispersion relation for the plane wave in vacuum. [2]

ii. Now, for a *linear* medium with relative permittivity ϵ_r and relative permeability μ_r, write down the dispersion relation and state how the speed of light is affected by the presence of the medium. [2]

iii. Describe what would happen to incoming plane waves if in some region either ϵ_r or μ_r were less than zero. [2]

(c) Consider electromagnetic plane waves with frequency ω propagating from vacuum towards an interface with a material.

i. How is the refractive index, n, defined in terms of the relative permittivity of the material, ϵ_r ? [2]

ii. Consider a material with relative permittivity:

$$\epsilon_r = 1 + \frac{\omega_a^2}{\omega_0^2 - \omega^2}$$

where ω_a and ω_0 are fixed, characteristic frequencies of the material, with $\omega_a > \omega_0$ and $\omega_0 > 0$. Describe what will happen to the electromagnetic waves in the *two* cases $\omega \to 0$ and $\omega \to \infty$. [4]

iii. Now consider a similar, but more complex, material with relative permittivity:

$$\epsilon_r = 1 + \frac{\omega_a^2(\omega_0^2 - \omega^2) + i\gamma\omega\omega_a^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

where γ is related to damping processes and is small but non-zero, and ω_a and ω_0 have the same meaning as before. Describe how the relative permittivity will vary with ω in the region $\omega \simeq \omega_0$. [Consider first $\omega = \omega_0$ and then values just either side of ω_0.]. [3]

iv. For this more complex material, it can be shown that the complex conductivity, $\hat{g}$, relates to the permittivity as:

$$\hat{g} = -i\omega\epsilon_0(\epsilon_r - 1)$$

Show that a material with non-zero γ and $\omega_0 = 0$ has a non-zero conductivity at $\omega = 0$ with a real part, and find the expression for the real part. [4]

v. What two general classes of materials are described by the two cases (ii) (non-zero ω_0 with zero γ) and (iii) (non-zero γ with $\omega_0 = 0$) you have considered above ? Justify your answer *briefly*. [3]