UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS3201**
ASSESSMENT PATTERN	:	**PHAS3201A**
MODULE NAME	:	**Electromagnetic Theory**
DATE	:	**21-May-08**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

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Answer ALL SIX questions in Section A and TWO questions from Section B

The numbers in square brackets at the right-hand side of the text indicate the provisional allocation of maximum marks per question or sub-section of a question.

You may find the following constants and theorems useful.

For any vector $\mathbf{F}$ and scalar φ,

$$\nabla \times \nabla\varphi = 0$$

For any vector function which can be written $\mathbf{F}(\mathbf{r}, t) = \mathbf{A} \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$ where $\mathbf{A}$ is a constant, then:

$$\begin{aligned} \nabla\cdot\mathbf{F} &= i\mathbf{k}\cdot\mathbf{F} \\ \nabla\times\mathbf{F} &= i\mathbf{k}\times\mathbf{F} \end{aligned}$$

Differential operators in spherical polar coordinates: $\mathbf{r} = (r, \theta, \phi), dv = r^2 \sin\theta dr d\theta d\phi$

$$\begin{aligned} \nabla\varphi &= \mathbf{i}_r\frac{\partial\varphi}{\partial r} + \mathbf{i}_\theta\frac{1}{r}\frac{\partial\varphi}{\partial\theta} + \mathbf{i}_\phi\frac{1}{r\sin\theta}\frac{\partial\varphi}{\partial\phi} \\ \nabla\times\mathbf{F} &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right]\mathbf{i}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi)\right]\mathbf{i}_\theta + \frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right]\mathbf{i}_\phi \end{aligned}$$

The Divergence theorem and Stokes' theorems are:

$$\begin{aligned} \int_V \nabla\cdot\mathbf{F}dv &= \oint_S \mathbf{F}\cdot\mathbf{n}da \\ \oint_C \mathbf{F}\cdot dl &= \int_S (\nabla\times\mathbf{F})\cdot\mathbf{n}da \end{aligned}$$

Useful four-vectors:

$$\begin{aligned} \partial_\mu &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{i}{c}\frac{\partial}{\partial t}\right) \\ j_\mu &= (J_x, J_y, J_z, ic\rho) \end{aligned}$$

SECTION A

1. Give equations to define the surface and bulk (or volume) current densities arising from a magnetisation $\mathbf{M}$. [2]

   Write a general current density, $\mathbf{J}$, as the sum of a free current density and a magnetisation current density, and using this and the first part of this question show that $\nabla\times\mathbf{B} = \mu_0\mathbf{J}$ can be re-arranged as $\nabla\times\mathbf{H} = \mathbf{J}_f$ with $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$. [4]

2. Starting from Faraday's law:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

show that, for time *invariant* (or static) fields, we can write $\mathbf{E} = \nabla F$ where F is a scalar function. [2]

If we write $\mathbf{B} = \nabla \times \mathbf{A}$, show that, for time *varying* fields, the expression for $\mathbf{E}$ changes to:

$$\mathbf{E} = \nabla G - \frac{\partial \mathbf{A}}{\partial t},$$

where G is a scalar function. [4]

3. Use the Maxwell equations:

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned}$$

to derive boundary conditions on the components of time *independent* fields $\mathbf{E}$ and $\mathbf{H}$ parallel to an interface between two linear media. What assumptions must you make about the interface ? [7]

4. Briefly explain the physical significance of the continuity equation

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

[2]

Show how the continuity equation can be written as the scalar product between two four-vectors. Explain the physical significance of this fact. [5]

5. The dispersion relation for electromagnetic waves of frequency ω and wavenumber k in a neutral plasma is written:

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right),$$

where ω_P is the plasma frequency. Explain, using appropriate equations, what happens to electromagnetic waves propagating towards the plasma when:

(a) $\omega < \omega_P$ [4]

(b) $\omega > \omega_P$ [4]

6. Give an expression for the Poynting vector $\mathbf{N}$ in terms of the time-varying electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. [2]

Explain what $\oint_S \mathbf{N} \cdot \mathbf{n} da$ represents physically. [2]

In spherical polar coordinates, if $\mathbf{E}$ lies along $\mathbf{i}_\phi$ and $\mathbf{B}$ lies along $\mathbf{i}_\theta$, what direction does $\mathbf{N}$ lie along ? [2]

SECTION B

7. (a) The potential at r due to charges q_1 and q_2 at $\mathbf{r}_1$ and $\mathbf{r}_2$ can be written as:

$$V(\mathbf{r}) = \frac{q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|}$$

What basic principle of electromagnetism has been assumed in this expression ? [2]

(b) The dipole moment for a simple dipole can be written as:

$$\mathbf{p} = q\mathbf{l}$$

Draw a sketch of the dipole with labels and explain the meaning of q and $\mathbf{l}$ in the formula. [4]

(c) The potential at r due to a dipole $\mathbf{p}$ at the origin when the distance $\mathbf{r}$ is large can be written:

$$V(\mathbf{r}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2},$$

Show, with the aid of a sketch, that this can be written as:

$$V(\mathbf{r}) = \frac{ql\cos\theta}{4\pi\epsilon_0 r^2}$$

if the dipole is aligned along $\mathbf{i}_z$. [2]

(d) For this configuration (i.e. with the dipole at the origin, lying along the z-axis), sketch the form of V:

i. At a fixed distance r from the dipole . [3]

ii. At a fixed polar angle θ at varying distances [3]

Using the form for grad in spherical polar coordinates given in the rubric, show that the field due to a dipole can be written:

$$\mathbf{E}(\mathbf{r}) = \frac{ql}{4\pi\epsilon_0 r^3}\left(2\cos\theta\mathbf{i}_r + \sin\theta\mathbf{i}_\theta\right)$$

[4]

(e) Consider a particle of charge e sufficiently far from the dipole that the field and potential given above are good approximations. Calculate the Lorentz force on the particle at the following points (N.B. the points are given in *Cartesian coordinates*):

i. (0,a,0) [3]

ii. (0,0,a) [3]

(f) Calculate the work done in moving the particle from the first point to the second along a path of constant radius a. Explain the sign of the answer. [6]

8. (a) The electric displacement field is written as:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

Define ϵ_0, $\mathbf{E}$ and $\mathbf{P}$. [3]

(b) Under some circumstances, it can be written $\mathbf{D} = \epsilon \mathbf{E}$. Explain how this happens, and define ϵ (give a formula). [2]

(c) The boundary conditions on $\mathbf{B}$ and $\mathbf{D}$ state that, in the absence of external charge at an interface, the perpendicular component of the field, $B^{\perp}$ or $D^{\perp}$, is conserved.

i. Show how these conditions can be derived using the divergence theorem and two of Maxwell's equations. [4]

ii. Show that a surface charge density of σ will change the boundary conditions on $\mathbf{D}$ to:

$$D_1^{\perp} - D_2^{\perp} = \sigma$$

[3]

iii. Why will the boundary condition on $\mathbf{B}$ never change ? [2]

(d) Consider a sphere of charge, radius R, with uniform charge density ρ.

i. Write ρ in terms of the total charge q and the radius of the sphere R. [2]

ii. Show that the electric field *inside* the sphere at a distance r from the centre can be written:

$$\mathbf{E} = \frac{qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}}$$

[2]

iii. Place the sphere at the origin, and show that, for points $\mathbf{r} = (0, y, 0)$ *inside* the sphere along the y-axis, the field can be written:

$$\mathbf{E}(\mathbf{r}) = \frac{y\rho}{3\epsilon_0}\mathbf{i}_y$$

[4]

iv. Place the sphere at a point $(0, d, 0)$ where $d < R$ and again for points *inside* the sphere along the y-axis show that the field can be written:

$$\mathbf{E}(\mathbf{r}) = \frac{(y-d)\rho}{3\epsilon_0}\mathbf{i}_y$$

[2]

v. Now, consider *two overlapping* spheres at $(0, 0, 0)$ and $(0, d, 0)$ with charge densities ρ and $-\rho$. Using these last two results and your formula for ρ from earlier, show that the field on the y-axis in the overlapping region can be written:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd}{R^3}\mathbf{i}_y$$

[6]

9. (a) The electric field in a plane wave is written as:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

i. Define k and ω [2]

ii. What is the relationship between the directions of k and E in free space ? Justify your answer with reference to one of Maxwell's equations. [4]

iii. Use Faraday's Law ($\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$) to show that the associated magnetic field can be written as:

$$\mathbf{B}(\mathbf{r}) = \frac{\mathbf{k}\times\mathbf{E}_0}{\omega} \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$$

Assume that the magnetic field has the form $\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$ [4]

(b) i. Write down the dispersion relation for the plane wave in vacuum. [2]

ii. Now, for a *linear* medium with relative permittivity ϵ_r and relative permeability μ_r, write down the dispersion relation and explain how the speed of light is affected. [2]

iii. Explain what would happen to incoming plane waves if either ϵ_r or μ_r were less than zero. [2]

(c) Consider electromagnetic plane waves propagating from vacuum towards the interface with a material whose relative permittivity can be written:

$$\hat{\epsilon}_r = 1 + \frac{\omega_P^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} + i\left(\frac{\omega_P^2\gamma\omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right),$$

where ω_P, γ and ω_0 are related to properties of the material. Note that $\omega_P > \omega_0$.

i. Consider first the case where $\omega_0 \gg 0$. Explain what will happen to the electromagnetic waves when $\omega \rightarrow 0$ and $\omega \rightarrow \infty$. [In this case, assume that $\gamma = 0$ and consider the refractive index]. [4]

ii. Now explain what will happen for $\omega \simeq \omega_0$. [Here, assume that γ has a small, non-zero value, and consider first $\omega = \omega_0$ and then what will happen either side of this value.]. [3]

iii. Within this model, it can be shown that the complex conductivity, $\hat{g}$, relates to the relative permittivity as:

$$\hat{g} = -i\epsilon_0\omega(\hat{\epsilon} - 1)$$

Show that a material with non-zero γ and $\omega_0 = 0$ has a conductivity at $\omega = 0$ with a real part. [4]

iv. What two general classes of materials are described by the two cases (i) (non-zero ω_0 with zero γ) and (iii) (non-zero γ with $\omega_0 = 0$) you have considered above ? Justify your answer *briefly*. [3]

10. (a) Write down the energy density u for electric and magnetic fields in terms of $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$. [2]

(b) The power radiated from a source at large distances is given as:

$$P_{\text{rad}} = \lim_{r \to \infty} \oint \mathbf{N} \cdot \mathbf{n} da,$$

where $\mathbf{N}$ is the Poynting vector. Why should the Poynting vector fall off *no faster* than $1/r^2$ for radiation of power ? What does this imply about the distance dependence of the individual fields ? [2]

(c) A small current loop carrying an alternating current of frequency ω can serve as a model for a magnetic dipole with dipole moment $\mathbf{m}(t) = m_0 \cos \omega t \mathbf{i}_z$. It has potentials which, at large distances, can be written in spherical polar coordinates as:

$$\begin{aligned} V(\mathbf{r}, t') &= 0 \\ \mathbf{A}(\mathbf{r}, t') &= -\frac{\mu_0 m_0 \omega}{4\pi c} \frac{\sin\theta}{r} \sin \omega t' \mathbf{i}_\phi \end{aligned}$$

with t' the *retarted* time.

i. Define the retarded time, t', and explain its meaning. [2]

ii. Show that $\partial t'/\partial r = -1/c$ and $\partial t'/\partial t = 1$. [3]

iii. Hence, using the form for curl in spherical polar coordinates given in the rubric, show that at large distances ($r \gg c/\omega$) the fields can be written:

$$\begin{aligned} \mathbf{E} &= \frac{\mu_0 m_0 \omega^2}{4\pi c} \frac{\sin\theta}{r} \cos \omega t' \mathbf{i}_\phi \\ \mathbf{B} &= -\frac{\mu_0 m_0 \omega^2}{4\pi c^2} \frac{\sin\theta}{r} \cos \omega t' \mathbf{i}_\theta \end{aligned}$$

Is there radiation from this system ? If so, in what direction ? [5]

(d) Consider an infinite line of charge with density λ (units of charge per unit length) lying along the z-axis. Using Gauss' Law, show that the electric field in cylindrical polar coordinates can be written:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 r} \mathbf{i}_r$$

[2]

(e) Ignoring relativistic effects, if the charges were moving at uniform velocity in the z-direction, would the electric field change ? Justify your answer. [2]

(f) Using $I = \lambda v$ show that the magnetic field for this situation can be written:

$$\mathbf{B} = \frac{\mu_0 \lambda v}{2\pi r} \mathbf{i}_\phi$$

[2]

(g) Show that the total energy density coming from both fields can be written:

$$u = \frac{1}{2\epsilon_0}\left(\frac{\lambda}{2\pi r}\right)^2\left[1 + \frac{v^2}{c^2}\right]$$

and comment briefly on the relative contributions of the two fields to the energy density. [6]

(h) Write down the Poynting vector for this system. Is there net energy flow through a closed cylindrical surface surrounding part of the wire ? [4]