DEPARTMENT OF PHYSICS & ASTRONOMY

PHAS3226 QUANTUM MECHANICS

Problem Paper 2 - solutions

Solutions to be handed in on Tuesday 2 November 2010

1. (Based on 2010 examination question)

(a) *Show that the operator $\hat{Q} = \hat{x}\hat{p}$ may be expressed in terms of the raising and lowering operators $\hat{a}_+$, $\hat{a}_-$,respectively, by .* [2]

$$\hat{Q} = i\frac{\hbar}{2}\left(\hat{a}_+^2 - \hat{a}_-^2 + 1\right)$$

The raising and lowering operators are $\hat{a}_+ = \left(\frac{1}{2\hbar m\omega}\right)^{1/2}(m\omega\hat{x} - i\hat{p})$ and $\hat{a}_- = \left(\frac{1}{2\hbar m\omega}\right)^{1/2}(m\omega\hat{x} + i\hat{p})$. By adding and then re-arranging we have

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}_+ + \hat{a}_-) \tag{1}$$

and by subtracting we have

$$\hat{p} = i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\hat{a}_+ - \hat{a}_-)\,. \tag{2}$$

Then

$$\hat{Q} = \hat{x}\hat{p} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}_+ + \hat{a}_-)\,i\left(\frac{m\hbar\omega}{2}\right)^{1/2}(\hat{a}_+ - \hat{a}_-)\,, \tag{3}$$

$$= i\frac{\hbar}{2}(\hat{a}_+ + \hat{a}_-)(\hat{a}_+ - \hat{a}_-) = i\frac{\hbar}{2}\left(\hat{a}_+^2 - \hat{a}_-^2 + \hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_-\right) = i\frac{\hbar}{2}\left(\hat{a}_+^2 - \hat{a}_-^2 + [\hat{a}_-, \hat{a}_+]\right) \tag{4}$$

But the basis commutator for $\hat{a}_+$, $\hat{a}_-$ is $[\hat{a}_-, \hat{a}_+] = 1$ so

$$\hat{Q} = i\frac{\hbar}{2}\left(\hat{a}_+^2 - \hat{a}_-^2 + 1\right). \tag{5}$$

(b) *Show that the matrix elements of the operator $\hat{Q} = \hat{x}\hat{p}$ are given by* [2]

$$\langle k\,|\hat{x}\hat{p}|\,n\rangle = i\frac{\hbar}{2}\left\{\sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} + \delta_{kn}\right\}.$$

[You may assume that the actions of $\hat{a}_+$, $\hat{a}_-$ on an harmonic oscillator basis energy eigenstates $|n\rangle$ are $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$ and $\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$.]

But $\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$ so

$$\hat{a}_+^2|n\rangle = \sqrt{n+1}\hat{a}_+|n+1\rangle = \sqrt{(n+2)(n+1)}|n+2\rangle \tag{6}$$

and

$$\hat{a}_-^2|n\rangle = \sqrt{n}\hat{a}_-|n-1\rangle = \sqrt{n(n-1)}|n-2\rangle \tag{7}$$

and hence

$$\hat{Q}|n\rangle = i\frac{\hbar}{2}\left\{\hat{a}_+^2 - \hat{a}_-^2 + 1\right\}|n\rangle = i\frac{\hbar}{2}\left\{\sqrt{(n+2)(n+1)}|n+2\rangle - \sqrt{n(n-1)}|n-2\rangle + |n\rangle\right\}. \tag{8}$$

Taking the scalar product with the bra $\langle k|$ and noting that $\langle k|n\rangle = \delta_{kn}$ gives the matrix elements of $\hat{Q} = \hat{p}\hat{x}$ as

$$\hat{Q}_{kn} = \langle k|\hat{x}\hat{p}|n\rangle = i\frac{\hbar}{2}\left\{\sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} + \delta_{kn}\right\}. \tag{9}$$

i. *Hence construct the matrix representing $\hat{Q}$ limiting it to a 5×5 matrix.* [3]
To construct the matrix representation for $\hat{Q}$ we take $k = 0, 1, 2, 3, 4, \ldots$, in turn, noting that the first term only contributes for $k = n+2$ the second term for $k = n-2$ and the last term for $k = n$. Thus the matrix for $\hat{Q}$ is (truncated to 5×5) is

$$\hat{Q} = i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 & -\sqrt{4}\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & \sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix}. \tag{10}$$

ii. *and deduce the matrix for $\hat{p}\hat{x}$, and show that $[\hat{x}, \hat{p}] = i\hbar$.* . [3, 2]

The Hermitian conjugate of $\hat{Q}$, i.e. $\hat{Q}^\dagger = (\hat{x}\hat{p})^\dagger = \hat{p}^\dagger\hat{x}^\dagger = \hat{p}\hat{x}$ as $\hat{x}$ and $\hat{p}$ are Hermitian operators. Thus the matrix for $\hat{p}\hat{x}$ is the Hermitian conjugate of the matrix for $\hat{Q}$. Since $Q^\dagger = \left(Q^T\right)^*$ i.e. the complex conjugation of the transposed matrix,

$$\hat{Q}^\dagger = \hat{p}\hat{x} = -i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & \sqrt{3}\sqrt{2} & 0 \\ -\sqrt{2} & 0 & 1 & 0 & \sqrt{4}\sqrt{3} \\ 0 & -\sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix}. \tag{11}$$

Thus in terms of matrices

$$\begin{aligned}\hat{x}\hat{p} - \hat{p}\hat{x} &= i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 & -\sqrt{4}\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & \sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix} \\ &\quad + i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & \sqrt{3}\sqrt{2} & 0 \\ -\sqrt{2} & 0 & 1 & 0 & \sqrt{4}\sqrt{3} \\ 0 & -\sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & -\sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix}\end{aligned}$$

$$\hat{x}\hat{p} - \hat{p}\hat{x} = i\frac{\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} = i\hbar. \tag{12}$$

(c) *The harmonic oscillator is in the quantum state specified by the normalized state vector*

$$|\psi\rangle = \frac{i}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle,$$

where $|n\rangle$ denotes the n-th energy eigenstate. Using the matrix representations of $\hat{Q}$ and $\hat{Q}^2$, or otherwise,

i. *calculate the expectation value of Q in the state $|\psi\rangle$,* [5]

The matrix representation of the state $|\psi\rangle$ is (up to the first five basis states)

$$|\psi\rangle = \begin{pmatrix} \frac{i}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{13}$$

and so the expectation value of $\hat{Q}$ in this state is

$$\left\langle\psi|\hat{Q}|\psi\right\rangle = i\frac{\hbar}{2}\begin{pmatrix} -\frac{i}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 & -\sqrt{4}\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & \sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{i}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = i\frac{\hbar}{2} \tag{14}$$

Note that operator $\hat{Q}$ is not an Hermitian operator.

The "otherwise" method makes use of the expression for the matrix elements. As $|\psi\rangle = \frac{i}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle$ then

$$\left\langle\psi|\hat{Q}|\psi\right\rangle = \left(-\frac{i}{\sqrt{3}}\langle 0| + \sqrt{\frac{2}{3}}\langle 1|\right)\hat{Q}\left(\frac{i}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right), \tag{15}$$

$$= \frac{1}{3}\left\langle 0|\hat{Q}|0\right\rangle - i\frac{\sqrt{2}}{3}\left\langle 0|\hat{Q}|1\right\rangle + i\frac{\sqrt{2}}{3}\left\langle 1|\hat{Q}|0\right\rangle + \frac{2}{3}\left\langle 1|\hat{Q}|1\right\rangle. \tag{16}$$

and recalling $\hat{Q}_{kn} = \langle k|\hat{x}\hat{p}|n\rangle = i\frac{\hbar}{2}\left\{\sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} + \delta_{kn}\right\}$ then

$$\left\langle\psi|\hat{Q}|\psi\right\rangle = i\frac{\hbar}{2}\left\{\frac{1}{3} + 0 + 0 + \frac{2}{3}\right\} = i\frac{\hbar}{2} \tag{17}$$

ii. *obtain* $\langle 1|\hat{Q}^2|1\rangle$. [3]

Squaring the matrix expression for $\hat{Q}$ gives

$$\hat{Q}^2 = i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 & -\sqrt{4}\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & \sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix} i\frac{\hbar}{2}\begin{pmatrix} 1 & 0 & -\sqrt{2} & 0 & 0 \\ 0 & 1 & 0 & -\sqrt{3}\sqrt{2} & 0 \\ \sqrt{2} & 0 & 1 & 0 & -\sqrt{4}\sqrt{3} \\ 0 & \sqrt{3}\sqrt{2} & 0 & 1 & 0 \\ 0 & 0 & \sqrt{4}\sqrt{3} & 0 & 1 \end{pmatrix}$$

$$= -\frac{\hbar^2}{4}\begin{pmatrix} -1 & 0 & -2\sqrt{2} & 0 & 2\sqrt{2}\sqrt{3} \\ 0 & -5 & 0 & -2\sqrt{2}\sqrt{3} & 0 \\ 2\sqrt{2} & 0 & -13 & 0 & -4\sqrt{3} \\ 0 & 2\sqrt{2}\sqrt{3} & 0 & -5 & 0 \\ 2\sqrt{2}\sqrt{3} & 0 & 4\sqrt{3} & 0 & -11 \end{pmatrix}. \tag{18}$$

and so

$$\langle 1|\hat{Q}^2|1\rangle = -\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 & -2\sqrt{2} & 0 & 2\sqrt{2}\sqrt{3} \\ 0 & -5 & 0 & -2\sqrt{2}\sqrt{3} & 0 \\ 2\sqrt{2} & 0 & -13 & 0 & -4\sqrt{3} \\ 0 & 2\sqrt{2}\sqrt{3} & 0 & -5 & 0 \\ 2\sqrt{2}\sqrt{3} & 0 & 4\sqrt{3} & 0 & -11 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$= \frac{5\hbar^2}{4}. \tag{19}$$

The "otherwise" method makes use of the completeness relation for the basis states $|n\rangle$, namely $\sum_n |n\rangle\langle n| = 1$, then

$$\langle 1|\hat{Q}^2|1\rangle = \langle 1|\hat{Q}\hat{Q}|1\rangle = \sum_n \langle 1|\hat{Q}|n\rangle\langle n|\hat{Q}|1\rangle. \tag{20}$$

From the expression for $\hat{Q}_{kn} = \langle k|\hat{x}\hat{p}|n\rangle = i\frac{\hbar}{2}\left\{\sqrt{(n+2)(n+1)}\delta_{k,n+2} - \sqrt{n(n-1)}\delta_{k,n-2} + \delta_{kn}\right\}$ we see that the only non-zero values occur for states $|n\rangle = |1\rangle$ and $|n\rangle = |3\rangle$, giving

$$\langle 1|\hat{Q}^2|1\rangle = \langle 1|\hat{Q}|1\rangle\langle 1|\hat{Q}|1\rangle + \langle 1|\hat{Q}|3\rangle\langle 3|\hat{Q}|1\rangle, \tag{21}$$

$$= \left(i\frac{\hbar}{2}\right)\left(i\frac{\hbar}{2}\right) + i\frac{\hbar}{2}\left(-\sqrt{3\times 2}\right)i\frac{\hbar}{2}\left(\sqrt{3\times 2}\right) = \frac{5}{4}\hbar^2. \tag{22}$$

2. *The raising and lowering angular momentum operators, $\hat{J}_+$, $\hat{J}_-$ are defined in terms of the Cartesian components $\hat{J}_x$, $\hat{J}_y$, $\hat{J}_z$ of angular momentum $\hat{J}$ by $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$; $\hat{J}_- = \hat{J}_x - i\hat{J}_y$.*

   Hence by adding, we have

$$\hat{J}_x = \frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right). \tag{23}$$

The action of $\hat{J}_+$, $\hat{J}_-$ on $|1,m\rangle$ are

$$\begin{aligned}\hat{J}_+|1,m\rangle &= \sqrt{2-m\,(m+1)}\hbar|1,m+1\rangle && (24)\\ \hat{J}_-|1,m\rangle &= \sqrt{2-m\,(m-1)}\hbar|1,m-1\rangle && (25)\end{aligned}$$

and so

$$\begin{aligned}\left\langle 1,m'|\hat{J}_+|1,m\right\rangle &= \sqrt{2-m\,(m+1)}\hbar\langle 1,m'|1,m+1\rangle = \sqrt{2-m\,(m+1)}\hbar\delta_{m'm+1}, && (26)\\ \left\langle 1,m'|\hat{J}_-|1,m\right\rangle &= \sqrt{2-m\,(m-1)}\hbar\langle 1,m'|1,m-1\rangle = \sqrt{2-m\,(m-1)}\hbar\delta_{m'm-1}, && (27)\\ \left\langle 1,m'|\hat{J}_x|1,m\right\rangle &= \frac{1}{2}\left\{\sqrt{2-m\,(m+1)}\hbar\delta_{m'm+1}+\sqrt{2-m\,(m-1)}\hbar\delta_{m'm-1}\right\}. && (28)\end{aligned}$$

(a) *Obtain the matrix representation of $\hat{J}_x$ for the state with $j=1$ in terms of the set of eigenstates of $\hat{J}_z$.* [6]

The matrix can be constructed by choosing $m'=1,0,-1$ and $m=1,0,-1$ in turn in eq(28) and noting that the Kronecker delta $\delta_{kn}=1$ if $k=n$ and $\delta_{kn}=0$ of $k\neq n$. Clearly the diagonal elements are zero and the table below can be built up,

	$m=1$	$m=0$	$m=-1$
$m'=1$	0	$\frac{\sqrt{2}}{2}$	0
$m'=0$	$\frac{\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$
$m'=-1$	0	$\frac{\sqrt{2}}{2}$	0

and so the matrix representation of $\hat{J}_x$ is $\hat{J}_x=\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$.

(b) *Solve the eigenvalue equation, i.e. $J_xc=\lambda c$, using the matrix representation of $\hat{J}_x$ and the basis states*

$$|j,m\rangle\equiv|1,1\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix};\quad |1,0\rangle=\begin{pmatrix}0\\1\\0\end{pmatrix};\quad |1,-1\rangle=\begin{pmatrix}0\\0\\-1\end{pmatrix};$$

i. *to find ALL the eigenvalues λ,* [3]

ii. *and ANY ONE of the eigenvectors c . (Make the first element of the eigenvector real and positive.)* [4]

Thus we have to solve the matrix equation $\hat{J}_x\mathbf{c}=\lambda\mathbf{c}$ for the eigenvectors $\mathbf{c}$. This equation has a non-trivial solution for $\mathbf{c}$ if

$$\det\left|\hat{J}_x-\lambda I_3\right|=0. \tag{29}$$

Writing $\lambda=\frac{\hbar}{\sqrt{2}}\mu$ we have to solve

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\lambda\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\frac{\hbar}{\sqrt{2}}\mu\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}, \tag{30}$$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix}-\mu&1&0\\1&-\mu&1\\0&1&-\mu\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=0 \tag{31}$$

This equation has a non-trivial solution for $\mathbf{c}$ if

$$\det\begin{vmatrix}-\mu&1&0\\1&-\mu&1\\0&1&-\mu\end{vmatrix}=0. \tag{32}$$

$$-\mu\begin{vmatrix} -\mu & 1 \\ 1 & -\mu \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & -\mu \end{vmatrix} = -\mu\left(\mu^2-1\right)+\mu = -\mu^3+2\mu = \mu\left(\mu^2-2\right)=0. \tag{33}$$

Thus $\mu = \pm\sqrt{2},\, 0$ and the eigenvalues are $\lambda = +\hbar,\ 0\hbar,\ -\hbar$ (As is to be expected as there is not difference between the x-axis and the z-axis for a free particle.)
To find the eigenvectors we have to find the components c_1, c_2, c_3.
For $\lambda = \hbar$, $(\mu = +\sqrt{2})$

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \hbar\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{34}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \sqrt{2}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \tag{35}$$

Multiplying out gives $\begin{pmatrix} c_2 \\ c_1+c_3 \\ c_2 \end{pmatrix} = \sqrt{2}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$ so

$$c_2 = \sqrt{2}c_1; \qquad c_1 + c_3 = \sqrt{2}c_2; \qquad c_2 = \sqrt{2}c_3, \tag{36}$$

so $c_1 = c_3$. Since the eigenvector must be normalized, i.e. $c_1^2 + c_2^2 + c_3^2 = 1$ then using the above $c_1^2 + 2c_1^2 + c_1^2 = 1$, and $c_1 = \pm\frac{1}{2}$. Taking $c_1 = \frac{1}{2}$ then the eigenvector for eigenvalue $+\hbar$ is

$$\mathbf{c}_{+1} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2}\sqrt{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}. \tag{37}$$

For eigenvalue $\lambda = -\hbar$ then we have

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = -\hbar\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \tag{38}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = -\sqrt{2}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}. \tag{39}$$

and

$$c_2 = -\sqrt{2}c_1; \qquad c_1 + c_3 = -\sqrt{2}c_2; \qquad c_2 = -\sqrt{2}c_3,$$

so as before $c_1 = c_3$. Normalizing, i.e. $c_1^2 + c_2^2 + c_3^2 = 1$ then using the above $c_1^2 + 2c_1^2 + c_1^2 = 1$, and $c_1 = \pm\frac{1}{2}$. Taking $c_1 = \frac{1}{2}$ then the eigenvector for eigenvalue $-\hbar$ is

$$\mathbf{c}_{-1} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2}\sqrt{2} \\ \frac{1}{2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}. \tag{40}$$

Finally for $\lambda = 0$ we have

$$\frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0. \tag{41}$$

Then

$$c_2 = 0; \qquad c_1 + c_3 = 0; \qquad c_2 = 0,$$

so $c_1 = -c_3$ and normalization gives $c_1^2 + c_2^2 + c_3^2 = 1 = 2c_1^2$ so $c_1 = \frac{1}{\sqrt{2}}$. Hence the eigenvector for eigenvalue $0\hbar$ is

$$\mathbf{c}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{42}$$

(c) *Solve the eigenvalue equation for* $\hat{J}_x$*, i.e.* $\hat{J}_x|\psi\rangle = \lambda|\psi\rangle$*, using the expansion of* $|\psi\rangle = c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle$ *in terms of the set* $|j,m\rangle \equiv |1,1\rangle,\ |1,0\rangle,\ |1,-1\rangle$ [3,4]

i. *to find ALL the eigenvalues* λ

ii. *and ANY ONE of the eigenstates* $|\psi\rangle$
Note that the states $|j,m\rangle$ *satisfy* $\langle j'm'|j,m\rangle = \delta_{j'j}\delta_{m'm}$ *and the actions of* $\hat{J}_+$*,* $\hat{J}_-$ *on the state* $|j,m\rangle$ *are*

$$\hat{J}_+|j,m\rangle = \sqrt{j(j+1)-m(m+1)}\hbar|j,m+1\rangle; \quad \hat{J}_-|j,m\rangle = \sqrt{j(j+1)-m(m-1)}\hbar|j,m-1\rangle.$$

We have to solve

$$\hat{J}_x|\psi\rangle = \lambda|\psi\rangle \tag{43}$$

Using the properties

$$\begin{aligned}
\hat{J}_+|1,m\rangle &= \sqrt{2-m(m+1)}\hbar|1,m+1\rangle &(44)\\
\hat{J}_-|1,m\rangle &= \sqrt{2-m(m-1)}\hbar|1,m-1\rangle &(45)
\end{aligned}$$

we have

$$\begin{aligned}
\hat{J}_+|1,1\rangle &= 0; \quad \hat{J}_+|1,0\rangle = \sqrt{2}\hbar|1,1\rangle; \quad \hat{J}_+|1,-1\rangle = \sqrt{2}\hbar|1,0\rangle, &(46)\\
\hat{J}_-|1,1\rangle &= \sqrt{2}\hbar|1,0\rangle; \quad \hat{J}_-|1,0\rangle = \sqrt{2}\hbar|1,-1\rangle; \quad \hat{J}_-|1,-1\rangle = 0. &(47)
\end{aligned}$$

giving

$$\hat{J}_x|1,1\rangle = \frac{\hbar}{\sqrt{2}}|1,0\rangle; \quad \hat{J}_x|1,0\rangle = \frac{\hbar}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right); \quad \hat{J}_x|1,-1\rangle = \frac{\hbar}{\sqrt{2}}|1,0\rangle; \tag{48}$$

Expressing $\hat{J}_x$ in terms of $\hat{J}_+$, $\hat{J}_-$ and expand $|\psi\rangle = c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle$,

$$\begin{aligned}
\frac{1}{2}\left(\hat{J}_+ + \hat{J}_-\right)\left(c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle\right) &= \lambda\left(c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle\right) &(49)\\
\frac{1}{2}\sqrt{2}\hbar\left(c_0|1,1\rangle + c_{-1}|1,0\rangle + c_1|1,0\rangle + c_0|1,-1\rangle\right) &= \lambda\left(c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle\right) &(50)\\
\frac{1}{2}\sqrt{2}\hbar\left(c_0|1,1\rangle + \left(c_{-1}+c_1\right)1,0\rangle + c_0|1,-1\rangle\right) &= \lambda\left(c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle\right) &(51)
\end{aligned}$$

Taking the scalar product with $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ in turn, or just noting that $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$ are linearly independent gives

$$\frac{1}{2}\sqrt{2}\hbar c_0 = \lambda c_1; \quad \frac{1}{2}\sqrt{2}\hbar\left(c_{-1}+c_1\right) = \lambda c_0; \quad \frac{1}{2}\sqrt{2}\hbar c_0 = \lambda c_{-1}. \tag{52}$$

Substituting the first and third equations into the second gives

$$\frac{1}{2}\sqrt{2}\hbar\left(\frac{1}{2\lambda}\sqrt{2}\hbar c_0 + \frac{1}{2\lambda}\sqrt{2}\hbar c_0\right) = \lambda c_0$$

$$\hbar^2 = \lambda^2 \tag{53}$$

Thus $\lambda = \pm\hbar$. These equations are also satisfied by $\lambda = 0$.
To find the eigenfunction, for $\lambda = \pm\hbar$ we have from eq(51)

$$\frac{1}{2}\sqrt{2}\hbar\left(c_0|1,1\rangle + \left(c_{-1}+c_1\right)1,0\rangle + c_0|1,-1\rangle\right) = \pm\hbar\left(c_1|1,1\rangle + c_0|1,0\rangle + c_{-1}|1,-1\rangle\right)$$

and

$$\frac{1}{2}\sqrt{2}\hbar c_0 = \pm\hbar c_1; \quad \frac{1}{2}\sqrt{2}\hbar\left(c_{-1}+c_1\right) = \pm\hbar c_0; \quad \frac{1}{2}\sqrt{2}\hbar c_0 = \pm\hbar c_{-1}$$

giving $c_1 = c_{-1}$ and $c_1 = \pm c_0/\sqrt{2}$. Using the normalization, $|c_1|^2 + |c_0|^2 + |c_{-1}|^2 = 1$ gives $|c_0|^2\left(\frac{1}{2}+1+\frac{1}{2}\right) = 1$ so $c_0 = \pm\frac{1}{\sqrt{2}}$. For the $\lambda = \hbar$ case taking $c_0 = \frac{1}{\sqrt{2}}$ gives

$$|\psi_1\rangle = \frac{1}{2}|1,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle = \frac{1}{2}\left(|1,1\rangle + \sqrt{2}|1,0\rangle + |1,-1\rangle\right), \tag{54}$$

the same as the eigenvector $\mathbf{c}_{+1} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$.

For the case $\lambda = -\hbar$ taking $c_0 = -\frac{1}{\sqrt{2}}$ gives

$$|\psi_{-1}\rangle = \frac{1}{2}|1,1\rangle - \frac{1}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle = \frac{1}{2}\left(|1,1\rangle - \sqrt{2}|1,0\rangle + |1,-1\rangle\right).$$

For $\lambda = 0$ then $c_0 = 0$ and $c_1 + c_{-1} = 0$. Normalization requires $|c_1|^2 + |c_0|^2 + |c_{-1}|^2 = 1$ so $2|c_1|^2 = 1$ and $c_1 = \frac{1}{\sqrt{2}}$. Hence the eigenfunction is $|\psi_0\rangle = \frac{1}{\sqrt{2}}|1,1\rangle - \frac{1}{\sqrt{2}}|1,-1\rangle = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right)$.

3. The basis states $|\alpha\rangle$ and $|\beta\rangle$ are the eigenstates of $\hat{S}_z$ with eigenvalues $\hbar/2$ and $-\hbar/2$ respectively.

(a) *Show that the state* $|\chi_{\hat{\mathbf{n}}}^+\rangle = \cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle$ *is an eigenstate of the spin operator* $\hat{S}_n = \hat{S}\cdot\hat{n} = \hat{S}_x\sin\theta + \hat{S}_z\cos\theta$, *the component of spin* S *of a spin-*$\frac{1}{2}$ *particle in the direction of the unit vector* $\hat{n} = (\sin\theta, 0, \cos\theta)$ *lying in the* x-z *plane, with eigenvalue* $+\hbar/2$. [7]

The easiest way to show that $\hat{S}_n|\chi_{\hat{\mathbf{n}}}^+\rangle = \frac{\hbar}{2}|\chi_{\hat{\mathbf{n}}}^+\rangle$ with $\hat{S}_n = \hat{S}_x\sin\theta + \hat{S}_z\cos\theta$ and $|\chi_{\hat{\mathbf{n}}}^+\rangle = \cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle$ is to substitute. Noting that $\hat{S}_x = \frac{1}{2}\left(\hat{S}_+ - \hat{S}_-\right)$ then

$$\left(\frac{1}{2}\left(\hat{S}_+ - \hat{S}_-\right)\sin\theta + \hat{S}_z\cos\theta\right)\left(\cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle\right)$$

$$\begin{aligned}
&= \frac{\hbar}{2}\sin\theta\cos(\theta/2)|\beta\rangle + \frac{\hbar}{2}\sin\theta\sin(\theta/2)|\alpha\rangle + \frac{\hbar}{2}\cos\theta\cos(\theta/2)|\alpha\rangle - \frac{\hbar}{2}\cos\theta\sin(\theta/2)|\beta\rangle, \\
&= \frac{\hbar}{2}\left(\cos\theta\cos(\theta/2) + \sin\theta\sin(\theta/2)\right)|\alpha\rangle + \frac{\hbar}{2}\left(\sin\theta\cos(\theta/2) - \cos\theta\sin(\theta/2)\right)|\beta\rangle, \\
&= \frac{\hbar}{2}\cos(\theta/2)|\alpha\rangle + \frac{\hbar}{2}\sin(\theta/2)|\beta\rangle = \frac{\hbar}{2}\left(\cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle\right) = \frac{\hbar}{2}|\chi_{\hat{\mathbf{n}}}^+\rangle.
\end{aligned} \tag{55}$$

(b) *A beam of electrons polarized with component of spin* $+\hbar/2$ *along the* z*-axis moves along the* y*-axis. The beam passes through a Stern-Gerlach magnet whose magnetic field is along a direction* $\hat{\mathbf{n}}$ *in the* x-z *plane at an angle* θ *to the* z*-axis. How many beams emerge? What are their relative intensities?* [6]

Initially the beam is in the state $|\alpha\rangle$. The Stern-Gerlach measures the component of spin along the direction $\hat{\mathbf{n}}$ in the x-z plane, i.e. it measures $\hat{S}_n$. Thus expand $|\alpha\rangle$ in terms of the eigenstates $|\chi_{\hat{\mathbf{n}}}^+\rangle$, and $|\chi_{\hat{\mathbf{n}}}^-\rangle$ of $\hat{S}_n$ with eigenvalues $+\hbar/2$ and $-\hbar/2$ respectively, as

$$|\alpha\rangle = a|\chi_{\hat{\mathbf{n}}}^+\rangle + b|\chi_{\hat{\mathbf{n}}}^-\rangle. \tag{56}$$

with $|a|^2 + |b|^2 = 1$ for normalization. The magnet splits the beam into 2 components.
The relative intensities are $|a|^2$ and $|b|^2$. But $a = \langle\chi_{\hat{\mathbf{n}}}^+|\alpha\rangle = \left(\langle\alpha|\cos(\theta/2) + \langle\beta|\sin(\theta/2)\right)|\alpha\rangle = \cos(\theta/2)$. Hence the relative probabilities are $|a|^2 = \cos^2(\theta/2)$ and $|b|^2 = 1 - |a|^2 = \sin^2(\theta/2)$.

(c) *If one of these emerging beams enters an ideal Stern-Gerlach filter which passes only electrons whose spin is in the* $+x$ *direction, what is the probability of each electron emerging from the filter?* [7]

There are two choices for the outgoing beams. One beam has electrons in the state $|\chi_{\hat{\mathbf{n}}}^+\rangle$ and the other in state $|\chi_{\hat{\mathbf{n}}}^-\rangle$. From above $|\chi_{\hat{\mathbf{n}}}^+\rangle = \cos(\theta/2)|\alpha\rangle + \sin(\theta/2)|\beta\rangle$ is an eigenstate of the spin operator $\hat{S}_n = \hat{S}_x \sin\theta + \hat{S}_z \cos\theta$, the component of spin S of a spin-$\frac{1}{2}$ particle in the direction of the unit vector $\hat{\mathbf{n}} = (\sin\theta, 0, \cos\theta)$ lying in the x-z plane. Then on choosing $\theta = \pi/2$, $\hat{S}_n = \hat{S}_x$ and the eigenstate for eigenvalue $\hbar/2$ along $+x$ direction is$|\chi^+_{\hat{\mathbf{x}}}\rangle = \cos(\pi/4)\,|\alpha\rangle + \sin(\pi/4)\,|\beta\rangle = \frac{1}{\sqrt{2}}\,(|\alpha\rangle + |\beta\rangle)$. Thus the probability that an electron in the state $|\chi^+_{\hat{\mathbf{n}}}\rangle$ is in the state $|\chi^+_{\hat{\mathbf{x}}}\rangle$ is

$$\begin{aligned} P_+ &= \left|\langle\chi^+_{\hat{\mathbf{x}}}|\chi^+_{\hat{\mathbf{n}}}\rangle\right|^2 = \left|\left[\frac{1}{\sqrt{2}}\left(\langle\alpha| + \langle\beta|\right)\right]\left[\cos(\theta/2)\,|\alpha\rangle + \sin(\theta/2)\,|\beta\rangle\right]\right|^2, && (57)\\ &= \frac{1}{2}\left(\cos(\theta/2) + \sin(\theta/2)\right)^2 = \frac{1}{2}\left(1 + 2\cos(\theta/2)\sin(\theta/2)\right), && (58)\\ P_+ &= \frac{1}{2}\left(1 + \sin\theta\right). && (59) \end{aligned}$$

If the state $|\chi^-_{\hat{\mathbf{n}}}\rangle$ is passed through the second magnet, then

$$P_- = \left|\langle\chi^+_{\hat{\mathbf{x}}}|\chi^-_{\hat{\mathbf{n}}}\rangle\right|^2 \tag{60}$$

The state $|\chi^-_{\hat{\mathbf{n}}}\rangle$ can be obtained by the transformation $\theta \to \theta + \pi$ in $|\chi^+_{\hat{\mathbf{n}}}\rangle$, then

$$|\chi^-_{\hat{\mathbf{n}}}\rangle = \cos(\theta/2 + \pi/2)\,|\alpha\rangle + \sin(\theta/2 + \pi/2)\,|\beta\rangle = -\sin(\theta/2)\,|\alpha\rangle + \cos(\theta/2)\,|\beta\rangle, \tag{61}$$

and

$$\begin{aligned} P_- &= \left|\langle\chi^+_{\hat{\mathbf{x}}}|\chi^-_{\hat{\mathbf{n}}}\rangle\right|^2 = \left|\left[\frac{1}{\sqrt{2}}\left(\langle\alpha| + \langle\beta|\right)\right]\left[-\sin(\theta/2)\,|\alpha\rangle + \cos(\theta/2)\,|\beta\rangle\right]\right|^2, && (62)\\ &= \frac{1}{2}\left(-\sin(\theta/2) + \cos(\theta/2)\right)^2 = \frac{1}{2}\left(1 - 2\cos(\theta/2)\sin(\theta/2)\right), && (63)\\ P_- &= \frac{1}{2}\left(1 - \sin\theta\right). && (64) \end{aligned}$$

Note that $P_+ + P_- = 1$.