PHAS3424 THEORY OF DYNAMICAL SYSTEMS

Dr Carla Figueira de Morisson Faria
Department of Physics and Astronomy
University College London*

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1. Introduction

Emphasis: Continuous Dynamical Systems and Chaos

- As much as possible, the examples and applications provided in this course will be drawn from physics.
- One should note, however, that dynamical systems are more general and may be found in other areas of knowledge, such as mathematics, biology, finance, etc.

1.1 Course Outline

Part I. Overview

The aim of this section is to place these systems in a broader context, and introduce several concepts which will be used throughout the course.

1. General Definitions and Concepts
2. Examples

*Many thanks to Aden Lam for providing the first typed version of these notes

Part II. Lagrangian and Hamiltonian Mechanics

The aim of this section is to bring all students to the same level of knowledge. We will keep this topic as brief as possible, and focus on the necessary concepts, so that all can follow the subsequent parts of the course. This is particularly important for physicists. It should take around two weeks.

1. Advantages/relation to Newtonian mechanics
2. Generalized coordinates
3. Derivation of Euler-Lagrange equations
   - The D'Alembert principle
   - The principle of the least action
4. Hamiltonian mechanics
   - Hamilton's equations
   - Time evolution of the Hamiltonian
   - Poisson bracket
   - Liouville's theorem
   - Applications

Part III. Nonlinear Dynamical Systems

This is the *core* of the course. We will focus on the necessary mathematical techniques in order to treat continuous dynamical systems, starting from linear systems and moving towards nonlinear dynamical systems.

1. Linear systems
   - Preliminary notions
   - Definition
   - Classification of fixed points
   - Non-simple linear systems
2. Analysis of nonlinear differential equations
   - Linearization at a fixed point
   - The linearization theorem
3. Stability of a point in phase space

   (a) Definition
   (b) Limit cycles
   (c) Structural stability of solutions
   (d) Liapunov stability theorem

4. Conservative systems
   (a) First integrals
      – Definition
      – Consequences
   (b) Applications

5. Bifurcations
   (a) Historic remarks and definition
   (b) Types of bifurcations
      – Turning point
      – Transcritical bifurcation
      – Pitchfork bifurcation
      – Hopf bifurcation

Part IV. Introduction to Chaos

This part of the course will be slightly reformulated with regard to what has been done last year, so the outline below is tentative.

(a) Generalities

(b) Characteristics

(c) The Poincaré-Bendixson theorem

(d) Attractors

(e) The Lorenz equations
   i. Generalities
   ii. Fixed points
   iii. Boundedness
   iv. Existence of attractors

(f) Discrete maps

1.2. Literature

- Lagrangian and Hamiltonian Mechanics
  - T. W. Kibble and F. H. Berkshire, *Classical Mechanics* (Imperial College Press, 2004): Very nice book for examples and quite complete.
  - Goldstein, Poole and Safko, *Classical Mechanics* (Addison Wesley, 2002): Very clear and more formal than Kibble and Berkhsire; nice outline; goes straight to the point. I am using it a lot!
- Dynamical systems in general
  - D. K. Arrowsmith, C. M. Place, *Dynamical Systems* (Chapman and Hall, 1992): Quite mathematical (some physicists may need to get used to the style), but very clear as far as linear dynamical systems and fixed points are concerned. I have been taking a lot of examples from here for the classification of fixed points, linearization and linear systems. Very nice examples and definitely worth using it.
  - S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Perseus, 1994)
  - P. G. Drazin, *Nonlinear Systems* (CUP, 1994): Very nice explanation of bifurcations and of the Lorenz equations!

Please note:

- This list is non-exhaustive and books may be added/deleted from it during the course.
- You will find all these books in the UCL library. There are many copies of Arrowsmith and Place and of Goldstein, Poole and Safko in the maths section.
- Links to supplementary readings/material will be added in the moodle page.

Part I. Overview: Dynamical Systems

1. General Definitions and Concepts

1. **Defn**: A dynamical system describes the evolution of some quantities $x_1, x_2, ..., x_n$, whose meaning will be specified later, of a physical, chemical, biological, etc, system as a function of time or a similar variable (for instance, temperature, density, etc).

   Key question: We would like to determine (i) how a particular state of the system evolves in time subject to specific rules and (ii) what kinds of solutions these rules allow.

2. **State of a system:** Described by a collection of a continuous (or discrete) parameters at a particular time $t_0 : x(t_0), x_2(t_0), ..., x_n(t_0)$.

3. **Phase space (or state space)**: The space $\chi(t, \vec{x})$ of all possible states of the system.

4. **Law of Evolution (or equation of motion)**: Using these laws, one can predictt $\chi(t, \vec{x})$ if we know the initial state $\chi(t_0, \vec{x}_0)$

   In general, a dynamical system is of the form

   $$\dot{\vec{x}} = \vec{F}(\vec{x}), \quad \text{with} \quad \vec{x} = (x_1, x_2, ..., x_n) \tag{1}$$

   Please note: This definition is applicable to a *continuous* dynamical system. The evolution of a *discrete* dynamical system is described by finite-difference equations.

2. Example: Hamiltonian Systems

They are a particular subclass of dynamical systems, which play a central role in physics

- **Variables:**
  - Coordinates: $\vec{q} = (q_1, ..., q_n)$
  - Momenta : $\vec{p} = (p_1, ..., p_n)$
- **Phase space:** $\mathrm{R}^{2n} = (\vec{q}, \vec{p})$

- **Laws of evolution:** Hamilton's Equations:

$$\frac{d\vec{q}}{dt} = \frac{\partial H(\vec{p},\vec{q})}{\partial \vec{p}}; \quad \frac{d\vec{p}}{dt} = -\frac{\partial H(\vec{p},\vec{q})}{\partial \vec{q}}. \tag{2}$$

  These laws of evolution can also be written in terms of components, i.e.,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{3}$$

  where $i = 1, 2, ...n$.

They describe how a Hamiltonian system evolves as a function of time.

Part II. Lagrangian and Hamiltonian Mechanics

1. Advantages with regard to Newtonian mechanics
2. Generalized coordinates
3. Derivation of Euler-Lagrange equations
   (a) The D'Alembert principle
   (b) The principle of the least action
4. Hamiltonian mechanics
   (a) Hamilton's equations
   (b) Time evolution of the Hamiltonian
   (c) Poisson bracket
   (d) Liouville's theorem
   (e) Applications

1 Advantages with regard to Newtonian Mechanics

Question: Given that we can determine a system's dynamics using Newton's equations of motion, why are the Lagrangian and Hamiltonian formulations useful or necessary?

1.1 Problem

Let us consider system of many particles, whose motion we wish to describe:

- Particles $i = 1, ..., N$
- Masses m_i
- Position vectors $\vec{r_i} = (x_i, y_i, z_i)$

Using Newton's equations of motion

$$m_i, \ddot{\vec{r_i}} = \vec{F_i}(\vec{r_1}, ..., \vec{r_N}, \dot{\vec{r_1}}, ..., \dot{\vec{r_N}}, t), \tag{4}$$

where $\ddot{\vec{r_i}}$ = acceleration of each particle and $\vec{F_i}(\vec{r_1}, ..., \vec{r_N}, \dot{\vec{r_1}}, ..., \dot{\vec{r_N}}, t)$ = sum of all forces acting upon each particle. These forces may be external and/or internal interactions between the particles.
This gives a system of $3N$ second-order differential equations, which must be solved. Let us focus on the strategy we adopt when using this formulation.

1.2 Strategy

1. Identify all forces along the x, y, z axes for each particle.
2. Using these forces, write Newton's equations of motion.
3. Identify all possible constraints.

   Meaning: Coordinates/velocities in this system may be inter-related such that the motion is *constrained*.

   Examples:

- Rigid bodies: the distances between the particles are kept unchanged.
- A particle placed on the surface of a sphere can only move if $r^2 - a^2 \geq 0$, where a is the radius of the sphere.

Difficulties posed by constraints:

(a) The coordinates $\vec{r_i}$, and thus the equations of motion, are no longer independent.

(b) The "forces of constraint" are problem-specific. These forces are only known by their effect on the motion.

4. Solve the problem.

Difficulties posed by the strategy: For complex problems (e.g., systems with many particles), the strategy adopted may be extremely cumbersome and hence lead to errors.

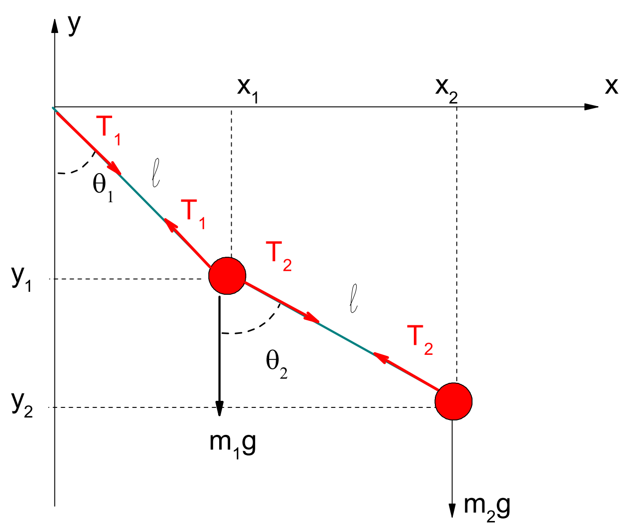
We will next discuss a more efficent way of addressing this type of problem, which will be developed for *holonomic* constraints.

Defn: If the constraint is such that $f(\vec{r_1}, \vec{r_2}, ..., t) = 0$, it is said to be *holonomic.*

For a discussion of other types of constraints and how to treat them see Goldstein, Poole and Safko, Classical Mechanics, in case you are interested.

1.3 Example

Let us consider two linked pendula of length l, composed of two particles of mass m_i $(i = 1, 2)$, whose motion occurs in the xy plane. We will write the equations of motion for each particle in this system using cartesian coordinates.



Particle 1:

$$m_1\ddot{x}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

$$m_1\ddot{y}_1 = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g$$

Particle 2:

$$m_2\ddot{x}_2 = -T_2 \sin\theta_2$$

$$m_2\ddot{y}_2 = T_2 \cos\theta_2 - m_2 g,$$

with $\sin\theta_1 = x_1/l$ and $\sin\theta_2 = (x_2 - x_1)/l$.

Constraints:

$$x_1^2 + y_1^2 = l^2$$

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l^2$$

(holonomic).

Issues:

1. In this simple case we have 2 forces of constraints - the lengths of the two pendulae are constrained to be l.

2. It may be easier to describe the dynamics of this system in terms of the two angles θ_1, θ_2 and the length l (given).

In order to avoid these issues, we intend to seek a new set of coordinates which do not exhibit these problems.

2 Generalized coordinates

A system of N particles free from constraints has $3N$ independent coordinates, i.e., it has $3N$ degrees of freedom. Suppose now there exists k holonomic constraints.

There are two ways of solving this problem:

1. Employ k equations determining the constraints to eliminate k variables.

2. Introduce $3N - k$ independent variables $q_1, q_2, ..., q_{3N-k}$ such that

$$\vec{r}_1 = \vec{r}_1(q_1, q_2, ..., q_{3N-k}, t) \tag{5}$$

...

$$\vec{r}_N = \vec{r}_N(q_1, q_2, ..., q_{3N-k}, t), \tag{6}$$

which contains the constraints in them implicitly.

Example: For the double pendulum discussed here, one may either consider x_1, x_2, y_1, y_2 and eliminate two coordinates, or consider the angles θ_1, θ_2 as variables from the start.

Please note: These coordinates do not necessarily have the dimensions of length.

3 Derivation of Euler-Lagrange's Equations

These equations may be derived either directly from Newton's equations of motion or from the principle of the least action. Here, we will address both situations.

3.1 D'Alembert's principle

- Convenient for us, as it starts from Newton's equations of motion.
- Uses a coordinate transformation from standard coordinates $\vec{r}_i$ to the generalized coordinates q_i.
- It is an important part of the derivation to assume the absence of constraints for the coordinates q_i.

Let us consider Newton's equation of motion

$$\vec{F}_i = m_i \ddot{\vec{r}}_i, \tag{7}$$

related to the $i-th$ particle in our system. This equation may be rewritten as

$$\vec{F}_i - m_i \ddot{\vec{r}}_i = 0. \tag{8}$$

Summing each of the individual forces, we may also write

D'Alembert's Principle

$$\sum_i (\vec{F}_i - m_i \ddot{\vec{r}}_i).\delta\vec{r}_i = 0, \tag{9}$$

where $\delta\vec{r}_i$ is a virtual displacement ("virtual" in the sense that it does not affect the system's forces and constraints). Physically, D'Alembert's principle states that the sum of the work by all internal forces in the system is vanishing.
We are now interested in

- eliminating the forces of constraint.
- considering all $3N-k$ independent degrees of freedom.

Hence, it is convenient (i) to assume that the work performed by the forces of constraint vanishes; (ii) to work in generalized coordinates.
Considering the transformation

$$\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_{3N-k}, t), \tag{10}$$

we find, using the chain rule, that

$$\delta\vec{r}_i = \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \tag{11}$$

and

$$\dot{\vec{r}}_i = \vec{v}_i = \frac{d\vec{r}_i}{dt} = \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}. \tag{12}$$

We will now insert Eq. (11) into Eq. (9). This gives

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{i,j} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j, \tag{13}$$

where

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \tag{14}$$

are the components of the *generalized force*, and

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j. \tag{15}$$

Please note: Q_j does not always have the dimensions of force, but $Q_j\delta q_j$ must always have the dimension of work.

Using the derivative of a product,

$$\sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\delta q_j = \sum_{i,j}\left[\frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\right]\delta q_j. \tag{16}$$

We will still perform some further rewriting so that the above equation is given in a more convenient form. We will use the fact that

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial \vec{v}_i}{\partial q_j} \tag{17}$$

in (12), so that

$$\frac{\partial \vec{v}_i}{\partial q_j} = \sum_k \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}. \tag{18}$$

Note as well that

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\frac{\partial \dot{q}_k}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j} \tag{19}$$

as $\partial\dot{q}_k/\partial\dot{q}_j = \delta k_j$. Bringing (18) and (19) together we may rewrite

$$\sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i,j}\left[\frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j}\right]. \tag{20}$$

Using now

$$m_i\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}m_i v_i^2\right) \tag{21}$$

and

$$m_i\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial q_j} = \frac{\partial}{\partial q_j}\left(\frac{1}{2}m_i v_i^2\right), \tag{22}$$

Eq. (9) can be rewritten as

$$\sum_i (\vec{F}_i - m\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j \right\} \delta q_j = 0, \tag{23}$$

where

$$T = \sum_i \frac{1}{2}m_i v_i^2 \tag{24}$$

is the system's kinetic energy.
We have assumed that the generalized coordinates q_j are all independent, i.e., that all constraints have been eliminated upon the variable transformation. This implies that any virtual displacement δq_j is independent of any other virtual displacement δq_k. Hence, Eq. (23) only holds if

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = 0. \tag{25}$$

Assuming further that $\vec{F}_i = -\nabla_i V$, where V does not depend on the generalized velocities, one may write

$$Q_j = -\sum_i \nabla_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}. \tag{26}$$

This gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0. \tag{27}$$

Since V does not depend on the generalized velocities, one may write Eq. (27)

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0. \tag{28}$$

Defining $L = T - V$ as the Lagrangian, Eq. (28) is given in its usual form, namely

Euler-Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial(T-V)}{\partial \dot{q}_j}\right)-\frac{\partial(T-V)}{\partial q_j}=0, \tag{29}$$

$j=1,...,3N-k$.

Please note: Nowhere in the derivation have we referred to any specific set of coordinates. Euler-Lagrange's equations are valid for any set of generalized coordinates that describe the $3N-k$ degrees of freedom of the system.

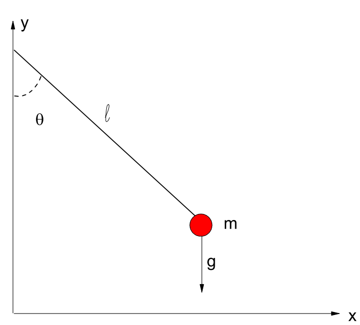
Examples:

1. Single particle moving in 1D, with coordinate x, in the potential $V(x)$

   (a) Lagrangian:
   $$L=\frac{1}{2}m\dot{x}^2-V(x),$$
   so that
   $$\frac{\partial L}{\partial \dot{x}}=m\dot{x};\qquad \frac{\partial L}{\partial x}=-\frac{\partial V}{\partial x}.$$

   (b) Euler-Lagrange equation:
   $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0\Rightarrow m\ddot{x}=-\frac{\partial V}{\partial x}.$$

2. Single pendulum



(a) Lagrangian:

$$L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta),$$

so that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}; \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta.$$

(b) Euler-Lagrange equation:

$$ml\ddot{\theta} = -mg\sin\theta$$

Equation of motion in θ.

Please note:

- θ is an angle, i.e., it is a generalized coordinate but it does not have the dimension of length.
- The angle θ is a more appropriate coordinate due to the constraint $(l-y)^2 + x^2 = l^2$.
- Neither the generalized momentum nor the generalized force have the correct dimensions of momentum and force. In fact,
  - Generalized momentum: $p_\theta = ml^2\dot{\theta}$ (angular momentum).
  - Generalized force: $Q_\theta = -mgl\sin\theta$ (torque).

Please note: The procedure in Examples 1 and 2 is less cumbersome than that used in Newtonian mechanics, as it suffices to write down the Lagrangian and apply Euler-Lagrange's equation. It is not necessary to identify all forces acting upon each particle of the system, etc.

3.2 Principle of the Least Action

This derivation of Euler-Lagrange's equations:

- employs concepts of variational calculus.
- is simpler than that of the previous section.
- introduces several concepts used in other areas, such as quantum mechanics, the theory of fields, etc.

Let us consider the Lagrangian:

$$L = L(\dot{q}_i, q_i, t) = T - V \tag{30}$$

and that the particle motion occurs between two fixed positions $q_i(t_1), q_i(t_2)$ for $(t_1 \leq t \leq t_2)$.
The action of the system is defined by

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt. \tag{31}$$

The motion of the system is such that S is stationary for the actual path of motion, implying

$$\delta S = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)dt = 0. \tag{32}$$

Let us now suppose that $q_i^{(0)}(t)$ represents the true path,and make infinitesimal variations around this: $q_i(t) = q_i^{(0)}(t) + \epsilon_i(t)$, with $\epsilon_i(t_1) = \epsilon_i(t_2) = 0$

$$\begin{aligned} S = S^{(0)} + \delta S &= \int_{t_1}^{t_2} L(q_i^{(0)} + \epsilon_i, \dot{q}_i^{(0)} + \dot{\epsilon}_i, t)dt \\ &= S^{(0)}(q_i^{(0)}, \dot{q}_i^{(0)}) + \int_{t_1}^{t_2} \sum_i [\frac{\partial L}{\partial \dot{q}_i}\dot{\epsilon}_i + \frac{\partial L}{\partial q_i}\epsilon_i]dt. \end{aligned} \tag{33}$$

Integrating the second term on the right-hand side by parts,

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}_i}\dot{\epsilon}_i dt = (\frac{\partial L}{\partial \dot{q}_i})\epsilon_i|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\epsilon_i dt. \tag{34}$$

The first term on the RHS vanishes as $\epsilon_i(t_1) = \epsilon_i(t_2) = 0$. This gives

$$S = S^{(0)}(q_i^{(0)}\dot{q}_i^{(0)}, t) + \int_{t_1}^{t_2} \sum_i \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} \right] \epsilon_i dt. \tag{35}$$

$$\delta S = \int_{t_1}^{t_2} \sum_i \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} \right] \epsilon_i dt. \tag{36}$$

This should be valid for arbitrary ϵ_i. Hence,

Euler-Lagrange Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i} \tag{37}$$

Please note: If L is independent of one of the coordinates q_k, then

$$\frac{\partial L}{\partial q_k} = 0, \qquad \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) = 0.$$

This implies that the generalized momenum p_k, with $\partial L/\partial \dot{q}_k = p_k$, is a conserved quantity.

Example: Particle in a central potential

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2, \qquad V = V(r)$$

Writing Lagrangian:

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

Recalling and using substitution $p_\theta = mr^2\dot{\theta}$

$$\frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{dp_\theta}{dt} = 0$$

Physical interpretation: The angular momentum is conserved.

4 Hamiltonian Mechanics

Instead of employing the generalized coordinates q_i and the generalized velocities $\dot{q}_i$, we use the generalized coordinates q_i and the generalized momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}$.

Advantages: The theoretical framework provided by the Hamiltonian formulation may be applied and/or extended to many areas of physics, such as chaos, quantum mechanics, statistical mechanics, etc.

We will start by defining a new function, the *Hamiltonian*, using a Legendre transformation

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i(p_i, q_i) - L(\dot{q}_i(p_i, q_i), q_i, t). \tag{38}$$

(see Goldstein for more details):

4.1 Hamilton's Equations

Aim: Find a simple set of equations of motion from H. Starting point: Differentiate the hamiltonian w.r.t. p_i, q_i.

$$\frac{\partial H}{\partial q_i} = \sum_j \left[\frac{\partial \dot{q}_i}{\partial q_j} p_i - \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial q_j} - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial q_j} \right] \tag{39}$$

Using $\partial L/\partial \dot{q}_i = p_i$, $\partial q_i/\partial q_j = \delta_{ij}$ and Euler-Lagrange equation,

$$\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} = -\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) = -\dot{p}_i \tag{40}$$

Applying the chain rule, we find

$$\frac{\partial H}{\partial p_i} = \sum_j \dot{q}_j \frac{\partial p_j}{\partial p_i} + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_i}, \tag{41}$$

where $\partial p_j/\partial p_i = \delta_{ij}$ and $\partial L/\partial \dot{q}_j = p_j$. The last terms in (41) cancel out, giving

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{42}$$

Hamilton's equations:

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i, \qquad \frac{\partial H}{\partial p_i} = \dot{q}_i. \tag{43}$$

Please note: While in the Lagrangian description we have $3N - k$ second-order differential equations in terms of the coordinates q_i, in the Hamiltonian formulation we have $2(3N - k)$ first-order differential equations in terms of p_i, q_i. These variables form the *phase space* of this type of system.

Example: Particle in a central potential
Lagrangian

$$L = T - V = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - V(r)$$

where we use the generalized coordinates $q_r = r$ and $q_\theta = \theta$. The generalized momenta are

$$p_r = \frac{\partial L}{\partial \dot{q}_r} = m\dot{r} \qquad p_\theta = \frac{\partial L}{\partial \dot{q}_\theta} = mr^2\dot{\theta}$$

Hamiltonian:

$$H(r, p_\theta, p_r) = \underbrace{p_r\dot{r}}_{\frac{p_r^2}{m}} + \underbrace{p_\theta\dot{\theta}}_{\frac{p_\theta^2}{mr^2}} - \underbrace{\frac{1}{2}m\dot{r}^2}_{\frac{p_r^2}{2m}} - \underbrace{\frac{1}{2}mr^2\dot{\theta}^2}_{\frac{p_\theta^2}{2mr^2}} + V(r)$$

This gives

$$H(r, p_\theta, p_r) = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r),$$

which is the total energy of the system.

Hamilton's equations:

$$\frac{\partial H}{\partial p_r} = \frac{p_r}{m} = \dot{r}; \qquad \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} = \dot{\theta}; \qquad \frac{\partial H}{\partial \theta} = 0 = \dot{p}_\theta; \qquad \frac{\partial H}{\partial r} = \frac{\partial V}{\partial r} - \frac{p_\theta^2}{mr^3} = -\dot{p}_r$$

Please note:

- If $H(p_i, q_i)$ is independent of q_i, p_i is conserved. This can be inferred from the Lagrangian formulation since $\dot{p}_i = \partial L/\partial q_i = -\partial H/\partial q_i$.
  Example: Particle in a central potential. In this case, $\partial H/\partial \theta = 0$, so that p_θ (the angular momentum) is conserved.

- If $\partial H/\partial t = 0$, H is a constant of motion. For the cases we have seen here (holonomic constraints, Lagrangian quadratic in $\dot{q}$) this constant is given by $H = T + V$, which is the total energy of the system.

- Not every dynamical system is a Hamiltonian system. A dynamical system given by the equations

  $$\dot{q} = F_1(q, p); \qquad \dot{p} = F_2(q, p)$$

  is a Hamiltonian system if

  $$\frac{\partial F_1}{\partial q} + \frac{\partial F_2}{\partial p} = 0$$

  .
  Example: 1D Harmonic Oscillator

  $$H(p, q) = \frac{p^2}{2m} + \frac{\omega^2}{2} q^2$$

  Hamilton's equations:

  $$\dot{p} = \underbrace{-\omega q}_{F_2(q,p)}; \qquad \dot{q} = \underbrace{\frac{p}{m}}_{F_1(q,p)}.$$

  This gives

  $$\frac{\partial F_1}{\partial q} = 0; \frac{\partial F_2}{\partial p} = 0 \to \frac{\partial F_1}{\partial q} + \frac{\partial F_2}{\partial p} = 0.$$

4.2 Time Evolution of H

Let us consider the Hamiltonian $H = (p_i, q_i, t)$ and compute

$$\frac{dH}{dt} = \sum_i \left[\frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i\right] + \frac{\partial H}{\partial t}$$

$$\frac{dH}{dt} = \sum_i \left[-\dot{p}_i\dot{q}_i + \dot{q}_i\dot{p}_i\right] + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If H or L has no explicit time dependence, then it is a conserved quantity.
We would like to show that this quantity is the total energy. Let us assume that

$$L = \sum_i \frac{1}{2}\dot{q}_i^2(q_i, p_i) f_i(\{q\}) - V(\{q\}), \tag{44}$$

where $\{q\} = \{q_1, q_2, ..., q_{3N-k}\}$ and f_i is a function of the coordinates q_i.
This gives

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = q_i f_i(\{q\}).$$

Using the Legendre transformation

$$H(p_i, q_i, t) = \sum_i p_i \dot{q}_i(p_i, q_i) - L,$$

$$H = \sum_i \dot{q}_i^2 f_i(\{q\}) - \frac{1}{2}\sum_i \dot{q}_i^2 f_i(\{q\}) + V(\{q\})$$

$$\boxed{H = \frac{1}{2}\sum_i \dot{q}_i^2 f_i(\{q\}) + V(\{q\}) = T + V} \tag{45}$$

Please note: Along a phase-space trajectory, if $\partial H/\partial t = 0$, $H(p_i, q_i)$ is conserved. This will be used in the construction of phase portraits for conservative systems, as we will find curves that follow $H(p_i, q_i) = E$.

4.3 Poisson Bracket

We will now see how H can help us (a) to describe the time evolution of a function $f(q_i, p_i, t)$ where f is an arbitrary quantity and (b) to find whether these quantities are preserved in time.

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial t}.$$

Using Hamilton's eqns:

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}.$$

Rewriting:

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}, \tag{46}$$

where

$$\{f, H\} = \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) \tag{47}$$

is the Poisson bracket $\{f, H\}$ of f and H.

If $\{f, H\} = 0$, then f is a constant of motion.

Analogy: Heisenberg's equation in Quantum Mechanics. The time evolution of an operator $\hat{f}$ is given by

$$\frac{d\hat{f}}{dt} = \frac{1}{i\hbar}[\hat{f}, \hat{H}] + \frac{\partial \hat{f}}{\partial t},$$

where $[\hat{f}, \hat{H}]$ is the commutator of $\hat{f}$ with the Hamiltonian operator $\hat{H}$.

4.3.1 Example

Particle in a spherically symmetric potential $V(r), r = \sqrt{x^2 + y^2 + z^2}$. We wish to show that the angular momentum component $L_x = yP_z - zP_y$ is a conserved quantity.

Poisson bracket:

$$\{L_x, H\} = \frac{\partial L_x}{\partial x}\frac{\partial H}{\partial p_x} - \frac{\partial L_x}{\partial p_x}\frac{\partial H}{\partial x} + \frac{\partial L_x}{\partial y}\frac{\partial H}{\partial p_y} - \frac{\partial L_x}{\partial p_y}\frac{\partial H}{\partial y} + \frac{\partial L_x}{\partial z}\frac{\partial H}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial H}{\partial z},$$

where

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(r).$$

Partial derivatives:

- Of L_x:

$$\frac{\partial L_x}{\partial x} = 0 = \frac{\partial L_x}{\partial p_x}, \frac{\partial L_x}{\partial y} = p_z, \frac{\partial L_x}{\partial p_y} = -z, \frac{\partial L_x}{\partial z} = -p_y, \frac{\partial L_x}{\partial p_z} = y$$

- of H:

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m}; \quad \frac{\partial H}{\partial p_z} = \frac{p_z}{m}; \quad \frac{\partial H}{\partial y} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} = \frac{\partial V}{\partial r}\frac{y}{r};$$

$$\frac{\partial H}{\partial z} = \frac{\partial V}{\partial r}\frac{z}{r}.$$

This gives

$$\{L_x, H\} = \cancel{p_z\frac{p_y}{m}} + z\frac{\partial V}{\partial y} - \cancel{p_y\frac{p_z}{m}} - y\frac{\partial V}{\partial z} = \frac{zy}{r}\frac{\partial V}{\partial r} - \frac{zy}{r}\frac{\partial V}{\partial r} = 0$$

Since $\partial L_x/\partial t = 0$, the full derivative $dL_x/dt = 0$ and L_x is a constant of motion.

Please note:

- A similar property can be obtained in Quantum Mechanics, using operators and commutators.
- When the number of conserved quantities is at least equal to the number of degrees of freedom, the system is integrable. Otherwise it is chaotic.

.

4.3.2 Properties of the Poisson Bracket

Let us consider the Poisson bracket for any two arbitrary functions $f(x_1, x_2)$ and $g(x_1, x_2)$,

$$\boxed{\{f, g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_1}}$$

Poisson brackets

- are linear

$$\{kf + \lambda g, h\} = \{kf, h\} + \{\lambda g, h\} = k\{f, h\} + \lambda\{g, h\},$$

where k and λ are arbitrary constants and $h(x_1, , , x_2)$ a third arbitrary function.

- are antisymmetric

$$\{f, g\} = -\{g, f\}$$

- satisfy the Jacobi identity

$$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0$$

- obey the Leibniz Rule

$$\{f,gh\}=g\{f,h\}+\{f,g\}h$$

Please note: Similar properties exist in quantum mechanics with regard to commutators.

4.4 Liouville's Theorem

Key idea: The ensemble of system points moving through phase space behave like an incompressible fluid in phase space.

This is useful for systems with many degrees of freedom, i.e., many particles, as in this case one may describe its evolution in terms of its phase-space density.

Starting point: The fluid velocity in an incompressible fluid satisfies $\nabla\cdot\vec{v}=0$. This argument may be generalised to n dimensions.

One may define the state of system in phase space as the vector

$$\vec{\chi}=(q_1,...,q_n,p_1,...,p_n),$$

so that the "velocity" in this case is

$$\dot{\vec{\chi}}=(\dot{q}_1,...,\dot{q}_n,\dot{p}_1,...,\dot{p}_n).$$

This gives

$$\nabla\cdot\dot{\vec{\chi}}=\frac{\partial\dot{q}_1}{\partial q_1}+...+\frac{\partial\dot{q}_n}{\partial q_n}+\frac{\partial\dot{p}_1}{\partial p_1}+...+\frac{\partial\dot{p}_n}{\partial p_n}=0.$$

Note that:

$$\frac{\partial\dot{q}_i}{\partial q_i}+\frac{\partial\dot{p}_i}{\partial p_i}=\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right)+\frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right)=0$$

for $i=1,...,n$.

$$\rightarrow\sum_{i=1}^{n}\frac{\partial\dot{q}_i}{\partial q_i}+\frac{\partial\dot{p}_i}{\partial p_i}=\nabla\cdot\dot{\vec{\chi}}=0 \qquad (48)$$

5 Applications

Aims: Using examples from Lagrangian & Hamiltonian mechanics, we intend to:

- Introduce concepts that will be subsequently addressed in a more systematic way such as fixed points, stable equilibrium, unstable equilibrium, etc.
- Highlight the relevance of a more systematic treatment.

5.1 Bead on a circular wire

Let us consider a bead sliding on a frictionless, rotating circular wire, as shown in Fig. 5.1.

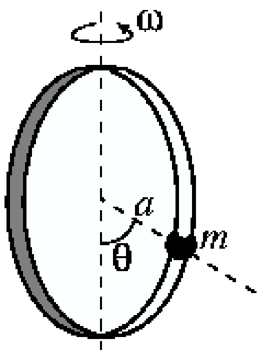


Figure 1: bead of mass m sliding on a frictionless wire of radius a, which rotates with constant angular velocity ω.

Assumptions:

- The wire rotates with constant angular velocity ω, with regard to the z-axis.
- The bead has mass m.
- The wire has radius a.
- For convenience, we have placed the origin at the centre of the wire loop.

5.1.1 General dynamics

In order to describe the dynamics of this system we must:

1. **Find a convenient set of coordinates:** spherical polar coordinates

- $r = \sqrt{x^2 + y^2 + z^2}$
- $\theta = \arctan \sqrt{x^2 + y^2}/(-z)$
- $\phi = \arctan y/x$

Please note: for convenience, θ is defined with regard to the *lower* z half axis.

2. **Find the constraints:**

$$\dot{\phi} = \omega, \qquad r = a$$

This implies that we just need one independent coordinate to describe this motion: the angle θ.

3. **Compute T, V and the Lagrangian L:**

$$V = mgz = -mga\cos\theta$$

$$T = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}m\omega^2 a^2 \sin^2\theta,$$

where the first term corresponds to the motion along θ, with the velocity $v_\theta = a\dot{\theta}$ and the second term corresponds to the motion along ϕ, with the velocity $v_\phi = \omega a \sin\theta = \omega\sqrt{x^2 + y^2}$.

$$\boxed{L = T - V = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mga\cos\theta}$$

4. **Compute Euler-Lagrange's equation with regard to θ:**

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

RHS:

$$\frac{\partial L}{\partial \theta} = -mga\sin\theta + ma^2\omega^2\sin\theta\cos\theta$$

LHS:

$$\frac{\partial L}{\partial \dot{\theta}} = ma^2\dot{\theta} = p_\theta, \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ma^2\ddot{\theta}$$

Equating LHS = RHS:

$$\boxed{ma^2\ddot{\theta} = -mga\sin\theta + ma^2\omega^2\sin\theta\cos\theta}$$

5. **Write Hamiltonian:**

   Using Legendre's transformation we have

$$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{ma^2} - \left[\frac{1}{2}\cancel{ma^2}\frac{p_\theta^2}{m^{\cancel{2}}a^{\cancel{4}2}} + \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta\right]$$

$$H = \frac{p_\theta^2}{2ma^2} - \frac{ma^2}{2}\omega^2\sin^2\theta - mga\cos\theta$$

6. **Hamilton's equations:**

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ma^2} \qquad \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$$

Please note:

(a) The Hamiltonian is a conserved quantity. In fact,

$$\frac{dH}{dt} = \frac{\partial H}{\partial p_\theta}\dot{p_\theta} + \frac{\partial H}{\partial \theta}\dot{\theta} = \dot{\theta}\dot{p_\theta} - \dot{p_\theta}\dot{\theta} = 0$$

This is expected as $\partial H/\partial t = 0$ (no explicit time dependence).

(b) The Hamiltonian is <u>not</u> the energy T+V. In fact,

$$E = T + V = \frac{p_\theta^2}{2ma^2} + \frac{ma^2}{2}\omega^2\sin^2\theta - mga\cos\theta$$

$\Rightarrow E \neq H$

<u>Explanation:</u>

- Physically, an <u>external</u> force is necessary to keep $\dot{\phi} = \omega$ constant. This implies that, during the process, energy is either given to or removed from the system.
- Note also that the Lagrangian L is not quadratic in the generalized velocities $\dot{q}_i$ (in this case $\dot{\theta}$). This was the assumption employed in Sec. 4.2 to prove that $H = T + V$. In that specific case, the generalized coordinates did not depend explicitly on time, while in this example we have the constraint $\phi = \omega t$.

5.1.2 Equilibrium points

These points require that the condition $\dot{p}_\theta = 0$ be satisfied. In this case,

$$ma^2\omega^2 \sin\theta \cos\theta - mga\sin\theta = 0$$

$$\left[a\omega^2 \cos\theta - g\right]\sin\theta = 0.$$

Solutions:

$$\sin\theta = 0, \qquad \cos\theta = \frac{g}{a\omega^2}.$$

Questions: Are these points stable or unstable? Why?
You will see that the available tools we have at the moment to tackle these questions come from Newtonian mechanics.

If $\sin\theta = 0$, then $\theta = \pi$ (Solution S_1) or $\theta = 0$ (Solution S_2). Both solutions are indicated in the figure below.

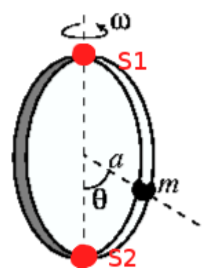


Figure 2: Equilibrium points S_1 and S_2 for the wire in a bead.

- Solution S_1: $\theta = \pi$. The bead sits at the top of the wire. This implies that small displacements from this position will make the bead go down under the joint influence of the gravitational and centrifugal forces. The bead is in **unstable equilibrium**.
- Solution S_2: $\theta = 0$. The bead sits at the bottom of the wire. Is this equilibrium point stable or unstable? Remember that both the gravitational and centrifugal forces are acting on the particle. We will see that this depends on the frequency ω, the radius a and the gravitational acceleration g via a third solution.

- Solution S_3: $\cos\theta_0 = g/(a\omega^2)$.
  - Components of the centrifugal and the gravitational forces in the direction of the wire cancel each other:
$$mg\sin\theta = ma\omega^2\sin\theta\cos\theta$$
$$\Rightarrow \cos\theta_0 = \frac{g}{a\omega^2}.$$
  - **Please note:**
    * $|\cos\theta_0| \leq 1$, so that θ_0 is only real if
$$\frac{g}{a\omega^2} \leq 1 \Rightarrow g \leq a\omega^2.$$
      If this does not hold, the bead will always be pushed towards $\theta = 0$, so that S_2 is **stable**. If $g < a\omega^2$, the bead will be pushed outwards until θ_0. This means that S_2 ($\theta = 0$) will become **unstable** and S_3 ($\theta = \arccos[g/(a\omega^2)]$) will be **stable**.
    * In principle, there is also a fourth solution at the angle $2\pi - \arccos[g/(a\omega^2)]$, on the other side of the bead, which will behave in the same way.
    * if $\omega \to \infty$ the equilibrium angle θ_0 will tend to $\pi/2$.

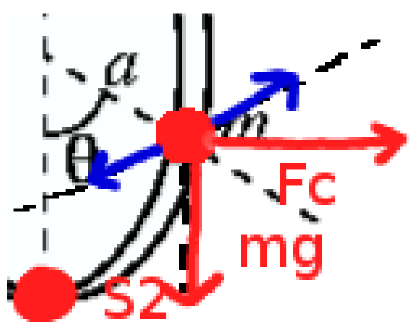


Figure 3: Forces acting upon the bead when it is at the equilibrium angle θ_0 related to the solution S_3. The red arrows indicate the gravitational force mg and the centrifugal forge F_c, while the blue arrows give their components in the direction of the wire. The solution S_2 is also indicated in the figure.

5.1.3 Concepts introduced in this example:

1. Fixed points: Solutions S_i, $(i = 1, 2, 3)$. At these points, the system is in equilibrium.

2. Stability/instability: Solutions S_i, $(i = 1, 2, 3)$ may be stable or unstable.

3. Bifurcations: The number of fixed points and their stability depends on a specific parameter (e.g., ω).

In the following parts of the course, we will deal with such concepts more systematically.