UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**PHAS3201**
ASSESSMENT PATTERN	:	**PHAS3201A**
MODULE NAME	:	**Electromagnetic Theory**
DATE	:	**18-May-09**
TIME	:	**10:00**
TIME ALLOWED	:	**2 Hours 30 Minutes**

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Answer ALL SIX questions in Section A and TWO questions from Section B

The numbers in square brackets at the right-hand side of the text indicate the provisional allocation of maximum marks per question or sub-section of a question.

You may find the following constants and theorems useful.

$$\begin{aligned}\text{Permeability of free space, } \mu_0 &= 4\pi \times 10^{-7}\,\mathrm{H/m}\\ \text{Speed of light in vacuo} &= 3.00 \times 10^{8}\,\mathrm{m/s}\end{aligned}$$

For any scalar G, $\nabla \cdot (\nabla G) = \nabla^2 G$.
For any vector $\mathbf{F}$, $\nabla \times \nabla \times \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$
In cylindrical polar coordinates for any scalar G:

$$\nabla G = \mathbf{i}_R \frac{\partial G}{\partial R} + \mathbf{i}_\phi \frac{1}{R}\frac{\partial G}{\partial \phi} + \mathbf{i}_z \frac{\partial G}{\partial z} \tag{1}$$

For any vector function which can be written $\mathbf{C}(\mathbf{r}, t) = \mathbf{D} \exp i\,(\mathbf{k}\cdot\mathbf{r} - \omega t)$ where $\mathbf{D}$ is a constant, then:

$$\begin{aligned}\nabla \cdot \mathbf{C} &= i\mathbf{k}\cdot\mathbf{C}\\ \nabla \times \mathbf{C} &= i\mathbf{k}\times\mathbf{C}\\ \nabla^2 \mathbf{C} &= -k^2\mathbf{C}\end{aligned}$$

The Divergence theorem and Stokes' theorems are:

$$\begin{aligned}\int_V \nabla \cdot \mathbf{F}\,dv &= \oint_S \mathbf{F}\cdot\mathbf{n}\,da\\ \oint_C \mathbf{F}\cdot d\mathbf{l} &= \int_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,da\end{aligned}$$

The co-variant differential four-vector:

$$\partial_\mu = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{1}{c}\frac{\partial}{\partial t}\right)$$

SECTION A

1. The Ampere-Maxwell equation in vacuum for time-independent fields can be written:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

Write the Ampere-Maxwell equation for linear dielectric and magnetic materials and time-dependent fields. Define any new terms you use. [3]

Show that the magnetic field $\mathbf{B} = 2x^2y\mathbf{i} + (3x^2 - 2xy^2)\mathbf{j}$ is compatible with the equation $\nabla \cdot \mathbf{B} = 0$, and find the current density which gives rise to the magnetic field. [4]

2. Sketch a hysteresis curve (B vs H) for a ferromagnetic material, starting with an unmagnetised material and going around a full saturation magnetisation loop. Define the remanence and the coercivity and show them on the $B-H$ curve. [4]

   Briefly describe important features for a ferromagnetic material to be used as:

   (a) an electromagnet [2]

   (b) a permanent magnet [2]

3. (a) The integral form of Gauss' law can be written:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = Q/\epsilon_0$$

   Define the symbols $\mathbf{E}$ and Q, and give the limits of the integral [3]

   (b) Obtain the differential form of Gauss' law and show how this leads to Poisson's equation $\nabla^2 V = 0$ for the scalar potential when there is no charge; use formulae from the rubric where necessary. [3]

4. (a) Define magnetisation M and explain *briefly* how the saturation of ferromagnetic materials arises. [3]

   (b) A neutron star, radius 25 km and mass 6×10^{30} kg, is made of neutrons (mass 1.67×10^{-27} kg with magnetic moment 9.6×10^{-27} A m^2). Calculate the saturation value of its magnetisation. [4]

5. (a) The plasma frequency is defined as:

$$\omega_P = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}$$

   Define the symbols N_e, m_e, e, ϵ_0. [4]

   (b) The dispersion relation for a plasma can be written $\omega^2 = \omega_P^2 + k^2 c^2$. Define k and show that the product of the phase and group velocities is $v_p v_g = c^2$. [2]

6. Two of Maxwell's equations can be written:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \cdot \mathbf{D} &= \rho_f \end{aligned}$$

   (a) Define D and ρ_f [2]

   (b) Use these equations to show that, under certain conditions, the boundary conditions on components of D perpendicular to an interface between two materials (labelled 1 and 2) can be written $D_1^\perp = D_2^\perp$. What physical conditions must apply for this to be true ? [4]

SECTION B

7. (a) The Ampere-Maxwell equation can be written:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

Find the integral form of the equation, defining any terms you may use. [2]

(b) Consider a parallel plate capacitor with large circular plates lying parallel to the $x - y$ plane with an applied potential which gives rise to an electric field $\mathbf{E}(r) = E_0 e^{i\omega t} \hat{\mathbf{z}}$.

Using the Ampere-Maxwell equation for time *dependent* fields, and considering a circular path, show that this induces a magnetic field with magnitude:

$$B(r) = \frac{1}{2} \mu_0 \epsilon_0 i \omega r E_0 e^{i\omega t},$$

where cylindrical polar coordinates are being used.

What is the direction of the magnetic field ? [8]

(c) What is the physical significance of the factor of i in this expression for the magnetic field ? [2]

(d) What is the direction of the Poynting vector in this region ? [2]

(e) Faraday's law of induction can be written in differential form as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Show that this can be written as $\oint \mathbf{E} \cdot d\mathbf{l} = -d\Phi/dt$. [2]

(f) The magnetic field in the capacitor will induce a changing electric field. Using Faraday's law, and by considering a rectangular loop lying in a radial plane perpendicular to the plates, show that, to first order in r, the total electric field can be written:

$$\mathbf{E}(r, t) = \left[1 - \left(\frac{kr}{2} \right)^2 \right] E_0 e^{i\omega t} \hat{\mathbf{z}},$$

where k is an appropriate wavevector. Define k. [8]

(g) Now the capacitor is filled with an imperfect dielectric with relative permittivity ϵ_r and conductivity g. Show that the induced magnetic field changes to:

$$B = \frac{1}{2} \mu_0 r (g + i\omega \epsilon_r \epsilon_0) E_0 e^{i\omega t}$$

[6]

8. (a) The contravariant current/charge density four-vector can be written as:

$$j^\mu = (J_1, J_2, J_3, c\rho),$$

where c is the speed of light in vacuum.
Define the terms J_1, J_2, J_3, ρ in the equation. [2]

(b) Show that it can be used with the covariant differential four-vector, given in the rubric, to write the continuity equation. Consider a set of charges and currents being observed in a reference frame S which are found experimentally to obey the continuity equation. What does this imply about these charges and currents in another reference frame S' ? [6]

(c) Consider a neutral wire, at rest in frame S, of cross-section A carrying a current I (evenly distributed across its area) lying along the x-axis.
Write down the current/charge density four vector for any point on the wire. [2]

(d) The Lorentz transform for a general contravariant four-vector ($x^\mu = (x^1, x^2, x^3, x^4)$) can be written as:

$$\begin{aligned} x'^1 &= \gamma\left(x^1 - \beta x^4\right) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \\ x'^4 &= \gamma\left(x^4 - \beta x^1\right), \end{aligned}$$

where $\beta = v/c, \gamma = 1/\sqrt{1-\beta^2}$. Write down the current/charge density four-vector relative to a reference frame S' moving with velocity $\mathbf{v} = (v, 0, 0)$ with respect to S, and show that the wire becomes charged, with charge density $\rho' = -\beta\gamma I/cA$. [4]

(e) A rectangular loop of wire, also with cross-section A, lying in the positive quadrant of the $x - y$ plane with side lengths a (along x) and b (along y) carries a current I (flowing clockwise when viewed from above). What is the magnetic dipole moment, $\mathbf{m}$? Be sure to give a vector answer. [2]

(f) Write down the current/charge density four-vector for each side of the loop in a frame S where the loop is at rest with one corner at the origin. [4]

(g) Using the Lorentz transforms, write down these four-vectors relative to the frame S' used above. Comment *briefly* on the charge densities: is the system still charge neutral ? [4]

(h) Explain why $\mathbf{m}' = \mathbf{m}$, where $\mathbf{m}'$ is the magnetic dipole moment in S' [2]

(i) The charged sides can be thought of as forming an electric dipole in S' (particularly far from the loop). By considering the total charge along each side as a single charge, and using the current/charge density four vector in S', show that the electric dipole moment can be written:

$$\mathbf{p}' = \frac{1}{c^2}\mathbf{m} \times \mathbf{v}.$$

[4]

9. (a) The following identity can be written for an electromagnetic field in a fixed volume V bounded by a surface S,

$$-\oint_S \mathbf{E}\times\mathbf{H}\cdot\mathbf{n}da = \frac{\partial}{\partial t}\int_V \frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})\,dv + \int_V \mathbf{J}\cdot\mathbf{E}dv$$

Give the physical significance of the two integrals on the right hand side, and hence give a physical interpretation of the Poynting vector, $\mathbf{N} = \mathbf{E}\times\mathbf{H}$. [6]

(b) A co-axial cable which lies along the z-axis consists of an inner cylindrical conductor, of radius a and an outer, earthed cylindrical conductor of radius b with the gap between the two filled with a dielectric. The inner conductor carries a current I when the potential difference between the two wires is V_{ab}.

i. Find $\mathbf{H}$ as a function of r for $a < r < b$, using the Ampere-Maxwell law. [4]

ii. Find $\mathbf{E}$ in terms of the potential $V(r)$ for $a < r < b$ using cylindrical polar coordinates. [N.B. You do not need to find an expression for V; however, you will need to think about what coordinates it depends on]. [4]

iii. Hence show that the magnitude of the Poynting vector can be written:

$$N = \frac{I}{2\pi r}\frac{dV}{dr}$$

What is its direction ? [4]

iv. Consider a cylindrical surface between the two conductors and co-axial with the cable. How much power is transmitted through the curved sides ? [1]

v. Write down an expression for the power, dP, transmitted through a small circular ring of width dr at radius r in one end of the surface. By integrating over the ends of the cylinder, show that the total power transmitted along the cable $P = VI$. Justify your answer carefully. [Remember that power is surface integral of the Poynting vector, and use cylindrical polar coordinates]. [7]

(c) The estimated safe upper limit for human exposure to electromagnetic radiation at microwave frequencies is 10 $\mathrm{Wm^{-2}}$. If the amplitude of the electric field from a mobile phone transmitter decays with distance d as $E(d) = 2500/d$ V/m, what is the closest safe distance of approach ? You may assume that $\epsilon_r = \mu_r = 1$ for air. [4]

10. (a) Consider a linear medium with finite conductivity g but *no* free (or external) charges. Write down the current density in the medium, $\mathbf{J}$, in terms of an applied electric field $\mathbf{E}$. [2]

(b) Write down Maxwell's equations for the medium. [6]

(c) Consider a plane wave whose electric field can be written:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$$

Show, using the Maxwell equations you have just written down, that the dispersion relation for this plane wave is:

$$\frac{k^2}{\omega^2} = \mu_0 \mu_r \epsilon_0 \epsilon_r \left(1 + i \frac{g}{\epsilon_0 \epsilon_r \omega}\right)$$

[4]

(d) State the limits in which the medium can be described as a poor conductor or a good conductor, referring to the dispersion relation. [2]

(e) What is a skin depth ? Show, using the dispersion relation above and substituting into the form for $\mathbf{E}$, that the skin depth for a *good* conductor can be written as:

$$\delta = \sqrt{\frac{2}{\omega \mu_0 \mu_r g}}$$

[6]

(f) Starting from the dispersion relation, and using an appropriate expansion for small quantities, show that a skin depth can also be defined for *poor* conductors, written as:

$$\delta = \frac{2}{g} \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0 \mu_r}}$$

[6]

(g) Calculate the conductivity of, and phase velocity in, a non-magnetic material whose skin depth is 2.1μm at 10^{11} Hz. [4]