Q1 2014

(1)

a) $\langle\psi|A|\phi\rangle = \langle A\psi|\phi\rangle = \langle\phi|A|\psi\rangle^*$

$= \langle\psi|A^*|\phi\rangle = \langle\phi|A|\psi\rangle^*$

$\langle\psi|\hat{A}|\chi\rangle = (\langle\chi|\hat{A}^\dagger|\psi\rangle)^*$

b) $\hat{A}$ is Hermitian, i.e. $\langle\psi|\hat{A}|\chi\rangle = (\langle\chi|\hat{A}|\psi\rangle)^*$

$\langle\phi_n|\hat{A}|\phi_n\rangle = a_n$

Hermitian conjugate of both sides $(\langle\phi_n|\hat{A}|\phi_n\rangle)^* = a_n^*$

$\langle\phi_n|\hat{A}|\phi_n\rangle = a_n^*$

$\Rightarrow a_n = a_n^*$

$\Rightarrow$ eigenvalues of $\hat{A}$, a Hermitian operator, are real

) $|\psi\rangle = \sum_n c_n|n\rangle$ ⇐ expansion of $|\psi\rangle$ in terms of the complete orthonormal states $\{|n\rangle\}$

where ~~$c_n = \langle\psi|n\rangle$~~

$c_n = \langle n|\psi\rangle$

$\langle m|n\rangle = \delta_{mn}$

where $\delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}$

$\Rightarrow |\psi\rangle = \sum_n \langle n|\psi\rangle|n\rangle$

$= \sum_n \langle\psi|n\rangle\langle n| = \left(\sum_n |n\rangle\langle n|\right)|\psi\rangle = |\psi\rangle$

$\Rightarrow \sum_n |n\rangle\langle n| = 1$

Q2

a) Two observables $\hat{A}$ & $\hat{B}$ are compatible if they share the same set of eigenvectors, i.e. $\hat{A}|\phi_n\rangle = a_n|\phi_n\rangle$ & $\hat{B}|\phi_n\rangle = b_n|\phi_n\rangle$.
This implies that if a measurement of A yielding a_n is followed by a measurement of B (yielding b_n), then a subsequent measurement of A should still yield a_n.

Commutation relation: $[\hat{A},\hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

Q2

b) $(\Delta\hat{A})^2 = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$ or $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$

difference of a measurement of $\hat{A}$ from the expectation value

so $(\Delta\hat{A})^2 = \hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2$

$\langle\Delta\hat{A}^2\rangle = \langle\psi|(\Delta\hat{A})^2|\psi\rangle$

$= \langle\psi|\hat{A}^2 - 2\hat{A}\langle\hat{A}\rangle + \langle\hat{A}\rangle^2|\psi\rangle$

$\langle\Delta\hat{A}^2\rangle = \langle\hat{A}^2\rangle - 2\langle\hat{A}\rangle^2 + \langle\hat{A}\rangle^2$

$\Rightarrow \langle\Delta\hat{A}^2\rangle = \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2$

c) $\Delta A\,\Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

The above relationship, i.e. the generalised uncertainty relationship, implies that for two non-commuting operators (e.g. $[\hat{L}_x,\hat{L}_y] = i\hbar L_z$), the uncertainty on the product of the two measurements is non-zero. This means that these two measurements cannot be exactly known at the same time in quantum mechanics

e.g $\Delta x\,\Delta p_x \geq \hbar/2$, $\Rightarrow$ we cannot know position & momentum of a particle with infinite precision. If we know x exactly, the momentum has ∞ uncertainty & the converse is also true.

Q3

a) $|\psi\rangle = \hat{A}|\phi\rangle$, $\{|n\rangle\}$ forms a complete orthonormal set

by the expansion postulate

$|\psi\rangle = \sum_m c_m|m\rangle$ & $|\phi\rangle = \sum_m d_m|m\rangle$

where $c_n = \langle n|\psi\rangle$ & $d_m = \langle m|\phi\rangle$

we can also expand one bases set $|\psi\rangle$ in terms of the other $|\phi\rangle$

$|\phi\rangle = \sum_n \quad |n\rangle = \sum_m \langle m|n\rangle|m\rangle$

where $\{|m\rangle\} \equiv \{|n\rangle\}$ but just with different labels

$|\psi\rangle = \hat{A}|\phi\rangle \longrightarrow \sum c_n|n\rangle =$

we can expand a single basis vector of $|\psi\rangle$, $|n\rangle$ in terms of $|\phi\rangle$

$c_n = \langle n|\psi\rangle$

so $c_n|n\rangle = \sum \hat{A} \sum_m$

Q3 redone

expansion postulate $|\psi\rangle = \sum_n c_n |n\rangle$

$$c_n = \langle n|\psi\rangle$$

c_n can be considered as a column vector

$$\underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

we consider $|\psi\rangle = \hat{A}|\phi\rangle$

$|\phi\rangle$ can also be expanded in the basis set $\{|n\rangle\}$. For convenience we use a different label m,

$$|\phi\rangle = \sum_m d_m |m\rangle \quad , \; d_m = \langle m|\phi\rangle$$

then ~~d_m $\langle m|$~~

$$c_n = \langle n|\psi\rangle = \langle n|\hat{A}|\phi\rangle$$
$$= \sum_m \langle n|\hat{A}|m\rangle d_m$$

& so $\quad c_n = \sum_m A_{nm} d_m$

where $\quad A_{nm} = \langle n|\hat{A}|m\rangle$

d_m too can be thought of as a column vector.

$$\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

& so we arrive at the matrix representation

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \\ A_{n1} & A_{n2} & & A_{nn} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$\underline{c} = \underline{\underline{A}}\,\underline{d}$$

(make $\underline{d} \equiv \underline{b}$ & be happy)

Q4

) Fermions are particles with half integer spins. The Pauli exclusion principle states that two identical particles of ½-integer spins (Fermions) cannot occupy the same quantum state simultaneously.

b) For these two electrons (two indistinguishable particles) we can consider the total wavefunctions (Ψ) to consist of a spatial part (ψ) & a spin part (χ) i.e $\Psi = \psi\chi$. We know that the Pauli Exclusion principle prevents two identical Fermions from occupying the same quantum state simultaneously, i.e. the total wavefunction of electron 1, Ψ_1, must be different from ~~electron~~ that of electron 2, Ψ_2 ($\Psi_1 \neq \Psi_2$). Since we are told that both electrons are in the same spin state ($\chi_1 = \chi_2$), then their respective spatial wave functions must be different ($\psi_1 \neq \psi_2$). More so, these two particles cannot occupy the same space simultaneously otherwise the Pauli Exclusion Principle would be violated & this implies that the probability density of finding the two electrons in the same point in space is zero.

Q5

a) $|\psi\rangle = \frac{\sqrt{3}}{2}|\phi_0\rangle + \frac{1}{2}|\phi_1\rangle$

& we know $\hat{H}|\phi_0\rangle = E_0|\phi_0\rangle$ (ground state)

$\hat{H}|\phi_1\rangle = E_1|\phi_1\rangle$ (first excited state)

we expect $\langle\psi|\psi\rangle = 1$ since we are told that the eigenvectors are normalised.

But this is easy to show explicitly

$\langle\psi|\psi\rangle = \left(\frac{\sqrt{3}}{2}\langle\phi_0| + \frac{1}{2}\langle\phi_1|\right)\left(\frac{\sqrt{3}}{2}|\phi_0\rangle + \frac{1}{2}|\phi_1\rangle\right)$

since we know that the eigenvectors must be orthonormal

then

$\langle\psi|\psi\rangle = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$

b) i) $P(E=E_0) = (\langle\phi_0|\psi\rangle)^2 = \frac{3}{4}$

$P(E=E_1) = (\langle\phi_1|\psi\rangle)^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

ii) energy expectation value $= \langle E\rangle = \langle\psi|\hat{H}|\psi\rangle$

$= \left(\frac{\sqrt{3}}{2}\langle\phi_0| + \frac{1}{2}\langle\phi_1|\right)\left(\frac{\sqrt{3}}{2}E_0|\phi_0\rangle + \frac{1}{2}|\phi_1\rangle\right)$

$= \frac{3}{4}E_0 + \frac{1}{4}E_1$

Q6

a) $\hat{S}_x = \frac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

$(|l, m_l\rangle)$

What basis? basis vectors $|\alpha\rangle = |1/2, 1/2\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|\beta\rangle = |1/2, -1/2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

Where m_l is the projection of spin onto the x-axis

b) $\hat{S}_x$ is diagonalised so ~~eigenvector~~ the eigenvalues are just the values of the diagonal elements & the eigenvectors are the column vectors of the operator matrix, i.e.,

$\hat{S}_x|\alpha\rangle = \cancel{\hat{S}_x|1/2, +1/2\rangle} = \frac{1}{2}\hbar|\alpha\rangle$

& $\hat{S}_x|\beta\rangle = -\frac{1}{2}\hbar|\beta\rangle$

explicitly

$\hat{S}_x|\alpha\rangle = \frac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\hbar\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\hbar|\alpha\rangle$

$\hat{S}_x|\beta\rangle = \frac{1}{2}\hbar\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\hbar\begin{pmatrix}0\\1\end{pmatrix} = -\frac{1}{2}\hbar|\beta\rangle$

c) Singlet

$\chi_s = \frac{1}{\sqrt{2}}\left(|\alpha\rangle|\beta\rangle - |\beta\rangle|\alpha\rangle\right)$

Triplet

$\chi_T = \frac{1}{\sqrt{2}}\left(|\alpha\rangle|\beta\rangle + |\beta\rangle|\alpha\rangle\right)$

$|\alpha\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ = 'spin up'

$|\beta\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ = 'spin down'

χ_s : singlet total spin wavefunction

χ_T : triplet total spin wave function

Section B

Q7 a) $\{|\chi_m\rangle\}$ & $\{|\psi_n\rangle\}$ are two complete, independent sets of orthonormal states

$|\Psi\rangle = \sum_m d_m|\chi_m\rangle$ ①

& $|\Psi\rangle = \sum_n c_n|\psi_n\rangle$ ②

d_m is given by $d_m = \langle\chi_m|\Psi\rangle$ ③

& c_n is given by $c_n = \langle\psi_n|\Psi\rangle$ ④

we can think of d_m & c_n as two column vectors

$\underline{c} = \begin{pmatrix}c_1\\c_2\\ \vdots\\ c_n\end{pmatrix}$ & $\underline{d} = \begin{pmatrix}d_1\\d_2\\ \vdots\\ d_m\end{pmatrix}$

again, we know that $d_m = \langle\chi_m|\Psi\rangle$ by ③

then using ①, $d_m = \langle\chi_m|\sum c_n|\psi_n\rangle$

Q7 (continued)

a) $d_m = \sum_n \langle \chi_m | \psi_n \rangle c_n$

we can express $\langle \chi_m | \psi_n \rangle = S_{mn}$

& so $d_m = \sum_n S_{mn} c_n$

which can be written in matrix form

$\underline{d} = \underline{\underline{S}}\, \underline{c}$ or explicitly
$$\begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1n} \\ S_{21} & S_{22} & \cdots & S_{2n} \\ \vdots & \vdots & \ddots & \\ S_{n1} & S_{n2} & \cdots & S_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

b) closure relation $\sum_n |n\rangle\langle n| = 1$

matrix representation of $\hat{A}$ in basis $\{|\chi\rangle\}$, A_χ, has matrix elements given by $\langle \chi_m | \hat{A} | \chi_n \rangle = (A_\chi)_{mn}$

& the matrix representation of $\hat{A}$ in the basis $\{|\psi\rangle\}$, A_ψ, has matrix elements given by $\langle \psi_m | \hat{A} | \psi_n \rangle = (A_\psi)_{mn}$

& we know the closure relationship $\sum_i |\psi_i\rangle\langle\psi_i| = 1$

$$\text{so } (A_\chi)_{mn} = \langle \chi_m | \hat{A} | \chi_n \rangle$$
$$(A_\chi)_{mn} = \sum_i \sum_j \langle \chi_m | \psi_i \rangle \langle \psi_i | \hat{A} | \psi_j \rangle \langle \psi_j | \chi_n \rangle$$
$$(A_\chi)_{mn} = \sum_i \sum_j S_{mi} (A_\psi)_{ij} S^*_{jn}$$
$$\Rightarrow \underline{\underline{A}}_\chi = S A_\psi S^{-1}$$
(S being unitary so $S^* = S^{-1}$)

c) $\hat{H} = \gamma (a \hat{S}_x + b \hat{S}_y + c \hat{S}_z)$, real coefficients & $a^2+b^2+c^2=1$, γ a real coefficient

$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{S}_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$(\frac{\hbar}{2}\sigma_z)$ $(\frac{\hbar}{2}\sigma_x)$ $(\frac{\hbar}{2}\sigma_y)$

& so
$$\hat{H} = \gamma \frac{\hbar}{2} \left(a \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$$
$$= \gamma \frac{\hbar}{2} \begin{pmatrix} c & a-ib \\ a+ib & -c \end{pmatrix}$$

Q7 Continued! (7)

d) eigenstates of $\hat{S}_y$: $|+_y\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle + i|\beta\rangle)$ (eigenvalues: $\hbar/2$

$|-_y\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle - i|\beta\rangle)$ $-\hbar/2$

$|+_y\rangle = S|\alpha\rangle$, $\alpha = \begin{pmatrix}1\\0\end{pmatrix}$ $\left(\beta = \begin{pmatrix}0\\1\end{pmatrix}\right)$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ i & i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$

$= \frac{1}{\sqrt{2}}(|\alpha\rangle + |\beta\rangle)$

$|-_y\rangle = S|\beta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}-i\\i\end{pmatrix}$

$S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$

$S^{-1}S = \frac{1}{2}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1 & -i\\ i & i\end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$ ✓

$|+_y\rangle = S$

$S^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$

$|+_y\rangle = S^{-1}|\alpha\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}(|\alpha\rangle + i|\beta\rangle)$

similarly $|-_y\rangle = S^{-1}|\beta\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}(|\alpha\rangle - i|\beta\rangle)$

e) $\hat{H}$ in $\{|+_y\rangle, |-_y\rangle\}$ basis

$H^{(y)} = S H^{(z)} S^{-1}$

$= \frac{1}{2}\begin{pmatrix}1 & -i\\ 1 & i\end{pmatrix}\begin{pmatrix}c & a-ib\\ a+ib & -c\end{pmatrix}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}\left(\gamma\frac{\hbar}{2}\right)$

$= \gamma\frac{\hbar}{2}\left(\frac{1}{2}\right)\begin{pmatrix}c-ia+b & a-ib+ic\\ c+ia-b & a-ib-ic\end{pmatrix}\begin{pmatrix}1 & 1\\ i & -i\end{pmatrix}$

$= \gamma\frac{\hbar}{2}\left(\frac{1}{2}\right)\begin{pmatrix}\cancel{c}-ia+b+ia+b\cancel{-c} & c-ia+b-ia-b+c\\ c+ia\cancel{-b}+ia\cancel{+b}+c & \cancel{c}+ia-b+ia-b\cancel{-c}\end{pmatrix}$

$= \gamma\frac{\hbar}{2}\begin{pmatrix}b & c-ia\end{pmatrix}$

Q7 (continued)

f) to get eigenvalues we solve the characteristic equation

$$|H^{(y)} - \lambda I| = 0$$

$$\gamma \frac{\hbar}{2} \begin{vmatrix} b-\lambda & c-ia \\ c+ia & -b-\lambda \end{vmatrix} = 0$$

$$-(b-\lambda)(b+\lambda) - (c+ia)(c-ia) = 0$$

$$b^2 - \lambda^2 + c^2 + a^2 = 0$$ but we know $a^2+b^2+c^2=1$

$$-\lambda^2 = -(a^2+b^2+c^2)$$

$$\lambda = \pm 1 \leftarrow \text{eigenvalues}$$

$$\begin{vmatrix} c-\lambda & a-ib \\ a+ib & -c-\lambda \end{vmatrix} = 0$$

$$c^2 - \lambda^2 + a^2 + b^2 = 0$$

$$\lambda^2 = 1$$

$$\lambda = \pm 1$$

$\boxed{\text{Same}}$?

g) $$|\psi_0\rangle^{(y)} = \left(\sqrt{\frac{1+b}{2}}\right)|+_y\rangle + \left(\frac{ia-c}{\sqrt{2(1+b)}}\right)|-_y\rangle$$

$$\langle \hat{S}_z \rangle = \langle \psi_0 | \hat{S}_z | \psi_0 \rangle$$

$$|\psi_0\rangle = \left(\frac{\sqrt{1+b}}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(|\alpha\rangle + i|\beta\rangle)$$

$$+ \left(\frac{ia-c}{\sqrt{2(1+b)}}\right) \left(\frac{1}{\sqrt{2}}\right)(|\alpha\rangle - i|\beta\rangle)$$

$$\Rightarrow \langle \hat{S}_z \rangle = \frac{1}{2}\Big[\frac{1+b}{2}\left(\langle\alpha|\hat{S}_z|\alpha\rangle + \langle\beta|\hat{S}_z|\beta\rangle\right)$$

$$\left(\frac{(-ia-c)(ia-c)}{2(1+b)}\right)\left(\langle\alpha|S_z|\alpha\rangle + \langle\beta|\hat{S}_z|\beta\rangle\right)$$

$$= \frac{1}{2}\left[\frac{1+b}{2}\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right) + \frac{a^2+iac}{2(1+b)}\left(\frac{\hbar}{2} - \frac{\hbar}{2}\right)\right]$$

... seems wrong

Q8 a) $E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle$ ①

the non-perturbed eigensystem is given as

~~$\hat{H}_0 |\psi_0\rangle = E_0$~~

$$\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$$

& the perturbed eigensystem is

$$\hat{H} |\psi_n\rangle = (\hat{H}_0 + \hat{H}') |\psi_n\rangle = E_n |\psi_n\rangle$$

in perturbation theory we assume that the eigenvalue & eigenvectors can be expanded in ~~terms of a~~ ~~p~~ power series in λ

i.e. $|\psi_n\rangle \simeq \lambda^0 |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle$

& $E_n^{(0)} \simeq \lambda^0 E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)}$

to the second order term. We will only use to the first order to show ①

we can now write out the eigensystem

$$(\hat{H}_0 + \lambda \hat{H}') (\lambda^0 |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle) = (\lambda^0 E_n^{(0)} + \lambda E_n^{(1)}) (\lambda^0 |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle)$$

from which we can write the following

λ^0: $\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$, which is what we expect

λ^1: $\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}' |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle$

take scalar product with $|\psi_n^{(0)}\rangle$

$$\langle \psi_n^{(0)} | \hat{H}_0 | \psi_n^{(1)} \rangle + \langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle = \underbrace{E_n^{(0)} \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle}_{=0 \text{ (orthonormality)}} + E_n^{(1)} \langle \psi_n^{(0)} | \psi_n^{(0)} \rangle$$

$$\underbrace{\langle \psi_n^{(1)} | \hat{H}_0 | \psi_n^{(0)} \rangle^*}_{=0 \text{ (orthonormality)}} + \langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle = E_n^{(1)}$$

$$\Rightarrow E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle$$

b) - Stark Effect in ground state hydrogen

- when the symmetries of the system 'does n't' match the symmetry of the perturbation.

i) $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$

$\hat{H} = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$, $V(x) = \begin{cases} 0 & 0 \leq x \leq a \\ \infty & x < 0 \text{ or } x > a \end{cases}$

Matrix elements given by $H_{mn} = \langle \psi_m | \hat{H} | \psi_n \rangle$

so $H_{mn} = \langle \psi_m | \hat{H} | H_n \rangle$

$= \cancel{\frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi}{a}x\right) dx}$

$= \frac{2}{a}\left[\frac{-\hbar^2}{2m}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\frac{d^2}{dx^2}\sin\left(\frac{n\pi}{a}x\right)dx + 0\right]$

where the P.E. term is zero since $V(x) = 0$ ~~when~~ in ~~when~~ $0 \leq x \leq a$ & $\psi(x) = 0$ elsewhere.

$H_{mn} = \frac{-\cancel{2}\hbar^2}{a\cancel{2}m}\int_0^a -\left(\frac{n\pi}{a}\right)^2 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx$

$H_{mn} = \frac{n^2\pi^2\hbar^2}{a^3 m}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx$ ✱

$\rightarrow H_{11} = \frac{\pi^2\hbar^2}{a^3 m}\int_0^a \sin^2\left(\frac{\pi}{a}x\right)dx$

$\cos 2\theta = 1 - 2\sin^2\theta$
$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$= \frac{\pi^2\hbar^2}{2ma^3}\int_0^a \left(1-\cos\left(\frac{2\pi}{a}x\right)\right)dx$

$= \frac{\pi^2\hbar^2}{2ma^2} - \frac{\pi^2\hbar^2}{2ma^3}\left(\frac{a}{2\pi}\right)\left[\sin\left(\frac{2\pi}{a}x\right)\right]_0^a$

$= \frac{\pi^2\hbar^2}{2ma^2}$

$H_{21} = \frac{\pi^2\hbar^2}{ma^3}\int_0^a \sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{a}x\right)dx$

$= \frac{\pi^2\hbar^2}{2ma^3}\int_0^a \left(\cos\left(\frac{\pi}{a}x\right) - \cos\left(\frac{3\pi}{a}x\right)\right)dx$

$= \frac{\pi^2\hbar^2}{2ma^3}\left[\left[\frac{a}{\pi}\sin\left(\frac{\pi}{a}x\right)\right]_0^a - \left[\frac{a}{3\pi}\sin\left(\frac{3\pi}{a}x\right)\right]_0^a\right] = 0$

$\cancel{H_{31} = \frac{\pi^2\hbar^2}{ma^3}\int_0^a}$

✱ $H_{mn} = \frac{n\pi^2\hbar^2}{a^3 m}\int_0^a \left(\cos\left(\frac{(m-n)\pi}{a}x\right) - \cos\left(\frac{(m+n)\pi x}{a}\right.\right.$

$= \frac{n\pi^2\hbar^2}{a^3 m}\left[\frac{a}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{a}x\right) - \frac{a}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{a}\right)\right]_0^a$

c) $\Rightarrow H_{mn} = \begin{cases} 0 \; ; & (m-n) = Z \;, \; Z \text{ an integer} \\ m=n, & \text{cannot be evaluated this way} \end{cases}$

$$H_{nn} = \frac{n^2\pi^2\hbar^2}{a^3 m}\int_0^a \sin^2\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^3}\int_0^a \left(1-\cos\left(\frac{2n\pi}{a}x\right)\right)dx$$

$$= \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Rightarrow \quad H_{11} = \frac{\pi^2\hbar^2}{2ma^2}, \quad H_{22} = \frac{4\pi^2\hbar^2}{2ma^2}, \quad H_{33} = \frac{9\pi^2\hbar^2}{2ma^2}, \quad H_{44} = \frac{16\pi^2\hbar^2}{2ma^2}$$

$$\cancel{H_{21} = H_{12} = H_{31} = H_{13} =} \; H_{21} = H_{31} = H_{41} = H_{12} = \dots = H_{mn} = 0 \;, \; m \neq n$$

$$\Rightarrow \quad H = \begin{pmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{pmatrix} = \begin{pmatrix} \frac{\pi^2\hbar^2}{2ma^2} & 0 & 0 & 0 \\ 0 & \frac{2\pi^2\hbar^2}{ma^2} & 0 & 0 \\ 0 & 0 & \frac{9\pi^2\hbar^2}{2ma^2} & 0 \\ 0 & 0 & 0 & \frac{8\pi^2\hbar^2}{ma^2} \end{pmatrix}$$

d) $V(x) = q\mathcal{E}x$ for $0 \leq x \leq a$ is added to the system

we know $\quad E_n^{(1)} = \langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle$

$$E_n^{(1)} = \frac{2}{a}\int_0^a q\mathcal{E}x \sin^2\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{q\mathcal{E}}{a}\int_0^a x\left(1-\cos\left(\frac{2n\pi x}{a}\right)\right)dx$$

$$= \frac{q\mathcal{E}}{a}\left[\frac{1}{2}a^2 - \left[\left(\frac{a}{2n\pi}\right)^2\cos\left(\frac{2n\pi}{a}x\right)\right]_0^a\right]$$

$$= \frac{q\mathcal{E}}{a}\left[\frac{1}{2}a^2 - \left(\frac{a}{2n\pi}\right)^2 \cancel{\left[\cos(2n\pi)-1\right]}\right] = \frac{aq\mathcal{E}}{2}$$

$$\cancel{\Rightarrow E_1^{(1)} = E_3^{(1)} = \frac{q\mathcal{E}}{2a}\left[a^2 - \frac{a^2}{2n^2\pi^2}\right.}$$

... no n dependence

so $\quad E_1^{(1)} = E_2^{(1)} = E_3^{(1)} = \frac{aq\mathcal{E}}{2} = E_4^{(1)}$

(19)

Q8 f) $E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}}$

to be done

Q9 show $[\hat{J}^2, \hat{J}_z] = 0$

we know

$[\hat{J}_x, \hat{J}_y] = \cancel{[\hat{J}_y, \hat{J}_z]} = \cancel{[\hat{J}_z, \hat{J}_x]} = i\hbar J_z$

$[A+B, C] = (A+B)C - C(A+B)$
$= AC - CA$
$+ BC - CB$
$= [A,C] + [B,C]$

we are also given

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

& $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

so $[\hat{J}^2, \hat{J}_z] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z, \hat{J}_z]$

$= [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z] + [\hat{J}_z^2, \hat{J}_z]$

~~we can show that for the general case~~

$\cancel{[\hat{J}_i^2, \hat{J}_z] = [\hat{J}_i \hat{J}_i, \hat{J}_z] = \hat{J}_i[\hat{J}_i, \hat{J}_z] + [\hat{J}_i, \hat{J}_z]\hat{J}_i}$

~~when $i = x, y$~~

~~then~~ $\cancel{[\hat{J}_i^2, \hat{J}_z] = i\hbar \hat{J}_i + i\hbar \hat{J}_i}$
$\cancel{= 2i\hbar \hat{J}_i}$

~~when $i = z$~~

~~then~~ $\cancel{[\hat{J}_i^2, \hat{J}_z] = 0}$

(13)

$$[\hat{J}^2, \hat{J}_z] = [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z] + [\hat{J}_z^2, \hat{J}_z]$$

$$= \hat{J}_x[\hat{J}_x, \hat{J}_z] + [\hat{J}_x, \hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y, \hat{J}_z] + \ldots$$

$$\ldots [\hat{J}_y, \hat{J}_z]\hat{J}_y + \underbrace{\hat{J}_z[\hat{J}_z, \hat{J}_z]}_{=0} + \underbrace{[\hat{J}_z, \hat{J}_z]\hat{J}_z}_{=0}$$

$$= -i\hbar \hat{J}_x\hat{J}_y - i\hbar \hat{J}_y\hat{J}_x + i\hbar \hat{J}_y\hat{J}_x + i\hbar \hat{J}_x\hat{J}_y$$

$$= 0$$

b) $\hat{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$

$\hat{J}_z |j, m_j\rangle = m_j\hbar |j, m_j\rangle$

$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ & $\hat{J}_- = \hat{J}_x - i\hat{J}_y$

$\cancel{\hat{J}_+ |j, m_j\rangle} = \hat{J}_+\hat{J}_z |j\, m_j\rangle = m_j\hbar \hat{J}_+ |j\, m_j\rangle$ (1)

we can establish the commutation relation

$$[\hat{J}_z, \hat{J}_+] = [\hat{J}_z, \hat{J}_x + i\hat{J}_y]$$

$$= [\hat{J}_z, \hat{J}_x] + [\hat{J}_z, i\hat{J}_y]$$

$$= i\hbar \hat{J}_y + i(-i\hbar \hat{J}_x)$$

$$[\hat{J}_z, \hat{J}_+] = \hbar \hat{J}_+$$

explicitly $\hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z = \hbar \hat{J}_+$

$$\Rightarrow \hat{J}_+\hat{J}_z = \hat{J}_z\hat{J}_+ - \hbar \hat{J}_+$$

& so $\hat{J}_+\hat{J}_z |j\, m_j\rangle = (\hat{J}_z\hat{J}_+ - \hbar\hat{J}_+)|j, m_j\rangle = m_j\hbar \hat{J}_+ |j, m_j\rangle$

re arranging gives $\Rightarrow \hat{J}_z(\hat{J}_+|j\, m_j\rangle) = (m_j+1)\hbar(\hat{J}_+|j\, m_j\rangle)$

~~but $\hat{J}_z|j, m_j+1\rangle = (m_j+1)\hbar|j, m_j+1\rangle$~~

$\Rightarrow \hat{J}_+$ as required

Q9 c) $\hat{J} = \hat{L} + \hat{S}$

$$\hat{J}^2 = (\hat{L}+\hat{S})(\hat{L}+\hat{S})$$
$$= \hat{L}^2 + \hat{S}^2 + 2\hat{L}\cdot\hat{S}$$
$$= \hat{L}^2 + \hat{S}^2 + 2\hat{L}_z\hat{S}_z + 2\hat{L}_x\hat{S}_x + 2\hat{L}_y\hat{S}_y$$

but recall

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \quad (1)$$
$$\hat{J}_- = \hat{J}_x - i\hat{J}_y \quad (2)$$

(1)+(2) : $\frac{1}{2}(\hat{J}_+ + \hat{J}_-) = \hat{J}_x$

& similarly from (1)−(2) :

$$\frac{1}{2i}(\hat{J}_+ - \hat{J}_-) = \hat{J}_y$$

so then

$$\hat{L}_x\hat{S}_x = \frac{1}{4}(\hat{L}_+ + \hat{L}_-)\cdot(\hat{S}_+ + \hat{S}_-)$$
$$= \frac{1}{4}(\hat{L}_+\hat{S}_+ + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + \hat{L}_-\hat{S}_-)$$

& $\hat{L}_y\hat{S}_y = -\frac{1}{4}(\hat{L}_+ - \hat{L}_-)(\hat{S}_+ - \hat{S}_-)$

$$= -\frac{1}{4}(\hat{L}_+\hat{S}_+ - \hat{L}_+\hat{S}_- - \hat{L}_-\hat{S}_+ + \hat{L}_-\hat{S}_-)$$

$$\Rightarrow \hat{J}^2 = \hat{L}^2 + \hat{S}^2 + 2\hat{L}_z\hat{S}_z + \frac{1}{2}(\cancel{\hat{L}_+\hat{S}_+} + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + \cancel{\hat{L}_-\hat{S}_-}) + \ldots$$
$$- \frac{1}{2}(\cancel{\hat{L}_+\hat{S}_+} - \hat{L}_+\hat{S}_- - \hat{L}_-\hat{S}_+ + \cancel{\hat{L}_-\hat{S}_-})$$

$$\Rightarrow \hat{J}^2 = \hat{L}^2 + \hat{S}^2 + \hat{L}_+\hat{S}_- + \hat{L}_-\hat{S}_+ + 2\hat{L}_z\hat{S}_z$$

d) $|lm, sm_s\rangle = |l, m\rangle |s, m_s\rangle$

$$\hat{J}^2 |ll, ss\rangle = \hat{J}^2(|l, m=l\rangle |s, m_s = s\rangle)$$

& $\hat{L}^2 |ll, ss\rangle = l(l+1)\hbar^2 |ll, ss\rangle$

$\hat{S}^2 |ll, ss\rangle = s(s+1)\hbar^2 |ll, ss\rangle$

$2\hat{L}_z\hat{S}_z |ll, ss\rangle = 2ls\hbar^2 |ll, ss\rangle$

we know that in general

$$\hat{J}_+ |j, m_j\rangle = [j(j+1) - m(m+1)]^{1/2}\hbar |j, m_j+1\rangle$$
$$\hat{J}_- |j, m_j\rangle = [j(j+1) - m(m-1)]^{1/2}\hbar |j, m_j-1\rangle$$

$$\hat{L}_+\hat{S}_- |ll, ss\rangle = \hat{L}_+|l, l\rangle \hat{S}_-|s, s\rangle$$
$$= \left(l(l+1) - l(l+1)\right)^{1/2} |l, l+1\rangle \left(s(s+1) - s(s-1)\right)^{1/2} |s, s-1\rangle$$
$$= 0$$

& similarly

$$\hat{L}_-\hat{S}_+ |ll, ss\rangle = \hat{L}_-|l, m_l = l\rangle \hat{S}_+|s, s\rangle$$
$$= \left(l(l+1) - l(l-1)\right)^{1/2} |l, l-1\rangle \left(s(s+1) - s(s+1)\right)^{1/2} |s, s+1\rangle$$
$$= 0$$

$$\Rightarrow \hat{J}^2 |ll, ss\rangle = \left(l(l+1) + s(s+1) + 2ls\right)\hbar^2 |ll, ss\rangle$$

$l = 1,\ s = 1/2$

$$\hat{J}^2 |1/2, -1/2\rangle = \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar^2 |1/2, -1/2\rangle$$
$$= \tfrac{3}{4}\hbar^2 |1/2, -1/2\rangle$$

& $\hat{J}_z |1/2, -1/2\rangle = -\frac{1}{2}\hbar |1/2, -1/2\rangle$

state ~~the~~ $|1/2, -1/2\rangle$ is a superposition of states

$|1\,1, 1/2\ 1/2\rangle,\ |1\,1, 1/2\ -1/2\rangle$

$|1\,0, 1/2\ 1/2\rangle,\ |1\,0, 1/2\ -1/2\rangle$

$|1\,-1, 1/2\ 1/2\rangle,\ |1\,-1, 1/2\ -1/2\rangle$

i.e. $|1/2, -1/2\rangle = a|1\,1, 1/2\ 1/2\rangle + b|1\,1, 1/2\ -1/2\rangle + \ldots$
$c|1\,0, 1/2\ 1/2\rangle + d|1\,0, 1/2\ -1/2\rangle + \ldots$
$e|1\,-1, 1/2\ 1/2\rangle + f|1\,-1, 1/2\ -1/2\rangle$

$\hat{S}_- |1/2, -1/2\rangle = 0$

$\hat{J}_- =$

$$\hat{J}_- = \hat{J}_x - i\hat{J}_y$$
$$= \hat{L}_x + \hat{S}_x - i(\hat{L}_y + \hat{S}_y)$$

so $\hat{J}_- |1\,1, 1/2\ 1/2\rangle =$

Q10

a) Show $\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq E_0$ ← variational principle

the generic eigensystem is

~~$\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$~~

~~where $|\psi_n\rangle$ is our normalisable function satisfying the boundary conditions~~

$$\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$$

where $\{|\phi_n\rangle\}$ is a complete set of orthonormal wavefunctions satisfying the eigensystem, i.e $\langle\phi_m|\phi_n\rangle = \delta_{mn}$

We can take $|\psi\rangle$, a normalisable function that satisfies the boundary conditions, as an approximate solution to the system.

Expanding $|\psi\rangle$ in terms of $\{|\phi_n\rangle\}$:

$$|\psi\rangle = \sum_n c_n|\phi_n\rangle$$

& so we can evaluate an expectation value of the hamiltonian for $|\psi\rangle$:

$$\langle E\rangle_\psi = \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\sum_n\sum_m c_m^* c_n\langle\phi_m|\hat{H}|\phi_n\rangle}{\sum_n|c_n|^2}$$

$$= \frac{\sum_n|c_n|^2E_n}{\sum|c_n|^2}$$

which is a normalised weighted sum of the energy eigenvalues of the Hamiltonian. By definition $E_n > E_0$ for any $n>0$

so $$\langle E\rangle_\psi - E_0 = \frac{\sum_n|c_n|^2E_n}{\sum_n|c_n|^2} - E_0 = \underbrace{\frac{\sum_n|c_n|^2(E_n-E_0)}{\sum_n|c_n|^2}}_{+\text{ve or } 0}$$

~~(unless $c_n=0$ for all $n>0$)~~

$$\Rightarrow \langle E\rangle_\psi \geq E_0$$

$$\Rightarrow \frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} \geq 0$$

b) This result can be used to put an upper-bound on the ground-state-energy. (19)
Since We Know that $\langle E\rangle_\psi \geq E_0$, if $\langle E_\psi\rangle$ ~~E~~ α is expressed in terms of a set of parameters, i.e., $\langle E\rangle_\psi = E(\alpha_1, \alpha_2, \dots, \alpha_n)$, then $\langle E\rangle_\psi$ can be minimised with respect to these parameters (through

solving $\frac{\partial E(\alpha_1,\alpha_2,\dots,\alpha_n)}{\partial \alpha_i} = 0$, $\alpha_i = \alpha_1, \alpha_2, \dots, \alpha_n$) & hence an upper bound on the ground state energy E_0 can be determined.

c) $E_n = \hbar\omega(n + 1/2)$ ← QHO

i) $|\psi\rangle = |0\rangle + \beta|1\rangle$

for normalisation, $A^2\langle\psi|\psi\rangle = 1$

$$A^2\left(\langle 0| + \beta^* \langle 1|\right)\left(|0\rangle + \beta|1\rangle\right) = 1$$

since β is real, $\beta^* = \beta$ & the eigenvectors are orthonormal i.e. $\langle n|m\rangle = \delta_{nm}$

then $A^2\langle\psi|\psi\rangle = A^2(1+\beta^2) = 1$

$$\Rightarrow A = \frac{1}{\sqrt{1+\beta^2}}$$

$\Rightarrow$ normalised wavefunction

$$|\psi\rangle = \frac{1}{\sqrt{1+\beta^2}}\left(|0\rangle + \beta|1\rangle\right)$$

ii) expectation value of Hamiltonian:

$$\langle E\rangle = \left(\frac{1}{\sqrt{1+\beta^2}}\right)^2 E_0 + \left(\frac{\beta}{\sqrt{1+\beta^2}}\right)^2 E_1$$

$$= \frac{E_0 + \beta^2 E_1}{1+\beta^2}$$

iii) $\hat{V}_\mathcal{E} = q\mathcal{E}\hat{x}$

The raising & lowering operators $\hat{a}_+$ & $\hat{a}_-$ are defined:

$$\hat{a}_\pm = \frac{1}{\sqrt{2}}\left[\alpha\hat{x} \mp \frac{i}{\hbar\alpha}\hat{p}\right]$$

$$\sqrt{2}\,\hat{a}_+ = \alpha\hat{x} - \frac{i}{\hbar\alpha}\hat{p} \quad (1)$$

$$\sqrt{2}\,\hat{a}_- = \alpha\hat{x} + \frac{i}{\hbar\alpha}\hat{p} \quad (2)$$

(1) + (2)

$$\frac{\sqrt{2}}{2\alpha}(\hat{a}_+ + \hat{a}_-) = \hat{x}$$

$$\Rightarrow \hat{V}_\mathcal{E} = \frac{q\mathcal{E}}{\sqrt{2}\,\alpha}(\hat{a}_+ + \hat{a}_-)$$

iv) $\beta = \frac{E_1 - E_0}{\gamma} - \sqrt{\frac{(E_1-E_0)^2}{\gamma^2} - 1}$, $\gamma = \frac{\sqrt{2}\, q\, \mathcal{E}}{\alpha}$ 20

$\langle E \rangle_{osc} = \frac{E_0 + \beta^2 E_1}{1+\beta^2}$ $\qquad \langle E \rangle_{\psi,\, total} = E(\beta) = \langle E \rangle_{osc} + \cancel{\langle E \rangle} \langle \hat{V}_{\mathcal{E}} \rangle$

$$\langle \hat{V}_{\mathcal{E}} \rangle = \left(\langle 0| + \beta^2 \langle 1| \right) \left(\frac{q\mathcal{E}}{\sqrt{2}\,\alpha} (\hat{a}_+ + \hat{a}_-) \right) \left(|0\rangle + \beta^2 |1\rangle \right) \frac{1}{1+\beta^2}$$

$$= \frac{q\mathcal{E}}{\sqrt{2}\,\alpha (1+\beta^2)} \left(\langle 0| + \beta^2 \langle 1| \right) \left(|1\rangle + \beta^2 |0\rangle \right)$$

$$= \frac{2 q \mathcal{E} \beta^2}{\sqrt{2}\, \alpha (1+\beta^2)} = \frac{\sqrt{2}\, q \mathcal{E}}{\alpha} \frac{\beta^2}{1+\beta^2}$$

$$\Rightarrow \langle E \rangle_{\psi,\, total} = E(\beta) = \frac{\alpha (E_0 + \beta^2 E_1) + \sqrt{2}\, q \mathcal{E} \beta^2}{\alpha (1+\beta^2)}$$

$$0 = \cancel{\frac{\partial \beta}{\partial \gamma}} \frac{\partial E}{\partial \beta} = \frac{(\alpha 2\beta E_1 + 2\sqrt{2}\, q \mathcal{E} \beta)(\alpha(1+\beta^2)) - 2\alpha\beta(\alpha(E_0 + \beta^2 E_1) + \sqrt{2}\, q \mathcal{E} \beta^2)}{\alpha^2 (1+\beta^2)^2}$$

$$\Rightarrow (\cancel{2}\alpha\cancel{\beta} E_1 + \cancel{2}\sqrt{2}\, q \mathcal{E} \cancel{\beta})(\cancel{\alpha}(1+\beta^2)) = \cancel{2}\cancel{\alpha}\cancel{\beta}(\alpha(E_0 + \beta^2 E_1) + \sqrt{2}\, q \mathcal{E} \beta^2)$$

$$(\alpha E_1 + \sqrt{2}\, q \mathcal{E})(1+\beta^2) = \alpha(E_0 + \beta^2 E_1) + \sqrt{2}\, q \mathcal{E} \beta^2$$

$$\alpha E_1 + \sqrt{2}\, q \mathcal{E} + \beta^2(\alpha E_1 + \sqrt{2}\, q \mathcal{E}) = \alpha E_0 + \beta^2 (\alpha E_1 + \sqrt{2}\, q \mathcal{E})$$

$\cancel{\beta^2(\alpha E_1}$

$\alpha(E_1 - E_0) + \sqrt{2}\, q \mathcal{E} = \beta^2 (\quad$ ... shit