All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Christoffel symbols:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\frac{\partial g_{da}}{\partial X^b} + \frac{\partial g_{bd}}{\partial X^a} - \frac{\partial g_{ab}}{\partial X^d}). \tag{1}$$

Geodesics parameterised by λ:

$$\frac{d^2X^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dX^b}{d\lambda}\frac{dX^c}{d\lambda} = 0. \tag{2}$$

Riemann curvature tensor:

$$R_{abc}{}^s = \frac{\partial\Gamma^s_{ac}}{\partial X^b} - \frac{\partial\Gamma^s_{bc}}{\partial X^a} + \Gamma^e_{ac}\Gamma^s_{be} - \Gamma^e_{bc}\Gamma^s_{ea}, \tag{3}$$

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc},$$

$$R_{abcd} + R_{bcad} + R_{cabd} = 0, \tag{4}$$

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{5}$$

Ricci tensor and Ricci scalar:

$$R_{mr} = R_{mnr}{}^n, \qquad R = g^{mr}R_{mr}. \tag{6}$$

Einstein tensor:

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R. \tag{7}$$

Schwarzschild metric line element: (with $c = 1$)

$$ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{8}$$

In equation (8) r_s is a constant.
Some physical constants:

$$c \simeq 2.99 \times 10^8 \text{ms}^{-1}, \qquad G \simeq 6.67 \times 10^{-11}\text{m}^3\text{s}^{-2}\text{kg}^{-1}$$

1. (a) State how a $\binom{1}{0}$-tensor V^b transforms under coordinate transformations between coordinates X^a and X'^a.

   (b) State how a $\binom{0}{1}$-tensor W_b transforms under coordinate transformations between coordinates X^a and X'^a.

   (c) Which of the following examples are correct tensor expressions? Explain your answers. (You may assume that V, W, T and K are all indeed valid tensors and in particular that μ is a scalar.)

      (i) $V^a V_b + \mu = T^a{}_b$

      (ii) $T_{ab}T^{ab} + T_{aa} = \mu$

      (iii) $T_a{}^b + W_a V^b = \mu K_a{}^b$

   (d) Let $A_{ab}{}^{cd}$ be a $\binom{2}{2}$-tensor such that $A_{ab}{}^{cd} = A_{ba}{}^{cd}$ and $A_{ab}{}^{cd} = -A_{ab}{}^{dc}$. Show that
$$A_{ab}{}^{ab} = 0.$$

   (e) Let $I_i{}^j$ be the $\binom{1}{1}$-tensor $I_i{}^j = \delta_i^j$, where δ_i^j is the Kronecker delta symbol. Show that the covariant derivative of $I_i{}^j$ is zero, that is,
$$\nabla_c I_i{}^j = 0.$$

2. (a) Let $L^a{}_b$ be a Lorentz transformation. Show that
$$\det L = \pm 1.$$

   (b) Suppose a space ship is moving with 3-velocity $\vec{v} = \{v_x, 0, 0\}$ with respect to the Earth, so the problem reduces to the case of one space and one time coordinate. Let x_S, t_S and x_E, t_E be the coordinates of the spaceship and of the Earth respectively. Assume that the speed of light satisfies $c = 1$ and that $|v_x| < 1$.

      (i) Starting with
$$\begin{pmatrix} t_S \\ x_S \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t_E \\ x_E \end{pmatrix}$$
and assuming that the ship's and Earth's origins coincide at an event p, show that $\alpha = \delta$ and $\beta = \gamma$.

      (ii) By considering the ship's origin in the Earth's frame show that $\beta = -\alpha v_x$.

      (iii) Show that $\alpha = \sqrt{\frac{1}{1-v_x^2}}$ by assuming *proper* Lorentz transformations.

   For the case $v_x = 1/\sqrt{3}$,

      (iv) Draw the spacetime diagram with axes t_E and x_E.

      (v) On this diagram draw the time axis (t_S) for the spaceship's rest frame and the space axis (x_S) in the spaceship's rest frame.

      (vi) Calculate the angle between these two lines.

3. Suppose g_{ij} is a metric tensor on an n-dimensional manifold.

   (a) How many independent components does the Riemann curvature tensor have in n dimensions?

   (b) The rest of this question is about the 2-dimensional ring torus with metric

   $$ds^2 = (c + a\cos(v))^2 du^2 + a^2 dv^2,$$

   where $c, a \in \mathbb{R}$ and $c > a$.

      (i) Compute all non-vanishing Christoffel symbols.

      (ii) Compute $R_{uvu}{}^{v}$ and $R_{vuv}{}^{u}$.

4. (a) Suppose g_{ij} is a metric tensor on an n-dimensional manifold. Let T^a be the tangent vector to a geodesic and ∇_a the covariant derivative. Show that

   $$T^a \nabla_a T^b = 0.$$

   (b) Given a covariant derivative ∇_a, show that formula (1) gives the unique connection coefficients for this covariant derivative that are symmetric, i.e. $\Gamma^c_{ab} = \Gamma^c_{ba}$, and satisfy the following condition:

   $$\nabla_a g_{bc} = 0.$$

   (c) Simplify the expressions

   $$R_{ab}{}^{ab}, \quad R_{abc}{}^{c},$$

   where indices have been raised and lowered using the metric g_{ij}.

   (d) For a constant $\lambda > 0$ let h_{ij} be the metric tensor $h_{ij} = \lambda g_{ij}$. If R, R_{ab}, G_{ij}, R_{abcd} and $R_{abc}{}^{d}$ are the curvature tensors for the metric g_{ij}, express the corresponding curvature tensors for the metric h_{ij}.

5. Let us consider a geodesic

$$t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda)$$

of the Schwarzschild metric given in equation (8).

(a) Using the Lagrangian approach, write down the geodesic equations for all coordinates.

(b) Assume that we are in the equatorial plane such that $\theta = \pi/2$. How many constants of motion does this system have? Rewrite the Lagrangian using these constants and show that the geodesics can be described similarly to a 1-dimensional classical mechanical system

$$\dot{r}^2/2 + V_{\text{eff}}(r) = C,$$

where C is a constant. Find $V_{\text{eff}}(r)$. What is the meaning of C in terms of the mechanical system?

(c) Now consider lightlike geodesics (massless particles). Sketch the effective potential and find its stationary point $r_\star$.

(d) Consider the vacuum field equations of general relativity in the presence of the cosmological constant Λ

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0.$$

Show that these equations are equivalent to $R_{ab} = \Lambda g_{ab}$.