MATH3305 — Problem Sheet 7 – Solutions

1. We take the trace of the field equation (apply g^{ab}) which gives

$$R - \frac{1}{2}4R + 4\Lambda = 0 \tag{17}$$

$$R = 4\Lambda. \tag{18}$$

Next eliminate R from the field equation

$$R_{ab} - \frac{1}{2}(4\Lambda)g_{ab} + \Lambda g_{ab} = 0 \tag{19}$$

$$R_{ab} = \Lambda g_{ab}. \tag{20}$$

Using the Ricci tensor components of the lecture we find the following field equations

$$e^{\nu(r)-a(r)}\left(\frac{1}{2}\nu''(r) + \frac{1}{4}\nu'(r)^2 + \frac{1}{r}\nu'(r) - \frac{1}{4}a'(r)\nu'(r)\right) = -\Lambda e^{\nu(r)} \tag{21}$$

$$-\frac{1}{2}\nu''(r) - \frac{1}{4}\nu'(r)^2 + \frac{1}{4}a'(r)\nu'(r) + \frac{1}{r}a'(r) = \Lambda e^{a(r)} \tag{22}$$

$$1 - e^{-a(r)} + \frac{1}{2}ra'(r)e^{-a(r)} - \frac{1}{2}r\nu'(r)e^{-a(r)} = \Lambda r^2. \tag{23}$$

From the first two field equation we get

$$e^{\nu}e^{a} = 1, \tag{24}$$

exactly like in the derivation of the Schwarzschild metric, see lecture notes.
Putting $e^{\nu} = e^{-a}$ into the third field equation gives the differential equation

$$1 - e^{\nu} - r\nu' e^{\nu} = \Lambda r^2 \tag{25}$$

which is solved by

$$e^{\nu} = 1 - \frac{C}{r} - \frac{\Lambda}{3}r^2. \tag{26}$$

Comparison with the Schwarzschild metric suggests $C = 2m$.

2. We begin by relabelling the time and radial coordinates by using τ and ρ instead of t and r

$$ds^2 = -d\tau^2 + \left(\frac{\mu/3}{\rho-\tau}\right)^{2/3} d\rho^2 + \left(\frac{9\mu}{8}(\rho-\tau)^2\right)^{2/3} d\Omega^2\,. \tag{27}$$

The term $r^2 d\Omega^2$ motivates the first coordinate transformation

$$r = \left((9\mu/8)(\rho-\tau)^2\right)^{1/3}\,, \quad \text{or} \quad r^{3/2} = \frac{3}{2}\sqrt{\mu/2}(\rho-\tau)\,, \tag{28}$$

which after differentiation yields

$$r^{1/2}dr = \sqrt{\mu/2}(d\rho - d\tau)\,, \quad \Rightarrow \quad d\tau = d\rho - \sqrt{2/\mu}\,r^{1/2}dr\,. \tag{29}$$

Next, this needs to be substituted into (27) and we find

$$\begin{aligned} ds^2 &= -\left(d\rho - \sqrt{2/\mu}\,r^{1/2}dr\right)^2 + \left(\frac{\mu}{3}\frac{3}{2}\sqrt{\mu/2}\,r^{-3/2}\right)^{2/3} d\rho^2 + r^2 d\Omega^2 \\ &= -\left(d\rho^2 - 2\sqrt{2/\mu}\,r^{1/2}d\rho\, dr + (2/\mu)r dr^2\right) + \frac{\mu}{2r}d\rho^2 + r^2 d\Omega^2 \\ &= -\left(1 - \frac{\mu}{2r}\right)d\rho^2 - \frac{2r}{\mu}dr^2 + 2\sqrt{\frac{2r}{\mu}}d\rho\, dr + r^2 d\Omega^2\,. \end{aligned} \tag{30}$$

Let us now write the metric g_{ab} in matrix form

$$g_{ab} = \begin{pmatrix} -\left(1-\frac{\mu}{2r}\right) & \sqrt{\frac{2r}{\mu}} & 0 & 0 \\ \sqrt{\frac{2r}{\mu}} & -\frac{2r}{\mu} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix}. \tag{31}$$

It is now fairly clear that we aim to find coordinates which diagonalise this matrix. Since our angular coordinates and also the radial coordinate are already in the correct form, we must introduce a new time coordinate. Let us try the following transformation

$$\begin{aligned} \rho &= t + f(r)\,, \quad d\rho = dt + f'(r)dr\,, \\ \Rightarrow \quad d\rho^2 &= dt^2 + 2f'dt\,dr + f'(r)^2 dr^2\,. \end{aligned} \tag{32}$$

where the prime is the derivative of f with respect to r. Then our metric becomes

$$\begin{aligned} ds^2 &= -\left(1-\frac{\mu}{2r}\right)\left(dt^2 + 2f'dt\,dr + f'(r)^2dr^2\right) - \frac{2r}{\mu}dr^2 \\ &\quad + 2\sqrt{\frac{2r}{\mu}}\left(dt + f'(r)dr\right)dr + r^2 d\Omega^2 \qquad (33) \\ &= -\left(1-\frac{\mu}{2r}\right)dt^2 + \left(2\sqrt{\frac{2r}{\mu}}f'(r) - \left(1-\frac{\mu}{2r}\right)f'(r)^2 - \frac{2r}{\mu}\right)dr^2 \\ &\quad + 2\left(\sqrt{\frac{2r}{\mu}} - \left(1-\frac{\mu}{2r}\right)f'(r)\right)dt\,dr + r^2d\Omega^2\,. \qquad (34) \end{aligned}$$

In order to make this line element diagonal, we need to choose $f'(r)$ such that the $dt\,dr$ vanishes. This means

$$f'(r) = \sqrt{\frac{2r}{\mu}}\left(1-\frac{\mu}{2r}\right)^{-1}. \tag{35}$$

We do not need to know $f(r)$ explicitly as only $f'(r)$ enters the transformed metric. One finds

$$\begin{aligned} \frac{2r}{\mu}\left(1-\frac{\mu}{2r}\right)^{-1} - \frac{2r}{\mu} &= \frac{2r}{\mu}\left(\left(1-\frac{\mu}{2r}\right)^{-1} - 1\right) \\ &= \frac{2r}{\mu}\left(\frac{\mu}{2r-\mu}\right) = \left(1-\frac{\mu}{2r}\right)^{-1}. \end{aligned} \tag{36}$$

Final result

$$ds^2 = -\left(1-\frac{\mu}{2r}\right)dt^2 + \left(1-\frac{\mu}{2r}\right)^{-1}dr^2 + r^2d\Omega^2\,, \tag{37}$$

We must set $\mu = 4M$.

5. We begin with recalling Verify that the following equation holds

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{F^2-1}{2} = \text{const.} \tag{38}$$

which was derived in the lecture notes, also $V_{\text{eff}} = -m/r = -r_s/(2r)$ was stated in the lectures. Assuming the particle is initially at rest at $r(0) = 3r_s$ ($\dot{r}(0) = 0$) yields

$$V_{\text{eff}}(3r_s) = -1/6 = \frac{F^2-1}{2} \tag{39}$$

which fixes F to be $F = \sqrt{2/3}$. Therefore,

$$\dot{r}^2 - r_s/r = -1/6 \tag{40}$$

$$\frac{dr}{d\lambda} = -\frac{1}{\sqrt{3}}\sqrt{\frac{3r_s}{r} - 1}, \tag{41}$$

where the minus sign describes in-falling particles. From the Lagrangian we know

$$\dot{t}(1 - r_s/r) = F. \tag{42}$$

Since F is now fixed we can use the chain rule

$$\frac{dr}{dt} = \frac{dr}{d\lambda}\frac{d\lambda}{dt} \tag{43}$$

which gives the second equation.
λ_0 is obtained by integration. A useful substitution is $3r_s/r = \sec^2 y$.
Near r_s one can Taylor expand the right-hand side of dr/dt and finds

$$\frac{dr}{dt} \simeq 1 - r_s/r, \tag{44}$$

which is easily integrated to yield

$$r \simeq r_s + c_1 e^{-t/r_s}. \tag{45}$$