MATH3305 — Problem Sheet 6 – Solutions

1. No written yet
2. Set $i=1, j=2, k=3$, then

$$\begin{aligned} \partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} &= 0\,, \\ \partial_1(-B_x) + \partial_2(-B_y) + \partial_3(-B_z) &= 0\,, \end{aligned} \tag{5}$$

which is $\operatorname{div} \boldsymbol{B} = 0$.
Set $i=0$, then

$$\begin{aligned} \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} &= -4\pi\rho\,, \\ \partial_1(-E_x) + \partial_2(-E_y) + \partial_3(-E_z) &= -4\pi\rho\,, \end{aligned} \tag{6}$$

which is equivalent to

$$\operatorname{div} \boldsymbol{E} = 4\pi\rho\,. \tag{7}$$

Next, setting $i=1$

$$\begin{aligned} \partial_0 F^{10} + \partial_2 F^{12} + \partial_3 F^{13} &= -4\pi j^1\,, \\ \partial_t(E_x) + \partial_2(-B_z) + \partial_3(B_y) &= -4\pi j^1\,. \end{aligned} \tag{8}$$

This is the x-component of the equation

$$\operatorname{curl} \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = 4\pi \boldsymbol{j}\,, \tag{9}$$

and likewise for the other components, thereby showing the second part.

3. No yet written
4. (i)

$$T_{ab} = \begin{pmatrix} \rho e^{\nu} & 0 & 0 & 0 \\ 0 & p e^{a} & 0 & 0 \\ 0 & 0 & p r^2 & 0 \\ 0 & 0 & 0 & p r^2 \sin^2\theta \end{pmatrix} \tag{10}$$

(ii)

$$T_a^b = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{11}$$

(iii) The only non-trivial component is when $b=1$, one gets

$$\begin{aligned} \nabla_a T^{a1} &= \partial_a T^{a1} + \Gamma^a_{ak} T^{k1} + \Gamma^1_{ak} T^{ak} \\ &= \partial_1 T^{11} + \Gamma^a_{a1} T^{11} + \Gamma^1_{00} T^{00} + \Gamma^1_{11} T^{11} + \Gamma^1_{22} T^{22} + \Gamma^1_{33} T^{33}\,. \end{aligned} \tag{12}$$

Now we are going back to Sheet 5 for the Christoffel symbol components and get

$$\begin{aligned}
\nabla_a T^{a1} &= \partial_r(pe^{-a}) + \left(\frac{2}{r} + \frac{1}{2}(\nu' + a')\right)(pe^{-a}) \\
&\quad + \frac{1}{2}\nu' e^{\nu-a}\rho e^{-\nu} + \frac{1}{2}a'(pe^{-a}) - re^{-a}(pr^{-2}) - r\sin^2\theta e^{-a}(pr^{-2}(\sin^2\theta)^{-1}) \\
&= p'e^{-a} - pa'e^{-a} + \frac{1}{2}(\nu' + a')pe^{-a} + \frac{1}{2}a'(pe^{-a}) + \frac{1}{2}\nu' e^{\nu-a}\rho e^{-\nu} \\
&= p'e^{-a} + \frac{1}{2}\nu' pe^{-a} + \frac{1}{2}\nu' e^{-a}\rho\,. \qquad (13)
\end{aligned}$$

Therefore $\nabla_a T^{ab} = 0$ implies

$$p' + \frac{1}{2}\nu'(p + \rho) = 0\,. \qquad (14)$$

5. $\bar{h}_{ab} = h_{ab} - \eta_{ab}h/2$ implies $h_{ab} = \bar{h}_{ab} - \eta_{ab}\bar{h}/2$ as was shown in the lecture. Now we can substitute

$$\begin{aligned}
R_{ab} &= \frac{1}{2}\left[\partial_{bs}h^s{}_a + \partial_{as}h^s{}_b - \partial_{ba}h - \Box h_{ab}\right] \\
&= \frac{1}{2}\left[\partial_{bs}(\bar{h}^s{}_a - \delta^s_a\bar{h}/2) + \partial_{as}(\bar{h}^s{}_b - \delta^s_b\bar{h}/2) - \partial_{ba}(-\bar{h}) - \Box(\bar{h}_{ab} - \eta_{ab}\bar{h}/2)\right] \\
&= \frac{1}{2}\left[\partial_{bs}\bar{h}^s{}_a - \partial_{ba}\bar{h}/2 + \partial_{as}\bar{h}^s{}_b - \partial_{ab}\bar{h}/2 + \partial_{ba}\bar{h} - \Box\bar{h}_{ab} + \eta_{ab}\Box\bar{h}/2\right] \\
&= \frac{1}{2}\left[\partial_{bs}\bar{h}^s{}_a + \partial_{as}\bar{h}^s{}_b - \Box\bar{h}_{ab} + \eta_{ab}\Box\bar{h}/2\right]\,, \qquad (15)
\end{aligned}$$

which is the desired result as shown in the lecture.