All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Formulae:

Christoffel symbols for the metric $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cp}\left(\frac{\partial g_{ap}}{\partial x^b} + \frac{\partial g_{bp}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^p}\right). \tag{1}$$

Geodesics parameterised by proper time τ (affinely parameterised if null)

$$V^a\nabla_a V^b = 0 \text{ or } \ddot{x}^b + \Gamma^b_{pq}\dot{x}^p\dot{x}^q = 0 \tag{2}$$

where $V^a = \dot{x}^a$ and dot denotes $\mathrm{d}/\mathrm{d}\tau$.
Geodesic deviation equation:

$$D_V^2 Y^d = R_{abc}{}^d V^a Y^b V^c \tag{3}$$

for a vector field Y defined along the geodesic, $D_V = V^a\nabla_a$.
Riemann curvature as a commutator:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^d = R_{abc}{}^d X^c. \tag{4}$$

Some symmetries of the curvature tensor

$$R_{abcd} + R_{bcad} + R_{cabd} = 0, \quad R_{abcd} = R_{cdab}. \tag{5}$$

Ricci tensor and scalar curvature (Ricci scalar):

$$r_{ac} = R_{abc}{}^b, \quad s = r_a{}^a. \tag{6}$$

1. The setting of this question is Minkowski space-time $\mathbb{M}$ with units chosen so that $c = 1$.

   (a) (i) In special relativity, what is the meaning of the term 'inertial oberver' **(3)**?

   (ii) Alice is an inertial observer in Minkowski space-time. If E is an event on her world-line, explain how Alice determines which distant events are simultaneous with E **(3)**. If Alice reckons the event F is simultaneous with E, explain how she defines the distance between E and F **(3)**.

   (b) In an inertial coordinate system with coordinates (t, x, y, z), two sprinters, Alice and Bob, are running in the plane $z = 0$ parallel to the y axis and separated by a distance d along the x axis, with speeds v and w, respectively, $v < w$. Alice crosses the starting line at $t = 0$, while Bob crosses the starting line at $t = T > 0$.

   Show that there is a Lorentz transformation of the form

$$L = \begin{pmatrix} \gamma(u) & \gamma(u)u & 0 & 0 \\ \gamma(u)u & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma(u) = (1 - u^2)^{-1/2},$$

   which can be used to define a new system of coordinates (t', x', y', z') in which Alice and Bob cross the starting line simultaneously if and only if $T < d$ **(6)**.

   If this condition is satisfied, write down the equations of the world-lines of Alice and Bob in both coordinate systems **(3 + 3)**. What is the distance, in the primed coordinate system, between the sprinters along the x' axis ?**(4)**

2. Let g_{ab} be a general space-time metric.

   (a) Show that for any event A, there exists a coordinate system x^a such that $\partial_a g_{bc} = 0$ at A, where $\partial_a = \partial/\partial x^a$. **(12)**

   (b) Show that in such a coordinate system

$$\Gamma_{ab}{}^{c} = 0 \text{ and } R_{abcd} = \frac{1}{2}[\partial_a\partial_c g_{bd} + \partial_b\partial_d g_{ac} - \partial_a\partial_d g_{bc} - \partial_b\partial_c g_{ad}]$$

   at the event A. **(6)**

   (c) Show that there does not exist a coordinate transformation that reduces the metric

$$\mathrm{d}s^2 = (1 + x^2)\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$$

   to the metric of Minkowski space. **(7)**

3. The 2-dimensional de Sitter metric is defined as

$$\mathrm{d}s^2 = \mathrm{d}u^2 - \cosh^2 u \mathrm{d}\varphi^2$$

with $u \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$.

(a) Find the geodesic equations and hence the Christoffel symbols for this metric. **(4 + 4 + 4)**

(b) Calculate $R_{010}{}^1$. **(4)**

(c) Find two conserved quantities for the geodesics. **(2+2)**
Show that along *non-radial* geodesics (i.e. φ *not* constant),

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 = M^2 - v^2$$

for some constant M, where $v = \tanh u$ **(2)**. Hence show that such geodesics satisfy

$$\tanh u = M \sin(\varphi - \varphi_0)$$

for some constant φ_0 **(2)**. For what values of M is the geodesic null **(1)**?
[The formula $R_{abc}{}^d = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^e_{bc}\Gamma^d_{ae} - \Gamma^e_{ac}\Gamma^d_{be}$ may be used without proof.]

4. Suppose that $(\mathscr{M}, g)$ is a curved space time, and that F is a tensor field of type $(0, 2)$, whose components satisfy $F_{ab} = -F_{ba}$. Suppose further that F_{ab} satisfies *Maxwell's equations*

$$\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} = 0, \quad \nabla^a F_{ab} = 0 \tag{7}$$

where ∇_a is the Levi-Civita connection.

(a) If Φ_a are the components of a covector field, obtain from (4) a formula for $(\nabla_a\nabla_b - \nabla_b\nabla_a)\Phi_c$ involving the Riemann curvature tensor, stating clearly any properties of ∇_a that you use **(6)**.

(b) Show that if $F_{ab} = \nabla_a\Phi_b - \nabla_b\Phi_a$ then the first equation of (7) is satisfied **(8)**.

(c) By imposing the second equation of (7), obtain a differential equation for Φ_a, and show that this reduces to

$$\Box\Phi_a = -r_{ab}\Phi^b$$

if $\nabla^a\Phi_a = 0$, where $\Box = \nabla^a\nabla_a$ **(2 + 4)**. [You may find it helpful to consider a contraction of the formula you obtained in part (a).]

(d) Suppose that $F_{ab} \neq 0$ at some event P in $\mathscr{M}$. Show that if T^a is a vector at P with the properties $F_{ab}T^a = 0$, $T_aF_{bc} + T_bF_{ca} + T_cF_{ab} = 0$ then $T_aT^a = 0$ **(5)**.

5. The Schwarzschild metric is

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

(a) Show that the coordinate transformation

$$v = t + r + 2m\log(r - 2m)$$

gives

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}v^2 - 2\mathrm{d}v\,\mathrm{d}r - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

and explain how this shows that $r = 2m$ is only a coordinate singularity of the metric **(5 + 3)**. State, giving a reason, whether $r = 0$ is a singularity that can be removed by a change of coordinates. [Further computation is not required.] **(2)**

(b) Show that the quantity

$$E = \left(1 - \frac{2m}{r}\right)\dot{v} - \dot{r}$$

is constant along radial timelike geodesics, where dot denotes derivative with respect to proper time **(5)**.

(c) Show that if $E = 1$, we have

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{\frac{2m}{r}}$$

and deduce that the particle reaches $r = 0$ in finite proper time **(3+2+5)**.

Solution 1. (a) (i) An inertial observer in SR is one who observes Newton's laws of motion to hold (in particular free particles move with constant velocity).

3 marks

(ii) Alice determines the simultaneity of a distant event F with E as follows: she sends a light signal to F at time t_1: E occurs at time t_2; she receives the light signal reflected at F at time t_3. Then E and F are simultaneous if $t_2 - t_1 = t_3 - t_2$.

3 marks

She then reckons the distance between and E and F is $c(t_2 - t_1) = c(t_3 - t_1)/2$.

3 marks

(b) The event E_1: 'Alice crosses the starting line' has coordinates $(0, 0, 0, 0)$ in the unprimed coord system. The event E_2: 'Bob crosses the starting line' has coords $(T, d, 0, 0)$. The displacement vector E_1E_2 is therefore $(T, d, 0, 0)$ and this is spacelike if and only if $T^2 - d^2 < 0$, i.e. $T < d$.

3 marks

We know that if the displacement vector between two events is spacelike, there is an inertial coordinate system in which the two events are simultaneous.

3 marks

[A solution involving changing coordinates with the proposed LT, involving the equation

$$\begin{pmatrix} 0 \\ d' \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(u) & \gamma(u)u & 0 & 0 \\ \gamma(u)u & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T \\ d \\ 0 \\ 0 \end{pmatrix}$$

is also OK. The top component of this equation yields

$$0 = \gamma T + \gamma u d = 0$$

so, not suprisingly, $u = -T/d$. [If $c \neq 1$, this would be $-c^2T/d$. Again a total of 6 points for this.]

In the unprimed coordinate system, the worldlines of A and B are respectively

$$\text{A} : \tau \mapsto (\tau, 0, \tau v, 0), \quad \text{B} : \tau \mapsto (\tau, d, (\tau - T)w, 0).$$

3 marks

To determine these worldlines in the primed coordinate system, we just multiply these parametersations by $L(u)$ with $u = -T/d$,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(u) & \gamma(u)u & 0 & 0 \\ \gamma(u)u & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ 0 \\ \tau v \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(u)\tau \\ -u\gamma(u)\tau \\ \tau v \\ 0 \end{pmatrix}$$

for A and

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma(u) & \gamma(u)u & 0 & 0 \\ \gamma(u)u & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ d \\ (\tau - T)w \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(u)[\tau - T] \\ \gamma(u)(u\tau + d) \\ (\tau - T)w \\ 0 \end{pmatrix}$$

for B.

3 marks

The distance along the x' axis is just the difference in the x' coordinates of simultaneous events on these world lines. Taking $\tau = 0$ on A's world line, the x'-coordinate is 0. Taking $\tau = T$ on B's world line, the x' coordinate is $\gamma(u)(uT + d)$. So the distance is

$$\gamma(u)(d - T^2/d) = \gamma(u)^{-1}d = \sqrt{1 - T^2/d^2}d.$$

4 marks

Solution 2. The first part is bookwork. If $\widetilde{g}_{ab}$ are the components of the metric in a coordinate system $\widetilde{x}^a$, wrt which A corresponds to $\widetilde{x}^a = 0$, then consider new coordinates of the form

$$\widetilde{x}^a = x^a + \frac{1}{2}Q^a_{pq}x^p x^q$$

where Q^a_{pq} is a collection of constants to be determined, symmetric in pq. We have

$$g_{ab}\mathrm{d}x^a\mathrm{d}x^b = \widetilde{g}_{ab}\mathrm{d}\widetilde{x}^a\mathrm{d}\widetilde{x}^b$$

and

$$\mathrm{d}\widetilde{x}^a = \mathrm{d}x^a + Q^a_{pq}x^p\mathrm{d}x^q.$$

Then

$$\begin{aligned} g_{ab}\mathrm{d}x^a\mathrm{d}x^b &= \widetilde{g}_{ab}(\mathrm{d}x^a + Q_{pq}{}^a x^p\mathrm{d}x^q)(\mathrm{d}x^b + Q^b_{rs}x^r\mathrm{d}x^s) \\ &= [\widetilde{g}_{ab} + Q_{pab}x^p + Q_{pba}x^p + O(x^2)]\mathrm{d}x^a\mathrm{d}x^b \end{aligned} \tag{8}$$

where the index on Q is lowered with $\widetilde{g}$. Then,

$$g_{ab} = \widetilde{g}_{ab} + x^p(Q_{pab} + Q_{pba}) + O(x^2).$$

Thus

$$\partial_c g_{ab} = \partial_c\widetilde{g}_{ab} + Q_{cab} + Q_{cba} + O(x).$$

So we just have to choose the components of Q so that

$$Q_{cab} + Q_{cba} = -\partial_c\widetilde{g}_{ab}(A).$$

The well known trick for this is to permute the indices cyclically and rearrange, getting

$$Q_{abc} = -\frac{1}{2}(\partial_a\widetilde{g}_{bc}(A) + \partial_b\widetilde{g}_{ac}(A) - \partial_c\widetilde{g}_{ab}(A)).$$

12 marks

From the definition of the curvature in the formulae on the front page,

$$\nabla_b X^d = \partial_b X^d + \Gamma_{bc}{}^d X^c,$$

and so

$$\nabla_a\nabla_b X^d = \partial_a\partial_b X^d + \partial_a\Gamma_{bc}{}^d X^c + O(x).$$

Skewing,

$$R_{abc}{}^d X^d = (\partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac})X^c \text{ at } A.$$

Now

$$\partial_a\Gamma^d_{bc} = \frac{1}{2}\partial_a[g^{ds}(\partial_b g_{cs} + \partial_c g_{bs} - \partial_s g_{bc})] = \frac{1}{2}g^{ds}[\partial_a\partial_b g_{cs} + \partial_a\partial_c g_{bs} - \partial_a\partial_s g_{bc}] \text{ at } A.$$

Hence at A,

$$R_{abc}{}^d = g^{ds}\frac{1}{2}\left(\partial_a\partial_c g_{bs} - \partial_a\partial_s g_{bc} - \partial_b\partial_c g_{as} + \partial_b\partial_s g_{ac}\right).$$

as required.

6 marks

For the given metric, we aim to show that the curvature is not zero at a point. The easiest route is to note that the metric is the Minkowski metric to $+O(x^2)$ at $(t, x, y, z) = 0$. So we can calculate the curvature at this point from the above formula. The second derivatives of the g_{ab} are all zero except for $\partial_1\partial_1 g_{00} = 2$. It follows that

$$R_{0101} = \frac{1}{2}\left(\partial_0\partial_0 g_{11} - \partial_0\partial_1 g_{01} + \partial_1\partial_1 g_{00} - \partial_0\partial_1 g_{01}\right) = -1.$$

Since the curvature of Minkowski space is zero and the curvature is a tensor, the given metric cannot be the Minkowski metric in disguise. [For full credit, I need to see clear statements including that the curvature of Minkowski space is zero and that the vanishing of a tensor at a point is independent of the coordinates.]

7 marks

Solution 3. (a) The Lagrangian for geodesics is

$$L = \frac{1}{2}(\dot{u}^2 - \cosh^2 u\dot{\varphi}^2).$$

Then

$$\frac{\partial L}{\partial \dot{u}} = \dot{u}, \ \frac{\partial L}{\partial \dot{\varphi}} = -\cosh^2 u\dot{\varphi}$$

and

$$\frac{\partial L}{\partial u} = -\sinh u \cosh u\dot{\varphi}^2, \quad \frac{\partial L}{\partial \varphi} = 0.$$

4 marks

Hence the geodesic equations are

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{u}} - \frac{\partial L}{\partial u} = 0 \text{ i.e. } \ddot{u} + \sinh u \cosh u\dot{\varphi}^2 = 0$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\varphi}} = 0 \text{ i.e. } -\cosh^2 u\ddot{\varphi} - 2\sinh u \cosh u\dot{u}\dot{\varphi} = 0.$$

4 marks

Comparing with

$$\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0,$$

we see from the first equation that

$$\Gamma^0_{11} = \sinh u \cosh u$$

and from the second that

$$\Gamma^1_{01} = \tanh u$$

(note no factor of 2). All other Γs are zero.

4 marks

(b) From the given formula,

$$\begin{aligned} R_{010}{}^1 &= \partial_0\Gamma^1_{10} - \partial_1\Gamma^1_{00} + \Gamma^e_{01}\Gamma^1_{0e} - \Gamma^e_{00}\Gamma^1_{1e} \\ &= \partial_u(\tanh u) + \Gamma^1_{01}\Gamma^1_{01} \\ &= \operatorname{sech}^2 u + \tanh^2 u \\ &= 1. \end{aligned}$$

(All other terms in the expression for R zero.)

4 marks

(c) For a non-radial geodesic, $\dot{\varphi} \neq 0$, and we use the fact that L is a constant, so

$$\dot{u}^2 - \cosh^2 u\dot{\varphi}^2 = \lambda \tag{9}$$

and also $\partial L/\partial\dot{\varphi}$ is a constant because L does not depend upon φ. So

$$\cosh^2 u\dot{\varphi} = J$$

(say).

For these two conserved quantities: 4 marks

What follows is very similar to deriving the orbits in Schwarzschild: divide the first equation by $\dot{\varphi}^2$ to get

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - \cosh^2 u = \lambda/\dot{\varphi}^2 = \lambda\cosh^4 u/J^2.$$

If $v = \tanh u$ as suggested in the question,

$$\frac{\mathrm{d}v}{\mathrm{d}\varphi} = \operatorname{sech}^2 u\frac{\mathrm{d}u}{\mathrm{d}\varphi}$$

and inserting into the previous equation

$$\cosh^4 u\left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 - \cosh^2 u = \lambda\cosh^4 u/J^2 \Rightarrow \left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 - \operatorname{sech}^2 u = \lambda/J^2$$

Finally, since $1 - \tanh^2 = \operatorname{sech}^2$, we get

$$\left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 - 1 + \tanh^2 u = \lambda/J^2 \Rightarrow \left(\frac{\mathrm{d}v}{\mathrm{d}\varphi}\right)^2 - 1 + v^2 = \lambda/J^2$$

which gives the required result with $M = \lambda/J^2 + 1$.

2 marks

Obtaining the solution by integration or verification:

$$v = M\sin(\varphi - \varphi_0)$$

2 marks

The geodesic is null if $\lambda = 0$ i.e. if $M = 1$.

1 mark

Solution 4. (a) We use the Leibnitz rule $\nabla_a(\Phi_c X^c) = (\nabla_a \Phi_c)X^c + \Phi_c \nabla_a X^c$ and the torsion free condition

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

to obtain

$$X^a(\nabla_a \nabla_b - \nabla_b \nabla_a)\Phi_a + \Phi_a(\nabla_a \nabla_b - \nabla_b \nabla_a)X^a = 0.$$

for all Φ and X. It follows that

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\Phi_c = -R_{abc}{}^d \Phi_d$$

6 marks

(b) If $F_{ab} = \nabla_a \Phi_b - \nabla_b \Phi_a$, then

$$\nabla_a F_{bc} = \nabla_a \nabla_b \Phi_c - \nabla_a \nabla_c \Phi_b$$

and

$$\begin{aligned}
\nabla_a F_{bc} + \nabla_b F_{ca} + \nabla_c F_{ab} &= \nabla_a \nabla_b \Phi_c - \nabla_a \nabla_c \Phi_b \\
&+ \nabla_b \nabla_c \Phi_a - \nabla_b \nabla_a \Phi_c \\
&+ \nabla_c \nabla_a \Phi_b - \nabla_c \nabla_b \Phi_a
\end{aligned}$$

and the terms can be regrouped to give three commutators on Φ:

$$-R_{abc}{}^d \Phi_d - R_{bca}{}^d \Phi_d - R_{cab}{}^d \Phi_d$$

and this vanishes by the given symmetry of the riemann curvature tensor.

[An alternative proof involves the observation that the skew-symmetrised derivatives can also be calculated (locally) with ∇ replaced by ∂ because of the symmetry of the connection. Then there are no curvature terms to worry about. This is completely acceptable if clearly explained.]

8 marks

(c) Indeed
$$\nabla^a(\nabla_a\Phi_b - \nabla_b\Phi_a) = 0 \Rightarrow \nabla^a\nabla_a\Phi_b - \nabla^a\nabla_b\Phi_a = 0$$
is a differential equation for Φ.

2 marks

Because
$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\Phi_c = -R_{abc}{}^d\Phi_d,$$
contracting on ac gives
$$\nabla^a\nabla_b\Phi_a - \nabla_b\nabla^a\Phi_a = -g^{ac}R_{abc}{}^d\Phi_d.$$
By the interchange symmetry of R, the RHS is equal to $-r_b{}^d\Phi_d$. The result follows.

4 marks

(d) This is very easy to the experienced, but will probably be found to be tricky: Contract the second equation with T^a, getting
$$T^aT_aF_{bc} + T_b(T^aF_{ca}) + T_c(T^aF_{ab}) = 0.$$
Because F is skew, the second condition means that both the second and third terms vanish, leaving
$$T^aT_aF_{bc} = 0.$$
It is given that at least one component vanishes at the given event P, so it follows that the scalar quantity (T^aT_a) must vanish there.

5 marks

Solution 5. (a) If v is as given,

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r + \frac{2m}{r-2m}\,\mathrm{d}r = \mathrm{d}t + \frac{1}{1-2m/r}\,\mathrm{d}r.$$

Now eliminate $\mathrm{d}t$ from the Schwarzschild metric in favour of $\mathrm{d}r$ and $\mathrm{d}v$. For example,

$$\begin{aligned}
\mathrm{d}v^2 &= \mathrm{d}t^2 + \frac{2}{1-2m/r}\mathrm{d}t\,\mathrm{d}r + \frac{1}{(1-2m/r)^2}\,\mathrm{d}r^2 \\
\Rightarrow (1-2m/r)\mathrm{d}v^2 &= (1-2m/r)\mathrm{d}t^2 + 2\mathrm{d}t\,\mathrm{d}r + \frac{1}{1-2m/r}\,\mathrm{d}r^2 \\
&= (1-2m/r)\mathrm{d}t^2 + 2\mathrm{d}r\left(\mathrm{d}v - \frac{1}{1-2m/r}\,\mathrm{d}r\right) + \frac{1}{1-2m/r}\,\mathrm{d}r^2 \\
&= (1-2m/r)\mathrm{d}t^2 + 2\mathrm{d}r\mathrm{d}v - \frac{1}{1-2m/r}\,\mathrm{d}r^2 \qquad (10)
\end{aligned}$$

So

$$(1-2m/r)\mathrm{d}t^2 - (1-2m/r)^{-1}\mathrm{d}r^2 = (1-2m/r)\mathrm{d}v^2 - 2\mathrm{d}r\mathrm{d}v,$$

and

$$\mathrm{d}s^2 = (1-2m/r)\mathrm{d}v^2 - 2\mathrm{d}r\mathrm{d}v - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

5 marks

The metric coefficients are now smooth near $r = 2m$ and the metric remains non-singular (and of the correct signature) there: the vanishing of the coefficient of $\mathrm{d}v^2$ does not matter.

3 marks

No coordinate transformation can remove the singularity at $r = 0$: this would follow by checking that scalar quantities made out of the curvature tensor blow up there.

2 marks

(b) For a radial timelike geodesic, parameterized by proper time,

$$(1-2m/r)\dot{v}^2 - 2\dot{v}\dot{r} = 1,$$

and because the Lagrangian is independent of v, the quantity

$$\frac{\partial L}{\partial \dot{v}} = 2(1-2m/r)\dot{v} - 2\dot{r}$$

is a constant. This is $2E$ from the question.

5 marks

(c) To get another equation use $L = 1/2$ for timelike geodesics, so

$$(1 - 2m/r)\dot{v}^2 - 2\dot{r}\dot{v} = 1$$

3 marks

If $E = 1$ and we combine the two equations, multiply the first by $(1 - 2m/r)$,

$$\begin{aligned}
(1 - 2m/r)^2\dot{v}^2 - 2(1 - 2m/r)\dot{v}\dot{r} &= 1 - 2m/r \\
\Rightarrow (E + \dot{r})^2 - 2(E + \dot{r})\dot{r} &= 1 - 2m/r \\
\Rightarrow (1 + \dot{r})^2 - 2(1 + \dot{r})\dot{r} &= 1 - 2m/r \\
\Rightarrow (1 + 2\dot{r} + \dot{r}^2) - 2\dot{r} - 2\dot{r}^2 &= 1 - 2m/r
\end{aligned}$$

Hence

$$\dot{r}^2 = 2m/r$$

from which the equation follows by taking square roots: the negative sign is chosen because r decreases as τ increases.

2 marks

Rearranging the differential equation and assuming that $r = r_0$ when $\tau = 0$,

$$\int_{r_0}^{r} \sqrt{r}\,\mathrm{d}r = -\sqrt{2m}\int_0^{\tau} \mathrm{d}\tau,$$

so

$$\frac{2}{3}(r_0^{3/2} - r^{3/2}) = \sqrt{2m}\tau.$$

Thus τ has a finite value when $r = 0$.

5 marks