MATH3305 — Problem Sheet 3 – Solutions

1. (i) The explicit formula for the Christoffel symbol components is

$$\Gamma^i_{jk} = \frac{1}{2} g^{is} \left(g_{sj,k} + g_{ks,j} - g_{jk,s} \right) ,$$

with $g_{ij} = \mathrm{diag}(1, r^2, r^2 \sin^2\theta)$, and $g^{ij} = \mathrm{diag}(1, r^{-2}, r^{-2} \sin^{-2}\theta)$. Careful use of this formula gives the following non-vanishing components of the Christoffel symbol

$$\begin{aligned}
&\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \\
&\Gamma^1_{22} = -r \qquad \Gamma^1_{33} = -r \sin^2\theta \\
&\Gamma^3_{23} = \cot\theta \qquad \Gamma^2_{33} = -\cos\theta \sin\theta .
\end{aligned}$$

(ii) The Lagrangian is $L = \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \dot{\phi}^2$ and the equations of motion are

$$\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{X}^i} = \frac{\partial L}{\partial X^i} \quad \text{with} \quad X^i = \{r, \theta, \phi\} .$$

The resulting geodesic equations are

$$\ddot{r} - r\dot{\theta}^2 - r \sin^2\theta \dot{\phi}^2 = 0 \tag{1}$$

$$\ddot{\theta} + \frac{2}{r} \dot{r}\dot{\theta} - \cos\theta \sin\theta \dot{\phi}^2 = 0 \tag{2}$$

$$\ddot{\phi} + 2 \cot\theta \dot{\theta}\dot{\phi} + \frac{2}{r} \dot{r}\dot{\phi} = 0 . \tag{3}$$

These can now be used to read off the Christoffel symbol components. For instance,

$$\ddot{\phi} + \Gamma^3_{ij} \dot{X}^i \dot{X}^j = 0 .$$

Now, one compares term with (3). There are no terms where i and j take the same value, therefore $\Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{33} = 0$. There is no term with $\dot{\theta}\dot{r}$, hence $\Gamma^3_{12} = \Gamma^3_{21} = 0$. For the non-vanishing terms we can read off $\Gamma^3_{13} = 1/r$ and $\Gamma^3_{23} = \cot\theta$, in agreement with the results of (i). The other components are done the same way.

3. The four explicit formulae are given by

$$\begin{aligned}
\nabla_a T^b_c &= \partial_a T^b_c + \Gamma^b_{as} T^s - \Gamma^s_{ac} T^b_s , \\
\nabla_a T^{bc} &= \partial_a T^{bc} + \Gamma^b_{as} T^{sc} + \Gamma^c_{as} T^{bs} , \\
\nabla_a T_{bc} &= \partial_a T_{bc} - \Gamma^s_{ab} T_{sc} - \Gamma^s_{ac} T_{bs} , \\
\nabla_a K_{cdb} &= \partial_a K_{cdb} - \Gamma^s_{ac} K_{sdb} - \Gamma^s_{ad} T_{csb} - \Gamma^s_{ab} T_{cds} .
\end{aligned}$$

4. Start with Lagrangian

$$L = -\frac{1}{t^4} \dot{t}^2 + \dot{x}^2 ,$$

which yields the equations of motion

$$\ddot{x} = 0 \qquad \ddot{t} - 2\frac{\dot{t}^2}{t} = 0 .$$

The first equations gives $x(\lambda) = c_1\lambda + c_2$ where c_i are constants. The second equation can be integrated by noting

$$\frac{\ddot{t}}{\dot{t}} = (\ln \dot{t})^{\cdot} = 2\frac{\dot{t}}{t} = 2(\ln t)^{\cdot}$$

which results in

$$\frac{1}{t} = c_3\lambda + c_4\,.$$

This finally suggests the better coordinate $y = 1/t$. Indeed,

$$y = \frac{1}{t} \qquad dy = -\frac{1}{t^2}dt \qquad dy^2 = \frac{1}{t^4}dt^2\,,$$

and hence

$$ds^2 = -\frac{1}{t^4}dt^2 + dx^2 = dx^2 - dy^2\,.$$