# 2012

# Model answers MATH3305

**Problem 1**

(a) $T'^a{}_b = \frac{\partial X'^a}{\partial X^c}\frac{\partial X^d}{\partial X'^b}T^c{}_d$

(b) $W'^a = \frac{\partial X'^a}{\partial X^b}W^b$.

(c)

$$\mu' = V'^a W'_a = \frac{\partial X'^a}{\partial X^b}V^b \frac{\partial X^c}{\partial X'^a}W_c = \delta^c_b V^b W_c = V^b W_b = \mu$$

(d) $g^{ab}$ denotes the inverse metric. $g_{ab}g^{bc} = \delta^c_a$. $g_{ab}g^{ab} = \delta^a_a = n$.

(e) First we must find the vector in cartesian coordinates.

$$T^a = \frac{dX^a}{d\lambda}$$

This gives the components $T^x = 1$ and $T^y = 1$. To find the transformation to polar coordinates we must employ the coordinate transform which they gave as their answer in part b).

$$r = \sqrt{x^2+y^2}$$
$$\theta = \arctan\left(\frac{y}{x}\right)$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$
$$\frac{\partial \theta}{\partial x} = \frac{-y}{\sqrt{x^2+y^2}}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{\sqrt{x^2+y^2}}$$

Putting these into $T'^a = \frac{\partial X'^a}{\partial X^b}T^b$, and substituting the values for $x = y = \lambda$,

$$T^r = \frac{x}{\sqrt{x^2+y^2}}T^x + \frac{y}{\sqrt{x^2+y^2}}T^y = \sqrt{2}$$
$$T^\theta = \frac{x}{\sqrt{x^2+y^2}}T^x - \frac{y}{\sqrt{x^2+y^2}}T^y = 0$$

**Problem 2**

(a) Well defined for all $y$ and all $z \neq 0$.

(b) Using the Lagrangian method we find that the geodesic equations are

$$\ddot{y} = 0, \tag{1}$$

$$\ddot{z} - \frac{1}{z}\dot{z}^2 = 0. \tag{2}$$

Comparing equation (2) with

$$\ddot{z} + \Gamma^z_{yy}\dot{y}\dot{y} + 2\Gamma^z_{yz}\dot{y}\dot{z} + \Gamma^z_{zz}\dot{z}\dot{z} = 0$$

gives

$$\Gamma^z_{zz} = -\frac{1}{z}, \quad \Gamma^z_{yy} = 0, \quad \Gamma^z_{yz} = 0. \tag{3}$$

A similar comparison for equation $\ddot{y} = 0$ shows that $\Gamma^z_{zz}$ is the only non-zero Christoffel symbol.

(c) Equation (1) gives

$$y(\lambda) = c_1\lambda + c_2. \tag{4}$$

Dividing equation (2) by $\dot{z}$ yields

$$\begin{aligned} &\frac{\ddot{z}}{\dot{z}} = \frac{\dot{z}}{z}, \\ \Rightarrow\ &\frac{d}{d\lambda}\ln(\dot{z}) = \frac{d}{d\lambda}\ln(z), \\ \Rightarrow\ &\ln(\dot{z}) = \ln(z) + c, \\ \Rightarrow\ &\dot{z} = \tilde{c}z \Rightarrow \ln(z) = c_3\lambda + c_4. \end{aligned}$$

This suggest the new coordinate $x = \ln(z)$. From $dx = \frac{dz}{z}$ we obtain $dz = zdx$ and

$$\begin{aligned} ds^2 &= dy^2 + \frac{1}{z^2}dz^2 \\ &= dx^2 + dy^2. \end{aligned}$$

Geodesics are straight lines.

(d) Since this is flat Euclidean 2-space $R_{abcd} = 0$, $R_{ab} = 0$ and $R = 0$.

## Problem 3

(a) The Riemann tensor and the scalar curvature tell you if the space is flat or curved. If and only if the space is flat will all components of the Riemann be zero everywhere on your manifold. The scalar curvature is a coordinate independent measure of curvature and therefore is linked to the intrinsic curvature of your space.

(b) Apply the commutation of the covariant derivative to the metric tensor.

$$
\begin{aligned}
(\nabla_a \nabla_b - \nabla_b \nabla_a)\, g_{cd} &= R_{abc}{}^{e} g_{ed} + R_{abd}{}^{e} g_{ce} \\
0 &= R_{abcd} + R_{abdc}
\end{aligned}
$$

(c) i Using the Lagrangian method we can pick out the Christoffel symbols. First $t$,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial t} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{t}} &= -2A(r)\dot{t} \\
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{t}} &= -2A'(r)\dot{t}\dot{r} - 2A(r)\ddot{t}
\end{aligned}
$$

and then $r$,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial r} &= -A'(r)\dot{t}^2 + B'(r)\dot{r}^2 \\
\frac{\partial \mathcal{L}}{\partial \dot{r}} &= 2B(r)\dot{r} \\
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{r}} &= 2B'(r)\dot{r}^2 + 2B(r)\ddot{t}
\end{aligned}
$$

Therefore the Christoffel symbols are

$$
\Gamma^t_{tr} = \frac{1}{2}\frac{A'(r)}{A(r)} \qquad \Gamma^r_{tt} = \frac{1}{2}\frac{A'(r)}{B(r)}
$$

$$
\Gamma^r_{rr} = \frac{1}{2}\frac{B'(r)}{B(r)}
$$

ii Now for the Riemann tensor. We input the Christoffel symbols into the following expression for the Riemann tensor

$$R_{abc}{}^{s} = \frac{\partial \Gamma^{s}_{ac}}{\partial X^{b}} - \frac{\partial \Gamma^{s}_{bc}}{\partial X^{a}} + \Gamma^{e}_{ac}\Gamma^{s}_{be} - \Gamma^{e}_{bc}\Gamma^{s}_{ea}$$

We know that we are actually interested in $R_{trtr} = R_{trt}{}^{r} g_{rr}$. Therefore we can compute first just $R_{trt}{}^{r}$

$$R_{trt}{}^{r} = \frac{\partial \Gamma^{s}_{tt}}{\partial r} - \frac{\partial \Gamma^{r}_{bt}}{\partial t} + \Gamma^{e}_{tt}\Gamma^{r}_{re} - \Gamma^{e}_{rt}\Gamma^{r}_{et}$$

$\frac{\partial \Gamma^{r}_{bt}}{\partial t} = 0$

$$R_{trt}{}^{r} = \frac{\partial \Gamma^{s}_{tt}}{\partial r} + \Gamma^{e}_{tt}\Gamma^{r}_{re} - \Gamma^{e}_{rt}\Gamma^{r}_{et}$$
$$R_{trt}{}^{r} = \frac{\partial \Gamma^{s}_{tt}}{\partial r} + \Gamma^{t}_{tt}\Gamma^{r}_{rt} + \Gamma^{r}_{tt}\Gamma^{r}_{rr} - \Gamma^{t}_{rt}\Gamma^{r}_{tt} - \Gamma^{r}_{rt}\Gamma^{r}_{rt}$$

Only the first, third and fourth terms are non-zero. Combining with $g_{r}r$ we have,

$$R_{trtr} = \frac{1}{4}\left[2A'' - A'\left(\frac{A'}{A} + \frac{B'}{B}\right)\right].$$

**Problem 4**

(a) We will show that

$$R_{abc}{}^{s} + R_{bca}{}^{s} + R_{cab}{}^{s} = 0, \tag{5}$$

whence the claim follows by contracting by $g_{sd}$ and using the definition $R_{abcd} = R_{abc}{}^{s} g_{sd}$. By definition

$$R_{abc}{}^{s} = \frac{\partial \Gamma^{s}_{ac}}{\partial X^{b}} - \frac{\partial \Gamma^{s}_{bc}}{\partial X^{a}} + \Gamma^{e}_{ac}\Gamma^{s}_{be} - \Gamma^{e}_{bc}\Gamma^{s}_{ea}. \tag{6}$$

Renaming indices according to $a \to b \to c \to a$ gives

$$R_{bca}{}^{s} = \frac{\partial \Gamma^{s}_{ba}}{\partial X^{c}} - \frac{\partial \Gamma^{s}_{ca}}{\partial X^{b}} + \Gamma^{e}_{ba}\Gamma^{s}_{ce} - \Gamma^{e}_{ca}\Gamma^{s}_{eb}, \tag{7}$$

and renaming indices according to the same rule again gives

$$R_{cab}{}^{s} = \frac{\partial \Gamma^{s}_{cb}}{\partial X^{a}} - \frac{\partial \Gamma^{s}_{ab}}{\partial X^{c}} + \Gamma^{e}_{cb}\Gamma^{s}_{ae} - \Gamma^{e}_{ab}\Gamma^{s}_{ec}. \tag{8}$$

Adding up equations (6),(7) and (8) gives equation (5).

(b) $R_{ab}{}^{ab} = g^{as} R_{abs}{}^{b} = g^{as} R_{as} = R$. Contracting identity $R_{abcd} = -R_{abdc}$ by $g^{cd}$ gives $R_{abc}{}^{c} = -R_{abd}{}^{d}$. Renaming $d \to c$ on the right hand side gives $R_{abc}{}^{c} = 0$.

(c) Contracting indices $b$ and $s$ in equation (5) gives

$$R_{asc}{}^{s} + R_{sca}{}^{s} + R_{cas}{}^{s} = 0.$$

The first term equals $R_{ac}$. The second term equals $R_{sca}{}^{s} = -R_{csa}{}^{s} = -R_{ca}$. The third term vanishes by part (b). Thus $R_{ab} = R_{ba}$ and since $g_{ij} = g_{ji}$ we have $G_{ij} = G_{ji}$.

The Einstein field equations read

$$G_{ij} = \frac{8\pi}{c^{2}} G T_{ij},$$

where $T_{ij}$ is the energy tensor. Symmetry $G_{ij} = G_{ji}$ implies that the energy tensor also has to satisfy symmetry $T_{ij} = T_{ji}$.

(d) Let $\widehat{R}$, $\widehat{R}_{ab}$, $\widehat{G}_{ij}$, $\widehat{R}_{abcd}$ and $\widehat{R}_{abc}{}^{d}$ be curvature tensors for metric $h_{ij}$. From $h^{ij} = \frac{1}{\lambda} g^{ij}$ we obtain that Christoffel symbols of $h_{ij}$ coincide with the Christoffel symbols of $g_{ij}$. Thus, by definition of $R_{abc}{}^{d}$ we have $\widehat{R}_{abc}{}^{d} = R_{abc}{}^{d}$, and

$$\widehat{R}_{abcd} = \lambda R_{abcd}, \quad \widehat{R}_{ab} = R_{ab}, \quad \widehat{R} = \frac{1}{\lambda} R, \quad \widehat{G}_{ab} = G_{ab}.$$

## Problem 5

(a) The Lagrangian is given by

$$L = \left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \tag{9}$$

The geodesic equations for $t(\lambda)$, $\theta(\lambda)$, $r(\lambda)$ and $\phi(\lambda)$ are

$$\begin{aligned}
t: &\quad 2\ddot{t}\left(1 - \frac{r_s}{r}\right) + 2\frac{r_s}{r^2}\dot{r}\dot{t} = 0, \\
\theta: &\quad \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{2}{r}\dot{r}\dot{\theta} = 0, \\
r: &\quad 2\left(1 - \frac{r_s}{r}\right)^{-1}\ddot{r} + \frac{r_s}{r^2}\dot{t}^2 - \frac{r_s}{\left(1 - \frac{r_s}{r}\right)^2 r^2}\dot{r}^2 - 2r\,\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2 = 0, \\
\phi: &\quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\phi}\dot{\theta}
\end{aligned}$$

where dot(s) denote differentiation(s) with respect to $\lambda$.

(b) Since $\frac{\partial L}{\partial t} = 0$ and $\frac{\partial L}{\partial \phi} = 0$, we have two constants of motion $E$ (energy) and $\ell$ (angular momentum) given by

$$\begin{aligned}
E &= \left(1 - \frac{r_s}{r}\right)\dot{t}, \\
\ell &= r^2\dot{\phi}.
\end{aligned}$$

Moreover, $L$ is a constant of motion.

Since $\theta(\lambda) = \frac{\pi}{2}$, the Lagrangian simplifies into

$$L = \left(1 - \frac{r_s}{r}\right) \dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1} \dot{r}^2 - r^2\dot{\phi}^2 \tag{10}$$

and using constants $E$, $L$ and $\ell$, equation (10) can be written as

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = C, \tag{11}$$

where

$$\begin{aligned}
V_{\text{eff}}(r) &= \frac{1}{2}\left(\frac{\ell^2}{r^2} - \frac{r_s\ell^2}{r^3} - \frac{r_s L}{r}\right), \\
C &= \frac{1}{2}\left(E^2 - L\right).
\end{aligned}$$

When we treat equation (11) as a 1-dimensional mechanical system the equation has the interpretation: RHS = kinetic energy $\frac{1}{2}\dot{r}^2$ plus potential energy $V_{\text{eff}}$, and LHS = total energy $C$ (constant).

(c) For a lightlike geodesic we have $L = 0$, and

$$\begin{aligned} V_{\text{eff}}(r) &= \frac{\ell^2}{2}\left(\frac{1}{r^2} - \frac{r_s}{r^3}\right), \\ C &= \frac{1}{2}E^2. \end{aligned}$$

The effective potential $V_{\text{eff}}$ has a *maximum* at $r_\star = \frac{3}{2}r_s$. See Figure 1.

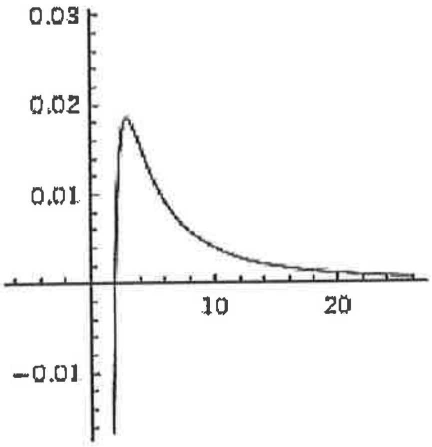


Figure 1: Effective potential $V_{\text{eff}}$.

(d) Trace the field equation which gives

$$R - \frac{1}{2}4R + 4\Lambda = 0 \tag{12}$$

$$R = 4\Lambda. \tag{13}$$

Next eliminate $R$ from the field equation

$$R_{ab} - \frac{1}{2}(4\Lambda)g_{ab} + \Lambda g_{ab} = 0 \tag{14}$$

$$R_{ab} = \Lambda g_{ab}. \tag{15}$$