2011

Model answers MATH3305

Problem 1

(a) (i) This is invalid as the two upper indices are contracted with each other on the LHS. Also, there is one free index labelled b on the LHS and two free indices labelled a and c on the RHS.

(ii) This is invalid as the summation of the LHS with respect to the dummy index b is unclear: there are three b's.

(iii) This is a well defined tensor expression.

(b) $\widetilde{R}_{abc}{}^{d} = \frac{\partial X^e}{\partial \widetilde{X}^a} \frac{\partial X^f}{\partial \widetilde{X}^b} \frac{\partial X^g}{\partial \widetilde{X}^c} \frac{\partial \widetilde{X}^d}{\partial X^h} R_{efg}{}^{h}$.

(c) Given V^a its norm is defined to be $|V| = \sqrt{g_{ab}V^aV^b}$. Since V^a is a tangent vector for a curve $\gamma = \{\cos(\lambda), 2\lambda\}$ then

$$V^a = \begin{pmatrix} -\sin(\lambda) \\ 2 \end{pmatrix}$$

Therefore, its norm is

$$\begin{aligned} |V| &= \sqrt{g_{ab}V^aV^b} \\ &= \left(g_{11}\left(V^1\right)^2 + 2g_{12}V^1V^2 + g_{22}\left(V^2\right)^2\right)^{\frac{1}{2}} \\ &= \left(\sin^2(\lambda) - 12\sin(\lambda) + 8\right)^{\frac{1}{2}} \end{aligned}$$

(d) We can expand any tensor T^{ab} into its symmetric and anti-symmetric parts: $T^{ab} = T^{(ab)} + T^{[ab]}$. It suffices for this problem to show $A_{ab}T^{(ab)} = 0$. This is easily done;

$$\begin{aligned} A_{ab}T^{(ab)} &= A_{ba}T^{(ba)} && \text{Relabel } a \to b\, b \to a \\ &= A_{ba}T^{(ab)} && \text{Symmetric} \\ &= -A_{ab}T^{(ab)} = 0 && \text{Anti}-\text{symmetric} \end{aligned}$$

Therefore, $A_{ab}T^{ab} = A_{ab}\left(T^{(ab)} + T^{[ab]}\right) = A_{ab}T^{[ab]}$

Problem 2

(a) The equation for a hypersphere with radius R in n-dimensions is

$$(X^1)^2 + (X^2)^2 + (X^3)^2 + \cdots + (X^n)^2 = R^2$$

(b) Using the transformations for spherical polar equations where

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}$$

then (a) is automatically satisfied for $n = 3$. Then by differentiating the coordinate relations above whilst keeping R constant we get,

$$\begin{aligned} dx &= R\cos\theta\cos\phi d\theta - R\sin\theta\sin\phi d\phi \\ dy &= R\cos\theta\sin\phi d\theta + R\sin\theta\cos\phi d\phi \\ dz &= -R\sin\theta d\theta \end{aligned}$$

Finally the metric for Euclidean 3-space is

$$ds^2 = dx^2 + dy^2 + dz^2$$

If we substitute in the values for dx, etc. We arrive at the result as required.

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

(c) We can use the Lagrangian approach,

$$L = R^2\left(\dot\theta^2 + \sin^2\theta\dot\phi^2\right)$$

Which gives the following geodesic equations,

$$\begin{aligned} \ddot\theta - \sin\theta\cos\theta\dot\phi^2 &= 0, && (1) \\ \ddot\phi + 2\cot\theta\dot\theta\dot\phi &= 0. && (2) \end{aligned}$$

Comparing equation (1) with

$$\ddot\theta + \Gamma^\theta_{\phi\phi}\dot\phi\dot\phi + 2\Gamma^\theta_{\phi\theta}\dot\phi\dot\theta + \Gamma^\theta_{\theta\theta}\dot\theta\dot\theta = 0$$

and equation (2) with

$$\ddot{\phi} + \Gamma^{\phi}_{\phi\phi}\dot{\phi}\dot{\phi} + 2\Gamma^{\phi}_{\phi\theta}\dot{\phi}\dot{\theta} + \Gamma^{\phi}_{\theta\theta}\dot{\theta}\dot{\theta} = 0$$

gives

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^{\theta}_{\phi\theta} = 0, \quad \Gamma^{\theta}_{\theta\theta} = 0. \tag{3}$$

$$\Gamma^{\phi}_{\phi\phi} = 0, \quad \Gamma^{\phi}_{\phi\theta} = \cot\theta, \quad \Gamma^{\phi}_{\theta\theta} = 0. \tag{4}$$

(d) In 2-dimensions the Riemann tensor only has one independent component: ${R_{\theta\phi\theta}}^{\phi}$ We can than explicitly compute this

$${R_{\theta\phi\theta}}^{\phi} = \Gamma^{\phi}_{\theta\theta,\phi} - \Gamma^{\phi}_{\phi\theta,\theta} + \Gamma^{e}_{\theta\theta}\Gamma^{\phi}_{\phi e} - \Gamma^{e}_{\phi\theta}\Gamma^{\phi}_{e\theta}$$

Immediately the first and third term are zero and finally we are left with (after differentiating),

$${R_{\theta\phi\theta}}^{\phi} = \csc^2\theta - \cot^2\theta$$

(e)

$$\begin{aligned} R_{ac} = {R_{abc}}^{b} &= {R_{a\theta c}}^{\theta} + {R_{a\phi c}}^{\phi} \\ &= R_{a\theta c\theta}g^{\theta\theta} + R_{a\phi c\phi}g^{\phi\phi} \end{aligned}$$

We have two non-zero components,

$$\begin{aligned} R_{\theta\theta} &= R_{\theta\phi\theta\phi}g^{\phi\phi} = R^2(1-\cos^2\theta)g^{\phi\phi} = 1 \\ R_{\phi\phi} &= R_{\theta\phi\theta\phi}g^{\theta\theta} = R^2(1-\cos^2\theta)g^{\theta\theta} = \sin^2\theta \end{aligned}$$

Finally we can compute R

$$\begin{aligned} R = R_{ab}g^{ab} &= R_{\theta\theta}g^{\theta\theta} + R_{\phi\phi}g^{\phi\phi} \\ &= \frac{1}{R^2} + \frac{\sin^2\theta}{R^2\sin^2\theta} = \frac{2}{R^2} \end{aligned}$$

Problem 3

(a) The Riemann tensor and the scalar curvature tell you if the space is flat or curved. If and only if the space is flat will all components of the Riemann be zero everywhere on your manifold. The scalar curvature is a coordinate independent measure of curvature and therefore is linked to the intrinsic curvature of your space.

(b) To show that $V_a W^a$ is constant along the curve, let us consider the quantity

$$T^a \nabla_a (V_b W^b)$$

We can rewrite this as follows

$$T^a \nabla_a (g_{bc} V^b W^c) = T^a V^b W^c \nabla_a g_{bc} + T^a g_{bc} W^c \nabla_a V^b + T^a g_{bc} V^b \nabla_a W^c$$

We order the terms such that

$$T^a V^b W^c \nabla_a g_{bc} + g_{bc} W^c T^a \nabla_a V^b + g_{bc} V^b T^a \nabla_a W^c \overset{!}{=} 0$$

(c) In n dimensions the Riemann curvature tensor has

$$\frac{1}{12} n^2 (n^2 - 1) \tag{5}$$

independent components.

(d) The Christoffel symbols can be gleaned from the equations of motion. Starting with the Lagrangian form of the line element we find.

In 2-dimensions the Riemann tensor only has one independent component: $R_{\theta\phi\theta}{}^{\phi}$ We can than explicitly compute this

$$L = \left(\frac{r}{r-1} \dot{r}^2 + r^2 \theta \dot{\phi}^2 \right)$$

Which gives the following geodesic equations,

$$\begin{aligned} \ddot{r} + \left(\frac{1}{2r} - \frac{1}{2(r-1)} \right) \dot{r}^2 - (r-1)\dot{\phi}^2 &= 0, & (6) \\ \ddot{\phi} + \frac{2}{r} \dot{r} \dot{\phi} &= 0. & (7) \end{aligned}$$

Comparing equation (6) with

$$\ddot{r} + \Gamma^{r}_{\phi\phi}\dot{\phi}\dot{\phi} + 2\Gamma^{r}_{\phi r}\dot{\phi}\dot{r} + \Gamma^{r}_{rr}\dot{r}\dot{r} = 0$$

and equation (7) with

$$\ddot{\phi} + \Gamma^{\phi}_{\phi\phi}\dot{\phi}\dot{\phi} + 2\Gamma^{\phi}_{\phi r}\dot{\phi}\dot{r} + \Gamma^{\phi}_{rr}\dot{r}\dot{r} = 0$$

gives

$$\Gamma^{r}_{\phi\phi} = -(r-1), \quad \Gamma^{r}_{\phi r} = 0, \quad \Gamma^{r}_{rr} = \frac{1}{2r} - \frac{1}{2(r-1)}. \tag{8}$$

$$\Gamma^{\phi}_{\phi\phi} = 0, \quad \Gamma^{\phi}_{\phi r} = \frac{1}{r}, \quad \Gamma^{\phi}_{rr} = 0. \tag{9}$$

$$R_{r\phi r}{}^{\phi} = \Gamma^{\phi}_{rr,\phi} - \Gamma^{\phi}_{\phi r,r} + \Gamma^{e}_{rr}\Gamma^{\phi}_{\phi e} - \Gamma^{e}_{\phi r}\Gamma^{\phi}_{er}$$

Immediately the first term is zero and finally we are left with (after differentiating),

$$R_{r\phi r}{}^{\phi} = -\frac{3}{2r^2} - \frac{1}{2(r^2-r)}$$

Problem 4

(a) With u^b and a^b defined as they are we can find the answer as follows,

$$\begin{aligned}
u^b a_b &= \eta_{bc} u^b a^c \\
&= \eta_{bc} u^b \frac{du^c}{d\lambda} \\
&= \frac{1}{2}\eta_{bc}\frac{d(u^c u^c)}{d\lambda} \\
&= \frac{1}{2}\frac{d(\eta_{bc} u^c u^c)}{d\lambda} \\
&= \frac{1}{2}\frac{d(1)}{d\lambda} = 0
\end{aligned}$$

(b) The defining relation for the Lorentz transformation is the preservation of the Minkowski metric; $L^a{}_b L^c{}_d \eta_{ac} = \eta_{bd}$. Taking the determinant of both sides we find,

$$\begin{aligned}
\mathrm{Det}(L^T.\eta.L) &= \mathrm{Det}(L) \\
\mathrm{Det}(L^T)\mathrm{Det}(\eta)\mathrm{Det}(L) &= \mathrm{Det}(\eta) \\
(\mathrm{Det}(L))^2 &= 1 \\
\Rightarrow \mathrm{Det}(L) &= \pm 1
\end{aligned}$$

(c) (i) The speed of light is an absolute constant. Therefore, both observers should have coordinates

$$\begin{pmatrix} t \\ t \end{pmatrix}$$

for a right moving photon and

$$\begin{pmatrix} t \\ -t \end{pmatrix}$$

for a left moving photon. Moreover, a Lorentz transform from one frame to the other should preserve this.

$$\begin{aligned}
\begin{pmatrix} t \\ x \end{pmatrix}_S &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}_E \\
\begin{pmatrix} t \\ x \end{pmatrix}_S &= \begin{pmatrix} (\alpha+\beta)t \\ (\gamma+\delta)t \end{pmatrix}_E
\end{aligned}$$

Since $t_S = x_S$ we have $\alpha + \beta = \gamma + \delta$. Similarly for a left moving photon we get the relation, $\alpha - \beta = -(\gamma - \delta)$. This gives as required $\alpha = \delta$ and $\beta = \gamma$.

(ii) In the Earth's reference frame, the ship will move in a straight line with equation $x = v_x t$,

$$\begin{aligned}
\begin{pmatrix} t \\ 0 \end{pmatrix}_S &= \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} t \\ v_x t \end{pmatrix}_E \\
\begin{pmatrix} t \\ 0 \end{pmatrix}_S &= \begin{pmatrix} (\alpha+\beta v_x)\,t \\ (\beta+\alpha v_x)\,t \end{pmatrix}_E
\end{aligned}$$

From the spatial component we can deduce $\beta = -\alpha v_x$

(iii) We now have the Lorentz transform in the form,

$$L^a{}_b = \begin{pmatrix} \alpha & -\alpha v_x \\ -\alpha v_x & \alpha \end{pmatrix}$$

We can therefore take the determinate and set it equal to one for proper Lorentz transformations.

$$\begin{aligned} \mathrm{Det}(L) &= \alpha^2 - \alpha^2 v_x^2 = 1 \\ \Rightarrow \alpha^2 &= \frac{1}{1-v_x^2} \\ \alpha &= \sqrt{\frac{1}{1-v_x^2}} \end{aligned}$$

(d) Taking

$$\begin{aligned} e^\phi &= \alpha(1+v_x) \\ &= \frac{1}{(1-v_x^2)^{\frac{1}{2}}}(1+v_x) \\ &= \frac{(1+v_x)}{((1-v_x)(1+v_x))^{\frac{1}{2}}} \\ &= \left(\frac{(1+v_x)}{(1-v_x)}\right)^{\frac{1}{2}} \end{aligned}$$

From this it is quite easy to see that $\cosh(\phi) = \alpha$ and $\sinh\phi = \alpha v_x$. Thus you arrive at the required result.

Problem 5

(a) The Lagrangian is given by

$$L = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \qquad (10)$$

The geodesic equations for $t(\lambda)$, $\theta(\lambda)$, $r(\lambda)$ and $\phi(\lambda)$ are

$$\begin{aligned}
t: &\quad 2\ddot{t}\left(1-\frac{r_s}{r}\right)+2\frac{r_s}{r^2}\dot{r}\dot{t}=0,\\
\theta: &\quad \ddot{\theta}-\sin\theta\cos\theta\dot{\phi}^2+\frac{2}{r}\dot{r}\dot{\theta}=0,\\
r: &\quad 2\left(1-\frac{r_s}{r}\right)^{-1}\ddot{r}+\frac{r_s}{r^2}\dot{t}^2-\frac{r_s}{\left(1-\frac{r_s}{r}\right)^2 r^2}\dot{r}^2-2r\dot{\theta}^2-2r\sin^2\theta\,\dot{\phi}^2=0,\\
\phi: &\quad \ddot{\phi}+\frac{2}{r}\dot{r}\dot{\phi}+2\cot\theta\dot{\phi}\dot{\theta}
\end{aligned}$$

where dot(s) denote differentiation(s) with respect to λ.

(b) Since $\frac{\partial L}{\partial t}=0$ and $\frac{\partial L}{\partial \phi}=0$, we have two constants of motion E (energy) and ℓ (angular momentum) given by

$$\begin{aligned}
E &= \left(1-\frac{r_s}{r}\right)\dot{t},\\
\ell &= r^2\dot{\phi}.
\end{aligned}$$

Moreover, L is a constant of motion.

Since $\theta(\lambda)=\frac{\pi}{2}$, the Lagrangian simplifies into

$$L=\left(1-\frac{r_s}{r}\right)\dot{t}^2-\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2-r^2\dot{\phi}^2 \tag{11}$$

and using constants E, L and ℓ, equation (11) can be written as

$$\frac{1}{2}\dot{r}^2+V_{\text{eff}}(r)=C, \tag{12}$$

where

$$\begin{aligned}
V_{\text{eff}}(r) &= \frac{1}{2}\left(\frac{\ell^2}{r^2}-\frac{r_s\ell^2}{r^3}-\frac{r_sL}{r}\right),\\
C &= \frac{1}{2}\left(E^2-L\right).
\end{aligned}$$

When we treat equation (12) as a 1-dimensional mechanical system the equation has the interpretation: RHS = kinetic energy $\frac{1}{2}\dot{r}^2$ plus potential energy V_{eff}, and LHS = total energy C (constant).

(c) For a lightlike geodesic we have $L = 0$, and

$$V_{\text{eff}}(r) = \frac{\ell^2}{2}\left(\frac{1}{r^2} - \frac{r_s}{r^3}\right),$$
$$C = \frac{1}{2}E^2.$$

The effective potential V_{eff} has a *maximum* at $r_\star = \frac{3}{2}r_s$. See Figure 1.

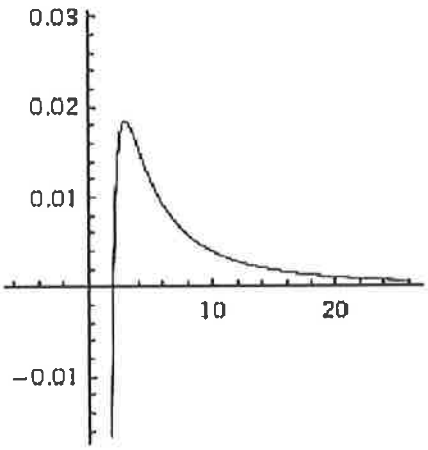

Figure 1: Effective potential V_{eff}.

(d) Trace the field equation which gives

$$R - \frac{1}{2}4R + 4\Lambda = 0 \tag{13}$$
$$R = 4\Lambda. \tag{14}$$

Next eliminate R from the field equation

$$R_{ab} - \frac{1}{2}(4\Lambda)g_{ab} + \Lambda g_{ab} = 0 \tag{15}$$
$$R_{ab} = \Lambda g_{ab}. \tag{16}$$