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QUANTUM PHYSICS 2016

~~[illegible]~~ (A)

(1) (a) $|\psi\rangle = \sum_n c_n |\phi_n\rangle$

$\langle\phi_m|\psi\rangle = \sum_n c_n \langle\phi_m|\phi_n\rangle = \sum_n c_n \delta_{mn}$

$c_n = \langle\phi_n|\psi\rangle$

When they ask to show → cannot simply write $c_n = \langle\phi_n|\psi\rangle$ must show working!

$|\psi\rangle = \sum_n c_n |\phi_n\rangle$

scalar product with $|\phi_m\rangle$

$\langle\phi_m|\psi\rangle = \sum_n c_n \langle\phi_m|\phi_n\rangle$

$\langle\phi_m|\psi\rangle = \sum_n c_n \delta_{mn}$

$\langle\phi_n|\psi\rangle = c_n$

(b) $|\psi\rangle = \sum_a d_a |\chi_a\rangle$ $\qquad d_a = \langle\chi_a|\psi\rangle$

Using $|\psi\rangle = \sum_n \langle\phi_n|\psi\rangle |\phi_n\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle$

$d_a = \langle\chi_a|\psi\rangle = \sum_n \langle\chi_a|\phi_n\rangle\langle\phi_n|\psi\rangle$

$$\boxed{d_a = \sum_n S_{an} c_n}$$

$|\psi\rangle = \sum_a d_a |\chi_a\rangle$

$d_a = \langle\chi_a|\psi\rangle$

$|\psi\rangle = \sum_n \langle\phi_n|\psi\rangle|\phi_n\rangle$

$|\psi\rangle = \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle$

since $d_a = \langle\chi_a|\psi\rangle \Rightarrow$

$\langle\chi_a|\psi\rangle = \sum_n \langle\chi_a|\phi_n\rangle\langle\phi_n|\psi\rangle$

$\langle\chi_a|\psi\rangle = \sum_n S_{an} c_n$

(2) (a) $\sum_n |n\rangle\langle n| = 1$

(b) Show how the unitary nature of the similarity transform (i.e. $SS^\dagger = I$) can be found using the closure relation.

$SS^\dagger$

$S = \sum_k \langle x_i|\phi_k\rangle \qquad S^\dagger = \sum_k \langle\phi_k|x_j\rangle$

$SS^\dagger = \sum_k \langle x_i|\phi_k\rangle\langle\phi_k|x_j\rangle$

$= \langle x_i| \sum_k |\phi_k\rangle\langle\phi_k| |x_j\rangle$

$= \langle x_i|x_j\rangle = \delta_{ij} \Rightarrow SS^\dagger = I$

$S = \sum_n \langle\chi_a|\phi_n\rangle$

$S^\dagger = \sum_n \langle\phi_n|\chi_b\rangle$

$SS^\dagger = \sum_n \langle\chi_a|\phi_n\rangle\langle\phi_n|\chi_b\rangle$

$= \langle\chi_a|\sum_n |\phi_n\rangle\langle\phi_n|\chi_b\rangle$

$= \langle\chi_a|\chi_b\rangle$

$= \delta_{ab}$ $\#$

$SS^\dagger = I$

(c) $\phi = \hat{A}\psi$

$b = Ac$

$|\phi\rangle = \sum_n b_n |n\rangle \qquad \psi = \sum_n c_n |n\rangle$

$A_{kj} = \langle k|\hat{A}|j\rangle$

$b_k = \sum A_{kj} c_j$

$\hat{A}|\phi_n\rangle = a_n |\phi_n\rangle$

quantities $A_{mn} = \langle\phi_m|\hat{A}|\phi_n\rangle$ are the matrix elements of operator $\hat{A}$ in the basis $\{\phi_n\}$.

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(3) (a) $\hat{A}^{\dagger} = (\hat{A}^{*})^{T}$

(b) $\hat{A}|\phi\rangle = a_n|\phi\rangle$ $\hat{A}^{\dagger}\langle\phi| = a_n^{*}\langle\phi|$

$\langle\phi|\hat{A}|\phi\rangle = a_n\langle\phi|\phi\rangle$

$\langle\phi|\hat{A}|\phi\rangle = \langle\hat{A}^{\dagger}\phi|\phi\rangle$ $= \langle\phi|\hat{A}^{\dagger}|\phi\rangle^{*}$ $= \langle\phi|\hat{A}|\phi\rangle^{*}$ $\hat{A}^{*}|\phi\rangle^{*} = a_n^{*}|\phi\rangle^{*}$

$= a_n^{*}\langle\phi|\phi\rangle$

$a_n\langle\phi|\phi\rangle = a_n^{*}\langle\phi|\phi\rangle$

$\therefore a_n$ is real.

(c) $\hat{A} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$\hat{p}$ and $\hat{x}$ give real eigenvalues.

$\therefore$ Eigenvalues of $\hat{p}^2$ and $\hat{x}^2$ must be real and >0

$\therefore$ Eigenvalue of $\hat{A}$ ~~is~~ must be greater than 0.

(4) (a) If state ϕ_n is an eigenstate of both $\hat{A}$ and $\hat{B}$ [ie. $\hat{A}\phi_n = a_n\phi_n$ and $\hat{B}\phi_n = b_n\phi_n$], the observable A and B have simultaneous eigenfunctions, the two observables are compatible.

$[\hat{A},\hat{B}] = 0$ "compatibility implies commuting operators".

(b) $(\Delta A)^2 = \langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle$

(c) $\Delta A \Delta B \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|$

~~The~~ If $\hat{A},\hat{B}$ commute, then the observable A and B can be measured exactly.

If $\hat{A},\hat{B}$ does not commute, then they ~~will be uncer~~ cannot be measured exactly and the uncertainty associated with those observable ~~will~~ $\Delta A\Delta B$ ~~will have a~~ can be a value and above.

(5) (a) Particles with antisymmetric wavefunctions are called fermions

fermions have spin, $s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots$ (odd half integer values)

Particles with symmetric wavefunctions are called bosons.

Bosons have spin, $s = 0, 1, 2, 3 \ldots$ integral values.

$\phi(1,2) = +\phi(2,1)$ $\Rightarrow$ Boson

$\phi(1,2) = -\phi(2,1)$ $\Rightarrow$ Fermion

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(b) → A system of two electrons is in a quantum state

→ Both electrons are in the spin-up state. → spin part is symmetric.

For fermions, the overall wavefunction must be antisymmetric.

Therefore, the spatial part of the wavefunction must be antisymmetric.

The spacial part of the wavefunction can be in the form $\frac{1}{\sqrt{2}}\left[\phi_a(r_1)\phi_b(r_2) - \phi_a(r_2)\phi_b(r_1)\right]$

At the same point $r_1 = r_2$

$\Rightarrow 0$.

Remember the overall wavefunction

$$\boxed{\Psi^{S/A}(1,2) = \frac{1}{\sqrt{2}}\left[\psi_a(1)\psi_b(2) \pm \psi_a(2)\psi_b(1)\right]}$$

When $a = b$

overall wavefunction goes to 0.

(6) (a) $|\alpha\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|\beta\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$S_y = \begin{pmatrix} \langle\alpha|\hat{S}_y|\alpha\rangle & \langle\alpha|\hat{S}_y|\beta\rangle \\ \langle\beta|\hat{S}_y|\alpha\rangle & \langle\beta|\hat{S}_y|\beta\rangle \end{pmatrix}$$

$|\alpha\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$

$|\beta\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$

A Captain's Product

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QM

SECTION B [2016]

Wherever in the notes, it says proof, understand and memorise it. It might come out

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(7) (a) · Hamiltonian, $\hat{H}$ possesses a complete set of orthonormal eigenstates $\{\phi_n\}$, $\phi_0, \phi_1, \phi_2 \ldots$ $\langle \phi_n | \phi_m \rangle = \delta_{nm}$ and corresponding energies $E_0, E_1, E_2 \ldots$ with the ground state energy $E_0 \leq E_1 \leq E_2$ and

$$\hat{H} |\phi_n\rangle = E_n |\phi_n\rangle$$

- Let $|\psi\rangle$ be the wavefunction, and expanding $|\psi\rangle$ in terms of the complete set $\{\phi_n\}$ as

$$|\psi\rangle = \sum_n c_n |\phi_n\rangle$$

- Then the expectation value of the energy in the state $|\psi\rangle$ is

$$\langle E \rangle_\psi = \frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle}$$

$$= \frac{\sum_n \sum_m c_m^* c_n \langle \phi_m | \hat{H} | \phi_n \rangle}{\sum_n \sum_m c_m^* c_n \langle \phi_m | \phi_n \rangle}$$ since $\hat{H}|\phi_n\rangle = E_n |\phi_n\rangle$

$$= \frac{\sum_n |c_n|^2 E_n}{\sum_n |c_n|^2}$$

$$\underbrace{\langle E \rangle_\psi - E_0}_{} = \frac{\sum_n |c_n|^2 E_n}{\sum_n |c_n|^2} - E_0 = \frac{\sum_n |c_n|^2 (E_n - E_0)}{\sum_n |c_n|^2}$$

$\langle E \rangle_\psi - E_0 \geq 0$ because every term on the RHS is positive.

$$\langle E \rangle_\psi \geq E_0$$

$$\frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0.$$

(b) Guess an approximate ground-state wavefunction, called the "trial wave-function". It can be used to calculate the expectation value of the Hamiltonian. To improve the estimate of the ground state energy, the trial wave-function contains variable parameter. The expectation value of the Hamiltonian is minimized wrt these parameters, and the minimum energy provides the best estimate of the true ground-state energy, within the given parameterized 'family' of wavefunctions.

$\int |\psi(x)|^2 \, dx = 1$

$$\boxed{\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 x^2} \qquad \boxed{\hat{p} = -i\hbar \frac{\partial}{\partial x}}$$

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(c) $\psi(x) = \begin{cases} A(c^2 - x^2), & |x| \le c \\ 0, & |x| > c \end{cases}$

(I) $$\int_{-c}^{c} A^2 (c^2 - x^2)(c^2 - x^2) \, dx = 1$$

$$A^2 \int_{-c}^{c} c^4 - 2x^2c^2 + x^4 \, dx = 1$$

$$A^2 \left[\frac{x^5}{5} - \frac{2x^3c^2}{3} + c^4 x \right]_{-c}^{c} = 1$$

$$\left(\frac{c^5}{5} - \frac{2c^5}{3} + c^5 \right) - \left(-\frac{c^5}{5} + \frac{2c^5}{3} - c^5 \right) = 1$$

$$A^2 \left(\frac{16}{15} c^5 \right) = 1$$

$$A = \sqrt{\frac{15}{16c^5}}$$

$$\boxed{\psi = \begin{cases} \sqrt{\frac{15}{16c^5}} (c^2 - x^2), & |x| \le c \\ 0 & |x| > c \end{cases}}$$

(II) Show $\rightarrow$ $E(c) = \frac{5}{4} \frac{\hbar^2}{mc^2} + \frac{m\omega^2 c^2}{14}$

$$E(c) = \langle \psi | \hat{H} | \psi \rangle$$

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (c^2 - x^2)$$

$$-\frac{\hbar^2}{2m} \frac{d}{dx} (-2x)$$

$$= \langle \psi | -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 | \psi \rangle$$

$$-\frac{\hbar^2}{2m}(-2)$$

$$\cancel{= \frac{15}{16c^5} \int_{-c}^{c} \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (c^2 - x^2)^2 + \frac{1}{2} m\omega^2 x^2 (c^2 - x^2)^2 \right) dx}$$

$$= \frac{15}{16c^5} \int_{-c}^{c} \left[(c^2 - x^2) \left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 \right) (c^2 - x^2) \right] dx$$

$$= \frac{15}{16c^5} \int_{-c}^{c} (c^2 - x^2) \left(\frac{\hbar^2}{m} + \frac{m\omega^2 x^2 c^2}{2} - \frac{m\omega^2 x^4}{2} \right) dx$$

$$= \frac{15}{16c^5} \int_{-c}^{c} \left(\frac{c^2\hbar^2}{m} + \frac{m\omega^2 x^2 c^4}{2} - \frac{m\omega^2 x^4 c^2}{2} - \frac{x^2\hbar^2}{m} - \frac{m\omega^2 x^4 c^2}{2} + \frac{m\omega^2 x^6}{2} \right) dx$$

$$= \frac{15}{16c^5} \left[\frac{c^2\hbar^2 x}{m} + \frac{m\omega^2 x^3 c^4}{6} - \frac{m\omega^2 x^5 c^2}{10} - \frac{x^3\hbar^2}{3m} - \frac{m\omega^2 x^5 c^2}{10} + \frac{m\omega^2 x^7}{14} \right]_{-c}^{c}$$

$$= \frac{15}{16c^5} \left\{ \left(\frac{c^3\hbar^2}{m} + \frac{c^7 m\omega^2}{6} - \frac{c^7 m\omega^2}{10} - \frac{c^3\hbar^2}{3m} - \frac{m\omega^2 c^7}{10} + \frac{m\omega^2 c^7}{14} \right) - \left(-\frac{c^3\hbar^2}{m} - \frac{c^7 m\omega^2}{6} + \frac{c^7 m\omega^2}{10} + \frac{c^3\hbar^2}{3m} + \frac{c^7 m\omega^2}{10} - \frac{c^7 m\omega^2}{14} \right) \right.$$

$\cancel{E_0 = \hbar\omega\left(\frac{1}{2}+n\right)}$

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$$= \frac{15}{16c^5}\left\{\left(\frac{2}{3}\frac{c^3\hbar^2}{m} + \frac{4}{105}c^7 m\omega^2\right) - \left(-\frac{2}{3}\frac{c^3\hbar^2}{m} - \frac{4}{105}m\omega^2\right)\right\}$$

$$= \frac{15}{16c^5}\left\{\frac{4}{3}\frac{c^3\hbar^2}{m} + \frac{8}{105}c^7 m\omega^2\right\}$$

$$E(c) = \frac{5}{4}\frac{\hbar^2}{mc^2} + \frac{m\omega^2}{14}c^2$$

(III) ~~Lowest energy:~~ $\cancel{E_0 = \frac{\hbar\omega}{2}}$

$$E(c) = \frac{5}{4}\frac{\hbar^2}{mc^2} + \frac{m\omega^2 c^2}{14}$$

$$\frac{5}{4}\frac{\hbar^2}{m}c^{-2}$$

$$\frac{dE}{dc} = -\frac{10}{4}\frac{\hbar^2}{mc^3} + \frac{2}{14}m\omega^2 c = 0$$

$$-2\cdot\left(\frac{5}{4}\frac{\hbar^2}{m}c^{-3}\right)$$

$$c\left(-\frac{10}{4}\frac{\hbar^2}{mc^4} + \frac{1}{7}m\omega^2\right) = 0$$

$$c = 0 \qquad \frac{1}{7}m\omega^2 = \frac{5}{2}\frac{\hbar^2}{mc^4}$$

$$\frac{2}{35}\frac{m^2\omega^2}{\hbar^2} = \frac{1}{c^4}$$

$$\frac{35\hbar^2}{2m^2\omega^2} = c^4$$

$$\boxed{c = \left(\frac{35}{2}\right)^{1/4}\left(\frac{\hbar}{m\omega}\right)^{1/2}}$$

$$E(c) = \frac{5}{4}\frac{\hbar^2}{m\left(\left(\frac{35}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)\right)} + \frac{m\omega^2}{14}\left[\left(\frac{35}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)\right]$$

$$= \frac{5}{4}\left(\frac{2}{35}\right)^{1/2}\hbar\omega + \frac{1}{14}\left(\frac{35}{2}\right)^{1/2}\hbar\omega$$

$$= 0.5976\ \hbar\omega$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} +$$

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(d) $\psi'(x) = (d^4 - x^4)$ $E(d) = \frac{45}{28}\frac{\hbar^2}{md^2} + \frac{15}{154}m\omega^2 d^2$

$$\frac{dE}{dd} = -2\left(\frac{45}{28}\right)\frac{\hbar^2}{md^3} + \frac{30}{154}m\omega^2 d = 0$$

$$d\left(-\frac{90}{28}\frac{\hbar^2}{md^4} + \frac{30}{154}m\omega^2\right) = 0$$

$$\frac{30}{154}m\omega^2 = \frac{90}{28}\frac{\hbar^2}{md^4}$$

$$d^4 = \frac{90\hbar^2(154)}{28m(30m\omega^2)}$$

$$d^4 = \frac{13860\,\hbar^2}{840\,m^2\omega^2}$$

$$d = \left(\frac{33}{2}\right)^{1/4}\left(\frac{\hbar}{m\omega}\right)^{1/2}$$

$$E(d) = \frac{45}{28}\frac{\hbar^2}{m\left(\left(\frac{33}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)\right)} + \frac{15}{154}m\omega^2\left(\frac{33}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right)$$

$$= \frac{45}{28}\left(\frac{2}{33}\right)^{1/2}\hbar\omega + \frac{15}{154}\left(\frac{33}{2}\right)^{1/2}\hbar\omega$$

$$= 0.791\ldots\hbar\omega.$$

(e) Actual ground state energy $\rightarrow \frac{1}{2}\hbar\omega$.

$A(c^2 - x^2)$ gives a better estimate because the actual ground state eigenfunction is $A e^{-\alpha^2 x^2/2}$ $\alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$.

$\downarrow$

Expanding this: $A\left(1 + \left(-\frac{m\omega}{2\hbar}x^2\right) + \frac{1}{2!}\left(-\frac{m\omega}{2\hbar}x^2\right)^2 + \ldots\right)$

$$A\left(1 - \frac{m\omega}{2\hbar}x^2 + \frac{m^2\omega^2 x^4}{8\hbar^2} + \ldots\right).$$

$A(c^2 - x^2)$ get first two terms correct. Therefore, it gives a ~~better~~ accurate energy.

$a_\pm = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega\hat{x} \mp i\hat{p}) = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} \mp i\frac{\hat{p}}{\hbar\alpha}\right)$ $\alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$.

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Show

(8) (a) $\hat{H} = \left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)\hbar\omega$

- Do $\hat{a}_+\hat{a}_-$
- The you will get it

(b) Show $[\hat{a}_-, \hat{a}_+] = 1$

$\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$

$\frac{1}{2}\left(\alpha\hat{x} + i\frac{\hat{p}}{\hbar\alpha}\right)\left(\alpha\hat{x} - i\frac{\hat{p}}{\hbar\alpha}\right) - \frac{1}{2}\left(\alpha\hat{x} - i\frac{\hat{p}}{\hbar\alpha}\right)\left(\alpha\hat{x} + i\frac{\hat{p}}{\hbar\alpha}\right)$

$\frac{1}{2}\left(\alpha^2\hat{x}^2 - \cancel{\alpha} i\frac{\hat{x}\hat{p}}{\hbar\cancel{\alpha}} + \frac{i\hat{p}\hat{x}}{\hbar} + \frac{\hat{p}^2}{\hbar^2\alpha^2}\right) - \frac{1}{2}\left(\alpha^2\hat{x}^2 + \frac{i}{\hbar}\hat{x}\hat{p} - \frac{i}{\hbar}\hat{p}\hat{x} + \frac{\hat{p}}{\hbar^2\alpha^2}\right)$

$\left(-\frac{i}{2\hbar}\hat{x}\hat{p} - \frac{i}{2\hbar}\hat{x}\hat{p}\right) + \left(\frac{i}{2\hbar}\hat{p}\hat{x} + \frac{i}{2\hbar}\hat{p}\hat{x}\right)$

$-\frac{i}{\hbar}\hat{x}\hat{p} + \frac{i}{\hbar}\hat{p}\hat{x}$

$\frac{i}{\hbar}(\hat{p}\hat{x} - \hat{x}\hat{p})$

$\frac{i}{\hbar}[\hat{p}, \hat{x}] = \frac{i}{\hbar}(-i\hbar) = 1.$

(c) $[\hat{H}, \hat{a}_\pm] = \pm\hat{a}_\pm\hbar\omega$

$\hat{H}|n\rangle = E_n|n\rangle$

$\hat{a}_+\hat{H}|n\rangle = E_n\hat{a}_+|n\rangle$

$(\hat{H}\hat{a}_+ - \hbar\omega\hat{a}_+)|n\rangle = E_n\hat{a}_+|n\rangle$

$\hat{H}\hat{a}_+|n\rangle = (E_n + \hbar\omega)\hat{a}_+|n\rangle$

$[\hat{H}, \hat{a}_+] = \hat{a}_+\hbar\omega$

$\hat{H}\hat{a}_+ - \hat{a}_+\hat{H} = \hbar\omega\hat{a}_+$

$\hat{H}\hat{a}_+ - \hbar\omega\hat{a}_+ = \hat{a}_+\hat{H}$

$\psi_{n_x, n_y, n_z}(x,y,z) = U_{n_x}(x)\, U_{n_y}(y)\, U_{n_z}(z)$

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(d) 2D → QHO.

Hamiltonian: $\hat{H} = \frac{1}{2m}(\hat{P}_x^2 + \hat{P}_y^2) + \frac{1}{2} m\omega^2 (\hat{x}^2 + \hat{y}^2)$

$|\psi\rangle = |n_x\rangle |m_y\rangle$

CHECK WHETHER NEED TO DERIVE EIGENENERGY FORMULA OR NOT!

$\hat{H}|\psi\rangle = E_{n_x m_y} |\psi\rangle$

$\hat{H} |n_x\rangle |m_y\rangle = E_{n_x m_y} |n_x\rangle |m_y\rangle$

In 2D, general formula for eigenenergy is $E_{n_x m_y} = (n_x + m_y + 1)\hbar\omega$

First 3 energy levels: $E_{0,0} = \hbar\omega$ corresponding eigenstates $\longrightarrow |0\rangle|0\rangle$

$E_{1,0} = E_{0,1} = 2\hbar\omega \rightarrow |1\rangle|0\rangle$ & $|0\rangle|1\rangle$

$E_{0,2} = E_{2,0} = E_{1,1} = 3\hbar\omega \rightarrow |0\rangle|2\rangle$ & $|2\rangle|0\rangle$ & $|1\rangle|1\rangle$

(e) show $\hat{x}^2 \hat{y}^2 = \frac{1}{4\alpha^4}(\hat{a}_{x+}^2 + \hat{a}_{x-}^2 + 2\hat{a}_{x+}\hat{a}_{x-} + 1)(\hat{a}_{y+}^2 + \hat{a}_{y-}^2 + 2\hat{a}_{y+}\hat{a}_{y-} + 1)$

where $\hat{a}_{x\pm}$ is a raising / lowering operator for eigenstates $|n_x\rangle$

$\hat{a}_{y\pm}$ is a raising / lowering operator for eigenstates $|m_y\rangle$

$\hat{a}_{x+}|n_x\rangle = \sqrt{n_x+1}\,|n_x+1\rangle$ $\hat{a}_{y+}|m_y\rangle = \sqrt{m_y+1}\,|m_y+1\rangle$

$\hat{a}_{x-}|n_x\rangle = \sqrt{n_x}\,|n_x-1\rangle$ $\hat{a}_{y-}|m_y\rangle = \sqrt{m_y}\,|m_y-1\rangle$

$\hat{x} = \frac{1}{\sqrt{2}\alpha}(\hat{a}_{x+} + \hat{a}_{x-})$

$\hat{x}^2 = \frac{1}{2\alpha^2}(\hat{a}_{x+} + \hat{a}_{x-})(\hat{a}_{x+} + \hat{a}_{x-})$

$\hat{y}^2 = \frac{1}{2\alpha^2}(\hat{a}_{y+}^2 + \hat{a}_{y-}^2 + \hat{a}_{y+}\hat{a}_{y-} + \hat{a}_{y-}\hat{a}_{y+})$

$= \frac{1}{2\alpha^2}(\hat{a}_{x+}^2 + \hat{a}_{x-}^2 + \hat{a}_{x+}\hat{a}_{x-} + \hat{a}_{x-}\hat{a}_{x+})$

$\hat{x}^2 = \frac{1}{2\alpha^2}(\hat{a}_{x+}^2 + \hat{a}_{x-}^2 + 2\hat{a}_{x+}\hat{a}_{x-} + 1)$ $\hat{y}^2 = \frac{1}{2\alpha^2}(\hat{a}_{y+}^2 + \hat{a}_{y-}^2 + 2\hat{a}_{y+}\hat{a}_{y-} + 1)$.

$\hat{x}^2 y^2 =$ ______ ✓

A Captain's Product

$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}_+ + \hat{a}_-) = \frac{1}{\sqrt{2}\alpha}(\hat{a}_+ + \hat{a}_-)$

$[\hat{a}_{x-}, \hat{a}_{x+}] = 1$

$\hat{a}_{x-}\hat{a}_{x+} - \hat{a}_{x+}\hat{a}_{x-} = 1$.

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(f) First order energy change formula : $\langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle$ $\hat{H}' = \gamma \hat{x}^2 \hat{y}^2$

~~which way should it be?~~ ASK!

For the first 3 eigenstates: $\langle n_x | \langle m_y | \gamma \hat{x}^2 \hat{y}^2 | n_x \rangle | m_y \rangle$

$$= \langle n_x | \langle m_y | \frac{\gamma}{4\alpha^4} (\hat{a}_{x+}^2 + \hat{a}_{x-}^2 + 2\hat{a}_{x+}\hat{a}_{x-} + 1)(\hat{a}_{y+}^2 + \hat{a}_{y-}^2 + 2\hat{a}_{y+}\hat{a}_{y-} + 1) | n_x \rangle | m_y \rangle$$

$$= \frac{\gamma}{4\alpha^4} \Big\{ \Big(\underset{0}{\langle n_x | \hat{a}_{x+}^2 | n_x \rangle} + \underset{0}{\langle n_x | \hat{a}_{x-}^2 | n_x \rangle} + 2 \langle n_x | \hat{a}_{x+} \hat{a}_{x-} | n_x \rangle + \underset{1}{\langle n_x | n_x \rangle} \Big) \times$$

$$\Big(\underset{0}{\langle m_y | \hat{a}_{y+}^2 | m_y \rangle} + \underset{0}{\langle m_y | \hat{a}_{y-}^2 | m_y \rangle} + 2\langle m_y | \hat{a}_{y+} \hat{a}_{y-} | m_y \rangle + \underset{1}{\langle m_y | m_y \rangle}$$

$$\hat{a}_{x+} | n_x \rangle = \sqrt{n_x + 1} \, | n_x + 1 \rangle$$

$$\hat{a}_{x+}^2 | n_x \rangle = \sqrt{n_x + 1} \sqrt{n_x + 2} \, | n_x + 2 \rangle$$

$$\langle n_x | \hat{a}_{x+}^2 | n_x \rangle = \sqrt{n_x + 1} \sqrt{n_x + 2} \, \delta_{n_x, n_x + 2}$$

$$\hat{a}_{x+} \hat{a}_{x-} | n_x \rangle = \sqrt{n_x} \, \hat{a}_{x+} | n_x - 1 \rangle$$

$$\langle n_x | \hat{a}_{x+} \hat{a}_{x-} | n_x \rangle = \sqrt{n_x} \sqrt{n_x} \, | n_x \rangle$$

$$= n_x \langle n_x | n_x \rangle .$$

$$E_{n_x m_y}^{(0)} = \frac{\gamma}{4\alpha^4} \Big\{ (2(n_x) + 1)(2(m_y) + 1) \Big\}$$

$$E_{0,0}^{(0)} = \frac{\gamma}{4\alpha^4}$$

$$E_{0,1}^{(0)} = E_{1,0}^{(0)} = \frac{3\gamma}{4\alpha^4}$$

(g) $E_n^{(1)} = \langle n_x | \langle m_y | \hat{H}' | n_x \rangle | m_y \rangle$

EVEN . SO THAT INTEGRAL DOESN'T GO TO 0 .

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$\hat{J}^2, \hat{J}$ $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z$ (z, y)

$$\boxed{\begin{aligned}[\hat{A}, \hat{B}\hat{C}] &= [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] \\ [\hat{A}\hat{B}, \hat{C}] &= \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}\end{aligned}}$$ → MEMORISE (NOT GIVEN IN 2016)

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(9) (a) $[\hat{J}^2, \hat{J}_x] = [\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_x]$

$= [\hat{J}_x^2, \hat{J}_x] + [\hat{J}_y^2, \hat{J}_x] + [\hat{J}_z^2, \hat{J}_x]$

$= \hat{J}_y[\hat{J}_y, \hat{J}_x] + [\hat{J}_y, \hat{J}_x]\hat{J}_y + \hat{J}_z[\hat{J}_z, \hat{J}_x] + [\hat{J}_z, \hat{J}_x]\hat{J}_z$

$= -i\hbar \hat{J}_y \hat{J}_z - i\hbar \hat{J}_z \hat{J}_y + i\hbar \hat{J}_z \hat{J}_y + i\hbar \hat{J}_y \hat{J}_z$

$= 0$

(b) $\hat{J}^2 |j, m_j\rangle = j(j+1)\hbar^2 |j, m_j\rangle$ $\hat{J}_z |j\, m_j\rangle = m_j \hbar |j\, m_j\rangle$

$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ $\hat{J}_- = \hat{J}_x - i\hat{J}_y$

Show that, provided that $-j \leq m_j \leq j-1$, the state $\hat{J}_+ |j, m_j\rangle$ is also an eigenvector of $\hat{J}_z$, with magnetic q.n equal to $m_j + 1$.

$\hat{J}_z |j, m_j\rangle = m_j \hbar |j, m_j\rangle$

$\hat{J}_+ \hat{J}_z |j, m_j\rangle = m_j \hbar \hat{J}_+ |j, m_j\rangle$

$(\hat{J}_z \hat{J}_+ - \hbar \hat{J}_+)|j, m_j\rangle = m_j \hbar \hat{J}_+ |j, m_j\rangle$

$\boxed{[\hat{J}_z, \hat{J}_\pm] = \pm\hbar \hat{J}_\pm}$

$[\hat{J}_z, \hat{J}_+] = +\hbar \hat{J}_+$

INCOMPLETE!!

$\hat{J}_z \hat{J}_+ |j, m_j\rangle = (m_j + 1)\hbar \hat{J}_+ |j, m_j\rangle$

$\hat{J}_z \hat{J}_+ - \hat{J}_+ \hat{J}_z = \hbar \hat{J}_+$

$\hat{J}_z \hat{J}_+ - \hbar \hat{J}_+ = \hat{J}_+ \hat{J}_z$

But also $\hat{J}_z |j, m+1\rangle = (m_j + 1)\hbar |j, m+1\rangle$

$\therefore$ $\hat{J}_+ |j, m_j\rangle$ must be a multiple of $|j, m+1\rangle$

$\hat{J}_+ |j, m\rangle = [j(j+1) - m_j(m_j+1)]^{1/2} \hbar |j, m_j + 1\rangle$

when $m_j = j$, $\hat{J}_+ |j, j\rangle = 0$

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(c) $\hat{J}_{\pm}|j,m_j\rangle = [j(j+1) - m_j(m_j \pm 1)]^{1/2} \hbar |j, m_j \pm 1\rangle$

$\hat{J}_{+}|j,m_j\rangle = [j(j+1) - m_j(m_j+1)]^{1/2} \hbar |j, m_j+1\rangle$

$\hat{J}_{-}|j,m_j\rangle = [j(j+1) - m_j(m_j-1)]^{1/2} \hbar |j, m_j-1\rangle$

$\hat{J}_{+}\hat{J}_{-}|j,m_j\rangle = [j(j+1) - m_j(m_j-1)]^{1/2} \hbar \hat{J}_{+}|j, m_j-1\rangle$

$\hat{J}_{+}\hat{J}_{-}|j,m_j\rangle = [j(j+1) - m_j(m_j-1)]^{1/2} \hbar [j(j+1) - (m_j-1)(m_j)]^{1/2} \hbar |j, m_j\rangle$

$\hat{J}_{+}\hat{J}_{-}|j,m_j\rangle = [j(j+1) - m_j(m_j-1)] \hbar^2 |j, m_j\rangle$

Condition : ~~$m_j = j$ again?~~

(d) Show $\hat{J}_{+}\hat{J}_{-} = \hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$.

$(\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y)$

$\hat{J}_x^2 - i\hat{J}_x\hat{J}_y + i\hat{J}_y\hat{J}_x + \hat{J}_y^2$ $\qquad \hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

$(\hat{J}^2 - \hat{J}_z^2) - i(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x)$ $\qquad [\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$.

$\hat{J}^2 - \hat{J}_z^2 + \hbar\hat{J}_z$.

Relate to the previous part:

$\hat{J}^2|j,m_j\rangle = j(j+1)|j,m_j\rangle$ $\qquad \hbar\hat{J}_z|j,m_j\rangle = m_j\hbar|j,m_j\rangle$

INCOMPLETE! $\hat{J}_z^2|j,m_j\rangle = m_j^2\hbar^2|j,m_j\rangle$

$j(j+1)\hbar^2 - m_j^2\hbar^2 + m_j\hbar^2$

$[j(j+1) - m_j(m_j+1)]\hbar^2$

(e) $\hat{J} = \hat{L} + \hat{S}$

$\hat{J}^2 = (\hat{L}+\hat{S})^2$

$= (\hat{L}+\hat{S}).(\hat{L}+\hat{S})$

$\hat{J}^2 = \hat{L}^2 + 2\hat{L}.\hat{S} + \hat{S}^2$

$\hat{L}.\hat{S} = \frac{1}{2}(\hat{J}^2 - \hat{S}^2 - \hat{L}^2)$

$l = 1 \quad s = \frac{1}{2} \quad j = \frac{1}{2}, \frac{3}{2}$

$(\hat{L}_z + \hat{S}_z)(\hat{L}_z + \hat{S}_z)$

$\hat{L}_z^2 + 2\hat{L}_z \cdot \hat{S}_z + \hat{S}_z^2$

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(f) $\hat{H} = \frac{\epsilon_1}{\hbar^2}(\hat{L} + \hat{S}) \cdot \hat{S} + \frac{\epsilon_2}{\hbar^2}(\hat{L}_z + \hat{S}_z)^2$

$= \frac{\epsilon_1}{\hbar^2}(\hat{L} \cdot \hat{S} + \hat{S}^2) + \frac{\epsilon_2}{\hbar^2}(\hat{L}_z + \hat{S}_z)^2$

$= \frac{\epsilon_1}{\hbar^2}\left(\frac{1}{2}(\hat{J}^2 - \hat{S}^2 - \hat{L}^2) + \hat{S}^2\right) + \frac{\epsilon_2}{\hbar^2}(\hat{L}_z + \hat{S}_z)^2$

$= \frac{\epsilon_1}{\hbar^2}\left(\frac{\hat{J}^2}{2} + \frac{\hat{S}^2}{2} - \frac{\hat{L}^2}{2}\right) + \frac{\epsilon_2}{\hbar^2}(\hat{L}_z + \hat{S}_z)^2$ ← NEED TO EXPAND OR NOT? ASK!

OR $\hat{L}_z + \hat{S}_z = \hat{J}_z$

$= \frac{\epsilon_1}{\hbar^2}\left(\frac{j(j+1)\hbar^2}{2} + \frac{s(s+1)\hbar^2}{2} - \frac{l(l+1)\hbar^2}{2}\right) + \frac{\epsilon_2}{\hbar^2}(m_j^2\hbar^2)$

$E = \epsilon_1\left(\frac{j(j+1)}{2} + \frac{s(s+1)}{2} - \frac{l(l+1)}{2}\right) + \epsilon_2 m_j^2$

When $l = 1$, $s = \frac{1}{2}$, $j = \frac{1}{2}$ or $\frac{3}{2}$

For $j = \frac{1}{2}$: $\quad E = -\frac{\epsilon_1}{4} + \frac{1}{4}\epsilon_2$

$m_j = \pm\frac{1}{2}$

~~When l = 1~~

For $j = \frac{3}{2}$: $\quad E = \frac{5}{4}\epsilon_1 + \frac{9}{4}\epsilon_2$

$m_j = \pm\frac{3}{2}, \pm\frac{1}{2} \rightarrow E = \frac{5}{4}\epsilon_1 + \frac{1}{4}\epsilon_2$. CHECK!

(g)

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(10) $\hat{H}_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$

(a) ~~Here~~ $\hat{H} = \hat{H}_0 + \lambda H'$ λ: To keep track of the order of perturbation.

$$(\hat{H}_0 + \lambda H')|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$$

Since the H' cause "small" changes to $\hat{H}_0$, in perturbation theory, the solution is expanded as a power series in the perturbation. For the theory to be valid, the series must be convergent. For the theory to be useful, the series must be rapidly convergent such that only the first few terms are important.

~~Eigen~~ (0) → in $E_n^{(0)}$: ~~[illegible]~~ known energy change.

$\psi_n^{(0)}$: ~~[illegible]~~ known eigenfunction.

$$E_n = \sum_{j=0}^{\infty} \lambda^j E_n^{(j)} = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$$

$$|\psi_n\rangle = \sum_{j=0}^{\infty} \lambda^j |\psi_n^{(j)}\rangle = |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \dots$$

(0) gives the order of change

λ^j ; j gives the order of perturbation.

$|\psi_n^{(1)}\rangle$: First order eigenfunction

$|\psi_n^{(2)}\rangle$: Second " "

$E_n^{(1)}$: First order energy change

$E_n^{(2)}$: Second order energy change.

(b) $(\hat{H}_0 + \lambda H')\left(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \dots\right) =$

$$\left(E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots\right)\left(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \dots\right)$$

Equating the coefficients of equal powers of λ.

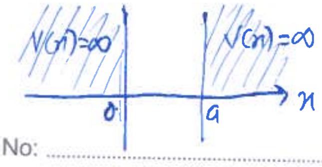
λ^0 ; $\hat{H}_0|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(0)}\rangle$

λ^1 ; $\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}'|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle$

$$(\hat{H}_0 - E_n^{(0)})|\psi_n^{(1)}\rangle = (E_n^{(1)} - \hat{H}')|\psi_n^{(0)}\rangle$$

Taking scalar product with $|\psi_n^{(0)}\rangle$

$\rightarrow E_n^{(1)} = \langle\psi_n^{(0)}|\hat{H}'|\psi_n^{(0)}\rangle$



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(c) Eigenvalues $E_n = \frac{\hbar^2 k_n^2}{2m}$ $k_n = \frac{n\pi}{a}$

READ THE LAST FEW PAGES OF THE NOTES AGAIN!

Eigenvectors $\psi_n(x) = \sqrt{\frac{2}{a}} \sin(k_n x)$

Two electrons in the same square well, with opposite spins, in singlet state.

→ Overall wavefunction must be antisymmetric.
Siglet state is antisymmetric.

∴ Spatial wavefunction must be symmetric on exchange of 2 particles.

$$\psi_n(x) = \frac{2}{a} \sin(k_{n_1} x_1) \sin(k_{n_2} x_2) \quad \text{Symmetric.}$$

$$E = \langle \psi_n | \hat{H}_1 + \hat{H}_2 | \psi_n \rangle = E_1 + E_2 = \frac{\hbar^2 \pi^2}{2ma^2} (n_1^2 + n_2^2)$$

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