# PST (The 2014 Exam)

# and some quasi-final recommendations

# A1

1. Sketch typical hysteresis loops for both a hard and a soft magnetic material. Identify your curves and indicate, for at least one, its remanence and coercivity.

Magnetism

# A1

1. Sketch typical hysteresis loops for both a hard and a soft magnetic material. Identify your curves and indicate, for at least one, its remanence and coercivity.

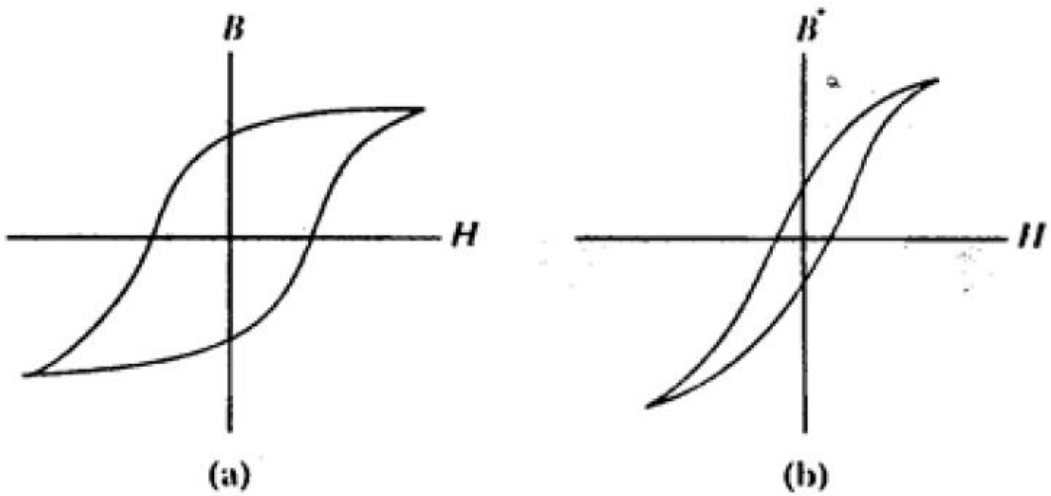



Figure 4.8: The hysteresis curves of (a) a hard and (b) a soft magnetic material.

# A1

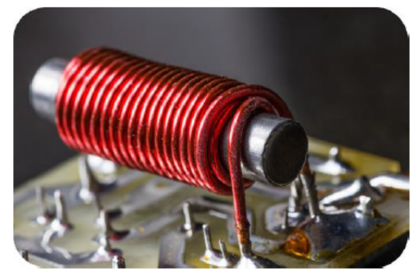
1. Sketch typical hysteresis loops for both a hard and a soft magnetic material. Identify your curves and indicate, for at least one, its remanence and coercivity.



## External magnetic field influence

This process of boundary shifting is not entirely reversible.

Once the external field is switched off, some of the original domains are restored, but many retain the last magnetized direction.

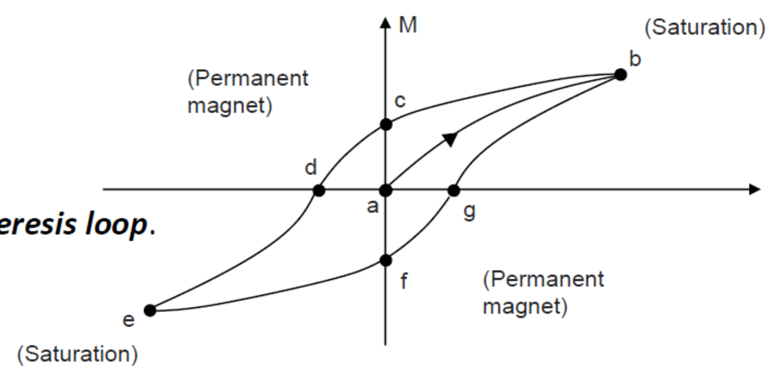


Consider an iron bar and a coil wrapped around it.

f) → g) and then b)

The loop can then be closed by raising the current as in the beginning to the saturation point (through the zero Magnetized point).

The path described is the ***hysteresis loop***.

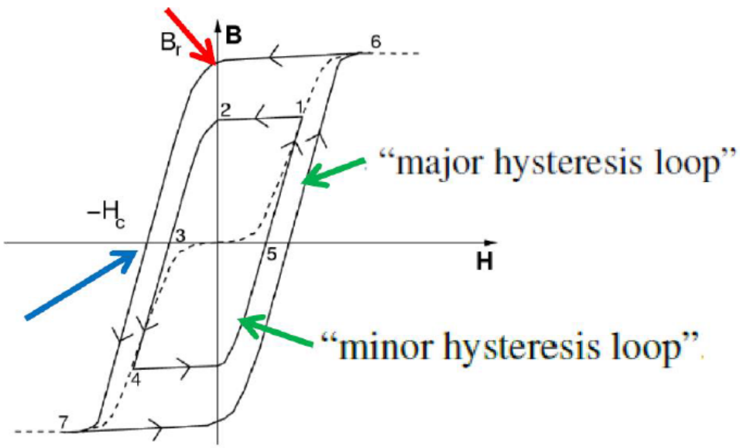


# A1

1. Sketch typical hysteresis loops for both a hard and a soft magnetic material. Identify your curves and indicate, for at least one, its remanence and coercivity.



## B-H curves for a ferromagnetic material




$\mathbf{H}_s$ ***Saturation intensity***

The magnetic intensity required to reach saturation

$\mathbf{B}_s$ ***Saturation induction***

Magnetic induction at saturation

$\mathbf{B}_r$ ***Remanence***

Magnetic induction value when H is returned to zero.

$\mathbf{H}_c$ ***Coercivity***

Value of magnetic intensity required to reduce B to zero after saturation.

# A2

2. Circularly polarized light is incident at the Brewster angle on a dielectric surface. With explanation, describe the polarization of the light that is reflected from this surface.

EM waves
Polarization
Brewster An.

**A2**

2. Circularly polarized light is incident at the Brewster angle on a dielectric surface. With explanation, describe the polarization of the light that is reflected from this surface.

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RECAP

## Linear Polarization

The polarization in the case of generic oscillation is just a combination of x and y:

$$\hat{\mathbf{s}} = (\cos\theta)\hat{\mathbf{x}} + (\sin\theta)\hat{\mathbf{y}}$$

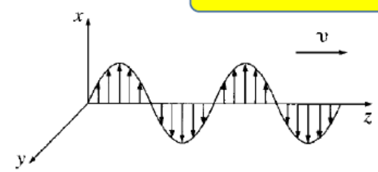
Note how this vector changes sign for $\theta = \pi$

This is a reminder that *polarization is defined with 180 degreed periodicity and identified in many contexts with a headless (or a double-headed) arrow.*

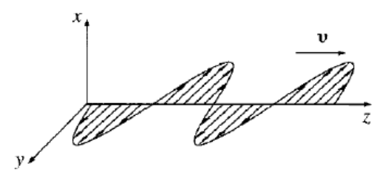
If we replace that expression for the s unit vector in the previous wave equation we get:

$$\tilde{\mathbf{f}}(z,t) = \tilde{A}\cos\theta\, e^{i(kz-\omega t+\varphi)}\hat{\mathbf{x}} + \tilde{A}\sin\theta\, e^{i(kz-\omega t+\varphi)}\hat{\mathbf{y}}$$

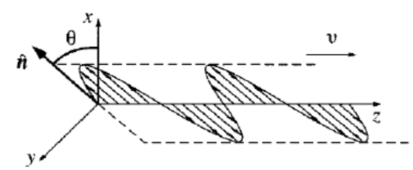
So polarization at an arbitrary angle is still a linear polarization obtained through the combination of two linear polarizations.




(a) Vertical polarization




(b) Horizontal polarization




(c) Polarization vector

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# A2

2. Circularly polarized light is incident at the Brewster angle on a dielectric surface. With explanation, describe the polarization of the light that is reflected from this surface.

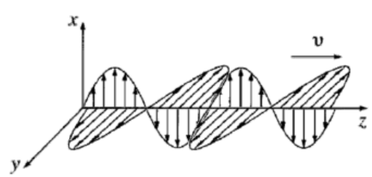


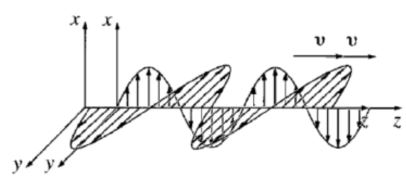
RECAP

## Circular Polarization

For completion, we should ask ourselves (given the field notation) what happens if we combine two linear polarizations which do not have the same additional phase constant term.

$$\tilde{\mathbf{f}}(z,t) = \tilde{A}\cos\theta\, e^{i(kz-\omega t+\varphi)}\hat{\mathbf{x}} + \tilde{A}\sin\theta\, e^{i(kz-\omega t+\psi)}\hat{\mathbf{y}}$$




Specifically we shift one wave by 90 degrees ( $\pi/2$ ) whereby now the value on one axis is =0 when the other is maximum.

# A2

2. Circularly polarized light is incident at the Brewster angle on a dielectric surface. With explanation, describe the polarization of the light that is reflected from this surface.

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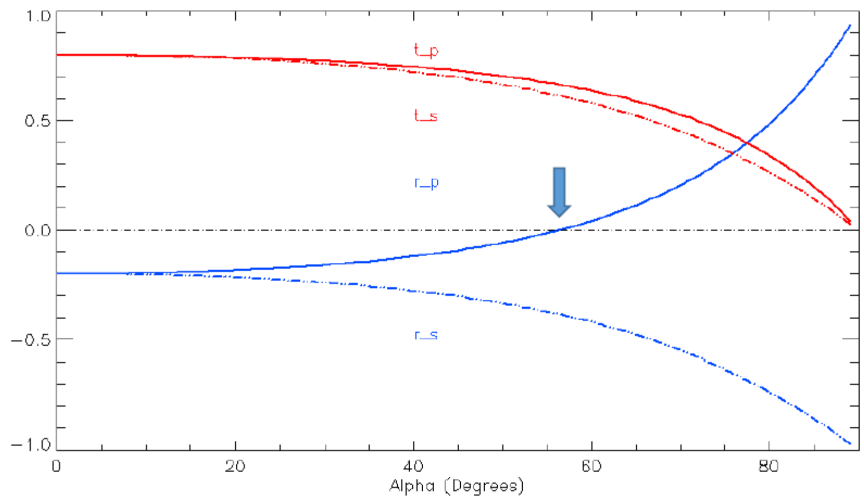
## Fresnel Relations – Special Angles (Brewster)

There is an angle $\alpha_B$ for which

$$\boxed{r_{\parallel} = \frac{n\cos\alpha' - n'\cos\alpha}{n\cos\alpha' + n'\cos\alpha} = 0}$$

$$n'\cos\alpha_B = n\cos\alpha'$$




We can use:

$$\boxed{r_{\parallel} = \frac{\tan(\alpha_B - \alpha')}{\tan(\alpha_B + \alpha')}}$$

Which suggest either $\alpha_B = \alpha'$

(which is only true at normal incidence)

or $\alpha_B + \alpha' = \pi/2$

$$\alpha' = \frac{\pi}{2} - \alpha_B$$

$$n'\cos\alpha_B = n\cos\alpha'$$

$$n'\cos\alpha_B = n\cos\left(\frac{\pi}{2} - \alpha_B\right)$$

$$n'\cos\alpha_B = n\sin\alpha_B$$

$$\frac{\sin\alpha_B}{\cos\alpha_B} = \frac{n'}{n}$$

$$\boxed{\alpha_B = \tan^{-1}\left(\frac{n'}{n}\right)}$$

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**A2**

2. Circularly polarized light is incident at the Brewster angle on a dielectric surface. With explanation, describe the polarization of the light that is reflected from this surface.

The Brewster angle is such that the parallel component of the incident polarization is not reflected.

As Circular polarization can be expressed as the sum of two linear polarizations, the axes can be chosen so that one of the two is the non-in-plane polarized radiation. This is the only component that will be reflected from the surface.

Hence the wave reflected will be linearly polarized.

# A3

3. (a) What is the ratio of the electrical energy to the magnetic energy stored in a unit volume of the atmosphere just above the Earth's surface, where the electric field strength is $100\ \mathrm{V\ m^{-1}}$ and the magnetic field strength is $5.0 \times 10^{-5}\ \mathrm{T}$?

   (b) What is an invariant property in electromagnetic relativity? Which of the following is an invariant quantity for electromagnetic waves?

   i. $\mathbf{E}$ and $\mathbf{B}$    ii. $|\mathbf{E}| - |\mathbf{B}|$    iii. $\mathbf{E} \cdot \mathbf{B}$    iv. $\mathbf{E} \times \mathbf{B}$

Maxwell Equations

# A3

3. (a) What is the ratio of the electrical energy to the magnetic energy stored in a unit volume of the atmosphere just above the Earth's surface, where the electric field strength is $100\ \mathrm{V\,m^{-1}}$ and the magnetic field strength is $5.0 \times 10^{-5}$ T?

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i. $\mathbf{E}$ and $\mathbf{B}$ ii. $|\mathbf{E}| - |\mathbf{B}|$ iii. $\mathbf{E} \cdot \mathbf{B}$ iv. $\mathbf{E} \times \mathbf{B}$

a) The ratio of the electrical and magnetic energy stored can be calculated as:

$$u_{\mathrm{elec}} = \frac{1}{2}\epsilon_0 E^2 = 0.5 \times 8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}} \times (100\ \mathrm{Vm^{-1}})^2 = 4.43 \times 10^{-8}\mathrm{J}.$$

and

$$u_{\mathrm{mag}} = \frac{B^2}{2\mu_0} = \frac{(5.0 \times 10^{-5}\ \mathrm{T})^2}{8\pi \times 10^{-7}\ \mathrm{N\,A^{-2}}} = 9.95 \times 10^{-4}\ \mathrm{J}.$$

So the ratio is $\dfrac{u_{\mathrm{elec}}}{u_{\mathrm{mag}}} = \dfrac{4.43 \times 10^{-8}\ \mathrm{J}}{9.95 \times 10^{-4}\ \mathrm{J}} = 4.4 \times 10^{-5}$.

# A3

3. (a) What is the ratio of the electrical energy to the magnetic energy stored in a unit volume of the atmosphere just above the Earth's surface, where the electric field strength is $100\ \mathrm{V\ m^{-1}}$ and the magnetic field strength is $5.0 \times 10^{-5}$ T?

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   i. $\mathbf{E}$ and $\mathbf{B}$ ii. $|\mathbf{E}| - |\mathbf{B}|$ iii. $\mathbf{E} \cdot \mathbf{B}$ iv. $\mathbf{E} \times \mathbf{B}$

b) An invariant quantity is a quantity that does not vary under a transformation (such as that from a reference frame to another)

The invariant quantity in question here is the scalar product in iii)

# A4

4. (a) Find the magnetic field **B** corresponding to the magnetic vector potential

$$\mathbf{A} = -\frac{1}{2}a\left(y^2\hat{\mathbf{x}} + z^2\hat{\mathbf{y}} + x^2\hat{\mathbf{z}}\right).$$

(b) What is the current density **J** associated with this magnetic field?

Vector potential

# A4

4. (a) Find the magnetic field $\mathbf{B}$ corresponding to the magnetic vector potential

$$\mathbf{A} = -\frac{1}{2}a\left(y^2\hat{\mathbf{x}} + z^2\hat{\mathbf{y}} + x^2\hat{\mathbf{z}}\right).$$

(b) What is the current density $\mathbf{J}$ associated with this magnetic field?

a) The magnetic field can be calculated from its relation to A:

$$\mathrm{B} = \boldsymbol{\nabla}\times\mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= -\frac{1}{2}a\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix}$$

$$= a(z\hat{\mathbf{x}} + x\hat{\mathbf{y}} + y\hat{\mathbf{z}}).$$

# A4

4. (a) Find the magnetic field $\mathbf{B}$ corresponding to the magnetic vector potential

$$\mathbf{A} = -\frac{1}{2}a\left(y^2\hat{\mathbf{x}} + z^2\hat{\mathbf{y}} + x^2\hat{\mathbf{z}}\right).$$

(b) What is the current density $\mathbf{J}$ associated with this magnetic field?

a) The magnetic field can be calculated from its relation to A:

$$\begin{aligned} \mathrm{B} = \boldsymbol{\nabla} \times \mathrm{A} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \\ &= -\frac{1}{2}a \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} \\ &= a(z\hat{\mathbf{x}} + x\hat{\mathbf{y}} + y\hat{\mathbf{z}}). \end{aligned}$$

(b) Using the differential version of Ampère's law,

$$\begin{aligned} \mathbf{J} &= \frac{1}{\mu_0}\boldsymbol{\nabla} \times \mathrm{B} \\ &= \frac{1}{\mu_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ az & ax & ay \end{vmatrix} \\ &= \frac{1}{\mu_0}(a\hat{\mathbf{x}} + a\hat{\mathbf{y}} + a\hat{\mathbf{z}}) \end{aligned}$$

# A5

5. In a linear dielectric medium, we can write

$$\sigma_P = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \text{and} \qquad \rho_P = -\boldsymbol{\nabla} \cdot \mathbf{P}.$$

(a) Briefly define the terms $\sigma_P$, $\rho_P$, and $\mathbf{P}$, and explain qualitatively, with the aid of a sketch, how $\sigma_P$ and $\rho_P$ arise at the atomic level.

(b) A certain medium demonstrates $\boldsymbol{\nabla} \cdot \mathbf{P} = 0$. What is the significance of this with respect to the charge density?

Polarization (of dielectrics) Fields in Matter

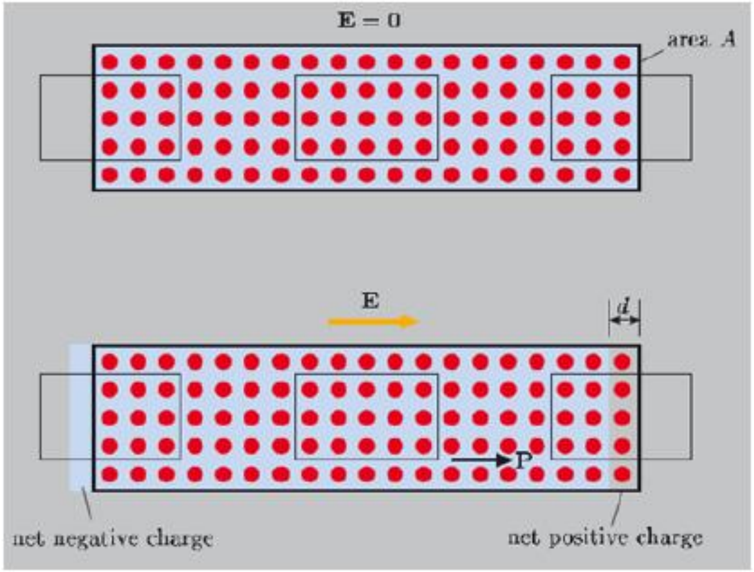


**Figure 2:** *Origin of surface charge density due to polarization, for Question 5a.*

# A5



## Polarizable Media

$$V = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\mathbf{P}(\boldsymbol{r}')\cdot\boldsymbol{n}}{|\boldsymbol{r}-\boldsymbol{r}'|}da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{-\boldsymbol{\nabla}\cdot\mathbf{P}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'$$

The first term looks like the potential of a surface charge $\sigma_b \equiv \mathbf{P}\cdot\boldsymbol{n}$

The second term looks like the potential of a volume charge $\rho_b \equiv -\boldsymbol{\nabla}\cdot\mathbf{P}$

$$V = \frac{1}{4\pi\epsilon_0}\left(\oint_S \frac{\sigma_b}{|\boldsymbol{r}-\boldsymbol{r}'|}da' + \int_V \frac{\rho_b}{|\boldsymbol{r}-\boldsymbol{r}'|}dv'\right)$$

$$V = \frac{1}{4\pi\epsilon_0}\int_V \frac{dq_P}{|\boldsymbol{r}-\boldsymbol{r}'|}$$

# A5

5. (a) $\sigma_P$: (bound) polarization surface charge density; the charge on the surface of a polarized object due to the polarization of the material. [The SI unit of bound surface charge density is C m$^{-2}$.]

$\rho_P$: (bound) polarization volume charge density; the macroscopic charge density within the volume of a dielectric due to non-uniform polarization of the material. In general, a dielectric will have a bound charge density given by this formula, where $\mathbf{P}$ is the polarization.

$\mathbf{P}$: polarization; when an electric field is applied to a non-conducting medium, small displacements of charge are caused through distortions of electron clouds or re-orientations of molecules. The magnitude is given by the electric dipole moment per unit volume of a dielectric material; the direction is a vector from the negative to the positive charge density.

Polarization reflects the fact that the atoms which make up the dielectric consist of separate positive (nucleus) and negative (electrons) charges. These respond differently to the electric field, leading to a *shift* in the overall charge distribution of the dielectric, while keeping it neutral. $\rho_P$ arises from the non-uniformity of $\mathbf{P}$ and $\sigma_P$ comes about from the shift in charge distribution when an electric field is applied as shown in Figure 2.

# A5

(b) When the polarization field in a dielectric is uniform then $\nabla \cdot \mathbf{P} = 0$ and the polarization volume charge density, $\rho_P$, is zero, although there still is a polarization surface charge density, $\sigma_P$. [2]

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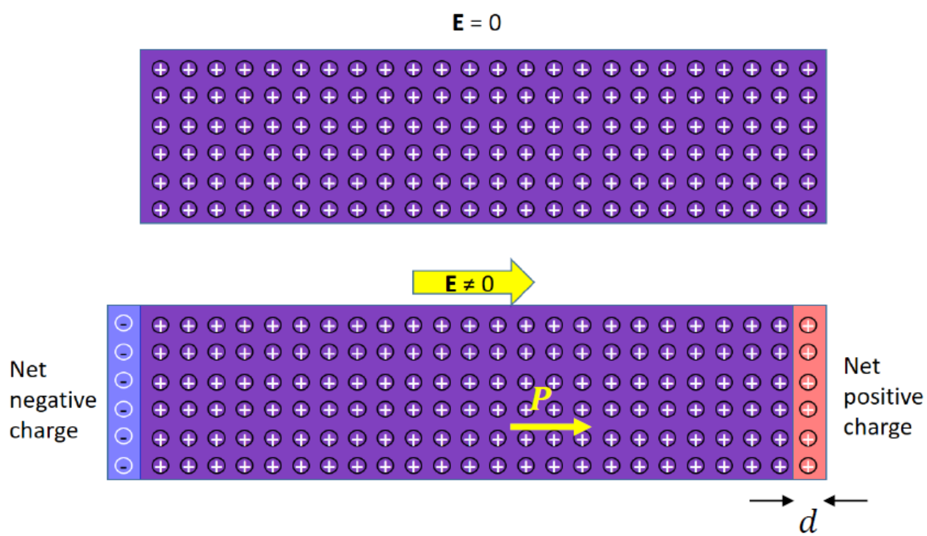
## Surface charge

$$\sigma_b \equiv \mathbf{P} \cdot \boldsymbol{n}$$

$$\rho_b \equiv -\boldsymbol{\nabla} \cdot \mathbf{P}$$



# A6

6. (a) Starting from

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{and} \qquad \nabla \cdot \mathbf{B} = 0,$$

show that we can write

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}.$$

Define $\phi$ and $\mathbf{A}$.

(b) Write down the conditions for

i. the Coulomb gauge; and,

ii. the Lorentz gauge.

Maxwell Equations Lorentz gauge

# A6s

6. (a) From $\boldsymbol{\nabla} \cdot \mathrm{B} = 0$, we have $\mathrm{B} = \boldsymbol{\nabla} \times \mathbf{A}$, so we can write

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t} \;\Rightarrow\; \boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial}{\partial t}\boldsymbol{\nabla} \times \mathbf{A} = 0 \;\Rightarrow\; \boldsymbol{\nabla} \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0.$$

For any vector, $\boldsymbol{\nabla} \times (\boldsymbol{\nabla}\mathbf{F}) = 0$, so we can write

$$E + \frac{\partial \mathbf{A}}{\partial t} = -\boldsymbol{\nabla}\phi,$$

or

$$E = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \text{as required.}$$

$\phi$: electrostatic scalar potential; $\mathbf{A}$: magnetic vector potential.

(b) i. Coulomb gauge: $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$.

ii. Lorentz gauge: $\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$;

[might also see it as $\partial_\mu a_\mu = 0$, or, $\boldsymbol{\nabla} \cdot \mathbf{A} + (1/c^2)\partial\phi/\partial t = 0$.]

# B1

Wave in conducting media

7. (a) Since $\tilde{\epsilon}$, the relative permittivity, is a complex quantity, we expect to find a complex refractive index, so let us write it as

$$n = n_{\text{real}} + n_{\text{imag}}\mathrm{i},$$

with $n^2 = \tilde{\epsilon}$.

i. Show, without approximation, that

$$n_{\text{real}}^2 = \frac{\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$
$$n_{\text{imag}}^2 = \frac{-\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$

ii. Hence, show that when $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$, $[\tilde{\epsilon}_{\text{imag}} > 0]$,

A. $n_{\text{real}} \approx \sqrt{\tilde{\epsilon}_{\text{real}}}$,

B. $n_{\text{imag}} \approx \dfrac{\tilde{\epsilon}_{\text{imag}}}{2\sqrt{\tilde{\epsilon}_{\text{real}}}}$.

iii. To what type of material, in general, does the condition $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$ relate?

(b) The physical electric field associated with a plane wave in a material has the form

$$\mathbf{E}_{\text{phys}} = E_0 \exp[-a_1 z] \cos(a_2 z - a_3 t)\hat{\mathbf{y}}.$$

Express each of the following quantities in terms of symbols that appear in the expression for $\mathbf{E}_{\text{phys}}$ and the speed of light in free space $c$ (no approximations required):

| | | | |
|---|---|---|---|
| i. $k_{\text{real}}$ | iii. $\omega$ | v. $n_{\text{imag}}$ | vii. $\tilde{\epsilon}_{\text{imag}}$ |
| ii. $k_{\text{imag}}$ | iv. $n_{\text{real}}$ | vi. $\tilde{\epsilon}_{\text{real}}$ | viii. $\mathbf{B}_{\text{phys}}$ |

Recall that $\tilde{\epsilon} = n^2$: this may help with a couple of these.

(c) i. A steak has relative permittivity $\tilde{\epsilon} = 40+12\mathrm{i}$ at 2.45 GHz, the frequency used in microwave ovens. Estimate the depth over which the amplitude of the electric field inside the steak falls to half of the external value.

ii. Polystyrene has $\tilde{\epsilon} = 1.03 + 3 \times 10^{-5}\mathrm{i}$ at 2.45 GHz. Demonstrate why it is a suitable container for use in a microwave oven.

# B1s

7. (a) Since $\tilde{\epsilon}$, the relative permittivity, is a complex quantity, we expect to find a complex refractive index, so let us write it as

$$n = n_{\text{real}} + n_{\text{imag}}\mathrm{i},$$

with $n^2 = \tilde{\epsilon}$.

i. Show, without approximation, that

$$n_{\text{real}}^2 = \frac{\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$
$$n_{\text{imag}}^2 = \frac{-\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$

(a) i. We can write

$$n^2 = n_{\text{real}}^2 - n_{\text{imag}}^2 + 2\mathrm{i}n_{\text{real}}n_{\text{imag}} = \tilde{\epsilon}_{\text{real}} + \mathrm{i}\tilde{\epsilon}_{\text{imag}}$$

to give

$$\tilde{\epsilon}_{\text{real}} = n_{\text{real}}^2 - n_{\text{imag}}^2 \quad \text{and} \quad \tilde{\epsilon}_{\text{imag}} = 2n_{\text{real}}n_{\text{imag}}$$

Substituting to eliminate $n_{\text{imag}}$ and $n_{\text{real}}$, respectively, gives

$$n_{\text{real}}^4 - \tilde{\epsilon}_{\text{real}}n_{\text{real}}^2 - \frac{\tilde{\epsilon}_{\text{imag}}^2}{4} = 0$$
$$n_{\text{imag}}^4 + \tilde{\epsilon}_{\text{real}}n_{\text{imag}}^2 - \frac{\tilde{\epsilon}_{\text{imag}}^2}{4} = 0$$

Solving these quadratics in $n_{\text{real}}^2$ and $n_{\text{imag}}^2$ gives the required

$$n_{\text{real}}^2 = \frac{\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$
$$n_{\text{imag}}^2 = \frac{-\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2}$$

# B1s

ii. Hence, show that when $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$, $[\tilde{\epsilon}_{\text{imag}} > 0]$,

A. $n_{\text{real}} \approx \sqrt{\tilde{\epsilon}_{\text{real}}}$,

B. $n_{\text{imag}} \approx \dfrac{\tilde{\epsilon}_{\text{imag}}}{2\sqrt{\tilde{\epsilon}_{\text{real}}}}$.

iii. To what type of material, in general, does the condition $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$ relate?

ii. A. With $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$ we have

$$
\begin{aligned}
n_{\text{real}}^2 &= \frac{\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2} \\
&\approx \frac{\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2}}{2} \\
&= \frac{2\tilde{\epsilon}_{\text{real}}}{2}
\end{aligned}
$$

So $n_{\text{real}} \approx \sqrt{\tilde{\epsilon}_{\text{real}}}$.

B. With $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$ we have

$$
\begin{aligned}
n_{\text{imag}}^2 &= \frac{-\tilde{\epsilon}_{\text{real}} + \sqrt{\tilde{\epsilon}_{\text{real}}^2 + \tilde{\epsilon}_{\text{imag}}^2}}{2} \\
&= \frac{-\tilde{\epsilon}_{\text{real}} + \tilde{\epsilon}_{\text{real}}\sqrt{1 + \left(\frac{\tilde{\epsilon}_{\text{imag}}}{\tilde{\epsilon}_{\text{real}}}\right)^2}}{2} \\
&\approx \frac{-\tilde{\epsilon}_{\text{real}} + \tilde{\epsilon}_{\text{real}}\left(1 + \frac{1}{2}\left(\frac{\tilde{\epsilon}_{\text{imag}}}{\tilde{\epsilon}_{\text{real}}}\right)^2\right)}{2} \\
&= \frac{\frac{1}{2}\frac{\tilde{\epsilon}_{\text{imag}}^2}{\tilde{\epsilon}_{\text{real}}}}{2}
\end{aligned}
$$

So $n_{\text{imag}} \approx \dfrac{\tilde{\epsilon}_{\text{imag}}}{2\sqrt{\tilde{\epsilon}_{\text{real}}}}$.

iii. In general, the condition $\tilde{\epsilon}_{\text{imag}} \ll \tilde{\epsilon}_{\text{real}}$ relates to a dielectric.

# B1s

(b) The physical electric field associated with a plane wave in a material has the form

$$\mathbf{E}_{\text{phys}} = E_0 \exp[-a_1 z] \cos(a_2 z - a_3 t)\hat{\mathbf{y}}.$$

Express each of the following quantities in terms of symbols that appear in the expression for $\mathbf{E}_{\text{phys}}$ and the speed of light in free space $c$ (no approximations required):

i. $k_{\text{real}}$
ii. $k_{\text{imag}}$
iii. $\omega$
iv. $n_{\text{real}}$
v. $n_{\text{imag}}$
vi. $\tilde{\epsilon}_{\text{real}}$
vii. $\tilde{\epsilon}_{\text{imag}}$
viii. $\mathbf{B}_{\text{phys}}$

Recall that $\tilde{\epsilon} = n^2$: this may help with a couple of these.

# B1s

(b) The physical electric field associated with a plane wave in a material has the form

$$\mathbf{E}_{\text{phys}} = E_0 \exp[-a_1 z] \cos(a_2 z - a_3 t)\hat{\mathbf{y}}.$$

Express each of the following quantities in terms of symbols that appear in the expression for $\mathbf{E}_{\text{phys}}$ and the speed of light in free space $c$ (no approximations required):

i. $k_{\text{real}}$
ii. $k_{\text{imag}}$
iii. $\omega$
iv. $n_{\text{real}}$
v. $n_{\text{imag}}$
vi. $\tilde{\epsilon}_{\text{real}}$
vii. $\tilde{\epsilon}_{\text{imag}}$
viii. $\mathbf{B}_{\text{phys}}$

Recall that $\tilde{\epsilon} = n^2$: this may help with a couple of these.

(b) i. $k_{\text{real}} = a_2$

ii. $k_{\text{imag}} = a_1$

iii. $\omega = a_3$

iv. $n_{\text{real}} = \frac{c}{\omega} k_{\text{real}} = \frac{a_2 c}{a_3}$

v. $n_{\text{imag}} = \frac{c}{\omega} k_{\text{imag}} = \frac{a_1 c}{a_3}$

vi. With $\tilde{\epsilon} = n^2 = n_{\text{real}}^2 - n_{\text{imag}}^2 + 2i n_{\text{real}} n_{\text{imag}}$, then
$\tilde{\epsilon}_{\text{real}} = n_{\text{real}}^2 - n_{\text{imag}}^2 = (a_2^2 - a_1^2)\, c^2/a_3^2.$

vii. $\tilde{\epsilon}_{\text{imag}} = 2 n_{\text{real}} n_{\text{imag}} = 2 a_1 a_2 c^2/a_3^2$

viii. Since $\hat{\mathbf{z}} = \hat{\mathbf{y}} \times (-\hat{\mathbf{x}})$,
$\mathbf{B}_{\text{phys}} = -\frac{E_0}{c} \exp[-a_1 z] \cos(a_2 z - a_3 t)\hat{\mathbf{x}}$

# B1s

(c) i. A steak has relative permittivity $\tilde{\epsilon} = 40+12\mathrm{i}$ at 2.45 GHz, the frequency used in microwave ovens. Estimate the depth over which the amplitude of the electric field inside the steak falls to half of the external value.

ii. Polystyrene has $\tilde{\epsilon} = 1.03 + 3 \times 10^{-5}\mathrm{i}$ at 2.45 GHz. Demonstrate why it is a suitable container for use in a microwave oven.

# B1s

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## P – 2s

SOL 2

Consider the steak as a semi-infinite slab.

The microwave amplitude within the steak will fall exponentially with distance "z" below the surface, as $e^{[-k_{imag}z]}$ and the depth "d" at which the amplitude has dropped to half of that on the surface is given by

$$e^{[-k_{imag}d]} = \frac{1}{2} \longrightarrow -k_{imag}d = -\ln(2)$$

$$d = \frac{\ln(2)}{k_{imag}}$$

$$k_{imag} = \frac{\omega n_{imag}}{c}$$

$$n_{imag} = \left(\frac{-\epsilon_{real} + \sqrt{\epsilon_{real}^2 + \epsilon_{imag}^2}}{2}\right)^{1/2}$$

$$= \left(\frac{-40 + \sqrt{40^2 + 12^2}}{2}\right)^{1/2} \cong 0.9384$$

$$d = \frac{c \cdot \ln(2)}{\omega n_{imag}} = \frac{3.00 \times 10^8 ms^{-1} \times \ln(2)}{0.9384 \times 2\pi \times 2.45 \times 10^9 Hz} \cong 1.44cm$$

7

# B1s

(c) i. A steak has relative permittivity $\tilde{\epsilon} = 40+12\mathrm{i}$ at 2.45 GHz, the frequency used in microwave ovens. Estimate the depth over which the amplitude of the electric field inside the steak falls to half of the external value. [4]

ii. Polystyrene has $\tilde{\epsilon} = 1.03 + 3 \times 10^{-5}\mathrm{i}$ at 2.45 GHz. Demonstrate why it is a suitable container for use in a microwave oven. [3]



## P – 2s

SOL 2

For polystyrene, $\epsilon_{imag} \ll \epsilon_{real}$ and we can use the approximation

$$n_{imag} \approx \frac{\epsilon_{imag}}{2\sqrt{\epsilon_{real}}} = \frac{3 \times 10^{-5}}{2\sqrt{1.03}} \cong 1.478 \times 10^{-5}$$

Then

$$d = \frac{c \cdot \ln(2)}{\omega n_{imag}} = \frac{3.00 \times 10^{8} ms^{-1} \times \ln(2)}{1.478 \times 10^{-5} \times 2\pi \times 2.45 \times 10^{9} Hz} \cong 9.1 \times 10^{2} m$$

# B2

8. (a) For an EM plane wave in vacuum with electric field $\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$, use one of Maxwell's equations to show that the magnetic field can be written:

$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}.$$

(b) Show that the magnitude of the Poynting vector is given by:

$$N = \sqrt{\frac{\epsilon_0}{\mu_0}} n E_0^2.$$

(c) An electromagnetic plane wave, $\mathbf{E} = \mathbf{E}_0 \exp \mathrm{i} (\mathbf{k} \cdot \mathbf{r} - \omega t)$, is incident on an interface between two media at an angle $\alpha$ from the normal to the interface. Some of the light is reflected at an angle $\alpha''$ from the normal and some is refracted at an angle $\alpha'$ to the normal. The ray travels from a medium with relative permittivity $\epsilon_r$ and index of refraction $n$ to one with $\epsilon_r'$ and $n'$. For both media, assume that the relative permeability is 1.

i. For which electromagnetic fields are the components parallel to an interface between two media continuous across that interface?

ii. Write down the equations which reflect this continuity of the parallel components of these fields across the interface.

iii. Show that the Fresnel relation for the reflection of the component $E_\parallel$, the component of the electric field parallel to the plane of incidence, is given by

$$r_\parallel = \frac{n' \cos\alpha - n \cos\alpha'}{n' \cos\alpha + n \cos\alpha'}.$$

(d) i. For a Hertzian dipole, with $r \gg \lambda$, we have

$$B_\phi \approx \frac{\mu_0}{4\pi} \frac{I_0 l}{r} \sin\theta \left( \frac{\omega}{c} \cos\omega\,(t - r/c) \right) \qquad E_\theta = \frac{l I_0 \sin\theta}{4\pi\epsilon_0} \frac{\omega \cos\omega\,(t - r/c)}{rc^2}.$$

Show that the time-averaged Poynting vector for a Hertzian dipole is given by

$$\overline{\mathbf{N}} = \frac{\omega^2 \eta^2}{2c\mu_0 r^2} \sin^2\theta\ \hat{\mathbf{r}},$$

where $\eta = \mu_0 I_0 l / 4\pi$.

ii. A Hertzian dipole is located at the origin of Cartesian coordinates and is aligned with the $y$-axis. The strength of the dipole is $3.0 \times 10^{-8}$ A m and its angular frequency is $2.0 \times 10^9\ \mathrm{s}^{-1}$. Calculate the time-averaged value of the Poynting vector at the point with Cartesian coordinates $(2.0, -2.0, 1.0)$ m.

Maxwell Equations

Poynting Vector

Reflection/Refraction at Interface with medium (Fresnel)

Hertzian dipole

# B2s

8. (a) For an EM plane wave in vacuum with electric field $\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$, use one of Maxwell's equations to show that the magnetic field can be written:

$$\mathbf{B} = \frac{\mathbf{k} \times \mathbf{E}}{\omega}.$$

(a) We will use Faraday's Law:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ i\mathbf{k} \times \mathbf{E} &= i\omega \mathbf{B} \\ \Rightarrow \mathbf{B} &= \frac{\mathbf{k} \times \mathbf{E}}{\omega} \end{aligned}$$

as required.

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$$N = \sqrt{\frac{\epsilon_0}{\mu_0}} n E_0^2.$$

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as required.

(b) Show that the magnitude of the Poynting vector is given by:

$$N = \sqrt{\frac{\epsilon_0}{\mu_0}}nE_0^2.$$

(b) We have that $\mathbf{N} = \mathbf{E}\times\mathbf{H}$. With $H = B/\mu_0$ and $B = kE/\omega = nE/c$ (using magnitudes throughout) we can write:

$$\begin{aligned} N &= |\mathbf{N}| = |\mathbf{E}\times\mathbf{H}| \\ &= \frac{1}{\mu_0}E_0\frac{nE_0}{c} \\ &= \frac{1}{\mu_0}E^2\sqrt{\epsilon_0\mu_0}n \\ &= \sqrt{\frac{\epsilon_0}{\mu_0}}nE_0^2 \end{aligned}$$

as required.

# B2s

(c) An electromagnetic plane wave, $\mathbf{E} = \mathbf{E}_0 \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$, is incident on an interface between two media at an angle $\alpha$ from the normal to the interface. Some of the light is reflected at an angle $\alpha''$ from the normal and some is refracted at an angle $\alpha'$ to the normal. The ray travels from a medium with relative permittivity $\epsilon_r$ and index of refraction $n$ to one with $\epsilon_r'$ and $n'$. For both media, assume that the relative permeability is 1.

i. For which electromagnetic fields are the components parallel to an interface between two media continuous across that interface?

(c) i. The components of **E** and **H** parallel to the interface are continuous across the interface.

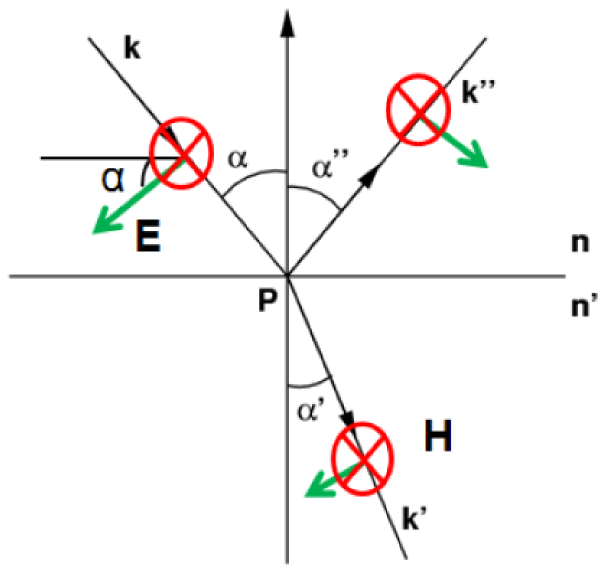



**Figure 3:** *Field scattering at an interface (figure not necessary).*

# B2s

ii. Write down the equations which reflect this continuity of the parallel components of these fields across the interface.

iii. Show that the Fresnel relation for the reflection of the component $E_{\|}$, the component of the electric field parallel to the plane of incidence, is given by

$$r_{\|} = \frac{n' \cos\alpha - n\cos\alpha'}{n'\cos\alpha + n\cos\alpha'}.$$

ii. The components parallel to the surface are:

$$E_0 \cos\alpha - E_0'' \cos\alpha = E_0' \cos\alpha',$$

and, as $\mathbf{H}$ is perpendicular to $\mathbf{E}$, it must lie entirely in the plane of the interface,

$$H_0 + H_0'' = H_0'.$$

# B2s

ii. Write down the equations which reflect this continuity of the parallel components of these fields across the interface.

iii. Show that the Fresnel relation for the reflection of the component $E_{\|}$, the component of the electric field parallel to the plane of incidence, is given by

$$r_{\|} = \frac{n' \cos\alpha - n\cos\alpha'}{n'\cos\alpha + n\cos\alpha'}.$$

iii. From

$$\begin{aligned} H_0 + H_0'' &= H_0' \\ \Rightarrow \sqrt{\frac{\epsilon}{\mu}} E_0 + \sqrt{\frac{\epsilon}{\mu}} E_0'' &= \sqrt{\frac{\epsilon'}{\mu'}} E_0' \end{aligned}$$

We assume that $\mu_r = \mu_r' = 1$ (pretty good unless we're dealing with ferromagnetic materials). With $\mu = \mu_0\mu_r$ and $\epsilon = \epsilon_0\epsilon_r$, we then get:

$$\sqrt{\epsilon_r} E_0 + \sqrt{\epsilon_r} E_0'' = \sqrt{\epsilon_r'} E_0'.$$

But we know that $\sqrt{\epsilon_r} = n$, so:

$$n\left(E_0 + E_0''\right) = n' E_0'.$$

Substituting for $E_0'$, we get, with $r_{\|} = E_0''/E_0$:

$$\begin{aligned} \frac{\cos\alpha}{n}\frac{E_0 - E_0''}{E_0 + E_0''} &= \frac{\cos\alpha'}{n'} \\ \Rightarrow n'\cos\alpha\left(1 - r_{\|}\right) &= n\cos\alpha'\left(1 + r_{\|}\right) \\ n'\cos\alpha - n\cos\alpha' &= r_{\|}\left[n\cos\alpha' + n'\cos\alpha\right] \\ r_{\|} &= \frac{n'\cos\alpha - n\cos\alpha'}{n\cos\alpha' + n'\cos\alpha}, \text{ as required.} \end{aligned}$$

# B2s

(d) i. For a Hertzian dipole, with $r \gg \lambda$, we have

$$B_\phi \approx \frac{\mu_0}{4\pi}\frac{I_0 l}{r}\sin\theta\left(\frac{\omega}{c}\cos\omega\left(t-r/c\right)\right) \qquad E_\theta = \frac{l I_0 \sin\theta}{4\pi\epsilon_0}\frac{\omega\cos\omega\left(t-r/c\right)}{rc^2}.$$

Show that the time-averaged Poynting vector for a Hertzian dipole is given by

$$\overline{\mathbf{N}} = \frac{\omega^2\eta^2}{2c\mu_0 r^2}\sin^2\theta\ \hat{\mathbf{r}},$$

where $\eta = \mu_0 I_0 l/4\pi$.

(d) i.

$$\begin{aligned}
\mathbf{N} &= \mathbf{E}\times\mathbf{H} \\
&= \mathbf{E}\times\frac{\mathbf{B}}{\mu_0} \\
&= E_\theta B_\phi\,(\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}) \\
&= \frac{\mu_0}{\epsilon_0}\left(\frac{I_0 l}{4\pi}\right)^2\frac{\omega^2}{r^2c^3\mu_0}\sin^2\theta\cos^2\omega(t-r/c)\,\hat{\mathbf{r}}
\end{aligned}$$

But the time-average of $\cos^2\omega(t-r/c)$ equals $1/2$ and using $c^2 = 1/\epsilon_0\mu_0$, we get

$$\begin{aligned}
\overline{\mathbf{N}} &= \frac{\mu_0^2}{2}\frac{\epsilon_0}{\epsilon_0}\left(\frac{I_0 l}{4\pi}\right)^2\frac{\omega^2}{r^2c\mu_0}\sin^2\theta\,\hat{\mathbf{r}} \\
&= \frac{\eta^2\omega^2}{2c\mu_0 r^2}\sin^2\theta\,\hat{\mathbf{r}}, \quad \text{as required.}
\end{aligned}$$

# B2s

ii. A Hertzian dipole is located at the origin of Cartesian coordinates and is aligned with the $y$-axis. The strength of the dipole is $3.0 \times 10^{-8}$ A m and its angular frequency is $2.0 \times 10^{9}$ s$^{-1}$. Calculate the time-averaged value of the Poynting vector at the point with Cartesian coordinates $(2.0, -2.0, 1.0)$ m.

ii. We can use the equation for $\overline{\mathbf{N}}$ from above.
The distance $r$ from the origin to the point $(2.0, -2.0, 1.0)$ m is $\sqrt{2.0^2 + 2.0^2 + 1.0^2} = 3.0$ m.
The angle $\theta$ is the angle between the direction of the dipole and the direction of the position vector of the measurement point, and since the dipole is aligned with the $y$-axis, this is given by $\cos\theta = y/r = -2.0/3.0$, so $\theta = 132°$.
Also, $\eta = \mu_0 I_0 l/4\pi$. Thus

$$\begin{aligned}\overline{\mathbf{N}} &= \frac{\omega^2\mu_0(I_0 l)^2}{32\pi^2 c r^2}\sin^2 132°\,\hat{\mathbf{r}} \\ &= \frac{(2.0\times10^9\ \mathrm{s}^{-1})^2 \times 4\pi\times10^{-7}\ \mathrm{N\,A}^{-2} \times (3.0\times10^{-8}\ \mathrm{A\,m})^2}{32\pi^2 \times 3.00\times10^8\ \mathrm{m\,s}^{-1} \times (3.0m)^2} \\ &\quad \times \sin^2 132°\,\hat{\mathbf{r}} \\ &= 2.9\times10^{-15}\ \mathrm{W\,m}^{-2}\,\hat{\mathbf{r}},\end{aligned}$$

where the unit vector $\hat{\mathbf{r}}$ that indicates the direction of power flow is in the direction of the position vector of the measurement point, that is,

$$\hat{\mathbf{r}} = \frac{1}{3}(2.0\hat{\mathbf{x}} - 2.0\hat{\mathbf{y}} + 1.0\hat{\mathbf{z}}).$$

# B3

9. (a) The current density 4-vector and the electromagnetic potential 4-vector can be written as

$$j_\mu = (J_1, J_2, J_3, ic\rho) ,$$

$$a_\mu = (A_1, A_2, A_3, i\phi/c) ,$$

where $\mu = 1, 2, 3, 4$.

Define the quantities $c$, $J_1$, $J_2$, $J_3$, $\rho$, $A_1$, $A_2$, $A_3$, and $\phi$ in these relationships.

Special Relativity

Lorentz Transformations

# B3s

9. (a) The current density 4-vector and the electromagnetic potential 4-vector can be written as

$$j_\mu = (J_1, J_2, J_3, ic\rho)\,,$$

$$a_\mu = (A_1, A_2, A_3, i\phi/c)\,,$$

where $\mu = 1, 2, 3, 4$.
Define the quantities $c$, $J_1$, $J_2$, $J_3$, $\rho$, $A_1$, $A_2$, $A_3$, and $\phi$ in these relationships.

(a) $c$: speed of light in vacuum;
$J_1$, $J_2$, $J_3$: the components of the current density vector $\mathbf{J}$;
$\rho$: free charge density;
$A_1$, $A_2$, $A_3$: the components of the magnetic vector potential $\mathbf{A}$; and,
$\phi$: the electric potential.

# B3s

(b) The Lorentz transform for the space-time components $x_\mu = (x_1, x_2, x_3, x_4)$ in frame $S$ to those in frame $S'$ moving with speed $v$ in the $x_3$ direction can be written

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_2 \\ x_3' &= \gamma x_3 + i\beta\gamma x_4 \\ x_4' &= \gamma x_4 - i\beta\gamma x_3 \end{aligned}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

What do the symbols $x_1$, $x_2$, $x_3$, and $x_4$ represent in terms of conventional space and time coordinates $x$, $y$, $z$, and $t$?

(b) $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$

# B3s

(c) An infinite cylinder with cross-sectional area $A$ and uniform charge density $\rho$ is at rest along the $x$-axis of an inertial frame $\Sigma$. A test charge $Q$ is stationary at point $(0, d, 0)$ outside the cylinder. A second frame, $\Sigma'$, with corresponding axes parallel to frame $\Sigma$, travels at speed $v$ along $\Sigma$'s $x$-axis.

i. A. Show that the electric force acting on charge $Q$ according to an observer in frame $\Sigma$ is given by

$$\frac{Q\rho A}{2\pi\epsilon_0 d}.$$

B. What is the magnetic force acting on charge $Q$ according to an observer in frame $\Sigma$?

(c) i. A. Since charge $Q$ is at rest in frame $\Sigma$, there is no magnetic force on it. The electric force is deduced by first using Gauss's law to determine the electric field, which must be radially outward from the cylinder. For a Gaussian surface in the form of a cylinder with radius $r$, length $l$, coaxial with the cylinder of charge, the radial field at the surface has magnitude $E$ given by

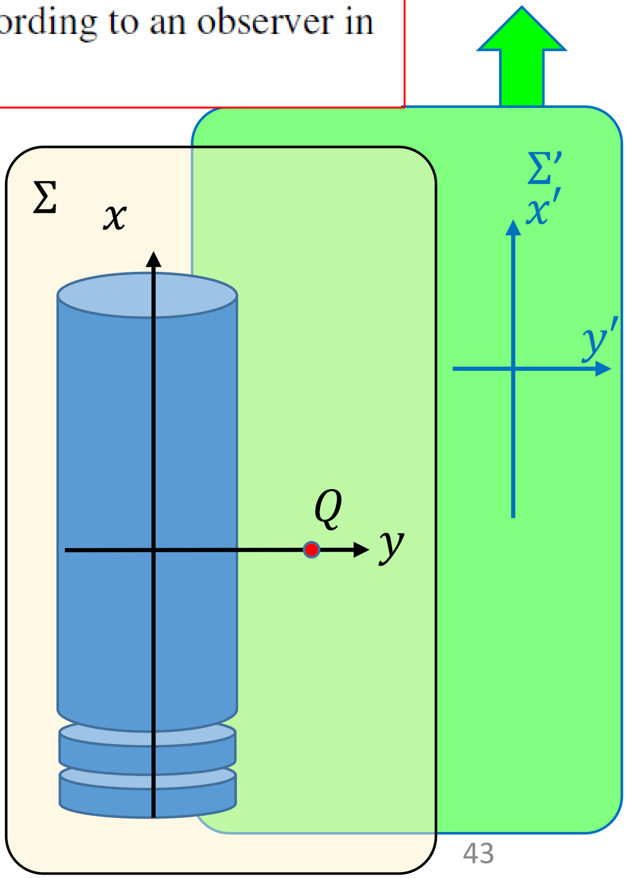
$$2\pi r l E = \frac{\rho A l}{\epsilon_0}, \quad \text{so} \quad E = \frac{\rho A}{2\pi\epsilon_0 r}.$$

The force on charge $Q$ at $(0, d, 0)$ therefore has magnitude

$$F_{\text{elec}} = QE = \frac{Q\rho A}{2\pi\epsilon_0 d},$$

and is in the $+y$-direction.

B. $F_{\text{elec}}$ is the only force on $Q$ observed in frame $\Sigma$ - there is no magnetic force.



# B3s

ii. What are the charge density and current density associated with the cylinder according to an observer in frame $\Sigma'$?

iii. Show that the electric and magnetic fields generated by the cylinder according to the observer in frame $\Sigma'$ are given by:

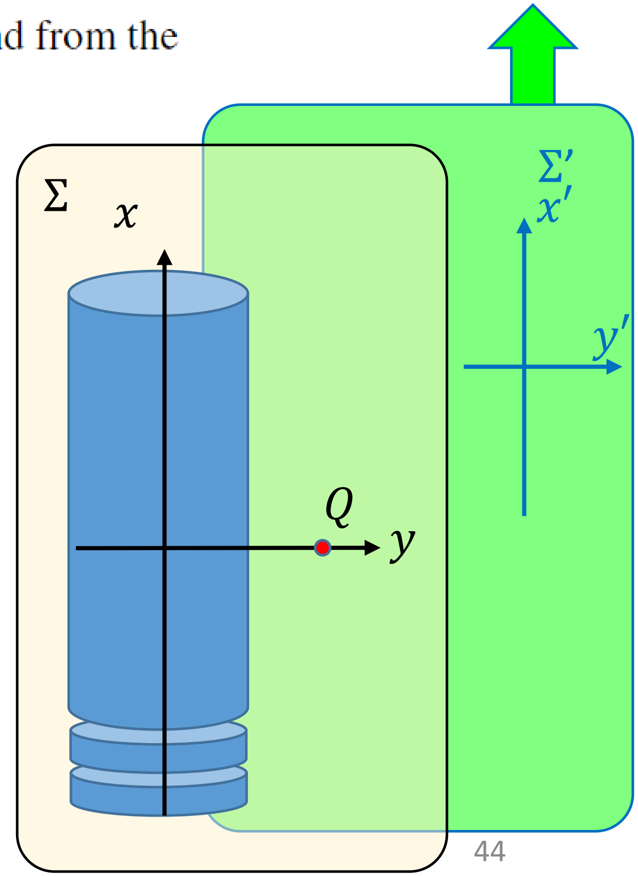
$$E' = \frac{\gamma\rho A}{2\pi\epsilon_0 r} \quad \text{and} \quad B' = -\frac{\mu_0\gamma v\rho A}{2\pi r}.$$

ii. The charge density $\rho'$ and current density $J'$ in frame $\Sigma'$ can be found from the Lorentz transformations:

$$\begin{aligned} \mathrm{i}c\rho' &= \gamma \mathrm{i}c\rho - \mathrm{i}\beta\gamma J_x \\ \rho' &= \gamma\rho - \frac{\beta\gamma}{c} J_x \\ &= \gamma\rho \ \text{ since the current density } J_x \text{ is zero in frame } \Sigma. \end{aligned}$$

$$\begin{aligned} J'_x &= \gamma J_x + \mathrm{i}\beta\gamma \mathrm{i}c\rho \\ &= -v\gamma\rho. \end{aligned}$$

$$\begin{aligned} J'_y &= J_y = 0 \\ J'_z &= J_z = 0 \end{aligned}$$



## B3s

ii. What are the charge density and current density associated with the cylinder according to an observer in frame $\Sigma'$?

iii. Show that the electric and magnetic fields generated by the cylinder according to the observer in frame $\Sigma'$ are given by:

$$E' = \frac{\gamma\rho A}{2\pi\epsilon_0 r} \quad \text{and} \quad B' = -\frac{\mu_0\gamma v\rho A}{2\pi r}.$$

iii. The electric field in frame $\Sigma'$ is deduced in the same way as in part (a), but using $\rho'$ in place of $\rho$.
Thus at distance $r$ from the cylinder,

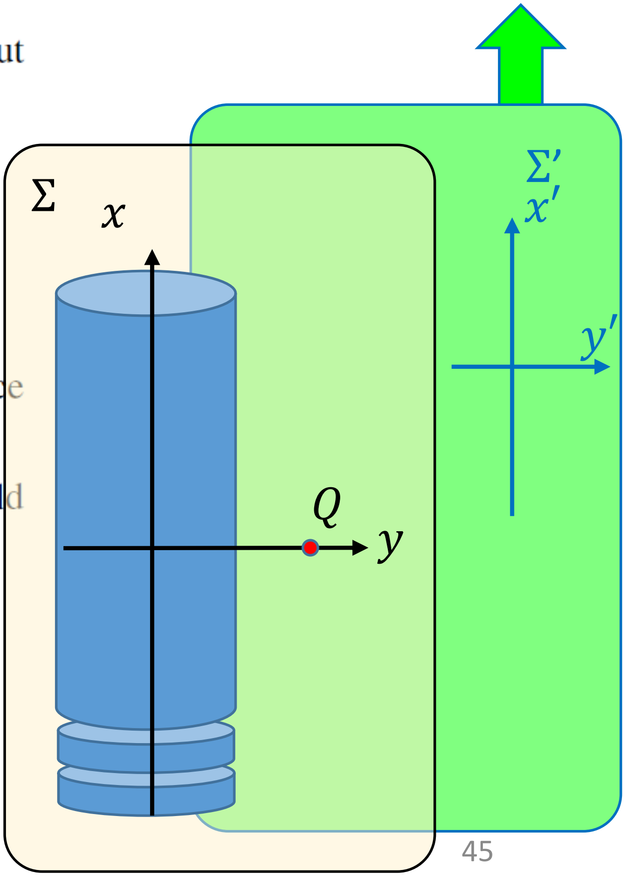
$$E' = \frac{\rho' A}{2\pi\epsilon_0 r} = \frac{\gamma\rho A}{2\pi\epsilon_0 r},$$

and is directed radially outwards.
The current $I'$ associated with current density $J'$ is $I' = J'A$; note that since dimensions transverse to the $x-$direction of motion are unchanged, $A' = A$.

At distance $r$ from a long straight wire carrying current $I'$, the magnetic field strength is

$$B' = \frac{\mu_0 I'}{2\pi r} = \frac{\mu_0 J' A}{2\pi r} = -\frac{\mu_0\gamma v\rho A}{2\pi r},$$

with direction given by the right-hand grip rule.



# B3s

iv. Show that the total force $F$ on $Q$ in frame $\Sigma$ is related to the total force $F'$ as observed in frame $\Sigma'$ by

$$F = \gamma F'.$$

iv. The test charge $Q$ is moving with speed $v$ in the $-x-$direction in frame $\Sigma$. The electric force on $Q$ has magnitude

$$F'_{\text{elec}} = QE' = \frac{Q\gamma\rho A}{2\pi\epsilon_0 d},$$

and is in the $+y'-$direction.
The test charge moves perpendicularly to the azimuthal magnetic field, so the magnitude of the magnetic force at distance $d$ from the cylinder is

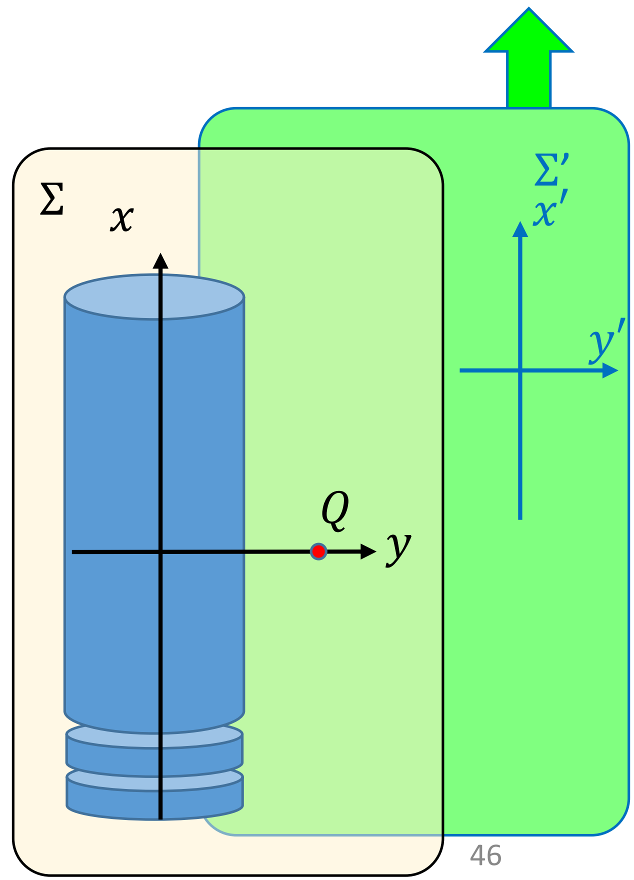
$$F'_{\text{mag}} = Qv|B'| = \frac{Qv\mu_0\gamma v\rho A}{2\pi d} = \frac{\mu_0 Q\gamma v^2\rho A}{2\pi d}.$$

Using the right-hand rule, with the velocity of $Q$ in the $-x-$direction and the magnetic field circulating in the anticlockwise direction about the $x-$axis when viewed in the direction of increasing $x$, we deduce that this force is in the $-y-$direction.
The net force in the $+y-$direction is therefore

$$\begin{aligned} F' &= F'_{\text{elec}} - F'_{\text{mag}} \\ &= \frac{Q\gamma\rho A}{2\pi\epsilon_0 d} - \frac{\mu_0 Q\gamma v^2\rho A}{2\pi d} \\ &= \frac{q\gamma\rho A}{2\pi d}\left(\frac{1}{\epsilon_0} - \mu_0 v^2\right) \end{aligned}$$

Now $c^2 = 1/\epsilon_0\mu_0$, so $\mu_0 = 1/\epsilon_0 c^2$, and the expression for the force can be rewritten as

$$F' = \frac{Q\gamma\rho A}{2\pi\epsilon_0 d}\left(1 - \frac{v^2}{c^2}\right) = \frac{Q\rho A}{2\pi\epsilon_0\gamma d} = \frac{F}{\gamma}, \quad \text{as required.}$$



# B10

10. (a) Describe the key characteristics of a plasma.

(b) The plasma frequency is defined as

$$\omega_{\mathrm{P}} = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}.$$

Define the symbols $N_e$, $\epsilon_0$, $m_e$ and $e$.

Plasma

Plasma frequency

Waves in plasma

# B10s

10. (a) Describe the key characteristics of a plasma.

(b) The plasma frequency is defined as

$$\omega_{\rm P} = \sqrt{\frac{N_e e^2}{m_e \epsilon_0}}.$$

Define the symbols $N_e$, $\epsilon_0$, $m_e$ and $e$.

Plasma

Plasma frequency

Waves in plasma

(a) A plasma consists of a group of slow-moving positive ions surrounded by an electron gas. [The fields influencing particle orbits are dominated by collective effects rather than binary (Coulomb) collisions.] The plasma is locally (if not globally) neutral and homogeneous.

(b) We have:

- $N_e$ is the electron density;
- $e$ is the electron charge;
- $m_e$ is the electron mass; and,
- $\epsilon_0$ is the permittivity of free space.

**B10s**

(c) An electromagnetic plane wave of frequency $\omega$ passing through a plasma will induce a polarization current density

$$\mathbf{J}_P = i\left(\frac{N_e e^2}{m_e \omega}\right)\mathbf{E}.$$

By considering the time derivative of the displacement field and using the definition of $\mathbf{D}$ in terms of the electric field $\mathbf{E}$ and the polarization $\mathbf{P}$, show that the relative permittivity, $\tilde{\epsilon}$, of a plasma can be written

$$\tilde{\epsilon} = 1 - \frac{\omega_P^2}{\omega^2}.$$

(c) We start with $\mathbf{D} = \epsilon_0 \tilde{\epsilon}\mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$ (with $\tilde{\epsilon}$ being the relative permittivity) and differentiate with respect to time:

$$\begin{aligned}
\frac{\partial \mathbf{D}}{\partial t} &= \epsilon_0 \epsilon_r \frac{\partial \mathbf{E}}{\partial t} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \\
\epsilon_0 \tilde{\epsilon}\frac{\partial \mathbf{E}}{\partial t} &= \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_P \\
-\epsilon_0 \tilde{\epsilon} i\omega \mathbf{E} &= -\epsilon_0 i\omega \mathbf{E} + i\left(\frac{N_e e^2}{m_e \omega}\right)\mathbf{E} \\
\tilde{\epsilon} &= 1 - \frac{1}{\omega^2}\left(\frac{N_e e^2}{m_e \epsilon_0}\right) = 1 - \frac{\omega_P^2}{\omega^2}
\end{aligned}$$

as required.

**B10s**

(d) What happens to the electric field amplitude as $\omega \to \omega_P$? What will happen physically in the plasma?

(e) Using the result of Question 10c, show that the linear dispersion relation is

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right).$$

(d) As we have $E_0 = D_0/(\epsilon_0\tilde{\epsilon})$ the electric field amplitude will grow extremely large as $\tilde{\epsilon} \to 0$. Physically, once the amplitude becomes large the resonance will cause further ionisation leading to dissipation of energy. [At the resonance, the system response is no longer linear so the given expressions fail.]

(e) The phase velocity $v_p = \omega/k = 1/\sqrt{\epsilon\mu}$. Now assuming that $\mu_r = 1$, we have:

$$\frac{\omega}{k} = \frac{1}{\sqrt{\tilde{\epsilon}\epsilon_0\mu_0}} \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{\epsilon_0\mu_0}}\frac{k}{\sqrt{\tilde{\epsilon}}} \quad \Rightarrow \quad k = \frac{\omega}{c}\sqrt{\tilde{\epsilon}},$$

so

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right),$$

as required, where we use $\sqrt{\epsilon_0\mu_0} = 1/c$.

**B10s**

(f) Hence, show that the group velocity for a material with this relative permittivity function is

$$v_{\rm group} = c\sqrt{1 - \frac{\omega_{\rm P}^2}{\omega^2}}.$$

(f) From $k^2 = \dfrac{\omega^2}{c^2}\left(1 - \dfrac{\omega_P^2}{\omega^2}\right)$, we can write

$$\omega = \sqrt{\omega_P^2 + k^2c^2}, \text{ so that}$$

$$v_{\rm group} = \frac{d\omega}{dk} = \frac{kc^2}{\sqrt{\omega_P^2 + k^2c^2}}$$

Also, with $k^2c^2 = \omega^2 - \omega_P^2$ and $kc = \sqrt{\omega^2 - \omega_P^2}$ we have

$$v_{\rm group} = c\sqrt{\frac{\omega^2 - \omega_P^2}{\omega^2}} = c\sqrt{1 - \frac{\omega_{\rm P}^2}{\omega^2}}, \text{ as required.}$$

**B10s**

(g) i. Show that the product of the phase and group velocities is $v_{\text{phase}} v_{\text{group}} = c^2$ .

ii. Comment on the significance of $v_{\text{phase}} v_{\text{group}} = c^2$ for the values that $v_{\text{group}}$ and $v_{\text{phase}}$ can take on.

(g) i. From

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{\omega_P^2}{\omega^2}\right),$$

we can write

$$v_{\text{phase}} = \frac{\omega}{k} = \frac{c}{\left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2}}.$$

Then, using the expression for $v_{\text{group}}$ from above, we have that

$$v_{\text{phase}} v_{\text{group}} = \frac{c}{\left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2}} \times c\left(1 - \frac{\omega_P^2}{\omega^2}\right)^{1/2} = c^2.$$

ii. Either $v_{\text{group}} = c = v_{\text{phase}}$, or $v_{\text{group}} < c$ and $v_{\text{phase}} > c$ with $v_{\text{group}} v_{\text{phase}} = c^2$.

# B10s

(h) A pulsar, which is $10^{19}$ m from Earth, emits pulses of electromagnetic radiation across a broad spectrum. The pulses travel to Earth through the interstellar medium, which can be regarded as a dilute plasma within which the number density of electrons and ions is about $3 \times 10^4\ \mathrm{m}^{-3}$. What is the difference in arrival time at Earth of the pulse detected with a radio telescope sensitive to frequencies around 100 MHz and the corresponding pulse detected with an optical telescope with a red filter?

(h) The group speed for a material with this relative permittivity function is given in Question 10f as

$$v_{\text{group}} = c\sqrt{1 - \frac{\omega_{\text{P}}^2}{\omega^2}}$$

and since $\omega \gg \omega_P$ for 100 MHz radiowaves and light, this can be approximated by

$$v_{\text{group}} \approx c\left(1 - \tfrac{1}{2}\tfrac{\omega_{\text{P}}^2}{\omega^2}\right).$$

The difference between the arrival times for the radiowave and red pulses is

$$\begin{aligned}
\Delta t &= \frac{D}{v_{\text{group radio}}} - \frac{D}{v_{\text{group red}}} \\
&= \frac{D}{c}\left[\left(1 - \tfrac{1}{2}\tfrac{\omega_P^2}{\omega_{\text{radio}}^2}\right)^{-1} - \left(1 - \tfrac{1}{2}\tfrac{\omega_P^2}{\omega_{\text{red}}^2}\right)^{-1}\right] \\
&\simeq \frac{D}{c}\left[\left(1 + \tfrac{1}{2}\tfrac{\omega_P^2}{\omega_{\text{radio}}^2}\right) - \left(1 + \tfrac{1}{2}\tfrac{\omega_P^2}{\omega_{\text{red}}^2}\right)\right] \\
&= \frac{D\omega_P^2}{2c}\left[\frac{1}{\omega_{\text{radio}}^2} - \frac{1}{\omega_{\text{red}}^2}\right].
\end{aligned}$$

Since $\omega_{\text{radio}} \ll \omega_{\text{red}}$, we can ignore the second term in brackets, so

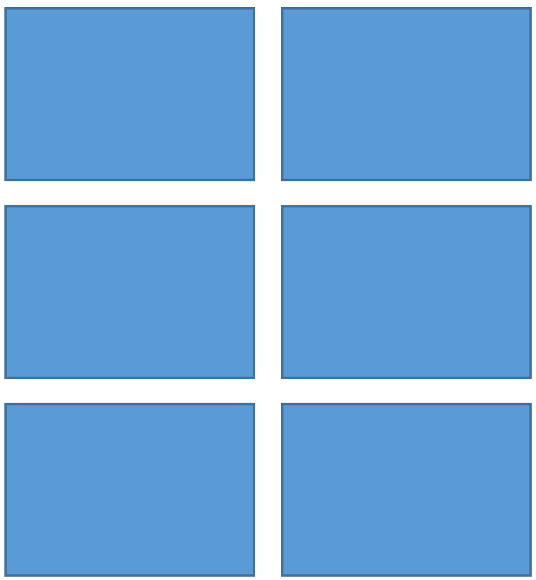
$$\Delta t = \frac{10^{19}\ \mathrm{m} \times (9.76 \times 10^3\ \mathrm{s}^{-1})^2}{2 \times 3.00 \times 10^8\ \mathrm{m\,s}^{-1} \times (2\pi \times 10^8\ \mathrm{Hz})^2} = 4.0\ \mathrm{s}$$

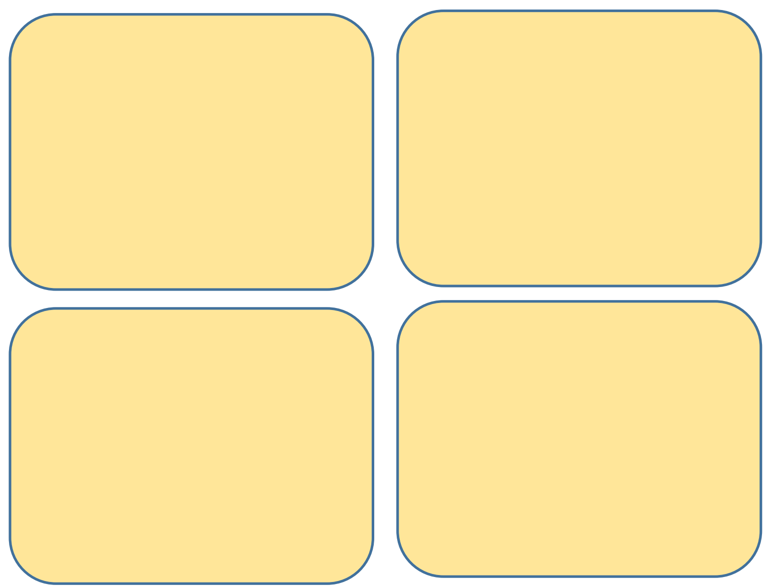
# General potential Exam Topics → Syllabus

# (it's on the PHAS3201 front page)

First six questions: Small problems, derivations, proof of a given topic on any of the syllabus topics

Part B: Two of Four on extended topics (headings of syllabus topics) and combinations of.





**Next time we meet: 26th April 2017 - REVISION LECTURE**

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**Happy Holidays!**