MATH3305 — Problem Sheet 7

Problems 1, 2, and 5 to be handed in **at the lecture** on Friday, 16 December 2016

1. Consider the vacuum field equations of general relativity in the presence of the the cosmological constant Λ

$$R_{ab} - \frac{1}{2} g_{ab} R + \Lambda g_{ab} = 0.$$

Show that these equations are equivalent to

$$R_{ab} = \Lambda g_{ab}.$$

Next solve the vacuum field equations with Λ, which results in

$$ds^2 = -\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2\right)} + r^2 d\Omega^2.$$

2. Consider the metric

$$ds^2 = -dt^2 + \left(\frac{\mu/3}{r-t}\right)^{2/3} dr^2 + \left(\frac{9\mu}{8}(r-t)^2\right)^{2/3} d\Omega^2 \,.$$

Show that this is the Schwarzschild metric by finding the coordinate transformation which transforms this metric into the standard form. Find the relationship between μ and mass parameter M.

3. The Schwarzschild metric can be written in the form

$$ds^2 = -\frac{\left(1 - \frac{M}{2r}\right)^2}{\left(1 + \frac{M}{2r}\right)^2} dt^2 + \left(1 + \frac{M}{2r}\right)^4 \left(dx^2 + dy^2 + dz^2\right) \,,$$

where $r^2 = x^2 + y^2 + z^2$ is the Euclidean distance from the origin. Find the coordinate transformation which transforms this metric into the standard form.

4. The spatial part of the Schwarzschild interior metric can be written in the form $ds^2 = dr^2/(1 - kr^2) + r^2 d\Omega^2$. Show that this is the metric of a 3-sphere.

5. Particle motion in the Schwarzschild spacetime. Consider the movement of a massive test particle that was initially at rest at $r(0) = 3r_s$ where r_s is the Schwarzschild radius. Show that

$$\frac{dr}{d\lambda} = -\frac{1}{\sqrt{3}} \sqrt{\frac{3r_s}{r} - 1}, \tag{5}$$

$$\frac{dr}{dt} = -\frac{1}{\sqrt{2}} \left(1 - \frac{r_s}{r}\right) \sqrt{\frac{3r_s}{r} - 1}.$$

Compute the proper time λ_0 the particles needs to reach to centre (Hint: Integrate (5), $\lambda_0 = 3\sqrt{3}\pi r_s/2$). Then solve the equation in terms of coordinate time t assuming $r \approx r_s$ and verify

$$r(t) \approx r_s + c_3 \exp(-c_4\, t/r_s).$$

Interpret your results!

6. [A classic] Consider a radio commentator falling radially into a Schwarzschild black hole. As he approaches the Schwarzschild radius, his broadcast wavelength strongly redshifts. The radio listener (far away from the black hole) observes the time dependence of this redshift $\lambda_{\mathrm{obs}}/\lambda_{\mathrm{e}} \approx \exp(-t/\mu)$. Find the relationship between μ and the mass of the black hole.

7. The de Sitter solution in Schwarzschild coordinates is given by

$$ds^2 = -\left(1 - \frac{\Lambda}{3}r^2\right) dt^2 + \left(1 - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2 d\Omega^2 \,.$$

Identify the radius r_Λ where the metric is singular when $\Lambda > 0$. Next, find the geodesic equations of the de Sitter solution and consider radial geodesics. Show that a freely falling observer starting at the origin with velocity v will cross the surface $r = r_\Lambda$ for finite affine parameter, thereby showing that r_Λ corresponds to a coordinate singularity.