All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Formulae:

Christoffel symbols for the metric $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cp}\left(\frac{\partial g_{ap}}{\partial x^b} + \frac{\partial g_{bp}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^p}\right). \tag{1}$$

Geodesics parameterised by proper time τ (affinely parameterised if null)

$$V^a\nabla_a V^b = 0 \text{ or } \ddot{x}^b + \Gamma^b_{pq}\dot{x}^p\dot{x}^q = 0 \tag{2}$$

where $V^a = \dot{x}^a$ and dot denotes $\mathrm{d}/\mathrm{d}\tau$.
Geodesic deviation equation:

$$D_V^2 Y^d = R_{abc}{}^d V^a Y^b V^c \tag{3}$$

for a vector field Y defined along the geodesic, $D_V = V^a\nabla_a$.
Riemann curvature as a commutator:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)X^d = R_{abc}{}^d X^c. \tag{4}$$

Some symmetries of the curvature tensor

$$R_{abcd} + R_{bcad} + R_{cabd} = 0, \quad R_{abcd} = R_{cdab}. \tag{5}$$

Ricci tensor and scalar curvature (Ricci scalar):

$$r_{ac} = R_{abc}{}^b, \quad s = r_a{}^a. \tag{6}$$

1. The setting of this question is Minkowski space-time $\mathbb{M}$ with units chosen so that $c = 1$. The Minkowski metric is denoted by η or η_{ab}.

   (a) Suppose that $\Gamma(\tau)$ is a curve describing the world-line of an observer. What condition on $V = \dot{\Gamma}$ corresponds to the speed of the observer being less than that of light in any inertial frame? What condition on V corresponds to τ being a *proper time* parameter?

   (b) Two inertial observers Alice and Bob have 4-velocities U and V respectively. What form do U and V take in Alice's rest frame? How are the components of V in this frame related to the velocity $\mathbf{v}$ of Bob relative to Alice?

   Show that
$$\eta(U, V) = \gamma(|\mathbf{v}|)$$
   where
$$\gamma(|\mathbf{v}|) = (1 - |\mathbf{v}|^2)^{-1/2}.$$

   (c) Suppose that Chris is a third inertial observer, with 4-velocity W. Show that if Chris's velocity relative to Alice is $\mathbf{w}$ then the velocity $\mathbf{u}$ of Chris relative to Bob satisfies
$$|\mathbf{u}|^2 = \frac{|\mathbf{v} - \mathbf{w}|^2 - |\mathbf{v}|^2|\mathbf{w}|^2 + (\mathbf{v} \cdot \mathbf{w})^2}{(1 - \mathbf{v} \cdot \mathbf{w})^2}.$$

2. Let $\mathbb{M}$ be Minkowski space with a given inertial coordinate system (t, x, y, z), with respect to which
$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2.$$
   New coordinates (T, X, Y, Z) are defined implicitly by
$$t = X \sinh T,\ x = X \cosh T,\ y = Y,\ z = Z.$$

   (a) Show that the Minkowski metric in the new coordinates takes the form
$$\mathrm{d}s^2 = X^2 \mathrm{d}T^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}Z^2. \qquad (*)$$

   (b) Calculate the Christoffel symbols Γ^c_{ab} for the metric (*) in the (T, X, Y, Z) coordinate system.

   (c) The scalar wave operator is defined to be
$$\Box u = \nabla^a \nabla_a u = \eta^{ab} \nabla_a \nabla_b u.$$
   Write out a formula for $\Box$ explicitly in terms of the partial derivatives $\partial/\partial T$, $\partial/\partial X$, $\partial/\partial Y$, $\partial/\partial Z$.

3. Consider the 2-dimensional metric

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{1+x^2} - (1+x^2)\mathrm{d}\varphi^2,$$

where $-\infty < x < \infty$, $0 < \varphi < 2\pi$.

(a) State what is the signature of this metric.

(b) Write down the Lagrangian and obtain the geodesic equations.

(c) Calculate all of the Christoffel symbols.

(d) Write down two conserved quantities for null geodesics.

(e) Assuming that $\dot{\varphi} \neq 0$, show that the null geodesics which satisfy $(x, \varphi) = (0, 0)$ when $\tau = 0$ are given parametrically by

$$x(\tau) = \pm J\tau, \ \varphi(\tau) = \arctan J\tau,$$

where J is a constant.

4. Let $(\mathscr{M}, g)$ be a curved space-time.

(a) Explain how the Leibniz rule is used to define the covariant derivative $\nabla_a T_b$ of a covector given a definition of $\nabla_a X^b$. Hence write the formula for $\nabla_a T_b$ in terms of $\partial_a T_b$ and the Γs.

(b) In a given coordinate system x^a, the components of g are (g_{ab}) and the components of T^a are *constant.* Show that

$$\nabla_a T_b = \frac{1}{2}(\partial_a T_b - \partial_b T_a) + \frac{1}{2} T^c \partial_c g_{ab}. \tag{7}$$

Deduce that if $T^c \partial_c g_{ab} = 0$, then T satisfies the Killing equation

$$\nabla_a T_b + \nabla_b T_a = 0.$$

(c) Continue with the assumption that the components of T^a are constant in the coordinate system x^a and $T^c \partial_c g_{ab} = 0$.

(i) Calculate $\nabla_c \nabla_a T^b$ in terms of the coordinate derivatives ∂_a and the Γs.

(ii) Show that $T^c \partial_c \Gamma^r_{pq} = 0$.

(iii) Hence or otherwise, show that

$$\nabla_c \nabla_a T_b = R_{abcd} T^d.$$

[The formula $R_{abc}{}^d = \partial_a \Gamma^d_{bc} - \partial_b \Gamma^d_{ac} + \Gamma^e_{bc}\Gamma^d_{ae} - \Gamma^e_{ac}\Gamma^d_{be}$ may be used without proof.]

5. Consider the spherically symmetric space-time metric

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - \frac{1}{A(r)}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2),$$

where $A(r) > 0$ is a smooth function in some region $r > r_0$.

(a) Derive the geodesic equations, and show that these equations allow solutions with θ constant and equal to $\pi/2$.

(b) Obtain the first-order system of equations for non-radial (i.e. with $\dot{\varphi} \neq 0$) null geodesics with $\theta = \pi/2$.

(c) Show that non-radial, equatorial null geodesics satisfy the equation

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = k - u^2 Q(u),$$

where $Q(u) = A(1/u)$ and $u = 1/r$, and k is a positive constant.

(d) Show that circular photon orbits with $r = 1/u_*$ are possible for any $u_* > 0$ satisfying

$$u_* Q'(u_*) + 2Q(u_*) = 0.$$

SOLUTIONS AND MARK SCHEME

Solution 1. (a) The condition is $\eta(V,V) > 0$ or $\eta_{ab}V^aV^b > 0$ (or even $(V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 > 0$). (Saying V is future-pointing is not required.)

3 marks

τ is a proper time parameter if $\eta(V,V) = 1$.

3 marks

(b) In A's rest frame,

$$U = (1, \mathbf{0}), \quad V = \gamma(|\mathbf{v}|)(1, \mathbf{v})$$

where $\mathbf{v}$ is Bob's velocity relative to Alice.

6 marks

From these expressions for U and V,

$$\eta(U,V) = \gamma(|\mathbf{v}|).$$

3 marks

(c) In A's rest frame, U and V are as above and $W = \gamma(|\mathbf{w}|)(1, \mathbf{w})$. Therefore

$$\begin{aligned} \gamma(|\mathbf{u}|) &= \eta(V,W) \\ &= \gamma(v)\gamma(w)(1 - \mathbf{v}\cdot\mathbf{w}). \end{aligned}$$

5 marks

Therefore, inverting and squaring

$$\begin{aligned} 1 - u^2 &= \frac{(1-v^2)(1-w^2)}{(1-\mathbf{v}\cdot\mathbf{w})^2} \\ \Rightarrow u^2 &= 1 - \frac{(1-v^2)(1-w^2)}{(1-\mathbf{v}\cdot\mathbf{w})^2} \\ &= \frac{1 - 2\mathbf{v}\cdot\mathbf{w} + (\mathbf{v}\cdot\mathbf{w})^2 - (1 - v^2 - w^2 + v^2w^2)}{(1-\mathbf{v}\cdot\mathbf{w})^2} \\ &= \frac{|\mathbf{v}-\mathbf{w}|^2 + \mathbf{v}\cdot\mathbf{w}^2 - v^2w^2}{(1-\mathbf{v}\cdot\mathbf{w})^2} \end{aligned}$$

as required.

5 marks

Parts (a) and (b) are bookwork or standard definitions. Part (c) was worked through in lectures, in the slightly different form of the relativistic velocity addition formula.

Solution 2. (a) We have

$$\mathrm{d}t = \mathrm{d}X \sinh T + X \cosh T \mathrm{d}T,\ \mathrm{d}x = \mathrm{d}X \cosh T + X \sinh T \mathrm{d}T,$$

so

$$\mathrm{d}t^2 - \mathrm{d}x^2 = (\mathrm{d}X \sinh T + X \cosh T \mathrm{d}T)^2 - (\mathrm{d}X \cosh T + X \sinh T \mathrm{d}T)^2 = X^2 \mathrm{d}T^2 - \mathrm{d}X^2$$

using $\cosh^2 - \sinh^2 = 1$. The formula for the metric follows.

6 marks

(b) Lagrangian

$$L = \frac{1}{2}(X^2 \dot{T}^2 - \dot{X}^2 - \dot{Y}^2 - \dot{Z}^2)$$

so

$$\frac{\partial L}{\partial \dot{T}} = X^2 \dot{T}, \qquad \frac{\partial L}{\partial \dot{X}} = -\dot{X}, \qquad \frac{\partial L}{\partial \dot{Y}} = -\dot{Y}, \qquad \frac{\partial L}{\partial \dot{Z}} = -\dot{Z};$$
$$\frac{\partial L}{\partial T} = 0, \qquad \frac{\partial L}{\partial X} = -X\dot{T}^2, \qquad \frac{\partial L}{\partial Y} = 0, \qquad \frac{\partial L}{\partial Z} = 0.$$

So the geodesic equations are

$$\begin{aligned} X^2 \ddot{T} + 2X\dot{X}\dot{T} &= 0, \\ \ddot{X} + X\dot{T}^2 &= 0, \\ -\ddot{Y} &= 0, \\ -\ddot{Z} &= 0. \end{aligned}$$

Hence the non-vanishing Γs are

$$\Gamma^0_{01} = \Gamma^0_{10} = 1/X,\ \Gamma^1_{00} = X.$$

10 marks

(c) We have

$$\nabla_a \partial_b u = \partial_a \partial_b u - \Gamma^c_{ab} \partial_c u,\ \text{so}\ \Box u = g^{ab} \partial_a \partial_b u - g^{ab} \Gamma^c_{ab} \partial_c u.$$

3 marks

In (T, X, Y, Z) coordinates,

$$g^{00} = 1/X^2, g^{ii} = -1$$

so the first term is

$$\frac{1}{X^2} \frac{\partial^2 u}{\partial T^2} - \frac{\partial^2 u}{\partial X^2} - \frac{\partial^2 u}{\partial Y^2} - \frac{\partial^2 u}{\partial Z^2}.$$

3 marks

Similarly $g^{ab}\Gamma^c_{ab} = -1/X\delta^c_1$, so the second term is

$$\frac{1}{X}\frac{\partial u}{\partial X}$$

Hence

$$\Box u = \frac{1}{X^2}\frac{\partial^2 u}{\partial T^2} - \frac{\partial^2 u}{\partial X^2} + \frac{1}{X}\frac{\partial u}{\partial X} - \frac{\partial^2 u}{\partial Y^2} - \frac{\partial^2 u}{\partial Z^2}$$

3 marks

This is an unseen question, exploring the students' familiarity with core techniques from the course. The wave operator has appeared in the notes. The first 16 marks on this question should be easy to get for any student who has learned this basic material.

Solution 3. (a) Signature is $(+,-)$.

2 marks

(b)

$$L = \frac{1}{2}\left(\frac{\dot{x}^2}{1+x^2} - (1+x^2)\dot{\varphi}^2\right).$$

Hence

$$\frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{1+x^2}, \qquad \frac{\partial L}{\partial \dot{\varphi}} = -(1+x^2)\dot{\varphi}$$
$$\frac{\partial L}{\partial x} = -\frac{x\dot{x}^2}{(1+x^2)^2} - x\dot{\varphi}^2, \qquad \frac{\partial L}{\partial \varphi} = 0$$

6 marks

Hence

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{\ddot{x}}{1+x^2} - \frac{x\dot{x}^2}{(1+x^2)^2} + x\dot{\varphi}^2 \tag{8}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = -(1+x^2)\ddot{\varphi} - 2x\dot{x}\dot{\varphi}. \tag{9}$$

Hence the geodesic equations are

$$\ddot{x} - \frac{x\dot{x}^2}{1+x^2} + x(1+x^2)\dot{\varphi}^2 = 0, \tag{10}$$
$$\ddot{\varphi} + \frac{2x}{1+x^2}\dot{x}\dot{\varphi} = 0. \tag{11}$$

4 marks

(c) So if $x = x^0$, $\varphi = x^1$, then the non-vanishing Γs are

$$\Gamma^0_{00} = -\frac{x}{1+x^2},\ \Gamma^0_{11} = x(1+x^2),\ \Gamma^1_{01} = \Gamma^1_{10} = \frac{x}{1+x^2}. \tag{12}$$

5 marks

(d) For null geodesics, the conserved quantities are $L = 0$ and $J = (1+x^2)\dot{\varphi}$.

3 marks

(e) Setting $L = 0$, we get

$$\frac{\dot{x}^2}{1+x^2} = (1+x^2)\dot{\varphi}^2 = \frac{J^2}{1+x^2}.$$

Hence $\dot{x} = \pm J$ and so $x(\tau) = \pm J\tau$ if $x = 0$ when $\tau = 0$.

From this,

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \frac{J}{1+J^2\tau^2} \tag{13}$$

and so

$$\varphi(\tau) = \arctan J\tau \text{ if } \varphi = 0 \text{ when } \tau = 0. \tag{14}$$

5 marks

Like Question 2, this one is about the application of core techniques to a metric that they've not seen before. The students have had plenty of practice in computing geodesics of 2-dimensional metrics.

Solution 4. (a) The Leibniz rule in the form

$$\nabla_a(X^b T_b) = X^b(\nabla_a T_b) + T_b(\nabla_a X^b)$$

is used. The LHS is a scalar, so equals

$$\partial_a(X^b T_b) = (\partial_a X^b)T_b + X^b(\partial_a T_b).$$

On the RHS,

$$T^b(\nabla_a X^b) = T^b(\partial_a X^b + \Gamma^b_{ac} X^c).$$

Hence

$$X^b(\nabla_a T_b) = X^b(\partial_a T_b - \Gamma^c_{ab} T_c)$$

and this being valid for arbitrary X means that

$$\nabla_a T_b = \partial_a T_b - \Gamma^c_{ab} T_c$$

6 marks

(b) Under the assumption $T_b = g_{bc}T^c$, where the T^c are constants, we have

$$\partial_a T_b = \partial_a(g_{bc}T^c) = T^c \partial_a g_{bc}.$$

Then

$$\begin{aligned}
\nabla_a T_b &= \partial_a T_b - \Gamma^c_{ab} T_c \\
&= T^c \partial_a g_{bc} - \frac{1}{2} g^{cs}(\partial_a g_{bs} + \partial_b g_{as} - \partial_s g_{ab}) T_c \\
&= T^c \partial_a g_{bc} - \frac{1}{2} T^s(\partial_a g_{bs} + \partial_b g_{as} - \partial_s g_{ab}) \\
&= \frac{1}{2}\left(T^c \partial_a g_{bc} - T^c \partial_b g_{ac}\right) + \frac{1}{2} T^s \partial_s g_{bc}.
\end{aligned}$$

4 marks

Using the formula $\partial_a T_b = T^c \partial_a g_{bc}$ we obtain

$$\nabla_a T_b = \frac{1}{2}(\partial_a T_b - \partial_b T_a) + \frac{1}{2} T^s \partial_s g_{ab}$$

as required.

2 marks

(c) Killing equation follows at once if $T^s \partial_s g_{ab} = 0$.

2 marks

(d) (i) Proceed as suggested, using the constancy of the components T^b.

$$\begin{aligned}\nabla_c \nabla_a T^b &= \nabla_c (\Gamma^b_{as} T^s) \\ &= (\partial_c \Gamma^b_{ad}) T^d + \Gamma^b_{cp} \Gamma^p_{ad} T^d - \Gamma^p_{ca} \Gamma^b_{pd} T^d \\ &= \left(\partial_c \Gamma^b_{ad} + \Gamma^p_{ad} \Gamma^b_{cp} - \Gamma^p_{ca} \Gamma^b_{pd}\right) T^d \end{aligned} \tag{15}$$

4 marks

(ii)

$$T^c \partial_c g_{ab} = 0 \Rightarrow \partial_e (T^c \partial_c g_{ab}) = 0 \Rightarrow T^c \partial_c (\partial_e g_{ab}) = 0$$

because the components T^c are constant. Since $T^c \partial_c g^{ab} = 0$ also, $T^c \partial_c \Gamma^r_{pq} = 0$.

3 marks

(iii) Looking at the given formula for curvature and comparing with (15), this is equal to

$$R_{cda}{}^b T^d + T^d \partial_d \Gamma^b_{ac}.$$

The last term is zero by (ii). And so

$$\nabla_c \nabla_a T_b = R_{cdab} T^d = R_{abcd} T^d$$

by the interchange symmetry of R given on the first page (5).

4 marks

This equation can also be obtained by differentiation of the Killing equation itself. A correct solution along those lines (which is longer if parts (i) and (ii) have been done) would be worth full marks.

Part (a) is bookwork. The rest of the problem is unseen, though the Killing equation appeared on Problem Set 5, with a derivation of the equation $\nabla_c \nabla_a T_b = R_{abcd} T^d$ by a different method.

Solution 5. (a)

$$L = \frac{1}{2}\left(A(r)\dot{t}^2 - \frac{1}{A(r)}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2\right)$$

Hence

$$\frac{\partial L}{\partial \dot{t}} = A(r)\dot{t},\ \frac{\partial L}{\partial \dot{r}} = -\frac{\dot{r}}{A(r)},\ \frac{\partial L}{\partial \dot{\theta}} = -r^2\dot{\theta},\ \frac{\partial L}{\partial \dot{\varphi}} = -r^2\sin^2\theta\dot{\varphi};$$

and

$$\begin{aligned}
\frac{\partial L}{\partial t} &= 0,\\
\frac{\partial L}{\partial r} &= \frac{1}{2}A'\dot{t}^2 + \frac{1}{2}\frac{A'}{A^2}\dot{r}^2 - r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2),\\
\frac{\partial L}{\partial \theta} &= -r^2\sin\theta\cos\theta\dot{\varphi}^2,\\
\frac{\partial L}{\partial \varphi} &= 0.
\end{aligned}$$

Hence the geodesic equations are

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(A(r)\dot{t}\right) = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{r}}{A(r)}\right) + \frac{1}{2}A'\dot{t}^2 + \frac{1}{2}\frac{A'}{A^2}\dot{r}^2 - r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2) = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\theta}\right) - r^2\sin\theta\cos\theta\dot{\varphi}^2 = 0,\\
&\frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\dot{\varphi}\right) = 0.
\end{aligned}$$

The equations are consistent with $\theta = \pi/2$, for then $\dot{\theta} = 0 = \ddot{\theta}$, $\cos\theta = 0$.

6 marks

(b) For null geodesics, $L = 0$, and we get the first order equations:

$$A(r)\dot{t} = E, \tag{16}$$

$$A(r)\dot{t}^2 - \frac{\dot{r}^2}{A(r)} - r^2\dot{\varphi}^2 = 0, \tag{17}$$

$$r^2\dot{\varphi} = J, \tag{18}$$

$$\theta = \pi/2. \tag{19}$$

where E and J are constants, with $J \neq 0$ for non-radial null geodesics.

6 marks

(c) A standard calculation is

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\frac{1}{r} = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{r^2\dot{\varphi}}\dot{r} = -\frac{1}{J}\dot{r}. \tag{20}$$

In (17), use (16) to eliminate $\dot{t}$ in favour of E, (18) to eliminate $\dot{\varphi}$ in favour of J and (20) to eliminate $\dot{r}$ in favour of $\mathrm{d}u/\mathrm{d}\varphi$. The result is

$$\frac{E^2}{A(r)} - \frac{J^2}{A(r)}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - u^2J^2 = 0.$$

Rearranging, and using $A(1/u) = Q(u)$, we get

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 = \frac{E^2}{J^2} - u^2Q(u)$$

which is of the desired form with $k = E^2/J^2$.

6 marks

(d) The second-order equation of motion is obtained by differentiating the previous equation,

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} = -uQ(u) - \frac{1}{2}u^2Q'(u).$$

4 marks

This equation has solutions with $u = u_*$ (constant) if and only if u_* is a zero of the function on the RHS, in other words if u_* is a (positive) zero of the function $u^2Q'(u) + 2uQ(u)$.

3 marks

Parts (a) and (b) of this question are standard and indeed are bookwork in the case of the Schwarzschild solution ($A(r) = 1 - 2m/r$). Circular photon (and particle) orbits were studied in the course, so part (c) is aimed at testing understanding of this in a slightly more general context than Schwarzschild.