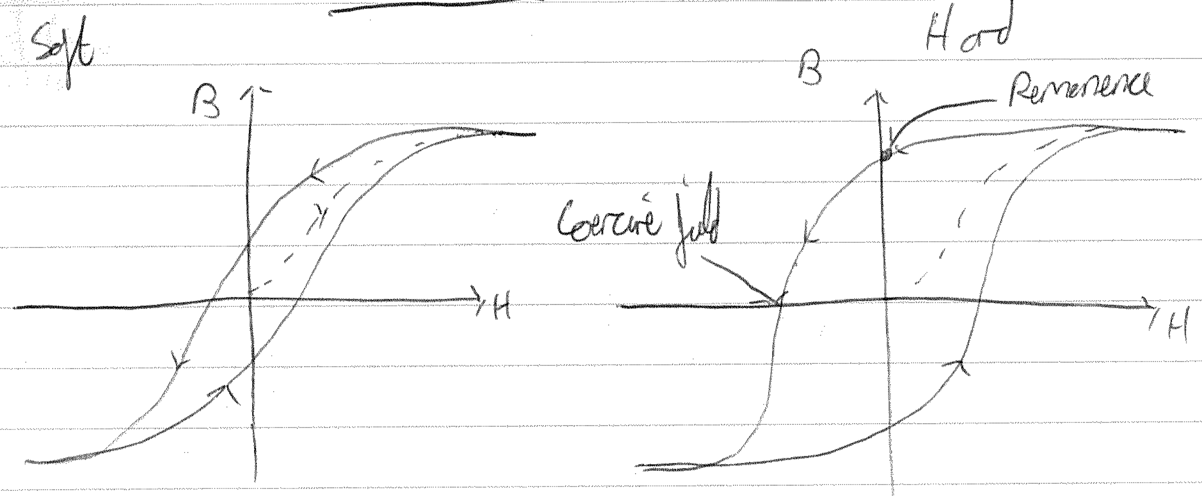
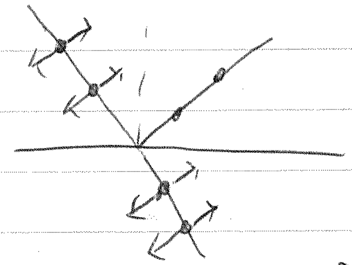
①

2015 EMT

1)



2) Circular light is 2 perpendicular waves with the same amplitude but $\frac{\pi}{2}$ out of phase. The 2 ~~polar~~ perpendicular directions are shown below for the incident wave denoted $\leftrightarrow x$ and $\cdot\, y$ polarisation



Because it is reflected at θ_B only the $\cdot\, y$ polarisation survives. Therefore the reflected beam is linearly polarized.

3a) $U_E = \frac{1}{2}\underline{E}\cdot\underline{D} = \frac{1}{2}\varepsilon_0 E^2$

$U_B = \frac{1}{2}\underline{B}\cdot\underline{H} = \frac{1}{2}\frac{B^2}{\mu_0}$

$\Rightarrow \frac{U_E}{U_B} = \frac{\varepsilon_0\mu_0 E^2}{B^2} = \frac{E^2}{c^2B^2} = \frac{100^2}{c^2\times(5\times10^{-5})^2} = 4.44\times10^{-5}$

b) Invariants are unchanged by a Lorentz transform in any direction

$\underline{E}\cdot\underline{B}$ is the only invariant in the list

4a) $\underline{B} = \nabla \times \underline{A} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} -\frac{1}{2}\alpha y^2 \\ -\frac{1}{2}\alpha z^2 \\ -\frac{1}{2}\alpha x^2 \end{pmatrix} = \begin{pmatrix} \partial_y(-\frac{1}{2}\alpha x^2) + \partial_z(\frac{1}{2}\alpha z^2) \\ \partial_z(-\frac{1}{2}\alpha y^2) + \partial_x(\frac{1}{2}\alpha x^2) \\ \partial_x(-\frac{1}{2}\alpha z^2) + \partial_y(\frac{1}{2}\alpha y^2) \end{pmatrix} = \begin{pmatrix} \alpha z \\ \alpha x \\ \alpha y \end{pmatrix}$

b) $\nabla \times \underline{B} = \mu_0 \underline{J} \Rightarrow \underline{J} = \frac{1}{\mu_0} \nabla \times \underline{B}$

$= \frac{1}{\mu_0} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} \alpha z \\ \alpha x \\ \alpha y \end{pmatrix} = \frac{1}{\mu_0} \begin{pmatrix} \partial_y(\alpha y) - \partial_z(\alpha x) \\ \partial_z(\alpha z) - \partial_x(\alpha y) \\ \partial_x(\alpha x) - \partial_y(\alpha z) \end{pmatrix} = \frac{\alpha}{\mu_0} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

5a) σ_p is the surface ~~current density~~ charge density

ρ_p is the volume charge density

$\underline{P}$ is the dipole moment per unit volume

In an applied field the centre of positive and negative charge in an atom or molecule can move apart creating a charge seperation and associated dipole moment.

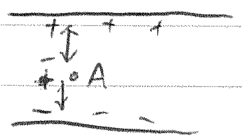


Diagram shows how surface charge density results from individual charge seperations.

If the charge seperation is a function of position the + and − at A may not cancel leaving a polarisation volume charge density.

b) $\nabla \cdot P = 0$ means $\rho_p = 0$ so there is no volume charge density

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6a) $\nabla \cdot \underline{B} = 0 \qquad \nabla \cdot \nabla \times \underline{A} = 0$ for any $\underline{A}$

$\Rightarrow \underline{B} = \nabla \times \underline{A}$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\nabla \times \left(\frac{\partial \underline{A}}{\partial t}\right)$$

$\Rightarrow \nabla \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = 0 \quad \Rightarrow \underline{E} + \frac{\partial \underline{A}}{\partial t} = -\nabla \phi$ since $\nabla \times \nabla f = 0$

$\Rightarrow \underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t}$

ϕ is electric scaler potential, $\underline{A}$ is magnetic vector potential

6i) Coulomb $\nabla \cdot \underline{A} = 0$

Lorenz $\nabla \cdot \underline{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$

7a) $n = n_{real} + n_{imag}\, i = \sqrt{\bar{\varepsilon}_{real} + \varepsilon_{imag}\, i}$

$(n_{real} + n_{imag}\, i)^2 = \bar{\varepsilon}_{real} + i\,\bar{\varepsilon}_{imag}$

$n_{real}^2 - n_{imag}^2 + 2i\, n_{real} n_{imag} = \bar{\varepsilon}_{real} + i\,\varepsilon_{imag}$

Real part : $n_{real}^2 - n_{imag}^2 = \bar{\varepsilon}_{real}$

Im part : $2 n_{real} n_{imag} = \bar{\varepsilon}_{imag}$

$\Rightarrow n_{real}^2 - \left(\frac{\bar{\varepsilon}_{imag}}{2 n_{real}}\right)^2 = \bar{\varepsilon}_{real}$

$n_{real}^4 - \bar{\varepsilon}_{real} n_{real}^2 - \frac{\bar{\varepsilon}_{imag}^2}{4} = 0$

$\Rightarrow n_{real}^2 = \frac{\bar{\varepsilon}_{real} \pm \sqrt{\bar{\varepsilon}_{real}^2 + \bar{\varepsilon}_{imag}^2}}{2}$ take + square root as $n_{real}^2 > 0$

$\Rightarrow n_{real}^2 = \bar{\varepsilon}_{real} + \sqrt{\bar{\varepsilon}_{real}^2 + \bar{\varepsilon}_{imag}^2}$

$$n_{imag}^2 = n_{real}^2 - \bar{\varepsilon}_{real} = \frac{-\bar{\varepsilon}_{real} + \sqrt{\bar{\varepsilon}_{real}^2 + \bar{\varepsilon}_{imag}^2}}{2}$$

ii) A) $\bar{\varepsilon}_{imag} << \bar{\varepsilon}_{real}$

$$n_{real}^2 \simeq \frac{\bar{\varepsilon}_{real} + \sqrt{\bar{\varepsilon}_{real}^2}}{2} = \bar{\varepsilon}_{real} \Rightarrow n_{real} \hat{\simeq} \hat{\varepsilon}_{real}$$

B) $$n_{imag}^2 = \frac{-\bar{\varepsilon}_{real} + \sqrt{\bar{\varepsilon}_{real}^2 + \bar{\varepsilon}_{imag}^2}}{2} \simeq \frac{-\bar{\varepsilon}_{real} + \bar{\varepsilon}_{real}\left(1 + \frac{\bar{\varepsilon}_{imag}^2}{\bar{\varepsilon}_{real}^2}\right)^{\frac{1}{2}}}{2}$$

$$\approx \frac{-\bar{\varepsilon}_{real} + \bar{\varepsilon}_{real}\left(1 + \frac{1}{2}\bar{\varepsilon}_{imag}^2 / \bar{\varepsilon}_{real}^2\right)}{2} = \frac{\bar{\varepsilon}_{imag}^2}{4\bar{\varepsilon}_{real}}$$

$$\Rightarrow n_{imag} \approx \frac{\bar{\varepsilon}_{imag}}{2\sqrt{\bar{\varepsilon}_{real}}}$$

iii) $\bar{\varepsilon}_{imag} << \bar{\varepsilon}_{real}$ for a good conductor

b) $E_{phys} = E_0 e^{-O_1 z} \cos(O_2 z - O_3 t)\,\hat{y}$

i) $E = E_0 e^{i(kx - \omega t)} = E_0 e^{-k_{imag} x} e^{i(k_{real} x - \omega t)}$

i) $k_{real} = O_2$

ii) $k_{imag} = O_1$

iii) $\omega = O_3$

iv) $\frac{\omega}{k} = \frac{c}{n} \Rightarrow n = \frac{c k}{\omega} \Rightarrow n_{real} = \frac{c\, k_{real}}{\omega} = \frac{c\, O_2}{O_3}$

v) $n_{imag} = \frac{c\, k_{imag}}{\omega} = \frac{c\, O_1}{O_3}$

vi) $\bar{\varepsilon} = n^2 = (n_{real} + i n_{imag})^2 = n_{real}^2 - n_{imag}^2 + 2i\, n_{real} n_{imag}$

$$\Rightarrow \bar{\varepsilon}_{real} = n_{real}^2 - n_{imag}^2 = \frac{c^2}{O_3^2}\left(O_2^2 - O_1^2\right)$$

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vii) $\bar{\varepsilon}_{imag} = 2 n_{real} n_{imag} = \frac{2c^2 a_1 a_2}{a_3^2}$

viii) $\underline{B}$ travels in same direction with same k vector and phase but perpendicular polarisation

$|\underline{B}| = \frac{|E|}{c}$

$$\Rightarrow \underline{B}_{phys} = \frac{E_0}{c} e^{-a_1 z} \cos(a_2 z - a_3 t)\,\hat{\underline{x}}$$

ci) $n = \sqrt{\varepsilon} = \sqrt{4\sqrt{109}\, e^{i\,0.2415}} = 6.46\, e^{0.1657 i}$

$= 6.392 + i\,0.9379$

$k = \frac{nw}{c}$ $\Rightarrow$ amplitude drops as $e^{-k_{imag} z} = e^{-\frac{w}{c} n_{imag} z}$

$= e^{-7.659 z}$

$e^{-7.659 d_{\frac{1}{2}}} = \frac{1}{2} \Rightarrow -7.659 d = -\ln 2$

$d_{\frac{1}{2}} = \frac{\ln 2}{7.659} = 0.0905\text{m} = 9\text{cm}$

ii) $\bar{\varepsilon}_{imag} << \varepsilon_{real} \Rightarrow n_{imag} \approx \frac{\varepsilon_{imag}}{2\sqrt{\varepsilon_{real}}} = \frac{3\times10^{-5}}{2\sqrt{1.03}} = 1.478\times10^{-5}$

Amplitude drops as $e^{-\frac{w n_{imag}}{c} z} = e^{-1.207\times10^{-4} z}$

$\Rightarrow$ Depth to drop to $\frac{1}{2}$ $d_{\frac{1}{2}} = \frac{\ln 2}{1.207\times10^{-4}} = 5743\text{m}$

For a 1cm thick container amplitude factor $= e^{-1.207\times10^{-4}\times0.01}$

$= 0.9999988$ $\Rightarrow$ negligible amplitude drop through container

so orange looks good inside

8a) $\nabla \times \underline{E} = -\dfrac{\partial B}{\partial t}$

$\Rightarrow \dfrac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}$

$= -i\, \underline{K} \times \underline{E}$

$\underline{B} = \int -i\, \underline{K} \times \underline{E}\, dt \qquad \underline{E} \propto e^{-i\omega t}$

$\Rightarrow \underline{B} = \dfrac{-i\, \underline{K} \times \underline{E}}{-i\omega} = \dfrac{\underline{K} \times \underline{E}}{\omega}$

b) $\underline{N} = \underline{E} \times \underline{H} = \dfrac{1}{\mu_0} \underline{E} \times \underline{B} = \dfrac{1}{\mu_0 \omega} \underline{E} \times \underline{K} \times \underline{E}$

Plane wave $\underline{K}$ and $\underline{E}$ are perpendicular, let $\underline{K}$ be in $\hat{\underline{x}}$, $\underline{E}$ in $\hat{y}$ without loss of generality

$\Rightarrow \quad \underline{E} \times \underline{K} \times \underline{E} = |K||E|^2\, \hat{y} \times \hat{x} \times \hat{y}$

$= |K||E|^2\, \hat{y} \times \hat{\underline{z}} = |K||E|^2 \hat{\underline{x}} = |E|^2 \underline{K}$

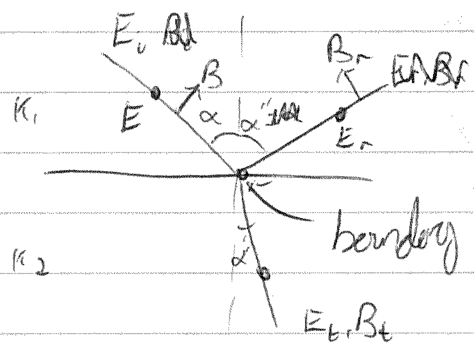
$\Rightarrow \underline{N} = \dfrac{|E|^2\, \underline{K}}{\mu_0 \omega} \qquad |\underline{N}| = N = \dfrac{|E|^2 |\underline{K}|}{\mu_0 \omega} \qquad$ use $\dfrac{\omega}{K} = \dfrac{c}{n}$

$= \dfrac{E_0^2\, K}{\mu_0 \omega} = \dfrac{E_0^2\, n}{\mu_0\, c} = \dfrac{E_0^2\, n}{\mu_0 \left(\frac{1}{\sqrt{\varepsilon_0 \mu_0}}\right)} = \sqrt{\dfrac{\varepsilon_0}{\mu_0}}\, E_0^2\, n$

c) i) $E_\parallel$ and $H_\parallel$ continuous across interface

Since $\mu_r = 1$ $\quad B_\parallel$ also continuous

ii)



$E_\parallel$ continuous $\Rightarrow$ ~~$E_i \sin\alpha + E_r \sin\alpha'' = E_t \sin\alpha'$~~

$E_i + E_r = E_t$ for polarisation shown

$B_\parallel$ continuous ~~$B_i \sin\alpha + B_r \sin\alpha'' = B_t \sin\alpha'$~~

$B_i \cos\alpha - B_r \cos\alpha'' = B_t \cos\alpha'$

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ii) $\cancel{r_\parallel = \frac{E_{r\parallel}}{E_{i\parallel}} = \frac{E_r \sin\alpha''}{E_i \sin\alpha}}$

Perpendicular component of B continuous $\Rightarrow B_i \cos\alpha - B_r \cos\alpha'' = B_t \cos\alpha'$

iii) $r_\parallel = \frac{E_r}{E_i}$ for polarisation in diagram

$$\frac{E}{B} = c = \frac{\omega}{k} \Rightarrow B = \frac{k}{\omega} E$$

$$B_i \cos\alpha - B_r \cos\alpha = B_t \cos\alpha' \quad \text{using } \alpha'' = \alpha$$

$$\frac{E_i k_1 \cos\alpha}{\omega} - \frac{E_r k_1 \cos\alpha}{\omega} = \frac{E_t k_2 \cos\alpha'}{\omega}$$

$$\Rightarrow k_1 (E_i - E_r) \cos\alpha = k_2 E_t \cos\alpha'$$

$$n = \frac{k}{\omega} c \Rightarrow n (E_i - E_r) \cos\alpha = n' E_t \cos\alpha'$$

Combine with $E_i + E_r = E_t$

$$\Rightarrow n E_i \cos\alpha - n E_r \cos\alpha = n' E_i \cos\alpha' + n' E_r \cos\alpha'$$

$$\Rightarrow E_i (n \cos\alpha - n' \cos\alpha') = E_r (n \cos\alpha + n' \cos\alpha')$$

$$r_\parallel = \frac{E_r}{E_i} = \frac{n \cos\alpha - n' \cos\alpha'}{n \cos\alpha + n' \cos\alpha'}$$

This is different to expression in paper but agrees with wikipedia

d i) $\underline{N} = E \times H = \frac{1}{\mu_0} E_\theta B_\phi \hat{r}$ as $\hat{\theta} \times \hat{\phi} = \hat{r}$

$$= \frac{1}{\mu_0} \frac{\mu_0 I_0^2 l^2}{16\pi^2 \varepsilon_0} \frac{\sin^2\theta\, \omega^2 \cos^2(\omega(t - \frac{r}{c}))}{r^2 c^3} \hat{r}$$

$$= \frac{\mu_0^2 I_0^2 l^2}{16\pi^2} \frac{\sin^2\theta}{\varepsilon_0 \mu_0^2} \frac{\omega^2}{r^2 c^3} \cos^2(\omega(t - \tfrac{r}{c})) \hat{r}$$

$$= \frac{\omega^2 m^2}{r^2} \frac{\sin^2\theta}{\frac{1}{c^2} c^3 \mu_0} \cos^2(\omega(t - \tfrac{r}{c})) \hat{r}$$

$$\langle \cos^2(\omega(t - \tfrac{r}{c})) \rangle = \frac{1}{2} \Rightarrow \langle N \rangle = \frac{\omega^2 m^2}{2 c r^2 \mu_0} \sin^2\theta\, \hat{r}$$

ii) Find angle ~~⊗~~ θ between dipole direction and position using dot product

$$\hat{\underline{y}} \cdot \underline{r} = |r| \cos\theta$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = -2 \quad \Rightarrow |r|\cos\theta = -2 \quad |r| = \sqrt{2^2+2^2+1} = 3$$

$$\Rightarrow \cos\theta = \frac{-2}{3} \quad \sin^2\theta = 1 - \frac{4}{9} = \frac{5}{9}$$

$$\Rightarrow \langle N \rangle = \frac{\omega^2 m^2}{2c\mu_0 3^2} \frac{5}{9} \hat{\underline{r}}$$

$$m = \frac{\mu_0 I_0 l}{4\pi} \qquad I_0 l = 3\times10^{-8}\,\mathrm{Am}$$

$$\Rightarrow m = \frac{4\pi\times10^{-7}\times3\times10^{-8}}{4\pi} = 3\times10^{-15}$$

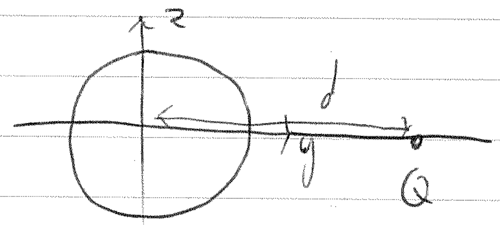
$$\langle N \rangle = \frac{(2\times10^{9})^2 \times (3\times10^{-15})^2}{2\times3\times10^{8}\times4\pi\times10^{-7}\times9} \times \frac{5}{9}\hat{\underline{r}}$$

$$= 2.95\times10^{-15}\hat{\underline{r}}\,\mathrm{Wm^{-2}}$$

9a) c speed of light
J_1, J_2, J_3 ~~as~~ are current density ($\mathrm{Am^{-2}}$) in x, y and z direction
A_1, A_2, A_3 are x, y, z components of 3vector magnetic potential
θ is electric scalar potential

b) x_1, x_2, x_3 represent x, y, z
x_4 is ict

c i) A)



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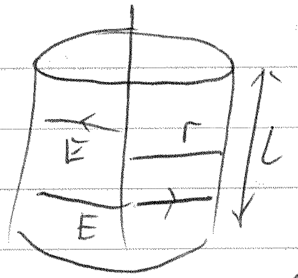


Gaussian cylinder around charged cylinder

$$\oint \underline{E}\cdot d\underline{S} = \frac{q_{enc}}{\varepsilon_0}$$

$$2\pi r l E = \frac{l A \rho}{\varepsilon_0}$$

$$\Rightarrow E = \frac{A\rho}{2\pi r \varepsilon_0}$$

$\Rightarrow$ Force on Q at $r=d$ $\quad F_{el} = \frac{Q\rho A}{2\pi\varepsilon_0 d}$

B) No currents $\Rightarrow \underline{B}=0$. no magnetic force

ii)
$$\begin{pmatrix} J_x' \\ J_y' \\ J_z' \\ ic\rho' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} J_x \\ J_y \\ J_z \\ ic\rho \end{pmatrix}$$

$$\Rightarrow ic\rho' = -i\beta\gamma \times 0 + \gamma i c\rho$$

$$\Rightarrow \rho' = \gamma\rho$$

$$J_y' = J_z' = 0$$

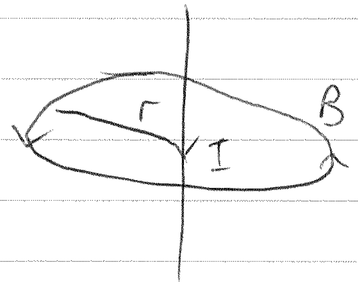
$$J_x' = i\beta\gamma\, ic\rho = -v\gamma\rho \quad \Rightarrow \underline{J}_{\Sigma'} = \begin{pmatrix} -v\gamma\rho \\ 0 \\ 0 \end{pmatrix}$$

iii) In Σ $\quad E = \frac{A\rho}{2\pi r\varepsilon_0}$ $\quad$ same setup in Σ' but different ρ'

In Σ' $\quad \rho' = \gamma\rho \Rightarrow \underline{E}' = \frac{A\gamma\rho}{2\pi\varepsilon_0 r}\hat{r}$

In Σ' $J_x = -v\gamma\rho$ for inside cylinder

Current carrying wire $\oint \underline{B}'\cdot d\underline{l} = \mu_0 I_{enc}$



$$\Rightarrow B' 2\pi r = \mu_0 A J_x = -\mu_0 A v\gamma\rho$$

$$\Rightarrow B' = -\frac{\mu_0 \gamma v \rho A}{2\pi r}\hat{\phi}$$

iv) In Σ total force $F = \frac{Q\rho A}{2\pi\varepsilon_0 d}$

In Σ' charge Q has velocity $-v\hat{x}$

$$\underline{F} = Q(\underline{E} + \underline{v}\times\underline{B})$$

$$= Q\left(\frac{A\gamma\rho}{2\pi\varepsilon_0 d}\hat{r} - v\hat{x}\times\left(-\frac{\mu_0\gamma v\rho A}{2\pi d}\right)\hat{\phi}\right)$$

In this coordinate system r, ϕ, x $\quad x\times\hat{\phi} = -$

$$\Rightarrow \underline{F}' = Q\left(\frac{A\gamma\rho}{2\pi\varepsilon_0 d} - \frac{\mu_0\gamma v^2\rho A}{2\pi d}\right)\hat{r}$$

$$= \underbrace{\frac{Q\rho A}{2\pi\varepsilon_0 d}}_{=F}\left(\gamma - \mu_0\gamma v^2\varepsilon_0\right)\hat{r}$$

$$\Rightarrow F' = F\gamma(1 - v^2\mu_0\varepsilon_0) \qquad \mu_0\varepsilon_0 = \frac{1}{c^2}$$

$$F' = F\gamma\left(1 - \frac{v^2}{c^2}\right) = \frac{F\gamma}{\gamma^2} = \frac{F}{\gamma}$$

$$\Rightarrow F = \gamma F'$$

6)

10a) Plasma is a gaseous mixture of negatively charged electrons and highly charged positive ions. Can be created at high temperature or in strong fields

b) ω_p is frequency above which plasma is transparent to light
N_e is number density of electrons
e is electron charge
m_e is electron mass
ε_0 is permittivity of free space.

c) Electron motion $\quad m_e \frac{d^2 z}{dt^2} = -eE_0 \sin\omega t$

$$\Rightarrow \quad z = \frac{eE_0}{\omega^2 m_e} \sin\omega t$$

Polarisation for each atom $\quad p = -ez = -\frac{e^2 E_0}{m_e \omega^2} \sin\omega t$

$$\Rightarrow \underline{P} = -N_e p = -\frac{N_e e^2}{m_e \omega^2} \underline{E}$$

~~And~~ $\underline{D} = \varepsilon_0 \varepsilon_r \underline{E} = \varepsilon_0 \underline{E} + \underline{P} \quad \Rightarrow \underline{P} = \varepsilon_0 (\varepsilon_r - 1) \underline{E}$

$$\Rightarrow \varepsilon_0 (\varepsilon_r - 1) = -\frac{N_e e^2}{m_e \omega^2}$$

$$\varepsilon_r = 1 - \frac{N_e e^2}{m_e \varepsilon_0 \omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

d) For $\omega \to \omega_p$ the wave is exciting resonant oscillations in the plasma as the wave is reinforcing plasma frequency oscill

e) $\frac{\omega}{k} = \frac{c}{n} \quad \Rightarrow \quad k = \frac{\omega n}{c}$

$$k^2 = \frac{\omega^2 n^2}{c^2} = \frac{\omega^2}{c^2} \tilde{\varepsilon} = \frac{\omega^2}{c^2} \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

f) $K = \frac{\omega}{c}\left(1-\frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}$

$V_g = \frac{d\omega}{dK}$ $\qquad \frac{dK}{d\omega} = \frac{1}{c}\left(1-\frac{\omega_p^2}{\omega^2}\right)^{\pm\frac{1}{2}}$

f) $V_g = \frac{d\omega}{dK}$ $\qquad K^2 = \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}$

$2K = \frac{2\omega}{c^2}\frac{d\omega}{dK}$ differentiate wrt K

$\Rightarrow V_g = \frac{K}{\omega}c^2 = \frac{c^2}{\omega}\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}}$

$= c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$

gi) $V_p = \frac{\omega}{K}$ and $V_g = \frac{K}{\omega}c^2 \Rightarrow V_p V_g = c^2$

ii) For $\omega > \omega_p$ $\quad 0 < \hat{\varepsilon} < 1 \Rightarrow n < 1 \quad V_p > c$

Get superluminal phase velocity for $\omega > \omega_p$ but group velocity must be subluminal since $V_p V_g = c^2$

For $\omega < \omega_p$ $\quad V_{group}$ complex $\Rightarrow V_{group}$ always $< c$

h) 2 frequencies $f_1 = 10^8 Hz$ $\quad f_2 = \frac{c}{700nm} = 4.3\times10^{14} Hz$

$V_{phase} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} = \frac{c}{\sqrt{1-\frac{\omega_p^2}{4\pi^2 f^2}}}$

$\omega_p = \sqrt{\frac{N_e e^2}{m_e \varepsilon_0}} = 9.76\times10^6\ s^{-1}$

$V_1 = \frac{c}{\sqrt{1-\frac{\omega_p^2}{4\pi^2 f_1^2}}}$ and $V_2 = \frac{c}{\sqrt{1-\frac{\omega_p^2}{4\pi^2 f_2^2}}}$

7)

$$\Delta t = \frac{d}{V_2} - \frac{d}{V_1} = \frac{d}{c}\left(\sqrt{1-\frac{\omega_p^2}{4\pi f_2^2}} - \sqrt{1-\frac{\omega_p^2}{4\pi f_1^2}}\right)$$

$$= \frac{d}{c} \times 1.207 \times 10^{-4}$$

$$= 4.02 \times 10^{6}\, s$$