MATH3305 — Problem Sheet 5 – Solutions

(i) The non vanishing Christoffel symbols are:

$$
\begin{aligned}
&\Gamma^0_{01} = \frac{1}{2}\nu'(r) && \\
&\Gamma^1_{00} = \frac{1}{2}\nu'(r)e^{\nu(r)-a(r)} && \Gamma^1_{11} = \frac{1}{2}a'(r) \\
&\Gamma^1_{22} = -re^{-a(r)} && \Gamma^1_{33} = -r\sin^2\theta e^{-a(r)} \\
&\Gamma^2_{12} = \frac{1}{r} && \Gamma^2_{33} = -\cos\theta\sin\theta \\
&\Gamma^3_{13} = \frac{1}{r} && \Gamma^3_{23} = \cot\theta.
\end{aligned}
$$

(ii) The trace terms follow directly and are:

$$
\begin{aligned}
&\Gamma^\sigma_{0\sigma} = 0 && \Gamma^\sigma_{1\sigma} = \frac{2}{r} + \frac{1}{2}(\nu'(r) + a'(r)) \\
&\Gamma^\sigma_{2\sigma} = \cot\theta && \Gamma^\sigma_{3\sigma} = 0.
\end{aligned}
$$

(iii) The Ricci tensor is defined by

$$
R_{\mu\nu} = \Gamma^\rho_{\mu\nu,\rho} - \Gamma^\rho_{\mu\rho,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu}. \tag{4}
$$

Its components are:

$$
\begin{aligned}
R_{00} &= e^{\nu(r)-a(r)}\left(\frac{1}{2}\nu''(r) + \frac{1}{4}\nu'(r)^2 + \frac{1}{r}\nu'(r) - \frac{1}{4}a'(r)\nu'(r)\right) \\
R_{11} &= -\frac{1}{2}\nu''(r) - \frac{1}{4}\nu'(r)^2 + \frac{1}{4}a'(r)\nu'(r) + \frac{1}{r}a'(r) \\
R_{22} &= 1 - e^{-a(r)} + \frac{1}{2}ra'(r)e^{-a(r)} - \frac{1}{2}r\nu'(r)e^{-a(r)} \\
R_{33} &= \sin^2\theta R_{22} \\
R_{ab} &= 0 \quad \text{if } a \neq b.
\end{aligned}
$$

(iv) Slightly lengthy, result is given in the question, it is

$$
\begin{aligned}
R = -\nu''(r)e^{-a(r)} - \frac{1}{2}\nu'(r)^2e^{-a(r)} + \frac{1}{2}a'(r)\nu'(r)e^{-a(r)} \\
+ \frac{2}{r^2} - \frac{2e^{-a(r)}}{r^2} + \frac{2}{r}a'(r)e^{-a(r)} - \frac{2}{r}\nu'(r)e^{-a(r)}.
\end{aligned}
$$

(v) Use the definition $G_{ab} = R_{ab} - g_{ab}R/2$ and the previous results, this should give

$$
\begin{aligned}
G_{00} &= -\frac{1}{r^2}e^{\nu(r)-a(r)}\left(1 - ra'(r) - e^{a(r)}\right) \\
G_{11} &= \frac{1}{r^2}\left(1 + r\nu'(r) - e^{a(r)}\right) \\
G_{22} &= r^2e^{-a(r)}\frac{1}{2}(\nu''(r) - \frac{1}{r}a'(r) + \frac{1}{r}\nu'(r) + \frac{1}{2}\nu'(r)^2 - \frac{1}{2}a'(r)\nu'(r)).
\end{aligned}
$$

(vi) This should be very easy, just compare the terms and note the pre-factor.

(vii) Begin with the given form and work backwards

$$\begin{aligned}
G_{00} &= \frac{1}{r^2}e^{\nu(r)}\frac{d}{dr}\left(r - re^{-a(r)}\right) \\
&= \frac{1}{r^2}e^{\nu(r)}\left(1 - e^{-a(r)} + ra'(r)e^{-a(r)}\right) \\
&= \frac{1}{r^2}e^{\nu(r)-a(r)}\left(e^{a(r)} - 1 + ra'(r)\right) \\
&= -\frac{1}{r^2}e^{\nu(r)-a(r)}\left(1 - ra'(r) - e^{a(r)}\right)
\end{aligned}$$

which is what we wanted to show.