Electrodynamics, Induction

and Maxwell Equations

Chapter 7,8 – D.J.Griffiths

RECAP

Electrodynamics recap:

Ohm's Law: Describes how a generic current flows.

$$\mathbf{J} = \sigma \mathbf{f}$$

Conductivity σ $[\sigma] = S/m$ 1 Siemens = 1 Ohm^{-1}

resistivity $\rho = \frac{1}{\sigma} = R\frac{A}{l}$ $[\rho] = \Omega m$

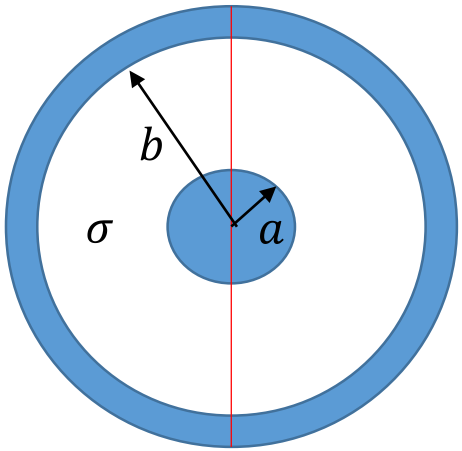
If the forces in question are electromagnetic (most cases of interest) $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

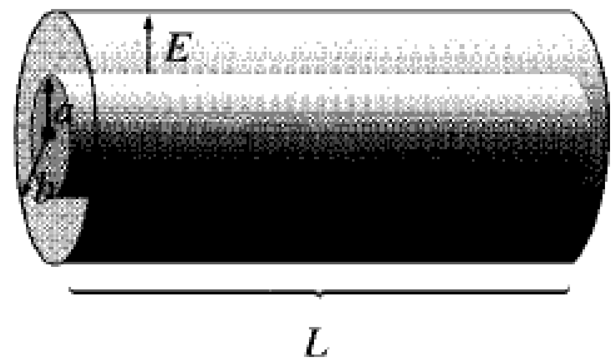
In the more familiar case of an element with two terminals and a potential difference between them: $\boxed{V = IR}$

Example 7.2

Two long cylinders (radii “a” and “b”) are separated by material of conductivity σ . If they are maintained at a potential difference V, what current flows from one to the other in a length L ?

$$V(a) - V(b) = V$$





Solution:

Ultimately we want to calculate the charge per unit time that flows through the cylinder radially outwards (or inwards depending on the sign of V).

The current I can be written as:

$$I = \int \mathbf{J} \cdot d\mathbf{a} = \sigma \int \mathbf{E} \cdot d\mathbf{a}$$

But the field on a conductor is given by the charge divided by ϵ_0

So if we considered some kind of linear charge density λ

The surface charge on the infinitesimal ring of the inner conductor da will be

$$\mathbf{E} = \frac{\lambda}{2\pi s\, \epsilon_0} \hat{\mathbf{s}}$$

$$I = \frac{\sigma \lambda \mathrm{L}}{\epsilon_0}$$

And the potential is

$$V = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = \frac{\lambda}{2\pi\, \epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$\boxed{I = \frac{2\pi\sigma \mathrm{L}}{\ln\left(b/a\right)} V}$$

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Drude model of Ohm's law

(full explanation requires quantum mechanics)
Paul Drude's classical approximation

"frequent collisions between electrons force a constant speed"

This can be considered as subsequent short bursts of acceleration (a) between collisions at a length λ

Time between collisions: $t = \sqrt{\frac{2\lambda}{a}}$ average speed: $\langle v \rangle = \frac{1}{2} a t = \sqrt{\frac{a\lambda}{2}}$

Thermal collision time: $t = \frac{\lambda}{v_{th}}$ $\longrightarrow$ $\langle v \rangle = \frac{a\lambda}{2v_{th}}$

$$\mathbf{J} = nfe\langle \mathbf{v} \rangle = \frac{nfe\lambda}{2v_{th}} \frac{\mathbf{F}}{m_e} = \frac{nfe^2\lambda}{2m_e v_{th}} \mathbf{E} \qquad \sigma \propto \frac{1}{v_{th}}$$

The power transformed into heat is the product of charge flowing per unit time (I) and work done per unit charge (V)

<u>Joule heating law</u> $P = VI = I^2R$

Electromotive Force

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(or e.m.f) is not a force but the integral of a force f pushing the charges
the work done per unit charge

$$\mathcal{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l}$$

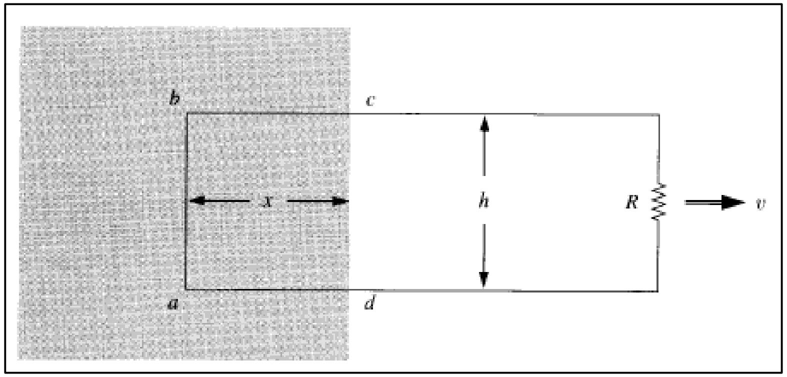
$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \mathcal{E} - Ir$$

Real emf sources have an internal resistance ”r” such that the potential difference between their terminals is

Motional emf

Generators exploit motional forces or the movement of a conductor in a magnetic field. (work performed by the “generator” of the motion)

$$\mathcal{E} \equiv \oint \mathbf{f}_{\mathbf{mag}} \cdot d\mathbf{l} = vBh$$



We looked at calculation of the energy involved or work performed

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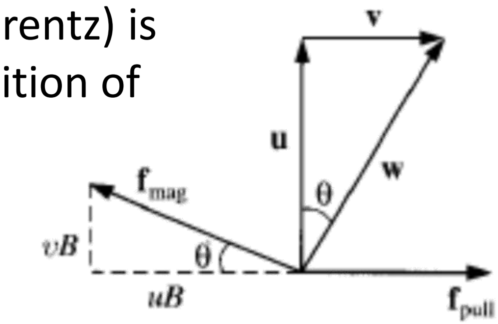
Key is to remember once the force on the motion of the charges (Lorentz) is calculated, the actual motion of the charged particles is the composition of the initial + induced motion.



$$\oint \mathbf{f}_{\mathbf{pull}} \cdot d\mathbf{l} = vBh = \mathcal{E}$$

We considered the magnetic flux defined by the loop and how this varies as changes is imposed:

$$\Phi \equiv \int \mathbf{B} \cdot d\mathbf{a}$$

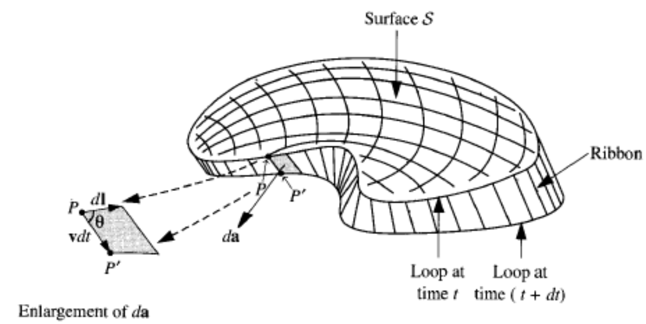
$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -Bhv = -\mathcal{E}$$

Flux Rule:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

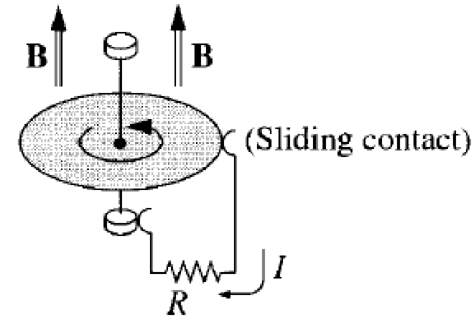
Also valid for generic surface

$$\frac{d\Phi}{dt} = \oint \mathbf{B} \cdot (\mathbf{v} \times d\mathbf{l}) \Rightarrow \boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$



Example 7.4

A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (Fig. 7.14). Find the current in the resistor.



Solution:

Consider Lorentz force acting on the infinitesimal element of the disc.

Generic element at a radial distance "s" from the axis has a tangential velocity $v = \omega s$

So Lorentz force acting on the element will be $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$

The resulting emf is:

$$\mathcal{E} \equiv \oint \mathbf{f}_{\mathbf{mag}} \cdot d\mathbf{l} = \int_0^a \omega s B ds = \omega B \frac{a^2}{2}$$

And the current is

$$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}$$

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}}$$

This case outlines that one has to make a choice to which definition of emf is most appropriate depending on the setup (here it's unclear what the flux of B is and where one would consider it).

But if we wanted to "assume" that there must be an equivalent… and use the definitions we have to infer it…

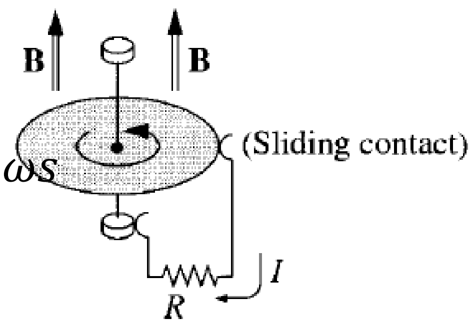
Example 7.4 (extra)

This case outlines that one has to make a choice to which definition of emf is most appropriate depending on the setup (here it’s unclear what the flux of B is and where one would consider it).

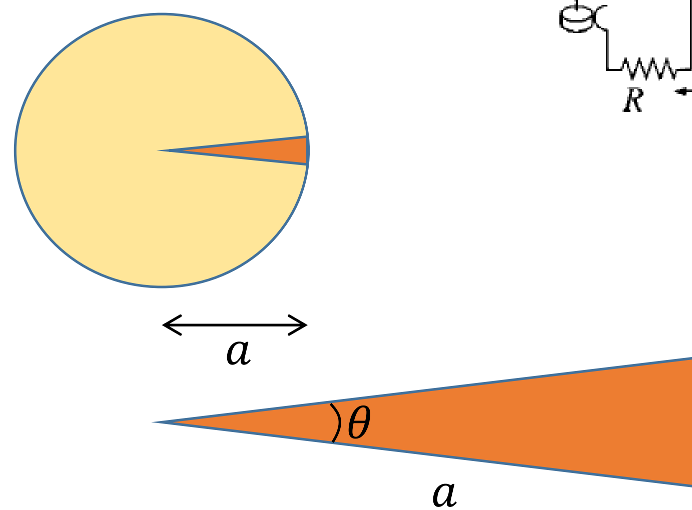
But if we wanted to “assume” that there must be an equivalent… and use the definitions we have to infer it…

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\omega B a^2}{2}$$

$$v = \omega s$$



$$\Phi(B) = \frac{\theta B a^2}{2}$$



In the limit for a small triangle $A \cong \dfrac{\theta a^2}{2}$

$\cong \theta a$

Faraday's Law

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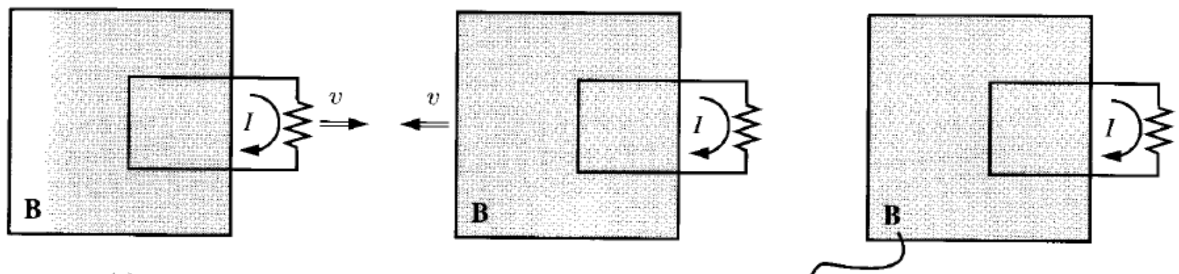


(a) (b) (c)

$$\boxed{\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}}$$

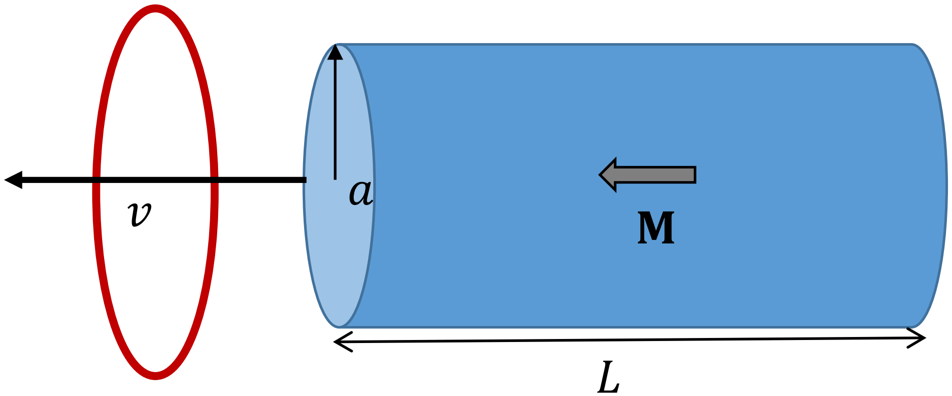
circuit is not changing (geometry)

integral form of Faraday's law which can be written in differential form applying Stokes:

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Example 7.5

A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.21). Graph the emf induced in the ring, as a function of time.



Solution:

The magnetic field is the same of a long solenoid with surface current

$$\mathbf{K}_\mathrm{b} = M\hat{\boldsymbol{\phi}}$$

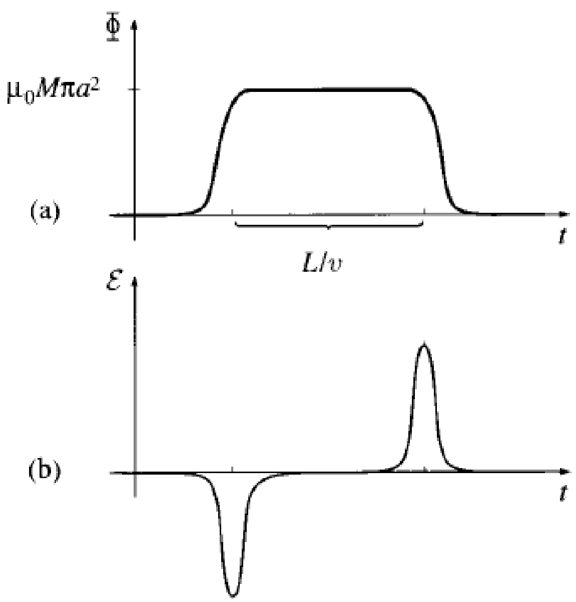
The field inside is

$$\mathbf{B} = \mu_0 \mathbf{M}$$

at the ends of the bar the radial component will be non zero.

Far away the flux will be zero… and start to build up to a maximum of

$$\mu_0 M \pi a^2$$



Induced Electric Field

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$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

There are two kinds of Electric fields:
-) those generated by electric charge
-) those associated to variations of magnetic fields

Faraday-induced electric fields are determined by $-\frac{\partial \mathbf{B}}{\partial t}$

In the same manner that magnetostatic fields are determined by $\mu_0 \mathbf{J}$

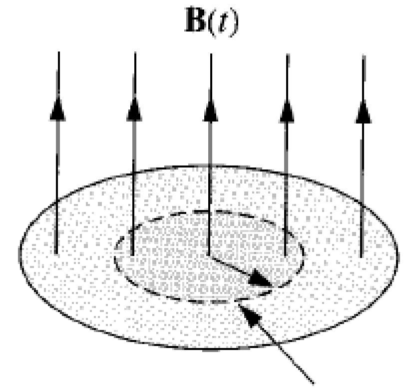
In another analogy, the rate of change of the magnetic flux plays the role of the enclosed current in the loop.

$$\oint \mathbf{E} \cdot d\mathbf{l} = \frac{d\Phi}{dt}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}$$

Example 7.7

A uniform magnetic field **B**(t), pointing straight up, fills the shaded circular region in figure.
If B is changing with time, what is the induced electric field?



Solution:

The electric field manifests itself in a tangential mode

(ANALOGY: magnetic field for a straight wire with current)

Consider an Amperian loop of radius "s" and apply Faraday:

$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}$$

Therefore:

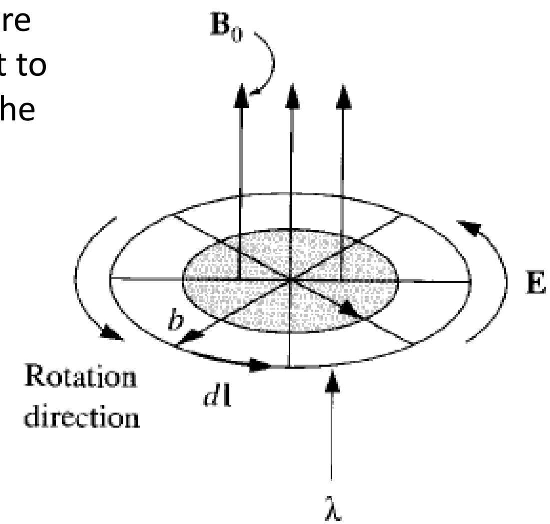
If B is increasing, E runs clockwise, as viewed from above.

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\boldsymbol{\phi}}$$

(<u>You can test this by considering one of two possible current directions and use the right hand screw rule to check if the resulting induced field is going against the increase of the magnetic field – as it should</u>)

Example 7.8

A line charge λ is glued onto the rim of a wheel of radius "b", which is then suspended horizontally as shown in figure, so that it is free to rotate (the spokes are made of some non-conducting material – wood, maybe). In the central region, out to radius "a", there is a uniform magnetic field B_0, pointing up. Now someone turns the field off. What happens?



Solution:

The variation of B induces an electric field, curling around the axis.
The electric field exerts a force on the charge at the rim and the wheel starts to turn. (charge is glued)

Lenz's law helps with the direction. (Motion is counterclockwise when viewed from above.

Faraday:

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

The torque on a length of segment dl is: $\mathbf{r} \times \mathbf{F} = b\lambda E$

The total torque on the wheel is:

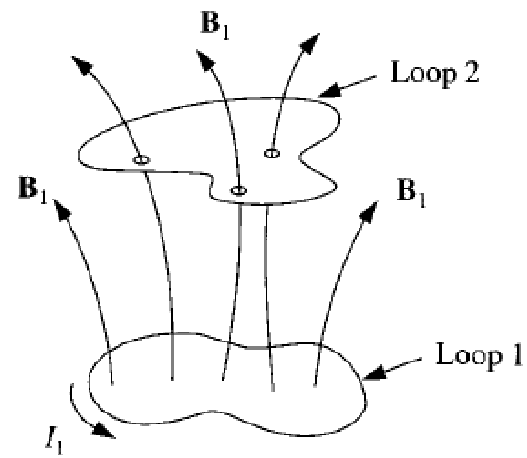
$$N = b\lambda \oint E dl = -b\lambda\pi a^2 \frac{dB}{dt}$$

The angular momentum imparted:

$$\int N\, dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0$$

Inductance

RECAP



Two loops of current, each generating a magnetic field in the other loop.

Biot-Savart $\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\imath}}}{\imath^2}$, $\mathbf{B_1} \propto I_1$

So $\mathbf{\Phi_2} \propto B_1 \propto I_1 \Rightarrow \mathbf{\Phi_2} = M_{21} I_1$

M21 is the coefficient of ***mutual inductance*** between the two loops.

Expressing B as $\mathbf{B} = \nabla \times \mathbf{A}$ and then using $\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\imath}$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\imath}.$$

and $M_{21} = M_{12} = M$

Self-Inductance

On the other hand, a current loop also induces an emf in the source loop itself proportional to the current.

$$\Phi = LI, \qquad \mathcal{E} = -L\frac{dI}{dt}.$$

L is called ***self-inductance*** or **<u>inductance</u>**.

It is measured in henries (H). [H] = Vs/A

Energy in Magnetic Fields

The energy used in a circuit is either lost (Joule heating) or stored (latent).

In a trip around the circuit, the work done on a unit charge is $-\mathcal{E}$

The total work can then be estimated from the amount of charge per unit time passing in the wire:

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

Setting energy at start =0, the overall work done will be $W = \frac{1}{2}LI^2$

So other than the current, the work depends on the geometry of the circuit.

Energy in Magnetic Fields

We saw earlier that the flux of **B** is equal to $\Phi = LI$ combining this with the nominal definition of the same flux:

$$\Phi = LI = \int_S \mathbf{B} \cdot d\mathbf{a} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint_P \mathbf{A} \cdot d\mathbf{l}$$

with P the perimeter of the loop and S any surface bound by P.

$$W = \frac{1}{2} LI^2 = \frac{1}{2} I \oint_P \mathbf{A} \cdot d\mathbf{l}$$

The current is travelling along dl so we can also write

$$W = \frac{1}{2} LI^2 = \frac{1}{2} \oint_P (\mathbf{A} \cdot \mathbf{I}) dl$$

…and it's generalization to volume currents.

$$W = \frac{1}{2} \int_V (\mathbf{A} \cdot \mathbf{J}) d\tau$$

Energy in Magnetic Fields

But we can use $\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J}$

To express the work entirely a function of the magnetic field.

$$W = \frac{1}{2\mu_0} \int_V \mathbf{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B}) d\tau$$

Product rule

$$\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \mathbf{A} \cdot (\boldsymbol{\nabla} \times \boldsymbol{B})$$

Allows us to write

$$W = \frac{1}{2\mu_0} \int_V [\mathbf{B} \cdot (\boldsymbol{\nabla} \times \mathbf{A}) - \boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B})] d\tau$$

$$W = \frac{1}{2\mu_0} \int_V B^2 d\tau - \frac{1}{2\mu_0} \int [\boldsymbol{\nabla} \cdot (\mathbf{A} \times \mathbf{B})] d\tau$$

Energy in Magnetic Fields

$$W = \frac{1}{2\mu_0}\int_V B^2 d\tau - \frac{1}{2\mu_0}\int [\nabla \cdot (\mathbf{A}\times\mathbf{B})] d\tau$$

$$W = \frac{1}{2\mu_0}\int_V B^2 d\tau - \frac{1}{2\mu_0}\oint_S (\mathbf{A}\times\mathbf{B}) \cdot d\mathbf{a}$$

The integration over volume and surface is to be done over the whole region where current is present, but then it can extend further (as J is zero elsewhere).

If we expand the volume out, the contribution from the first integral increases at the expense of the second (A and B decrease)… to the point where they go to zero at infinity:

$$W = \frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau$$

Regardless if the energy is stored in the unit volume of magnetic field $\dfrac{B^2}{2\mu_0}$

or in $\dfrac{\mathbf{A}\cdot\mathbf{J}}{2}$

Energy in Magnetic Fields

Why is energy required to set up a field which does no work?

Recall that the energy is stored (not used).

Also, as the magnetic field is created B is changing and this requires an electric field. This electric field ***can*** do work.

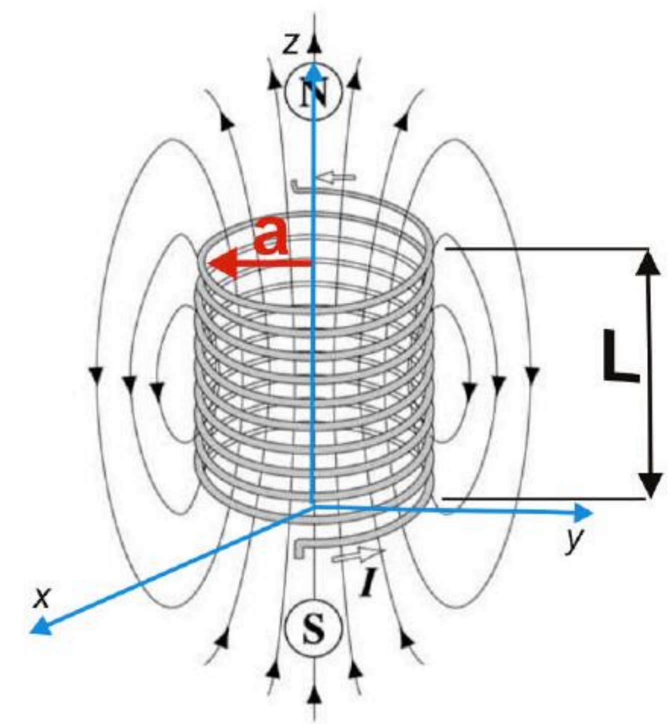
Possibly one of the last similarities with the electric field:

$$W_{\text{elec}} = \frac{1}{2}\int (V\rho)\, d\tau = \frac{\epsilon_0}{2}\int E^2\, d\tau,$$

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A}\cdot\mathbf{J})\, d\tau = \frac{1}{2\mu_0}\int B^2\, d\tau.$$

Electrodynamics
B and H in infinite solenoid compared to uniformly magnetized bar

Consider a tightly wound coil carrying a current *I*



Current : I
Length: L
Number of coils (or turns): N
Radius: a

$$A_\phi = \frac{\mu_0 I A}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{(R^2+a^2+z^2-2aR\cos\phi')^{1/2}}$$

Consider a piece of width "dz" :

$$A_\phi = \frac{\mu_0 I a^2}{4}\frac{R}{(a^2+z^2)^{3/2}} \qquad R \ll a$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{1}{R}\begin{vmatrix}\mathbf{i}_R & R\mathbf{i}_\phi & \mathbf{i}_z \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_R & RA_\phi & A_z\end{vmatrix}$$

$$dB_R = \frac{\mu_0 I a^2}{4}\frac{-3zR}{(a^2+z^2)^{3/2}}$$
$$dB_z = \frac{\mu_0 I a^2}{4}\frac{2}{(a^2+z^2)^{3/2}}$$
$$dB_\phi = 0$$

Electrodynamics
B and H in infinite solenoid compared to uniformly magnetized bar

For the full solenoid we need to integrate:

$$\mathbf{B}(\zeta) = \int_0^L \frac{N}{L}\mathbf{dB}(z-\zeta)dz$$

- On the axis, the B field is always radial (R=0)
- At the centre of any solenoid, the B field is always
- For an infinite solenoid, the B field is always axial
- We don't expect the field to remain axial near the ends of a solenoid

$$B_R(\zeta) = \frac{\mu_0 I a^2 N}{4}\int_0^L \frac{-3(z-\zeta)R}{(a^2+(z-\zeta)^2)^{5/2}}dz \Rightarrow 0$$

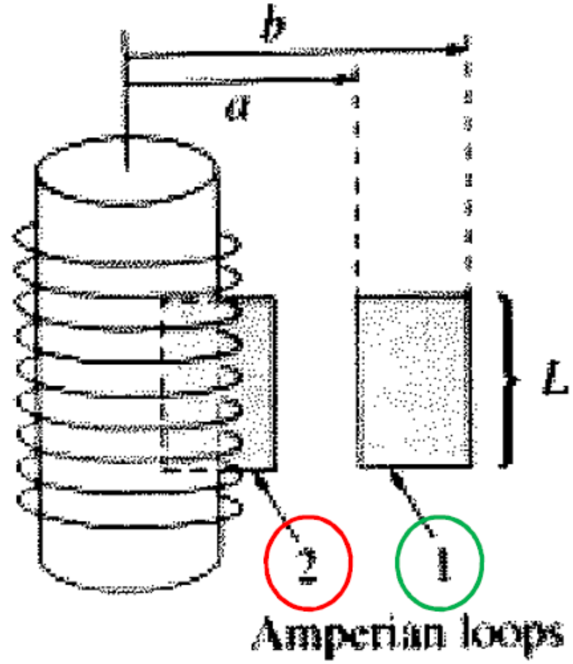
$$B_z(\zeta) = \frac{\mu_0 I a^2 N}{4}\int_0^L \frac{2}{(a^2+(z-\zeta)^2)^{3/2}}dz \Rightarrow z-\zeta = a\ \tan\theta$$

$$= \frac{\mu_0 I a^2 N}{4}\int_{\theta_1}^{\theta_2}(2/a^2)\cos\theta\, d\theta = \frac{\mu_0 I N}{2L}(\sin\theta_2 - \sin\theta_1) \qquad \theta_2\,,\theta_1 \Rightarrow \frac{\pi}{2}\,, -\frac{\pi}{2}$$

$$B_z(\zeta) = \frac{\mu_0 I N}{L}\ as\ L\to\infty$$

Electrodynamics
B and H in infinite solenoid compared to uniformly magnetized bar

Ampere's Law:



1

$$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0$$

$$B(a) = B(b)$$

The field is zero everywhere

2

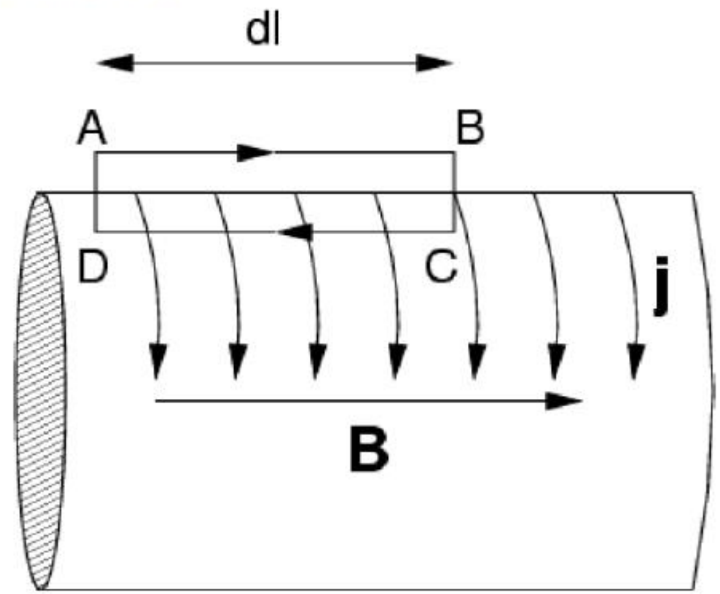
$$\oint \mathbf{B} \cdot d(l) = BL = \mu_0 I_{enc} = \mu_0 NI$$

So:

- Far from ends, field is axial
- $\nabla \times \boldsymbol{B} = \mu_0 \mathbf{J}$ but **J** = 0 inside the solenoid so the field must be uniform $$B_z = \mu_0 NI/L$$
- Outside a long solenoid, $B_z \Rightarrow 0$

Electrodynamics
B and H in infinite solenoid compared to uniformly magnetized bar

Bar magnet



- Assume *uniform* magnetization, $\mathbf{M} = (0, 0, M_z)$
- There will be an associated surface magnetization current, $\mathbf{j}_m$
- This will be $\mathbf{j}_m = (0, M_z, 0)$ in cylindrical polar coordinates
- Compare this with $\mathbf{j}_f = NI/L$ in the solenoid (*free* current)

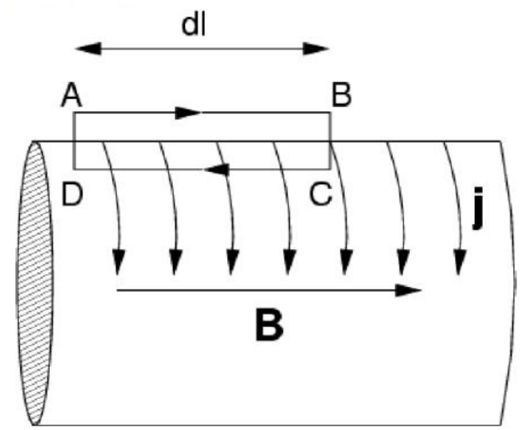
$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{loop}$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mathbf{B}_{out} \cdot AB + \mathbf{B}_{in} \cdot CD = \mu_0 j dl$$

$$B_z dl = \mu_0 j dl$$

We can use the same geometry for the solenoid and the bar magnet

Electrodynamics
B and H in infinite solenoid compared to uniformly magnetized bar



Bar magnet

For the solenoid / For the bar magnet

$$j = \frac{NI}{L}\ , \qquad B_z = \mu_0 nI/L \qquad\qquad j = M_z\ , \qquad B_z = \mu_0 M$$

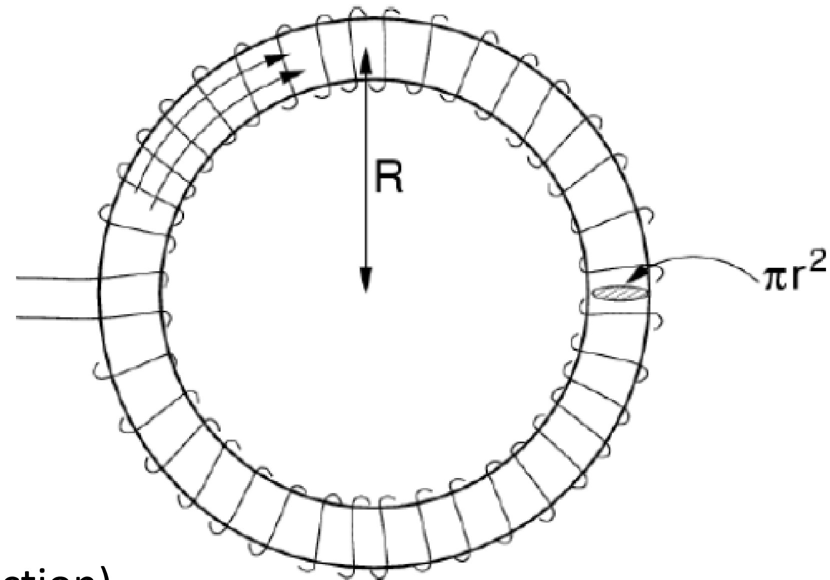
$$M = 0\ , \qquad \mathbf{H} = \mathbf{B}/\mu_0 \qquad\qquad \mathbf{M} = \frac{\mathbf{B}}{\mu_0}\ , \mathbf{H} = 0$$

Putting the two together (solenoid on bar magnet) gives an electromagnet with $j = j_f + j_m$

We find $B_z = \mu_0 \left(\frac{NI}{L} + M_z \right)$ $\qquad H_z = \frac{NI}{L}$ $\qquad \boxed{B_z = \mu_0 (H_z + M_z)}$

Electrodynamics
Toroid

Winding on torus



A toroidal closed loop

Closed lines of B

Assume radius of ring R >> r (raidus of cross section)

N turns total, current I

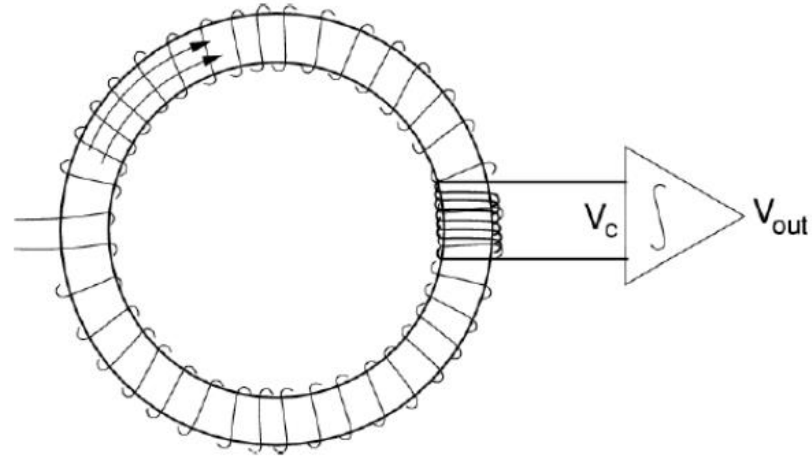
$$\oint \mathbf{H} \cdot d\mathbf{l} = H2\pi R = NI$$

$$H = \frac{NI}{2\pi R}$$

Electrodynamics
Toroid and Fluxmeter

Fluxmeter

Wind an extra coil, with n_c turns over the magnetising coil



Connect these to a fluxmeter (op-amp circuit with low impedence R_c)

$$V_{out} = K \int_0^t I_c dt$$

I_c flows because V_c is induced by the changing B in the toroid

$$\Phi(t) = B(t)A = \pi r^2 B(t)$$

Faraday law gives the voltage around one turn as:

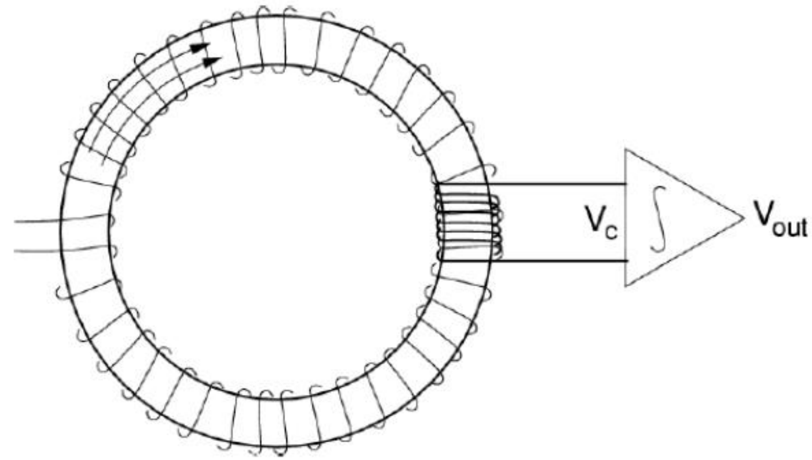
$$\Delta V = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi(t)}{dt} = -\pi r^2 \frac{dB}{dt}$$

so $$V_c(t) = n_c \Delta V = \pi r^2 n_c \frac{dB}{dt} = I_c R_c$$

$$V_{out}(t) = \frac{n_c \pi r^2 K}{R_c} \int_0^t \frac{dB}{dt} dt$$

Electrodynamics
Torus

Fluxmeter



$$V_{out}(t) = \frac{n_c \pi r^2 K}{R_c} \int_0^t \frac{dB}{dt} dt$$

$$V_{out}(t) = \left(\frac{n_c \pi r^2 K}{R_c}\right)(B(t) - B(0))$$

$$V_{out}(t) = C\Delta B(t)$$

We impose **H** via current in a toroidal loop

Measure **B** via fluxmeter output

This provides direct evidence of **B** and **H** and so **M** as well

Electrodynamics – Energy Density in solenoid

In a circuit with a resistance R we have:

$$V + \mathcal{E} = IR\frac{dI}{dt} \qquad \mathcal{E} = -\frac{d\Phi}{dt}$$

The work done moving dq:

$$Vdq = VIdt = -\mathcal{E}Idt + I^2Rdt$$

Ignoring Ohmic losses for now:

$$dW_b = Id\Phi$$

(energy required to maintain the current I)

Generalizing:

$$dW_b = \sum_{i=1}^{n} I_i d\Phi_i$$

If start from a zero current and increase it parametrically (α) we can rewrite the differential expression as

$$I_i' = \alpha I_i$$

$$d\Phi_i = \Phi_i d\alpha$$

$$\int dW_b = \int_0^1 d\alpha \sum_{i=1}^{n} I_i \alpha \Phi_i$$

$$W_b = \sum_i I_i \Phi_i \int_0^1 \alpha d\alpha = \frac{1}{2}\sum_i I_i \Phi_i$$

Electrodynamics – Energy Density in solenoid

We have a total energy: $$W = \frac{1}{2}\sum_i I_i \Phi_i$$

Consider each turn as a circuit
$$\Phi_i = \Phi = \pi r^2 B$$
$$\sum_i I_i = NI$$

From an Amperian loop, $NI = HL$ $\qquad V = \pi r^2 L$

$$W = \frac{1}{2}\sum_i I_i \pi r^2 B = \frac{\pi r^2 B}{2}\sum_i I_i = \frac{\pi r^2 BNI}{2} = \frac{\pi r^2 BHL}{2} = \frac{1}{2}HBV$$

Energy density is $$U = \frac{1}{2}HB$$

More generally: $$U = \frac{1}{2}\mathbf{H}\cdot\mathbf{B}$$

Electrodynamics – Energy Density in solenoid

More generally for a single circuit:

$$\Phi_i = \int_{S_i} \boldsymbol{B} \cdot \boldsymbol{n} da = \oint_{C_i} \boldsymbol{A} \cdot d\boldsymbol{l}_i$$

For which the total energy is

$$W = \frac{1}{2} \sum_i \oint_{C_i} I_i \, \mathbf{A} \cdot d\mathbf{l_i}$$

As we are interested in the magnetic energy density in a general medium:

$$I_i d\mathbf{l_i} \rightarrow \mathbf{J} dv \qquad \sum_i \oint_{C_i} \rightarrow \int_V$$

$$U = \frac{1}{2} \int_V \mathbf{J} \cdot \mathbf{A} dv$$

Using: $\nabla \times \mathbf{H} = \mathbf{J}$

$$\nabla \cdot (\mathbf{A} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H})$$

$$\begin{aligned} U &= \frac{1}{2} \int_V \boldsymbol{H} \cdot (\nabla \times \boldsymbol{A}) dv - \frac{1}{2} \int_V \nabla \cdot (\boldsymbol{A} \times \boldsymbol{H}) dv \\ &= \frac{1}{2} \int_V \boldsymbol{H} \cdot \boldsymbol{B} dv - \int_S (\boldsymbol{A} \times \boldsymbol{H}) \cdot \boldsymbol{n} da \end{aligned}$$

$$U = \frac{1}{2} \mathbf{H} \cdot \mathbf{B}$$

As the volume considered goes to infinity

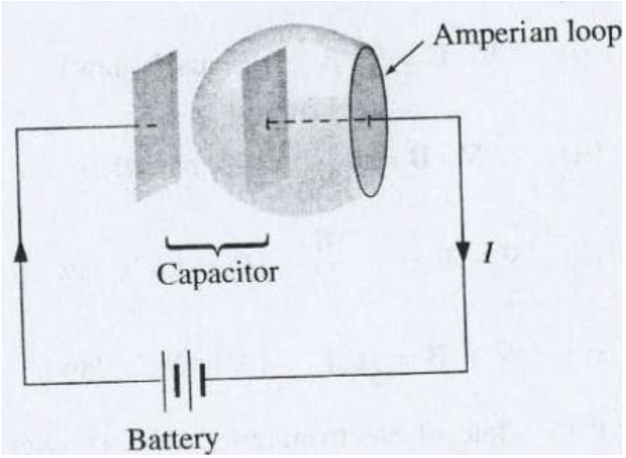
Summary of Linear Media

- Linear: χ_e is independent of $\mathbf{E}$ (or χ_m of $\mathbf{B}$)
- Isotropic: $\mathbf{P}$ is parallel to $\mathbf{E}$ (or $\mathbf{M}$ to $\mathbf{H}$)
- $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$
- $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$ so $\mathbf{D} = \epsilon\mathbf{E}$, with $\epsilon = \epsilon_0\,(1 + \chi_e)$
- $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$
- $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
- $\mathbf{M} = \chi_m\mathbf{H}$ so $\mathbf{B} = \mu_0\mu_r\mathbf{H}$ with $\mu_r = 1 + \chi_m$
- $\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f$

Summary of Non-Linear Media

- Unpaired electrons give *intrinsic* moment
- There is a *short-range* force which aligns these spins
- If parallel, *ferromagnetic* ordering
- If anti-parallel, *anti-ferromagnetic* ordering
- Local domains of aligned atoms form (up to microns across)
- Long-range forces arrange these opposed to each other
- Highly non-linear B vs. H curves: *hysteresis*
- Energy density, $U = \frac{1}{2}\mathbf{B} \cdot \mathbf{H}$

Displacement current



To account for time-varying electric fields in Ampere's Law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da$$

Consider a closed surface integral over the current density

The current passing through an element of area, "da" is

$$dI = \mathbf{J} \cdot \mathbf{n} da$$

$$I = -\oint \mathbf{J} \cdot \mathbf{n} da = -\int \boldsymbol{\nabla} \cdot \mathbf{J} dv$$

$$I = \frac{dQ}{dt} = \frac{d}{dt}\int_V \rho dv = \int_V \frac{\partial \rho}{\partial t} dv$$

Ampere Maxwell's Equation

$$\int_V \frac{\partial \rho}{\partial t} dv = -\int \nabla \cdot \mathbf{J} dv$$

$$\int_V \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} \right) dv = 0$$

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Continuity equation

$$\nabla \times \mathbf{H} = \mathbf{J}_f$$

$$\nabla \cdot (\nabla \times \mathbf{H})(= 0) = \nabla \cdot \mathbf{J}_f$$

$$\nabla \cdot \mathbf{D} = \rho_f$$

$$\nabla \cdot \mathbf{J}_f + \frac{\partial (\nabla \cdot \mathbf{D})}{\partial t} = 0$$

$$\nabla \cdot \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) = 0$$

Ampere Maxwell equation

$$\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H}$$

Maxwell's Equation

Differential Form | **Integral Form**

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \qquad \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{n} da$$

Ampère-Maxwell

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da = -\frac{d\Phi}{dt}$$

Faraday

$$\nabla \cdot \mathbf{D} = \rho_f \qquad \oint_S \mathbf{D} \cdot \mathbf{n} da = \int_v \rho_f dv$$

Coulomb-Gauss

$$\nabla \cdot \mathbf{B} = 0 \qquad \oint_S \mathbf{B} \cdot \mathbf{n} da = 0$$

Biot-Savart+

Maxwell's Equation

Assume a uniform, linear, isotropic medium

- Then $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

assume that the medium has uniform conductivity g, so that $\mathbf{J}_f = g\mathbf{E}$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times (\nabla \times \mathbf{H}) = \nabla \times \left(\mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t} \right)$$

$$\nabla (\nabla \cdot \mathbf{H}) - \nabla^2 \mathbf{H} = g \nabla \times \mathbf{E} + \epsilon \frac{\partial}{\partial t} \nabla \times \mathbf{E}$$

$$\nabla \cdot \mathbf{H} = 0 \qquad\qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla^2 \mathbf{H} - g\mu \frac{\partial \mathbf{H}}{\partial t} - \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

As $g \to 0$ (a non-conducting medium) $\boxed{\nabla^2 \mathbf{H} = \epsilon\mu \frac{\partial^2 \mathbf{H}}{\partial t^2}}$

Maxwell's Equation

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} \qquad \mathbf{B} = \mu \mathbf{H}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla(\cancel{\nabla \cdot \mathbf{E}}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial \mathbf{J}_f}{\partial t} - \mu \frac{\partial^2 \mathbf{D}}{\partial t^2}$$

no free charges (so $\nabla \cdot \mathbf{E} = 0$) $\qquad \mathbf{J}_f = g\mathbf{E} \qquad \mathbf{D} = \epsilon \mathbf{E}$

$$\nabla^2 \mathbf{E} - g\mu \frac{\partial \mathbf{E}}{\partial t} - \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

if $g \to 0$ we find:

$$\boxed{\nabla^2 \mathbf{E} = \epsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2}}$$

the speed of the wave is $c = 1/\sqrt{\epsilon\mu}$

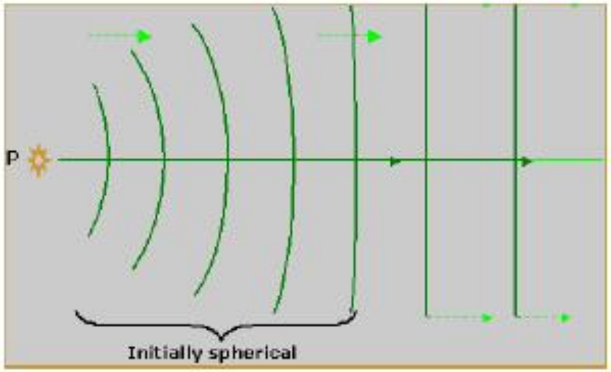
$\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{B} = \mu \mathbf{H}$

The solutions will be plane waves:

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k}_H \cdot \mathbf{r} - \omega_H t)}$$

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\mathbf{k}_E \cdot \mathbf{r} - \omega_E t)}$$

Introduction to Plane Waves



$$\mathbf{C} = \mathbf{C}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)}$$

$$\frac{\partial \mathbf{C}}{\partial t} = -i\omega\mathbf{C}$$

$$\nabla\times\mathbf{C} = i\mathbf{k}\times\mathbf{C}$$

$$\nabla\cdot\mathbf{C} = i\mathbf{k}\cdot\mathbf{C}$$

$$\nabla\cdot\mathbf{C} = i\left(C_{0x}k_x + C_{0y}k_y + C_{0z}k_z\right)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t+\phi)} = i\mathbf{k}\cdot\mathbf{C}$$

$$\mathbf{k} = (0, 0, k)$$

$$\mathbf{H}(\mathbf{r}, t) = \mathbf{H}_0 e^{i(\mathbf{k}_H\cdot\mathbf{r}-\omega_H t)} \qquad \nabla^2\mathbf{H} = \epsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}$$

$$\nabla^2\mathbf{H} = -k^2\mathbf{H}$$

$$\partial^2\mathbf{H}/\partial t^2 = -\omega^2\mathbf{H}$$

if $k^2/\omega^2 = \epsilon\mu$, then a plane wave solves the equation for $\mathbf{H}$

phase velocity is $c = 1/\sqrt{\epsilon\mu}$