PHAS3201: Theory of Dynamical Systems

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Contents

1 Preliminaries

A dynamical system describes the evolution of a set of continuous/discrete quantities $\mathbf{x} = (x_1, x_2, ..., x_n)$, which are typically functions of time (and other parameters). The state of a system is just the collection of these quantities at a given time, so $\mathbf{x}_{t_0}$ is one state, and $\mathbf{x}_{t_1}$ is another. The space of all possible states of a system is the phase/state space $\chi(t, \mathbf{x})$.

1.1 Generalised Coordinates

If a system has ν degrees of freedom, then $n = \nu$ independent coordinates are required to fully specify it. Imposing a constraint is equivalent to removing a degree of freedom from the system. Thus, for each constraint, two of these coordinates can be 'rolled into one' and turned into a generalised coordinate.

More specifically, if a constraint is *holonomic* then it is of the form:

$$f(\mathbf{r}, t) = 0$$

Where $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_n)$, and each $\mathbf{r}_i$ is the position vector of particle i.

Since a holonomic constraint can be rewritten so that one of the coordinates equals some combination of all the rest, we can eliminate one coordinate from the set. For k holonomic constraints, we can remove k coordinates from the original set. Redefining these new coordinates as 'generalised coordinates' $\mathbf{q} = (q_1, q_2, ..., q_n)$ allows us to work with $\nu - k$ coordinates that are free from constraints[1]!

Typically, we work with N particles with $3N$ degrees of translational freedom, and thus we need $3N - k$ generalised coordinates.

For example, a bob on the end of a pendulum in 3-D space is positioned by $\mathbf{r} = (x, y, z)$, but it is subject to the constraints $x^2 + y^2 + z^2 = l^2$ where l is the length of the pendulum, and $z = 0$. Thus we can eliminate two coordinates and just use $\mathbf{q} = (\theta)$.

1.2 Classifying Dynamical Systems

A dynamical system is *linear* if:

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = A\mathbf{x}$$

Where A is a matrix. A dynamical system is *simple* if A is non-singular. It follows that the only fixed point of a linear system is the origin.

1.2.1 Jordan Forms

2 Lagrangian and Hamiltonian Dynamics

2.1 The Problem With Newton

Newtonian dynamics requires us to - for each particle: find all the forces acting on it, and all the constraints it must obey. We must then construct $3N$ coupled second order differential equations of the form:

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}(\mathbf{x}, \dot{\mathbf{x}}, t)$$

And then solve them.

Clearly this is mental, and impractical/analytically impossible for any large system of particles (or for any system that has particles that are heavily coupled to each other). Thus we need to use new methods of solving for equations of motion that don't involve computation times on the order of the lifetime of the universe.

[1] The constraints are implicit in the way we construct these coordinates, so technically they're still wrapped up in there, but we don't have to deal with them directly anymore.

2.2 Lagrangian Dynamics

Lagrangian dynamics revolves around a quantity called the Lagrangian[2] $\mathcal{L}$, defined as follows:

$$\mathcal{L} = \mathcal{T} - \mathcal{V} \equiv \mathcal{L}\left(q_i, \dot{q}_i, t\right)$$

Where $\mathcal{T}$ is the kinetic energy of the system, and $\mathcal{V}$ is the potential energy of the system. All we do to solve for the equations of motion for particles in a system is plug this into the Euler-Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Where q_i is the generalised coordinate we want to solve for.

Note that the Euler-Lagrange equation has the form of Newtonian motion:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = F_i$$

Where $p_i = \partial_{\dot{q}_i}\mathcal{L}$ is the generalised momentum, and $F_i = \partial_{q_i}\mathcal{L}$ is the generalised force. Note that the units of these two quantities need not be the units of momentum/force respectively - they are *generalised* and so may have weird units in the context of the given system.

These two quantities are important, and you should be able to immediately retrieve them from the Lagrangian.

Note that if $\mathcal{L}$ is independant of one of the coordinates q_i then the generalised momentum is conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_i}}_{\text{constant}} - \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{0} = 0$$

2.2.1 Principle of Least Action

The equation above stems from calculus of variations, the integral we are trying to minimise is:

$$\mathcal{A} = \int \mathcal{L} dt$$

Where $\mathcal{A}$ is the *action* of a system. Basically, it is assumed that physical systems minimise their action, so by finding the Lagrangian that minimises $\mathcal{A}$, we can in turn find out exactly how the system evolves. The Euler-Lagrange equation can be taken as a given. For a proper derivation from the principle of least action, see the appendix.

2.2.2 DERIVATIONS - FILL THIS IN

2.3 Hamiltonian Dynamics

Instead of using the generalised coordinates q_i and their generalised velocities $\dot{q}_i$, we can use the generalised coordinates q_i and the generalised momenta $p_i = \partial_{\dot{q}_i}\mathcal{L}$. This framework is called Hamiltonian Dynamics/the Hamiltonian formulation. We start by defining the Hamiltonian $\mathcal{H}$ using a legendre transform of the Lagrangian:

$$\mathcal{H} = p_i \dot{q}_i - \mathcal{L}$$

Where we sub into $\mathcal{L}$:

$$\dot{q}_i \equiv \dot{q}_i\left(q_i, p_i\right)$$

Which we can obtain from the generalised momentum equation from the previous subsection.

Hamilton's equations are as follows:

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}$$

[2] Wow, shocker.

And their solutions yield the equations of motion of the system. Please note that if $\mathcal{H}$ has no explicit time dependence, then it is a conserved quantity:

$$\frac{\partial \mathcal{H}}{\partial t} = \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0$$

Moreover, this conserved quantity is the total energy of the system:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = 0 \rightarrow \mathcal{H} = \mathcal{T} + \mathcal{V}$$

2.3.1 Derivation of Hamilton's Equations

The derivation of these equations is exceptionally straightforward. For the derivation of $\dot{q}_i$:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial p_i} &= \frac{\partial}{\partial p_i}\left(p_j \dot{q}_j - \mathcal{L}\right) \\
&= \underbrace{\frac{\partial p_j}{\partial p_i}}_{\delta_{ij}} \dot{q}_j + p_j \frac{\partial \dot{q}_j}{\partial p_i} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial p_i} \\
&= \dot{q}_i + p_j \frac{\partial \dot{q}_j}{\partial p_i} - p_j \frac{\partial \dot{q}_j}{\partial p_i} \\
\frac{\partial \mathcal{H}}{\partial p_i} &= \dot{q}_i
\end{aligned} \tag{1}$$

Similarly, for the derivation of $\dot{p}_i$:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial q_i} &= \frac{\partial}{\partial q_i}\left(p_j \dot{q}_j - \mathcal{L}\right) \\
&= \underbrace{\frac{\partial p_j}{\partial q_i}}_{0} \dot{q}_j + p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_j} \underbrace{\frac{\partial q_j}{\partial q_i}}_{\delta_{ij}} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial q_i} \\
&= p_j \frac{\partial \dot{q}_j}{\partial q_i} - \frac{\partial \mathcal{L}}{\partial q_i} - p_j \frac{\partial \dot{q}_j}{\partial q_i} \\
&= -\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\frac{\mathrm{d}}{\mathrm{d}t} p_i \\
-\frac{\partial \mathcal{H}}{\partial q_i} &= \dot{p}_i
\end{aligned} \tag{2}$$

2.3.2 Time Evolution of $\mathcal{H}$

2.4 Liouville's Theorem

3 Linearisation