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Quantum 16

A1a) $|\psi\rangle = \sum_n c_n |\phi_n\rangle \qquad c_n = \langle \phi_n | \psi\rangle$

by using $|\psi\rangle = I|\psi\rangle$

$= \sum_n |\phi_n\rangle\langle\phi_n|\psi\rangle = \sum_n |\phi_n\rangle c_n$

b) $|\psi\rangle = \sum_a d_a |\chi_a\rangle = \sum_n c_n |\phi_n\rangle$

$\cancel{|\chi_a\rangle = S_{an}|\phi_n\rangle} \quad \chi = S\phi S^{-1}$

Take $\langle \chi_b|$

$\Rightarrow d_b = \sum_n c_n \langle \chi_b|\phi_n\rangle \Rightarrow d_a = \sum_n c_n \langle \chi_a|\phi_n\rangle$

$= \sum_n c_n S_{an}$

2a) $\sum_n |\phi_n\rangle\langle\phi_n| = \mathbb{1}$

b) $(SS^\dagger)_{ij} = \sum_k \langle \chi_i|\phi_k\rangle\langle\phi_k|\chi_j\rangle$

$= \langle \chi_i| \left(\sum_k |\phi_k\rangle\langle\phi_k| \right) |\chi_j\rangle$

$= \langle \chi_i|\chi_j\rangle = \delta_{ij} \Rightarrow SS^\dagger = I$

c) $\hat{A}|\psi_n\rangle = o_n|\psi_n\rangle$

$|\psi\rangle = \sum_n c_n|\phi_n\rangle$

$\Rightarrow \hat{A}|\psi\rangle = \sum_n c_n \hat{A}|\phi_n\rangle = \sum_n c_n o_n |\phi_n\rangle$

Can write $\hat{A}$ as $\begin{pmatrix} o_0 & & & \\ & o_1 & & \\ & & o_2 & \\ & & & o_3 \end{pmatrix}$ where $|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \end{pmatrix}$

3a) $\hat{A}^{+} = \hat{A}$ ~~where~~ $\hat{A}^{+} = (A^{*})^{T}$

b) $\hat{A} = \hat{A}^{+}$ $\hat{A}|\psi_n\rangle = a_n|\psi_n\rangle \Rightarrow \langle\psi_n|\hat{A}|\psi_n\rangle = a_n$

Take Hermitian conjugate $\langle\psi_n|\hat{A}^{+} = \langle\psi_n|A = \langle\psi_n|a_n^{*}$

$\Rightarrow \langle\psi_n|A|\psi_n\rangle = a_n^{*}$

$\Rightarrow a_n = a_n^{*}$ a_n real

c) $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

~~Eigenvalue~~ $\hat{p}$ and $\hat{x}$ give real eigenvalues $\Rightarrow \hat{p}^2$ eigenvalue and $\hat{x}^2 > 0$ $\Rightarrow$ eigenvalue of $\hat{H} > 0$

4a) Eigenvalue associated with A and B can be simultaneously know exactly. $[\hat{A}, \hat{B}] = 0$ for compatible observers

b) $\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2}$

c) This means the values of observables associated with 2 incompatible observers cannot be known exactly e.g momentum and position can't be known exactly

5a) Fermions must have an overall ~~a~~ antisymmetric wavefunction for many ~~partic~~ and have half integer spin. Bosons have symmetric wavefn and integer spin. For many particle systems interchange of particles give symmetry

$\psi_a(\underline{r}_1)\psi_b(\underline{r}_2) = \pm\,\psi_a(\underline{r}_2)\psi_b(\underline{r}_1)$ + for bosons, − for fermions

(2)

5b) Overall state must be antisymmetric
Spin part is symmetric therefore spatial part must be antisymmetric

Spatial state ψ_a and ψ_b - must be negative on interchange of particles

$\psi_a(\underline{r}_1)\,\psi_b(\underline{r}_2) = -\psi_a(\underline{r}_2)\,\psi_b(\underline{r}_1)$

Same place $\Rightarrow \underline{r}_1 = \underline{r}_2 \quad \Rightarrow \psi_a(\underline{r}_1)\psi_b(\underline{r}_1) = -\psi_a(\underline{r}_1)\psi_b(\underline{r}_1)$

$\Rightarrow \psi_a(\underline{r}_1)\psi_b(\underline{r}_1) = 0$

6a) Use basis where eigenvectors are $\uparrow \begin{pmatrix}1\\0\end{pmatrix}$ and $\downarrow \begin{pmatrix}0\\1\end{pmatrix}$

S_y in this basis is $\frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$

b) Eigenvalues $\pm\frac{\hbar}{2}$

c) Each of the spin eigenstates can be written in terms of eigenstates in other x.y.z e.g. ~~NB~~ $|\uparrow\rangle_z = a|\uparrow\rangle_x + b|\downarrow\rangle_x$
where $|a|^2 = |b|^2 = \frac{1}{2}$ in all cases

$\Rightarrow$ 2 beams will emerge from second splitter with magnitudes $\frac{1}{\sqrt{2}}$

7a) $|\psi\rangle$ has energy eigenvalues $|\phi_n\rangle$ with $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$

$\Rightarrow |\psi\rangle = \sum_{n=0}^{\infty} a_n|\phi_n\rangle$ ~~where $\sum_{n=0}^{\infty}|a_n|^2 \neq 1$~~

$\Rightarrow \langle\psi|\hat{H}|\psi\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\langle\phi_m|a_m^* a_n\hat{H}|\phi_n\rangle = \sum_{n=0}^{\infty}|a_n|^2 E_n$

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\sum_{n=0}^{\infty}|a_n|^2 E_n}{\sum_{n=0}^{\infty}|a_n|^2} \quad \text{Since } E_n \geq E_0$$

$$\frac{\sum_{n=0}^{\infty}|a_n|^2 E_n}{\sum_{n=0}^{\infty}|a_n|^2} \geq E_0$$ with equality for $a_0 = 1$ and $a_n = 0$ for $n > 0$

b) For a Hamiltonian $\hat{H}$ come up with a trial wave function $|\psi_{trial}\rangle$

Now $\frac{\langle\psi_t|\hat{H}|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \geq E_0$ giving an upper bound on E_0. Can have parameters in $|\psi_{trial}\rangle$ and can then minimise E_{trial}

c) $$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1$$

$$\Rightarrow \int_{-c}^{c} A^2(c^2-x^2)^2 = 1$$

$$\int_{-c}^{c} A^2(c^4 - 2c^2x^2 + x^4)\,dx = A^2\left[c^4x - \frac{2c^2x^3}{3} + \frac{x^5}{5}\right]_{-c}^{c}$$

$$= A^2\left(c^5 - \frac{2c^5}{3} + \frac{c^5}{5} + c^5 - \frac{2c^5}{3} + \frac{c^5}{5}\right) = \frac{16c^5A^2}{15} = 1$$

$$\Rightarrow A = \sqrt{\frac{15}{16c^5}}$$

(3)

(ii) $E(c) = \langle \psi | \hat{H} | \psi \rangle = \langle \psi | -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m\omega^2 x^2 |\psi\rangle$

$$= \int_{-c}^{c} A^2 (c^2-x^2)\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2 x^2}{2}\right)(c^2-x^2)\,dx$$

$$= \int_{-c}^{c} A^2(c^2-x^2)\left(\frac{\hbar^2}{m} + \frac{m\omega^2x^2c^2}{2} - \frac{m\omega^2x^4}{2}\right)dx$$

$$= \int_{-c}^{c}\frac{15}{16c^5}\left(\frac{\hbar^2c^2}{m} + \frac{m\omega^2x^2c^4}{2} - \frac{m\omega^2x^4c^2}{2} - \frac{\hbar^2x^2}{m} - \frac{m\omega^2x^4c^2}{2} + \frac{m\omega^2x^6}{2}\right)dx$$

$$= \frac{15}{16c^5}\left[\frac{\hbar^2c^2x}{m} + \frac{m\omega^2x^3c^4}{6} - \frac{\hbar^2x^3}{3m} - \frac{m\omega^2c^2x^5}{5} + \frac{m\omega^2x^7}{14}\right]_{-c}^{c}$$

$$= \frac{15}{16c^5}\left(\frac{2\hbar^2c^3}{m} + \frac{m\omega^2c^7}{3} - \frac{2\hbar^2c^3}{3m} - \frac{2m\omega^2c^7}{5} + \frac{m\omega^2c^7}{7}\right)$$

$$= \frac{15}{16c^5}\left(\frac{4}{3}\frac{\hbar^2c^3}{m} + \frac{8}{105}m\omega^2c^7\right)$$

$$= \frac{5\hbar^2}{4mc^2} + \frac{m\omega^2c^2}{14}$$

(iii) $\frac{dE}{dc} = -\frac{5\hbar^2}{2mc^3} + \frac{m\omega^2c}{7} = 0$

$$\frac{m\omega^2c^4}{7} = \frac{5\hbar^2}{2m} \Rightarrow c = \left(\frac{35\hbar^2}{2m^2\omega^2}\right)^{\frac{1}{4}} = \left(\frac{35}{2}\right)^{\frac{1}{4}}\left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}$$

$$E_{min} = \frac{5\hbar^2}{4m}\left(\frac{2}{35}\right)^{\frac{1}{2}}\frac{m\omega}{\hbar} + \frac{m\omega^2}{14}\left(\frac{35}{2}\right)^{\frac{1}{2}}\frac{\hbar}{m\omega}$$

$$= \hbar\omega\left(\frac{5}{4}\left(\frac{2}{35}\right)^{\frac{1}{2}} + \frac{1}{14}\left(\frac{35}{2}\right)^{\frac{1}{2}}\right) = 0.598\,\hbar\omega$$

d) $\frac{dE}{dd} = -\frac{45\hbar^2}{14md^3} + \frac{15}{77}m\omega^2 d = 0 \Rightarrow d = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}\left(\frac{45\times77}{14\times15}\right)^{\frac{1}{4}} = \left(\frac{\hbar}{m\omega}\right)^{\frac{1}{2}}\left(\frac{33}{2}\right)$

$$E_{min} = \hbar\omega\left(\frac{45}{28}\left(\frac{2}{33}\right)^{\frac{1}{2}} + \frac{15}{154}\left(\frac{33}{2}\right)^{\frac{1}{2}}\right) = 0.791\,\hbar\omega$$

e) $A(c^2-x^2)$ gives lower energy $\Rightarrow$ closer to ground state
Actual ground state has energy $\frac{\hbar\omega}{2}$ $\Rightarrow$ $\leq 10\%$ off

Actual ground state $Ae^{-m\omega x^2/2\hbar} = A\left(1 - \frac{m\omega x^2}{2\hbar} + \frac{m^2\omega^2}{8\hbar^2}x^4 + \dots\right)$
$\Rightarrow A(c^2-x^2)$ will get first 2 terms correct which gives more accurate energy.

8a) $(a_+a_- + \frac{1}{2})\hbar\omega = \left(\frac{1}{2m\hbar\omega}(m\omega\hat{x} - i\hat{p})(m\omega\hat{x} + i\hat{p}) + \frac{1}{2}\right)\hbar\omega$

$$= \left(\frac{1}{2m\hbar\omega}\left(m^2\omega^2\hat{x}^2 + im\omega\underbrace{(\hat{x}\hat{p} - \hat{p}\hat{x})}_{i\hbar} + \hat{p}^2\right) + \frac{1}{2}\right)\hbar\omega$$

$$= \left(\frac{1}{2m\omega\hbar}\left(m^2\omega^2\hat{x}^2 - \hbar m\omega + \hat{p}^2\right) + \frac{1}{2}\right)\hbar\omega$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega\hat{x}^2 = \hat{H}$$

b) $[a_-, a_+] = [\frac{1}{2m\hbar\omega}(m\omega\hat{x} + i\hat{p}), (m\omega\hat{x} - i\hat{p})]$

$$= \frac{1}{2m\hbar\omega}\left(m^2\omega^2\underbrace{[\hat{x},\hat{x}]}_{0} - im\omega[\hat{x},\hat{p}] + im\omega[\hat{p},\hat{x}] - \underbrace{[\hat{p},\hat{p}]}_{0}\right)$$

$$= \frac{1}{2m\hbar\omega}\left(-im\omega\, i\hbar + im\omega(-i\hbar)\right) = 1$$

c) $\hat{H}\hat{a}_+|n\rangle = ([\hat{H}, a_+] + a_+\hat{H})|n\rangle$
~~$= (a_+\hbar\omega + a_+(a_+a_- + \frac{1}{2})\hbar\omega)|n\rangle$~~

$$= (a_+\hbar\omega + a_+\hat{H})|n\rangle$$
$$= (a_+\hbar\omega + a_+E_n)|n\rangle = (E_n + \hbar\omega)\hat{a}_+|n\rangle$$

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d) $|\psi\rangle = |n_x\rangle|m_y\rangle \qquad E_{n,m} = (n_x + m_y + 1)\hbar\omega$

$E_{0,0} = (0+0+1)\hbar\omega = \hbar\omega$

$E_{0,1} = E_{1,0} = (1+0+1)\hbar\omega = 2\hbar\omega \qquad |1\rangle|0\rangle$ and $|0\rangle|1\rangle$

$E_{0,2} = E_{2,0} = E_{1,1} = 3\hbar\omega \qquad |1\rangle|1\rangle$ and $|0\rangle|2\rangle$ and $|2\rangle|0\rangle$

e) $\hat{x}^2\hat{y}^2$

$a_{+,x} + a_{-,x} = \frac{2m\omega}{\sqrt{2m\hbar\omega}}\hat{x} \Rightarrow \hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(a_{+,x} + a_{-,x})$

$\Rightarrow \hat{x}^2\hat{y}^2 = \frac{\hbar^2}{4m^2\omega^2}(a_{x+} + a_{x-})^2(a_{y+} + a_{y-})^2$

$= \frac{\hbar^2}{4m^2\omega^2}(a_{x+}^2 + a_{x-}^2 + a_{x+}a_{x-} + a_{x-}a_{x+})(a_{y+}^2 + a_{y-}^2 + a_{y+}a_{y-} + a_{y-}a_{y+})$

$[a_-, a_+] = 1 \Rightarrow a_-a_+ = a_+a_- + 1$

$\hat{x}^2\hat{y}^2 = \frac{1}{4\alpha^4}(a_{x+}^2 + a_{x-}^2 + 2a_{x+}a_{x-} + 1)(a_{y+}^2 + a_{y-}^2 + 2a_{y+}a_{y-} + 1)$

f) $\Delta E = \langle\psi|\Delta H|\psi\rangle$

$= \langle n_x|\langle m_y|\frac{\gamma\hbar^2}{4m^2\omega^2}(a_{x+}^2 + a_{x-}^2 + 2a_{x+}a_{x-} + 1)(a_{y+}^2 + a_{y-}^2 + 2a_{y+}a_{y-} + 1)|n_x\rangle|m_y\rangle$

$\langle n_x|a_{x+}^2|n_x\rangle = 0 \quad$ as $\langle n_x|n_{x+2}\rangle = 0$

Non-zero contributions from a_+a_- and 1

$= \frac{\gamma\hbar^2}{4m^2\omega^2}\langle n_x|\langle m_y|(2a_{x+}a_{x-} + 1)(2a_{y+}a_{y-} + 1)|n_x\rangle|m_y\rangle$

$= \frac{\gamma\hbar^2}{4m^2\omega^2}(2n_x + 1)(2m_y + 1)$

$\Delta E_{0,0} = \frac{\gamma\hbar^2}{4m^2\omega^2} \qquad \Delta E_{1,0} = \Delta E_{0,1} = \frac{3\gamma\hbar^2}{4m\omega^2}$

g) $|\psi_n'\rangle = |\psi_n\rangle + \gamma \sum \frac{\langle \psi_m | \hat{x}^2 \hat{y}^2 | \psi_n \rangle}{E_n - E_m}$

As energies are same for 2nd and 3rd eigenstate ~~the~~ the denominator goes to 0. We ~~can~~ therefore have to diagonalise our basis states before calculating the effect of the perturbation - this is degenerate perturbation theory

9a) $[\hat{J}_x, \hat{J}_y] = i\hbar \hat{J}_z \qquad [\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} J_k$

$[\hat{J}_x, \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2] = [\hat{J}_x, \hat{J}_y^2] + [\hat{J}_x, \hat{J}_z^2]$ as $[\hat{J}_x, \hat{J}_x^2] = 0$

$= J_x J_y J_y - J_y J_y J_x + J_x J_z J_z - J_z J_z J_x$

$= J_x J_y J_y \underbrace{- J_y J_x J_y + J_y J_x J_y}_{+0} - J_y J_y J_x + J_x J_z J_z \underbrace{- J_z J_x J_z + J_z J_x J_z}_{+0} - J_z J_z J_x$

$= [J_x, J_y] J_y + J_y [J_x, J_y] + [J_x, J_z] J_z + J_z [J_x, J_z]$

$= i\hbar (J_z J_y + J_y J_z - J_y J_z - J_z J_y) = 0$

b) ~~Also eigenvector of J_z if J_+ and J_z are compatible observables~~

~~$[J_+, J_z$~~

b) $J_z J_+ |j m_j\rangle$

$[J_z, J_+] = J_z J_+ - J_+ J_z = [J_z, J_x] + i[J_z, J_y]$

$= i\hbar J_y + \hbar J_x = \hbar J_+$

$\Rightarrow J_z J_+ |j m_j\rangle = (J_+ J_z + \hbar J_+) |j m_j\rangle$

$= (J_+ m_j \hbar + \hbar J_+) |j m_j\rangle = \underbrace{(m_j + 1)\hbar}_{\text{eigenvalue}} |j m_j\rangle$

(5)

Therefore J_+ promotes $|j m_j\rangle \to |j, m_j+1\rangle$

There is no state for $m_j = j+1$ therefore $J_+|j,j\rangle = 0$

c) $J_+ J_- |j m_j\rangle$

$= J_+ \left[j(j+1) - m_j(m_j-1)\right]^{\frac{1}{2}} \hbar \, |j, m_j - 1\rangle$

$= \hbar^2 \left[j(j+1) - m_j(m_j-1)\right]^{\frac{1}{2}} \left[j(j+1) - (m_j-1)\cancel{+}m_j\right]^{\frac{1}{2}} |j, m_j\rangle$

$= \hbar^2 \left[j(j+1) - m_j(m_j-1)\right] |j, m_j\rangle$

Doesn't apply for $m_j = -j$ as action of first operator (J^-) will give 0

d) $J_+ J_- = (J_x + i J_y)(J_x - i J_y)$
$= \hat{J}_x^2 + \hat{J}_y^2 + i\hat{J}_y \hat{J}_x - i \hat{J}_x \hat{J}_y$

$= \hat{J}^2 - \hat{J}_x^2 + i[J_y, J_x] = \hat{J}^2 - J_x^2 + \hbar \hat{J}_z$

$\Rightarrow (\hat{J}^2 - \hat{J}_x^2 + \hbar \hat{J}_z)|j, m_j\rangle = \hbar^2\left[j(j+1) - m_j(m_j-1)\right]|j, m_j\rangle$

$\hat{J}^2 |j, m_j\rangle = \hbar^2 j(j+1)|j, m_j\rangle$ and $\hbar J_z |m_j, J_z\rangle = \hbar^2 m_j |m_j J_z\rangle$

$\Rightarrow \hat{J}_x^2 |j, m_j\rangle = m_j^2 |j, m_j\rangle$

e) $\hat{\underline{J}} = \hat{\underline{L}} + \hat{\underline{S}} \Rightarrow \hat{\underline{J}} \cdot \hat{\underline{J}} = (\hat{\underline{L}} + \hat{\underline{S}}) \cdot (\hat{\underline{L}} + \hat{\underline{S}})$

$J^2 = L^2 + S^2 + 2\underline{L} \cdot \underline{S}$

$\Rightarrow \hat{\underline{L}} \cdot \hat{\underline{S}} = \frac{1}{2}(J^2 - L^2 - S^2)$

f) $\hat{H} = \frac{\varepsilon_1}{\hbar^2}\left(\frac{1}{2}(J^2 - S^2 - L^2) + S^2\right) + \frac{\varepsilon_2}{\hbar^2}\left(L_z + S_z\right)^2$

$J = L \oplus S = 1 \oplus \frac{1}{2} = \frac{1}{2} \oplus \frac{3}{2}$

$\Rightarrow \hat{H} = \frac{\varepsilon_1}{\hbar^2}\left(\frac{J^2}{2} + \frac{S^2}{2} - \frac{L^2}{2}\right) + \frac{\varepsilon_2}{\hbar^2} J_z^2$

$= \varepsilon_1\left(\frac{j(j+1)}{2} + \frac{s(s+1)}{2} - \frac{l(l+1)}{2}\right) + \varepsilon_2 m_j^2$

$j = \frac{1}{2}$ $\quad m_j = \pm\frac{1}{2}$ $\quad \hat{H} = \varepsilon_1\left(\frac{3}{8} + \frac{3}{8} - 1\right) + \frac{\varepsilon_2}{4} = \frac{1}{4}\left(-\varepsilon_1 + \varepsilon_2\right)$

$s = \frac{1}{2}$ $\quad l = 1$

$j = \frac{3}{2}$ $\quad m_j = \pm\frac{3}{2}$ $\quad \hat{H} = \varepsilon_1\left(\frac{5}{8} + \frac{3}{8} - 1\right) + \varepsilon_2\frac{9}{4} = \frac{9}{4}\varepsilon_2$

$m_j = \pm\frac{1}{2}$ $\quad \hat{H} = \varepsilon_1\left(\frac{5}{8} + \frac{3}{8} - 1\right) + \frac{\varepsilon_2}{4} = \frac{\varepsilon_2}{4}$

g) Could use $|s, m_s, l, m_l\rangle$

$\Rightarrow \hat{H}|\psi\rangle = \varepsilon_1\left(\frac{j(j+1)}{2} + \frac{s(s+1)}{2} - \frac{l(l+1)}{2}\right) + \varepsilon_2\left(m_l + m_s\right)^2$

10a) Let small change $\hat{H}' = \lambda H_1$ where λ tracks order
Can expand E_n as $E_n = \sum_{i=0}^{\infty} \lambda^i E_n'^{(i)}$

and $|\psi_n\rangle = \sum_{i=0}^{\infty} \lambda^i |\psi_n^{(0)}\rangle$

where $^{(i)}$ superscript gives order of change - power of λ gives the order e.g. 1st order energy change $\cancel{\Delta} E_n^{(1)} = \lambda E_n'^{(1)}$

b) $\Rightarrow \hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

$$(\hat{H}_0 + \lambda \hat{H}_1)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \dots) = (E_n'^{(0)} + \lambda E_n'^{(1)} + \lambda^2 E_n'^{(2)})(|\psi_n^{(0)}\rangle + \dots$$

1st order $\Rightarrow$ equate λ terms

$$\hat{H}_1|\psi_n^{(0)}\rangle + \hat{H}_0|\psi_n^{(1)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n'^{(1)}|\psi_n^{(0)}\rangle$$

Take $\langle\psi_n^{(0)}|$ $\qquad E_n^{(0)}\langle\psi_n^{(0)}|$

$$\langle\psi_n^{(0)}|\hat{H}_1|\psi_n^{(0)}\rangle + \langle\psi_n^{(0)}|\hat{H}_0|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + E_n'^{(1)}$$

$$\Rightarrow E_n'^{(1)} = \langle\psi_n^{(0)}|\hat{H}_1|\psi_n^{(0)}\rangle$$

$$E_n^{(1)} = \lambda E_n'^{(1)} = \langle\psi_n^{(0)}|\lambda\hat{H}_1|\psi_n^{(0)}\rangle = \langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle$$

c) Overall wavefunction must be antisymmetric. In singlet state which is antisymmetric $\Rightarrow$ spatial wavefunction must be symmetric on interchange of 2 particles

$\Rightarrow \psi_1(x) = \frac{2}{a}\sin(k_1 x_1)\sin(k_1 x_2)$ symmetric

$$E = \langle\psi_1|H_1+H_2|\psi_1\rangle = E_1 + E_2 = \frac{\hbar^2\pi^2}{2ma^2}(n_1^2 + n_2^2)$$

This is ~~fata~~ symmetric form for when $n_1 \neq n_2$

$$\psi_2(x) = \frac{2}{a}\left(\frac{1}{\sqrt{2}}\sin(k_1x_1)\sin(k_2x_2) + \frac{1}{\sqrt{2}}\sin(k_1x_2)\sin(k_2x_1)\right)$$

Still gives $E = \frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2)$

as $E = \langle \psi_2 | H_1 + H_2 | \psi_2 \rangle$

$$= \left\langle \tfrac{1}{\sqrt{2}}\phi_1^{n_1}\phi_2^{n_2} + \tfrac{1}{\sqrt{2}}\phi_1^{n_2}\phi_2^{n_1} \middle| H_1+H_2 \middle| \tfrac{1}{\sqrt{2}}\phi_1^{n_1}\phi_2^{n_2} + \tfrac{1}{\sqrt{2}}\phi_1^{n_2}\phi_2^{n_1} \right\rangle$$

$$= \frac{1}{2}\left(\langle \phi_1^{n_1}\phi_2^{n_2} | H_1+H_2 | \phi_1^{n_1}\phi_2^{n_2}\rangle + \langle \phi_1^{n_2}\phi_2^{n_1} | H_1+H_2 | \phi_1^{n_2}\phi_2^{n_1}\rangle\right)$$

$$= \frac{1}{2}(E_1+E_2+E_1+E_2) = \frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2)$$

$$\psi_3 = \frac{2}{a}\sin(k_2x_1)\sin(k_2x_2)$$

d) $\psi_1 = \frac{2}{a}\sin(k_1x_1)\sin(k_1x_2)\,\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)$

$$\psi_2 = \frac{2}{a}\left(\frac{1}{\sqrt{2}}\sin(k_1x_1)\sin(k_2x_2) + \frac{1}{\sqrt{2}}\sin(k_2x_1)\sin(k_1x_2)\right)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)$$

$$\psi_3 = \frac{2}{a}\sin(k_2x_1)\sin(k_2x_2)\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)$$

e) Now spin symmetric so must be spatially antisymmetric
$\Rightarrow n_1 \neq n_2$

First 3 ~~states~~ energy levels

$n_1=1$	$n_2=2$	$E = 5\frac{\hbar^2\pi^2}{2ma^2}$
$n_1=1$	$n_2=3$	$E = 9$ "
$n_1=2$	$n_2=3$	$E = 13$ "

$$\psi_1 = \frac{2}{a}\left(\frac{1}{\sqrt{2}}\sin(k_1x_1)\sin(k_2x_2) - \frac{1}{\sqrt{2}}\sin(k_2x_1)\sin(k_1x_2)\right)$$

spin triplet
$$\begin{cases} |\uparrow\rangle_1|\uparrow\rangle_2 \\ \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2\right) \\ |\downarrow\rangle_1|\downarrow\rangle_2 \end{cases}$$

$$\psi_2 = \frac{2}{a}\left(\frac{1}{\sqrt{2}}\sin(k_1x_1)\sin(k_3x_2) - \frac{1}{\sqrt{2}}\sin(k_3x_1)\sin(k_1x_2)\right)$$ spin triplet

⑦

$$\psi_3 = \frac{2}{a}\left(\frac{1}{\sqrt{2}}\sin(k_2 x_1)\sin(k_3 x_2) - \frac{1}{\sqrt{2}}\sin(k_3 x_1)\sin(k_2 x_2)\right)$$ spin triplet

f) $$E_n^{(1)} = \left\langle \frac{2}{a}\sin k_1 x_1 \sin k_1 x_2 \middle| V_0 a^2 \delta(x_1 - \tfrac{a}{2})\delta(x_2 - \tfrac{a}{3}) \middle| \frac{2}{a}\sin k_1 x_1 \sin k_2 x_2 \right\rangle$$

$$= \frac{4V_0}{\cancel{a^2}} \int_{x_1=0}^{a}\int_{x_2=0}^{a} \sin^2(k_1 x_1)\sin^2(k_1 x_2)\,\delta(x_1 - \tfrac{a}{2})\delta(x_2 - \tfrac{a}{3})\,dx_1\,dx_2$$

$$= 4V_0 \sin^2\left(\frac{k_1 a}{2}\right)\sin^2\left(\frac{k_1 a}{3}\right) \qquad k_1 = \frac{\pi}{a}$$

$$= 4V_0 \sin^2\left(\frac{\pi}{2}\right)\sin^2\left(\frac{\pi}{3}\right) = 3V_0$$

$$\Rightarrow E_n = \frac{\hbar^2\pi^2}{2ma^2}(1^2+1^2) + 3V_0 = \frac{\hbar^2\pi^2}{ma^2} + 3V_0$$