CHAPTER 7

The Schwarzschild metric and black holes

In this Chapter we shall give a first example of a non-trivial solution of Einstein's vacuum equations, $r_{ab} = 0$. This is the Schwarzschild metric. It is spherically symmetric and is the GR version of the newtonian potential $-m/r$ due to a mass m at the origin.

We shall study the timelike and null geodesics in the Schwarzschild metric. To a first approximation, there are timelike geodesics which correspond to closed elliptical orbits, as in Newton's theory. When GR corrections are taken into account, it will be seen that the orbits do not close up: this is the famous precession of the perihelion[1] which is measurable in the case of the orbit of Mercury. This was one of the first observational verifications of GR.

Null geodesics represent the paths taken by light-rays, and we shall see that there is a bending effect. This effect was observed by Eddington during the solar eclipse of 1919 (and is responsible for gravitational lensing). This effect provides another observational verification for GR.

The Minkowski metric is defined initially in a set $r > 2m$ in units with $G = c = 1$. However, it turns out that this is a defect of the coordinates rather than of the metric itself. After a change of coordinates the metric continues through this surface. This surface is now intepreted as the event horizon of a black hole.

7.1. Spherically symmetric, static metrics

A function on $\mathbb{R}^3$ is spherically symmetric if it is a function only of r, the distance from the origin. It is natural to study such things using spherical polars. We have seen that the flat metric in $\mathbb{R}^3$ in spherical polars is

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2), \tag{7.1.1}$$

where θ is the colatitude (i.e. latitude, but measured from the north pole rather than the equator) and φ is longitude.

The Minkowski metric in these coordinates is

$$\mathrm{d}s_\eta^2 = \mathrm{d}t^2 - \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2). \tag{7.1.2}$$

Let us write

$$\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2 \tag{7.1.3}$$

which is the round metric on the unit sphere $x^2+y^2+z^2 = 1$ in $\mathbb{R}^3$. This will save writing later.

A spherically symmetric, static[2] metric is obtained from this by introducing functions of r as coefficients

$$\mathrm{d}s^2 = A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - C(r)r^2\,\mathrm{d}\omega^2. \tag{7.1.4}$$

where we require $A > 0$, $B > 0$ and $C > 0$ in the region of interest. The dependence of these functions only on r encodes the spherical symmetry of the metric and also its time-invariance. One can, of course, consider more general metric forms, but that is beyond the scope of this course.

[1] closest point to the sun

[2] i.e. with coefficients independent of t

7.2. Schwarzschild

The Schwarzschild metric[3] is an idealization of Einstein's vacuum equations which describes the field due to a point mass or perhaps outside a star. It is the general-relativistic analogue of the newtonian potential $-m/r$ which is the potential describing the gravitational field due to a point-mass m in Newton's theory of gravitation.

We shall not derive this in full. We start with the form (7.1.4). This should approach the Minkowski metric when r is large, so we assume

$$A(r) \sim 1,\ B(r) \sim 1, C(r) \sim 1 \text{ as } r \to \infty. \tag{7.2.1}$$

PROPOSITION 7.2.1. *By a change of r variables, $\rho = f(r)$, (7.1.4) can be made to take the form*

$$\mathrm{d}s^2 = \alpha(\rho)\mathrm{d}t^2 - \beta(\rho)\mathrm{d}\rho^2 - \rho^2\mathrm{d}\sigma^2. \tag{7.2.2}$$

PROOF. If we define $\rho = \sqrt{C(r)}r$, then we shall get the coefficient of $\mathrm{d}\sigma^2$ correct. Since $C > 0$ this is certainly invertible for large enough r, so we define

$$\alpha(\rho) = A(r(\rho)).$$

For the $\mathrm{d}r$ term,

$$B(r)\mathrm{d}r^2 = B(r)\frac{\mathrm{d}r}{\mathrm{d}\rho}^2 \mathrm{d}\rho^2$$

so

$$\beta(\rho) = B(r(\rho))\frac{\mathrm{d}r}{\mathrm{d}\rho}^2.$$

This completes the proof. □

We use this proposition, then rechristen ρ as r. So we may as well look at metrics in the slightly simpler form

$$A(r)\mathrm{d}t^2 - B(r)\mathrm{d}r^2 - r^2\mathrm{d}\omega^2 \tag{7.2.3}$$

We saw in the previous chapter that in the weak field limit, the component g_{00} should be matched with twice the newtonian potential computed by an observer with 4-vector $(1, \mathbf{0})$, up to an additive constant. So the simplest possible guess for $A(r)$ is $1 - 2m/r$: the value 1 comes from the required asymptotic form of the metric.

It turns out that there is a choice of B which then gives a metric which satisfies Einstein's equations where the metric is defined:

THEOREM 7.2.2. *The* Schwarzschild metric

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\sigma^2,\ (r > 2m) \tag{7.2.4}$$

satisfies $r_{ab} = 0$.

We shall not prove this in full. You can see most of the details in Woodhouse.
We shall, however, record the geodesic equations and the Γs for this metric

PROPOSITION 7.2.3. *The geodesic equations for the metric (7.1.4) are*

$$\begin{aligned}
&\ddot{t} + \frac{A'}{A}\dot{t}\dot{r} = 0 \text{ or } \frac{\mathrm{d}}{\mathrm{d}\tau}\left(A\dot{t}\right) = 0\\
&\ddot{r} + \frac{A'}{2B}\dot{t}^2 + \frac{B'}{2B}\dot{r}^2 - \frac{r}{B}\dot{\theta}^2 - \frac{r}{B}\sin^2\theta\dot{\varphi}^2 = 0\\
&\ddot{\theta} + (2/r)\dot{\theta}\dot{r} - \sin\theta\cos\theta\dot{\varphi}^2 = 0\\
&\ddot{\varphi} + \frac{2}{r}\dot{r}\dot{\varphi} + 2\cot\theta\dot{\theta}\dot{\varphi} = 0 \text{ or } \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\sin^2\theta\dot{\varphi}\right) = 0.
\end{aligned}$$

[3]Named in honour of Karl Schwarzschild, 1873–1916

Proof. We know the drill by now...

The Lagrangian for the geodesics is

$$L = \frac{1}{2}\left(A\dot{t}^2 - B\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)\right). \tag{7.2.5}$$

Hence

$$\begin{aligned}
\frac{\partial L}{\partial \dot{t}} &= A\dot{t} & \frac{\partial L}{\partial t} &= 0\\
\frac{\partial L}{\partial \dot{r}} &= -B\dot{r} & \frac{\partial L}{\partial r} &= \frac{1}{2}\left(A'\dot{t}^2 - B'\dot{r}^2 - 2r(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)\right).\\
\frac{\partial L}{\partial \dot{\theta}} &= -r^2\dot{\theta} & \frac{\partial L}{\partial \theta} &= -r^2\sin\theta\cos\theta\dot{\varphi}^2\\
\frac{\partial L}{\partial \dot{\varphi}} &= -r^2\sin^2\theta\dot{\varphi} & \frac{\partial L}{\partial \varphi} &= 0
\end{aligned}$$

Here dot is differentiation with respect to τ, prime is differentiation with respect to r. Next,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}u}(A\dot{t}) &= A(\ddot{t} + (A'/A)\dot{r}\dot{t})\\
\frac{\mathrm{d}}{\mathrm{d}u}(-B\dot{r}) &= -B(\ddot{r} + (B'/B)\dot{r}^2)\\
\frac{\mathrm{d}}{\mathrm{d}u}(-r^2\dot{\theta}) &= -r^2(\ddot{\theta} + (2/r)\dot{r}\dot{\theta})\\
\frac{\mathrm{d}}{\mathrm{d}u}(-r^2\sin^2\theta\dot{\varphi}) &= -r^2\sin^2\theta(\ddot{\varphi} + (2/r)\dot{r}\dot{\varphi} + 2\cot\theta\,\dot{\theta}\dot{\varphi})
\end{aligned}$$

and combining these with the previous calculations we get the equations of the Proposition. □

Proposition 7.2.4. *The non-zero Christoffel symbols for the metric (7.1.4) are as follows:*

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{A'}{2A};$$

$$\Gamma^1_{00} = \frac{A'}{2B} \qquad \Gamma^1_{11} = \frac{B'}{2B} \qquad \Gamma^1_{22} = -\frac{r}{B} \qquad \Gamma^1_{33} = -\frac{r\sin^2\theta}{B}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta.$$

Proof. These are read off from the geodesic equations in the usual way. □

7.2.1. Curvature computations. Recall:

$$R_{abc}{}^d = \partial_a\Gamma^d_{bc} - \partial_b\Gamma^d_{ac} + \Gamma^d_{ap}\Gamma^p_{bc} - \Gamma^d_{bp}\Gamma^p_{ac}$$

and

$$r_{ac} = R_{abc}{}^b = R_{a0c}{}^0 + R_{a1c}{}^1 + R_{a2c}{}^2 + R_{a3c}{}^3. \tag{7.2.6}$$

To give a flavour of these calculations, let us compute $R_{a0c}{}^0$. We shall show:

$$R_{101}{}^0 = \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{A'B'}{4AB} \tag{7.2.7}$$

and

$$R_{a0c}{}^0 = 0 \text{ for all other } ac. \tag{7.2.8}$$

We shall also compute $R_{121}{}^2$ and $R_{131}{}^3$ and obtain

$$r_{11} = \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{rB} \tag{7.2.9}$$

by putting $a = c = 1$ in (7.2.6).

We have

$$R_{a0c}{}^0 = \partial_a\Gamma^0_{0c} - \partial_0\Gamma^0_{ac} + \Gamma^0_{as}\Gamma^s_{0c} - \Gamma^0_{0s}\Gamma^s_{ac}.$$

We may as well assume $a \leqslant c$ because this is symmetric in ac.

Now $\partial_0 = \partial_t$ and none of the Γ depends upon t. Also,

$$\Gamma^0_{01} = \Gamma^0_{10} = A'/2A, \Gamma^0_{ab} = 0 \text{ all other } ab.$$

So

$$R_{a0c}{}^0 = \partial_a \Gamma^0_{0c} + \Gamma^0_{as}\Gamma^s_{0c} - \frac{A'}{2A}\Gamma^0_{ac}.$$

We know $R_{00c}{}^d = 0$, so start with $a = 1$

$$R_{10c}{}^0 = \partial_1 \Gamma^0_{0c} + \Gamma^0_{1s}\Gamma^s_{0c} - \frac{A'}{2A}\Gamma^1_{1c}.$$

If $c = 2, 3$ every term is zero, because the only non-vanishing Γ^1_{ab} is

$$\Gamma^1_{11} = \frac{B'}{2B}.$$

So

$$\begin{aligned} R_{101}{}^0 &= \partial_1 \Gamma^0_{01} + \Gamma^0_{1s}\Gamma^s_{01} - \frac{A'}{2A}\Gamma^1_{11} \\ &= \left(\frac{A'}{2A}\right)' + (\Gamma^0_{10})^2 - \frac{A'B'}{4AB} \\ &= \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{A'B'}{4AB} \end{aligned}$$

and

$$R_{a0c}{}^0 = 0 \text{ for all other } ac.$$

We have

$$R_{121}{}^2 = \partial_1\Gamma^2_{21} - \partial_2\Gamma^2_{11} + \Gamma^2_{1s}\Gamma^s_{21} - \Gamma^2_{2s}\Gamma^s_{11}$$

Looking at the non-vanishing Γs, this comes down to

$$\begin{aligned} R_{121}{}^2 &= -\frac{1}{r^2} + \Gamma^2_{12}\Gamma^2_{21} - \Gamma^2_{21}\Gamma^1_{11} \\ &= -\frac{1}{r^2} + \frac{1}{r^2} - \frac{B'}{2Br} \\ &= -\frac{B'}{2Br}. \end{aligned}$$

Similarly

$$\begin{aligned} R_{131}{}^3 &= \partial_1\Gamma^3_{31} - \partial_3\Gamma^3_{11} + \Gamma^3_{1s}\Gamma^s_{13} - \Gamma^3_{3s}\Gamma^s_{11} \\ &= -\frac{1}{r^2} + \Gamma^3_{13}\Gamma^3_{13} - \Gamma^3_{31}\Gamma^1_{11} \\ &= -\frac{B'}{2rB} \end{aligned}$$

because $\Gamma^3_{13} = 1/r$ and $\Gamma^1_{11} = B'/2B$.

Hence

$$\begin{aligned} r_{11} &= R_{101}{}^0 + R_{111}{}^1 + R_{121}{}^2 + R_{131}{}^3 \\ &= R_{101}{}^0 + R_{121}{}^2 + R_{131}{}^3 \\ &= \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{rB}. \end{aligned}$$

Similar computations yield

PROPOSITION 7.2.5. *The non-vanishing components of the Ricci tensor of the spherically symmetric metric (7.1.4) are*

$$\begin{aligned}
r_{00} &= -\frac{A''}{2B} + \frac{A'B'}{4B^2} + \frac{A'^2}{4AB} - \frac{A'}{rB}, \\
r_{11} &= \frac{A''}{2A} - \frac{(A')^2}{4A^2} - \frac{A'B'}{4AB} - \frac{B'}{rB}, \\
r_{22} &= \frac{rA'}{2AB} - \frac{rB'}{2B^2} + \frac{1}{B} - 1 \\
r_{33} &= \sin^2\theta r_{22}.
\end{aligned}$$

Now we can verify that the Schwarzschild metric of Theorem 7.2.2 does indeed satisfy the Einstein vacuum equations $r_{ab} = 0$.

Eliminating the A'' terms between r_{00} and r_{11} gives $AB' + BA' = 0$ so AB is constant. And this should be 1 by the boundary condition.

Inserting $B = 1/A$, $B' = -A'/A^2$,

$$\begin{aligned}
r_{00} &= -\frac{1}{2}AA'' - \frac{1}{r}AA' \\
r_{11} &= \frac{1}{A^2}r_{00} \\
r_{22} &= rA' + A - 1 \\
r_{33} &= \sin^2\theta(rA' + A - 1).
\end{aligned}$$

Solving $r_{22} = 0$ gives $A = 1 - 2m/r$, where m is a constant and one checks that this also solves the $r_{00} = 0$ equation. Hence we arrive at the *Schwarzschild metric*

$$\mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2).$$

where $m > 0$ and, for the moment anyway, $r > 2m$.

Woodhouse, GR, Sect. 7.1–7.2

7.3. Physical consequences

7.3.1. Gravitational time dilatation: 'heavy clocks run slowly'. Suppose that Alice and Bob have positions $r = r_A$ and $r = r_B$ in the Schwarzschild space-time (angular positions also fixed) with both r_A and $r_B > 2m$. How do they compare the rates at which their ideal clocks run?

Imagine two 'ticks' of Alice's ideal clock, separated by a small proper time interval $\delta\tau_A$. To compare, we assume that Alice's clock emits photons at each of the two ticks. Bob receives these photons and in particular can record the elapsed time between receiving the first and second photon. This gives him a time interval $\delta\tau_B$, and the ratio $\delta\tau_A/\delta\tau_B$ is the amount by which Alice's clock appears to run slowly as compared with Bob's.

Let's do it. Suppose first that δt_A is the difference in t-coordinate between the two ticks of Alice's clock and suppose that δt_B is the difference in t-coordinates of when the two photons are received by Bob. (NB the Schwarzschild time-coordinate is *NOT* proper time for either Alice or Bob!) Then it is pretty clear that $\delta t_A = \delta t_B$ because the metric coefficients are all independent of t. We shall make this computation explicitly below, just to be sure. Thus we need to see how δt_A is related to $\delta\tau_A$ and similarly for t_B and τ_B.

Now Alice's world line has the simple form $\tau_A \mapsto (U\tau_A, r_A, \theta_A, \varphi_A)$, where r_A, θ_A and φ_A are constants, and τ_A is a proper time parameter if the associated velocity 4-vector

$$U\frac{\partial}{\partial t} \tag{7.3.1}$$

is of unit length, i.e. $g(U\partial_t, U\partial_t) = 1$. This entails

$$(1 - 2m/r_A)U^2 = 1, \text{ so } U = \frac{\mathrm{d}t}{\mathrm{d}\tau_A} = (1 - 2m/r_A)^{-1/2}. \tag{7.3.2}$$

The corresponding equation holds with A replaced by B, so

$$\delta\tau_B = \frac{d\tau_B}{dt}\delta t_B = \frac{d\tau_B}{dt}\left(\frac{d\tau_A}{dt}\right)^{-1}\delta\tau_A \tag{7.3.3}$$

Hence

$$\frac{\delta\tau_B}{\delta\tau_A} = \sqrt{\frac{1-2m/r_B}{1-2m/r_A}} \tag{7.3.4}$$

Thus if Alice is nearer to $r = 2m$ then this factor is greater than one, and so Bob will record a *longer* elapsed time between two ticks of Alice's clocks, this becoming (in principle) infinite as r_A approaches $2m$[4] .

REMARK 7.3.1. Note that observers with constant (r, θ, φ) coordinates in Schwarzschild are *not* freely falling.

It is interesting to compute the trajectory of a photon sent by Alice at $(t_A, r_A, \theta_0, \varphi_0)$ to Bob at $(t_B, r_B, \theta_0, \varphi_0)$. This is a *radial*[5] *null geodesic* for the Schwarzschild metric.

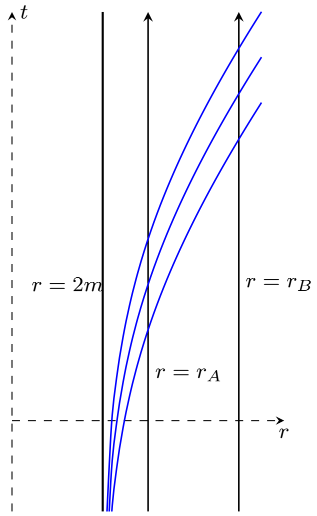


FIGURE 1. Diagam showing trajectories of photons between Alice and Bob in the Schwarzschild metric. The curved blue trajectories are radial null geodesics, the three solid vertical lines are the event horizon $r = 2m$ and the worldlines $r = r_A$ and $r = r_B$ of Alice and Bob.

Such a null geodesic has a parameterization

$$\tau \longmapsto (t(\tau), r(\tau), \theta_0, \varphi_0)$$

where θ_0 and φ_0 are constants. Being null means

$$\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 = 0 \tag{7.3.5}$$

so

$$\frac{dr}{dt} = (1-2m/r) \tag{7.3.6}$$

(plus sign if photon is travelling outwards as t increases).

Hence

$$dt = \frac{r\,dr}{r-2m} \tag{7.3.7}$$

[4]We shall later identify the surface $r = 2m$ with the event horizon of a black hole

[5]i.e. θ and φ are constant along the geodesic

and

$$t_B - t_A = r_B - r_A + 2m \log \frac{r_B - 2m}{r_A - 2m} \tag{7.3.8}$$

for a photon travelling from

$$(t_A, r_A, \theta_0, \varphi_0) \text{ to } (t_B, r_B, \theta_0, \varphi_0). \tag{7.3.9}$$

Hence $\delta t_A = \delta t_B$ as previously argued.

7.3.2. Geodesics in Schwarzschild. The Lagrangian L for geodesics in Schwarzschild is

$$L = \frac{1}{2}\left((1-2m/r)\dot{t}^2 - (1-2m/r)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)\right).$$

We take the parameter to be proper time τ and $\dot{}$ to mean differentiation wrt to τ.

We have conserved quantities

$$E = (1-2m/r)\dot{t}, J = r^2 \sin\theta\dot{\varphi} \tag{7.3.10}$$

and

$$L = 1/2 \text{ for timelike, } \; L = 0 \text{ for null geodesics.} \tag{7.3.11}$$

REMARK 7.3.2. The conserved quantity E is the total energy of our particle, assumed to have unit rest-mass. (Not the gravitating one, the one that's orbiting.) Remember that total energy is a relative concept in relativity: if a particle has 4-momentum P and an observer has 4-velocity U, then $g(U, P)$ is the total energy of the particle (in units with $c = 1$, and as measured by our observer). If the freely falling particle has unit rest-mass, and the observer is taken to be fixed with respect to the Schwarzschild coordinates, i.e. his/her 4-velocity is

$$U = (1-2m/r)^{-1/2}(1, \mathbf{0})$$

then

$$g_{ab}U^aV^b = (1-2m/r)^{1/2}\dot{t} = (1-2m/r)^{-1/2}E.$$

On the other hand $g_{ab}U^aV^b$ is also to be interpreted as $\gamma(v)$, just as in SR, where v is the relative speed of the two observers. Thus

$$E = (1-2m/r)^{1/2}\gamma(v) = (1-2m/r)^{1/2}(1-v^2)^{-1/2} \simeq 1 + v^2/2 - m/r. \tag{7.3.12}$$

if m/r is small and so is v. Recall that $G = 1$, $c = 1$ and the mass of the particle is 1; if we restored units, and the particle has rest-mass μ, say, then this becomes

$$E \simeq \mu c^2 + \frac{1}{2}\mu v^2 - \frac{Gm\mu}{r} \tag{7.3.13}$$

The terms here are the rest-energy of the particle, its kinetic energy and its gravitational potential energy. So this approximation is in perfect agreement with newtonian gravity and special relativity: the first term in (7.3.13) is the rest-energy of the mass μ as predicted by special relativity, the second term is its kinetic energy and the third is its gravitational potential energy (according to newtonian gravity).

PROPOSITION 7.3.3. *Equatorial*[6] *timelike geodesics in Schwarzschild are given by the equations*

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = E^2 - (1-2m/r) \; \textit{(radial geodesics)} \tag{7.3.14}$$

and by

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - 3mu^2 = \frac{m}{J^2} \; \textit{(non-radial geodesics)}, \tag{7.3.15}$$

where $u = 1/r$ and the angular momentum $J = r^2\dot{\varphi} \neq 0$ is a constant. The equation (7.3.15) has the first integral

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 - 2mu^3 = \frac{E^2-1}{J^2} + \frac{2m}{J^2}u, \tag{7.3.16}$$

[6] i.e. with $\theta = \pi/2$

Similarly,

PROPOSITION 7.3.4. *Radial null geodesics in Schwarzschild are given by (7.3.5–7.3.9). Non-radial, equatorial null geodesics in Schwarzschild satisfy*

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - 3mu^2 = 0 \tag{7.3.17}$$

which has the first integral

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 - 2mu^3 = \frac{E^2}{J^2}. \tag{7.3.18}$$

PROOF. We have seen that for Schwarzschild geodesics, with $\theta = \pi/2$ we have the conserved quantities

$$E = (1 - 2m/r)\dot{t}, J = r^2\dot{\varphi} \tag{7.3.19}$$

and the further conservation equation

$$(1 - 2m/r)\dot{t}^2 - (1 - 2m/r)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = 2L. \tag{7.3.20}$$

where $L = 1/2$ for timelike and $L = 0$ for null geodesics. In the radial case, $\dot{\varphi} = 0$ and for timelike geodesics, (7.3.20) gives

$$E^2 - \frac{\mathrm{d}r}{\mathrm{d}\tau}^2 = 1 - \frac{2m}{r}, \tag{7.3.21}$$

from which (7.3.14) follows at once.

In the non-radial case, we follow the same moves that led to newtonian orbits (see problem set 1). We set $u = 1/r$. Then

$$(1 - 2mu)^{-1}E^2 - (1 - 2mu)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = 2L \tag{7.3.22}$$

Divide through by $J^2 = r^4\dot{\varphi}^2$:

$$(1 - 2mu)^{-1}\frac{E^2}{J^2} - (1 - 2mu)^{-1}u^4\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 - u^2 = \frac{2L}{J^2} \tag{7.3.23}$$

Now

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\varphi} \tag{7.3.24}$$

and substituting this in to (7.3.23) gives

$$\frac{1}{1 - 2mu}\frac{E^2}{J^2} - \frac{1}{1 - 2mu}\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 - u^2 = \frac{2L}{J^2} \tag{7.3.25}$$

Rearranging this yields

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 - 2mu^3 = \frac{E^2 - 2L}{J^2} + \frac{4mL}{J^2}u \tag{7.3.26}$$

The results in the Propositions follow from this by setting $L = 1/2$ for timelike and $L = 0$ for null. The second-order equation follows by differentiation with respect to φ, and cancelling u_φ. □

REMARK 7.3.5. For newtonian gravity, the equation for orbits (see homework problem 1.10) are

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = \frac{m}{J^2} \tag{7.3.27}$$

with first integral

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 = \frac{A}{J^2} + \frac{2m}{J^2}u \tag{7.3.28}$$

Thus the GR correction to this equation is the $-2mu^3$ term on the LHS of (7.3.26).

Remember that $u = 1/r$ so large r corresponds to small u and the effect of the cubic correction term is stronger for small radii. Remember also that the Schwarzschild metric appears only to be OK for $r > 2m$ which corresponds to $0 < u < 1/2m$.

7.3.3. Circular timelike orbits and the precession of perihelion. You can find an extensive analysis of the timelike geodesics in Schwarzschild in Woodhouse's book, Chapter 8. We shall just look at circular orbits and small perturbations of them. This already leads to the precession of perihelion, which I may have mentioned was one of the first verifications of GR.

Consider a circular timelike orbit in Schwarzschild. For such an orbit, evidently $u_{\varphi\varphi} = 0$, $u_\varphi = 0$. Setting $u_{\varphi\varphi} = 0$ in (7.3.15) gives the equation

$$3mu^2 - u + m/J^2 = 0 \tag{7.3.29}$$

so solving the quadratic,

$$u = \frac{1 \pm \sqrt{1 - 12m^2/J^2}}{6m} \tag{7.3.30}$$

Thus we have circular orbits if $J^2 > 12m^2$. For m small, the larger value of u is approximately $1/3m$ and is just less than this value. The smaller value of u is

$$u = u_0 = \frac{1 - \sqrt{1 - 12m^2/J^2}}{6m} \simeq \frac{m}{J^2} \tag{7.3.31}$$

if m/J^2 is small, and this is the newtonian value of the radius of a circular orbit for given m and J (cf. (7.3.27))

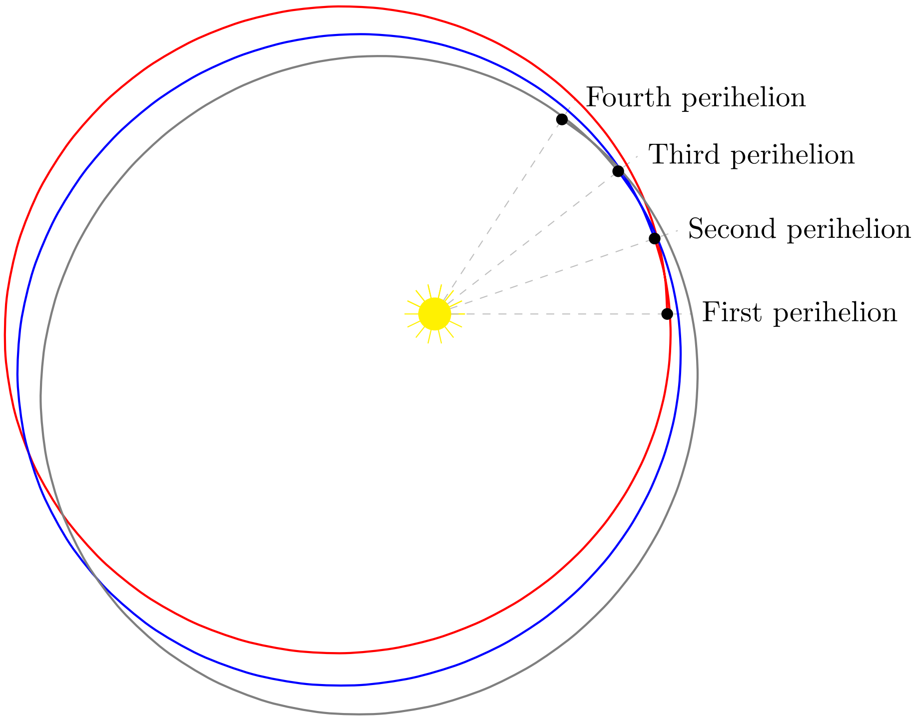


FIGURE 2. Plot of $r = 1/(0.1 + 0.03\cos(0.95\varphi))$, showing the precession of the perihelion. The succesive perihelia are shown and occur at $\varphi = 0, 2\pi/0.95, 4\pi/0.95, 6\pi/0.95, \ldots$. The red arc is the part of the orbit from $\varphi = 0$ to $\varphi = 2\pi/.95$; the blue arc is the part of the orbit from $\varphi = 2\pi/0.95$ to $4\pi/0.95$; the grey arc is the part of the orbit from $\varphi = 4\pi/0.95$ to $6\pi/0.95$

We consider a small perturbation of this orbit, $u(r) = u_0 + v(\varphi)$ where v is supposed to be very small. Inserting into the equation of motion (7.3.15) gives

$$\frac{\mathrm{d}^2 v}{\mathrm{d}\varphi^2} + v - 6mu_0 v = O(v^2)$$

Neglecting the quadratic term, v satisfies simple harmonic motion as a function of φ, but with period $2\pi/\sqrt{1-6mu_0}$. Thus the solution has the shape

$$u(\varphi) = u_0 + \varepsilon\cos(\sqrt{1-6mu_0}\varphi + \varphi_0) \tag{7.3.32}$$

The *perihelion* is the point on an orbit closest to the sun. This is the *largest* value of u (remembering the reciprocal relationship $u = 1/r$). To maximize u, and taking $\varphi_0 = 0$ for simplicity, we need $\cos = 1$ and so the perihelia occur at

$$\varphi = 0, \frac{2\pi}{\sqrt{1-6mu_0}}, \frac{4\pi}{\sqrt{1-6mu_0}}, \dots .$$

The gap between these angles is $> 2\pi$, which means that the perihelion *advances* in successive orbits: see Figure 2.

If we restore the units, within this approximation we get a perihelion advance of

$$\frac{6Gm\pi}{r_0c^2}$$

which for Mercury comes out to be approximately $40''$ *per century.* In fact, Mercury's perihelion advances even without GR, the perturbation being due to the gravitational interaction with the other planets. However, the observed value was different from all calculations based on newtonian gravity. The correction due to GR accounts precisely for this anomaly.

7.3.4. Photon trajectories: gravitational bending of light. From Proposition 7.3.4, non-radial photon trajectories satisfy:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u - 3mu^2 = 0. \tag{7.3.33}$$

We note first, that circular orbits exist if $u - 3mu^2 = 0$, i.e. $u = 1/3m$. The existence of circular photon orbits shows clearly that light is affected by gravity in Einstein's theory.

These orbits are unstable. Indeed, trying $u = \frac{1}{3m} + v$. Then

$$\frac{\mathrm{d}^2v}{\mathrm{d}\varphi^2} = v + O(v^2)$$

and $v \sim e^{\pm\varphi}$, so these perturbed solutions tend to grow exponentially.

Let us consider instead the trajectory of a photon which comes in from infinity (in a straight line with respect to the asymptotic coordinate system) and passes near to our gravitating object.

Recall that in polar coordinates, straight lines not through the origin are given by equations of the form

$$r\cos(\varphi - \varphi_0) = C. \tag{7.3.34}$$

Indeed, remembering $x = r\cos\varphi$, $y = r\sin\varphi$, (7.3.34) is equivalent to

$$r\cos\varphi\cos\varphi_0 - r\sin\varphi\sin\varphi_0 = C \text{ i.e. } x\cos\varphi_0 - y\sin\varphi_0 = C. \tag{7.3.35}$$

Thus this straight line is inclined at an angle φ_0 to the y-axis. In terms of the reciprocal coordinate $u = 1/r$, (7.3.34) takes the form

$$u = \alpha\cos(\varphi - \varphi_0) \tag{7.3.36}$$

Let us seek an approximate solution of (7.3.33) which corresponds to a photon approaching the sun from infinity along a line parallel to the y-axis. Thus we seek a solution

$$u(\varphi) = \alpha\cos\varphi + v(\varphi) \tag{7.3.37}$$

where $v(\varphi)$ is small and

$$v(-\pi/2) = 0 \tag{7.3.38}$$

so that as $\varphi \to -\pi/2$, $u(\varphi) \to 0$ and $r \to \infty$. (This corresponds to being parallel to being asymptotically parallel to the y-axis and with $y \sim -\infty$.

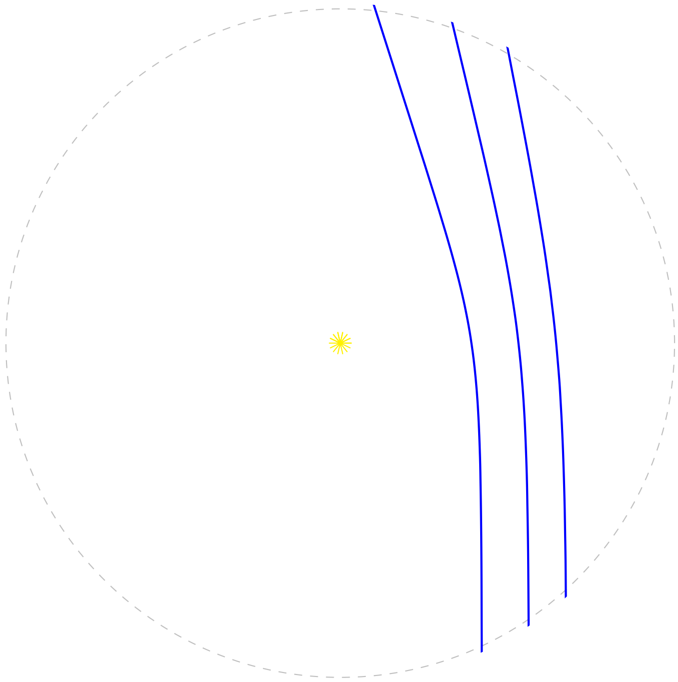


FIGURE 3. Bending of light by a star: plots of $u = \alpha\cos\varphi + \alpha^2(1+\sin\varphi)^2$ for $\alpha = 0.08, 0.06, 0.05$

This is now an exercise in differential equations. It is better to work with the first integral (7.3.18)

$$\left(\frac{\mathrm{d}u}{\mathrm{d}\varphi}\right)^2 + u^2 - 2mu^3 = \frac{E^2}{J^2}.$$

We have to follow our noses and compute:

$$u' = -\alpha\sin\varphi + v'. \tag{7.3.39}$$

Substituting this into (7.3.18) assuming that v is of order α^2, so that

$$(\alpha\cos\varphi + v(\varphi))^3 = \alpha^3\cos^3\varphi + O(\alpha^4) \tag{7.3.40}$$

we obtain, after some algebra,

$$\alpha^2 - 2\alpha\sin\varphi\, v' + 2\alpha\cos\varphi\, v - 2m\alpha^3\cos^3\varphi = E^2/J^2 + O(\alpha^4). \tag{7.3.41}$$

Hence

$$\alpha^2 = E^2/J^2 \tag{7.3.42}$$

and

$$v'\sin\varphi - v\cos\varphi = -m\alpha^2\cos^3\varphi \tag{7.3.43}$$

We use the integrating factor method to solve this. The integrating factor is $1/\sin^2\varphi$, so

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{v}{\sin\varphi}\right) = -m\alpha^2\frac{\cos^3\varphi}{\sin^2\varphi} = -m\alpha^2\left(\frac{\cos\varphi}{\sin^2\varphi} - \cos\varphi\right) \tag{7.3.44}$$

Hence

$$v(\varphi) = C\sin\varphi + m\alpha^2(1+\sin^2\varphi).$$

Applying the boundary condition $v(-\pi/2) = 0$, we get $C = 2m\alpha^2$ and

$$v(\varphi) = m\alpha^2(1+\sin\varphi)^2.$$

We conclude that

$$u(\varphi) = \alpha\cos\varphi + m\alpha^2(1+\sin\varphi)^2. \tag{7.3.45}$$

is an approximate null geodesic in Schwarzschild for α small.

We calculate the angle of deflection by looking for the values of φ for which $u = 0$. We already know one of them: $\varphi = -\pi/2$. We expect the other one to be approximately $\varphi = \pi/2$ (since this would correspond to zero deflection). So try to solve $u(\pi/2 - \lambda) = 0$, assuming λ is small. Substituting this into (7.3.45),

$$u(\pi/2 - \lambda) = 0 \Leftrightarrow \alpha \sin\lambda + m\alpha^2(1 + \cos\lambda)^2 = 0 \tag{7.3.46}$$

and putting $\sin\lambda \simeq \lambda$ $\cos\lambda \simeq 1$,

$$\lambda \simeq -4\alpha m. \tag{7.3.47}$$

So the asymptotic direction of the light-ray is approximately $\pi/2 + 4m\alpha$, showing that the photon has been deflected by an angle $4m\alpha$ due to the gravitational pull of the star.

Again, putting in the units, the deflection is approximately $4mG/Dc^2$, where D is the 'impact parameter' (the smallest value of r on the trajectory).

This was observed by Eddington during the 1919 total eclipse of the sun.

7.4. Extensions of Schwarzschild: introduction to black holes

We are going to consider the significance of the set $r = 2m$ in Schwarzschild. This is a 3-dimensional surface inside the 4-dimensional space-time. If we fix t we have a two-dimensional sphere of radius $2m$, so the surface as a whole looks like a cylinder of some kind.

We do have to worry about $r = 2m$, because a particle in free fall along a radial timelike geodesic will reach this set in finite proper time. Indeed, recall that for such a geodesic,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = -\sqrt{E^2 - 1 + \frac{2m}{r}} \tag{7.4.1}$$

for some constant $E \geqslant 1$. If $\tau = 0$ when $r = R$, this gives

$$\tau(r) = \int_r^R \frac{\mathrm{d}s}{\sqrt{E^2 - 1 + 2m/s}}. \tag{7.4.2}$$

So the particle reaches $r = 2m$ at proper time $\tau(2m) < \infty$. This means that the problem of understanding the metric in the vicinity of $r = 2m$ is of real physical relevance.

7.4.1. Toy examples.

Example 7.4.1. The metric

$$\mathrm{d}s_1^2 = \frac{\mathrm{d}t^2}{2t} + 2t\,\mathrm{d}\theta^2, \quad (0 < x < \infty) \tag{7.4.3}$$

is singular at $t = 0$. However, if we define

$$\mathrm{d}r = \frac{\mathrm{d}t}{\sqrt{2t}}, \quad \text{so } r = \sqrt{2t},$$

the metric becomes

$$\mathrm{d}s_1^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 = \mathrm{d}x^2 + \mathrm{d}y^2$$

($x = r\cos\theta, y = r\sin\theta$). The origin $(x, y) = (0, 0)$ corresponds to the singularity $t = 0$ of (7.4.3): despite appearances, the metric is perfectly good there.

Example 7.4.2. Consider the metric

$$\mathrm{d}s_2^2 = e^{2u}\mathrm{d}u^2 + e^{2v}\mathrm{d}v^2, \quad -\infty < u, v < \infty. \tag{7.4.4}$$

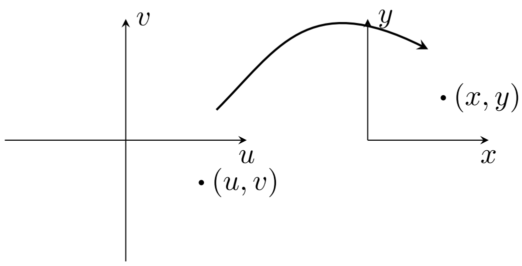
There might appear to be nothing to say about this but if we put

$$x = e^u, y = e^v \text{ so}$$

mapping uv plane to xy quadrant, then

$$\mathrm{d}s_2^2 = \mathrm{d}x^2 + \mathrm{d}y^2 \;\; 0 < x, y < \infty$$

and the metric extends (as the flat metric) to the whole xy plane.



Definition 7.4.3. In cases where a metric is ill-defined at a point, but after a change of coordinates it becomes well defined, we say that we have a *coordinate singularity.*

A coordinate singularity is not a true geometric singularity: it just corresponds to looking at the metric in a poor choice of coordinates.

We shall now see that the surface $r = 2m$ is a coordinate singularity rather than a metric singularity of Schwarzschild by exhibiting new coordinates in which the singularity disappears!

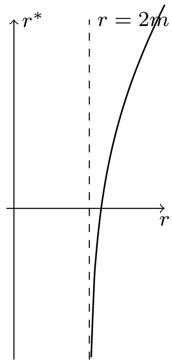
7.4.2. Eddington–Finkelstein coordinates. Define

$$r^* = r + 2m\log|r - 2m| \tag{7.4.5}$$

so

$$\frac{\mathrm{d}r^*}{\mathrm{d}r} = 1 + \frac{2m}{r-2m} = \frac{r}{r-2m} = \left(1 - \frac{2m}{r}\right)^{-1}. \tag{7.4.6}$$

The graph of r^* as a function of r is shown in the diagram:



Define new variables

$$v = t + r^*, w = t - r^* \tag{7.4.7}$$

Proposition 7.4.4. *Changing variables from (t, r) to (v, r), Schwarzschild becomes*

$$\mathrm{d}s^2 = (1 - 2m/r)\mathrm{d}v^2 - \mathrm{d}r\,\mathrm{d}v - \mathrm{d}v\,\mathrm{d}r - r^2\mathrm{d}\omega^2.$$

Note that this is a non-degenerate Lorentzian metric (signature $+---$) even when $r = 2m$.

Proof. Since $v = t + r^*$,

$$\mathrm{d}v = \mathrm{d}t + \mathrm{d}r^* = \mathrm{d}t + (1 - 2m/r)^{-1}\mathrm{d}r \tag{7.4.8}$$

we have

$$\begin{aligned}(1 - 2m/r)\mathrm{d}t^2 &= (1 - 2m/r)(\mathrm{d}v - (1 - 2m/r)^{-1}\,\mathrm{d}r)^2 &(7.4.9)\\ &= (1 - 2m/r)\mathrm{d}v^2 - \mathrm{d}r\mathrm{d}v - \mathrm{d}v\mathrm{d}r + (1 - 2m/r)^{-1}\mathrm{d}r^2 &(7.4.10)\end{aligned}$$

so

$$(1 - 2m/r)\mathrm{d}t^2 - (1 - 2m/r)^{-1}\mathrm{d}r^2 = (1 - 2m/r)\mathrm{d}v^2 - \mathrm{d}r\mathrm{d}v - \mathrm{d}v\mathrm{d}r. \tag{7.4.11}$$

The metric

$$(1 - 2m/r)\mathrm{d}v^2 - \mathrm{d}r\,\mathrm{d}v - \mathrm{d}v\,\mathrm{d}r$$

has matrix form

$$\begin{bmatrix}\mathrm{d}v & \mathrm{d}r\end{bmatrix}\begin{bmatrix}1 - \frac{2m}{r} & -1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\mathrm{d}v\\ \mathrm{d}r\end{bmatrix}$$

showing clearly that it is non-singular near $r = 2m$. □

Let us discuss what's happened here. To avoid later confusion, we shall rechristen the r coordinate here ρ. Thus (suppressing θ and φ) we have two coordinate systems: the original (t, r) and the new (v, ρ) with

$$v = t + r + 2m \log |r - 2m|, \quad \rho = r. \tag{7.4.12}$$

By the laws for change of vector fields,

$$\frac{\partial}{\partial t} = \frac{\partial v}{\partial t}\frac{\partial}{\partial v} + \frac{\partial \rho}{\partial t}\frac{\partial}{\partial \rho} = \frac{\partial}{\partial v} \tag{7.4.13}$$

and[7]

$$\frac{\partial}{\partial r} = \frac{\partial v}{\partial r}\frac{\partial}{\partial v} + \frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} = (1 - 2m/r)^{-1}\frac{\partial}{\partial v} + \frac{\partial}{\partial \rho}. \tag{7.4.14}$$

The following picture may help to visualize what's going on:

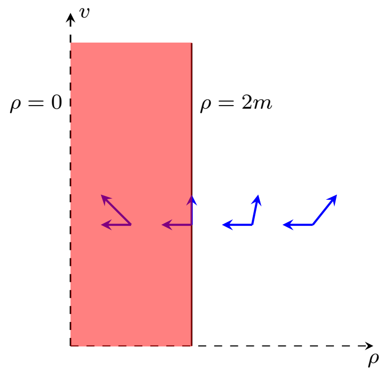


FIGURE 4. The change to Eddington–Finkelstein coordinates. The hypersurface $\rho = 2m$ is shown as are four pairs of future-pointing null vectors. One of the null vectors is always pointing inwards (toward $r = \rho = 0$). The other is pointing outward for $r > 2m$, is tangent to $r = 2m$ for a point on this hypersurface and then points inward for $r < 2m$.

Future-pointing radial null vectors in the original coordinates are (positive multiples of)

$$\frac{\partial}{\partial t} \pm (1 - 2m/r)\frac{\partial}{\partial r}. \tag{7.4.15}$$

Using (7.4.13), in the new coordinates these become

$$\frac{\partial}{\partial v} \pm \left(\frac{\partial}{\partial v} + (1 - 2m/r)\frac{\partial}{\partial \rho}\right). \tag{7.4.16}$$

Hence a pair of radial future-pointing null vectors in the new coordinates is:

$$2\partial_v + (1 - 2m/r)\partial_\rho, -\partial_\rho. \tag{7.4.17}$$

To summarize, a change of variables has enabled us to extend the Schwarzschild metric through the hypersurface $r = 2m$. When so extended, the metric is defined for $0 < r < 2m$, but *light emitted from any point in this region can never escape*. For this reason $r = 2m$ is called the event horizon.

REMARK 7.4.5. The radius $r = 2m$ is called the *Schwarzschild radius*. For our sun this is about 3 kilometres, well inside the sun itself. In particular the Schwarzschild metric is not valid there, because matter is present in this region. The above discussion is only of significance if matter is so highly concentrated that the region $r = 2m$ is contained in a region of empty space. The region $r < 2m$, to which we have now extended the Schwarzschild metric, is then called the *black hole region* of the space-time.

[7]this equation would be confusing if we had not renamed r as ρ!

7.4.3. What happens near the event horizon? Suppose Alice and Bob are near a black hole described by the Schwarzschild metric. Alice is sitting at $r = R$ and the unfortunate Bob[8] falls through $r = 2m$. How can we analyze this?

If Bob is freely falling, radially, so $\dot\theta = \dot\varphi = 0$, the Lagrangian in Eddington–Finkelstein coordinates is

$$L = \frac{1}{2}((1 - 2m/r)\dot v^2 - 2\dot v\dot r). \tag{7.4.18}$$

Here L is independent of v and so

$$\frac{\partial L}{\partial \dot v} = (1 - 2m/r)\dot v - \dot r \tag{7.4.19}$$

is a constant, F, say. Also $L = 1/2$ along a timelike geodesic parameterized by proper time.

Suppose that $\tau = 0$ when $r = 2m$. For small τ, these equations reduce to

$$-\dot r = F, -\dot v\dot r = 1/2 \tag{7.4.20}$$

So near the event horizon $r = 2m$, Bob's world line is

$$v \simeq \tau/2F, r \simeq 2m - F\tau, \text{ for small } \tau. \tag{7.4.21}$$

Thus Bob doesn't notice anything particularly strange in crossing the event horizon: though in reality, for a typical sized black hole, the tidal forces (the difference between the force felt on your head and your feet) get a bit strong well before you encounter the event horizon.

What does Alice see of Bob's descent? To answer this question, suppose she is sitting at $r = R > 2m$, and receives a photon from Bob at every tick of his clock. Then A's world-line is $\tau_A \mapsto (V\tau_A, R)$, her velocity 4-vector is $(V, 0)$ and so τ_A is proper time if this has length^2 equal to 1 with respect to our metric. This gives $V = (1 - 2m/R)^{-1/2}$ (cf. §7.3.1). So

$$v(\tau_A) = (1 - 2m/R)^{-1/2}\tau_A, \; r(\tau_A) = R \text{ along A's worldline.} \tag{7.4.22}$$

A photon emitted by Bob's clock at proper time $\tau_B < 0$ and heading out to Alice will satisfy

$$(1 - 2m/r)\dot v^2 - 2\dot r\dot v = 0 \tag{7.4.23}$$

with initial conditions

$$v(0) = \frac{\tau_B}{2F}, r(0) = 2m - F\tau_B. \tag{7.4.24}$$

Dividing by $\dot r\dot v$, (7.4.23) gives

$$\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{2r}{r - 2m} \tag{7.4.25}$$

and so, integrating,

$$v(r) = 2(r + 2m\log(r - 2m)) + c \tag{7.4.26}$$

for some integration constant. Inserting the initial conditions, we get

$$v(r) = 2[r - 2m + 2m\log\frac{r - 2m}{F|\tau_B|} + (F + \frac{1}{4F})\tau_B]. \tag{7.4.27}$$

So the photon emitted by Bob at his proper time τ_B is received by Alice when $r = R$, so

$$v(R) = 2\left[R - 2m + 2m\log\frac{R - 2m}{F|\tau_B|} + (F + \frac{1}{4F})\tau_B\right]. \tag{7.4.28}$$

and by (7.4.22) the corresponding value of A's proper time is

$$\tau_A = 2\sqrt{1 - \frac{2m}{R}}\left[R - 2m + 2m\log\frac{R - 2m}{F|\tau_B|} + (F + \frac{1}{4F})\tau_B\right]. \tag{7.4.29}$$

In particular, $\tau_A \to +\infty$ as $\tau_B \to 0$ (from below):

$$\tau_A \sim 4m\sqrt{1 - \frac{2m}{R}}\log\frac{1}{|\tau_B|} \text{ as } \tau_B \to 0.$$

Alice sees Bob 'frozen' at the point at which he crosses $r = 2m$: she sees his clock run more and more slowly, and never sees it reach $\tau_B = 0$.

[8]Perhaps Bob's a robot

7.4.4. The full extension of Schwarzschild: Kruskal coordinates. To understand the structure of the extended Schwarzschild space-time more fully, pass from (t, r) to (v, w), where

$$v = t + r + 2m\log(r - 2m), \quad w = t - r - 2m\log(r - 2m). \tag{7.4.30}$$

(This is more symmetrical than changing to (v, r) coordinates as we did in the previous section.) A simple calculation shows

$$\frac{1}{2}(\mathrm{d}v\mathrm{d}w + \mathrm{d}w\mathrm{d}v) = \mathrm{d}t^2 - (\mathrm{d}r^*)^2 = \mathrm{d}t^2 - (1 - 2m/r)^{-2}\mathrm{d}r^2, \tag{7.4.31}$$

So the Schwarzschild metric, in these coordinates, has the form

$$\mathrm{d}s^2 = (1 - 2m/r)\frac{1}{2}(\mathrm{d}v\mathrm{d}w + \mathrm{d}w\mathrm{d}v) - r^2\mathrm{d}\omega^2, -\infty < v, w < \infty. \tag{7.4.32}$$

where r is defined implicitly by v and w through

$$\frac{1}{2}(v - w) = r + 2m\log(r - 2m), r > 2m \tag{7.4.33}$$

Note that as r goes from $2m$ to ∞ so the RHS of (7.4.33) goes from $-\infty$ to ∞ and so given any value of $(v - w)/2$, there's a unique value $r > 2m$ which solves (7.4.33). This metric degenerates at $r = 2m$, which corresponds $v - w \to -\infty$. There is a remarkable trick—analogous to what happened in Example 7.4.2 above—which allows us to extend this metric through $r = 2m$.

We set

$$v = 4m\log v', \quad w = -4m\log(-w')$$

or

$$v' = \exp(v/4m), \quad w' = -\exp(-w/4m).$$

This maps the whole (v, w) plane to the quadrant $\{v' > 0, w' < 0\}$ in the $(v,' w')$ plane (compare Example 7.4.2). Note that for r just a bit bigger than $2m$, v has some finite value, $w \to +\infty$, so v' has some positive value and w' is just less than zero.

Then

$$\begin{aligned} \mathrm{d}v\mathrm{d}w &= -16m^2\frac{\mathrm{d}v'}{v'}\frac{\mathrm{d}w'}{w',} & (7.4.34)\\ v'w' &= -\exp((v - w)/4m) & (7.4.35)\\ &= -\exp(r/2m + \log(r - 2m)) = -e^{r/2m}(r - 2m). & (7.4.36) \end{aligned}$$

(Here we write $\mathrm{d}v\mathrm{d}w$ for $(\mathrm{d}v\mathrm{d}w + \mathrm{d}w\mathrm{d}v)/2$ for brevity.)

Hence

PROPOSITION 7.4.6. *The exterior region $r > 2m$ of the Schwarzschild metric corresponds to the region $v' > 0, w' < 0$ with the metric*

$$g^* = \frac{16m^2}{r}e^{-r/2m}\mathrm{d}v'\mathrm{d}w' - r^2\mathrm{d}\omega^2$$

where r is defined implicitly as a function of (v', w') by

$$e^{r/2m}(r - 2m) = -v'w'. \tag{7.4.37}$$

REMARK 7.4.7. This extension was obtained in 1960 by Kruskal. The crucial point is that g^* is a well-defined lorentzian metric wherever $r > 0$. From (7.4.37)

$$r > 0 \Leftrightarrow e^{r/2m}(r - 2m) > -2m \Leftrightarrow -v'w' > -2m.$$

Thus the metric is defined in the set $v'w' < 2m$, but the coordinates (v', w') can have either sign.

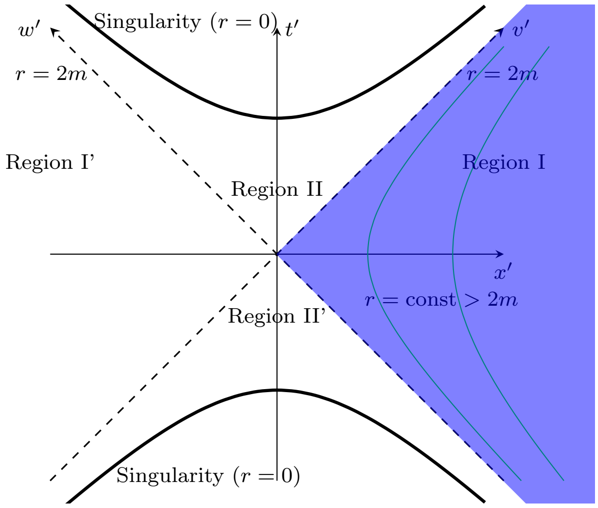
REMARK 7.4.8. If you prefer something looking more obviously Lorentzian, define

$$t' = v' + w', x' = v' - w', \text{ so } (\mathrm{d}t')^2 - (\mathrm{d}x')^2 = 4\mathrm{d}v'\mathrm{d}w'$$

so

$$g^* = \frac{4m^2}{r}e^{-r/2m}[(\mathrm{d}t')^2 - (\mathrm{d}x')^2] - r^2\mathrm{d}\omega^2$$

The following picture, the *Kruskal diagram* should help.



Note:

- Radial null geodesics are given by
$$v' = \tau, w', \theta, \varphi \text{ constant},$$
and
$$w' = \tau, v', \theta, \varphi \text{ constant}.$$
(i.e. they are always at 45° in the Kruskal diagram).
- The event horizon $r = 2m$ is given by $v'w' = 0$: the two dashed lines;
- The singularity at $r = 0$ is given by $v'w' = 2m$: the two thick hyperbolae at top and bottom of the picture.
- Region I is the domain of the original Schwarzschild metric, $r > 2m$.
- Region II is the region in which the Eddington–Finkelstein coordinates describe the metric: Region II is the region interior to the event horizon $r = 2m$, i.e. the black hole region. The boundary between regions I and II is the positive v'-axis.

We emphasise that the (v', w') axes are at 45° in this diagram. Also that the hyperbola at the top of the picture is the true singularity of the metric and represents the black hole itself. The ultimate fate of every particle or photon in region II (inside the event horizon) is to end up terminated by this singularity.

The singularity is the set $v'w' = 2m$ and is shielded from view by the event horizon. GR and all known laws of physics break down at the singularity.

7.5. Gravitational collapse

There are two problems with the Schwarzschild solution as a realistic model of a gravitating object such as a star.

Most stars are rotating; and the Schwarzschild solution is a vacuum solution, so one needs a separate description of the metric along the world-line of the star itself.

The first problem has been solved in the sense that the *Kerr metric* is a solution of Einstein's vacuum equations $r_{ab} = 0$ which contains an additional parameter corresponding to angular momentum. This metric is beyond the scope of this course...

For the second problem, consider the Schwarzschild radius $r_s = 2m$. Putting back in the units,
$$r_s = \frac{2Gm}{c^2}.$$
For typical objects, this radius is inside the object: e.g., for the Earth it is about 1 cm and for the sun, about 3 km. So (ignoring angular momentum) for these objects, it is a question of 'grafting' part of the exterior Schwarzschild metric to another metric which describes the matter content.

7.6. The life and death of stars

Stars like our sun manage to maintain their size against the pull of gravity because of the pressure generated by the nuclear fusion reactions going on its core. When the fuel runs out (to simplify a complicated story), a star of the mass of our sun is expected to settle down as a white dwarf: not very bright, density approx 10^9 kg/m^3.

For stars whose mass is between about 1.5 times and 3.2 times the mass of the sun, it is expected that the final state will be a *neutron star*. Such an object consists mainly of neutrons crushed together at almost unimaginable densities in their core of around 10^{17} kg/m^3.

It seems that for stars of mass more than about 8 times that of the sun, no known physical process can overcome the pull of gravity once the fuel runs out. The star collapses inside its Schwarzschild radius, an event horizon is formed, and inside that a singularity - which has no satisfactory physical description.

7.7. Some figures, or a tale of 3 black holes

(with apologies to Kip Thorne)

In his popular book ‘Black holes and time warps’, Kip Thorne starts by describing a space mission to explore black holes of different sizes.

- **Hades:** $10\times$ solar mass: $r_s = 30$ km. Tidal forces already perceptible in an orbit of circumference 10^5 km; at 3000 km the tidal force is more than $15g$!
- **Sagittario (centre of Milky Way)** 10^6 solar masses, $r_s = 3 \times 10^7$ km. (about $8\times$ moon’s orbit around earth). To hover at $1.0001r_s$ would require a force of $1.5 \times 10^8 g$.
- **‘Gargantua’** 15×10^{12} solar masses. $r_s = 4.5\times10^{13}$ km $= 5$ light years. For such a hole, no perceptible tidal forces even on an orbit whose circumference is $1.0001\times$ horizon circumference. To hold in this position, 10g acceleration is required. All physical experiments are entirely consistent with being in a 10g gravitational field. You could slip through the horizon with without any ill effects (apart from never being able to get out again).

  From a larger distance, such a large hole would have a major light-bending effect on stars in the line of sight. See a recent movie for a simulation of this.