CHAPTER 2

Minkowski Spacetime and Special Relativity

2.1. Introduction

In this chapter we try to explain how Newton's basic laws had to be changed to take into account the famous 'null result' of the Michelson–Morley experiment which appeared to show that the the speed of light was the same independent of the motion of the light-source.

In newtonian physics, there are such things as particles, masses and so on. The basic tenet of newtonian physics, in modern language, is that there are *inertial frames of reference*, and an observer at rest in such a frame observes free particles to remain at rest or travel at constant speeds. S/he also observes the famous law $\mathbf{F} = m\mathbf{a}$ to hold for masses acted upon by forces. These inertial frames are such that if R is such a frame and R' is moving with constant velocity (i.e. constant speed and direction) with respect to R, then R' is also an inertial frame. (See Woodhouse, Special Relativity, Ch. 1 for more details. Those with an interest in history may be interested in the end of Sect. 1.4, which indicates that Newton was aware of the idea of relativity, but preferred to present things in a different way.)

Suppose that Alice and Bob are observers at rest respectively in R and R', and that R' is moving at constant speed relative to R. With suitable orientation of the axes, we'd expect

$$x' = x - vt, y' = y, z' = z \tag{2.1.1}$$

where v is the relative speed of the frames. (If Bob is sitting at $(x', y', z') = (0, 0, 0)$, then Alice gives Bob's coordinates as $x = vt, y = z = 0$.)

This transformation (often known as the Galilean transformation between inertial frames) is clearly incompatible with the idea that the speed of light is the same in all inertial frames. The idea that this should be the case—and more generally that all physics should appear the same to all (inertial) observers—is the first of Einstein's famous postulates:

1. The laws by which the states of physical systems undergo change are not affected, whether these changes of state be referred to the one or the other of two systems of coordinates in uniform translatory motion.
2. As measured in any inertial frame of reference, light is always propagated in empty space with a definite velocity c that is independent of the state of motion of the emitting body.

In slogan form:

1. The laws of physics are the same in all inertial frames of reference.
2. The speed of light in free space has the same value c in all inertial frames of reference.

If the laws of physics are the same for any inertial frame, then it follows that no experiment can be performed that will single out one frame as preferred above all others. In particular, there can be no 'absolute standard of rest' and only *relative motion* is physically meaningful.

Note that the constancy of the speed of light means that something has to go wrong with the 'obvious' change of coordinates (2.1.1). More explicitly, this change of coordinates is not compatible with Maxwell's equations of electrodynamics.

It is convenient to make certain subsidiary assumptions explicit as well:

- P1 Free particles and photons (light particles) appear to inertial observers to travel in straight lines at constant speeds.
- P2 Photons appear to travel at the same speed c to all inertial observers.
- P3 The standard clock of one inertial observer appears to any other observer to run at a constant rate.

P4 Free particles cannot travel faster than c.

Note that in P3, although we assume that all clocks are observed by inertial observers to run at constant rates, we do not assume that these rates are all the same: if Alice and Bob are inertial observers moving relative to each other and are carrying identical clocks, Alice will observe the 'ticks' of Bob's clock to be at equal intervals as measured by her clock. However, the intervals between these two sets of 'ticks' are not assumed to be the same.

Consideration of Maxwell's equations, and the wave equation satisfied by EM radiation, suggest that it is sensible to consider a 4-dimensional vector space as the natural arena for physics. We need to give this space a structure which allows us to capture the constancy of the speed of light.

Definition 2.1.1. Minkowski Spacetime $\mathbb{M}$ (or sometimes just Minkowski Space) is a four-dimensional affine space equipped with with a Lorentzian symmetric bilinear form η. 'Lorentzian' means non-degenerate, signature $(+,-,-,-)$.

The points of $\mathbb{M}$ are called *events* (as they are localized in both space and time). If we pick a particular event E in $\mathbb{M}$, we denote the set of position vectors $\overrightarrow{EP}$, $P \in \mathbb{M}$, by M (or M_E). For the most part we can blur the distinction between M and $\mathbb{M}$—it is, after all, just the choice of an event—but for a complete understanding, it's worth making the effort to keep them separate.

Having chosen E to be an origin, we can now choose a basis (e_0, e_1, e_2, e_3) of M and expand the position vector of the general point P as a combination

$$\overrightarrow{EP} = cte_0 + xe_1 + ye_2 + ze_3. \tag{2.1.2}$$

for real numbers (t, x, y, z). We may suppose that η is diagonal with respect to this basis, with one 1 and three -1s on the diagonals, so that

$$\eta(\overrightarrow{EP}, \overrightarrow{EP}) = c^2t^2 - x^2 - y^2 - z^2.$$

(Here c is going to be the speed of light and is included for dimensional consistency to convert the time coordinate t into a quantity with the units of length.) We interpret t and (x, y, z) as respectively the time and space (or temporal and spatial, if you want to use the correct adjectives) coordinates. Then

$$\eta(\overrightarrow{EP}, \overrightarrow{EP}) = 0 \Leftrightarrow c^2t^2 = x^2 + y^2 + z^2$$

and this is the case if and only if something travelling at speed c, starting from $(0, 0, 0)$ at time 0 reaches (x, y, z) at time t.

If we take c to be the speed of light, then the geometry of $\mathbb{M}$ with its bilinear from η has the geometry of 'light-rays' built into it: two events E and F can be connected by a light-ray if and only if $\eta(\overrightarrow{EF}, \overrightarrow{EF}) = 0$.

In what follows, we answer the various questions about how P1–P4 are captured by the geometry of $\mathbb{M}$ and η.

From now on we usually take the speed of light c to be equal to 1.

This can be regarded as a choice of units. (For example, if distance is measured in light-years, and time in years, then $c = 1$.) In any given formula, you can always see where the factors of c should go by dimensional analysis.

2.2. What are the inertial coordinate systems?

Hypothesis 2.2.1. The inertial coordinate systems are those obtained as above by fixing a particular event E as an origin and introducing coordinates corresponding to a basis (e_0, e_1, e_2, e_3) of M with respect to which the components of η are

$$\eta(e_0, e_0) = 1, \eta(e_1, e_1) = \eta(e_2, e_2) = \eta(e_3, e_3) = -1,\ \eta(e_a, e_b) = 0 \text{ for } a \neq b. \tag{2.2.1}$$

Note that there are many choices of inertial coordinate systems. From the mathematical point of view, this is because there are many different choices of basis of M with respect to which η takes standard diagonal form. From the physical point of view this is because there are many different inertial observers, all on an equal footing.

Remark 2.2.2. Note that if X and Y are two vectors in M, and if (e_0, e_1, e_2, e_3) is a basis as in (2.2.1) then if

$$X = X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3, \quad Y = Y^0 e_0 + Y^1 e_1 + Y^2 e_2 + Y^3 e_3$$

we have

$$\eta(X, Y) = X^0 Y^0 - X^1 Y^1 - X^2 Y^2 - X^3 Y^3.$$

2.3. Worldlines

Recall that anything—particle, observer, photon—which exists for an extended period of time, is described in Minkowski spacetime by a *worldline*. This is a curve in $\mathbb{M}$ consisting of all the events through which our particle, observer, photon passes.

For example, suppose Alice is at rest at the spacial origin of an inertial coordinate system (t, x, y, z). The events on Alice's world line have coordinates of the form $(t, 0, 0, 0)$, t being the time on the clock that Alice has beside her.

More generally, Alice observes a particle by noting its (x, y, z) coordinates for different times t. In other words she observes the particle's world-line in the form of a curve

$$\Gamma(t) = (t, x(t), y(t), z(t))$$

in the given coordinates.

It is often useful to 'decouple' the parameter which parameterises the curve from an observer's time coordinate, replacing the above by the more general form

$$\Gamma(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau))$$

so that all 4 coordinates depend on the parameter τ.

Example 2.3.1. If Alice is an inertial observer who sets up an inertial coordinate system as above with herself at the (spatial) origin $x = y = z = 0$, then her worldline will be

$$t(\tau) = \tau, x(\tau) = y(\tau) = z(\tau) = 0.$$

Definition 2.3.2. For the worldline $\Gamma(\tau)$ of a particle, observer or photon in $\mathbb{M}$, $\mathrm{d}\Gamma/\mathrm{d}\tau$ is called the *velocity 4-vector*.

The use of the term 4-vector is traditional. It helps to distinguish this vector from 'ordinary' velocity vectors: e.g. the velocity vector of a particle as measured by an observer.

Note that in terms of the original parameterization, $\Gamma(t) = (t, x(t), y(t), z(t))$,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \left(1, \frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right).$$

and the *spatial part* of this is the 3-vector

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right).$$

which is the instanteous velocity of the particle as calculated by Alice when her clock says time t.

For now, we shall mainly be concerned with straight, constant-speed worldlines: i.e. where Γ has the form

$$\Gamma(\tau) = X + V\tau \tag{2.3.1}$$

where X and V are constant vectors in M. (Here again we are regarding $\mathbb{M}$ and M as the same by choice of an event E of $\mathbb{M}$ corresponding to the zero-vector in M.

We now have to see how P2 and P4 are to be interpreted: photons travel at speed $c = 1$ as measured by any inertial observer, and no particle is ever observed to travel faster than light.

2.3.1. What are the photon worldlines? Suppose that a photon is emitted by a laser at the event with coordinates $(0,0,0,0)$ and passes through the event (t,x,y,z), relative to the above inertial coordinate system. In other words, at the later time t, its spatial coordinates are (x,y,z). Then the distance covered is $\sqrt{x^2+y^2+z^2}$ but this must be equal to t as the speed is 1. In particular, if E and P are two events on the worldline of a (free) photon, then $\overrightarrow{EP}$ is *null* in the sense that $\eta(\overrightarrow{EP},\overrightarrow{EP})=0$. [NB, a null vector need not be the zero vector!]

The following definition is useful:

DEFINITION 2.3.3. Two events P and Q are *null-separated* if the displacement vector $X=\overrightarrow{PQ}$ is null, i.e. $\eta(X,X)=0$.

REMARK 2.3.4. This definition depends only upon the events P and Q, and the form η; it does not depend upon any choice of inertial basis or coordinate system.

To flesh this remark out: We saw by calculation in a particular inertial frame, that if P and Q are two events on a photon worldline, then $\overrightarrow{PQ}$ is a null vector. But the latter is a statement purely about the geometry of $\mathbb{M}$: it uses only the basic facts that given any two events we have a displacement vector, and that we can feed vectors to η. In particular all inertial observers agree about when a pair of events are null separated, and hence the speed of light is the same for all such observers.

This leads us to the following

HYPOTHESIS 2.3.5. The worldline of a photon has the form

$$\Gamma(\tau)=X+N\tau, \tag{2.3.2}$$

where X and N are constant vectors and N is *null*, i.e. $\eta(N,N)=0$.

This hypothesis is justified, to some extent, by the following:

PROPOSITION 2.3.6. *If P_1 and P_2 are any events on the worldline* (2.3.2)*, then P and Q are null-separated.*

PROOF. If P_1 and P_2 correspond to parameter values τ_1 and τ_2, then

$$\overrightarrow{P_1P_2}=(X+N\tau_2)-(X+N\tau_1)=(\tau_2-\tau_1)N. \tag{2.3.3}$$

Now, by the bilinearity of η,

$$\eta((\tau_2-\tau_1)N,(\tau_2-\tau_1)N)=(\tau_2-\tau_1)^2\eta(N,N)=0. \tag{2.3.4}$$

□

In summary, we have seen that if inertial coordinate systems are defined as in Hypothesis 2.2.1 and free photon worldlines are as in Hypothesis 2.3.5, then all inertial observers agree on the speed at which photons travel.

2.3.2. What are the free particle worldlines? Return to the two events E and P at the beginning of the previous section, and suppose now that they are on the worldline of a particle travelling at uniform speed v with $0\leqslant v<1$. Then we must have

$$\sqrt{x^2+y^2+z^2}=|vt|<|t| \tag{2.3.5}$$

and so

$$t^2-x^2-y^2-z^2>0. \tag{2.3.6}$$

DEFINITION 2.3.7. A vector $X\in M$ is *timelike* if $\eta(X,X)>0$. Two events P and Q are *timelike separated* if the displacement vector $\overrightarrow{PQ}$ is timelike.

We now make the free-particle hypothesis:

HYPOTHESIS 2.3.8. The worldline of a free particle has the form

$$\Gamma(\tau) = X + V\tau \tag{2.3.7}$$

where X and V are constant vectors of M and V is *timelike*. The parameter τ is called *proper time* if $\eta(V, V) = 1$. In this case, if P_1 and P_2 are two events on the worldline with parameter values τ_1 and τ_2 respectively, then $\tau_2 - \tau_1$ is interpreted as the elapsed time between these two events as measured by a clock carried by an observer on this worldline.

REMARK 2.3.9. As for null-vectors, the notion of a vector being time-like is independent of any choice of observer or coordinate system. In particular if one inertial observer thinks that a particle is travelling at speed less than that of light, all observers will agree on this.

REMARK 2.3.10. Note the analogy between a curve being parameterized by proper time here and the idea of unit-speed curves being parameterized by arc-length for 'ordinary' curves in euclidean space.

As in the case of photon worldlines, we started in Alice's coordinate system (t, x, y, z), and calculated that the events E and P are on the worldline of a particle moving at speed < 1 if (and only if) the displacement vector $\overrightarrow{EP}$ is timelike. This, however, is a statement which is independent of any particular choice of inertial coordinate system. Thus it must be the case that Bob, with an inertial coordinate system (t', x', y', z') will also calculate that E and P are events on the worldline of a particle moving at speed less than 1.

2.4. Why do clocks carried by inertial observers all go at uniform rates?

Let us remember that the postulate P3 states that if Alice and Bob are inertial observers, possibly in relative motion, then if Alice looks at Bob's clock, she will see that it is ticking at a uniform rate, but that this rate may be different from the rate at which her own (identical) clock is ticking.

Suppose that Bob has worldline

$$\Gamma(\tau) = V\tau \tag{2.4.1}$$

(so that he passes through the event E at parameter value $\tau = 0$). Recall the hypothesis that τ is proper time (i.e. the time as measured on the clock he's carrying with him) if $\eta(V, V) = 1$.

In Alice's coordinate system (t, x, y, z), this has the form

$$t(\tau) = V^0\tau, x(\tau) = V^1\tau, y(\tau) = V^2\tau, z(\tau) = V^3\tau. \tag{2.4.2}$$

where

$$(V^0)^2 - (V^1)^2 - (V^2)^2 - (V^3)^2 = 1. \tag{2.4.3}$$

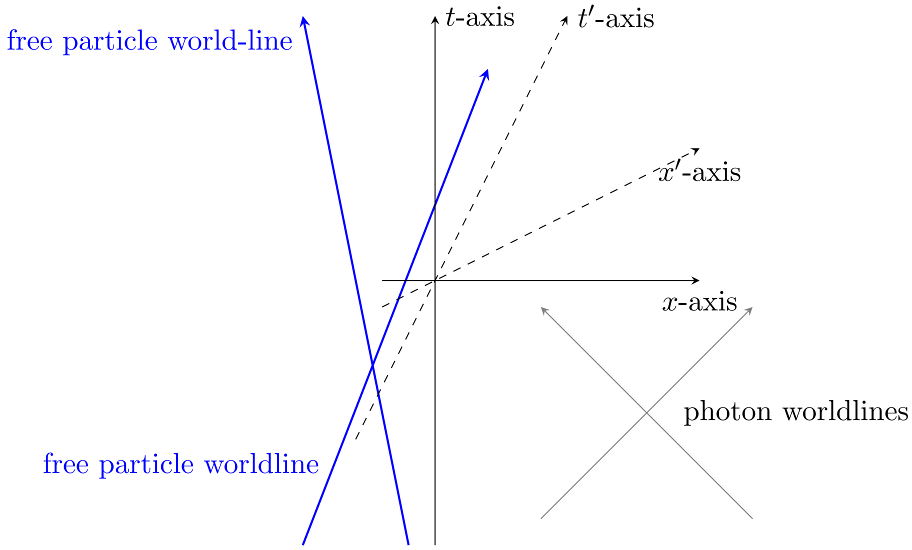
In particular $V^0 \neq 0$ and we find the time τ measured on Bob's clock is related to Alice's time coordinate t by the fixed multiple V^0. So with all our hypotheses made about what inertial frames are, we see that each of P1—P4 are now satisfied.

2.5. Spacetime diagrams

Minkowski spacetime can be pictured by suppressing one or two of the spatial dimensions and drawing a picture with time going up the page and x or x and y going across.

Suppressing y and z, leaving just a space variable x and a time variable t in play, a typical spacetime diagram is shown below. I've drawn in the worldlines of two free massive particles, two photon worldlines and the axes of two inertial coordinate systems.

Note that the photon worldlines are at 45° and that the free particles have worldlines inclined at less than 45° to the vertical.



2.6. Time dilatation—'moving clocks run slowly'

The calculation of the previous section allows us to understand and quantify the sense in which moving clocks run slowly. We imagine our two inertial observers, Alice and Bob, with their identical clocks, one moving relative to the other. The slogan 'moving clocks run slowly' actually means that Alice will observe Bob's clock runing more slowly then her clock (and symmetrically, Bob will observe Alice's clock running slowly, by the same factor).

Based on our hypotheses, we can work out exactly what is going on here. In the previous section, we had that Alice set up inertial coordinates (t, x, y, z), with her worldline being $x = y = z = 0$. The time coordinate t is time as measured on her clock.

We saw that if Bob's worldline is $\Gamma(\tau) = V\tau$, with $\eta(V, V) = 1$, then the components (v^1, v^2, v^3) of Bob's velocity with respect to Alice's coordinates are related to Bob's 4-velocity vector by

$$v^i = V^i/V^0, \ (i = 1, 2, 3).$$

At this point it is convenient to use ordinary euclidean 3-dimensional vector notation $\mathbf{v}$ for the three dimensional vector with components (v^1, v^2, v^3). We can write

$$(V^0, V^1, V^2, V^3) = V^0(1, \mathbf{v})$$

from which

$$\eta(V, V) = (V^0)^2(1 - |\mathbf{v}|^2) = 1.$$

Thus

$$V^0 = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}.$$

The RHS here is usually denoted by $\gamma(v)$ (where as usual $v = |\mathbf{v}|$),

$$\gamma(v) = \frac{1}{\sqrt{1 - |\mathbf{v}|^2}}. \tag{2.6.1}$$

This has to be replaced by

$$\gamma(v) = \frac{1}{\sqrt{1 - |\mathbf{v}|^2/c^2}}. \tag{2.6.2}$$

in units in which $c \neq 1$. Notice that

$$\gamma(v) \geqslant 1 \tag{2.6.3}$$

with equality if and only if $\mathbf{v} = \mathbf{0}$. Moreover $\gamma(v) \to \infty$ as $v \to c$.

Now the relation

$$t = V^0\tau = \gamma(v)\tau \tag{2.6.4}$$

encodes the time dilatation or 'moving clocks run slowly': indeed, if P_1 and P_2 are two events on Bob's world line which occur at parameter values τ_1 and τ_2, then he reckons that the time difference is $\tau_2 - \tau_1$. Alice however, reckons that the time difference between these two events is $\gamma(v)(\tau_2 - \tau_1)$, so the time elapsed is greater, according to Alice, by a factor of $\gamma(v)$.

2.7. Simultaneity and distance

We have made a hypothesis about how coordinates introduced by diagonalizing η are inertial coordinates introduced by inertial observers and how unit speed straight lines are worldlines parameterized by proper time.

There is a good question, though, which is how would an observer actually try to set up coordinates without appealing to an absolute standard of rest. So suppose Alice wants to set up such coordinates.

Suppose that F is any event in $\mathbb{M}$. Alice is travelling on her straight world line which doesn't pass through F. She sends out light signals and lets them scatter off F. She finds that the signal emitted at time τ_1 on her clock scatters off F and is picked up by her at time τ_2. She infers two things: that the distance to F is $c(\tau_2 - \tau_1)/2$. And that F should have time coordinate $\frac{1}{2}(\tau_1 + \tau_2)$. This is the radar method of assigning times and measuring distances.

So using only allowable methods, she assigns a position and time coordinate to F.

Let's see how all this looks in terms of trajectories and worldlines.

If Alice's trajectory is $\Gamma(\tau) = U\tau$, and F has position vector Y relative to the chosen origin, the photon trajectory is

$$\sigma \mapsto U\tau_1 + N\sigma$$

on the outward leg and

$$\sigma' \mapsto U\tau_2 + N'\sigma'$$

on the return leg. Here N and N' are null vectors, and we may suppose that the parameters σ and σ' are chosen so that these trajectories hit Y at $\sigma = 1$, $\sigma' = 1$:

$$U\tau_1 + N = Y = U\tau_2 + N' \tag{2.7.1}$$

The displacement vector from E to F is

$$X := \overrightarrow{EF} = Y - \frac{1}{2}(\tau_1 + \tau_2)U = \frac{1}{2}(\tau_2 - \tau_1)U + N' = \frac{1}{2}(\tau_1 - \tau_2)U + N \tag{2.7.2}$$

We also assume $\eta(U, U) = 1$ so that Alice's worldline is parameterised by her proper time τ.

Proposition 2.7.1. $\eta(U, \overrightarrow{EF}) = 0$.

Proof. From (2.7.1),

$$U(\tau_1 - \tau_2) = N' - N. \tag{2.7.3}$$

Take the η-inner product of this with N and with N' to get

$$\begin{aligned} (\tau_1 - \tau_2)\eta(U, N) &= \eta(N, N') & (2.7.4)\\ (\tau_1 - \tau_2)\eta(U, N') &= -\eta(N, N') & (2.7.5) \end{aligned}$$

where we've used

$$\eta(U, U) = 1,\ \eta(N, N) = \eta(N', N') = 0.$$

We can get our hands on $\eta(N, N')$ by squaring (2.7.3)

$$\eta(N' - N, N' - N) = (\tau_1 - \tau_2)^2\eta(U, U) \text{ so } -2\eta(N, N') = (\tau_1 - \tau_2)^2. \tag{2.7.6}$$

Combining with (2.7.4),

$$\eta(U, N) = -\frac{1}{2}(\tau_1 - \tau_2). \tag{2.7.7}$$

Now we calculate

$$\begin{aligned}\eta(U, \overrightarrow{EF}) &= \eta\left(U, N + \frac{1}{2}(\tau_1 - \tau_2)U\right) \\ &= \eta(U, N) + \frac{1}{2}(\tau_1 - \tau_2) \\ &= -\frac{1}{2}(\tau_1 - \tau_2) + \frac{1}{2}(\tau_1 - \tau_2) \\ &= 0\end{aligned}$$

as required. □

If you didn't like that proof, here is another. There are often several different ways to accomplish the same thing.

PROOF. The idea of this proof is to write everything in terms of the null vectors N and N'. It is perhaps a more symmetrical proof than the previous one. From (2.7.3), we obtain

$$U = \frac{N' - N}{\tau_1 - \tau_2}. \tag{2.7.8}$$

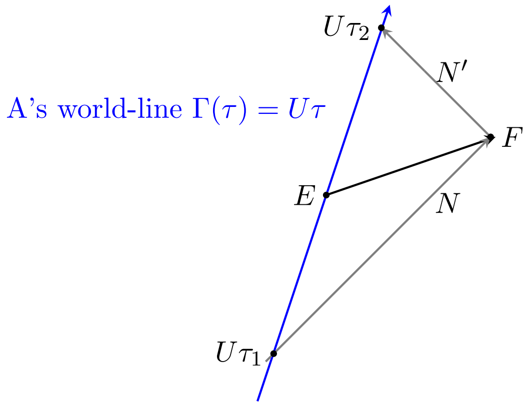
We also have the two formulae for $\overrightarrow{EF}$ in (2.7.2). Adding these, we get

$$2\overrightarrow{EF} = N + N'. \tag{2.7.9}$$

Now we calculate

$$\eta(U, \overrightarrow{EF}) = \frac{1}{2(\tau_1 - \tau_2)}\eta(N + N', N' - N) = 0. \tag{2.7.10}$$

by using again the bilinearity of η to expand the RHS. □



DEFINITION 2.7.2. If Alice is moving uniformly with 4-velocity vector U, then she reckons two events E and F to be simultaneous if

$$\eta(U, \overrightarrow{EF}) = 0.$$

These events then have a well-defined spatial separation d, where

$$d^2 = -\eta(\overrightarrow{EF}, \overrightarrow{EF}).$$

However we look at it, the key point is that if we have two observers, Alice and Bob, moving relative to each other, then they will generally disagree about which pairs of distant events are simultaneous.

From the mathematical or geometric point of view, they have different 4-velocity vectors U and V. If F and G are distant events, Alice thinks they are simultaneous if $\eta(U, \overrightarrow{FG}) = 0$, while Bob thinks they are simultaneous if $\eta(V, \overrightarrow{FG}) = 0$. These are different conditions, and if one is satisfied, then there is no guarantee that the other one will also be.

DEFINITION 2.7.3. Let P and Q be two particles. If Alice is an inertial observer with 4-velocity U, she measures the distance between these two particles at time τ on her clock by

- Finding events F on the world-line of P and G on the worldline of Q which are simultaneous with the event E at time τ on her worldline; in other words, finding F and G such that $\eta(U, \overrightarrow{EF}) = 0$, $\eta(U, \overrightarrow{EG}) = 0$.
- Calculating the distance as

$$d = \sqrt{-\eta(\overrightarrow{FG}, \overrightarrow{FG})}.$$

2.8. Length contraction

Let's push this operational definition of distance or length to see how an inertial observer will measure the length of a moving rod. Suppose that we have a rod of length d, whose endpoints have world-lines

$$\alpha(\tau) = V\tau, \beta(\tau') = D + V\tau'$$

where $\eta(D, V) = 0$. The length of the rod should be defined as the length of the rod as measured by an observer at rest with respect to the rod. For such an observer, with 4-velocity V, we ask: which pairs of events $\alpha(\tau)$ and $\beta(\tau')$ are simultaneous. Plugging in the defintion we need

$$\eta(V, D + V(\tau' - \tau)) = \eta(V, D) + \tau' - \tau = 0.$$

By assumption $\eta(V, D) = 0$, so we get 0 if and only if $\tau' = \tau$. For these simultaneous events, the relative position vector $\overrightarrow{EF}$ is equal to D, independently of τ. So the length of the rod is $\sqrt{-\eta(D, D)}$.

To be more concrete, in the rest-frame of the rod, if it is lying along the x-axis, we'd have

$$\alpha(\tau) = (\tau, 0, 0, 0), \ \beta(\tau) = (\tau, d, 0, 0).$$

If Alice's worldline is $\Gamma(\tau) = U\tau$, to measure the length, she has to find events E and F on the world-lines that she considers to be simultaneous.

If these are $V\tau$ and $D + V\tau'$, then the displacement vector is

$$X = D + V(\tau' - \tau) \tag{2.8.1}$$

To satisfy the simultaneity condition (Definition 2.7.3) we need to solve

$$\eta(U, D + V(\tau' - \tau)) = 0 \tag{2.8.2}$$

which gives $\tau' - \tau = -\eta(U, D)/\eta(U, V)$.

Hence the relative position vector X between these simultaneous events will be

$$X = D - \frac{\eta(U, D)}{\eta(U, V)}V. \tag{2.8.3}$$

Thus we compute

$$\eta(X, X) = \eta(D, D) - 2\frac{\eta(U, D)\eta(V, D)}{\eta(U, V)} + \frac{\eta(U, D)^2}{\eta(U, V)^2} = \eta(D, D) + \frac{\eta(U, D)^2}{\eta(U, V)^2}, \tag{2.8.4}$$

making heavy use of the bilinearity of η, and the fact $\eta(V, D) = 0$.

Hence Alice calculates the length of the rod as

$$d' = \sqrt{-\eta(D, D) - \frac{\eta(U, D)^2}{\eta(U, V)^2}}. \tag{2.8.5}$$

Recall that $-\eta(D, D) = d^2$ is the square of the length of the rod as measured in its rest-frame, so $d' \leqslant d$ in general.

To understand this calculation, we may work explicitly in the frame in which the rod is at rest. Then

$$V = (1, 0, 0, 0), D = (0, d, 0, 0). \tag{2.8.6}$$

Alice's 4-velocity vector U has the form

$$U = \gamma(u)(1, \mathbf{u}) = \gamma(u)(1, u_1, u_2, u_3) \tag{2.8.7}$$

where $\mathbf{u}$ is the velocity of Alice relative to Bob and (u_1, u_2, u_3) are its components in Bob's coordinates and

$$\gamma(u) = \frac{1}{\sqrt{1 - |\mathbf{u}|^2}}. \tag{2.8.8}$$

as in (2.6.1). Hence

$$\eta(U, D) = -\gamma(u) du_1, \eta(U, V) = \gamma(u) \tag{2.8.9}$$

and so from (2.8.5), we find

$$d' = d\sqrt{1 - u_1^2}$$

This is the famous *Lorentz–Fitzgerald length contraction*: if the component u_1 of the relative velocity of the observer in the direction of the rod is non-zero, then the observer judges the length of the rod to be less than d by the factor $\sqrt{1 - u_1^2}$. Note that there is no length contraction if the observer is moving at right-angles to the rod.

2.9. Lorentz transformations

The set of linear transformations of M which *preserve* η is called the Lorentz group. This means, concretely, the set of linear maps $L : M \to M$ such that

$$\eta(LX, LY) = \eta(X, Y)$$

for all $X, Y \in M$. Even more concretely, if we choose a diagonalizing basis (e_0, e_1, e_2, e_3) of M, then M is identified with $\mathbb{R}^4$, L becomes a real 4×4 matrix and the condition is

$$L^t \eta L = \eta, \eta = \mathrm{diag}(1, -1, -1, -1).$$

The group of all Lorentz transformations, or the Lorentz group, is also denoted by $O(1, 3)$. We shall use Lorentz transformations mainly to compare different inertial frames with the same event in $\mathbb{M}$ as origin. More precisely, suppose that Alice introduces an inertial basis (e_0, e_1, e_2, e_3) and Bob introduces an inertial basis $(\widetilde{e}_0, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3)$. Their coordinates are respectively (t, x, y, z) and $(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$.

Because (e_i) and $(\widetilde{e}_i)$ are both diagonalizaing bases, there is a Lorentz transformation L with the property

$$(\widetilde{e}_0, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3) = (e_0, e_1, e_2, e_3)L \tag{2.9.1}$$

This is shorthand for expressing the primed basis vectors as linear combinations of the unprimed ones.

Note that the matrix product

$$\begin{pmatrix} \widetilde{e}_0^t \\ \widetilde{e}_1^t \\ \widetilde{e}_2^t \\ \widetilde{e}_3^t \end{pmatrix} \eta(\widetilde{e}_0, \widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3) \tag{2.9.2}$$

has as its ab component the scalar $\eta(\widetilde{e}_a, \widetilde{e}_b)$. Hence, as $(\widetilde{e}_a)$ are diagonalizing, this matrix is diagonal, with entries

$$\eta(\widetilde{e}_0, \widetilde{e}_0) = 1, \eta(\widetilde{e}_1, \widetilde{e}_1) = \eta(\widetilde{e}_2, \widetilde{e}_2) = \eta(\widetilde{e}_3, \widetilde{e}_3) = -1$$

But substituting in terms of e and L,

$$L^t e^t \eta e L = \eta, \text{ so } L^t \eta L = \eta,$$

confirming the relevance of the Lorentz group for changing frame between observers.

Suppose that our frames are related by (2.9.1). Multiplying on the right by the column vector $(\widetilde{t}, \widetilde{x}, \widetilde{y}, \widetilde{z})$, we get

$$\widetilde{t}\widetilde{e}_0 + \widetilde{x}\widetilde{e}_1 + \widetilde{y}\widetilde{e}_2 + \widetilde{z}\widetilde{e}_3 = (e_0, e_1, e_2, e_3)L \begin{pmatrix} \widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} \tag{2.9.3}$$

so that by definition of (t, x, y, z),

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = L \begin{pmatrix} \widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} \tag{2.9.4}$$

Note the usual confusing point that the L appears multiplying the untilded e's in (2.9.1) but the tilded coordinates in (2.9.4).

2.9.0.1. *Examples.* A particular example (with $c = 1$) is

$$L = \begin{pmatrix} \gamma(v) & \gamma(v)v & 0 & 0 \\ \gamma(v)v & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9.5}$$

This is often referred to as the standard 2D Lorentz transformation. It is of course, four-dimensional, but $y = \widetilde{y}$ and $z = \widetilde{z}$, so all the action is going on in the way the (t, x) and $(\widetilde{t}, \widetilde{x})$ variables are related to each other. To save writing, we'll ignore the y, z and $\widetilde{y}$, $\widetilde{z}$ variables in the rest of the discussion of this example.

Here, as before, $\gamma(v) = 1/\sqrt{1 - v^2}$. Suppose Bob is sitting at the origin of the $\tilde{}$ coordinate system. Then his world line is $\widetilde{x} = 0$. Inserting $(\widetilde{t}, 0)$ into the coordinate transformation, we see that

$$t = \gamma(v)\widetilde{t}, x = \gamma(v)v\widetilde{t}.$$

This gives Bob's worldline, as a parameterized curve in Alice's coordinate system (the parameter being $\widetilde{t}$). Since $x/t = v$ for this curve, we see that Bob is moving at speed v in the direction of Alice's positive x-axis. The conclusion is that this Lorentz transformation corresponds precisely to two inertial observers one moving at speed v relative to the other.

It is of interest to derive this from the postulates P1—P4 and the relativity principle R. I shall omit this here: you can find it in Woodhouse, SR (new edition, §§4.4–4.6.)

Consideration of this transformation gives a different way to derive the standard counterintuitive properties of SR: time dilatation, length contraction, and so on.

2.9.1. The Lorentz and Poincaré Groups. The Poincaré group is the Lorentz group with (4-dimensional) translations included. Thinking in terms of coordinate transformations, the typical element of the Poincaré group has the form

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = L \begin{pmatrix} \widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} + \begin{pmatrix} p \\ q \\ r' \\ s \end{pmatrix} \tag{2.9.6}$$

where (p, q, r, s) are constants and L is a Lorentz transformation.

From a more sophisticated point of view, it is the natural symmetry group of the affine space $\mathbb{M}$, preserving the bilinear form η on the set M_E of all position vectors relative to E.

Remark 2.9.1. The 3-dimensional euclidean group is contained in the Poincaré group, via

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} \widetilde{t} \\ \widetilde{x} \\ \widetilde{y} \\ \widetilde{z} \end{pmatrix} + \begin{pmatrix} 0 \\ q \\ r \\ s \end{pmatrix} \tag{2.9.7}$$

where R is a 3-dimensional orthogonal transformation.

2.10. Orientation and time-orientation

Just as the reflections are isometries of euclidean space that do not seem to be realized physically, so there are some Lorentz transformations that are not so relevant as others. The physically most relevant Lorentz transformations are those that preserve the spatial orientation (rotations rather than reflections) and a *time orientation* or time's 'arrow'.

Definition 2.10.1. A time orientation of M or $\mathbb{M}$ is a choice of one half or 'nappe[1]' of the cone of timelike vectors in M. Accordingly every timelike-vector is either *future-pointing* (if it is in the chosen nappe) or *past-pointing* (if it is in the other nappe).

An inertial basis (e_0, e_1, e_2, e_3) is time-oriented if the timelike vector e_0 is future-pointing.

Lemma 2.10.2. *Let X and Y be timelike vectors. Then $\eta(X, Y) \neq 0$. Moreover, $\eta(X, Y) > 0$ if X and Y are in the same nappe of the cone and $\eta(X, Y) < 0$ otherwise.*

Proof. If X is timelike, then we know that $\eta(X, X) > 0$. On the orthogonal space $\Sigma = \{S : \eta(X, S) = 0\}$, η is negative definite, because its signature is one plus and three minuses. Thus any vector S which is η-orthogonal to X must satisfy $\eta(S, S) < 0$. So $\eta(X, Y) \neq 0$ if X and Y are timelike.

The last part is perhaps most easily seen by choosing a frame in which $X = \lambda e_0$, for some $\lambda > 0$. If the components of Y are (Y^0, Y^1, Y^2, Y^3), then $\eta(X, Y) = \lambda Y^0$. Clearly X and Y are in the same nappe if $Y^0 > 0$. $\square$

One can show that if a Lorentz transformation maps one future-pointing timelike vector to a future-pointing timelike vector, then it maps all future-pointing timelike vectors to future-pointing timelike vectors.

Definition 2.10.3. A Lorentz transformation is *orthochronous* if it maps the future-pointing nappe to the future-pointing nappe, in other words, if it preserves the time-orientation. The subgroup of such transformations is denoted $O_+(1, 3)$.

Definition 2.10.4. A Lorentz transformation is called *proper* if its determinant is 1. The group of proper, orthochronous Lorentz transformations is denoted $SO_+(1, 3)$.

We note that spatial reflections are excluded from $SO_+(1, 3)$, and so is any transformation that reverses the arrow of time. Thus this seems to be the most physically appropriate subgroup of $O(1, 3)$.

Alongside these restricted groups, we should also restrict the allowable inertial bases. We say that a basis is oriented and time-oriented if e_0 is future-pointing and (e_1, e_2, e_3) is right-handed. Then $SO_+(1, 3)$ maps any oriented and time-oriented basis to another such basis, and conversely any two such bases are related by an element of $SO_+(1, 3)$.

Remark 2.10.5. It is worth mentioning that X is future-pointing (timelike or null) if and only if $-X$ is past-pointing (timelike or null).

2.10.1. Causality in Special Relativity. If $\mathbb{M}$ is given a time-orientation, then the non-zero null vectors also fall into two distinct sets, the future-pointing and the past-pointing. A null vector N is future-pointing if $\eta(X, N) > 0$ for any given future-pointing timelike X. Similarly a null vector is past-pointing if $\eta(X, N) < 0$ for future-pointing timelike X. (Of course, if N is null future-pointing, then $\eta(Y, N) < 0$ if Y is timelike past-pointing.

A future-pointing null vector is in the boundary of the future-pointing nappe of the cone—it is a limiting case of future-pointing timelike vectors. For example, if we consider

$$X_t = e_0 + te_1,$$

where e_0 is future-pointing timelike and e_1 is spacelike (i.e. $\eta(e_1, e_1) = -1$), then

$$\eta(X_t, X_t) = 1 - t^2$$

is timelike future-pointing if $|t| < 1$ and null future-pointing if $t = \pm 1$.

Let E and F be two events in $\mathbb{M}$. The event E (for example an explosion or a light signal) can only have a causal effect on F if the displacement vector $\overrightarrow{EF}$ is timelike or null future-pointing. This condition guarantees that E and F can be connected by a particle travelling in a straight line at speed $\leqslant c = 1$ (the speed of light) *and* that F is in the future of E.

The set of all events F which can be affected causally effected by a given event E is the set

$$\mathrm{Fut}(E) = \{F \in \mathbb{M} \text{ such that } \overrightarrow{EF} \text{ is timelike or null future-pointing}\} \tag{2.10.1}$$

[1]Dictionary definition: In geometry, a nappe is half of a double cone

The set $\mathrm{Fut}(E)$ can be pictured as the solid half-cone whose boundary is the set of future-pointing null vectors emanating from E.

Similarly, the set of all events G which can affect or influence E is

$$\begin{aligned}\mathrm{Past}(E) &= \{G \in \mathbb{M} \text{ such that } \overrightarrow{GE} \text{ is timelike or null future-pointing}\} \\ &= \{G \in \mathbb{M} \text{ such that } \overrightarrow{EG} \text{ is timelike or null past-pointing}\} \qquad (2.10.2)\end{aligned}$$

This can be pictured as the solid half-cone whose boundary is the set of past-pointing null vectors emanating from E.

So although space and time are mixed up in the geometry of special relativity, there is still a well defined notion of causality.

We have defined timelike and null vectors. If a vector is not timelike or null, it is called spacelike:

DEFINITION 2.10.6. If $X \in M$, then X is called *spacelike* if $\eta(X, X) < 0$. Two events E and F in $\mathbb{M}$ are said to be *spacelike separated* if $\eta(\overrightarrow{EF}, \overrightarrow{EF}) < 0$.

We end by noting the following:

PROPOSITION 2.10.7. *Suppose that E and F are events such that $\overrightarrow{EF}$ is future-pointing timelike. Then there exist inertial coordinates such that E has coordinates* $(0, 0, 0, 0)$ *and F has coordinates* $(t, 0, 0, 0)$ *with $t > 0$.*

Suppose that E and F are spacelike separated events. Then there exists an inertial frame with respect to which E and F are simultaneous (e.g. E has coordinates $(0, 0, 0, 0)$ *and F has coordinates* $(0, d, 0, 0)$*. Moreover, there exist other coordinate systems in which E occurs before F.*

PROOF. The first follows from the basic fact that given if X is any future-pointing timelike vector, then there is an oriented and time-oriented basis (e_0, e_1, e_2, e_3) with respect to which η is diagonal, and such that X is a positive multiple of e_0. In such a basis, $\overrightarrow{EF} = (t, 0, 0, 0)$, where $t > 0$, and if we choose the origin so that the coordinates of E are $(0, 0, 0, 0)$, then the coordinates of F will be $(t, 0, 0, 0)$.

Similarly if $\eta(\overrightarrow{EF}, \overrightarrow{EF}) < 0$, we can pick a multiple e_1 of $\overrightarrow{EF}$ such that $\eta(e_1, e_1) = -1$. We extend this to a diagonalizing (oriented and time-oriented) basis of η, and then $\overrightarrow{EF}$ has the desired form. In particular, it is η-orthogonal to e_0 and so these events will be judged simultaneous by an observer with 4-velocity e_0.

For the last part, let $V = e_0 + \lambda e_1$. Then

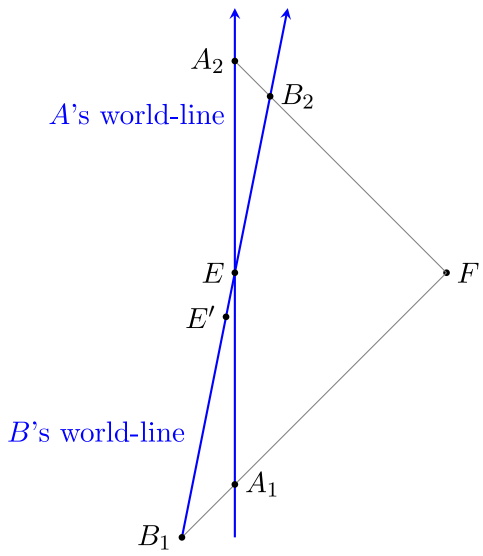
$$\eta(V, e_1) = -\lambda.$$

So if V is the 4-velocity vector of an observer, Bob, he will reckon that F happens after E if $\lambda < 0$ and that F happens before E if $\lambda > 0$. $\square$

REMARK 2.10.8. The above makes complete sense from the point of view of the radar method. See the picture below. Consider two inertial observers, Alice and Bob, and suppose that E is an event on both of their worldlines. If we suppose that Alice judges E and F to be simultaneous, this means that Alice bounces a light signal off F, then if she sends it out at τ_1 and receives it at τ_2, she assigns E time $(\tau_1 + \tau_2)/2$. In the diagram, Alice sends her light signal out at event A_1, and receives it at A_2, and E is the midpoint of the segment A_1A_2.

Now if Bob is heading towards F, it is clear that the light signal he needs to send to bounce of F has to be transmitted at event B_1 and received at event B_2. The event on his worldline that he judges to be simultaneous with F is therefore the midpoint of B_1B_2, shown as E'. It is clear from the geometry that the segment EB_1 is longer than EB_2, so E' will be, as shown, before E on his worldline.

Similarly, if B is heading away from F, the event he judges simultaneous with F will be later, on his worldline, then the event E.



2.10.2. Spatial and temporal components. We have seen that inertial observers, and free particles and photons have straight lines, and the basic feature of a worldline is the 4-velocity vector. It is often annoying to choose a full inertial basis to solve particular problems, but it is important to split Minkowski vectors into their *spatial* and *temporal* components with respect to a particular timelike vector.

Suppose that V is a timelike future-pointing 4-vector. Then we can write any vector X in terms of its components parallel to and η-orthogonal to V. That is

$$X = \lambda V + Y, \text{ where } \eta(V, Y) = 0. \tag{2.10.3}$$

Taking the scalar product with V,

$$\eta(V, X) = \lambda \eta(V, V) \text{ so } \lambda = \frac{\eta(V, X)}{\eta(V, V)}. \tag{2.10.4}$$

Then

$$Y = X - \frac{\eta(V, X)}{\eta(V, V)} V. \tag{2.10.5}$$

More concretely, relative to an inertial basis in which V is a positive multiple of e_0,

$$V = (V^0, \mathbf{0}), \quad X = (X^0, \boldsymbol{\xi}) \tag{2.10.6}$$

where

$$\eta(V, X) = V^0 X^0, \eta(V, V) = (V^0)^2 \tag{2.10.7}$$

and

$$Y = (0, \boldsymbol{\xi}). \tag{2.10.8}$$

Here $\mathbf{0}$ is the ordinary 3-dimensional zero-vector and $\boldsymbol{\xi}$ is also a euclidean 3-vector.

It is worth spelling out that if X and Z are any two Minkowski vectors with components

$$X = (X^0, \boldsymbol{\xi}), Z = (Z^0, \boldsymbol{\zeta})$$

then

$$\eta(X, Z) = X^0 Z^0 - \boldsymbol{\xi} \cdot \boldsymbol{\zeta}.$$

A photon's velocity 4-vector will take the form

$$\omega(1, \mathbf{e}) \tag{2.10.9}$$

where $\omega > 0$ (the photon is travelling forward in time) and $\mathbf{e}$ is a unit vector. For physical reasons, ω is identified with the frequency of the photon as measured by an observer with 4-velocity V.

EXAMPLE 2.10.9. Relative velocity. Suppose that Alice and Bob are inertial observers with 4-velocity vectors U and V, with $\eta(U,U) = \eta(V,V) = 1$.

In Alice's rest-frame,

$$U = (1, \mathbf{0}), V = \gamma(1, \mathbf{v}) \tag{2.10.10}$$

for some constant γ. This is, of course, the γ factor again, because the condition $\eta(V,V) = 1$ says

$$\gamma^2(1 - |\mathbf{v}|^2) = 1.$$

Thus $\mathbf{v}$ is the velocity vector of Bob as measured by Alice.

REMARK 2.10.10. What we've just seen is a **very useful** way of calculating γ-factors: if Alice and Bob are inertial observers with 4-velocities U and V, then the γ-factor of their relative speed is equal to $\eta(U,V)$.

We shall use this in the following:

EXAMPLE 2.10.11. Alice, Bob and Chris are inertial observers with 4-velocity vectors U, V, and W, respectively, so $\eta(U,U) = \eta(V,V) = \eta(W,W) = 1$. Suppose that Bob reckons that Chris's (relative) speed is w and Alice reckons Bob's (relative) speed is u. What does Alice reckon that Chris's speed (relative to her) is?

Call the unknown speed ζ. Then from the above, the γ-factor of ζ is $\eta(U,W)$,

$$\eta(U,W) = \gamma(\zeta). \tag{2.10.11}$$

What we know is that

$$\eta(U,V) = \gamma(u),\ \eta(V,W) = \gamma(w). \tag{2.10.12}$$

To get a handle on this it turns out to be simplest to work in B's rest-frame. Then

$$U = \gamma(u)(1, \mathbf{u}),\ \ V = (1, \mathbf{0}),\ \ W = \gamma(w)(1, \mathbf{w}) \tag{2.10.13}$$

where $\mathbf{u}$ is the velocity vector of Alice relative to Bob and $\mathbf{w}$ is the velocity vector of Chris relative to Bob.

Then

$$\gamma(\zeta) = \eta(U,W) = \gamma(u)\gamma(w)(1 - \mathbf{u}\cdot\mathbf{w}) \tag{2.10.14}$$

This is an answer, but it is instructive to rearrange it a bit. By squaring and taking the reciprocal,

$$1 - \zeta^2 = \frac{(1-u^2)(1-v^2)}{(1-\mathbf{u}\cdot\mathbf{w})^2} \tag{2.10.15}$$

Hence

$$\begin{aligned}
\zeta^2 &= \frac{(1-\mathbf{u}\cdot\mathbf{w})^2 - (1-u^2)(1-w^2)}{(1-\mathbf{u}\cdot\mathbf{w})^2} && (2.10.16)\\
&= \frac{1 - 2\mathbf{u}\cdot\mathbf{w} + (\mathbf{u}\cdot\mathbf{w})^2 - 1 + u^2 + w^2 - u^2w^2}{(1-\mathbf{u}\cdot\mathbf{w})^2} && (2.10.17)\\
&= \frac{(u^2 - 2\mathbf{u}\cdot\mathbf{w} + w^2) + \mathbf{u}\cdot\mathbf{w}^2 - u^2w^2}{(1-\mathbf{u}\cdot\mathbf{w})^2} && (2.10.18)\\
&= \frac{|\mathbf{u}-\mathbf{w}|^2 + \mathbf{u}\cdot\mathbf{w}^2 - u^2w^2}{(1-\mathbf{u}\cdot\mathbf{w})^2} && (2.10.19)
\end{aligned}$$

This remarkably complicated formula nonetheless reproduces the classical answer $|\mathbf{u}-\mathbf{w}|^2$ to a first approximation if u and w are much less than the light-speed $c = 1$.

2.11. Interstellar travel

Consider the following simplified model of a voyage made by a space-ship from earth. We will make it more complicated when we discuss uniform acceleration in the next Chapter. A spaceship sets off from the earth at constant speed v to travel to a star distance D from the earth. In the earth's frame of reference, the travel time is clearly

$$T = 2D/v. \tag{2.11.1}$$

Let E be the event 'spaceship leaves earth' let A be the event 'spaceship reaches destination' and let R be the event 'spaceship arrives back at earth'. We assume that the space ship travels at constant speed v on outward and return trip. Continuing the idealization, let's suppose that the earth is inertial, with 4-velocity U. Taking E to be the origin of $\mathbb{M}$, the world-line of the earth is then $\tau \mapsto \tau U$. On the outward leg, the spaceship's trajectory is $\sigma \mapsto \sigma V_1$ and on the inward leg it is $\mu \mapsto TV_1 + \mu V_2$, where TV_1 is the displacement vector from E to the 'arrival' event A.

We may choose the frame so $U = (1,0,0,0)$, $V_1 = \gamma(v)(1,v,0,0)$, $V_2 = \gamma(v)(1,-v,0,0)$. By geometry (see the picture below, in which the y and z directions have been suppressed)

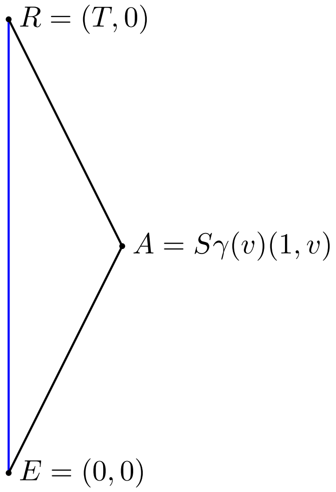
$$T(1,0) = S\gamma(v)(1,v) + S\gamma(v)(1,-v). \tag{2.11.2}$$

Hence $T = 2S\gamma(v)$ and so

$$2S = T\gamma(v) = \frac{2D}{v\gamma(v)}. \tag{2.11.3}$$

This quantity is the elapsed time from the point of view of the astronauts, as measured by their on-board clocks. As v approaches the light speed 1, so D/v approaches D and $1/\gamma(v) \to 0$. Thus $S \to 0$ as $v \to 1$.

The conclusion is that with a fast enough space ship, astronauts could complete a trip to a stars hundreds of light-years away within their life-spans. However, none of their friends would be alive when they got back to Earth...



2.12. Summary of key notation and definitions

Definition 2.12.1. If $X \in M$ is a vector, we say that X is
— *timelike* if $\eta(X,X) > 0$;
— *null* if $\eta(X,X) = 0$;
— *spaceelike* if $\eta(X,X) < 0$.

If X is the displacement vector $\overrightarrow{EF}$ from an event E to an event F in $\mathbb{M}$, we have the following corresponding definitions

Definition 2.12.2. Let E and F be events in M and let $X = \overrightarrow{EF}$ be the displacement vector. Then

— E and F are timelike separated if X is timelike;
— E and F are null separated if X is null;
— E and F are spacelike separated if X is spacelike.

We usually assume that M and hence $\mathbb{M}$ are given a time-orientation. Then for timelike and null vectors, we can say whether they are future or past-pointing. 4-velocity vectors of genuine particles are then taken to be future-pointing. If E and F are timelike or null separated and $\overrightarrow{EF}$ is future-pointing, then we assume that something happening at event E can have a causal influence on what happens at F: and only in this case. In particular, space-like separated events cannot causally affect each other (as faster-than-light travel would be needed). And also E cannot affect F even if they are timelike or null separated if $\overrightarrow{EF}$ is past-pointing.

A velocity 4-vector U is a timelike (future-pointing) vector with $\eta(U, U) = 1$. If Alice has velocity 4-vector U, then in her rest-frame, $U = (1, \mathbf{0})$ and the velocity 4-vector V of a particle is written $V = \gamma(v)(1, \mathbf{v})$ where $\mathbf{v}$ is the relative velocity of another particle as measured by Alice. In particular $\eta(U, V) = \gamma(v)$ gives the γ-factor of the relative velocity $\mathbf{v}$.

Remark 2.12.3. If you compare the way I have set out SR with what is in Woodhouse, you will notice that throughout his book, 4-vectors are denoted X^a, Y^a, V^a, etc., and that the Minkowski metric is denoted by g or g_{ab}. The point is that X^a is the set of components (X^0, X^1, X^2, X^3) in a basis (e_0, e_1, e_2, e_3) which is not necessarily explicitly mentioned. Thus my X corresponds to Woodhouse's X^a via

$$X = X^0 e_0 + X^1 e_1 + X^2 e_2 + X^3 e_3.$$

He also uses the summation convention over repeated upstairs and downstairs indices (which we'll come back to) and would write the RHS $X^a e_a$. I have avoided the use of the summation convention so far, though there will be no escape when we come to GR.

I have chosen η rather than g as notation for the Minkowski metric in order to reserve g as notation for the curved metrics that we'll use in GR.

CHAPTER 3

Further topics in Special Relativity

3.1. Non-inertial observers: acceleration

Particles acted upon by forces will not have straight worldlines in $\mathbb{M}$. We want to capture two key notions in the straight-line case: that of not travelling faster than the speed of light (which we continue to take to be 1 most of the time) and that of proper time parameter, namely the time as measured by a clock travelling along the worldline.

HYPOTHESIS 3.1.1. If $\Gamma(\tau)$ is the worldline of any massive particle, then τ is a proper time parameter along Γ if the velocity vector

$$V(\tau) = \frac{\mathrm{d}\Gamma}{\mathrm{d}\tau}$$

is timelike, future-pointing, and satisfies

$$\eta(V(\tau), V(\tau)) = 1$$

for all τ.

We also make an assumption about how such an accelerating observer will judge simultaneity:

HYPOTHESIS 3.1.2. If Alice is an observer with worldline $\tau \mapsto \Gamma(\tau)$ and E is an event on her worldline with parameter value τ_1, say, then an event F is judged by Alice to be simultaneous with E if $\eta(V(\tau_1), \overrightarrow{EF}) = 0$.

Another way of thinking about this is as follows. If Bob is an inertial observer with 4-velocity $V(\tau_1)$ then he will judge E and F to be simultaneous if $\eta(V(\tau_1), \overrightarrow{EF}) = 0$. So we are saying that Alice's notion of simultaneity at the event $E = \Gamma(\tau_1)$ should be the same as the notion of simultaneity of an inertial observer (Bob) who has the same 4-velocity as she does, at the event E.

From now on we shall use dot to denote differentiation with respect to τ. Thus

$$V(\tau) = \dot{\Gamma}(\tau). \tag{3.1.1}$$

The acceleration vector $A = \ddot{\Gamma} = \dot{V}$ is the second derivative of the parameterized curve. Note the following calculation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta(V, V) = 2\eta(V, \dot{V}) = 0 \tag{3.1.2}$$

The zero on the RHS is because $\eta(V(\tau), V(\tau))$ is constantly equal to 1!

Thus A is η-orthogonal to V and in particular is space-like. In particular in the frame in which V has components $(1, 0, 0, 0)$, A will have components $(0, \mathbf{a})$ and this acceleration is what the particle actually feels. So the *magnitude* of the acceleration felt is

$$a = \sqrt{-\eta(\ddot{\Gamma}(\tau), \ddot{\Gamma}(\tau))}$$

EXAMPLE 3.1.3. **Constant acceleration in a plane** Consider an accelerating particle in a plane, which we may as well take to be the (t, x) plane. Then $\Gamma(\tau)$ has the form

$$(t(\tau), x(\tau)). \tag{3.1.3}$$

The proper time condition is

$$\dot{t}^2 - \dot{x}^2 = 1 \tag{3.1.4}$$

Constant acceleration is the condition

$$\ddot{t}^2 - \ddot{x}^2 = -a^2 \tag{3.1.5}$$

where a is a constant. A trick to solve this is to suppose

$$\dot{t} = \cosh u(\tau), \dot{x} = \sinh u(\tau). \tag{3.1.6}$$

Then

$$\ddot{t} = \dot{u}\sinh u, \ddot{t}\dot{u}\cosh u. \tag{3.1.7}$$

Substituting in to (3.1.5)

$$\dot{u}^2 = a^2 \tag{3.1.8}$$

So $u = a(\tau - \tau_0)$ (if $a > 0$, so the curve is future-pointing). Integrating the equations

$$\dot{t} = \cosh a\tau, \quad \dot{x} = \sinh a\tau, \tag{3.1.9}$$

yields

$$t(\tau) = \frac{1}{a}\sinh a\tau, \quad x(\tau) = \frac{1}{a}(\cosh a\tau - 1) \tag{3.1.10}$$

choosing the constants of integration so that the particle is at $(t, x) = (0, 0)$ when $\tau = 0$.

3.1.1. Interstellar travel revisited. Suppose that a spaceship starts from rest and its engines deliver uniform acceleration a. What happens?

One thing is that the velocity remains below that of light, as it must. For large τ,

$$t(\tau) \sim \frac{1}{2a}e^{a\tau}, x(\tau) \sim \frac{1}{2a}e^{a\tau} \tag{3.1.11}$$

which is a parameterization (not by proper time) of the null ray $t = x$.

[The trajectory is a hyperbola.]

The relation $t = \cosh a\tau$ relates the time τ which elapses on board the ship, compared with that t measured by clocks left behind on earth.

To reach distance D, you have to solve $D = \frac{1}{a}(\cosh a\tau - 1)$. If D is reasonably large, $\sinh a\tau \simeq \exp(a\tau)/2$ so

$$\tau = \frac{1}{a}\log(2D). \tag{3.1.12}$$

This logarithmic relationship means that in principle, with modest accelerations from rest, a uniformly accelerating spaceship can cover interstellar distances in reasonable times (as measured by the astronauts). For example, suppose that we measure distance in light-years and time in years. There are 10^{16} metres in a light-year and 3×10^7 seconds in a year.

So the acceleration due to gravity, $10\text{m}s^{-2}$, is equal to $10 \times 10^{-16} \times (3 \times 10^7)^2$ light-years per year^2. This is (miraculously) approximately 1. So according to the above, a spaceship accelerating so that the astronauts would feel a earth's gravity on board would cover distance D light-years in τ years, where $\tau \sim \log(2D)$, if D is reasonably large. If $D = 100$, then $\tau = \log(200) \simeq 5.3$ years.

3.1.2. Relativistic motion in a circle. (Cf. Woodhouse, SR, p. 111).

Suppose that a particle moves on a circle

$$x(t) = R\cos\omega t, y(t) = R\sin\omega t, z(t) = 0.$$

In other words,

$$\Gamma(t) = (t, R\cos\omega t, R\sin\omega t, 0)$$

We do not claim t is proper time, and the first thing to work out is the relation between t and τ. We have

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = (1, -R\omega\sin\omega t, R\omega\cos\omega t, 0)$$

which has Minkowski length-squared equal to $1 - R^2\omega^2$. Thus

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \sqrt{\eta(\mathrm{d}\Gamma/\mathrm{d}t, \mathrm{d}\Gamma/\mathrm{d}t)} = \sqrt{1 - R^2\omega^2}.$$

Hence the 4-velocity vector of the particle is

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\tau} = \frac{\mathrm{d}\Gamma}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{\sqrt{1-R^2\omega^2}}(1, -\omega R\sin\omega t, \omega R\cos\omega t, 0).$$

Then the *acceleration* is

$$\frac{\mathrm{d}^2\Gamma}{\mathrm{d}\tau^2} == \frac{1}{1-R^2\omega^2}(0, -\omega^2 R\cos\omega t, -\omega^2 R\sin\omega t, 0).$$

The Minkowski length-squared of this is

$$-\frac{\omega^4 R^2}{(1-R^2\omega^2)}$$

and so the acceleration felt by the particle is

$$a = \frac{\omega^2 R}{1-R^2\omega^2}$$

This is larger than the non-relativistic value by the factor $1/(1-R^2\omega^2)$.

3.2. Momentum and energy: $E = mc^2$

In this section we shall consider collisions between particles (including photons) in special relativity.

We have to make some physical assumptions. We assume that massive particles have a well-defined mass (the rest-mass). If a particle has 4-velocity V, with $\eta(V, V) = 1$, then the 4-momentum of the particle is defined to be

$$P = mV. \tag{3.2.1}$$

Suppose we have k particles with 4-momenta $P_1, \dots, P_k$ which interact (for example in the LHC) and after the interaction there are m outgoing particles with momenta $Q_1, \dots, Q_m$. The basic assumption is the *conservation of total* 4*-momentum*

$$Q_1 + \cdots + Q_m = P_1 + \cdots + P_k. \tag{3.2.2}$$

If Alice is an inertial observer with 4-velocity U, and we have a particle of rest-mass m and with 4-velocity $P = mV$, then Alice can look at the spatial and temporal parts of P. If she measures the velocity of the particle in her rest frame as $\mathbf{v}$, then

$$P = m\gamma(v)(1, \mathbf{v}) \tag{3.2.3}$$

where as usual

$$\gamma(v) = \frac{1}{\sqrt{1-v^2}},\ v = |\mathbf{v}|. \tag{3.2.4}$$

Expanding this using the binomial expansion for small v,

$$\gamma(v) = 1 + \frac{v^2}{2} + O(v^4)$$

we see that

$$P \simeq m(1 + v^2/2)(1, \mathbf{v}) = (m + mv^2/2, m\mathbf{v}) + O(v^3).$$

It is good to restore c here, in which case we'd get

$$P \simeq (mc^2 + mv^2/2, m\mathbf{v}) + O(v^3/c).$$

Now the term $mv^2/2$ appearing here is the classical kinetic energy of a particle of mass m and moving at speed v. The spatial component $m\mathbf{v}$ is just the classical momentum.

Einstein's conclusion from these considerations was the 'equivalence' of mass and energy: that a particle of rest-mass m should have total energy mc^2: an observer with 4-velocity U assigns total energy $\eta(U, P)$ to a particle with 4-momentum P. The very surprising conclusion is that a free particle of mass m has to be assigned energy mc^2 by an inertial observer for whom the particle is at rest.

EXAMPLE 3.2.1. **Elastic collisions** A particle of rest mass m, velocity $\mathbf{u}$ relative to an inertial frame, collides with a particle of rest mass m, which is at rest. After the collision, the velocities of the particles are $\mathbf{v}$ and $\mathbf{w}$. If θ is the angle between $\mathbf{v}$ and $\mathbf{w}$, show that

$$\cos\theta = \frac{1}{vw}(1 - 1/\gamma(v))(1 - 1/\gamma(w)). \tag{3.2.5}$$

Let the incoming momenta be $P_1 = mU_1, P_2 = mU_2$, and the outgoing momenta be $Q_1 = mV_1$ and $Q_2 = mV_2$. In the rest frame of the at-rest particle, we have

$$P_1 = m\gamma(u)(1, \mathbf{u}), P_2 = (m, \mathbf{0}), Q_1 = m\gamma(v)(1, \mathbf{v}), Q_2 = m\gamma(w)(1, \mathbf{w}). \tag{3.2.6}$$

Conservation of momentum says

$$P_1 + P_2 = Q_1 + Q_2. \tag{3.2.7}$$

We can get some information by taking the η-square of each side,

$$\eta(P_1 + P_2, P_1 + P_2) = \eta(Q_1 + Q_2, Q_1 + Q_2) \tag{3.2.8}$$

which yields

$$2m^2 + 2\eta(P_1, P_2) = 2m^2 + 2\eta(Q_1, Q_2) \tag{3.2.9}$$

since $\eta(P_1, P_1) = \eta(P_2, P_2) = \eta(Q_1, Q_1) = \eta(Q_2, Q_2) = m^2$. Using (3.2.6) to compute the cross-terms in (3.2.9), we get

$$m^2\gamma(u) = m^2\gamma(v)\gamma(w)(1 - \mathbf{v}\cdot\mathbf{w}). \tag{3.2.10}$$

This is useful because $\mathbf{v}\cdot\mathbf{w} = vw\cos\theta$, and $\cos\theta$ is what we are looking for. On the other hand, we need to eliminate $\gamma(u)$. This can be done by taking the scalar product with P_2, or, in down-to-earth terms, just by inspecting the temporal component of the conservation equation (3.2.7), which gives

$$\gamma(u) + 1 = \gamma(v) + \gamma(w) \tag{3.2.11}$$

Combining this with (3.2.10) gives

$$1 - vw\cos\theta = \frac{\gamma(v) + \gamma(w) - 1}{\gamma(v)\gamma(w)} \tag{3.2.12}$$

so

$$\begin{aligned}
vw\cos\theta &= 1 - \frac{\gamma(v) + \gamma(w) - 1}{\gamma(v)\gamma(w)} \\
vw\cos\theta &= \frac{\gamma(v)\gamma(w) - \gamma(v) - \gamma(w) + 1}{\gamma(v)\gamma(w)} \\
&= \frac{(\gamma(v) - 1)(\gamma(w) - 1)}{\gamma(v)\gamma(w)} \\
&= (1 - 1/\gamma(v))(1 - 1/\gamma(w)).
\end{aligned} \tag{3.2.13}$$

This is the result. Notice that $v\gamma(v) = v/\sqrt{1 - v^2} = \sqrt{\gamma^2(v) - 1}$, so a nice way of writing this is

$$\begin{aligned}
\cos\theta &= \frac{(\gamma(v) - 1)(\gamma(w) - 1)}{\sqrt{\gamma(v)^2 - 1}\sqrt{\gamma(w)^2 - 1}} \\
&= \sqrt{\frac{\gamma(v) - 1}{\gamma(v) + 1}}\sqrt{\frac{\gamma(w) - 1}{\gamma(w) + 1}}
\end{aligned} \tag{3.2.14}$$

If v and w are small,

$$\gamma(v) - 1 \simeq v^2/2, \ \gamma(v) + 1 \simeq 2$$

and so $\cos\theta \simeq 0$. The fact that the outgoing trajectories are at $90°$ is a standard consequence of conservatio of energy in newtonian mechanics.

Relativistically, however, $\gamma(v) > 1$ and $\gamma(w) > 1$, so $\cos\theta$ is strictly less than $90°$. Such trajectories are observed in high energy particle interactions (e.g. in the LHC) and provide confirmation of special relativity (or more precisely of the conservation of 4-momentum).

3.3. Momentum of photons

Photons are massless, but they do have momentum. To any photon is associated a null vector, K, say. If Alice is an inertial observer with 4-velocity U, then in Alice's rest frame, we have

$$L = \omega(1, \mathbf{e}), \tag{3.3.1}$$

where $\mathbf{e}$ is a unit vector. A natural assumption (by considering solutions of the wave equation) is that ω is the (angular) frequency of the photon, as measured by Alice. We make this assumption without further justification. We then assume that the momentum of the photon is

$$\hbar L \tag{3.3.2}$$

where $\hbar \simeq 1.05 \times 10^{-34} Js$ is Planck's constant.

By way of partial justification (at least if you know a tiny bit of quantum mechanics): recall that the temporal component (relative to Alice's frame) of the 4-momentum of a massive particle is the energy of the particle as measured by Alice. Thus a reasonable requirement of the 4-momentum of a photon is that it should be a multiple of its 4-velocity, such that its temporal component is the energy that Alice would measure. With K as in (3.3.1) the temporal component of (3.3.2) is $\hbar\omega$. That the energy of a photon is given by $E = \hbar\omega$ is a basic principle of quantum mechanics.

It is now natural to assume that 4-momentum is also conserved in collisions involving photons.

Example 3.3.1. A photon with frequency ω collides with an electron at rest in an inertial frame. After the collision, the frequency of the electron is ω'. Obtain a relation between the scattering angle of the photon, the frequencies and the rest-mass of the electron.

The initial momenta (in the electron's rest frame) are

$$P_1 = \hbar\omega(1, \mathbf{e}), P_2 = (m, \mathbf{0}) \tag{3.3.3}$$

The final momenta are

$$Q_1 = \hbar\omega'(1, \mathbf{e}'), Q_2 = m\gamma(v)(1, \mathbf{v}). \tag{3.3.4}$$

The momentum conservation equation is

$$P_1 + P_2 = Q_1 + Q_2. \tag{3.3.5}$$

Since we want to know $\cos\theta$, we want to square (3.3.5) in such a way as to get $\mathbf{e} \cdot \mathbf{e}'$ as a cross term. For this purpose we rearrange it so that both photon momenta are on the same side of the equation:

$$P_1 - Q_1 = Q_2 - P_2, \tag{3.3.6}$$

which gives

$$\eta(P_1 - Q_1, P_1 - Q_1) = -2\eta(Q_1, P_1) = \eta(Q_2 - P_2, Q_2 - P_2) = 2m^2 - 2m^2\gamma(v). \tag{3.3.7}$$

Simplifying,

$$\hbar^2\omega\omega'(1 - \cos\theta) = m^2(\gamma(v) - 1). \tag{3.3.8}$$

Looking at the temporal component of (3.3.5), we find

$$\hbar\omega + m = m\gamma(v) + \hbar\omega' \Rightarrow m(\gamma(v) - 1)) = \hbar(\omega - \omega') \tag{3.3.9}$$

so finally

$$\hbar\omega\omega'(1 - \cos\theta) = m(\omega - \omega') \tag{3.3.10}$$

Remark 3.3.2. This process is known as *Compton scattering*.