Chapter 2 - Macroscopic Fields

Maxwells equations have two major variants: the *microscopic* set of Maxwells equations uses total charge and total current including the difficult-to-calculate atomic level charges and currents in materials. The *macroscopic* set of Maxwells equations defines two new auxiliary fields that can sidestep having to know these atomic sized charges and currents. Unlike the microscopic equations, "Maxwells macroscopic equations", also known as Maxwells equations in matter, factor out the bound charge and current to obtain equations that depend only on the free charges and currents. These equations are more similar to those that Maxwell himself introduced. The cost of this factorization is that additional fields need to be defined: the displacement field **D**0 which is defined in terms of the electric field **E** and the polarization **P** of the material, and the magnetic-**H** field, which is defined in terms of the magnetic-**B** field and the magnetization **M** of the material. In this chapter, we will look at these macroscopic fields, **D** and **H**.

2.1 PHAS 2201 - Year 2 Electromagnetism recall

We begin this section of the course by going over electrostatic concepts which should be very familiar from PHAS2201, including Gauss law and the effect of dielectrics on capacitance.

Electrostatics

- We start with a single charge, q, at $\mathbf{r}'$:

$$\mathbf{E}(\mathbf{r}) = q(\mathbf{r} - \mathbf{r}')/(4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|^3) \tag{1}$$

- Taking a surface integral gives $\oint_S \mathbf{E} \cdot \mathbf{n} da = q/\epsilon_0$
- Increasing the number of charges, and using the principle of superposition, we get:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \int \rho dv/\epsilon_0 \tag{2}$$

- This leads directly to Gauss' law: $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$

Dielectrics

A dielectric is an electrical **insulator** that can be polarized by an applied electric field. When a dielectric is placed in an electric field, electric charges do not flow through the material, as in a conductor, but only slightly shift from their average equilibrium positions causing dielectric polarization. Because of dielectric polarization, positive charges are displaced toward the field and negative charges shift in the opposite direction. This creates an internal electric field which reduces the overall field within the dielectric itself.

- Recall that capacitance is defined by $Q = C\Delta V$

- Capacitance *changes* when a dielectric is added:

$$C_{dielectric} = \kappa C_{vacuum} \tag{3}$$

- A dielectric has no free charges: an insulator
- The polarization is $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ (and is defined as dipole moment per unit volume)
- This gives the susceptibility, χ_e
- The dielectric constant is $\kappa = 1 + \chi_e$

Polarization reflects the fact that the atoms which make up the dielectric consist of separate positive (nucleus) and negative (electrons) charges. These respond differently to the electric field, leading to a *shift* in the overall charge distribution of the dielectric, while keeping it neutral. We will consider the *microscopic* origin of polarization in detail in next section of the course.

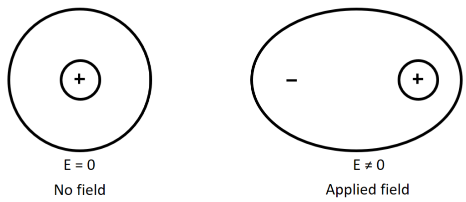

Figure 1: Figure 2.1: Electronic polarization occurs due to displacement of the centre of the negatively charged electron cloud relative to the positive nucleus of an atom by the electric field.

2.2 Electric Field in Dielectric Media

- We want to develop a theory for electric fields in the presence of polarized media
- We will start by considering the field *outside* a piece of polarized dielectric
- This will introduce the ideas of *polarization* charge densities
- Then we will move onto the field *inside* a piece of polarized dielectric
- We will find a useful reformulation of Gauss' Law

We start by finding the potential at a point $\mathbf{r}$ due to a small volume of polarized material at a point $\mathbf{r}'$. We will then integrate this over the entire piece of dielectric material. First, note that the potential at $\mathbf{r}$ due to a dipole at $\mathbf{r}'$ is:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{4}$$

Recall that $\mathbf{p} = q\mathbf{d} and that \mathbf{P} = \mathbf{p}/\delta v$. Then we use the fact that the polarization is the dipole moment per unit volume to write:

$$\Delta\phi(\mathbf{r}) = \frac{\boldsymbol{\Delta} v'\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3} \tag{5}$$

When we take the limit $\Delta v \to 0$ and sum over the elements, we get an expression for the total potential:

$$\phi(\mathbf{r}) = \int_V \frac{dv'\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3} \tag{6}$$

We use the gradient of $1/|\mathbf{r}-\mathbf{r}'|$, derived as (worth remembering!):

$$\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \tag{7}$$

to transform this:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv' \tag{8}$$

Using the formula for $\nabla\cdot(\phi\mathbf{F})$ from the Mathematical Identities,

$$\nabla\cdot(\phi\mathbf{F}) = (\nabla\phi)\cdot\mathbf{F} + \phi\nabla\cdot\mathbf{F} \tag{9}$$

and rearranging (we want $\mathbf{F}\cdot\nabla\phi$) we can write, with $\mathbf{F} = \mathbf{P}(\mathbf{r}')$ and $\phi = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_V \left[\nabla\cdot\left(\frac{\mathbf{P}(\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla\cdot\mathbf{P}(\mathbf{r}')\right]dv' \tag{10}$$

Finally, we use the divergence theorem on the first term $[\int_V \nabla\cdot\mathbf{F}dv = \oint_S \mathbf{F}\cdot\mathbf{n}da]$, to give the potential outside a polarized dielectric object:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\oint_S \frac{\mathbf{P}(\mathbf{r}')\cdot\mathbf{n}}{|\mathbf{r}-\mathbf{r}'|}da' + \frac{1}{4\pi\epsilon_0}\int_V \frac{-\nabla\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv' \tag{11}$$

- The *surface* polarization charge density is defined:

$$\sigma_P = \mathbf{P}\cdot\mathbf{n} \tag{12}$$

- The *volume* polarization charge density is defined:

$$\rho_P = -\nabla \cdot \mathbf{P} \tag{13}$$

- We can write the potential as:

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\oint_S \frac{\sigma_P}{|\mathbf{r}-\mathbf{r}'|}da' + \int_V \frac{\rho_P}{|\mathbf{r}-\mathbf{r}'|}dv'\right) \tag{14a}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{dq_P}{|\mathbf{r}-\mathbf{r}'|} \tag{14b}$$

For uniform polarization, $\nabla \cdot \mathbf{P} = 0$, so there is no bound charge within the material, but there will be bound charge on the surface.

Bound charge: The charge within a material that is unable to move freely through the material. Small displacements of bound charge are responsible for polarization of a material by an electric field.

Free charge: The charge in a conducting material associated with the conduction electrons that are free to move throughout the material. These electrons can carry electric current.

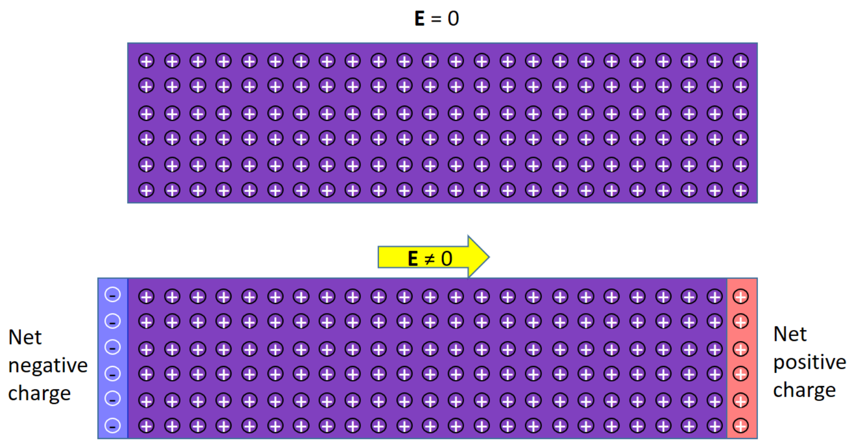

Figure 2: Origin of surface charge density due to polarization.

We have considered the field due to a polarized dielectric, but only *outside* the dielectric. What is the field *inside* a polarized dielectric?

- Consider three (small) charged conductors embedded in a dielectric
- They have charges q_1, q_2 and q_3 (sum to Q)
- Now use Gauss Law:

$$\oint_S \mathbf{E} \cdot \mathbf{n} da = \frac{1}{\epsilon_0}(Q + Q_P) \tag{15}$$

We start by noting that:

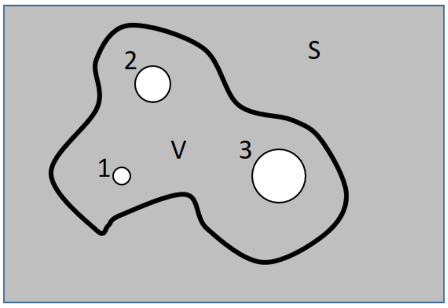

Figure 3: Sketch of three conductors embedded in a dielectric.

$$Q_P = \int_{S_1+S_2+S_3} \mathbf{P}\cdot\mathbf{n}da + \int_V -\nabla\cdot\mathbf{P}dv \tag{16}$$

It is important to realize that the *arbitrary* bounding surface S does not enter into this integral because there is no polarization charge density on it (it is not a real surface). We use the divergence theorem $[\int_V \nabla\cdot\mathbf{F}dv = \oint_S \mathbf{F}\cdot\mathbf{n}da]$ to transform the second integral into a surface integral. But we must take care: this time, we must include the surface S because it bounds the volume V. It is also important to understand the directions of the surface normals. Explicitly, this gives:

$$\begin{aligned} Q_P &= \int_{S_1+S_2+S_3} \mathbf{P}\cdot\mathbf{n}da - \oint_S \mathbf{P}\cdot\mathbf{n}da - \int_{S_1+S_2+S_3} \mathbf{P}\cdot\mathbf{n}da && \text{(17a)}\\ &= -\oint_S \mathbf{P}\cdot\mathbf{n}da && \text{(17b)}\end{aligned}$$

Now we can use this is in Gauss law inside the dielectric, which was given as Eq. 15:

$$\oint_S \mathbf{E}\cdot\mathbf{n}da = \frac{1}{\epsilon_0}Q - \frac{1}{\epsilon_0}\oint_S \mathbf{P}\cdot\mathbf{n}da \tag{18}$$

After a little manipulation, we can rewrite this in terms of the free or external charge, Q.

- Using the divergence theorem yet again, we find that:

$$\begin{aligned} Q &= \oint_S (\epsilon_0\mathbf{E}+\mathbf{P})\cdot\mathbf{n}da && \text{(19a)}\\ \mathbf{D} &= \epsilon_0\mathbf{E}+\mathbf{P} && \text{(19b)}\end{aligned}$$

  becomes

$$\int_V \rho(v)dv = \int \nabla\cdot\mathbf{D}dv \tag{20}$$

- The *electric displacement* $\mathbf{D}$ is the field whose divergence is the free (or external) charge density

- So, if we consider a charge density, and use the divergence theorem, we get:

Divergence of D

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho(\boldsymbol{r}) \tag{21}$$

External charge

- We have talked about *free* or *external* charge (as opposed to the bound charge)
- With a dielectric, the difference is clear
- Charge added from outside (*external* charge) is different to *polarization* charge
- But it is *not* free to move
- For a conductor, charge is free to move around
- It is important to be aware of the difference between charge added and charge already present
- In general, the polarization $\mathbf{P}$ is a function of the material and the external field $\mathbf{E}$
- We write $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ in *linear, isotropic, homogeneous* media
- In these media, as χ_e (the electric susceptibility) is constant:

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon \mathbf{E} \tag{22}$$

- We call $\epsilon = \epsilon_0(1 + \chi_e)$ the permittivity, and ϵ/ϵ_0 the *relative* permittivity or *dielectric constant*
- Linear: $\mathbf{P}$ depends linearly on $\mathbf{E}$
- Homogeneous: χ_e does not vary with position
- Isotropic: $\mathbf{P}$ and $\mathbf{E}$ are parallel

[Non-examinable material] [It is important to realize that a sufficiently strong electric field can break apart the charges in a material which form the microscopic dipoles. At this point, called dielectric breakdown, all approximations discussed to this point are invalid. For air, whose dielectric constant is 1.0006, the maximum field sustainable without breakdown is around $3 \times 10^6 V/m$. The reason that we refer to an isotropic dielectric for the relation $\mathbf{P} = \epsilon_0 \chi_e(\mathbf{E})\mathbf{E}$ is that it implies that the polarization has the same direction as the external field. This is a good approximation for most media, but it is necessary in some media to replace this with a tensor relationship, where the two vectors are not in the same direction. This type of behaviour is more common in magnetic materials, which we will come to.]

Energy density

- What is the energy density of an electric field?
- We will consider this in two ways:
  - Charge flowing into a capacitor;
  - Adding a small charge to a field.
- The final result is the same:

Energy density of an Electric Field

$$u = \frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E} \tag{23}$$

Considering a capacitor first, we assume that it is in the process of being charged. If we start with the expression for power (which is rate of change of energy with time) for a current $I(t)$ flowing at a voltage $V(t)$ at time $t, P(t) = V(t)I(t)$.

Then the energy is:

$$W = \int P(t)dt = \int V(t)I(t)dt = \int \frac{Q(t)}{C}\frac{dQ}{dt}dt = \frac{1}{2}\frac{Q^2}{C} \tag{24}$$

For a parallel plate capacitor with plates of area A separated by a distance d, we know that the capacitance is given by $C = \frac{\epsilon A}{d}$. Using $V = Q/C$, we find that the electric field can be written:

$$E = \frac{V}{d} = \frac{Q}{Cd} = \frac{Qd}{\epsilon Ad} = \frac{Q}{\epsilon A} \tag{25}$$

Of course, as $D = \epsilon E$, we find that $D = \frac{Q}{A}$. So the energy density is given by:

$$u = \frac{W}{Ad} = \frac{1}{2}\frac{Q^2}{CAd} \tag{26a}$$

$$= \frac{1}{2}\frac{Q^2}{\epsilon A^2} \tag{26b}$$

$$= \frac{1}{2}\mathbf{D}\cdot\mathbf{E} \tag{26c}$$

Another (more general) way to reach the same formula is to consider the work done bringing a charge from infinity to the point where the energy density is required. We know that the energy of a point charge, q, in a potential ϕ is $W = q\phi$. This can be generalized for a charge distribution given by the charge density $\rho(\mathbf{r})$:

$$W = \int_V \rho\phi dv \tag{27}$$

Now, what would be change in electrostatic energy when adding a small amount of charge, $\delta\rho$? We use our recent result for Gauss theorem, $\nabla \cdot \mathbf{D} = \rho$:

$$\delta W = \int_V \delta\rho\phi dv \tag{28a}$$
$$\delta\rho = \nabla\cdot\delta\mathbf{D} \tag{28b}$$
$$\phi(\nabla \cdot \mathbf{D}) = \nabla\cdot(\phi\mathbf{D}) - \mathbf{D}\cdot(\nabla\phi) \tag{28c}$$
$$\delta W = \int_V \nabla\cdot(\phi\delta\mathbf{D}) - \delta\mathbf{D} \cdot \nabla\phi dv \tag{28d}$$
$$\delta W = \int_S \phi\delta\mathbf{D} \cdot \mathbf{n} da - \int_V \delta\mathbf{D} \cdot \nabla\phi dv \tag{28e}$$

where we have used the divergence theorem on the first part of the integral in the final line. But we know that $\mathbf{E} = -\nabla\phi$, and we can notice that the first term will fall off rapidly with distance ($\mathbf{D}$ with $1/r^2$ and ϕ with $1/r$). This means that we can write overall, as the volume being integrated tends to infinity:

$$\delta W = \int_V \delta\mathbf{D} \cdot \mathbf{E} dv \tag{29}$$

Now, if we assume a linear, dielectric medium, we know that $\mathbf{D} = \epsilon\mathbf{E}$, and we can integrate over the field going from 0 to $\mathbf{D}$:

$$\delta W = \int_0^D \delta W = \int_0^D \int_V \delta\mathbf{D} \cdot \mathbf{E} dv \tag{30}$$

We can write:

$$W = \frac{1}{2} \int_0^E \int_V \epsilon\delta(E^2) dv = \frac{1}{2} \int_V \epsilon E^2 dv \tag{31}$$

This of course gives us the result we derived above, namely $u = \mathbf{E} \cdot \mathbf{D}/2$.

3 Magnetic Field Revision

An important point to note as we start the area of magnetic fields is that this is where the essential link between electric fields and magnetic fields (leading to the unified area of electromagnetism) becomes apparent. Thus far we have considered electrostatics only.

- The magnetic field at $\mathbf{r}_2$ due to a circuit at $\mathbf{r}_1$, in both integral and differential forms:

Biot-Savart Field Law

$$\mathbf{B}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} I_1 \oint_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} \tag{32}$$

$$d\mathbf{B}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} I_1 \frac{d\mathbf{l}_1 \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} \tag{33}$$

- Note that this is empiricially derived.
- For a current density, we find:

$$\mathbf{B}(\mathbf{r}_2) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}_1) \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} dv_1 \tag{34}$$

- This implies that $\nabla_2 \cdot \mathbf{B} = 0$, which indicates a lack of magnetic monopoles.

We can show the last statement using the mathematical identity for $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - (\nabla \times \mathbf{G}) \cdot \mathbf{F}$:

$$\nabla_2 \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int_V \nabla_2 \cdot \frac{\mathbf{J}(\mathbf{r}_1) \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} dv_1 \tag{35a}$$

$$= \frac{\mu_0}{4\pi} \int_V -\mathbf{J}(\mathbf{r}_1) \cdot \left(\nabla_2 \times \frac{\mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} \right) dv_1 \tag{35b}$$

where, since we are taking the divergence at point $\mathbf{r}_2$, the term involving $\nabla_2 \times \mathbf{J}(\mathbf{r}_1)$ is zero. But now we can use two identities:

1. $\nabla(1/r_{12}) = \mathbf{r}_{12}/|r_{12}|^3$
2. $\nabla \times (\nabla \phi) = 0$

This shows that the integral on the right-hand size of equation 35a is zero, and hence there are no magnetic monopoles (though note that we started from just this assumption: that the magnetic field arises from the line integral around a circuit!).

- The original, integral form of Ampres Law is:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{36}$$

  where the current is that flowing through the area enclosed by the path.

- The differential form comes from writing $I = \int_S \mathbf{J} \cdot \mathbf{n} da$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{37}$$

- But we have to account for time-varying $\mathbf{E}$:

Ampere-Maxwell Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{38}$$

We can understand why this is incomplete by considering a capacitor being charged with a constant current, I. Using Ampres law (in original form) we see:

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da \tag{39}$$

Now consider a loop, C, around the wire leading to one plate of the capacitor, and two different surfaces, as shown in **??**:

1. A surface cutting the wire
2. A surface passing between the plates of the capacitor, and not cutting the wire

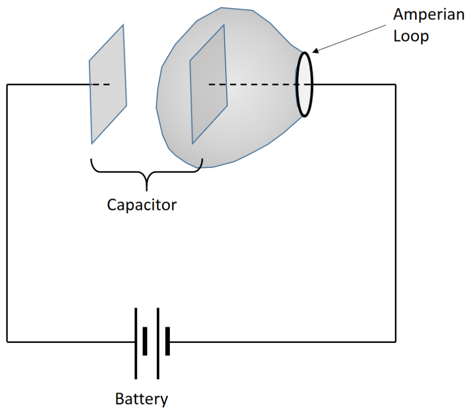

Figure 4: Amperian loops on a charging capacitor.

It is clear that these will give two different answers for the integral over the current density: in the first, the answer will be I, and in the second it will be zero. This is clearly wrong, as Ampres law insists that the choice of surface be arbitrary. The resolution to the problem, using the continuity equation, will be considered later, in Chapter 5, on Maxwells Equations.

Faraday's Law

- Electromotive force (emf) is equivalent to a potential difference
- Often encountered in terms of circuits, with inductance
- Around a circuit, the emf, ℓ, is defined by:

$$\mathcal{E} = \oint_C \mathbf{E} \cdot d\mathbf{l} \tag{40}$$

- Faradays Law (integral form):

$$\mathcal{E} = -\frac{d\Phi}{dt} \tag{41}$$

We define the magnetic flux, Φ, as:

$$\Phi = \int_S \mathbf{B} \cdot \mathbf{n} da \tag{42}$$

in other words the magnetic field crossing a surface. Now, using the definition of emf we can relate the electric field to the derivative of the magnetic field:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} da \tag{43}$$

Provided that the circuit being considered does not change with time, we can take the time derivative inside the integral. We can also use Stokes theorem [$\oint C \mathbf{F} \cdot d\mathbf{l} = \int S \nabla \times \mathbf{F} \cdot \mathbf{n} da$] on the line integral of $\mathbf{E}$ to obtain the surface integral of $\nabla \times \mathbf{E}$:

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} da = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da \tag{44}$$

Since this must be true for all fixed surfaces S, we find:

The differential form of Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{45}$$

- When the magnetic field is static, this reduces to the conservative field $\mathbf{E}, \nabla \times \mathbf{E} = 0$
- Notice the minus sign: Lenzs law states that any induced magnetic field opposes the change in flux that induced it

4 Magnetic Vector Potential

The solution of many electrostatic problems is made easier by working in terms of the potential rather than the electric field directly. The same idea can be applied to the magnetic field, though the eventual solution is rather more complex.

- Since $\nabla \times \nabla \phi = 0$ we know that we can write $\mathbf{E} = -\nabla \phi$when $\partial \mathbf{B} / \partial t = 0$
- Similarly, we know that $\nabla \cdot \mathbf{B} = 0$
- The relevant identity here is $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

- We can then write generally:

The Magnetic Vector Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{46}$$

where $\mathbf{A}$ is the vector potential

When we consider the form of the vector potential, it should be immediately apparent (by analogy with the electric field as gradient of the potential) that there is a freedom in choosing it:

$$\mathbf{A}' \rightarrow \mathbf{A} + \nabla f \tag{47}$$

for any scalar function f results in the same $\mathbf{B}$ field since $\nabla\times(\nabla f) = 0$. This invariance under a transformation is called gauge invariance. It should not be surprising: the electrostatic potential, ϕ, is not defined up to an arbitrary additive constant (and all potentials are actually *potential differences.*

There are different ways of choosing the vector potential which help with different situations. Consider a situation where the electric field does not change with time. Then we write Ampres Law as:

$$\nabla \times \mathbf{B} = \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J} \tag{48a}$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} \tag{48b}$$

- The Coulomb gauge is: $\nabla \cdot \mathbf{A} = 0$
- It leads to the following expression for the vector potential: $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$
- By analogy with Poissons equation, $\nabla^2 V = -\rho/\epsilon_0$, we can write:

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \int_V \frac{\mathbf{J}(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} d\mathbf{r}_2 \tag{49}$$

- The current density determines the vector potential
- There are other choices of gauge, for instance, the Lorentz gauge is $\nabla \cdot \mathbf{A} = -\mu_0\epsilon_0(\partial V/\partial t)$
- Gauge invariance is a more general phenomenon
- Solving for vector potential is (generally) harder than solving for the electrostatic potential
- The electric field can no longer be expressed as the gradient of a scalar potential if there is a time-varying $\mathbf{B}$ field:

$$\mathbf{E}(t) = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \tag{50}$$

This last change can be seen rather easily. Consider the Maxwell equation for the curl of the electric field:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{51}$$

and substitute in the form of $\mathbf{B} = \nabla \times \mathbf{A}$:

$$\nabla \times \mathbf{E} + \frac{\partial}{\partial t}\nabla \times \mathbf{A} = 0 \tag{52}$$

The vector $\mathbf{E} + \partial\mathbf{A}/\partial t$ has zero curl. We know from identities that it can be written as a gradient of a scalar:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla\phi \tag{53}$$

So, rearranging, we find that $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$

5 Magnetic Intensity

As we saw with the electric field, $\mathbf{E}$, the introduction of a medium other than vacuum results in changes to Maxwells equations. These changes can be handled by using an alternative field which includes the effects of the medium implicitly. We will now do the same for magnetic fields. A word of caution: non-linear magnetic media are much more common than non-linear electric media; we will deal with these rather interesting materials in Chapter 4 on Ferromagnetism.

- We introduced the polarization of a dielectric material, $\mathbf{P} \propto \mathbf{E}$
- Similarly, we introduce a quantity, proportional to the magnetic induction $\mathbf{B}$
- This is the magnetization, $\mathbf{M}$
- It describes the response of a material to the magnetic induction
- Electrons can be modelled as moving in loops around atoms: we can use the magnetic dipole to model the response

Let us consider the vector potential at a point $\mathbf{r}_1$ due to a small volume of magnetised material at a point $\mathbf{r}_2$ (we will see later that this is given by the expression below). This small volume will have magnetic moment $\mathbf{\Delta m} = \mathbf{M}(\mathbf{r}_2)\delta V_2$. Then we can write:

$$\begin{aligned}
\mathbf{A}(\mathbf{r}_1) &= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{\Delta m} \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} && \text{(54a)}\\
&= \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{M}(\mathbf{r}_2) \times \mathbf{r}_{12}}{|\mathbf{r}_{12}|^3} dV_2 && \text{(54b)}\\
&= \frac{\mu_0}{4\pi}\int_V \mathbf{M}(\mathbf{r}_2) \times \nabla_2 \frac{1}{\mathbf{r}_{12}} dV_2 && \text{(54c)}
\end{aligned}$$

Now we use the expansion of $\nabla \times (\phi \mathbf{f})$, with $\mathbf{F} = \mathbf{M}$ and $\phi = 1/r_{12}$ to write

$$\mathbf{F} \times \nabla\phi = \phi\nabla \times \mathbf{F} - \nabla \times (\phi\mathbf{F}) \tag{55a}$$

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \left[\frac{\nabla_2 \times \mathbf{M}(\mathbf{r}_2)}{r_{12}} - \nabla_2 \times \left(\frac{\mathbf{M}(\mathbf{r}_2)}{r_{12}}\right)\right] dV_2 \tag{55b}$$

Now we use the theorem $\int_V \nabla \times \mathbf{F} dV = \int_S \mathbf{n} \times \mathbf{F} da$ to write:

$$\mathbf{A}(\mathbf{r}_1) = \frac{\mu_0}{4\pi}\int_V \left[\frac{\nabla_2 \times \mathbf{M}(\mathbf{r}_2)}{r_{12}} dV_2 - \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{n} \times \mathbf{M}(\mathbf{r}_2)}{r_{12}} da_2 \tag{56a}$$

$$= \frac{\mu_0}{4\pi}\int_V \left[\frac{\nabla_2 \times \mathbf{M}(\mathbf{r}_2)}{r_{12}} dV_2 + \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{M} \times \mathbf{n}(\mathbf{r}_2)}{r_{12}} da_2 \tag{56b}$$

- This then leads us to the **magnetization current densities**:
- We formally define:

$$\mathbf{J}_M = \nabla \times \mathbf{M} \tag{57}$$

$$\mathbf{j}_M = \mathbf{M} \times \mathbf{n} \tag{58}$$

- $\mathbf{J}_M$ is the volume magnetization current density
- $\mathbf{j}_M$ is the surface magnetization current density

It is clear that there will be no bound current density where the magnetization is uniform. So within the bulk of the rod there is a bound current density given by $\mathbf{J}_M = \nabla \times \mathbf{M}$, and at the surface there is a bound surface current per unit length given by $\mathbf{j}_M = \mathbf{M} \times \mathbf{n}$ is a unit vector in the direction of the outward normal to the surface. $\mathbf{J}_M$ is a current per unit area, where the area is perpendicular to the direction of flow, and jM is a current per unit length, where the length is in the plane of the surface and perpendicular to the direction of the surface current. These bound currents are the net effect of the microscopic currents associated with magnetic dipoles.

- We move on to considering how linear magnetic media behave . . .
- We know that $\nabla \times \mathbf{B} = \mu_0\mathbf{J}$
- We also have $\mathbf{J} = \mathbf{J}_M + \mathbf{J}_f$
- Here $\mathbf{J}_f$ is due to the motion of free charges, and $\mathbf{J}_M = \nabla \times \mathbf{M}$
- So $\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M})$ or $\nabla \times (\frac{\mathbf{B}}{\mu_0} - M) = \mathbf{J}_f$
- We then define $\mathbf{H}$, the magnetic *intensity*, as

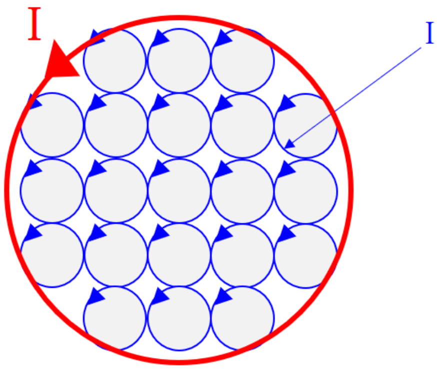

Figure 5: Origins of the magnetization surface current.

Magnetic Intensity

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \tag{59}$$

This yields $\nabla \times \mathbf{H} = \mathbf{J}_f$

The magnetic intensity serves a similar purpose to the electric displacement, in accounting for the response of the medium as well as the magnetic induction. We can rewrite this, using Stokes theorem:

$$\int_S \nabla \times \mathbf{H} \cdot \mathbf{n} da = \int_S \mathbf{J}_f \cdot \mathbf{n} da \tag{60a}$$

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_f \cdot \mathbf{n} da (= I_f) \tag{60b}$$

This tells us that the integral of the intensity along a closed loop is equal to the current flowing across the surface defined by that loop. It also gives the units as amperes per metre (the same units as the magnetization). It is important to note that the three quantities that we have defined so far (the magnetic induction, **B**, the magnetization, **M** and the magnetic intensity, **H**) are not necessarily parallel; this will be important when considering ferromagnetism in particular.

Magnetic susceptibility

- For a linear, isotropic material, we assert (based on experimental observations):

$$M = \chi_m \mathbf{H} \tag{61}$$

  where χ_m is the *magnetic susceptibility*

- We can write $B = \mu_0(1 + \chi_m)\mathbf{H}$
- If $chi_m > 0$ we have a paramagnetic material

- If $\chi_m < 0$ we have a diamagnetic material
- Note that χ_m can depend on temperature, but is generally small for these materials (less than 10^{-5})

6 Interfaces and Boundary Conditions

- Understanding how the different field vectors change at interfaces is important
- We need to consider both medium/vacuum and medium/medium interfaces
- We will consider the electric and magnetic fields in two groups:
  - D and B together
  - E and H together
- We want to know what is conserved

Normal components

- First notice that we can write similar equations for **D** and **B**:

$$\nabla \cdot \mathbf{D} = \rho_f \tag{62}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{63}$$

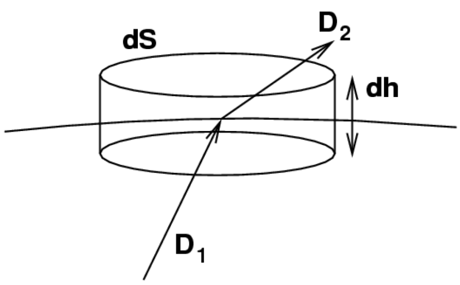

Figure 6: Small cylinder at interface

- Consider an interface with no free charges
- Consider the small cylinder of Fig. 6, height dh, area da.

Gauss theorem tells us:

$$\int V\nabla \cdot \mathbf{D} dv = \int_V \rho_f dv \tag{64}$$

$$\rightarrow \oint_S \mathbf{D} \cdot \mathbf{n} da = \int_V \rho_f dv \tag{65}$$

For the magnetic field, we find: I

$$\oint_S \mathbf{B} \cdot \mathbf{n} da = 0 \tag{66}$$

What is the flux of **D** through the box? Take the limit $dh \rightarrow 0$, and for an interface with no free charge we find:

$$\oint_S \mathbf{D} \cdot \mathbf{n} da = \mathbf{D}_2 \cdot \mathbf{n} da - \mathbf{D}_1 \cdot \mathbf{n} da \tag{67a}$$

$$\mathbf{D}_2 \cdot \mathbf{n} = \mathbf{D}_1 \cdot \mathbf{n} \tag{67b}$$

$$\mathbf{D}_{1\perp} = \mathbf{D}_{2\perp} \tag{67c}$$

$$\mathbf{B}_{1\perp} = \mathbf{B}_{2\perp} \tag{67d}$$

where the opposite signs on the displacement vectors come from their opposing directions (compared to the surface normals). This implies that the normal components of **D** are continuous across an interface with no free charges, while the normal components of B are always continuous. This means that lines of **D** and **B** are conserved at an interface with no free charges. Note that, in fact

$$\mathbf{D}_{2\perp} - \mathbf{D}_{1\perp} = \sigma_f \tag{68}$$

Tangential components

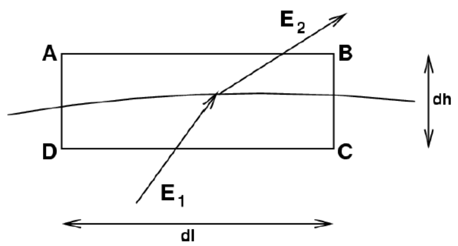

Figure 7: Small loop at interface.

- First notice that we can write similar equations for **E** and **H**:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{69}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{70}$$

- Consider an interface with no free current
- Consider the small loop of 7, height dh, length dl.

Stokes theorem tells us

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} da = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da \tag{71a}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} da \tag{71b}$$

Taking the limit $dh \to 0$, $da = dldh \to 0$ we find:

$$\int_S \nabla \times \mathbf{E} \cdot \mathbf{n} da = \oint_C \mathbf{E} \cdot d\mathbf{l} \tag{72}$$

But this can be written as $\mathbf{E}_2 \cdot \vec{AB} + \mathbf{E}_1 \cdot \vec{CD}$. As the vectors from A to B and from C to D have opposite directions, we write:

$$\mathbf{E}_1 \cdot d\mathbf{l} = \mathbf{E}_2 \cdot d\mathbf{l} \tag{73a}$$

$$\mathbf{E}_{1\parallel} = \mathbf{E}_{2\parallel} \tag{73b}$$

And for an interface with no free surface current (surface magnetization currents are irrelevant) we have a similar result for H:

$$\mathbf{H}_1 \cdot d\mathbf{l} = \mathbf{H}_2 \cdot d\mathbf{l} \tag{74a}$$

$$\mathbf{H}_{1\parallel} = \mathbf{H}_{2\parallel} \tag{74b}$$

This implies that the tangential components of the E and H fields are conserved subject to the conditions explained above. This means that field lines are not conserved in general across the interface for these fields.

- Normal components of **B** are continuous across an interface
- Normal components of **D** are continuous across an interface with no free charges
- Tangential components of **E** are continuous across an interface
- Tangential components of **H** are continuous across an interface with no free currents
- Field lines of **E** and **H** are not conserved across interfaces in general

7 Summary of Linear Media

- Linear: χ_e is independent of **E** (or χ_m of B)
- Isotropic: **P** is parallel to **E** (or **M** to **H**)
- Homogeneous: χ_e is position independent
- $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$

- $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$ so $\mathbf{D} = \epsilon \mathbf{E}$, with $\epsilon = \epsilon_0(1 + \chi_e)$
- $\nabla \cdot \mathbf{D} = \rho_f$
- $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
- $\mathbf{M} = \chi_m \mathbf{H}$ so $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$ with $\mu_r = 1 + \chi m$
- $\nabla \times \mathbf{H} = \mathbf{J}_f$
- continuous across an interface:
  - $B_\perp$
  - $D_\perp$ (when $\underline{\text{n}}$o free charges)
  - $E_\parallel$
  - $H_\parallel$ (when $\underline{\text{n}}$o free currents)