2013

Model answers MATH3305

Problem 1

(a) $V'^a = \frac{\partial X'^a}{\partial X^b} V^b$

(b) $W'_a = \frac{\partial X^b}{\partial X'^a} W_b$.

(c) (i) Can't add a scalar to a type $\binom{1}{1}$ expression.

(ii) Can't contract two lower indices (or two upper) with each other.

(iii) This is a well defined tensor expression.

(d)

$$\begin{aligned} A_{ab}{}^{ab} &= A_{ba}{}^{ba} && \text{Relabel } a \to b\, b \to a \\ &= A_{ab}{}^{ba} && \text{Symmetric} \\ &= -A_{ab}{}^{ab} = 0 && \text{Anti}-\text{symmetric} \end{aligned}$$

(e) Expanding out the expression on the left hand side we get,

$$\begin{aligned} \nabla_c I_i{}^j &= \partial_c I_i{}^j - \Gamma^k_{ci} I_k{}^j + \Gamma^j_{ck} I_i{}^k \\ &= 0 - \Gamma^k_{ci}\delta_k{}^j + \Gamma^j_{ck}\delta_i{}^k \\ &= -\Gamma^j_{ci} + \Gamma^j_{ci} \\ &= 0 \end{aligned}$$

Problem 2

(a) The defining relation for the Lorentz transformation is the preservation of the Minkowski metric; $L^a{}_b L^c{}_d \eta_{ac} = \eta_{bd}$. Taking the determinant of both sides we find,

$$\begin{aligned}
\text{Det}(L^T.\eta.L) &= \text{Det}(L) \\
\text{Det}(L^T)\text{Det}(\eta)\text{Det}(L) &= \text{Det}(\eta) \\
(\text{Det}(L))^2 &= 1 \\
\Rightarrow \text{Det}(L) &= \pm 1
\end{aligned}$$

(b) (i) The speed of light is an absolute constant. Therefore, both observers should have coordinates

$$\begin{pmatrix} t \\ t \end{pmatrix}$$

for a right moving photon and

$$\begin{pmatrix} t \\ -t \end{pmatrix}$$

for a left moving photon. Moreover, a Lorentz transform from one frame to the other should preserve this.

$$\begin{aligned}
\begin{pmatrix} t \\ x \end{pmatrix}_S &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} t \\ t \end{pmatrix}_E \\
\begin{pmatrix} t \\ x \end{pmatrix}_S &= \begin{pmatrix} (\alpha+\beta)t \\ (\gamma+\delta)t \end{pmatrix}_E
\end{aligned}$$

Since $t_S = x_S$ we have $\alpha + \beta = \gamma + \delta$. Similarly for a left moving photon we get the relation, $\alpha - \beta = -(\gamma - \delta)$. This gives as required $\alpha = \delta$ and $\beta = \gamma$.

(ii) In the Earth's reference frame, the ship will move in a straight line with equation $x = v_x t$,

$$\begin{aligned}
\begin{pmatrix} t \\ 0 \end{pmatrix}_S &= \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} t \\ v_x t \end{pmatrix}_E \\
\begin{pmatrix} t \\ 0 \end{pmatrix}_S &= \begin{pmatrix} (\alpha+\beta v_x)\,t \\ (\beta+\alpha v_x)\,t \end{pmatrix}_E
\end{aligned}$$

From the spatial component we can deduce $\beta = -\alpha v_x$

(iii) We now have the Lorentz transform in the form,

$$L^a{}_b = \begin{pmatrix} \alpha & -\alpha v_x \\ -\alpha v_x & \alpha \end{pmatrix}$$

We can therefore take the determinate and set it equal to one for proper Lorentz transformations.

$$\begin{aligned} \mathrm{Det}(L) &= \alpha^2 - \alpha^2 v_x^2 = 1 \\ \Rightarrow \alpha^2 &= \frac{1}{1 - v_x^2} \\ \alpha &= \sqrt{\frac{1}{1 - v_x^2}} \end{aligned}$$

(iv,v) See attached diagram.

(vi) We know that $x_E = Vt_E$ and thus $\alpha = \arctan(1/\sqrt{3})$. Therefore,

$$\begin{aligned} \beta &= \pi/2 - 2 \times \alpha \\ &= \pi/2 - 2 \times \frac{\pi}{3} \\ &= \pi/6 \end{aligned}$$

Problem 3

(a) The number of indenpendent components of the Riemann curvature tensor R_{abcd} in n-dimensions is given by

$$\frac{1}{12}n^2(n^2-1)$$

!

(b) i Using the Lagrangian method we can pick out the Christoffel symbols. First u,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial u} &= 0 \\
\frac{\partial \mathcal{L}}{\partial \dot{u}} &= 2(c+a\cos(v))^2\dot{u} \\
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{u}} &= 2\left(2(-a\sin(v))\dot{u}\dot{v} + (c+a\cos(v))\ddot{u}\right)
\end{aligned}$$

and then v,

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial v} &= -2a(c+a\cos(v))\sin(v)\dot{u}^2 \\
\frac{\partial \mathcal{L}}{\partial \dot{v}} &= 2a^2 v \\
\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{v}} &= 2a^2\ddot{v}
\end{aligned}$$

Therefore the Christoffel symbols are

$$\Gamma^u_{\ uv} = -\frac{a\sin(v)}{c+a\cos(v)} \qquad \Gamma^v_{\ uu} = \frac{\sin(v)(c+a\cos(v))}{a}$$

ii Now for the Riemann tensor. We input the Christoffel symbols into the following expression for the Riemann tensor

$$R_{abc}{}^{s} = \frac{\partial \Gamma^s_{\ ac}}{\partial X^b} - \frac{\partial \Gamma^s_{\ bc}}{\partial X^a} + \Gamma^e_{\ ac}\Gamma^s_{\ be} - \Gamma^e_{\ bc}\Gamma^s_{\ ea}$$

First $R_{uvu}{}^{v}$

$$R_{uvu}{}^{v} = \frac{\partial \Gamma^{v}_{uu}}{\partial v} - \frac{\partial \Gamma^{v}_{vu}}{\partial u} + \Gamma^{e}_{uu}\Gamma^{v}_{ve} - \Gamma^{e}_{vu}\Gamma^{v}_{eu}$$

$\frac{\partial \Gamma^{v}_{vu}}{\partial u} = 0$ and also the third term is piecewise zero,

$$\begin{aligned} R_{uvu}{}^{v} &= \frac{\partial \Gamma^{v}_{uu}}{\partial v} - \Gamma^{u}_{vu}\Gamma^{v}_{uu} \\ &= \frac{\partial}{\partial v}\left[(c + a\cos(v))\sin(v)\right] + \sin(v)^2 \\ &= \frac{\cos(v)}{a}(c + a\cos(v)) \end{aligned}$$

Second $R_{vuv}{}^{u}$

$$R_{vuv}{}^{u} = \frac{\partial \Gamma^{u}_{vv}}{\partial u} - \frac{\partial \Gamma^{u}_{uv}}{\partial v} + \Gamma^{e}_{vv}\Gamma^{u}_{ue} - \Gamma^{e}_{uv}\Gamma^{u}_{ev}$$

$\frac{\partial \Gamma^{u}_{vv}}{\partial u} = 0$ and also the third term is piecewise zero,

$$\begin{aligned} R_{uvu}{}^{v} &= -\frac{\partial \Gamma^{u}_{uv}}{\partial v} - \Gamma^{u}_{uv}\Gamma^{u}_{uv} \\ &= -\frac{\partial}{\partial v}\left[\frac{-a\sin(v)}{c + a\cos(v)}\right] - \frac{a^2\sin(v)^2}{(c + a\cos(v))^2} \\ &= \frac{a\cos(v)}{c + a\cos(v)} \end{aligned}$$

Problem 4

(a) Let γ be a curve with affine parametrisation $X^a(\lambda)$. The tangent vector to γ is given by $T^a = dX^a/d\lambda$. Moreover, $\nabla_a T^b$ can be written as follows

$$\nabla_a T^b = \partial_a T^b + \Gamma^b_{ac} T^c \tag{1}$$

$$T^a \partial_a T^b = T^a \frac{\partial T^b}{\partial X^a} = \frac{\partial X^a}{\partial \lambda}\frac{\partial T^b}{\partial X^a} = \frac{\partial T^b}{\partial \lambda} = \frac{\partial^2 X^b}{\partial \lambda^2} \tag{2}$$

and therefore the equation of parallel transport becomes

$$\frac{\partial^2 X^b}{\partial \lambda^2} + \Gamma^b_{ac} T^a T^c = \frac{\partial^2 X^b}{\partial \lambda^2} + \Gamma^b_{ac} \frac{\partial X^a}{\partial \lambda}\frac{\partial X^c}{\partial \lambda} = 0 \tag{3}$$

which is the geodesic equation.

(b) Use the action of ∇_a on g_{bc} and try to solve the resulting equation for C^a_{bc}.

$$\nabla_a g_{bc} = g_{bc,a} - C^d_{ab} g_{dc} - C^d_{ac} g_{bd} = 0 \tag{4}$$

$$\nabla_c g_{ab} = g_{ab,c} - C^d_{ca} g_{db} - C^d_{cb} g_{ad} = 0 \tag{5}$$

$$\nabla_b g_{ca} = g_{ca,b} - C^d_{bc} g_{da} - C^d_{ba} g_{cd} = 0 \tag{6}$$

Let us consider the following combination (4) + (5) − (6) of the metricity conditions

$$g_{bc,a} + g_{ab,c} - g_{ca,b} - 2C^d_{ca} g_{db} = 0 \tag{7}$$

Apply g^{bm} to this equation and we find

$$\delta^m_d C^d_{ca} = \frac{1}{2} g^{mb}(g_{bc,a} + g_{ab,c} - g_{ca,b}) \tag{8}$$

Finally we rename indices $c \to b, a \to c, m \to a, b \to d$ and arrive at

$$C^a_{bc} = \Gamma^a_{bc} = \frac{1}{2} g^{ad}(g_{db,c} + g_{cd,b} - g_{bc,d}) \tag{9}$$

Hence, C^a_{bc} is uniquely fixed to be the Christoffel symbol and therefore ∇_a is unique. Note that Γ^a_{bc} is often called the connection.

(c) $R_{ab}{}^{ab} = g^{as} R_{abs}{}^{b} = g^{as} R_{as} = R$. Contracting identity $R_{abcd} = -R_{abdc}$ by g^{cd} gives $R_{abc}{}^{c} = -R_{abd}{}^{d}$. Renaming $d \to c$ on the right hand side gives $R_{abc}{}^{c} = 0$.

(d) Let $\widehat{R}$, $\widehat{R}_{ab}$, $\widehat{G}_{ij}$, $\widehat{R}_{abcd}$ and $\widehat{R}_{abc}{}^{d}$ be curvature tensors for metric h_{ij}. From $h^{ij} = \frac{1}{\lambda} g^{ij}$ we obtain that Christoffel symbols of h_{ij} coincide with the Christoffel symbols of g_{ij}. Thus, by definition of $R_{abc}{}^{d}$ we have $\widehat{R}_{abc}{}^{d} = R_{abc}{}^{d}$, and

$$\widehat{R}_{abcd} = \lambda R_{abcd}, \quad \widehat{R}_{ab} = R_{ab}, \quad \widehat{R} = \frac{1}{\lambda} R, \quad \widehat{G}_{ab} = G_{ab}.$$

Problem 5

(a) The Lagrangian is given by

$$L = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \tag{10}$$

The geodesic equations for $t(\lambda)$, $\theta(\lambda)$, $r(\lambda)$ and $\phi(\lambda)$ are

$$\begin{aligned}
t: &\quad 2\ddot{t}\left(1 - \frac{r_s}{r}\right) + 2\frac{r_s}{r^2}\dot{r}\dot{t} = 0, \\
\theta: &\quad \ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{2}{r}\dot{r}\dot{\theta} = 0, \\
r: &\quad 2\left(1 - \frac{r_s}{r}\right)^{-1}\ddot{r} + \frac{r_s}{r^2}\dot{t}^2 - \frac{r_s}{\left(1 - \frac{r_s}{r}\right)^2 r^2}\dot{r}^2 - 2r\,\dot{\theta}^2 - 2r\sin^2\theta\,\dot{\phi}^2 = 0, \\
\phi: &\quad \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\phi}\dot{\theta}
\end{aligned}$$

where dot(s) denote differentiation(s) with respect to λ.

(b) Since $\frac{\partial L}{\partial t} = 0$ and $\frac{\partial L}{\partial \phi} = 0$, we have two constants of motion E (energy) and ℓ (angular momentum) given by

$$\begin{aligned}
E &= \left(1 - \frac{r_s}{r}\right)\dot{t}, \\
\ell &= r^2\dot{\phi}.
\end{aligned}$$

Moreover, L is a constant of motion.

Since $\theta(\lambda) = \frac{\pi}{2}$, the Lagrangian simplifies into

$$L = \left(1 - \frac{r_s}{r}\right)\dot{t}^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\phi}^2 \tag{11}$$

and using constants E, L and ℓ, equation (11) can be written as

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = C, \tag{12}$$

where

$$\begin{aligned}
V_{\text{eff}}(r) &= \frac{1}{2}\left(\frac{\ell^2}{r^2} - \frac{r_s\ell^2}{r^3} - \frac{r_s L}{r}\right), \\
C &= \frac{1}{2}\left(E^2 - L\right).
\end{aligned}$$

When we treat equation (12) as a 1-dimensional mechanical system the equation has the interpretation: RHS = kinetic energy $\frac{1}{2}\dot{r}^2$ plus potential energy V_{eff}, and LHS = total energy C (constant).

(c) For a lightlike geodesic we have $L = 0$, and

$$\begin{aligned} V_{\text{eff}}(r) &= \frac{\ell^2}{2}\left(\frac{1}{r^2} - \frac{r_s}{r^3}\right), \\ C &= \frac{1}{2}E^2. \end{aligned}$$

The effective potential V_{eff} has a *maximum* at $r_\star = \frac{3}{2}r_s$. See Figure 1.

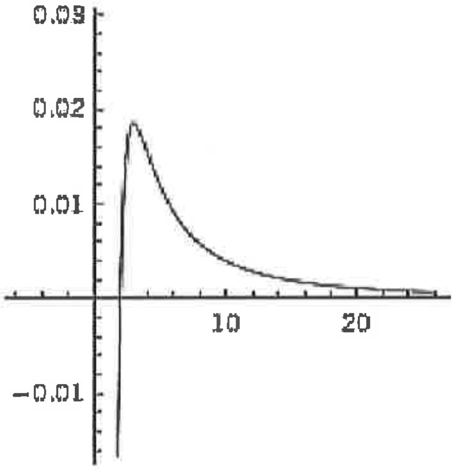

Figure 1: Effective potential V_{eff}.

(d) Trace the field equation which gives

$$R - \frac{1}{2}4R + 4\Lambda = 0 \tag{13}$$

$$R = 4\Lambda. \tag{14}$$

Next eliminate R from the field equation

$$R_{ab} - \frac{1}{2}(4\Lambda)g_{ab} + \Lambda g_{ab} = 0 \tag{15}$$

$$R_{ab} = \Lambda g_{ab}. \tag{16}$$