All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Symmetrisation and skew-symmetrisation:

$$T_{(ab)} := \frac{1}{2}(T_{ab} + T_{ba}), \qquad T_{[ab]} := \frac{1}{2}(T_{ab} - T_{ba}). \tag{1}$$

Christoffel symbol:

$$\Gamma^c_{ab} = \frac{1}{2} g^{cd}(g_{da,b} + g_{bd,a} - g_{ab,d}). \tag{2}$$

Geodesics parameterised by λ:

$$\frac{d^2X^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dX^b}{d\lambda}\frac{dX^c}{d\lambda} = 0. \tag{3}$$

Riemann curvature tensor:

$$R_{mnr}{}^s = \Gamma^s_{mr,n} - \Gamma^s_{nr,m} + \Gamma^a_{mr}\Gamma^s_{an} - \Gamma^a_{nr}\Gamma^s_{am}, \tag{4}$$

$$R_{abcd} = -R_{bacd}, \qquad R_{abcd} = -R_{abdc}, \qquad R_{abcd} = R_{cdab}$$

$$R_{dabc} + R_{dcab} + R_{dbca} = 0. \tag{5}$$

Ricci tensor and Ricci scalar:

$$R_{mr} = R_{mnr}{}^n, \qquad R = g^{mr}R_{mr}. \tag{6}$$

Schwarzschild-de Sitter line element:

$$ds^2 = -\left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)dt^2 + \left(1 - \frac{2m}{r} - \frac{\Lambda}{3}r^2\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{7}$$

1. (a) Describe how a contravariant vector V^a transforms under coordinate transformations between coordinates x^a and x'^a.

   (b) Describe how a covariant vector W_a transforms under coordinate transformations between coordinates x^a and x'^a.

   (c) Let $M^a{}_{bc}$ be a rank 3 tensor. Show that $M^a{}_{ac}$ transforms like a covariant vector under coordinate transformations.

   (d) Let $\mathcal{M}$ be a N-dimensional manifold and let g_{ab} be a metric tensor. What is meant by g^{ab}? Compute $g_{ab}g^{bc}$ and $g_{ab}g^{ab}$.

   (e) Consider a 3-dimensional manifold and the rank 3 tensor T_{abc}. How many independent components has T_{abc} in general? How many components has $T_{[abc]}$, a totally skew-symmetric tensor?

   (f) Let $\mathcal{M}$ be a 4-dimensional manifold. Consider the rank 2 tensor T_{ab}. Show that the following is a valid equation

$$\operatorname{tr} T g_{ab} + g_{ac}g_{bd}T^{cd} = T_{(ab)} + \frac{1}{8}g^{cd}g_{cd}T_{ab} + g^{cd}T_{(cd)}g_{ab} - \frac{1}{2}T_{ba}.$$

2. (a) Consider a 2-dimensional manifold $\mathcal{M}$ with the following line element

$$ds^2 = dy^2 + z^4 dz^2.$$

   For which values of y and z is this line element well defined?

   (b) Find the non-vanishing Christoffel symbols using Eq. (2).

   (c) Obtain the geodesic equations parametrised by λ, say.

   (d) Solve the geodesic equations and suggest an improved coordinate system. What is the metric in the new coordinates? What lines describe the geodesics geometrically?

   (e) What can you say about the Riemann curvature tensor, the Ricci tensor and the Ricci scalar of this manifold?

3. Let K_{abcd} be a rank 4 tensor with the following symmetry properties:

$$\begin{aligned} K_{abcd} + K_{bacd} &= 0, \\ K_{abcd} + K_{abdc} &= 0, \\ K_{dabc} + K_{dcab} + K_{dbca} &= 0. \end{aligned} \tag{$*$}$$

(a) Show that these three properties imply the following additional symmetry

$$K_{abcd} = K_{cdab}.$$

[Hint: Start by writing out $(*)$ four times with permuted indices.]

Let $\mathcal{L}_v$ be the Lie derivative with respect to the vector v^a. For $f \in C^\infty(\mathcal{M})$ it is defined by $\mathcal{L}_v f = v^a \nabla_a f$. Moreover,

$$\mathcal{L}_v w^a = v^b \nabla_b w^a - w^b \nabla_b v^a,$$

and it satisfies the Leibnitz rule.

(b) Show that $\mathcal{L}_v u_a = v^b \nabla_b u_a + u_b \nabla_a v^b$.

4. The most general static and spherically symmetric metric is given by

$$ds^2 = -a(r)dt^2 + b(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

and results in the following two Einstein tensor components

$$G^t_t = -\frac{b'(r)}{rb(r)^2} + \frac{1}{r^2 b(r)} - \frac{1}{r^2}, \qquad G^r_r = \frac{a'(r)}{ra(r)b(r)} + \frac{1}{r^2 b(r)} - \frac{1}{r^2},$$

where the prime denotes differentiation with respect to r.

(a) Solve the static and spherically symmetric field equations with cosmological term.

(b) Discuss the importance of that solution when $\Lambda = 0$: Considering the limit $r \to \infty$, what does this metric describe?

(c) Now set $m = 0$ (but $\Lambda \neq 0$) and consider the spatial part of the metric only

$$ds^2_{\text{spatial}} = \frac{dr^2}{1 - (\Lambda/3)r^2} + r^2 d\Omega^2.$$

Suggest an improved coordinate system to remove the coordinate singularity at $r = \sqrt{3/\Lambda}$.

5. (a) Consider the Lagrangian $L = g_{ab}\dot{X}^a\dot{X}^b$ and show that the resulting Euler-Lagrange equations are the geodesic equations.

   (b) Consider the Schwarzschild-de Sitter line element (7) with $m = 0$ and derive the geodesic equations using the Lagrangian approach for massless and massive particles. You may assume $\theta = \pi/2$.

   (c) How many constants of motion has this system and what is their physical interpretation? Rewrite the Lagrangian using these constants and show that the geodesics can be described analogous to a 1-dimensional classical mechanical system

$$\dot{r}^2/2 + V_{\text{eff}}(r) = C,$$

   where C is a constant. Find $V_{\text{eff}}(r)$. What is the meaning of C?