EMAG

1a) $\frac{1}{2}\begin{pmatrix}x^3\\y^3\\z^3\end{pmatrix} = -\nabla\left(\frac{x^4+y^4+z^4}{8}\right) = -\nabla V$

Gradient of a scalar potential => Can represent electrostatic field

b) $\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_0} \Rightarrow Q_{tot} = \int \varepsilon_0 \nabla\cdot\underline{E}\, dV$

$\nabla\cdot\underline{E} == \frac{3}{2}(x^2+y^2+z^2) = \frac{3}{2}r^2$

$\Rightarrow Q_{tot} = \int_0^a \varepsilon_0 \frac{3}{2} r^2 \, 4\pi r^2 dr$

$= 6\varepsilon_0\pi \int_0^a r^4 dr = \frac{6\varepsilon_0\pi a^5}{5}$

2a) Invariant quantities are unchanged by a Lorentz transformation

i) Charge on electron is invariant

ii) Speed of light is invariant

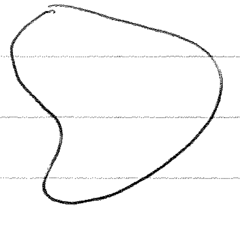
iii) Charge density not invariant as volume changes but charge same

iv) $\underline{E}\cdot\underline{B}$ is invariant

bi) Spacelike $dr^2 > c^2dt^2 \Rightarrow ds^2 > 0$

ii) Timelike $dr^2 < c^2dt^2 \Rightarrow ds^2 < 0$

3) $\nabla\cdot\underline{E} = \frac{\rho}{\varepsilon_0}$ Take surface with enclosed volume



$\int \nabla\cdot\underline{E}\, dV = \int \frac{\rho}{\varepsilon_0} dV$

Divergence ~~Stokes~~ theorem $\int \nabla\cdot\underline{E}\, dV = \oint \underline{E}\cdot d\underline{s}$

$\Rightarrow \oint \underline{E}\cdot d\underline{s} = \frac{1}{\varepsilon_0}\int \rho\, dV = \frac{Q_{enc}}{\varepsilon_0}$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$ Take surface integral

$$\int (\nabla \times \underline{E}) \cdot d\underline{S} = -\frac{\partial}{\partial t} \int \underline{B} \cdot d\underline{S}$$

Stokes theorem $\int (\nabla \times \underline{E}) \cdot d\underline{S} = \oint \underline{E} \cdot d\underline{l}$

$$\Rightarrow \oint \underline{E} \cdot d\underline{l} = -\frac{\partial}{\partial t} \int \underline{B} \cdot d\underline{S} = -\frac{\partial \Phi}{\partial t} \qquad \Phi = \text{flux} = \int \underline{B} \cdot d\underline{S}$$

4a) $t' = t - \frac{|\underline{r}|}{c}$ a wave takes time $\frac{|\underline{r}|}{c}$ to reach observer due to finite speed. Therefore an observer experiences the charge distribution at a time retarded by $\frac{|\underline{r}|}{c}$

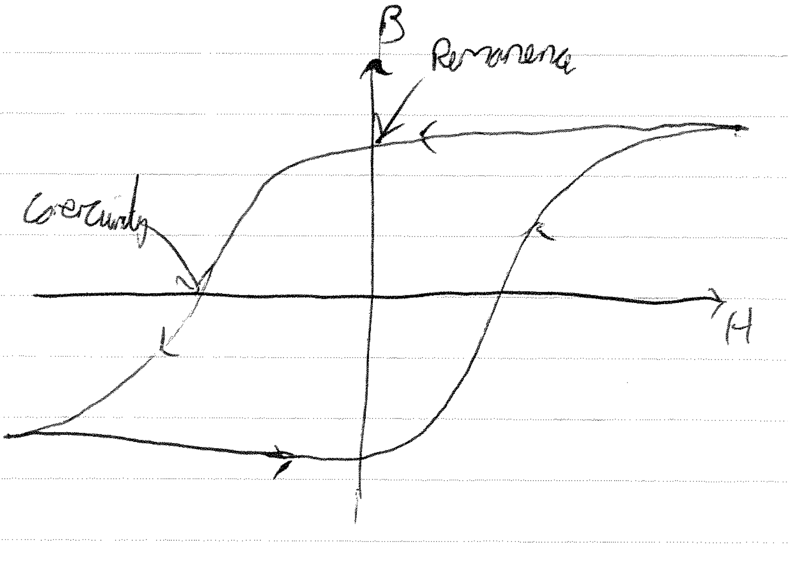
b) $$\frac{\partial f(t')}{\partial t} = \frac{\partial f(t')}{\partial t'} \frac{\partial t'}{\partial t} \qquad t' = t - \frac{\cancel{|r|}}{c} \Rightarrow \frac{\partial t'}{\partial t} = 1$$

$$= \frac{\partial f(t')}{\partial t'}$$

$$\frac{\partial f(t')}{\partial r} = \frac{\partial f(t')}{\partial t'} \frac{\partial t'}{\partial r} \qquad \frac{\partial t'}{\partial r} = \frac{-1}{c}$$

$$= \frac{-1}{c} \frac{\partial f(t')}{\partial t'}$$

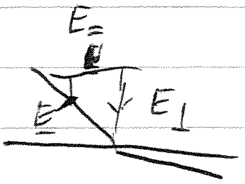
5a)



5b) Permanent magnets are not in an external field so want a large remanence. They should also remain mostly magnetised when experiencing a small field so want a good coercivity

They all have similar remanences so pick highest coercivity
=> material C as will remain magnetised the longest

6a)



$r_{\parallel}$ gives amplitude reflection of parallel component of wave and $r_{\perp}$ gives transverse reflection amplitude

n is refractive index of dielectric $n = \sqrt{\varepsilon_r}$

b) $\theta_r = \theta_i$
$\sin\theta_i = n\sin\theta_t$

c) $\underline{k}$ is in direction of travel for each of the 3 waves

$|k_i| = |k_r| \qquad |k_t| = n|k_i|$

7a i) $\frac{w}{k} = \frac{c}{n} \Rightarrow k = \frac{nw}{c} = 3.76\times10^{6} + 1.285\times10^{-4} i$

ii) $\underline{E} = E_0 e^{i(kx-wt)}\hat{\underline{z}} = E_0 e^{i(3.76\times10^{6}x - 9.4\times10^{14}t)} e^{-1.285\times10^{-4}x}\hat{\underline{z}}$

iii) After 1km Amplitude $= E_0 e^{-1.285\times10^{-4}\times10^{3}} = 0.879E_0$

iv) $V_p = \frac{w}{k} = \frac{c}{n} = \frac{3\times10^{8}}{1.2+4.1\times10^{-11}i} = \frac{3\times10^{8}(1.2-4.1\times10^{-11}i)}{(1.2+4.1\times10^{-11}i)(1.2-4.1\times10^{-11}i)}$

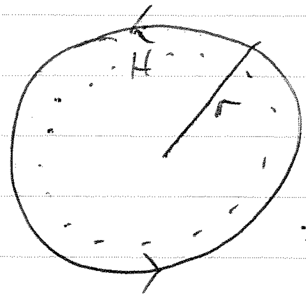
$= 2.5\times10^{8} - 8.54\times10^{-3}i \ \mathrm{ms^{-1}}$

7b) Linear - $P \propto E$

Isotropy - polarization independent of direction of $\underline{E}$

Homogeneity - the polarisation is the same everywhere in material

ii) Amperes law $\oint \underline{H} \cdot d\underline{l} = I_{enc}$



Take circular path radius r

$\underline{H}$ is in $\hat{\underline{\theta}}$ direction $\Rightarrow \underline{H} \cdot d\underline{l} = Hdl$

$\Rightarrow \oint \underline{H} \cdot d\underline{l} = 2\pi r H$

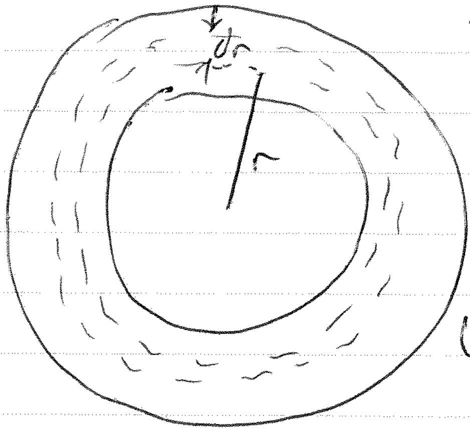
$I_{enc} = NI$ as N turns each with current $\underline{I}$

$\Rightarrow \quad 2\pi r \underline{H} = NI \hat{\underline{\theta}} \qquad \underline{H} = \frac{NI}{2\pi r} \hat{\underline{\theta}}$

iii) $\underline{B} = \mu_0 \mu_r \underline{H} = \frac{\mu_0 \mu_r NI}{2\pi r} \hat{\underline{\theta}}$

$\underline{M} = \chi \underline{H} = (\mu_r - 1)\underline{H} = \frac{(\mu_r - 1)NI}{2\pi r} \hat{\underline{\theta}}$

iv) $U = \frac{\underline{B} \cdot \underline{H}}{2} = \frac{\mu_0 \mu_r N^2 I^2}{4\pi^2 r^2}$



$\Rightarrow U = \int \frac{\underline{B} \cdot \underline{H}}{2} dV = \int \frac{\mu_0 \mu_r N^2 I^2}{8\pi^2 r^2} dV$

Ring of height a, at radius r, width dr

$dV = 2\pi r \, dr \, a$

$U = \int_{r=R}^{R+a} \frac{\mu_0 \mu_r N^2 I^2}{8\pi^2 r^2} 2\pi r a \, dr$

③

$$U = \frac{\mu_0 \mu_r N^2 I^2 a}{4\pi} \int_{r=R}^{R+a} \frac{1}{r}\, dr$$

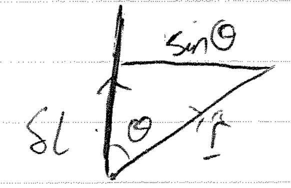
$$= \frac{\mu_r \mu_0 N^2 I^2 a}{4\pi} \ln\left(\frac{R+a}{R}\right)$$

8ai) A) $\omega \to 0$ small r $\frac{\omega}{r} = 0$

$$\underline{B}(\underline{r}) = \frac{\mu_0 I_0 \delta l}{4\pi} \sin\theta \left(0 + \frac{1}{r^2}\right) e^{i(kr-0)} \hat{\underline{\phi}}$$

$$= \frac{\mu_0 I}{4\pi} \frac{1}{r^2} \sin\theta\, \delta l\, e^{ikr} \hat{\underline{\phi}}$$

$$\delta l \times \hat{\underline{r}} = \delta l \sin\theta \hat{\underline{\phi}}$$



$\Rightarrow \underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi} \frac{1}{r^2}$ ~~sin~~ $e^{ikr}\, \delta \underline{l} \times \hat{\underline{r}}$ $\quad r \to 0 \quad e^{ikr} \to 1$

$$\underline{B}(\underline{r}) = \frac{\mu_0 I}{4\pi} \frac{\delta \underline{l} \times \hat{\underline{r}}}{r^2}$$

B) Large r ~~$\underline{B}(\underline{r}) = \frac{\mu_0 I_0}{4\pi} \delta l \sin\theta ($~~

r large $\frac{-i\omega}{rc}$ dominant term compared to $\frac{1}{r^2}$

$$\Rightarrow \underline{B}(\underline{r}) = \frac{\mu_0 I_0 \delta l}{4\pi} \sin\theta \left(\frac{-i\omega}{rc}\right) \underbrace{e^{i(kr-\omega t)}} \hat{\underline{\phi}}$$

← just gives phase

$\Rightarrow B \propto \frac{1}{r}$ $\Rightarrow$ radiation field

ii) Radiation term dominant when $\frac{\omega}{rc} \gg \frac{1}{r^2}$

$\frac{\omega}{c} \gg \frac{1}{r}$ $\quad \frac{2\pi}{\lambda} \gg \frac{1}{r}$ $\Rightarrow$ $r \gg \lambda$ for radiation field

iii) Radiation fields $\underline{B}(\underline{r}) = \mu_0 \frac{I_0 \delta l}{4\pi} \sin\theta \left(\frac{-i\omega}{rc}\right) e^{i(kr-\omega t)} \underline{\hat{\phi}}$

$$\underline{E} = \frac{\mu_0 I_0 \delta l}{4\pi} \sin\theta \left(\frac{-i\omega}{r}\right) e^{i(kr-\omega t)} \underline{\hat{\theta}}$$

$$\underline{N} = \underline{E} \times \underline{H} = \underline{\underline{E \times B}} \qquad \underline{\hat{\theta}} \times \underline{\hat{\phi}} = -\underline{\hat{r}}$$

$$= \overset{\mu_0}{\cancel{4}} \mu_0 \frac{I_0^2 \delta l^2}{16\pi^2} \sin^2\theta \left(\frac{\omega^2}{r^2 c}\right) e^{2i(kr-\omega t)} \underline{\hat{r}}$$

$$P(t) = |\underline{E} \times \underline{H}| = \frac{\mu_0 I_0^2 \delta l^2 \omega^2 \sin^2\theta}{16\pi^2 r^2 c} \cos^2(\cancel{2i}kr-\omega t) \underline{\hat{r}}$$

$$\Rightarrow \langle P \rangle = \frac{\mu_0 I_0^2 \delta l^2 \omega^2 \sin^2\theta}{32\pi^2 r^2 c} \underline{\hat{r}}$$

b) $$\underbrace{\frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}} E_0^2}_{\bar{N}} = \frac{\mu_0 I_0^2 \delta l^2 \omega^2 \sin^2\theta}{32\pi^2 r^2 c}$$

$$\Rightarrow \overset{\bar{N}}{\cancel{E_0^2}} \propto \frac{\sin^2\theta}{r^2}$$

$$\Rightarrow \overset{\bar{N}}{\cancel{E_0}} = A \frac{\sin^2\theta}{r^2}$$

$$3 \times 10^{-3} = A \frac{\sin^2 30}{5^2} \qquad \Rightarrow A = 3 \times 10^{-3} \times 100 = 0.3$$

$$\bar{N} = 0.3 \frac{\sin^2\theta}{r^2}$$

$r = 20m$ maximum $\bar{N}$ for $\sin\theta = 1$ $\theta = 90°$

$$\bar{N} = \frac{0.3}{20^2} = 7.5 \times 10^{-4} \ Wm^{-2}$$

8(i) $j_\mu = (c\rho, -\underline{J})$ $\quad \underline{J} = 0$

$\rho = \frac{\lambda}{\sigma}$ $\quad j_\mu = \left(\frac{c\lambda}{\sigma}, 0, 0, 0\right)$ inside rod

$A_\alpha = \left(\frac{\phi}{c}, -\underline{A}\right) = \left(\frac{\lambda}{2\pi\varepsilon_0 c}\ln\left(\frac{R}{r}\right), 0, 0, 0\right)$ outside rod

ii) Lorentz transform in x_3 direction leaves x_1 and x_2 unchanged meaning $\sigma' = \sigma$

$$j'_\mu = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} c\rho \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma c\rho \\ -\gamma\beta c\rho \end{pmatrix} = \begin{pmatrix} \gamma\frac{c\lambda}{\sigma} \\ -\gamma\beta\frac{c\lambda}{\sigma} \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c\rho' \\ -J_x \\ -J_y \\ -J_z \end{pmatrix} = \begin{pmatrix} \gamma\frac{c\lambda}{\sigma} \\ -\gamma\beta\frac{c\lambda}{\sigma} \\ 0 \\ 0 \end{pmatrix}$$

$$a_\mu' = \begin{pmatrix} \phi'/c \\ -A_x \\ -A_y \\ -A_z \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & & \\ -\gamma\beta & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\begin{pmatrix} \frac{\lambda}{2\pi\varepsilon_0 c}\ln\left(\frac{R}{r}\right) \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\gamma\lambda}{2\pi\varepsilon_0 c}\ln\left(\frac{R}{r}\right) \\ -\frac{\gamma\beta\lambda}{2\pi\varepsilon_0 c}\ln\left(\frac{R}{r}\right) \\ 0 \\ 0 \end{pmatrix}$$

9(i) Plasma is an ionized gas consisting of positive ions and free electrons such that there is no overall charge.

ii) $\omega_p = \sqrt{\frac{10^{12}(1.6\times10^{-19})^2}{9.11\times10^{-31}\varepsilon_0}} = 5.63\times10^{7}\ s^{-1}$

iii A) $\omega << \omega_p$ $\quad k^2 < 0 \Rightarrow$ K imaginary - wave decays as it enters plasma

B) $\omega >> \omega_p$ $\quad k^2 >> 0$ some reflection and some transmission with no decay (Travelling waves)

iv A) $\omega < \omega_p$

$$K = \frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

$$= i\frac{\omega}{c}\sqrt{\frac{\omega_p^2}{\omega^2}-1} = \frac{i\omega}{c}\frac{\omega_p}{\omega}\sqrt{1-\frac{\omega^2}{\omega_p^2}} = \frac{i\omega_p}{c}\sqrt{1-\frac{\omega^2}{\omega_p^2}}$$

$$\Rightarrow E = E_0 e^{i(kz-\omega t)} = E_0 e^{-i\omega t} e^{-\frac{\omega_p}{c}\sqrt{1-\frac{\omega^2}{\omega_p^2}}\,z}$$

$\Rightarrow$ Amplitude falls as $e^{-\frac{\omega_p}{c}\sqrt{1-\frac{\omega^2}{\omega_p^2}}\,z}$

B) $\exp\left(-\frac{2\pi\times10^8}{c}\sqrt{1-0.99^2}\times0.5\right) = 0.863$ for $\omega = 0.99\omega_p$

$\frac{\omega}{\omega_p} << 1$ $\quad \exp\left(-\frac{\omega_p z}{c}\right) = \exp\left(-\frac{2\pi\times10^8}{c}\times0.5\right) = 0.351$

bi) Skin effect is tendency of an alternating current to become distributed within a conductor such that the current density is largest near the surface and decreases going deeper into the conductor.

ii) $K = \omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}\sqrt{1+\frac{ig}{\varepsilon_0\varepsilon_r\omega}}$

Good conductor $\Rightarrow g >> \omega\varepsilon_r\varepsilon_0$

$$k = \omega\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}\sqrt{\frac{ig}{\varepsilon_0\varepsilon_r\omega}}\left(1+\frac{\varepsilon_0\varepsilon_r\omega}{ig}\right)^{\frac{1}{2}}$$ expand to first order in $\frac{\varepsilon_0\varepsilon_r\omega}{ig}$

$$= \sqrt{i\omega\mu_0\mu_r g}\left(1+\frac{\varepsilon_0\varepsilon_r\omega}{2ig}\right)$$

$$= \frac{(1+i)}{\sqrt{2}}\sqrt{\omega\mu_0\mu_r g}\left(1+\frac{\varepsilon_0\varepsilon_r\omega}{2ig}\right)$$

$\leftarrow g >> \omega\varepsilon_r\varepsilon_0 \Rightarrow$ to 0

⑤

$$k = \sqrt{\frac{\omega \mu_0 \mu_r g}{2}} \; (1+i)$$

$$\underline{E} = \underline{E}_0 \, e^{i(kz-\omega t)} = e^{i\left(\sqrt{\frac{\omega\mu_0\mu_r g}{2}}\, z - \omega t\right)} \; e^{-\sqrt{\frac{\mu_0\mu_r g\omega}{2}}\, z}$$

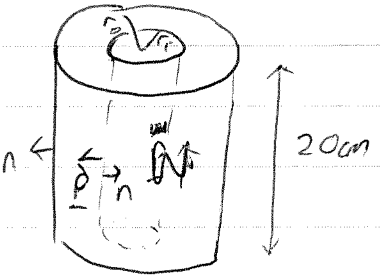
$$\underbrace{}_{e^{-\frac{z}{\delta}}}$$

$$\delta = \sqrt{\frac{2}{\mu_0 \mu_r g \omega}}$$

iii) Using previous expression $k = \sqrt{\frac{\omega\mu_0\mu_r g}{2}} \, (1+i) = \frac{1+i}{\delta}$

$V_\phi = \frac{\omega}{Re\{k\}} = \frac{\omega}{1/\delta} = \omega\delta = 2\pi \times 8\times10^{10} \times 3\times10^{-6} = 1.51\times10^{6} \, ms^{-1}$

10ai)



~~Total polarisation $\underline{P} =$~~

Surface charge density $\sigma = \underline{P} \cdot \hat{n}$

Inner wall $\sigma_i = -\frac{A}{r_i^2}$ Outer wall $\sigma = \frac{A}{r_o^2}$

$\Rightarrow$ Inner surface charge $= 2\pi r_i l \left(-\frac{A}{r_i^2}\right) = -1.68\times10^{-8} C$

$q_0 = 2\pi r_0 l \left(\frac{A}{r_0^2}\right) = 6.28\times10^{-9} C$

ii) Overall charge $= 0 \Rightarrow q_{volume} + q_{inner} + q_{outer} = 0$

$\Rightarrow q_{volume} = 1.68\times10^{-8} C - 6.28\times10^{-9} C = 1.052\times10^{-8} C$

iii) $\underline{D} = \varepsilon_0 \underline{E} + \underline{P}$

$= \underline{P}$ at $r = 4.5cm$ $\underline{D} = \frac{4\times10^{-10}}{(4.5\times10^{-3})^2} = 1.98\times10^{-7} \; C/m^2$

bi) $\underline{M} = 500{,}000(a-r)\hat{\underline{z}}$ A/m

Surface current density $J_{surface} = \underline{M} \times \hat{\underline{n}}$

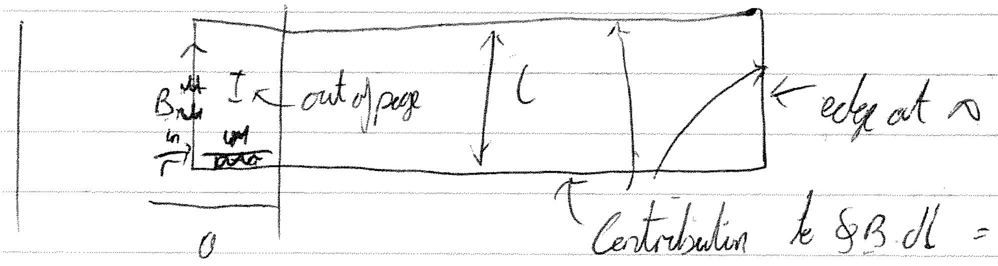
$= 500{,}000\,(a-a)\ \hat{\underline{z}} \times \hat{\underline{r}} = 0$

Volume current density $\underline{J}_v = \nabla \times \underline{M} = \nabla \times 500{,}000\,(a-r)\,\hat{\underline{z}}$

$$= -\frac{\partial}{\partial r}\left(500{,}000(a-r)\right)\hat{\underline{\phi}}$$

$$= 500{,}000\,\hat{\underline{\phi}}$$

ii) ~~$\oint \underline{B}_{out} \cdot d\underline{l} = \mu_0 I_{enc}$~~ Take rectangular path



← edge at ∞

Contribution to $\oint B.dl = 0$ on these edges

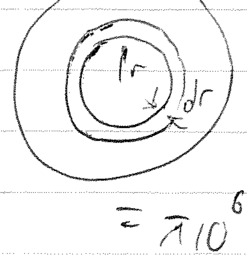
$$\oint \underline{B}_{in} \cdot d\underline{l} = \mu_0 I_{enc}$$

$$B_{in} l = l(a-r)\,500{,}000$$

$$\underline{B_{in}}(r) = (a-r)\,500{,}000\,\hat{z}\ T$$

$J_{surface} = 0$ $\quad B_{out} = 0$

iii) Flux in ring at r width dr $= dB = 2\pi r\,dr\,(a-r)\,500{,}000$

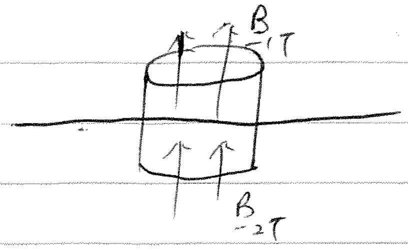


$$\Rightarrow \int \underline{B} \cdot d\underline{A} = \int_{r=0}^{a} 10^6 \pi (ar - r^2)\,dr$$

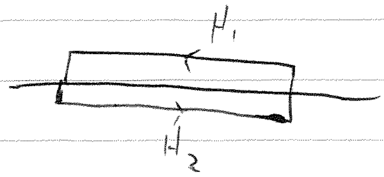
$$= \pi 10^6 \left(\frac{a^3}{2} - \frac{a^3}{3}\right) = \frac{\pi 10^6 a^3}{6} = \frac{4\pi}{3}\ Tm^2$$

⑥

c) Boundary conditions



$\oint \underline{B}\cdot d\underline{A} = 0 \quad A(B_{1T} - B_{2T}) = 0 \Rightarrow B_T$ is continuous



$\oint \underline{H}\cdot d\underline{l} = 0 \quad \Rightarrow H_{1\parallel} = H_{2\parallel}$

Ampere's law

$B_{1\perp} = B_{2\perp} \Rightarrow B_1 \sin\theta_1 = B_2 \sin\theta_2$

$\mu_1 B_{1=} = \mu_2 B_{2=} \Rightarrow \mu_1 B_1 \cos\theta_1 = \mu_2 B_2 \cos\theta_2$

÷ by each other $\dfrac{\tan\theta_1}{\mu_1} = \dfrac{\tan\theta_2}{\mu_2} \Rightarrow \dfrac{\mu_2}{\mu_1} = \dfrac{\tan\theta_2}{\tan\theta_1}$