

PHAS3201: Electromagnetic Theory

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April 17, 2018

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1 Important Equations to Know Right off the Bat

These are not the only equations you need to know, but they are the most important. Each of these is likely to be involved in section A question of the form: ‘Calculate the <insert weird name from the 1800s here> using <insert generic looking symbol with many different meanings here>’. You must know the exact equation for the job. That is what this section is for.

We will be given the necessary vector triple products, but not necessarily the Maxwell equations or Stokes’/Divergence theorem.

1.1 Mathematical Trivia

- **Divergence Theorem**

$$\int_V \nabla \cdot \mathbf{A} dV = \oint_S \mathbf{A} \cdot d\mathbf{S}$$

- **Stokes’ Theorem**

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\mathbf{r}$$

1.2 Definitions

1.2.1 Relationships Between Fields

$$\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu_0 \mu_r (\mathbf{H} + \mathbf{M})$$

The following equations are implied by Gauss’Laws (Maxwell 1/2), and properties of vector (triple) products. It is worth understanding their motivation.

$$\mathbf{A} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

1.2.2 Linear, Isotropic, Homogeneous Shortcuts

For linear, isotropic, homogeneous materials: we can make some nifty approximations. Namely:

- Polarization is linear w.r.t. applied electric field:

$$\mathbf{P} = \epsilon_0 \epsilon_r \chi_e \mathbf{E}$$

- Magnetization is linear w.r.t. applied magnetic field:

$$\mathbf{M} = \chi_m \mathbf{H}$$

- Substitution of the above two approximations into the field equations above yields:

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \epsilon_r (1 + \chi_e) \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mu_r (1 + \chi_m) \mathbf{H}$$

1.2.3 Capacitance

DO THIS

1.2.4 Energy Crap

- **Electric Field Energy Density**

$$u_e = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$$

- **Magnetic Field Energy Density**

$$u_m = \frac{1}{2} \mathbf{H} \cdot \mathbf{B}$$

- **Poynting Vector** (also denoted as \mathbf{S})

$$\mathbf{N} = \frac{1}{\mu_0 \mu_r} \mathbf{E} \times \mathbf{B}$$

1.3 Maxwell and Related

1.3.1 Maxwell

- Gauss' Electric Law

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

- Gauss' Magnetic Law

$$\nabla \cdot \mathbf{B} = 0$$

- Faraday's Law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- Maxwell-Ampere Law

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

1.3.2 Biot-Savart Law

$$\int d\mathbf{B} = \int \frac{\mu_0}{4\pi} I \frac{d\mathbf{l} \times \mathbf{r}}{|\mathbf{r}|^3}$$

1.4 Polarization and Current Density Equations

\mathbf{P} is the polarization density of a material, and is defined for a homogeneous material as:

$$\mathbf{P} = \frac{\mathbf{p}}{V}$$

Where \mathbf{p} is the overall polarization, and V is the volume.

1.4.1 Electric

- Surface polarization charge density (this is very easy for simple shapes and clearly defined polarisation distributions):

$$\sigma_P = \mathbf{P} \cdot \mathbf{n}$$

- Volume polarization charge density:

$$\rho_P = -\nabla \cdot \mathbf{P}$$

- Polarization charge (get it from ρ_P and Gauss' Electric Law):

$$Q_P = -\oint_S \mathbf{P} \cdot d\mathbf{A}$$

- Current Density due to electric polarization:

$$\mathbf{J}_{\text{electric}} = \frac{\partial \mathbf{P}}{\partial t}$$

1.4.2 Magnetic

\mathbf{M} is the magnetization density of a material, and is defined for a homogeneous material as:

$$\mathbf{M} = \frac{\mathbf{m}}{V}$$

Where \mathbf{m} is the overall magnetization, and V is the volume.

- Surface Current Density

$$J_{\text{magnetic}_S} = \mathbf{M} \times \hat{\mathbf{n}}$$

- Volume Current Density

$$J_{\text{magnetic}_V} = \nabla \times \mathbf{M}$$

1.4.3 Current Density

$$\begin{aligned}\mathbf{J} &= \mathbf{J}_{\text{free}} + \mathbf{J}_{\text{electric}} + \mathbf{J}_{\text{magnetic}_V} \\ &= \mathbf{J}_{\text{free}} + \frac{\partial \mathbf{P}}{\partial t} + \nabla \times \mathbf{M}\end{aligned}$$

Where σ is the electric conductivity of the material, and \mathbf{J}_{free} has been approximated as linearly proportional to the applied field (free electric charges driven by applied field). This is appropriate for linear, isotropic, homogeneous materials.

GOTTA KNOW PHYSICAL INTUITION FOR ALL OF THESE!!!

1.5 Gauges

- Coulomb Gauge

$$\nabla \cdot \mathbf{A} = 0$$

- Lorentz Gauge

$$\nabla \cdot \mathbf{A} = \frac{1}{c^2} \frac{\partial \phi}{\partial t}$$

2 Important Proofs, Derivations, and Demonstrations

These are things you need to be capable of justifying. A question of the form: ‘Show that <insert strange expression here> using <insert slightly less strange expression here>’ is the general gist of this section. Enjoy the following nightmare.

2.1 Continuous Components of Fields

HOW DO I DO THIS FUCK

3 Multipole Stuff

We can expand the field or potential of an arbitrary charge distribution in terms of multipoles (the monopole, dipole, quadrupole, etc.). Dipoles are specifically useful for modelling the effect of an electric field on a dielectric.

3.1 The Monopole

From really far away, a charge distribution of any shape behaves like a point charge with effective charge Q . This can be found by taking the integral of the charge distribution over all appropriate space:

$$Q = \int_V \rho(\mathbf{r}) dV$$

3.2 The Electric Dipole

Defined as a pair of equal and opposite charges q and $-q$ separated by a small distance \mathbf{l} , or by using the charge density ρ_P in a small volume:

$$\mathbf{p} = q\mathbf{l} = \int \mathbf{r}_P \rho_P dV$$

The potential at any point \mathbf{r} due to a dipole is:

$$\mathbf{V}(\mathbf{r}) = \frac{ql \cos \theta}{4\pi\epsilon_0 r^2} = \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

The dipole is the first correction to the monopole approximation of a charge distribution.

3.2.1 Properties of Dipole Field and Potential

By definition:

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right)$$

Evaluating this yields:

$$\begin{aligned}E_r &= \frac{\partial V}{\partial r} = -\frac{ql\cos\theta}{2\pi\epsilon_0 r^3} \\E_\theta &= \frac{1}{r}\frac{\partial V}{\partial \theta} = -\frac{ql\sin\theta}{4\pi\epsilon_0 r^3} \\E_\phi &= \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} = 0\end{aligned}$$

Which basically dictates how the potential falls off: as $1/r^3$ for the field, and $1/r^2$ for the potential.

3.2.2 Dielectric Materials as a Bunch of Dipoles

Dielectrics have no free charges (are neutral overall), and consist of nuclei and electrons which deform under a uniform applied field into dipoles (approximately).

DO WE NEED TO KNOW EQUATIONS FOR THIS?

3.3 The Magnetic Dipole

Ch3, WHAT IS THIS?

3.4 Hertzian Dipole

KNOW THIS DO IT!

4 Magnetism in General

Above a certain temperature, thermal excitations mean that the magnetic moments of electrons (electron spins) in a material are 'jiggling' about, and are randomly aligned. They can be realigned by a strong applied field, but will not naturally align unless the material is brought back below this 'Curie temperature' T_C . For this reason: above T_C , all materials become paramagnetic. This temperature is approximately defined as:

$$T_C = \frac{C}{\chi_m}$$

Where C is the Curie constant, and χ_m is the magnetic susceptibility of the material.

4.1 Types of Magnetism

- **Paramagnetism** occurs when the material has naturally randomly aligned electron spins. These spins align parallel to an applied field ($\chi_m > 0$).
- **Diamagnetism** occurs when the material has all electrons paired, opposes applied field ($\chi_m < 0$). **CONCEPTUAL EXPLANATION** It is the weakest of all magnetic effects, and so only occurs appreciably in purely diamagnetic materials below the Curie temperature.
- **Ferromagnetism** occurs below the Curie temperature, when individual domains within the material have the spins of all their constituent electrons aligned. Overall, the ferromagnet has no magnetic moment, but each individual domain does¹. These domains align with the applied field. The differences between 'hard' and 'soft' ferromagnets are clear when looking at Hysteresis. This type of magnetism is by far the strongest. Ferromagnets keep their magnetic overall magnetization after the applied field is removed.

¹Domains are on the order of $1 - 10\mu\text{m}$ across.

- **Anti-Ferromagnetism** occurs below the Curie temperature, when individual domains within the material have the spins of all their constituent electrons aligned anti-parallel. Overall, the ferromagnet has no magnetic moment, and neither does each individual domain. These domains and their constituents align with an applied field, but do not retain this magnetization when the applied field is removed.
- **Ferrimagnetism** is similar to Anti-Ferromagnetism, but the anti-alignment doesn't completely cancel out. One of the directions has a stronger overall magnetic moment than the other. Similarly to ferromagnets, they retain this overall magnetization when the applied field is removed.

4.1.1 Hysteresis

Below the critical temperature of a ferromagnet (the temperature which destroys the magnetic behaviour), the application of an increasing magnetic field \mathbf{H} will slowly make all the domains align - giving the material an overall magnetization. When the applied field is removed, this overall magnetization weakens, but sticks around. This behavior, and subsequent behavior corresponding to the reversal of the applied field yields what is called a 'Hysteresis loop', the main features of which are outlined below.

FINISH THIS, HARD VS SOFT, REMENANCE, COERCIVITY, SATURATION.

5 Poynting Vector

The Poynting Vector \mathbf{N} (sometimes \mathbf{S} in other literature) is defined as:

$$\mathbf{N} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

And is the directional energy transferred per unit area per unit time. It's units are W/m^2 . Using plane wave solutions for $\mathbf{E} = \mathbf{E}_0 \exp[-i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$ and using Faraday's Law, it is possible to show that the time-average of \mathbf{N} is:

$$\langle \mathbf{N} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0 \epsilon_r}{\mu_0 \mu_r}} E_0^2 \hat{\mathbf{k}}$$

5.1 EM Pressure

The momentum carried by a single photon is:

$$p_i = \frac{E_i}{c}$$

The average momentum carried by a whole bunch of photons is:

$$\left\langle \sum_i p_i \right\rangle = \frac{\langle \sum_i E_i \rangle}{c}$$

So the change of momentum per unit area per unit time is:

$$\Delta p = \frac{\langle \mathbf{N} \rangle}{c}$$

Which (for $\mu_r = \epsilon_r = 1$, and reflected photons) is:

$$\Delta p = \epsilon_0 E_0^2$$

Or half that for absorbed photons.

6 Plasma

Plasma is defined as a fluid in which there are an appreciable number of free moving charged particles. The simplest plasma is one which is neutral overall.

6.1 Plasma Frequency

A block of fluid of surface area A in which all electrons are displaced by the vector \mathbf{x} is equivalent to a capacitor with charge Q , where:

$$Q = N_e A e \mathbf{x}$$

Where N_e is the number of electrons per unit volume, and e is the electric charge. Moreover, each electron/proton pair now forms a dipole of length $|\mathbf{x}|$, yielding a polarization density of:

$$\mathbf{P} = N_e e \mathbf{x}$$

Such that (using the equation for the electric field of a capacitor):

$$\mathbf{E} = \frac{C}{\varepsilon_0 A} = \frac{Q}{\varepsilon_0 A} = \frac{N_e e}{\varepsilon_0 A}$$

FINISH THIS

7 Special Relativity

ALL OF IT, Standard questions involve parallel and perpendicular \mathbf{B} and \mathbf{E} fields, along with all the standard transformation stuff. Velocities which remove both fields etc.

8 Common Stuff

Linear material with conductivity g , plane wave solution.

Continuity equation.

$\mathbf{E} \cdot \mathbf{B}$ invariant for EM waves?

Know Coulomb and Lorentz Gauges.