Machine Learning: Mathematical Background

Linear Algebra

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Lecture Overview

1 Lecture Overview

2 Linear Spaces & Linear Mappings

3 Geometrical Structure

4 Linear Equations

5 Matrix Decompositions

6 Quadratic Forms

7 Useful Results

8 Summary

9 Appendix: Further Proofs

Maths & Machine Learning

- Much of machine learning is concerned with:
  - Solving systems of linear equations ⟶ **Linear Algebra**
  - Minimising cost functions (a scalar function of several variables that typically measures how poorly our model fits the data).
    To this end we are often interested in studying the continuous change of such functions ⟶ **(Differential) Calculus**
  - Characterising uncertainty in our learning environments stochastically ⟶ **Probability**
  - Drawing conclusions based on the analysis of data ⟶ **Statistics**

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Learning Outcomes for Today’s Lecture

- By the end of this lecture you should be familiar with some fundamental objects in and results of **Linear Algebra**
- For the most part we will concentrate on the statement of results which will be of use in the main body of this module
- However we will not be so concerned with the proof of these results

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Vector Spaces

- Setting in which **linear algebra** takes place
- A **vector space**, V, is a set, the elements of which are called **vectors**, denoted by bold lower case letters, e.g. $\mathbf{x}$, $\mathbf{y}$ etc.
- Two operations are defined on a vector space:
  - **Vector Addition**
  - **Scalar Multiplication**
- For our purposes a **scalar** is a real number, usually denoted by a lower case letter

Vector Spaces

- V must satisfy:

  1. **Additive Closure**: if $\mathbf{x}, \mathbf{y} \in V$ then $\mathbf{x} + \mathbf{y} \in V$
  2. **Scalar Closure**: if $\alpha \in \mathbb{R}$ and $\mathbf{x} \in V$ then $\alpha\mathbf{x} \in V$
  3. **Identity Element of Addition**: $\exists$ a **zero vector**, $\mathbf{0}$, such that: $\mathbf{x} + \mathbf{0} = \mathbf{x}, \quad \forall \mathbf{x} \in V$
  4. **Inverse Element of Addition**: $\exists$ an **additive inverse**, $-\mathbf{x}$, for each $\mathbf{x} \in V$, such that: $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
  5. **Identity Element of Scalar Multiplication**: $\exists$ a **multiplicative identity**, 1, such that: $1\mathbf{x} = \mathbf{x}, \quad \forall \mathbf{x} \in V$
  6. **Commutativity**: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}, \quad \forall \mathbf{x}, \mathbf{y} \in V$
  7. **Associativity**: $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$; and $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}, \quad \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\alpha, \beta \in \mathbb{R}$
  8. **Distributivity**: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$; and $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}, \quad \forall \mathbf{x}, \mathbf{y} \in V$ and $\alpha, \beta \in \mathbb{R}$

Linear Independence, Span, Basis & Dimension

- A set of vectors $\mathbf{v}_1, ..., \mathbf{v}_n \in V$ is **linearly independent** if:

$$\alpha_1\mathbf{v}_1 + ... + \alpha_n\mathbf{v}_n = \mathbf{0} \quad \implies \quad \alpha_1 = ... = \alpha_n = 0$$

- The **span** of $\mathbf{v}_1, ..., \mathbf{v}_n \in V$, is the set of all vectors that can be expressed as a linear combination of them:

$$\text{span}(\mathbf{v}_1, ..., \mathbf{v}_n) = \{\mathbf{v} \quad | \quad \mathbf{v} = \beta_1\mathbf{v}_1 + ... + \beta_n\mathbf{v}_n \quad \forall\beta_1, ..., \beta_n \in \mathbb{R}\}$$

- A **basis** for V is a set of vectors which are linearly independent and which span the whole of V
  - So every linearly independent set of vectors forms a basis for its span

- A **dimension** of V, $\dim(V)$, is the number of vectors in a basis

Euclidean Space

- So far our language has been abstract, but we will be interested in a particular vector space: the **Euclidean space**, $\mathbb{R}^n$
- Here the vectors are *n*-tuples of real numbers defined as **column vectors**, e.g.:

$$\mathbf{x} = \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{bmatrix}$$

- Similarly a **row vector** is defined as:

$$\mathbf{x}^T = [x_1, x_2, ..., x_n]$$

- Note: On occasion we use $[\mathbf{x}]_i$ as an alternative to x_i

Euclidean Space: Addition & Multiplication

- **Vector Addition**:

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \cdot \\ \cdot \\ x_n + y_n \end{bmatrix}$$

- **Scalar Multiplication**:

$$\alpha \mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \cdot \\ \cdot \\ \alpha x_n \end{bmatrix}$$

Euclidean Space: The Standard Basis

- One particular basis in $\mathbb{R}^n$ is the **standard basis**:

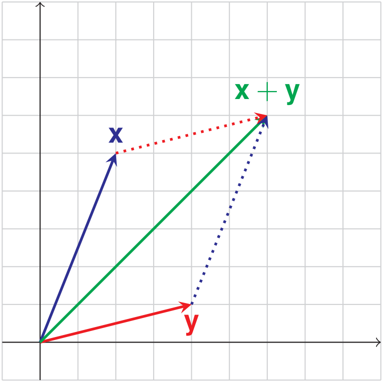
$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots \quad , \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

- Any vector **x** can be expressed in the standard basis:

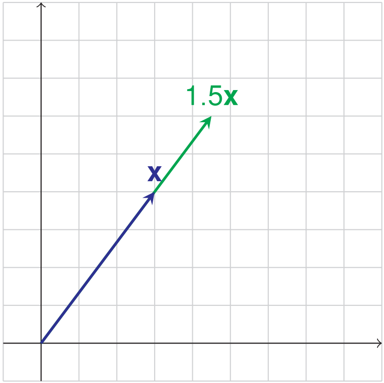
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n$$

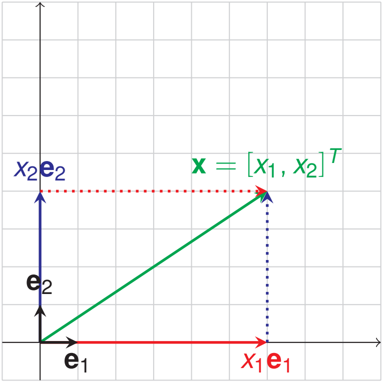
Euclidean Space: Vector Addition in $\mathbb{R}^2$



Euclidean Space: Scalar Multiplication in $\mathbb{R}^2$



Euclidean Space: The Standard Basis in $\mathbb{R}^2$



Subspaces

- If V is a vector space then $S \subseteq V$ is a **subspace** of V if:
  1. $\mathbf{0} \in S$
  2. S is closed under addition: $\mathbf{x}, \mathbf{y} \in S \implies (\mathbf{x} + \mathbf{y}) \in S$
  3. S is closed under scalar multiplication: $\mathbf{x} \in S, \alpha \in \mathbb{R} \implies \alpha\mathbf{x} \in S$
- So if U and W are subspaces of V then so is their sum:

$$U + W = \{\mathbf{u} + \mathbf{w} | \mathbf{u} \in U, \mathbf{w} \in W\}$$

- If $U \cap W = \{\mathbf{0}\}$ then $U + W$ is a **direct sum**, written as: $U \oplus W$
- It can be shown that:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$$

- And in particular:

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

Affine Subspaces

- If V is a vector space, $\mathbf{x}_0$ is a vector within V, $\mathbf{x}_0 \in V$, and $U \subseteq V$ is a subspace of V, then the subset L is an **affine subspace** or **linear manifold** of V if:

$$\begin{aligned} L &= \mathbf{x}_0 + U \\ &= \{\mathbf{v} \in V \mid \exists \mathbf{u} \in U : \mathbf{v} = \mathbf{x}_0 + \mathbf{u}\} \subseteq V \end{aligned}$$

Where:
U is called the **direction space**
$\mathbf{x}_0$ is called the **support point**

Affine Subspaces

- Note that an affine subspace for which $\mathbf{x}_0 \notin U$ will exclude the null vector, $\mathbf{0}$, and hence is **not** a vector space

  Such an affine subspace does not go through the origin

Affine Subspaces: Parametric Spaces

- Affine subspaces can described by **parameters**:
- If L is a k-dimensional affine space and $\{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k\}$ is a basis of U, then every element $\mathbf{x} \in L$ can be described by:

$$\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{x}_1 + \ldots + \lambda_k \mathbf{x}_k$$

Where $\{\lambda_i \in \mathbb{R}\}_{i=1}^{k}$ are a set **parameters**.

Affine Subspaces: Lines, Planes & Hyperplanes

- **Lines** are 1-dimensional affine subspaces, with elements $\mathbf{y} \in \mathbb{R}^n$, described as:

  $$\mathbf{y} = \mathbf{x}_0 + \lambda \mathbf{x}_1$$

  Where $\lambda \in \mathbb{R}$, $U = \text{span}(\mathbf{x}_1) \subseteq \mathbb{R}^n$

- **Planes** are 2-dimensional affine subspaces, with elements $\mathbf{y} \in \mathbb{R}^n$, described as:

  $$\mathbf{y} = \mathbf{x}_0 + \lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2$$

  Where $\lambda_1, \lambda_2 \in \mathbb{R}$, $U = \text{span}(\mathbf{x}_1, \mathbf{x}_2) \subseteq \mathbb{R}^n$

- **Hyperplanes** are $(n-1)$-dimensional affine subspaces in $\mathbb{R}^n$, with elements $\mathbf{y} \in \mathbb{R}^n$, described as:

  $$\mathbf{y} = \mathbf{x}_0 + \sum_{i=1}^{n-1} \lambda_i \mathbf{x}_i$$

  Where $\{\lambda_i\}_{i=1}^{n-1} \in \mathbb{R}$, $U = \text{span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{n-1}) \subseteq \mathbb{R}^n$

Matrices

- Denote by a bold upper case letter, e.g. $\mathbf{A}$

- A matrix, $\mathbf{A} \in \mathbb{R}^{n \times m}$ is an $n \times m$ array of numbers with elements $\left\{A_{ij}\right\}_{i,j=1}^{n,m}$, i.e.:

$$\mathbf{A} = \underline{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ A_{21} & A_{22} & \dots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nm} \end{bmatrix}$$

- Note: On occasion we use $[\mathbf{A}]_{ij}$ as an alternative to A_{ij}

Linear Maps

- A **linear map** is a function, $f : V \rightarrow W$ where V and W are vector spaces, that satisfies:

  1. $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y}), \quad \forall\ \mathbf{x}, \mathbf{y} \in V$
  2. $f(\alpha \mathbf{x}) = \alpha f(\mathbf{x}), \quad \forall\ \mathbf{x} \in V, \forall\ \alpha \in \mathbb{R}$

Matrices as Linear Maps

- In particular, suppose V and W are finite-dimensional vector spaces with bases $\{\mathbf{v}_i\}_{i=1}^m$ and $\{\mathbf{w}_i\}_{i=1}^n$ respectively, then every matrix $\mathbf{A} \in \mathbb{R}^{n\times m}$ induces a linear map, $f : \mathbb{R}^m \to \mathbb{R}^n$ given by:

  $$f(\mathbf{x}) = \mathbf{Ax}$$

  where $\mathbf{x} \in V$ and $f(\mathbf{x}) \in W$, and where:

  $$[\mathbf{Ax}]_i = \sum_{j=1}^m A_{ij}x_j \qquad \text{for: } i = 1, ..., n$$

- For $n = 3, m = 2$:

$$f(\mathbf{x}) = \mathbf{Ax} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \\ A_{31}x_1 + A_{32}x_2 \end{bmatrix}$$

Matrix Addition

- We can define a **matrix addition** operation for $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{R}^{n \times m}$ such that:

  $$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

  By:

  $$C_{ij} = A_{ij} + B_{ij}$$

- For $n = 3$, $m = 2$:

$$\begin{aligned}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \\ A_{31} + B_{31} & A_{32} + B_{32} \end{bmatrix}\end{aligned}$$

Matrix Addition

- This definition implies that for $\mathbf{x} \in \mathbb{R}^m$:

$$\begin{aligned} [\mathbf{Cx}]_i = \sum_{j=1}^{m} C_{ij} x_j &= \sum_{j=1}^{m} (A_{ij} + B_{ij}) x_j \\ &= \sum_{j=1}^{m} A_{ij} x_j + \sum_{j=1}^{m} B_{ij} x_j \\ &= [\mathbf{Ax}]_i + [\mathbf{Bx}]_i \end{aligned}$$

- Hence the definition implies that:

$$\mathbf{Cx} = (\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{Ax} + \mathbf{Bx}$$

- Thus matrix addition offers a more efficient mechanism for adding **Ax** and **Bx**

Matrix Multiplication

- We can define a **matrix multiplication** operation for $\mathbf{A} \in \mathbb{R}^{n \times l}$, $\mathbf{B} \in \mathbb{R}^{l \times m}$, $\mathbf{C} \in \mathbb{R}^{n \times m}$ such that:

  $$\mathbf{C} = \mathbf{AB}$$

  By:

  $$C_{ik} = \sum_{k=1}^{l} A_{ik} B_{kj}$$

- For $n = 3, m = 2, l = 3$:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \\ C_{31} & C_{32} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\ A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} & A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \end{bmatrix}$$

- And in general $\mathbf{BA} \neq \mathbf{AB}$

Matrix Multiplication

- This definition implies that for $\mathbf{x} \in \mathbb{R}^m$:

$$\begin{aligned}
[\mathbf{Cx}]_i = \sum_{j=1}^{m} C_{ij} x_j = \sum_{j=1}^{m} \sum_{k=1}^{l} A_{ik} B_{kj} x_j &= \sum_{k=1}^{l} A_{ik} \sum_{j=1}^{m} B_{kj} x_j \\
&= \sum_{k=1}^{l} A_{ik} [\mathbf{y}]_k \\
&= [\mathbf{z}]_i
\end{aligned}$$

  Where: $\mathbf{y} = \mathbf{Bx}$ and $\mathbf{z} = \mathbf{Ay}$
- Hence the definition implies that:

$$\mathbf{Cx} = \mathbf{ABx} = \mathbf{A}(\mathbf{Bx})$$

- Thus matrix multiplication offers a mechanism for performing the operations $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^l$ followed by $\mathbf{A} : \mathbb{R}^l \to \mathbb{R}^n$

Scalar Multiplication

- We can define a **scalar multiplication** operation for $\mathbf{A}, \mathbf{C} \in \mathbb{R}^{n \times m}$ and $\alpha \in \mathbb{R}$ such that:

  $$\mathbf{C} = \alpha \mathbf{A}$$

  By:

  $$C_{ij} = \alpha A_{ij}$$

- For $n = 3$, $m = 3$:

$$\alpha \mathbf{A} = \begin{bmatrix} \alpha A_{11} & \alpha A_{12} & \alpha A_{13} \\ \alpha A_{21} & \alpha A_{22} & \alpha A_{23} \\ \alpha A_{31} & \alpha A_{32} & \alpha A_{33} \end{bmatrix}$$

Scalar Multiplication

- This definition implies that for $\mathbf{x} \in \mathbb{R}^m$:

$$\begin{aligned}[\mathbf{Cx}]_i = \sum_{j=1}^{m} C_{ij} x_j = \sum_{j=1}^{m} \alpha A_{ij} x_j &= \alpha \sum_{j=1}^{m} A_{ij} x_j \\ &= \alpha [\mathbf{Ax}]_i\end{aligned}$$

- Hence the definition implies that:

$$(\alpha \mathbf{A})\mathbf{x} = \alpha(\mathbf{Ax})$$

Matrix Transpose

- If $\mathbf{A} \in \mathbb{R}^{n\times m}$ then its **transpose**, $\mathbf{A}^T \in \mathbb{R}^{m\times n}$ is defined by:

$$A^T{}_{ij} = A_{ji} \qquad \forall\, i, j$$

- Therefore:

  1. $(\mathbf{A}^T)^T = \mathbf{A}$
  2. $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  3. $(\alpha\mathbf{A})^T = \alpha\mathbf{A}^T$
  4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$

Matrix Identity

- The **identity matrix**, $\mathbf{I}$, is a **square** matrix with 1's on the diagonal and zeroes elsewhere

- $\mathbf{I}_n$ is the $n \times n$ identity matrix:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

- For $\mathbf{A} \in \mathbb{R}^{n \times m}$ then:

$$\mathbf{I}_n\mathbf{A} = \mathbf{A}\mathbf{I}_m = \mathbf{A}$$

Matrix Inverse

- For a square matrix, **A**, its **inverse** (if it exists) satisfies:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \mathbf{A}\mathbf{A}^{-1}$$

- If $\mathbf{A}^{-1}$ does not exist then we say that **A** is **singular**

- For square matrices **A**, **B** with inverses $\mathbf{A}^{-1}$, $\mathbf{B}^{-1}$ respectively, then:

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

- In particular, for $\mathbf{A} \in \mathbb{R}^{2\times 2}$:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Longrightarrow \quad \mathbf{A}^{-1} = \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Where $a, b, c, d \in \mathbb{R}$

Symmetric Matrices

- A square matrix, $\mathbf{A} \in \mathbb{R}^{n \times n}$, is **symmetric** if $\mathbf{A}^T = \mathbf{A}$

Nullspace & Range

- If **A** is a linear map, $\mathbf{A} : V \to W$ then the **nullspace**, or **kernel**, of **A** is:

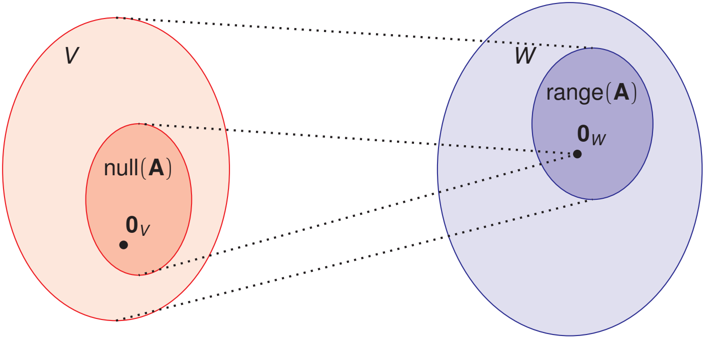
$$\text{null}(\mathbf{A}) = \{\mathbf{v} \in V | \mathbf{A}\mathbf{v} = \mathbf{0}\}$$

- and the **range**, or **image**, of **A** is:

$$\text{range}(\mathbf{A}) = \{\mathbf{w} \in W | \exists \mathbf{v} \in V \text{ such that } \mathbf{A}\mathbf{v} = \mathbf{w}\}$$

Nullspace & Range

$\mathbf{A} : V \rightarrow W$



Columnspace & Rowspace

- The **columnspace** of a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ is the span of its m columns (which are all vectors in $\mathbb{R}^n$)
- The **rowspace** of a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ is the span of its n rows (which are all row vectors in $\mathbb{R}^m$)
- So:

$$\begin{aligned} \text{columnspace}\left(\mathbf{A}\right) &= \text{range}\left(\mathbf{A}\right) \\ \text{rowspace}\left(\mathbf{A}\right) &= \text{range}\left(\mathbf{A}^T\right) \end{aligned}$$

- Recall from our definitions of **Span**, **Basis** and **Dimension** that the dimension of the columnspace (rowspace) is equal to the number of **linearly independent** vectors amongst the columns (rows) of **A**

Rank

- It can be shown (See Appendix for proof) that dimension of the columnspace is equal to the dimension of the rowspace, and this dimension is known as the **rank** of **A**:

$$\text{rank}(\mathbf{A}) = \dim\left(\text{range}(\mathbf{A})\right)$$

- So the **rank** of **A** is equal to the number of linearly independent vectors amongst the columns (rows) of **A**

- Of course, $\dim\left(\text{range}(\mathbf{A})\right) \leqslant m$ and $\dim\left(\text{range}(\mathbf{A}^T)\right) \leqslant n$, so:

$$\text{rank}(\mathbf{A}) \leqslant \min(n, m)$$

Rank

- For a matrix $\mathbf{A} \in \mathbb{R}^{n \times l}$ and a matrix $\mathbf{B} \in \mathbb{R}^{l \times m}$:

$$\text{rank}(\mathbf{AB}) \leqslant \text{rank}(\mathbf{A})$$

  - **Proof:**
    Recall:

$$\begin{aligned} \text{rank}(\mathbf{AB}) &= \dim\left(\text{range}(\mathbf{AB})\right) \\ \text{rank}(\mathbf{A}) &= \dim\left(\text{range}(\mathbf{A})\right) \end{aligned}$$

    Now, consider any vector $\mathbf{y} \in \text{range}(\mathbf{AB})$, then by the definition of the range there exists some $\mathbf{x}$ such that:

$$\begin{aligned} & \mathbf{y} = (\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) = \mathbf{Az} \qquad \text{where:} \quad \mathbf{z} = \mathbf{Bx} \\ \implies \quad & \mathbf{y} \in \text{range}(\mathbf{A}) \\ \implies \quad & \text{range}(\mathbf{AB}) \subseteq \text{range}(\mathbf{A}) \\ \implies \quad & \dim(\text{range}(\mathbf{AB})) \leqslant \dim(\text{range}(\mathbf{A})) \\ \implies \quad & \text{rank}(\mathbf{AB}) \leqslant \text{rank}(\mathbf{A}) \end{aligned}$$

Rank

- For a matrix $\mathbf{A} \in \mathbb{R}^{n \times l}$ and a matrix $\mathbf{B} \in \mathbb{R}^{l \times m}$:

$$\text{rank}(\mathbf{AB}) \leqslant \min\left(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\right)$$

  - **Proof:**
    We know that:

$$\begin{aligned} \text{rank}(\mathbf{AB}) &\leqslant \text{rank}(\mathbf{A}) \\ \text{rank}(\mathbf{AB}) &\leqslant \text{rank}(\mathbf{B}) \end{aligned}$$

    Since these hold simultaneously:

$$\implies \quad \text{rank}(\mathbf{AB}) \leqslant \min\left(\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\right)$$

Nullity

- The dimension of the nullspace of $\mathbf{A} \in \mathbb{R}^{n \times m}$ is known as the **nullity** of $\mathbf{A}$:

$$\text{nullity}(\mathbf{A}) = \dim\left(\text{null}(\mathbf{A})\right)$$

- **Rank-Nullity Theorem**:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = m$$

(See Appendix for proof)

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Norms

- A **norm** equips a vector space with a notion of **length**
- More formally a norm on a real vector space, V, is a function, $\| \cdot \| : V \to \mathbb{R}$ that satisfies the following, $\forall$ $\mathbf{x}, \mathbf{y} \in V$ and $\forall$ $\alpha \in \mathbb{R}$:

1. $\|\mathbf{x}\| \geqslant 0$ with equality iff $\mathbf{x} = \mathbf{0}$
2. $\|\alpha \mathbf{x}\| = |\alpha| \|\mathbf{x}\|$
3. $\|\mathbf{x} + \mathbf{y}\| \leqslant \|\mathbf{x}\| + \|\mathbf{y}\|$

Norms

- Examples of norms include:

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{where: } p \geqslant 1$$

Inner Product

- An **inner product** equips a vector space with a notion of **similarity**

- More formally an inner product on a real vector space, V, is a function, $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ that satisfies the following, $\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$ and $\forall\ \alpha \in \mathbb{R}$:

  1. $\langle\mathbf{x}, \mathbf{x}\rangle \geqslant 0$ with equality iff $\mathbf{x} = \mathbf{0}$
  2. $\langle\mathbf{x}+\mathbf{y}, \mathbf{z}\rangle = \langle\mathbf{x}, \mathbf{z}\rangle + \langle\mathbf{y}, \mathbf{z}\rangle$; and $\langle\alpha\mathbf{x}, \mathbf{y}\rangle = \alpha\langle\mathbf{x}, \mathbf{y}\rangle$
  3. $\langle\mathbf{x}, \mathbf{y}\rangle = \langle\mathbf{y}, \mathbf{x}\rangle$

- An inner product on V induces a norm on V:

$$\|\mathbf{x}\| = \sqrt{\langle\mathbf{x}, \mathbf{x}\rangle}$$

Orthonormality

- Two vectors $\mathbf{x}, \mathbf{y}$ are said to be **orthogonal** if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$
- If two orthogonal vectors $\mathbf{x}, \mathbf{y}$ have unit length, i.e. $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$, then they are described as **orthonormal**

Scalar Product

- For Euclidean space (i.e. $V = \mathbb{R}^n$) then the standard inner product is the **dot product** or **scalar product**:

$$\begin{aligned}\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} &= \sum_{i=1}^{n} x_i y_i \\ &= \mathbf{x}^T \mathbf{y}\end{aligned}$$

- And the norm induced by this inner product is the **two-norm**, $\| \cdot \|_2$

- In addition for Euclidean space we can define the notion of **angle**, θ, between two vectors $\mathbf{x}$, $\mathbf{y}$ via the dot product:

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\|_2 \|\mathbf{y}\|_2 \cos\theta$$

Orthonormality in $\mathbb{R}^n$

- Members of the standard basis in $\mathbb{R}^n$, $\{\mathbf{e}_i\}_{i=1}^n$ are **orthonormal**
- For example, (for $n = 2$):

$$\mathbf{x} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 8\mathbf{e}_1 + 4\mathbf{e}_2$$

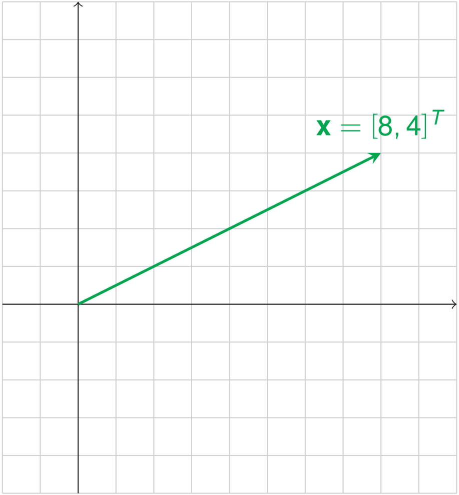
- But other orthonormal bases exist, for example, (for $n = 2$):

$$\mathbf{b}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \qquad \mathbf{b}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

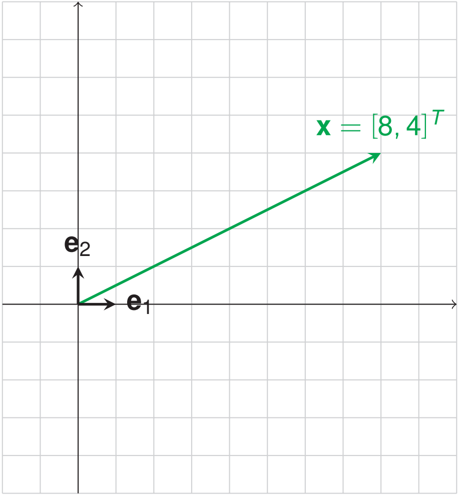
And this basis yields the following expression:

$$\mathbf{x} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} = 6\sqrt{2}\mathbf{b}_1 + 2\sqrt{2}\mathbf{b}_2$$

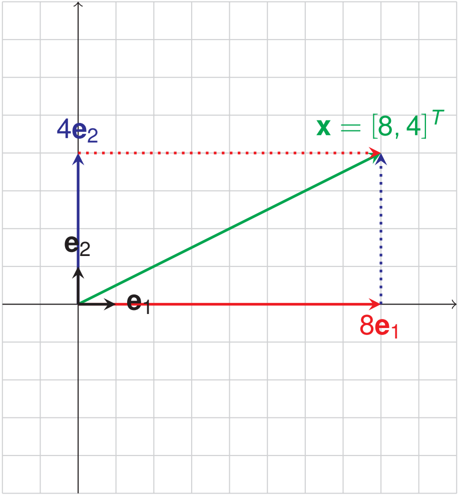
Orthonormality in $\mathbb{R}^2$



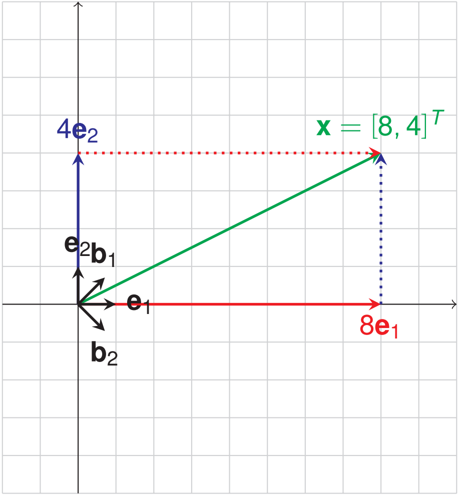
Orthonormality in $\mathbb{R}^2$



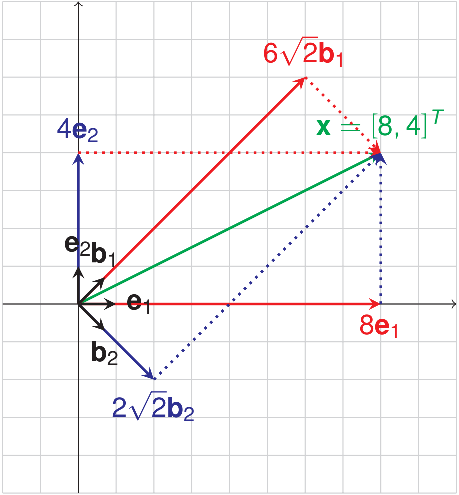
Orthonormality in $\mathbb{R}^2$



Orthonormality in $\mathbb{R}^2$



Orthonormality in $\mathbb{R}^2$



Orthogonal Matrices

- A square matrix, $\mathbf{Q} \in \mathbb{R}^{n \times n}$, is **orthogonal** if its columns are orthonormal:

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$$

- Therefore:

$$\mathbf{Q}^T = \mathbf{Q}^{-1}$$

- Orthogonal matrices preserve inner products:

$$(\mathbf{Qx}) \cdot (\mathbf{Qy}) = (\mathbf{Qx})^T(\mathbf{Qy}) = \mathbf{x}^T\mathbf{Q}^T\mathbf{Q}\mathbf{y} = \mathbf{x}^T\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$$

- ...consequently they preserve 2-norms

- So multiplication by an orthogonal matrix can be considered to be a mapping that preserves vector length, but **rotates** or **reflects** the vector about the origin

Projections

- In general a **projection**, $\mathbf{P}_U : V \rightarrow U$, is a linear mapping from a vector space, V, to a subspace of V, $U \subseteq V$, such that:

$$\mathbf{P}_U \mathbf{P}_U = \mathbf{P}_U$$

- We will have particular interest in **orthogonal projections** in Euclidean space, which map a vector $\mathbf{x} \in \mathbb{R}^n$ to a vector $\mathbf{P}_U \mathbf{x} \in U \subseteq \mathbb{R}^n$, that is 'closest' to $\mathbf{x}$, such that $\|\mathbf{x} - \mathbf{P}_U \mathbf{x}\|_2$ is minimal.

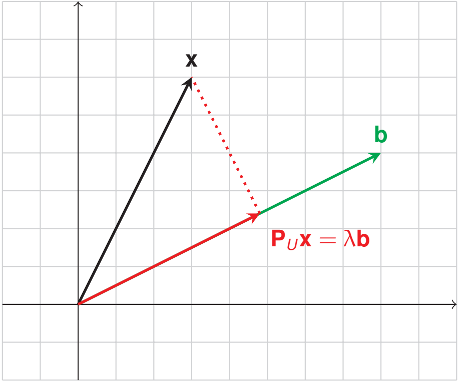
Orthogonal Projections onto Lines

- Consider a 1-dimensional subspace passing through the origin (i.e. a line), described by the basis vector, **b**

- An orthogonal projection of a vector $\mathbf{x} \in \mathbb{R}^n$ onto this line must satisfy:

  1. $\mathbf{P}_U\mathbf{x} = \lambda\mathbf{b}$ for some: $\lambda \in \mathbb{R}$
  2. $\|\mathbf{x} - \mathbf{P}_U\mathbf{x}\|_2^2$ must be minimal

Orthogonal Projections onto Lines



Orthogonal Projections onto Lines

- Thus we seek λ that satisfies:

$$\underset{\lambda}{\operatorname{argmin}} \|\mathbf{x} - \lambda \mathbf{b}\|_2^2$$

- Differentiating with respect to λ and seeking stationary points implies:

$$\begin{aligned} 2(\mathbf{x} - \lambda \mathbf{b}) \cdot \mathbf{b} &= 0 \\ \implies \qquad \lambda &= \frac{\mathbf{x} \cdot \mathbf{b}}{\|\mathbf{b}\|_2^2} \end{aligned}$$

Orthogonal Projections onto Lines

- This tells us that $(\mathbf{x} - \mathbf{P}_U)$ is orthogonal to **b** since:

$$\begin{aligned}(\mathbf{x} - \mathbf{P}_U) \cdot \mathbf{b} &= \mathbf{x} \cdot \mathbf{b} - \lambda \|\mathbf{b}\|_2^2 \\ &= \mathbf{x} \cdot \mathbf{b} - \frac{\mathbf{x} \cdot \mathbf{b}}{\|\mathbf{b}\|_2^2} \|\mathbf{b}\|_2^2 = 0\end{aligned}$$

- Furthermore, we can derive an explicit form for the matrix $\mathbf{P}_U$:

$$\begin{aligned}\mathbf{P}_U \mathbf{x} &= \lambda \mathbf{b} = \mathbf{b} \lambda \\ &= \mathbf{b} \frac{\mathbf{x} \cdot \mathbf{b}}{\|\mathbf{b}\|_2^2} = \mathbf{b} \frac{\mathbf{b}^T \mathbf{x}}{\|\mathbf{b}\|_2^2} = \frac{\mathbf{b}\mathbf{b}^T}{\|\mathbf{b}\|_2^2} \mathbf{x}\end{aligned}$$

- Thus:

$$\mathbf{P}_U = \frac{\mathbf{b}\mathbf{b}^T}{\|\mathbf{b}\|_2^2}$$

Orthogonal Projections onto Subspaces

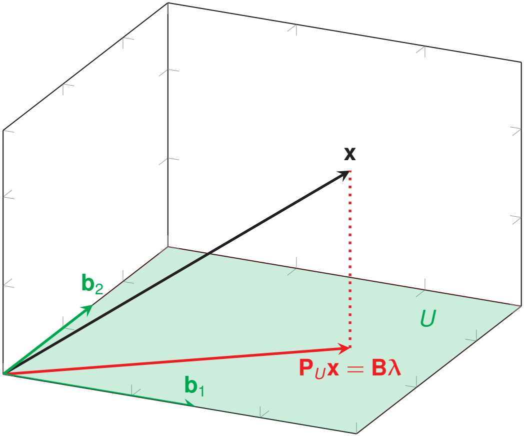
- Consider an m-dimensional subspace, $U \subseteq \mathbb{R}^n$, passing through the origin, described by the basis vectors, $\{\mathbf{b}_i\}_{i=1}^{m}$

- An orthogonal projection of a vector $\mathbf{x} \in \mathbb{R}^n$ onto this subspace must satisfy:

  1. $\mathbf{P}_U\mathbf{x} = \sum_{i=1}^{m} \lambda_i \mathbf{b}_i = \mathbf{B}\boldsymbol{\lambda}$
     Where:
     $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_m] \in \mathbb{R}^{n \times m}$, $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T \in \mathbb{R}^m$

  2. $\|\mathbf{x} - \mathbf{P}_U\mathbf{x}\|_2^2$ must be minimal

Orthogonal Projections onto Subspaces: 2-D Example



Orthogonal Projections onto Subspaces

- Thus we seek λ that satisfies:

$$\underset{\boldsymbol{\lambda}}{\operatorname{argmin}} \|\mathbf{x} - \mathbf{B}\boldsymbol{\lambda}\|_2^2$$

- Differentiating with respect to $\boldsymbol{\lambda}$ and seeking stationary points implies:

$$\begin{aligned} 2\mathbf{B}^T(\mathbf{x} - \mathbf{B}\boldsymbol{\lambda}) &= 0 \\ \implies \qquad \boldsymbol{\lambda} &= (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{x} \end{aligned}$$

(These are known as the **normal equations**, they will be discussed in more detail in the context of **Linear Regression**)

Orthogonal Projections onto Subspaces

- This tells us that $(\mathbf{x} - \mathbf{P}_U)$ is orthogonal to $\mathbf{b}_i$ for all i, since:

$$\begin{aligned}(\mathbf{x} - \mathbf{P}_U) \cdot \mathbf{b}_i &= \mathbf{x} \cdot \mathbf{b}_i - \mathbf{B}\boldsymbol{\lambda} \cdot \mathbf{b}_i \\ &= \mathbf{x} \cdot \mathbf{b}_i - \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{x} \cdot \mathbf{b}_i = 0\end{aligned}$$

- Furthermore, we can derive an explicit form for the matrix $\mathbf{P}_U$:

$$\begin{aligned}\mathbf{P}_U\mathbf{x} &= \mathbf{B}\boldsymbol{\lambda} \\ &= \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{x}\end{aligned}$$

- Thus:

$$\mathbf{P}_U = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$$

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Solving Linear Equations

- Let's say we want to solve the following system of linear equations:

$$\mathbf{Ax} = \mathbf{b}$$

where $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{b} \in \mathbb{R}^n$ are both known, and $\mathbf{x} \in \mathbb{R}^m$ is unknown

- This is equivalent to a set of n equations:

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + ... + A_{1m}x_m &= b_1 \\ A_{21}x_1 + A_{22}x_2 + ... + A_{2m}x_m &= b_2 \\ &\vdots \\ A_{n1}x_1 + A_{n2}x_2 + ... + A_{nm}x_m &= b_n \end{aligned}$$

Solving Linear Equations

- This system of equations can have:
  - A **unique solution**, **no solutions**, or **infinitely many solutions**
  - But it cannot have more than 1 and less than an infinite number of solutions. Why?
  - If $\mathbf{x}_A$ and $\mathbf{x}_B$ are solutions, then $\mathbf{x}_C = \alpha\mathbf{x}_A + (1-\alpha)\mathbf{x}_B$ must also be a solution, for all α
- In order to solve this system we need to:
  - Check how many **distinct** simultaneous equations we have in our system
  - Check the **consistency** of these equations
  - Compare the **number** of these equations to the number of unknowns in our system

Solving Linear Equations

- To check how many distinct equations we have:
  - Form the **augmented matrix**, $\mathbf{A}|\mathbf{b} = [\mathbf{A}, \mathbf{b}] \in \mathbb{R}^{n\times(m+1)}$
  - Recall that the number of linearly independent rows in a matrix is equal to the matrix rank, so:

$$\begin{aligned}\#\text{ distinct equations} &= \#\text{ linearly independent rows in } \mathbf{A}|\mathbf{b}\\ &= \operatorname{rank}(\mathbf{A}|\mathbf{b}) \leqslant \min(n, m+1)\end{aligned}$$

- To check the consistency of these equations:
  - Note that the number of distinct ‘left-hand sides’ of these equations should equal the number of these equations for consistency:

$$\begin{aligned}\#\text{ distinct ‘LHS’ of equations} &= \#\text{ linearly independent rows in } \mathbf{A}\\ &= \operatorname{rank}(\mathbf{A}) \leqslant \min(n, m)\end{aligned}$$

Solving Linear Equations

- If $\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}|\mathbf{b})$:
    - The equations are **inconsistent**
    - We have no solutions
- If $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) < m$:
    - There are too few equations to fully specify the number of unknowns
    - The system is **underdetermined**
    - We have infinitely many solutions
- If $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}|\mathbf{b}) = m \leqslant n$:
    - There are at least the same number of *consistent* equations as unknowns
    - We have a unique solution

Solving Linear Equations

		UNDER-DETERMINED	**EXACTLY DETERMINED**	**OVER-DETERMINED**
		$\text{rank}(\mathbf{A}\vert\mathbf{b}) < m$	$\text{rank}(\mathbf{A}\vert\mathbf{b}) = m$	$\text{rank}(\mathbf{A}\vert\mathbf{b}) = m$
		$n \overset{\leq}{\underset{\geq}{=}} m$	$n = m$	$n > m$
CONSISTENT	$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}\vert\mathbf{b})$	∞	1	1
INCONSISTENT	$\text{rank}(\mathbf{A}) < \text{rank}(\mathbf{A}\vert\mathbf{b})$	0	0	0

Table: Number of solutions to $\mathbf{Ax} = \mathbf{b}$

Solving Linear Equations: An Algebraic Solution

- Let us try to solve our system directly using the machinery of linear algebra:

$$\begin{aligned} \mathbf{Ax} &= \mathbf{b} \\ \implies \qquad \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

  ...if $\mathbf{A}^{-1}$ exists!

- It turns out that $\mathbf{A}^{-1}$ exists iff $\mathbf{A}$ is square and full rank [see **Invertible Matrix Theorem**]
- If $\mathbf{A}$ is square and of full rank then of course $\text{rank}(\mathbf{A}) = m$
- So the algebraic solution, if it exists, yields a unique solution

Solving Linear Equations: Gauss-Jordan Elimination

- **Gauss-Jordan Elimination**, or **row reduction**, is an algorithm for solving a system of linear equations
- It aims to re-write the linear equations encoded in the augmented matrix in **reduced row echelon form** (rref)

Solving Linear Equations: Gauss-Jordan Elimination

- For a matrix, **A**, the rref is a matrix, rref$(\mathbf{A})$, of similar dimensions that satisfies:
  - Leftmost non-zero element in each row is 1.
    This element is called a **pivot**
  - Any column can have at most 1 pivot
  - If a column has a pivot then the rest of the elements in the column are 0
  - For any two pivots, P_1 and P_2, if P_2 is to the right of P_1, then P_2 is below P_1
  - Any rows consisting of only zeroes are in the bottom of the matrix

Gauss-Jordan Elimination: Row Operations

- Gauss-Jordan elimination operations used to obtain rref:
  - Switch two rows
  - Multiply a row by a non-zero constant
  - Add a scalar multiple of one row to any other row
- Clearly each of these operations will not alter the essential information encoded in a system of linear equations
- Futhermore, $\text{rank}(\mathbf{A}) = \text{rank}(\text{rref}(\mathbf{A}))$, and this is equal to the number of non-zero rows in $\text{rref}(\mathbf{A})$

Gauss-Jordan Elimination: Solutions

- The rref of the augmented matrix, rref$(\mathbf{A}|\mathbf{b})$, can be can be converted back to a set of equations and the solutions can then be read off easily

- Note that, rref$(\mathbf{A})$, is simply the matrix obtained by removing the last column from rref$(\mathbf{A}|\mathbf{b})$

Gauss-Jordan Elimination: Solutions

- If rref$(\mathbf{A})$ contains a row of zeroes, and the equivalent row in rref$(\mathbf{A}|\mathbf{b})$ contains a non-zero in the last entry of that row:
    - The equations are **inconsistent**
    - We have no solutions
    - (Recall: rank$(\mathbf{A}) <$ rank$(\mathbf{A}|\mathbf{b})$)

Gauss-Jordan Elimination: Solutions

- Otherwise, check if rref$(\mathbf{A}|\mathbf{b})$ contains a row of zeroes, and remove the equivalent rows from rref$(\mathbf{A})$:
  - If we are **not** left with the identity matrix, then:
    - There are too few equations to fully specify the number of unknowns
    - The system is **underdetermined**
    - We have infinitely many solutions
    - (Recall: rank$(\mathbf{A}) = $ rank$(\mathbf{A}|\mathbf{b}) < m$)
  - If we are left with the identity matrix, then:
    - There are the same number of consistent equations as unknowns
    - We have a unique solution
    - (Recall: rank$(\mathbf{A}) = $ rank$(\mathbf{A}|\mathbf{b}) = m$)

Gauss-Jordan Elimination: Example

- Solve the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 6 \\ 2x_1 + x_2 + 8x_3 &= 8 \\ x_1 + 2x_2 + 7x_3 &= 10 \\ -x_1 + x_2 - x_3 &= 2 \end{aligned}$$

- We can write the augmented matrix as follows:

$$\mathbf{A}|\mathbf{b} = \begin{bmatrix} 1 & 1 & 5 & 6 \\ 2 & 1 & 8 & 8 \\ 1 & 2 & 7 & 10 \\ -1 & 1 & -1 & 2 \end{bmatrix}$$

Gauss-Jordan Elimination: Example

- Perform the following Gauss-Jordan Elimination operations:
  1. (Row 2 - Row 1) ← Row 1
  2. (2 × Row 1 - Row 2) ← Row 2
  3. (Row 2 + Row 3 - 3 × Row 1) ← Row 3
  4. (Row 4 + 2 × Row 2 - 3 × Row 1) ← Row 4

- This results in the following rref matrix:

$$\text{rref}(\mathbf{A}|\mathbf{b}) = \begin{bmatrix} \mathbf{1} & 0 & 3 & 2 \\ 0 & \mathbf{1} & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- From which we note that we have infinitely many solutions

Gauss-Jordan Elimination: Example

- Converting back to equations:

$$\begin{aligned} x_1 + 3x_3 &= 2 \\ x_2 + 2x_3 &= 4 \end{aligned}$$

- Which results in the following **parametric solution**:

$$\begin{aligned} x_1 &= 2 - 3\lambda \\ x_2 &= 4 - 2\lambda \\ x_3 &= \lambda \end{aligned}$$

Where $\lambda \in \mathbb{R}$

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Determinants

- The **determinant** is associated with square matrices only. For some square matrix $\mathbf{A}$ the determinant is denoted by $\det(\mathbf{A})$ or $|\mathbf{A}|$

- It is a function which maps a matrix to a real scalar

- Geometrically the determinant of $\mathbf{A} \in \mathbb{R}^{n \times n}$ can be thought of as a signed volume of the *n*-dimensional paralleltope that results from the action of $\mathbf{A}$ on the unit cube

Determinants

- For $\mathbf{A} \in \mathbb{R}^{2\times 2}$:

$$\det(\mathbf{A}) = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{21}A_{12}$$

- For $\mathbf{A} \in \mathbb{R}^{3\times 3}$:

$$\det(\mathbf{A}) = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11}\begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12}\begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13}\begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

- For $\mathbf{A} \in \mathbb{R}^{n\times n}$:

$$\det(\mathbf{A}) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} |\mathbf{M}_{ij}|$$

where $|\mathbf{M}_{ij}|$ is the determinant of the $(n-1)\times(n-1)$ matrix that results from **A** by removing the *i*-th row and the *j*-th column

Determinants

- **Properties**:

$$\det(\mathbf{I}) = 1$$

$$\det\left(\mathbf{A}^{T}\right) = \det(\mathbf{A})$$

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

$$\det\left(\mathbf{A}^{-1}\right) = (\det(\mathbf{A}))^{-1}$$

$$\det(\alpha\mathbf{A}) = \alpha^{n}\det(\mathbf{A})$$

Traces

- The **trace** is associated with square matrices only. For some square matrix $\mathbf{A} \in \mathbb{R}^n$ the trace is denoted by $\mathrm{tr}(\mathbf{A})$
- It is a function which maps a matrix to a real scalar, and is defined as the sum of the diagonal elements of $\mathbf{A}$:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$$

Traces

- **Properties**:

$$\operatorname{tr}(\mathbf{A}+\mathbf{B}) = \operatorname{tr}(\mathbf{A}) + \operatorname{tr}(\mathbf{B}) \qquad \text{where: } \mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times n}$$

$$\operatorname{tr}(\alpha\mathbf{A}) = \alpha\operatorname{tr}(\mathbf{A}) \qquad \text{where: } \alpha \in \mathbb{R},\ \mathbf{A} \in \mathbb{R}^{n\times n}$$

$$\operatorname{tr}(\mathbf{I}_n) = n$$

$$\operatorname{tr}(\mathbf{AB}) = \operatorname{tr}(\mathbf{BA}) \qquad \text{where: } \mathbf{A} \in \mathbb{R}^{n\times k}, \mathbf{B} \in \mathbb{R}^{k\times n}$$

Invertible Matrix Theorem

- Let $\mathbf{A} \in \mathbb{R}^{n \times n}$, then the following statements are equivalent:
  - $\mathbf{A}$ is invertible
  - $\text{rank}(\mathbf{A}) = n$
  - $\det(\mathbf{A}) \neq 0$
  - The columns of $\mathbf{A}$ are linearly independent
  - The rows of $\mathbf{A}$ are linearly independent
  - $\nexists\ \mathbf{x}$ such that $\mathbf{Ax} = \mathbf{0}$
  - ...and many more...

Eigenvectors & Eigenvalues

- For a square matrix, $\mathbf{A} \in \mathbb{R}^{n \times n}$, we say that $\mathbf{x}$ is an **eigenvector** of $\mathbf{A}$ corresponding to an **eigenvalue**, λ, if:

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

- If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$, then $(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$
- Therefore if $(\mathbf{A} - \lambda\mathbf{I})^{-1}$ exists, then the unique solution is $\mathbf{x} = \mathbf{0}$
- Therefore for non-trivial solutions we must demand that $\nexists(\mathbf{A} - \lambda\mathbf{I})^{-1}$
- Therefore for non-trivial solutions $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ [by the **Invertible Matrix Theorem**]
- $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ is known as the **characteristic polynomial** of $\mathbf{A}$
  - Its (possibly non-unique) roots determine the possible eigenvalues of our problem

Eigendecomposition

- The **eigen-** or **spectral decomposition** of a square matrix, $\mathbf{A} \in \mathbb{R}^{n \times n}$, with n linearly independent eigenvectors, $\{\mathbf{q}_i\}_{i=1}^n$ states that:

  $$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$$

  Where:

  - $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n]$;
  - $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ where λ_i is the eigenvalue associated with the eigenvector $\mathbf{q}_i$

- From this it follows that:

  $$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$$

Eigendecomposition: Symmetric Matrices

- If $\mathbf{A} \in \mathbb{R}^{n \times n}$ is (real and) symmetric, then there exists an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{A}$, $\left\{\mathbf{q}_i | \mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}, \quad \forall\, j\right\}_{i=1}^{n}$, and the associated eigenvalues are real

- From this it follows that the eigendecomposition of a real symmetric matrix, $\mathbf{A}$, is given by:

  $$\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T$$

  Where:

  - $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n]$;
  - $\boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$ where λ_i is the eigenvalue associated with the eigenvector $\mathbf{q}_i$

Eigendecomposition: Geometric Intuition

- Consider the linear map, $\mathbf{Q} \in \mathbb{R}^{n \times n}$, described above, and the set of standard basis vectors which span $\mathbb{R}^n$, $\{\mathbf{e}_i\}_{i=1}^n$
- Now consider this map acting on the *i*-th such vector:

$$\mathbf{Q}\mathbf{e}_i = \sum_{j=1}^{n} \mathbf{q}_j [\mathbf{e}_i]_j = \mathbf{q}_i$$

- Thus $\mathbf{Q}$ maps $\mathbf{e}_i$ to the corresponding eigenvector of $\mathbf{A}$.
- Similarly, the matrix $\mathbf{Q}^{-1}$ maps the *i*-th eigenvector of $\mathbf{A}$ to the corresponding standard basis vector:

$$\mathbf{Q}^{-1}\mathbf{q}_i = \mathbf{e}_i$$

Eigendecomposition: Geometric Intuition

- Now we consider the linear map, $\mathbf{A} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}$, as performing the following operations when it acts on an eigenvector of $\mathbf{A}$:

  1. A transformation from the basis spanned by the eigenvectors to that spanned by the standard basis vectors:

  $$\mathbf{Q}^{-1}\mathbf{q}_i = \mathbf{e}_i$$

  2. A scaling of the resulting standard basis vector by the corresponding eigenvalue:

  $$\boldsymbol{\Lambda}\mathbf{Q}^{-1}\mathbf{q}_i = \boldsymbol{\Lambda}\mathbf{e}_i = \lambda_i\mathbf{e}_i$$

  3. A basis change of the resulting scaled vector back to the basis spanned by the eigenvectors:

  $$\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}\mathbf{q}_i = \mathbf{Q}\lambda_i\mathbf{e}_i = \lambda_i\mathbf{Q}\mathbf{e}_i = \lambda_i\mathbf{q}_i$$

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Quadratic Forms

- Assume a square matrix, $\mathbf{M} \in \mathbb{R}^{n \times n}$. Then the **quadratic form** of **M** is defined by:

$$\mathbf{x}^T \mathbf{M} \mathbf{x}$$

Quadratic Forms

- Note also that it is possible to write any matrix, **M**, as follows:

  $$\mathbf{M} = \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^T\right) + \frac{1}{2}\left(\mathbf{M} - \mathbf{M}^T\right) = \mathbf{A} + \mathbf{B}$$

  Where:
  $\mathbf{A} = \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^T\right)$ is a **symmetric** matrix
  $\mathbf{B} = \frac{1}{2}\left(\mathbf{M} - \mathbf{M}^T\right)$ is an **antisymmetric** matrix

- Note further that because $\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T\mathbf{M}^T\mathbf{x}$, then:

  $$\mathbf{x}^T\mathbf{M}\mathbf{x} = \mathbf{x}^T\mathbf{A}\mathbf{x}$$

- In other words it is always possible to express the quadratic form of a general square matrix, **M**, as the quadratic form of an associated symmetric matrix, $\mathbf{A} = \frac{1}{2}\left(\mathbf{M} + \mathbf{M}^T\right)$

Definiteness of Matrices

- A real symmetric matrix, **A**, is said to be:
  - **Positive Semidefinite** (psd), written as $\mathbf{A} \succeq 0$, iff $\mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0, \forall \mathbf{x} \neq \mathbf{0}$
  - **Positive Definite** (pd), written as $\mathbf{A} \succ 0$, iff $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0, \forall \mathbf{x} \neq \mathbf{0}$
  - **Negative Semidefinite** (nsd), written as $\mathbf{A} \preceq 0$, iff $\mathbf{x}^T\mathbf{A}\mathbf{x} \leqslant 0, \forall \mathbf{x} \neq \mathbf{0}$
  - **Negative Definite** (nd), written as $\mathbf{A} \prec 0$, iff $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0, \forall \mathbf{x} \neq \mathbf{0}$
  - **Indefinite** otherwise

PSD & PD Matrices: Properties

- For a real symmetric matrix, **psd** matrix, $\mathbf{A} \in \mathbb{R}^{n\times n}$, with (real) eigenvalues, $\{\lambda_i\}_{i=1}^n$:

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0, \quad \forall \mathbf{x} \neq \mathbf{0} \quad &\iff \quad \{\lambda_i \geqslant 0\}_{i=1}^n \\ &\implies \quad \det(\mathbf{A}) = \prod_{i=1}^n \lambda_i \geqslant 0 \end{aligned}$$

- For a real symmetric matrix, **pd** matrix, $\mathbf{A} \in \mathbb{R}^{n\times n}$, with (real) eigenvalues, $\{\lambda_i\}_{i=1}^n$:

$$\begin{aligned} \mathbf{x}^T\mathbf{A}\mathbf{x} > 0, \quad \forall \mathbf{x} \neq \mathbf{0} \quad &\iff \quad \{\lambda_i > 0\}_{i=1}^n \\ &\implies \quad \det(\mathbf{A}) = \prod_{i=1}^n \lambda_i > 0 \\ &\implies \quad \textbf{A is invertible} \end{aligned}$$

PSD & PD Matrices: Properties

- For positive semidefinite property:
  - **Proof:** *(if)*

$$\begin{aligned} & \mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0, \quad \forall \mathbf{x} \neq \mathbf{0} \\ \implies \quad & \mathbf{q}_i^T\mathbf{A}\mathbf{q}_i \geqslant 0, \quad \text{where } \mathbf{q}_i \text{ is the } i\text{-th eigenvector of } \mathbf{A} \\ \implies \quad & \lambda_i\|\mathbf{q}_i\|_2^2 \geqslant 0, \quad \text{where } \lambda_i \text{ is the } i\text{-th eigenvalue of } \mathbf{A} \\ \implies \quad & \lambda_i \geqslant 0 \end{aligned}$$

  - **Proof:** *(only if)*

$$\begin{aligned} & \lambda_i \geqslant 0 \\ \implies \quad & \sum_{i=1}^{n} \lambda_i a_i^2 \geqslant 0, \quad \text{where } \mathbf{a} = [a_1, ..., a_n]^T \\ \implies \quad & \mathbf{a}^T\boldsymbol{\Lambda}\mathbf{a} \geqslant 0, \quad \text{where } \boldsymbol{\Lambda} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_n) \\ & \text{In particular, for } \mathbf{a} = \mathbf{Q}^T\mathbf{x}, \text{ where } \mathbf{x} \neq \mathbf{0} \text{ and } \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n]\text{:} \\ \implies \quad & \mathbf{x}^T\mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^T\mathbf{x} \geqslant 0, \quad \forall \mathbf{x} \neq \mathbf{0} \\ \implies \quad & \mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0, \quad \forall \mathbf{x} \neq \mathbf{0}, \quad \text{by the Spectral Theorem} \end{aligned}$$

PSD Matrices: Further Properties

- For a real, symmetric, **psd** matrix, $\mathbf{A} \in \mathbb{R}^{n\times n}$, we can write:

$$\mathbf{A} = \begin{bmatrix} \mathbf{B} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{D} \end{bmatrix}$$

where: $\mathbf{B} \in \mathbb{R}^{u\times u}$, $\mathbf{D} \in \mathbb{R}^{v\times v}$, $\mathbf{C} \in \mathbb{R}^{u\times v}$, and $u + v = n$.

- Consider the quadratic form induced by some non-trivial vector, $\mathbf{x} = [\mathbf{a}^T, \mathbf{b}^T]^T$, where $\mathbf{a} \in \mathbb{R}^u$ and $\mathbf{b} \in \mathbb{R}^v$:

$$\begin{aligned} & \mathbf{x}^T\mathbf{A}\mathbf{x} \geqslant 0 \\ \implies \quad & \mathbf{a}^T\mathbf{B}\mathbf{a} + \mathbf{b}^T\mathbf{C}^T\mathbf{a} + \mathbf{a}^T\mathbf{C}\mathbf{b} + \mathbf{b}^T\mathbf{D}\mathbf{b} \geqslant 0 \\ \implies \quad & \mathbf{a}^T\mathbf{B}\mathbf{a} + 2\mathbf{a}^T\mathbf{C}\mathbf{b} + \mathbf{b}^T\mathbf{D}\mathbf{b} \geqslant 0 \end{aligned}$$

PSD Matrices: Further Properties (Cont.)

- This must hold for all **x**, in particular:

$$\begin{aligned} \mathbf{a} \neq \mathbf{0}, \mathbf{b} = \mathbf{0} \quad &\implies \mathbf{a}^T \mathbf{B} \mathbf{a} \geqslant 0 \\ \mathbf{a} = \mathbf{0}, \mathbf{b} \neq \mathbf{0} \quad &\implies \mathbf{b}^T \mathbf{D} \mathbf{b} \geqslant 0 \end{aligned}$$

- Thus, if **A** is positive semidefinite, then all its block diagonal matrices are all also psd
- (Similar results hold if **A** is positive definite, negative semidefinite, or negative definite)

Geometry of Positive Definite Quadratic Forms

- Let's examine the behaviour of $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is real, symmetric, and $\mathbf{A} \succ 0$
- Since $\mathbf{A}$ is **positive definite** then we can write:

$$\mathbf{A} = \mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}$$

  where:

  - $\mathbf{A}^{\frac{1}{2}} = \mathbf{Q}\boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{Q}^T, \qquad \mathbf{A}^{\frac{1}{2}} = (\mathbf{A}^{\frac{1}{2}})^T$
  - $\mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_n]$ for $\{\mathbf{q}_i | \mathbf{q}_i \cdot \mathbf{q}_j = \delta_{ij}, \quad \forall\, j\}_{i=1}^n$
  - $\boldsymbol{\Lambda}^{\frac{1}{2}} = \text{diag}(\sqrt{\lambda_1}, \sqrt{\lambda_2}, ..., \sqrt{\lambda_n})$ where λ_i is the eigenvalue associated with the eigenvector $\mathbf{q}_i$

- Furthermore:

$$\mathbf{A}^{\frac{1}{2}} = (\mathbf{A}^{\frac{1}{2}})^T, \qquad \mathbf{A}^{-\frac{1}{2}} = (\mathbf{A}^{\frac{1}{2}})^{-1} = \mathbf{Q}\boldsymbol{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^T$$

Geometry of Positive Definite Quadratic Forms

- Consider the contour $f(\mathbf{x}) = c$, where $c \in \mathbb{R}$:

$$\begin{aligned} c &= \mathbf{x}^T \mathbf{A} \mathbf{x} \\ &= \mathbf{x}^T \mathbf{A}^{\frac{1}{2}} \mathbf{A}^{\frac{1}{2}} \mathbf{x} \\ &= \|\mathbf{A}^{\frac{1}{2}} \mathbf{x}\|_2^2 \\ &= \|\mathbf{z}\|_2^2 \end{aligned}$$

  where $\mathbf{z} = \mathbf{A}^{\frac{1}{2}} \mathbf{x}$

- So the radius of $\mathbf{z}$ lies on a hypersphere of radius $\sqrt{c}$

Geometry of Positive Definite Quadratic Forms

- We can write this as $\mathbf{z} = \sqrt{c}\widehat{\mathbf{z}}$, where $\|\widehat{\mathbf{z}}\|_2 = 1$, so:

$$\begin{aligned}\mathbf{x} &= \mathbf{A}^{-\frac{1}{2}}\mathbf{z} \\ &= \mathbf{Q}\boldsymbol{\Lambda}^{-\frac{1}{2}}\mathbf{Q}^T\sqrt{c}\widehat{\mathbf{z}} \\ &= \sqrt{c}\mathbf{Q}\boldsymbol{\Lambda}^{-\frac{1}{2}}\widetilde{\mathbf{z}}\end{aligned}$$

  where $\widetilde{\mathbf{z}} = \mathbf{Q}^T\widehat{\mathbf{z}}$

- Note that:

$$\|\widetilde{\mathbf{z}}\|_2^2 = \widehat{\mathbf{z}}^T\mathbf{Q}\mathbf{Q}^T\widehat{\mathbf{z}} = \widehat{\mathbf{z}}^T\widehat{\mathbf{z}} = \|\widehat{\mathbf{z}}\|_2^2 = 1$$

- So the contour defined by $f(\mathbf{x}) = c$ is satisfied by:

$$\mathbf{x} = \sqrt{c}\mathbf{Q}\boldsymbol{\Lambda}^{-\frac{1}{2}}\widetilde{\mathbf{z}} \qquad \text{such that: } \|\widetilde{\mathbf{z}}\|_2 = 1$$

Geometry of Positive Definite Quadratic Forms

- **Interpretation**:
  - Start with a locus of points, $\widetilde{\mathbf{z}}$ that sit on the unit hypersphere, $\|\widetilde{\mathbf{z}}\|_2 = 1$
  - Premultiply $\widetilde{\mathbf{z}}$ by $\sqrt{c}\mathbf{\Lambda}^{-\frac{1}{2}}$:
    - This scales the axis lengths in proportion to the inverse square roots of the eigenvalues of **A**
    - This results in an axis-aligned **ellipsoid**
  - Premultiply $\sqrt{c}\mathbf{\Lambda}^{-\frac{1}{2}}\widetilde{\mathbf{z}}$ by **Q**:
    - Since **Q** is orthogonal, recall that it preserves length and angle
    - This rotates or reflects the ellipsoid
    - The axes are now aligned with the eigenvectors of **Q** since:

$$\mathbf{Q}\mathbf{e}_i = \sum_{j=1}^{n} [\mathbf{e}_i]_j \mathbf{q}_j = \mathbf{q}_i$$

Geometry of Positive Definite Quadratic Forms

- So the contours of $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ are ellipsoids such that:
  - The axes point in the direction of the eigenvectors of **A**
  - The radii are proportional to the inverse square root of the corresponding eigenvalues of **A**

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Some Useful Results

1. For any matrix $\mathbf{X} \in \mathbb{R}^{n \times m}$, the matrix $\mathbf{X}^T\mathbf{X} \in \mathbb{R}^{m \times m}$ is symmetric
2. $\mathbf{X}^T\mathbf{X}$ is always positive semidefinite for all $\mathbf{X}$
    - **Proof:**

$$\mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{a} = \|\mathbf{X}\mathbf{a}\|_2^2 \geqslant 0, \qquad \forall\, \mathbf{a} \neq \mathbf{0}$$

Some Useful Results

3. $\mathrm{rank}(\mathbf{X}^T\mathbf{X}) = m$ if and only if $\mathbf{X}^T\mathbf{X} \succ 0$
   - **Proof:** (*only if*)

$$\begin{aligned} & \mathrm{rank}(\mathbf{X}^T\mathbf{X}) = m \\ \Longrightarrow\quad & \nexists\, \mathbf{a} \text{ such that } \mathbf{X}^T\mathbf{X}\mathbf{a} = \mathbf{0} && \text{[by \textbf{Linear Independence}]} \\ \Longrightarrow\quad & \mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{a} = \|\mathbf{X}\mathbf{a}\|_2^2 > 0, \qquad \forall\, \mathbf{a} \neq \mathbf{0} \\ \Longrightarrow\quad & \mathbf{X}^T\mathbf{X} \succ 0 \end{aligned}$$

   - **Proof:** (*if*)

$$\begin{aligned} & \mathbf{X}^T\mathbf{X} \succ 0 \\ \Longrightarrow\quad & \|\mathbf{X}\mathbf{a}\|_2^2 > 0, \qquad \forall\, \mathbf{a} \neq \mathbf{0} \\ \Longrightarrow\quad & \nexists\, \mathbf{a} \text{ such that } \mathbf{X}\mathbf{a} = \mathbf{0} \\ \Longrightarrow\quad & \nexists\, \mathbf{a} \text{ such that } \mathbf{X}^T\mathbf{X}\mathbf{a} = \mathbf{0} \\ \Longrightarrow\quad & \mathrm{rank}(\mathbf{X}^T\mathbf{X}) = m && \text{[by \textbf{Invertible Matrix Theorem}]} \end{aligned}$$

Some Useful Results

4. $\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X})$
   - **Proof:**
     For any vector $\mathbf{a} \in \text{null}(\mathbf{X}^T\mathbf{X})$:

$$\begin{aligned} \mathbf{X}^T\mathbf{X}\mathbf{a} = \mathbf{0} &\implies \mathbf{a}^T\mathbf{X}^T\mathbf{X}\mathbf{a} = 0 \\ &\implies \|\mathbf{X}\mathbf{a}\|_2^2 = 0 \\ &\implies \mathbf{X}\mathbf{a} = \mathbf{0} \end{aligned}$$

For any vector $\mathbf{b} \in \text{null}(\mathbf{X})$:

$$\mathbf{X}\mathbf{b} = \mathbf{0} \implies \mathbf{X}^T\mathbf{X}\mathbf{b} = \mathbf{0}$$

So, any **a** which belongs to the nullspace of $\mathbf{X}^T\mathbf{X}$ also belongs to the nullspace of $\mathbf{X}$ and vice-versa, so:

$$\text{nullity}(\mathbf{X}^T\mathbf{X}) = \text{nullity}(\mathbf{X})$$

By the **Rank-Nullity Theorem**:

$$\text{rank}(\mathbf{X}) + \text{nullity}(\mathbf{X}) = m$$

$$\text{rank}(\mathbf{X}^T\mathbf{X}) + \text{nullity}(\mathbf{X}^T\mathbf{X}) = m$$

So: $\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X})$

Some Useful Results

5. $\text{rank}(\mathbf{X}^T\mathbf{X}) \leqslant \min(n, m)$
   - **Proof:**
     Recall $\text{rank}(\mathbf{X}) \leqslant \min(n, m)$

6. $(\mathbf{X}^T\mathbf{X})^{-1}$ exists if and only if $\text{rank}(\mathbf{X}^T\mathbf{X}) = m$
   - **Proof:**
     By **Invertible Matrix Theorem**

Some Useful Results

7. $\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) \quad \forall\ \mathbf{y}$
   - **Proof:**

     Recall:

     $$\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X}) = \text{rank}(\mathbf{X}^T)$$

     Because $(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y})$ has more columns than $\mathbf{X}^T\mathbf{X}$ then:

     $$\text{rank}(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) \geqslant \text{rank}(\mathbf{X}^T\mathbf{X}) \tag{1}$$

     Observe that: $(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) = \mathbf{X}^T[\mathbf{X},\mathbf{y}]$:

     $$\begin{aligned} \implies \quad \text{rank}(\mathbf{X}^T[\mathbf{X},\mathbf{y}]) &\leqslant \min(\text{rank}(\mathbf{X}^T), \text{rank}([\mathbf{X},\mathbf{y}])) \\ &= \text{rank}(\mathbf{X}^T) \\ &= \text{rank}(\mathbf{X}^T\mathbf{X}) \qquad (2) \end{aligned}$$

     Combining (1) and (2):

     $$\begin{aligned} &\text{rank}(\mathbf{X}^T\mathbf{X}) \leqslant \text{rank}(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) \leqslant \text{rank}(\mathbf{X}^T\mathbf{X}) \\ \implies \quad &\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) \end{aligned}$$

Some Useful Results

- To sum up:
  - $\mathbf{X}^T\mathbf{X}$ is symmetric
  - $\mathbf{X}^T\mathbf{X} \succeq 0$
  - $\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X}) \leqslant \min(n, m)$
  - $\exists(\mathbf{X}^T\mathbf{X})^{-1} \iff (\text{rank}(\mathbf{X}^T\mathbf{X}) = m) \iff \mathbf{X}^T\mathbf{X} \succ 0$
  - $\text{rank}(\mathbf{X}^T\mathbf{X}) = \text{rank}(\mathbf{X}^T\mathbf{X}|\mathbf{X}^T\mathbf{y}) \quad \forall\ \mathbf{y}$

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Summary

- **Linear Algebra** is an essential tool that helps us deal with systems of linear equations
- We have introduced the setting and some structure for the operation of linear algbera
- Building on this we have introduced some key results which will be of direct use in machine learning

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Columspace & Rowspace: Revisited

- Recall that the dimension of the columnspace is equal to the dimension of the rowspace.

  For a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$ this can be expressed as:

  $$\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T)$$

Columspace & Rowspace: Revisited

- **Proof:**

Let the row rank of $\mathbf{A}$ be r, thus $\text{rank}(\mathbf{A}^T) = r$.
And let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r$ be a basis of the rowspace of $\mathbf{A}$.

Now, consider a set of scalar coefficients, $c_1, c_2, \ldots, c_r$, such that:

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \ldots + c_r\mathbf{A}\mathbf{x}_r \\ &= \mathbf{A}(c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_r\mathbf{x}_r) = \mathbf{A}\mathbf{v} \end{aligned}$$

Where: $\mathbf{v} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_r\mathbf{x}_r$

Now, $\mathbf{v}$ is a linear combination of vectors in the row space of $\mathbf{A}$.
Thus, $\mathbf{v} \in \text{rowspace}(\mathbf{A})$ (**Fact 1**).

Furthermore, since $\mathbf{A}\mathbf{v} = \mathbf{0}$, then $\mathbf{v}$ is orthogonal to every row vector of $\mathbf{A}$.
Thus, $\mathbf{v}$ is orthogonal to every vector in $\text{rowspace}(\mathbf{A})$ (**Fact 2**).

Columspace & Rowspace: Revisited

- **Proof (Cont.):**

  Facts 1 & 2 can only hold simultaneously if $\mathbf{v}$ is orthogonal to itself, and this occurs only if $\mathbf{v} = \mathbf{0}$.

  Therefore: $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_r\mathbf{x}_r = \mathbf{0}$

  But $\{\mathbf{x}_i\}_{i=1}^{r}$ are linearly independent, so $c_1 = c_2 = \ldots = c_r = 0$.

  Thus we have shown that:

$$c_1\mathbf{A}\mathbf{x}_1 + c_2\mathbf{A}\mathbf{x}_2 + \ldots + c_r\mathbf{A}\mathbf{x}_r = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = \ldots = c_r = 0$$

  Thus, by the definitive property of linear independence, $\{\mathbf{A}\mathbf{x}_i\}_{i=1}^{r}$ are linearly independent.

Columspace & Rowspace: Revisited

- **Proof (Cont.):**

  Now, each $\mathbf{Ax}_i$ is a vector in the columnspace of $\mathbf{A}$, so $\{\mathbf{Ax}_i\}_{i=1}^r$ is a set of r linearly independent vectors in the columnspace of $\mathbf{A}$. Thus:

$$\begin{aligned} & \dim(\text{columnspace}(\mathbf{A})) \geqslant r \\ \Longrightarrow \quad & \text{rank}(\mathbf{A}) \geqslant r \\ \Longrightarrow \quad & \text{rank}(\mathbf{A}) \geqslant \text{rank}(\mathbf{A}^T) \end{aligned}$$

  Analagously, we can prove:

$$\text{rank}(\mathbf{A}^T) \geqslant \text{rank}(\mathbf{A})$$

  Since these inequalities hold simultaneously, this implies:

$$\begin{aligned} & \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A}^T) \\ \Longrightarrow \quad & \dim(\text{columnspace}(\mathbf{A})) = \dim(\text{rowspace}(\mathbf{A})) \end{aligned}$$

Rank-Nullity Theorem: Revisited

- Recall the Rank-Nullity Theorem for a matrix $\mathbf{A} \in \mathbb{R}^{n \times m}$:

$$\text{rank}(\mathbf{A}) + \text{nullity}(\mathbf{A}) = m$$

Rank-Nullity Theorem: Revisited

- **Proof:**

Consider the matrix equation $\mathbf{Ax} = \mathbf{0}$, where $\mathbf{x} \in \mathbb{R}^m$.

From our discussion of Gauss-Jordan elimination, recall that:

$$\text{rank}(\text{rref}(\mathbf{A})) = \text{rank}(\mathbf{A}) = r$$

Thus $\text{rref}(\mathbf{A})$ has only r non-zero rows, and hence $(m - r)$ of the variables in the solution $\mathbf{x}$ are free.

But the number of free variables is the number of free parameters in a general parametric solution of $\mathbf{Ax} = \mathbf{0}$.

And the number of free parameters defines the dimension of the space spanned by the solutions.

Recall that the dimension of the space spanned by the solutions is, definitively, the dimension of the null space of $\mathbf{A}$.

In other words the nullity of $\mathbf{A}$ is equal to $(m - r)$:

$$\text{nullity}(\mathbf{A}) = m - r$$