MATH3305 — Problem Sheet 2 – Solutions

1. One needs the definitions of contravariant and covariant vectors. Then

$$\mu' = V'^a W'_a = \frac{\partial X'^a}{\partial X^c} V^c \frac{\partial X^d}{\partial X'^a} W_d = \frac{\partial X^c}{\partial X^d} V^c W_d = \delta^d_c V^c W_d = V^c W_c = \mu \,.$$

   In the second step, we used the chain rule. Since $\mu' = \mu$, it is a scalar.

2. (i) We start with $dx = \sin\theta\cos\phi dr + r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi$, and similarly for $dy = \sin\theta\sin\phi dr + r\cos\theta\sin\phi d\theta + r\sin\theta\cos\phi d\phi$ and $dz = \cos\theta dr - r\sin\theta d\theta$. Then all three quantities are squared and we arrive at

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \,.$$

   (ii) Hence the metric tensor in matrix form is

$$g_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

   (iii) The inverse metric is

$$g^{bc} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2}\theta \end{pmatrix}$$

   (iv) Both matrices are diagonal, so we immediately see that $g_{ab} g^{bc} = \delta^c_a$.

3. Notes:

   (a) equation is correct, left-hand side is a scalar, right hand side is summed over 4 indices

   (b) We have 3 c's; 3 a's and 3 b's, so this is not well defined

   (c) Tensor minus scalar is ill defined

   (d) Equation is correct.