Model answers MATHM305

1. (a)

$$V'^a = \frac{\partial X'^a}{\partial X^b} V^b. \tag{1}$$

4 points (SEEN)

(b)

$$V'_a = \frac{\partial X^b}{\partial X'^a} V_b. \tag{2}$$

4 points (SEEN)

(c)

$$\begin{aligned} M'^a{}_{ab} &= \frac{\partial X'^a}{\partial X^c} \frac{\partial X^d}{\partial X'^a} \frac{\partial X^e}{\partial X'^b} M^c{}_{de} = \frac{\partial X^d}{\partial X^c} \frac{\partial X^e}{\partial X'^b} M^c{}_{de} \\ &= \delta^d_c \frac{\partial X^e}{\partial X'^b} M^c{}_{de} = \frac{\partial X^e}{\partial X'^b} M^c{}_{ce} = \frac{\partial X^e}{\partial X'^b} M^a{}_{ae} \end{aligned} \tag{3}$$

4 points (UNSEEN)

(d) g^{ab} denotes the inverse metric. $g_{ab}g^{bc} = \delta^c_a$. $g_{ab}g^{ab} = \delta^a_a = N$.

3 points (SEEN)

(e) In general T_{abc} has $3 \times 3 \times 3 = 27$ independent components. For $T_{[abc]}$, write explicitely $T_{[123]}$, so there is only one component.

5 points (UNSEEN)

(f)

$$\operatorname{tr} T g_{ab} + g_{ac} g_{bd} T^{cd} = T_{(ab)} + \frac{1}{8} g^{cd} g_{cd} T_{ab} + g^{cd} T_{(cd)} g_{ab} - \frac{1}{2} T_{ba}. \tag{4}$$

Note that $g_{ac}g_{bd}T^{cd} = T_{ab}$, $g^{cd}g_{cd} = 4$ and $g^{cd}T_{(cd)} = \operatorname{tr} T$. Therefore,

$$\operatorname{tr} T g_{ab} + T_{ab} = T_{(ab)} + \frac{1}{2} T_{ab} + \operatorname{tr} T g_{ab} - \frac{1}{2} T_{ba}. \tag{5}$$

Now the trace terms cancel and we have

$$T_{ab} = T_{(ab)} + \frac{1}{2}T_{ab} - \frac{1}{2}T_{ba}. \tag{6}$$

which equals

$$T_{ab} = T_{(ab)} + T_{[ab]}. \tag{7}$$

5 points (UNSEEN)

2. (a) Well defined for all y and z.

3 points (UNSEEN)

(b)

$$\Gamma^z_{zz} = \frac{2}{z}, \tag{8}$$

all others vanish.

5 points (UNSEEN)

(c)

$$\ddot{y} = 0, \qquad \ddot{z} + \frac{2}{z}\dot{z}^2 = 0. \tag{9}$$

3 points (UNSEEN)

(d)

$$\ddot{y} = 0 \Rightarrow y = c_1\lambda + c_2, \tag{10}$$

$$\ddot{z} + \frac{2}{z}\dot{z}^2 = 0. \tag{11}$$

Devision by $\dot{z}$ yields

$$\frac{\ddot{z}}{\dot{z}} = -2\frac{\dot{z}}{z}, \tag{12}$$

$$\frac{d}{d\lambda}\ln(\dot{z}) = -2\frac{d}{d\lambda}\ln(z), \tag{13}$$

$$\ln(\dot{z}) = -2\ln(z) + c, \tag{14}$$

$$\dot{z} = \tilde{c}z^{-2} \Rightarrow \frac{1}{3}z^3 = c_3\lambda + c_4. \tag{15}$$

6 points (UNSEEN)

This suggests the new coordinate $x = z^3/3$. The new metric reads

$$ds^2 = dx^2 + dy^2, \tag{16}$$

and hence geodesics are straight lines.

4 points (UNSEEN)

(e) Since this is flat Euclidean 2-space $R_{abcd} = 0$, $R_{ab} = 0$ and $R = 0$.

2 points (SEEN)

3. (a) Write out the condition $(*)$ four times

$$K_{dabc} + K_{dcab} + K_{dbca} = 0 \tag{17}$$
$$K_{cdab} + K_{cbda} + K_{cabd} = 0 \tag{18}$$
$$K_{bcda} + K_{bacd} + K_{bdac} = 0 \tag{19}$$
$$K_{abcd} + K_{adbc} + K_{acdb} = 0 \tag{20}$$

5 points (UNSEEN)

Next, identify the terms needed for the proof

$$K_{dabc} + \underline{K_{dcab}} + K_{dbca} = 0 \tag{21}$$
$$\underline{K_{cdab}} + K_{cbda} + K_{cabd} = 0 \tag{22}$$
$$K_{bcda} + \underline{K_{bacd}} + K_{bdac} = 0 \tag{23}$$
$$\underline{K_{abcd}} + K_{adbc} + K_{acdb} = 0 \tag{24}$$

and also see if all other terms might cancel if these four equations are combined linearly. 5 points (UNSEEN)

Using the first two properties we have

$$-K_{adbc} + \underline{K_{cdab}} + K_{dbca} = 0 \tag{25}$$
$$\underline{K_{cdab}} + K_{cbda} + K_{cabd} = 0 \tag{26}$$
$$-K_{cbda} - \underline{K_{abcd}} - K_{dbac} = 0 \tag{27}$$
$$\underline{K_{abcd}} + K_{adbc} + K_{acdb} = 0 \tag{28}$$

In order for the underlined terms not to cancel, we consider (first) - (second) - (third) + (fourth) equation. 4 points (UNSEEN)

All non-underlined terms now cancel: first-first with fourth-second; first-fourth with third-fourth; second-second with third-first. This yields

$$2R_{abcd} - 2R_{cdab} = 0 \tag{29}$$

and the identity follows. 4 points (UNSEEN)

(b) Using the Leibnitz rule on the scalar $w^a u_a$ gives

$$\mathcal{L}_v(w^a u_a) = u_a \mathcal{L}_v w^a + w^a \mathcal{L}_v u_a. \tag{30}$$

On the other hand,

$$\mathcal{L}_v(w^a u_a) = v^b \nabla_b (w^a u_a). \tag{31}$$

4 points (UNSEEN)

Using the definition of $\mathcal{L}_v w^a$ we find

$$u_a(v^b \nabla_b w^a - w^b \nabla_b v^a) + w^a \mathcal{L}_v u_a = v^b w^a \nabla_b u_a + v^b u_a \nabla_b w^a. \tag{32}$$

Hence, we find

$$-u_a w^b \nabla_b v^a + w^a \mathcal{L}_v u_a = v^b w^a \nabla_b u_a, \tag{33}$$

and therefore

$$w^a \mathcal{L}_v u_a = u_b \nabla_a v^b + v^b \nabla_b u_a, \tag{34}$$

3 points (UNSEEN)

4.

$$G^t_t = -\frac{b'(r)}{rb(r)^2} + \frac{1}{r^2 b(r)} - \frac{1}{r^2}, \tag{35}$$

$$G^r_r = \frac{a'(r)}{ra(r)b(r)} + \frac{1}{r^2 b(r)} - \frac{1}{r^2}, \tag{36}$$

(a) The vacuum field equations with cosmological term are

$$G^a_b + \Lambda \delta^a_b = 0. \tag{37}$$

$$G^t_t - G^r_r = 0 \Rightarrow \frac{a'}{a} + \frac{b'}{b} = 0, \tag{38}$$

from which one finds $a = 1/b$ and the constant of integration can be set to one by a rescaling of the time coordinate.

5 points (UNSEEN)

To solve the $G^t_t + \Lambda = 0$ equation, note that

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r}{b}\right) = -\frac{b'}{rb^2} + \frac{1}{r^2 b}. \tag{39}$$

Hence, we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r}{b}\right) = \frac{1}{r^2} - \Lambda, \tag{40}$$

$$\frac{r}{b} = r - \frac{\Lambda}{3}r^3 - C, \tag{41}$$

$$\frac{1}{b} = 1 - \frac{C}{r} - \frac{\Lambda}{3}r^2, \tag{42}$$

where C is a constant of integration which one identifies with the mass parameter $C = 2m$.

8 points (UNSEEN)

(b) With $\Lambda = 0$ this is the Schwarzschild solution. In the limit $r \to \infty$ the Schwarzschild metric approaches Minkowski spacetime in spherical polar coordinates. It describes the exterior gravitational field of a static and spherically symmetric body. It is the most important known exact vacuum solution of the Einstein field equations.

5 points (UNSEEN/SEEN)

(c) With $r = 1/\sqrt{\Lambda/3}\sin\chi$ we have

$$1 - \frac{\Lambda}{3}r^2 = 1 - \sin^2\chi = \cos^2\chi, \tag{43}$$

$$dr = \sqrt{3/\Lambda}\cos\chi d\chi, \tag{44}$$

$$dr^2 = 3/\Lambda\cos^2\chi d\chi^2. \tag{45}$$

Therefore

$$ds^2_{\text{spatial}} = \frac{3}{\Lambda}\left[d\chi^2 + \sin^2\chi d\Omega^2\right] \tag{46}$$

which is the line-element of a 3-sphere. 7 points (UNSEEN)

5. (a) The Euler-Lagrange equations are given by

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{X}^c} = \frac{\partial L}{\partial X^c} \tag{47}$$

$$\frac{\partial L}{\partial X^c} = \frac{\partial g_{ab}}{\partial X^c}\dot{X}^a\dot{X}^b = g_{ab,c}\dot{X}^a\dot{X}^b \tag{48}$$

$$\frac{\partial L}{\partial \dot{X}^c} = g_{ab}\dot{X}^a\delta^b_c + g_{ab}\delta^a_c\dot{X}^b = g_{ac}\dot{X}^a + g_{cb}\dot{X}^b = 2g_{ca}\dot{X}^a \tag{49}$$

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{X}^c} = 2g_{ca,b}\dot{X}^b\dot{X}^a + 2g_{ca}\ddot{X}^a \tag{50}$$

Hence, we find the following equations of motion

$$g_{ab,c}\dot{X}^a\dot{X}^b = 2g_{ca}\ddot{X}^a + g_{ca,b}\dot{X}^b\dot{X}^a + g_{cb,a}\dot{X}^b\dot{X}^a \tag{51}$$

which after sorting the terms leads to

$$2g_{ca}\ddot{X}^a + (g_{ca,b} + g_{cb,a} - g_{ab,c})\dot{X}^a\dot{X}^b = 0 \tag{52}$$

Next, we apply g^{cd} to this latter equation and find

$$2\delta^d_a\ddot{X}^a + g^{cd}(g_{ca,b} + g_{bc,a} - g_{ab,c})\dot{X}^a\dot{X}^b = 0 \tag{53}$$

$$\ddot{X}^d + g^{dc}(g_{ca,b} + g_{bc,a} - g_{ab,c})\dot{X}^a\dot{X}^b = 0. \tag{54}$$

6 points (SEEN)

(b)

$$L = -(1-\Lambda r^2/3)\dot{t}^2 + \dot{r}^2/(1-\Lambda r^2/3) + r^2\dot{\phi}^2, \tag{55}$$

where $L = 0$ for massless particles and $L = -1$ for massive particles. The geodesic equations are

$$t: \quad -2\frac{d}{d\lambda}\left((1-\Lambda r^2/3)\dot{t}\right) = 0, \tag{56}$$

$$\phi: \quad 2\frac{d}{d\lambda}\left(r^2\dot{\phi}\right) = 0, \tag{57}$$

$$r: \quad 2\frac{d}{d\lambda}\left(\dot{r}/(1-\Lambda r^2/3)\right) \tag{58}$$

$$= 2\Lambda r/3\dot{t}^2 + \dot{r}^2/(1-\Lambda r^2/3)^{-2}(2\Lambda r/3) + 2r\dot{\phi}^2. \tag{59}$$

7 points (UNSEEN)

(c) The system has two constants of motion

$$-2(1-\Lambda r^2/3)\dot{t} = -2E, \qquad \text{energy} \tag{60}$$

$$2r^2\dot{\phi} = 2\ell, \qquad \text{angular momentum.} \tag{61}$$

4 points (UNSEEN)

The Lagrangian can be written in the following form

$$\dot{r}^2/2 + V_{\text{eff}}(r) = C, \tag{62}$$

where

$$V_{\text{eff}} = \frac{1}{2}\left(1-\frac{\Lambda}{3}r^2\right)\left(\frac{\ell^2}{r^2}-L\right) \tag{63}$$

Hence one finds $C = E^2/2$. One should read the equation as kinetic energy plus potential energy equals total energy ($= C$).

8 points (UNSEEN/SEEN)