1

<u>Quantum 1S</u>

1a) $|\psi\rangle = \sum_n a_n |n\rangle$ where $\sum_n |a_n|^2 = 1$

Take $\langle m|$

$\langle m|\psi\rangle = \sum_n a_n \langle m|n\rangle = \sum_n a_n S_{mn} = a_m$

$\Rightarrow a_n = \langle n|\psi\rangle$

b) $|\psi\rangle = \sum_n b_n |n\rangle = \sum_a c_a |a\rangle$

Take $\langle a'|$

$\sum_n b_n \langle a'|n\rangle = c_{a'}$ $\quad a' \to a$

$\Rightarrow c_a = \sum_n b_n \langle a|n\rangle = \sum_n b_n S_{an}$

2a) $\langle f|Ag\rangle = \langle Af|g\rangle \Rightarrow A = A^\dagger$ where $A^\dagger = (A^T)^*$

b) $A|n\rangle = a_n |n\rangle \Rightarrow \langle m|A|n\rangle = a_n \langle m|n\rangle$

$A|m\rangle = a_m |m\rangle \Rightarrow \langle m|A^\dagger = \langle m| a_m^*$

$\Rightarrow \langle m|A = \langle m| a_m^* \Rightarrow \langle m|A|n\rangle = a_m^* \langle m|n\rangle$

$\Rightarrow a_n \langle m|n\rangle = a_m^* \langle m|n\rangle$ eigenvalues real

$(a_n - a_m^*)\langle m|n\rangle = 0 \quad a_n \neq a_m^*$ then $\langle m|n\rangle = 0$

3a) $A|\psi\rangle = a|\psi\rangle \qquad B|\psi\rangle = b|\psi\rangle$

$\Rightarrow \hat{A}\hat{B}|\psi\rangle = \hat{A}b|\psi\rangle = ab|\psi\rangle$

$\hat{B}\hat{A}|\psi\rangle = \hat{B}a|\psi\rangle = ab|\psi\rangle$

$\Rightarrow \hat{A}\hat{B}|\psi\rangle = \hat{B}\hat{A}|\psi\rangle \quad \Rightarrow [\hat{A},\hat{B}]|\psi\rangle 0 \quad [\hat{A},\hat{B}] = 0$

b) When operators commute $\Delta A \Delta B = 0$ so ~~these~~ observables can be known exactly. For non-commuting operators ~~no~~ neither observable can be known exactly

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle[A,B]\rangle\right|$$

4a) $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$

Cyclic relation $[L_i, L_j] = \varepsilon_{ijk}\, i\hbar L_k$

b) $[\hat{L}_+, \hat{L}_z] = [\hat{L}_x + i\hat{L}_y, \hat{L}_z]$

$= [L_x, L_z] + i[L_y L_z]$

$= -i\hbar L_y + i\, i\hbar L_x = -i\hbar L_y - \hbar L_x = -\hbar \hat{L}_+$

5a) A ~~particle~~ Particle of spin $\frac{1}{2}$ are fired into an inhomogeneous magnetic field such that beams will emerge in different directions for each magnetic moment. For a spin $\frac{1}{2}$ particle 2 beams will emerge. This is because there are $2S+1$ spin states. The important result was that there were an even number of beams. Before S-G experiment only integer orbital angular momentum was known so $\frac{1}{2}$ integer spin came as a surprise

bi) Along x spin would be measured as $+x$ and $-x$ with equal probability

ii) Along z spin will be measured as $+z$

②

6b) ~~(N)~~ $\dfrac{\langle \psi_{trial} | \hat{H} | \psi_{trial} \rangle}{\langle \psi | \psi \rangle} \geq E_0$

By creating a trial ψ, based on parameters, the above expression can be calculated and minimised to give an upper bound on the ground state energy. A trial wavefunction could have form ~~α~~ $|0\rangle + \beta|1\rangle$ for a perturbed harmonic oscillator

~~a~~

a) In general, an observables expectation value can be calculated as a function of parameters in a trial wave function. For different purposes this can be maximised or minimised to give limiting properties of a system: The Hartree-Fock and Ritz method both use this principle

7a) λ is a parameter used to keep track of the order of the corrections. It allows us to create power series for the energy and wave-function in terms of λ. Using these power series in our Hamiltonian and equating powers of λ allows us to calculate our different order corrections

The first order correction is the energy term preceded by a coefficient of λ and second order λ^2.

b) $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} \ldots \quad \psi_n = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \ldots$

$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$

$(H_0 + \lambda H')(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \ldots) = (E_n^{(0)} + \lambda E_n^{(1)} + \ldots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \ldots)$

~~Pa~~ λ terms $H'|\psi_n^{(0)}\rangle + H_0|\psi_n^{(1)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle$

$\langle \psi_n^{(0)} | H' | \psi_n^{(0)} \rangle + \langle \psi_n^{(0)} | H_0 | \psi_n^{(1)} \rangle = E_n^{(0)} \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle + E_n^{(1)}$

$\langle\psi_n^{(0)}|H_0 = E_n^{(0)}\langle\psi_n^{(0)}|$

$\Rightarrow \langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle + E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}||\psi_n^{(1)}\rangle + E_n^{(1)}$

$\Rightarrow E_n^{(1)} = \langle\psi_n^{(0)}|H'|\psi_n^{(0)}\rangle$

c) Ground state of harmonic oscillator - perturbation of $\propto \hat{x}$ will have no first order correction as ground state is even and perturbation odd

In general if wavefunction and perturbation have opposite parities the first order correction is 0

d) Since $E_m^{(0)} > E_n^{(0)}$ and modulus numerator is positive always the second order correction to the ground state is negative as denominator is negative for all terms in sum

e) $E_n^{(1)} = \langle\psi_{nlm}| -\frac{GM}{r}|\psi_{nlm}\rangle$

$= -GM\left\langle\frac{1}{r}\right\rangle_{nlm} = -\frac{GM}{n^2} = \frac{-4.41\times10^{-40}}{n^2}$

40 orders of magnitude smaller than energy level $\Rightarrow$ insignificant

In 2nd order proportional to G^2M^2 so will be even less significant

f) $H' = \gamma\hat{x} = \frac{\gamma}{\sqrt{2}\alpha}(a_+ + a_-) = \gamma\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$

$E_n^{(1)} = \langle 0|\gamma\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)|0\rangle = \langle 0|\gamma\sqrt{\frac{\hbar}{2m\omega}}(|1\rangle + 0) = 0$

(3)

$\langle 1|\, \gamma\sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)|0\rangle = \langle 1|\, \gamma\sqrt{\frac{\hbar}{2m\omega}}\,|1\rangle = \gamma\sqrt{\frac{\hbar}{2m\omega}}$

$\langle m|H'|0\rangle = 0$ for $m \geq 2$ as ladder operator only increase $|0\rangle$ to $|1\rangle$

$$\Rightarrow E_n^{(2)} = \frac{|\langle 1|H'|0\rangle|^2}{E_0 - E_1} = \frac{\gamma^2 \hbar / 2m\omega}{\frac{\hbar\omega}{2} - \frac{3\hbar\omega}{2}}$$

$$= \frac{-\gamma^2}{2m\omega^2}$$

8a) Fermions have half-integer spin, bosons integer spin. For multiple particle systems, ~~fermions~~ on interchange of any 2 particles the wave function must be antisymmetric for fermions and symmetric for bosons

$\psi_1(x_1)\psi_2(x_2) = \pm\, \psi_1(x_2)\psi_2(x_1)$ + for bosons
- for fermions

b) No 2 fermions can have the same overall wavefunction (ie occupy same quantum state)

If they had same ψ $\psi_1(x_1)\psi_1(x_2) = -\psi_1(x_2)\psi_1(x_1)$
as antisymmetric $\Rightarrow \psi_1(x_1)\psi_1(x_2) = 0$ no wavefunction

c) ~~Generally~~ $S_1 = S_2 = \frac{1}{2}$
Overall spin quantum number $S = S_1 \oplus S_2 = 0, 1$

$S = 0$ $m_s = 0$ $\Rightarrow$

$|0\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|\downarrow\rangle_2 - |\downarrow\rangle_1|\uparrow\rangle_2\right)$ overall antisymmetric singlet state

S^2 eigenvalue $\hbar^2 S(S+1) = 0$
S_z eigenvalue $\hbar m_s = 0$

$S=1$ $m_s=-1,0,1$ triplet state all symmetric

$|1\,-1\rangle = |\downarrow\rangle_1|\downarrow\rangle_2$

$|1\,0\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1|\downarrow\rangle_2 + |\downarrow\rangle_1|\uparrow\rangle_2)$

$|1\,1\rangle = |\uparrow\rangle_1|\uparrow\rangle_2$

S^2 eigenvalues are $2\hbar^2$ for all triplet states
S_z eigenvalues $-\hbar, 0, \hbar$ respectively

Spatial part of wavefunction must be symmetric for singlet and antisym for triplet so wavefunction is antisymmetric overall

d) $\hat{H} = B\hat{S}_z + \gamma\left(\frac{\hat{S}_{+1}+\hat{S}_{-1}}{2}\right)\left(\frac{\hat{S}_{+2}-\hat{S}_{-2}}{2i}\right)$

$= B\hat{S}_z + \frac{\gamma}{4i}(\hat{S}_{+1}+\hat{S}_{-1})(\hat{S}_{+2}-\hat{S}_{-2})$

$S_z = S_{z1} + S_{z2}$
$S_+|\beta\rangle = \hbar|\alpha\rangle$
$S_-|\alpha\rangle = \hbar|\beta\rangle$

$\hat{H}|\alpha\rangle_1|\alpha\rangle_2 = \left(BS_z + \frac{\gamma}{4i}(S_{+1}+S_{-1})(S_{+2}-S_{-2})\right)|\alpha\rangle_1|\alpha_2\rangle$

$= B\hbar|\alpha\rangle_1|\alpha\rangle_2 + \frac{\gamma}{4i}(0+\hbar|\beta\rangle_1)(0-\hbar|\beta\rangle_2)$

$= B\hbar|\alpha\rangle_1|\alpha\rangle_2 - \frac{\hbar^2\gamma}{4i}|\beta\rangle_1|\beta\rangle_2$

$\hat{H}|\alpha\rangle_1|\beta\rangle_2 = 0 + \frac{\gamma}{4i}(0+\hbar|\beta\rangle_1)(\hbar|\alpha\rangle_2 + 0) = \frac{\hbar^2\gamma}{4i}|\beta\rangle_1|\alpha\rangle_2$

$\hat{H}|\beta\rangle_1|\alpha\rangle_2 = 0 + \frac{\gamma}{4i}(\hbar|\alpha\rangle_1+0)(0-\hbar|\beta\rangle_2) = -\frac{\hbar^2\gamma}{4i}|\alpha\rangle_1|\beta\rangle_2$

$\hat{H}|\beta_1\rangle|\beta\rangle_2 = -\hbar B|\beta_1\rangle|\beta\rangle_2 + \frac{\gamma}{4i}(\hbar|\alpha\rangle_1+0)(\hbar|\alpha\rangle_2 - 0)$

$= -\hbar B|\beta\rangle_1|\beta_2\rangle + \frac{\gamma\hbar^2}{4i}|\alpha\rangle_1|\alpha\rangle_2$

Using basis $\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$ $|\alpha\rangle_1 |\alpha\rangle_2$, $|\alpha\rangle_1 |\beta\rangle_2$, $|\beta\rangle_1 |\alpha\rangle_2$, $|\beta\rangle_1 |\beta\rangle_2$

$$\hat{H}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = B\hbar|\alpha\rangle_1|\alpha\rangle_2 - \frac{\hbar^2}{4i}\gamma|\beta\rangle_1|\beta_2\rangle = \begin{pmatrix}B\hbar\\0\\0\\-\frac{\hbar^2\gamma}{4i}\end{pmatrix}$$

Left hand column of $\hat{H}$

Same with other vectors $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \ldots$

$$\text{gives } \hat{H} = \begin{pmatrix} B\hbar & 0 & 0 & \frac{\gamma\hbar^2}{4i} \\ 0 & 0 & -\frac{\gamma\hbar^2}{4i} & 0 \\ 0 & \frac{\gamma\hbar^2}{4i} & 0 & 0 \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 & -B\hbar \end{pmatrix}$$

e) $\det(H - \lambda I) = 0$

$$\begin{vmatrix} B\hbar-\lambda & 0 & 0 & \frac{\gamma\hbar^2}{4i} \\ 0 & -\lambda & -\frac{\gamma\hbar^2}{4i} & 0 \\ 0 & \frac{\gamma\hbar^2}{4i} & -\lambda & 0 \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 & -B\hbar-\lambda \end{vmatrix} = 0$$

$$= (B\hbar-\lambda)\begin{vmatrix} -\lambda & -\frac{\gamma\hbar^2}{4i} & 0 \\ \frac{\gamma\hbar^2}{4i} & -\lambda & 0 \\ 0 & 0 & -B\hbar-\lambda \end{vmatrix} + \frac{\gamma\hbar^2}{4i}\begin{vmatrix} 0 & -\lambda & -\frac{\gamma\hbar^2}{4i} \\ 0 & \frac{\gamma\hbar^2}{4i} & -\lambda \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 \end{vmatrix}$$

$$= (B\hbar-\lambda)\left(-\lambda^2(B\hbar+\lambda) - \frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2}{4i}(B\hbar+\lambda)\right) + \frac{\gamma\hbar^2}{4i}\left(-\lambda^2\frac{\gamma\hbar^2}{4i} - \frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2}{4i}\right)$$

$$= -\lambda^2B^2\hbar^2 + \lambda^4 + (B\hbar-\lambda)(B\hbar+\lambda)\frac{\gamma^2\hbar^4}{16} + \frac{\lambda^2\gamma^2\hbar^4}{16} - \frac{\gamma^4\hbar^8}{256} = 0$$

$$\lambda^4 - \lambda^2B^2\hbar^2 + \frac{\gamma^2B^2\hbar^6}{16} - \cancel{\frac{\gamma^2\hbar^4\lambda^2}{16}} + \cancel{\frac{\gamma^2\hbar^4\lambda^2}{16}} - \frac{\gamma^4\hbar^8}{256} = 0$$

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Using basis $\begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$ $\begin{matrix} |\alpha\rangle_1|\alpha\rangle_2 \\ |\alpha\rangle_1|\beta\rangle_2 \\ |\beta\rangle_1|\alpha\rangle_2 \\ |\beta\rangle_1|\beta\rangle_2 \end{matrix}$

$$\hat{H}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = B\hbar|\alpha\rangle_1|\alpha\rangle_2 - \frac{\hbar^2}{4i}\gamma|\beta\rangle_1|\beta_2\rangle = \begin{pmatrix} B\hbar \\ 0 \\ 0 \\ -\frac{\hbar^2\gamma}{4i} \end{pmatrix}$$

Left hand column of $\hat{H}$

Same with other vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \dots$

gives $$\hat{H} = \begin{pmatrix} B\hbar & 0 & 0 & \frac{\gamma\hbar^2}{4i} \\ 0 & 0 & -\frac{\gamma\hbar^2}{4i} & 0 \\ 0 & \frac{\gamma\hbar^2}{4i} & 0 & 0 \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 & -B\hbar \end{pmatrix}$$

e) $\det(\hat{H}-\lambda\underline{I})=0$ $$\begin{vmatrix} B\hbar-\lambda & 0 & 0 & \frac{\gamma\hbar^2}{4i} \\ 0 & -\lambda & -\frac{\gamma\hbar^2}{4i} & 0 \\ 0 & \frac{\gamma\hbar^2}{4i} & -\lambda & 0 \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 & -B\hbar-\lambda \end{vmatrix} = 0$$

$$= (B\hbar-\lambda)\begin{vmatrix} -\lambda & -\frac{\gamma\hbar^2}{4i} & 0 \\ \frac{\gamma\hbar^2}{4i} & -\lambda & 0 \\ 0 & 0 & -B\hbar-\lambda \end{vmatrix} + \frac{\gamma\hbar^2}{4i}\begin{vmatrix} 0 & -\lambda & -\frac{\gamma\hbar^2}{4i} \\ 0 & \frac{\gamma\hbar^2}{4i} & -\lambda \\ -\frac{\gamma\hbar^2}{4i} & 0 & 0 \end{vmatrix}$$

$$= (B\hbar-\lambda)\left(-\lambda^2(B\hbar+\lambda) - \frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2(B\hbar+\lambda)}{4i}\right) + \frac{\gamma\hbar^2}{4i}\left(-\lambda^2\frac{\gamma\hbar^2}{4i} - \frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2}{4i}\frac{\gamma\hbar^2}{4i}\right)$$

$$= -\lambda^2B^2\hbar^2 + \lambda^4 + (B\hbar-\lambda)(B\hbar+\lambda)\frac{\gamma^2\hbar^4}{16} + \frac{\lambda^2\gamma^2\hbar^4}{16} - \frac{\gamma^4\hbar^8}{256} = 0$$

$$\lambda^4 - \lambda^2B^2\hbar^2 + \frac{\gamma^2B^2\hbar^6}{16} - \cancel{\frac{\gamma^2\hbar^4\lambda^2}{16}} + \cancel{\frac{\gamma^2\hbar^4\lambda^2}{16}} - \frac{\gamma^4\hbar^8}{256} = 0$$

$$\lambda^4 - \beta^2\hbar^2\lambda^2 + \frac{\gamma^2\beta^2\hbar^6}{16} - \frac{\gamma^4\hbar^8}{256} = 0$$

$$\lambda^2 = \frac{\beta^2\hbar^2 \pm \sqrt{\beta^4\hbar^4 + \frac{\gamma^4\hbar^8}{64} - \frac{\gamma^2\beta^2\hbar^6}{4}}}{2}$$

$$= \frac{\beta^2\hbar^2 \pm \sqrt{(\beta^2\hbar^2 - \frac{\gamma^2\hbar^4}{8})^2}}{2}$$

$$= \frac{\beta^2\hbar^2}{2} \pm \left(\frac{\beta^2\hbar^2 - \frac{\gamma^2\hbar^4}{8}}{2}\right)$$

$$\lambda^2 = \beta^2\hbar^2 - \frac{\gamma^2\hbar^4}{16} \qquad \lambda_{1,2} = \pm\hbar\sqrt{\beta^2 - \frac{\gamma^2\hbar^2}{16}}$$

$$\lambda^2 = \frac{\gamma^2\hbar^4}{16} \qquad \lambda_{3,4} = \pm\frac{\gamma\hbar^2}{4}$$

9a) $$[a_-, a_+] = \left[\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\right]$$

$$= \frac{m\omega}{2\hbar}\left(\frac{i}{m\omega}[\hat{p},\hat{x}] - \frac{i}{\omega}m[\hat{x},\hat{p}]\right)$$

$$= \frac{m\omega}{2\hbar}\left(\frac{i}{m\omega}(-i\hbar) - \frac{i}{m\omega}(i\hbar)\right)$$

$$= \frac{1}{2\hbar}(2\hbar) = 1$$

b) $$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$= \frac{1}{2m} i\sqrt{\frac{\hbar m\omega}{2}}\, i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)(a_+ - a_-) + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega}(a_+ + a_-)(a_+ + a_-)$$

$$= -\frac{\hbar\omega}{4}(a_+^2 + a_-^2 - a_+a_- - a_-a_+) + \frac{\hbar\omega}{4}(a_+^2 + a_-^2 + a_+a_- + a_-a_+)$$

$$= \frac{\hbar\omega}{4}(2a_+a_- + 2a_-a_+) = \frac{\hbar\omega}{4}(2a_+a_- + 2(1 + a_+a_-))$$

$$= \frac{\hbar\omega}{4}\left(4a_+a_- + 2\right) = \hbar\omega\left(a_+a_- + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$$

$$\hat{N} = \hat{a}_+\hat{a}_-$$

c) $[\hat{H}, \hat{a}_\pm] = \hbar\omega[a_+a_-, a_\pm]$ use $[AB,C] = A[B,C] + [A,C]B$

$$= \hbar\omega\left(a_+[a_-, a_\pm] + [a_+, a_\pm]a_-\right)$$

$$= \hbar\omega\left(a_+ \tfrac{1}{0} + \tfrac{0}{-1} a_-\right)$$

$$= \cancel{\pm\hbar\omega} \pm \hbar\omega a_\pm$$

$$\hat{H}|\psi\rangle = E|\psi\rangle$$

$$\hat{H} a_\pm|\psi\rangle = \left(a_\pm \hat{H} \pm \hbar\omega a_\pm\right)|\psi\rangle$$

$$= (E \pm \hbar\omega) a_\pm |\psi\rangle$$ eigenvalue $E \pm \hbar\omega$

d) $\hat{B} = xp + px + \hbar = \cancel{2}xp + xp - i\hbar + \hbar = 2xp - i\hbar + \hbar$

$$= 2\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\, i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) + \hbar(1-i)$$

$$= i\hbar\left(a_+^2 - a_-^2 - a_+a_- + a_-a_+\right) + \hbar(1-i)$$

$$= i\hbar\left(a_+^2 - a_-^2 + [a_-, a_+]\right) + \hbar(1-i)$$

$$= i\hbar\left(a_+^2 - a_-^2\right) + i\hbar + \hbar - i\hbar = i\hbar\left(a_+^2 - a_-^2\right) + \hbar$$

e) $\hat{B}|0\rangle = \sqrt{2}\, i\hbar|2\rangle + \hbar|0\rangle$

$\hat{B}|1\rangle = \sqrt{6}\, i\hbar|3\rangle + \hbar|1\rangle$

$\hat{B}|2\rangle = 2\sqrt{3}\, i\hbar|4\rangle - \sqrt{2}\, i\hbar|0\rangle + \hbar|2\rangle$

$\hat{B}|3\rangle = 2\sqrt{5}\, i\hbar|5\rangle - \sqrt{6}\, i\hbar|1\rangle + \hbar|3\rangle$

$$\Rightarrow B = \begin{pmatrix} \hbar & 0 & -\sqrt{2}\, i\hbar & 0 \\ 0 & \hbar & 0 & -\sqrt{6}\, i\hbar \\ \sqrt{2}\, i\hbar & 0 & \hbar & 0 \\ 0 & \sqrt{6}\, i\hbar & 0 & \hbar \end{pmatrix}$$

f) $E_n^{(1)} = \langle \psi_n^{(0)} | \gamma \hat{B} | \psi_n^{(0)} \rangle$

$$E_0^{(1)} = \cancel{\hbar}\langle 0 | \gamma \hat{B} | 0 \rangle = (1 \quad 0 \quad 0 \quad 0)\, \gamma \begin{pmatrix} \hbar \\ 0 \\ \sqrt{2} i \hbar \\ 0 \end{pmatrix} = \gamma \hbar$$

~~E~~ All first order corrections are just diagonal of matrix
$\Rightarrow E_n^{(1)} = \gamma \hbar$ for all n

$\Rightarrow$ Perturbation small if $\gamma << (\frac{1}{2} + n)$

g) $$\beta B^2 = \frac{\gamma}{\hbar} \hbar^2 \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix}$$

$$= \cancel{\hbar}\gamma \hbar \begin{pmatrix} 3 & 0 & -2i\sqrt{2} & 0 \\ 0 & 7 & 0 & -2i\sqrt{6} \\ 2i\sqrt{2} & 0 & 3 & 0 \\ 0 & 2i\sqrt{6} & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle 0 | \beta B^2 | 0 \rangle = \gamma \hbar (1 \quad 0 \quad 0 \quad 0) \begin{pmatrix} 3 \\ 0 \\ 2i\sqrt{2} \\ 0 \end{pmatrix} = 3\gamma\hbar$$

3 times larger than 1st order correction

+6

use $[AB, C] = A[B, C] + [A, C]B$

10a) $[J_y J_y, J_z] = J_y [J_y, J_z] + [J_y, J_z] J_y$

$= J_y i\hbar J_x + i\hbar J_x J_y = i\hbar (J_y J_x + J_x J_y)$

b) $[J_z^2, J_z] = 0$

$[J_x^2, J_z] = J_x [J_x, J_z] + [J_x, J_z] J_x$
$= -i\hbar J_x J_y - i\hbar J_y J_x$

$\Rightarrow [J^2, J_z] = [J_x^2 + J_y^2 + J_z^2, J_z]$

$= i\hbar (J_y J_x + J_x J_y) - i\hbar (J_x J_y + J_y J_x) = 0$

$\hat{J}^2$ and $\hat{J}_z$ are compatible operators $\Rightarrow$ eigenvectors of $\hat{J}_z$ are also eigenvectors of J^2

c) $\hat{J}^2 |j, m_j\rangle = \hbar^2 j(j+1) |j, m_j\rangle$

$\hat{J}_z |j, m_j\rangle = m_j \hbar |j, m_j\rangle$

d) $[J_\pm, J_z] = \mp \hbar J_\pm$ from 4b

$J_z J_\pm |j, m_j\rangle = (J_\pm J_z \pm \hbar J_\pm) |j m_j\rangle$

$= (m_j \hbar \pm \hbar) J_\pm |j m_j\rangle$

Eigenvalue $(m_j \pm 1)\hbar$

e) If there were a state where m_j exceeded j then J_z^2 would be greater than J^2 which is already physically impossible. Therefore, ~~as~~ because J_+ increases

f) $E_n^{(1)} = \langle \psi_n^{(0)} | \gamma \hat{B} | \psi_n^{(0)} \rangle$

$$E_0^{(1)} = \langle 0 | \gamma \hat{B} | 0 \rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \gamma \begin{pmatrix} \hbar \\ 0 \\ \sqrt{2} i \hbar \\ 0 \end{pmatrix} = \gamma \hbar$$

~~E~~ All first order corrections are just diagonal of matrix
$\Rightarrow E_n^{(1)} = \gamma \hbar$ for all n

$\Rightarrow$ Perturbation small if $\gamma << (\frac{1}{2} + n)$

g) $$\beta B^2 = \frac{\gamma}{\hbar} \hbar^2 \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix}$$

$$= \gamma \hbar \begin{pmatrix} 3 & 0 & -2i\sqrt{2} & 0 \\ 0 & 7 & 0 & -2i\sqrt{6} \\ 2i\sqrt{2} & 0 & 3 & 0 \\ 0 & 2i\sqrt{6} & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\langle 0 | \beta B^2 | 0 \rangle = \gamma \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 2i\sqrt{2} \\ 0 \end{pmatrix} = 3 \gamma \hbar$$

3 times larger than 1st order correction

m_j but j remains the same, there must be a state ~~wen~~ where $J_+ |j\, m_j\rangle = 0$ to stop J_z^2 being greater than J^2

The same applies in the opposite direction, if $m_j < -j$ $J_z^2 > J^2$ which is not allowed.

$$J_z |j, j\rangle = \hbar j \; |j, j\rangle$$

$$J_z |j\, -j\rangle = -\hbar j \; |j, -j\rangle$$

f) $$J_y = \frac{J_+ - J_-}{2i}$$

$$\Rightarrow \hat{J}_y |\phi\rangle = \frac{1}{2i} (J_+ - J_-)\left(a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle \right)$$

$$= \frac{\hbar}{2i}\left[a\left(0 - \sqrt{2}|1,0\rangle\right) + b\left(\sqrt{2}|1,1\rangle - \sqrt{2}|1,-1\rangle\right) + c\left(\sqrt{2}|1,0\rangle + 0\right) \right]$$

$$= \frac{\hbar}{2i}\left[b\sqrt{2}|1,1\rangle + (c-a)\sqrt{2}|1,0\rangle - b\sqrt{2}|1,-1\rangle \right]$$

g) ~~Eigenvectors~~ Eigenvalues $\hbar, 0, -\hbar$

$\hbar$: $a = \frac{b}{\sqrt{2}i}$ $\quad c = \frac{-b}{\sqrt{2}i}$ $\quad \Rightarrow |1,1\rangle_y = d\left(|1,1\rangle + \sqrt{2}i|1,0\rangle - |1,-1\rangle\right)$

$d = \frac{1}{\sqrt{1+2+1}} = \frac{1}{2}$ $\quad |1,1\rangle_y = \frac{1}{2}|1,1\rangle + \frac{i}{\sqrt{2}}|1,0\rangle - \frac{1}{2}|1,-1\rangle$

0: $c = a$ $\quad b = 0$ $\quad |1,0\rangle_y = \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{\sqrt{2}}|1,-1\rangle$

-1: $a = \frac{-b}{\sqrt{2}i}$ $\quad c = \frac{b}{\sqrt{2}i}$ $\quad |1,-1\rangle_y = \frac{-1}{2}|1,1\rangle + \frac{i}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$