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QUANTUM PHYSICS 2015

(1) (a) $|\psi\rangle = \sum_n c_n |n\rangle$

To find the coefficient, take the inner product with a bra $\langle m|$

$$\langle m|\psi\rangle = \sum_n c_n \langle m|n\rangle$$

$$= \sum_n c_n \delta_{mn}$$

$$\therefore c_n = \langle n|\psi\rangle$$

(b) Transform coefficients in the basis $|n\rangle$ to basis $|a\rangle$ using similarity transform.

$$|\psi\rangle = \sum_a d_a |a\rangle \qquad c_n = \langle n|\psi\rangle$$

$$|\psi\rangle = \sum_a \langle a|\psi\rangle |a\rangle$$

$$\boxed{c_n = \langle n|\psi\rangle}$$

Second method: $|\psi\rangle = \sum_n c_n |n\rangle = \sum_n \langle n|\psi\rangle |n\rangle = \sum_n |n\rangle\langle n|\psi\rangle$

closure relation : $\sum_n |n\rangle\langle n| = 1$

$$|\psi\rangle = \sum_a d_a |a\rangle$$

$$d_a = \langle a|\psi\rangle$$

$$|\psi\rangle = \sum_n |n\rangle\langle n|\psi\rangle$$

$$\langle a|\psi\rangle = \sum_n \langle a|n\rangle\langle n|\psi\rangle$$

since $d_a = \langle a|\psi\rangle$,

$$d_a = \sum_n \langle a|n\rangle\langle n|\psi\rangle$$

$$d_a = \sum_n \langle a|n\rangle . c_n$$

$$\boxed{d_a = \sum_n S_{an} c_n}$$

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(2) (a) Hermition operater : $\langle\phi|\hat{A}|\psi\rangle = \langle\hat{A}^{+}\phi|\psi\rangle = \langle\psi|\hat{A}^{+}|\phi\rangle^{*}$

If $\hat{A}$ is Hermitian, then $\hat{A} = \hat{A}^{+}$

(b) $\hat{A}|\phi_n\rangle = a_n|\phi_n\rangle$

$\hat{A}|\phi_m\rangle = a_m|\phi_m\rangle \rightarrow \hat{A}^{+}\langle\phi_m| = a_m^{*}\langle\phi_m|$

$\langle\phi_m|\hat{A}|\phi_n\rangle = a_n\langle\phi_m|\phi_n\rangle$

$\langle\hat{A}^{+}\phi_m|\phi_n\rangle$

$a_m^{*}\langle\phi_m|\phi_n\rangle = a_n\langle\phi_m|\phi_n\rangle$

$(a_m^{*} - a_n)\langle\phi_m|\phi_n\rangle = 0$

If $a_m^{*} \neq a_n$, then $\langle\phi_m|\phi_n\rangle = 0$. $\therefore$ The eigenvectors of a Hermitian operator for different eigenvalues are orthogonal.

(3) (a) $\hat{A}|\phi_n\rangle = a_n|\phi_n\rangle$ $\quad [\hat{A},\hat{B}] = 0$

$\hat{B}|\phi_n\rangle = b_n|\phi_n\rangle$ $\quad \hat{A}\hat{B} = \hat{B}\hat{A}$

$\hat{A}|\phi_n\rangle = a_n|\phi_n\rangle$

~~$\hat{B}\hat{A}$~~

$\hat{B}\hat{A}|\phi_n\rangle = a_n\hat{B}|\phi_n\rangle = a_n b_n|\phi_n\rangle$

$\hat{B}|\phi_n\rangle = b_n|\phi_n\rangle$

$\hat{A}\hat{B}|\phi_n\rangle = b_n\hat{A}|\phi_n\rangle = b_n a_n|\phi_n\rangle$

$$(\hat{A}\hat{B} - \hat{B}\hat{A})|\phi_n\rangle = \underbrace{(b_n a_n - a_n b_n)}_{=0}|\phi_n\rangle$$

$\therefore$ If two operators share eigenvectors, they must commute.

(b) $$\boxed{(\Delta A)(\Delta B) \geq \frac{1}{2}|\langle[\hat{A},\hat{B}]\rangle|}$$ → NOT GIVEN! MEMORISE.

If the operators commute, ie. $[\hat{A},\hat{B}] = 0$, then $(\Delta A)(\Delta B) = 0$. Therefore, observables can be measured exactly.

If the operators does not commute, neither observable can be known exactly.

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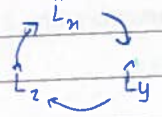
(1) (a) $[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$



Cyclic Permutation

Commutator relation

$$\boxed{\hat{L} \times \hat{L} = i\hbar \hat{L}}$$

(b) $[\hat{L}_+, \hat{L}_z]$ $\qquad \hat{L}_+ = \hat{L}_x + i\hat{L}_y$

$[\hat{L}_x + i\hat{L}_y, \hat{L}_z]$

$[\hat{L}_x, \hat{L}_z] + i[\hat{L}_y, \hat{L}_z]$

$-i\hbar \hat{L}_y + i[i\hbar \hat{L}_x]$

$-\hbar \hat{L}_x - i\hbar \hat{L}_y$

$-\hbar(\hat{L}_x + i\hat{L}_y)$

$-\hbar \hat{L}_+$

$[\hat{L}_+, \hat{L}_z] = -\hbar \hat{L}_+$

(5) (a) For particles of $s = 1/2$:

The particle is directed along the x-axis and passes between the poles of a magnet with a very non-uniform field along the z-axis.

NOT ENOUGH FOR 4 MARKS

Potential energy : $V = g_s \frac{\mu_B}{\hbar} \underline{s} \cdot \underline{B}$

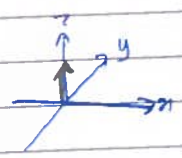
Force : $F_z = -g_s \frac{\mu_B}{\hbar} S_z \frac{dB}{dz}$

$S_z \to +\hbar/2$: force is upwards

$S_z \to -\hbar/2$: force is downwards.

For particles of spin $\frac{1}{2}$, 2 beams will be seen after passing through the magnetic field. This is becau

there are $2s+1$ states.

(b) 

If the initial beam was polarised so that all spins were alligned $+z$,

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(6) (a) Variational principle :

Words : The expectation value of the Hamiltonian , evaluated in an arbitrary state , is always greater than or equal to the ground - state energy .

Particularly suitable for calculating the ground state energies of a system .

The expectation value of the Hamiltonian is minimized with respect to variational parameter and the minimum energy provides the best estimate of the true ground - state energy , within the given parameterized 'family' of wavefunctions .

$$\langle E \rangle_\psi \geq E_0$$

Expectation value of the energy in the state $|\psi\rangle$

$$\boxed{\frac{\langle \psi | \hat{H} | \psi \rangle}{\langle \psi | \psi \rangle} \geq E_0}$$

E_0 : Ground state energy
$\hat{H}$: Hamiltonian

$$|\psi\rangle = \sum_n c_n |\phi_n\rangle .$$

(b) To obtain an approximate value for the ground state energy of a quantum system , one need to guess an approximate ground-state wavefunction , called the "trial wavefunction" , which is used to calculate the expectation value of the Hamiltonian . To improve the estimate of the ground - state energy , the trial wave-function contains variable parameters called variational parameters . The expectation value of the Hamiltonian is minimized wrt these parameters , and the min. ~~ene~~ energy provides the best estimate of the true ground - state energy , within the given parameterized 'family' of wavefunctions .

SECTION B

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(c) $\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

(a) $\hat{H} = \hat{H}_0 + \lambda \hat{H}'$

- In perturbation theory, the solution is expanded as a power series in the perturbation.
- For theory to be ~~useful~~ valid, the series must be convergent.
- For the theory to be useful, series must be rapidly convergent.
- λ is used to keep track of the order of the perturbation.
- ~~It allow us to create power series for the energy and Hamiltonian and equating~~

$$\cancel{(\hat{H}_0 + \lambda \hat{H}')} \quad E_n = \sum_{j=0}^{\infty} \lambda^j E_n^{(j)}$$

$$= E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$$

$$|\psi_n\rangle = \sum_{j=0}^{\infty} \lambda^j |\psi_n^{(j)}\rangle = |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots$$

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$$

$$(\hat{H}_0 + \lambda \hat{H}')(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots)$$

$$= (E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots)(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots)$$

- ~~Equating~~ Coefficients of equal powers of λ must be same on both sides.

$\downarrow$

λ^1: Gives the first order

λ^2: Gives the second order.

(b) $$\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}' |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle$$

$$(\hat{H}' - E_n^{(1)}) |\psi_n^{(0)}\rangle = (E_n^{(0)} - \hat{H}_0) |\psi_n^{(1)}\rangle$$

Taking the scalar product with $|\psi_n^{(0)}\rangle$

$$\langle \psi_n^{(0)} | (\hat{H}' - E_n^{(1)}) |\psi_n^{(0)}\rangle = \langle \psi_n^{(0)} | (E_n^{(0)} - \hat{H}_0) |\psi_n^{(1)}\rangle$$

$\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

RHS is $\boxed{\text{zero}}$ because $\langle \psi_n^{(0)} | \hat{H}_0 | \psi_n^{(1)}\rangle = \langle \psi_n^{(1)} | \hat{H}_0 | \psi_n^{(0)}\rangle^* = E_n^{(0)} \langle \psi_n^{(0)} | \psi_n^{(1)}\rangle$

$$\boxed{\langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)}\rangle = E_n^{(1)}}$$

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- For the case of Harmonic oscillator,

(c) When perturbation, $\hat{H}' = \gamma \hat{x}$, the first order energy change will be 0.

- This is because the ground-state wavefunction of the harmonic oscillator is an even function

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

and the pertubation term is odd.

(d) $$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle \psi_m^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

If n is the ground state, $E_n^{(0)} - E_m^{(0)} < 0$ (always) and thus $E_n^{(2)} < 0$.

(e) In AU: Eigenvalues for the hydrogen atom, $E_{n\ell m} = -\frac{1}{(2n^2)}$

Hamiltonian, $\hat{H}_0 = \frac{\hat{P}^2}{2} - \frac{1}{r}$

Perturbation, $\hat{H}' = -\frac{GM}{r}$

Given: the expectation value for a hydrogenic eigenstate $|\psi\rangle_{n\ell m}$

$$\left\langle \frac{1}{r} \right\rangle_{n\ell m} = \frac{1}{n^2}$$

First order energy change, $E_0^{(1)} = \langle \psi^{(0)}_{n} | -\frac{GM}{r} | \psi^{(0)}_{n} \rangle$

$$= -GM \langle \psi_n^{(0)} | \frac{1}{r} | \psi_n^{(0)} \rangle$$

$G \simeq 2.4 \times 10^{-43}$ a.u

$M \approx 1836$ au.

$$= -\frac{GM}{n^2}$$

$$= \frac{-4.4064 \times 10^{-40}}{n^2}$$

Comparing this to the energy level of the 1s orbital, $E_{100} = -\frac{1}{2}$

$$E_0^{(1)} = \frac{-4.4064 \times 10^{-40}}{(1)^2}$$

The first order energy change is very small. (≈ 40 times smaller)

In the second order, it will be even more insignificant as the second order energy change will be proportional to G^2M^2.

quantum harmonic oscillator

No:

$\hat{H}_0 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

$E_n = \hbar\omega\left(n+\frac{1}{2}\right)$

$\alpha = \sqrt{\frac{m\omega}{\hbar}}$

$\hat{x} = \left(\frac{\hbar}{2m\omega_0}\right)(\hat{a}_+ + \hat{a}_-)$

$\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$

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$\langle n|n+1\rangle\delta_{n,n}$

$\langle\hat{a}_+\rangle = \langle n|\hat{a}_+|n\rangle = \sqrt{n+1}\,\langle n|n+1\rangle =$

$\langle\hat{a}_-\rangle = \langle n|\hat{a}_-|n\rangle = \sqrt{n}\,\langle n|n-1\rangle =$

$\langle n|n-1\rangle\delta_{n,n}$

(f) $\hat{H}' = \gamma\hat{x}$

$\hat{H}' = \left(\frac{\gamma\hbar}{2m\omega_0}\right)(\hat{a}_+ + \hat{a}_-) \quad \alpha \quad \gamma\left(\frac{\hbar}{2m\omega_0}\right)^{1/2}(\hat{a}_+ + \hat{a}_-)$

First order energy change, $E_0^{(1)} = \langle\psi_0^{(0)}|\hat{H}'|\psi_0^{(0)}\rangle$

$= \langle 0|\hat{H}'|0\rangle$

$= \langle 0|\left(\frac{\gamma\hbar}{2m\omega_0}\right)(\hat{a}_+ + \hat{a}_-)|0\rangle$

$= \left(\frac{\gamma\hbar}{2m\omega_0}\right)\left\{\langle 0|\hat{a}_+|0\rangle + \langle 0|\hat{a}_-|0\rangle\right\}$

$= \left(\frac{\gamma\hbar}{2m\omega_0}\right)\left\{ 0 \right\}$

$E_0^{(1)} = 0.$

Second order energy change, $E_0^{(2)} = \sum_{m\neq n}\frac{|\langle\psi_m^{(0)}|\hat{H}'|\psi_0^{(0)}\rangle|^2}{E_0^{(0)} - E_m^{(0)}}$

$= \sum_{m\neq n}\frac{|\langle\psi_m^{(0)}|\gamma\hat{x}|\psi_0^{(0)}\rangle|^2}{E_0^{(0)} - E_m^{(0)}}$

$\boxed{n=0}$ and $\boxed{m\neq n}$

For $m=1$: $= \frac{\left|\langle\psi_1^{(0)}|\left(\frac{\gamma\hbar}{2m\omega_0}\right)(\hat{a}_+ + \hat{a}_-)|\psi_0^{(0)}\rangle\right|^2}{E_0^{(0)} - E_1^{(0)}}$

For $m \geq 2$: $\langle m|\gamma\hat{x}|0\rangle = 0$

~~because ladder operator can~~ only do $|0\rangle \rightarrow |1\rangle$.

$= \frac{\left|\langle 1|\left(\frac{\gamma\hbar}{2m\omega_0}\right)(\hat{a}_+ + \hat{a}_-)|0\rangle\right|^2}{E_0^{(0)} - E_1^{(0)}}$

$\therefore \; E_0^{(2)} = \cancel{\frac{\gamma^2\hbar^2}{4m^2\omega_0^2\,\hbar\omega}}$

$= \frac{\left|\langle 1|\left(\frac{\gamma\hbar}{2m\omega_0}\right)\hat{a}_+|0\rangle + \cancel{\langle 1|\left(\frac{\gamma\hbar}{2m\omega_0}\right)\hat{a}_-|0\rangle}^{0}\right|^2}{E_0^{(0)} - E_1^{(0)}}$

WRONG ANSWER

WHY NOT SQUARE

ASK!

$= \frac{\left(\frac{\gamma\hbar}{2m\omega_0}\right)^2}{\frac{\hbar\omega}{2} - \frac{3\hbar\omega}{2}} = -\frac{\left(\frac{\gamma\hbar}{2m\omega_0}\right)^2}{\hbar\omega}$ $\quad \gamma^2$

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$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ $\alpha = \sqrt{\frac{m\omega}{\hbar}}$ $\hat{a}_\pm = \frac{1}{\sqrt{2}}\left(\alpha\hat{x} \mp \frac{i}{\hbar\alpha}\hat{p}\right)$

$\hat{H}|n\rangle = (n+\frac{1}{2})\hbar\omega|n\rangle$ $\hat{a}_+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

No: $\hat{a}_-|n\rangle = \sqrt{n}\,|n-1\rangle$ Date:

(9) (a) Show $[\hat{a}_-, \hat{a}_+] = 1$

$$\boxed{\begin{aligned}[\hat{x}, \hat{p}_x] &= i\hbar \\ [\hat{x}, \hat{x}] &= 0 \\ [\hat{p}, \hat{p}] &= 0\end{aligned}}$$

$$[\hat{a}_-, \hat{a}_+] = \left[\frac{1}{\sqrt{2}}\left(\alpha\hat{x} + \frac{i}{\hbar\alpha}\hat{p}\right), \frac{1}{\sqrt{2}}\left(\alpha\hat{x} - \frac{i}{\hbar\alpha}\hat{p}\right)\right]$$

$$= \frac{1}{2}\left[\left(\alpha\hat{x} + \frac{i}{\hbar\alpha}\hat{p}\right), \left(\alpha\hat{x} - \frac{i}{\hbar\alpha}\hat{p}\right)\right]$$

$$= \frac{1}{2}\left\{\left[\alpha\hat{x}, -\frac{i}{\hbar\alpha}\hat{p}\right] + \left[\frac{i}{\hbar\alpha}\hat{p}, \alpha\hat{x}\right]\right\}$$

$$= \frac{1}{2}\left\{\left(-\frac{i}{\hbar}\right)[\hat{x}, \hat{p}] + \frac{i}{\hbar}[\hat{p}, \hat{x}]\right\}$$

$$= \frac{1}{2}\left\{\left(-\frac{i}{\hbar}\right)(i\hbar) + \frac{i}{\hbar}(-i\hbar)\right\}$$

$$[\hat{a}_-, \hat{a}_+] = 1$$

(b) $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$

Showing Hamiltonian can be written as $\hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right)$

MEMORISE!

$\boxed{\hat{N} = \hat{a}_+\hat{a}_-}$

$$\hat{a}_+\hat{a}_- = \left[\frac{1}{\sqrt{2}}\left(\alpha\hat{x} - \frac{i}{\hbar\alpha}\hat{p}\right)\right]\left[\frac{1}{\sqrt{2}}\left(\alpha\hat{x} + \frac{i}{\hbar\alpha}\hat{p}\right)\right]$$

$$\hat{a}_+\hat{a}_- = \frac{1}{\hbar\omega}\hat{H} - 1$$

$$= \frac{1}{2}\left(\alpha\hat{x} - \frac{i}{\hbar\alpha}\hat{p}\right)\left(\alpha\hat{x} + \frac{i}{\hbar\alpha}\hat{p}\right)$$

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)$$

$$\cancel{= \frac{1}{2}\left[\alpha^2\hat{x}^2 + \frac{1}{\hbar^2\alpha^2}\hat{p}^2\right]}$$

$\alpha = \left(\frac{m\omega}{\hbar}\right)^{1/2}$

$$\boxed{\hat{N} \equiv \hat{a}_+\hat{a}_-}$$

$$= \frac{1}{2}\left(\alpha^2\hat{x}^2 + \alpha\hat{x}\frac{i}{\hbar\alpha}\hat{p} - \frac{i}{\hbar\alpha}\hat{p}\alpha\hat{x} + \frac{1}{\hbar^2\alpha^2}\hat{p}^2\right)$$

$\alpha^2 = \left(\frac{m\omega}{\hbar}\right)$

$$= \frac{1}{2}\left(\alpha^2\hat{x}^2 + \frac{1}{\hbar^2\alpha^2}\hat{p}^2 + \frac{i}{\hbar}\left(\hat{x}\hat{p} - \hat{p}\hat{x}\right)\right)$$

$[\hat{x}, \hat{p}] = i\hbar$

$$= \frac{1}{2}\left(\alpha^2\hat{x}^2 + \frac{1}{\hbar^2\alpha^2}\hat{p}^2 + \frac{i}{\hbar}[\hat{x}, \hat{p}]\right)$$

$$= \frac{1}{2}\left(\alpha^2\hat{x}^2 + \frac{\hat{p}^2}{\hbar^2\alpha^2} - 1\right) = \frac{1}{2}\left(\frac{m\omega}{\hbar}\hat{x}^2 + \frac{\hat{p}^2}{\hbar m\omega} - 1\right)$$

$$= \frac{1}{\hbar\omega}\left(\frac{1}{2}m\omega^2\hat{x}^2 + \frac{\hat{p}^2}{2m}\right) - \frac{1}{2} = \frac{1}{\hbar\omega}\hat{H} - 1$$

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(c) (I) $[\hat{H}, \hat{a}_\pm] = \pm\hbar\omega\hat{a}_\pm$

$$\left[\hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right), \hat{a}_\pm\right]$$

$$\hbar\omega\left[\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right), \hat{a}_\pm\right] = \hbar\omega[\hat{a}_+\hat{a}_-, \hat{a}_\pm]$$

$\hbar\omega[\hat{a}_+\hat{a}_-, \hat{a}_+]$

~~$\hbar\omega[\hat{a}_+\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_+\hat{a}_-]$~~

$\hbar\omega\{\hat{a}_+[\hat{a}_-, \hat{a}_+] + \underbrace{[\hat{a}_+, \hat{a}_+]}_{=0}\hat{a}_-\}$

$\hbar\omega\hat{a}_+$

$\hbar\omega[\hat{a}_+\hat{a}_-, \hat{a}_-]$

$\hbar\omega\{\hat{a}_+[\hat{a}_-, \hat{a}_-] + [\hat{a}_+, \hat{a}_-]\hat{a}_-\}$

$-\hbar\omega\hat{a}_-$

(II) $\hat{H}|\phi\rangle = E|\phi\rangle$

$\hat{H}(a_\pm|\phi\rangle) = \_\_ (a_+|\phi\rangle)$

~~$\hat{H}(a_+|\phi\rangle)$~~

$[\hat{H}, \hat{a}_+] = \hbar\omega\hat{a}_+$

$\hat{H}\hat{a}_+ - \hat{a}_+\hat{H} = \hbar\omega\hat{a}_+$

$\hat{H}\hat{a}_+ - \hbar\omega\hat{a}_+ = \hat{a}_+\hat{H}$

$\hat{H}|\phi\rangle = E|\phi\rangle$

$\hat{a}_+\hat{H}|\phi\rangle = E\hat{a}_+|\phi\rangle$

$(\hat{H}\hat{a}_+ - \hbar\omega\hat{a}_+)|\phi\rangle = E\hat{a}_+|\phi\rangle$

$\hat{H}\hat{a}_+|\phi\rangle - \hbar\omega\hat{a}_+|\phi\rangle = E\hat{a}_+|\phi\rangle$

$\hat{H}|\phi\rangle = E|\phi\rangle$

$\hat{H}\hat{a}_\pm|\phi\rangle = E\hat{a}_\pm|\phi\rangle$

$(\hat{H}\hat{a}_\pm - \hat{a}_\pm\hat{H}) = \pm\hbar\omega\hat{a}_\pm$

$\hat{H}\hat{a}_\pm = \pm\hbar\omega\hat{a}_\pm + \hat{a}_\pm\hat{H}$

$(\pm\hbar\omega\hat{a}_\pm + \hat{a}_\pm\hat{H})|\phi\rangle = E\hat{a}_\pm|\phi\rangle$

$\pm\hbar\omega\hat{a}_\pm|\phi\rangle + \hat{a}_\pm\hat{H}|\phi\rangle = E\hat{a}_\pm|\phi\rangle$

$\pm\hbar\omega\hat{a}_\pm|\phi\rangle + \hat{a}_\pm E|\phi\rangle = E\hat{a}_\pm|\phi\rangle$

$(E\pm\hbar\omega)\hat{a}_\pm|\phi\rangle = \hat{H}\hat{a}_\pm|\phi\rangle$

MEMORISE

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}(\hat{a}_+ + \hat{a}_-) = \frac{1}{\sqrt{2}\alpha}(\hat{a}_+ + \hat{a}_-)$$

$$\hat{p} = i\left(\frac{m\omega\hbar}{2}\right)^{1/2}(\hat{a}_+ - \hat{a}_-) = \frac{i\hbar\alpha}{\sqrt{2}}(\hat{a}_+ - \hat{a}_-)$$

$[\hat{x}, \hat{p}] = i\hbar$

$\hat{x}\hat{p} - \hat{p}\hat{x} = i\hbar$

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(d) $\hat{B} = \hat{x}\hat{p} + \hat{p}\hat{x} + \hbar$ in terms of $\hat{a}_\pm$ and $\hbar$

$\hat{B} = 2\hat{p}\hat{x} + i\hbar + \hbar$

$$\hat{B} = 2\left(\frac{i\hbar\alpha}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}\alpha}\right)(\hat{a}_+ - \hat{a}_-)(\hat{a}_+ + \hat{a}_-) + i\hbar + \hbar$$

$$= i\hbar\left[\hat{a}_+^2 + \hat{a}_+\hat{a}_- - \hat{a}_-\hat{a}_+ - \hat{a}_-^2\right] + i\hbar + \hbar$$

$[\hat{a}_-, \hat{a}_+] = 1$

$\hat{a}_-\hat{a}_+ - \hat{a}_+\hat{a}_- = 1$

$\hat{a}_-\hat{a}_+ = 1 + \hat{a}_+\hat{a}_-$

$$= i\hbar\left[\hat{a}_+^2 - \hat{a}_-^2 + \hat{a}_+\hat{a}_- - \hat{a}_+\hat{a}_- - I\right] + i\hbar + \hbar$$

$$= i\hbar\left[\hat{a}_+^2 - \hat{a}_-^2\right] + \hbar$$

(e) $\hat{B} = i\hbar[\hat{a}_+^2 - \hat{a}_-^2] + \hbar$ matrix representation for $\hat{B}$ $\rightarrow$ 4 x 4 matrix

$\langle n|\hat{B}|n\rangle \Rightarrow$

$\hat{B}|n\rangle = \left(i\hbar[\hat{a}_+^2 - \hat{a}_-^2] + \hbar\right)|n\rangle$

$\langle n|\hat{B}|n\rangle = \langle n|\left(i\hbar[\hat{a}_+^2 - \hat{a}_-^2] + \hbar\right)|n\rangle$

$= i\hbar\langle n|\hat{a}_+^2|n\rangle - i\hbar\langle n|\hat{a}_-^2|n\rangle + \hbar\langle n|n\rangle$

$\hat{a}_+|n\rangle = \sqrt{n+1}|n+1\rangle$

$\hat{a}_+^2|n\rangle = \sqrt{n+1}\,\hat{a}_+|n+1\rangle$

$\hat{a}_+^2|n\rangle = \sqrt{n+1}\sqrt{n+2}|n+2\rangle$

$\langle n|\hat{a}_+^2|n\rangle = \sqrt{n+1}\sqrt{n+2}\langle n|n+2\rangle$

$= \sqrt{n+1}\sqrt{n+2}\,\delta_{n,n+2}$

$\hat{a}_-|n\rangle = \sqrt{n}|n-1\rangle$

$\hat{a}_-^2|n\rangle = \sqrt{n}\,\hat{a}_-|n-1\rangle$

$\hat{a}_-^2|n\rangle = \sqrt{n}\sqrt{n-1}|n-2\rangle$

$\langle n|\hat{a}_-^2|n\rangle = \sqrt{n}\sqrt{n-1}\langle n|n-2\rangle$

$= \sqrt{n}\sqrt{n-1}\,\delta_{n,n-2}$

$$\langle n|\hat{B}|n\rangle = i\hbar\sqrt{n+1}\sqrt{n+2}\,\delta_{n,n+2} - i\hbar\sqrt{n}\sqrt{n-1}\,\delta_{n,n-2} + \hbar\,\delta_{nn}$$

$$= i\hbar\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2}\sqrt{3} & 0 & 0 \end{pmatrix} - i\hbar\begin{pmatrix} 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \sqrt{3}\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \hbar\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \hbar & 0 & -\hbar i\sqrt{2} & 0 \\ 0 & \hbar & 0 & -\hbar i\sqrt{6} \\ \hbar i\sqrt{2} & 0 & \hbar & 0 \\ 0 & \hbar i\sqrt{6} & 0 & \hbar \end{pmatrix}$$

$\rightarrow$ ANSWER IS SLIGHTLY DIFFERENT

CHECK!

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$$E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}' | \psi_n^{(0)} \rangle$$

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(f) $\hat{H} = \gamma \hat{B}$

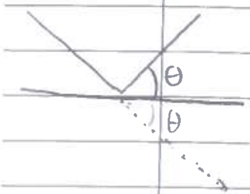
$$\hat{B} = \hbar \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix}$$

$$E_n^{(1)} = \langle \psi_n^{(0)} | \gamma \hat{B} | \psi_n^{(0)} \rangle$$

$$E_0^{(1)} = \gamma\hbar \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \gamma\hbar \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{E_0^{(1)} = \gamma\hbar}$$



$$E_1^{(1)} = \gamma\hbar \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & i\sqrt{6} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{E_1^{(1)} = \gamma\hbar}$$

$\therefore$ All the first order corrections $\rightarrow \gamma\hbar$

Condition for γ that will make the perturbation small is

$E_n^{(1)} = \gamma\hbar$ for $n = 0, 1, 2, 3$

?

(g) $\hat{H}' = \beta \hat{B}^2$ with $\beta = \frac{\gamma}{\hbar}$

$$\beta \hat{B}^2 = \beta\hbar^2 \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -i\sqrt{2} & 0 \\ 0 & 1 & 0 & -i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} = \beta\hbar^2 \begin{pmatrix} 1+2 & 0 & 2i\sqrt{2} & 0 \\ 0 & 7 & 0 & -2i\sqrt{6} \\ 2i\sqrt{2} & 0 & 3 & 0 \\ 0 & 2i\sqrt{6} & 0 & 7 \end{pmatrix}$$

$$\beta\hat{B} = \beta\hbar^2 \begin{pmatrix} 1 & 0 & i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2i\sqrt{2} & 0 \\ 0 & -5 & 0 & 2i\sqrt{6} \\ 2i\sqrt{2} & 0 & -1 & 0 \\ 0 & 2i\sqrt{6} & 0 & -5 \end{pmatrix} \rightarrow$$

using the $\hat{B}$ from the numerical solution

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$$E_0^{(1)} = \beta\hbar^2 \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 & -2i\sqrt{2} & 0 \\ 0 & 7 & 0 & -2i\sqrt{6} \\ 2i\sqrt{2} & 0 & 3 & 0 \\ 0 & 2i\sqrt{6} & 0 & 7 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \beta\hbar^2 \begin{pmatrix} 3 & 0 & -2i\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\beta\hbar^2 \begin{pmatrix} -1 & 0 & 2i\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \beta\hbar^2 (3) \quad \times \xrightarrow{\text{why?}} \quad = -\beta\hbar^2$$

~~using the~~ using the B from the numerical solution

$$E_0^{(2)} = \gamma\hbar \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & i\sqrt{2} & 0 \\ 0 & 1 & 0 & i\sqrt{6} \\ i\sqrt{2} & 0 & 1 & 0 \\ 0 & i\sqrt{6} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad ?$$

$$E_n^{(2)} = \sum_{k \neq n} \frac{|\langle \psi_k^{(0)} | H' | \psi_n^{(0)} \rangle|^2}{(E_n^{(0)} - E_k^{(0)})} \qquad \frac{1}{2}\hbar\omega - \frac{3}{2}\hbar\omega$$

$E_n = (n + \frac{1}{2})\hbar\omega$

$E_0^{(0)} = \frac{1}{2}\hbar\omega$

$E_1^{(0)} = \frac{3}{2}\hbar\omega$.

$$= \gamma\hbar \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{|\gamma\hbar|^2}{\hbar\omega}$$

$$= \frac{\gamma^2\hbar}{\omega}$$

INCOMPLETE!

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commutator relationship: $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

$[\hat{A}, \hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$

$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$

$\hat{J}_\pm |j,m_j\rangle = [j(j+1) - m_j(m_j \pm 1)]^{1/2}\hbar |j, m_j \pm 1\rangle$

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$[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z$

$\hat{J}_x \to \hat{J}_y \to \hat{J}_z \to \hat{J}_x$

show

(10) (a) $[\hat{J}_y^2, \hat{J}_z] = i\hbar(\hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y)$

$$[\hat{J}_y\hat{J}_y, \hat{J}_z] = \hat{J}_y[\hat{J}_y, \hat{J}_z] + [\hat{J}_y, \hat{J}_z]\hat{J}_y$$

$$= \hat{J}_y(i\hbar\hat{J}_x) + (i\hbar\hat{J}_x)\hat{J}_y$$

$$= i\hbar(\hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y)$$

(b) $[\hat{J}^2, \hat{J}_z]$

$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2$

$$[\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2, \hat{J}_z]$$

$$= [\hat{J}_x^2, \hat{J}_z] + [\hat{J}_y^2, \hat{J}_z] + [\hat{J}_z^2, \hat{J}_z]$$

$$= -i\hbar(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x) + i\hbar(\hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y)$$

$$= 0$$

$[\hat{J}_y^2, \hat{J}_z] = i\hbar(\hat{J}_y\hat{J}_x + \hat{J}_x\hat{J}_y)$

$\cancel{[\hat{J}_z^2, \hat{J}_x] = i\hbar(\hat{J}_z\hat{J}_y + \hat{J}_y\hat{J}_z)}$

$\cancel{[\hat{J}_x^2, \hat{J}_y] = i\hbar(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x)}$

$$[\hat{J}_x^2, \hat{J}_z] = [\hat{J}_x\hat{J}_x, \hat{J}_z]$$

$$= \hat{J}_x[\hat{J}_x, \hat{J}_z] + [\hat{J}_x, \hat{J}_z]\hat{J}_x$$

$$= -i\hbar\hat{J}_x\hat{J}_y + (-i\hbar\hat{J}_y)\hat{J}_x$$

$$= -i\hbar(\hat{J}_x\hat{J}_y + \hat{J}_y\hat{J}_x)$$

$\therefore$ $\hat{J}^2, \hat{J}_z$ are compatible operators. Eigenvectors of $\hat{J}^2$ are also eigenvectors of $\hat{J}_z$.

$\boxed{[\hat{J}^2, \hat{J}_z] = 0}$

$$[\hat{J}_z^2, \hat{J}_z] = [\hat{J}_z\hat{J}_z, \hat{J}_z]$$

$$= \hat{J}_z[\hat{J}_z, \hat{J}_z] + [\hat{J}_z, \hat{J}_z]\hat{J}_z$$

$$= 0.$$

(c) $\hat{J}^2|j,m_j\rangle = j(j+1)\hbar^2|j,m_j\rangle$ $\qquad$ $\hat{J}_z|j,m_j\rangle = m_j\hbar|j,m_j\rangle$

(d) $\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ $\qquad$ $\hat{J}_- = \hat{J}_x - i\hat{J}_y$

Show that $\hat{J}_\pm|j,m_j\rangle$ is also an eigenvector of $\hat{J}_z$.

$$\hat{J}_z|j,m_j\rangle = m_j\hbar|j,m_j\rangle$$

$$\hat{J}_\pm\hat{J}_z|j,m_j\rangle = m_j\hbar\hat{J}_\pm|j,m_j\rangle$$

$$(\hat{J}_z\hat{J}_\pm \mp \hbar\hat{J}_\pm)|j,m_j\rangle = m_j\hbar\hat{J}_\pm|j,m_j\rangle$$

$$\hat{J}_z(\hat{J}_\pm|j,m_j\rangle) = (m_j\hbar \pm \hbar)(\hat{J}_\pm|j,m_j\rangle)$$

$$\hat{J}_z(\hat{J}_\pm|j,m_j\rangle) = \hbar(m_j \pm 1)(\hat{J}_\pm|j,m_j\rangle)$$

$[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm$

$\hat{J}_z\hat{J}_\pm - \hat{J}_\pm\hat{J}_z = \pm\hbar\hat{J}_\pm$

$\hat{J}_z\hat{J}_\pm \mp \hbar\hat{J}_\pm = \hat{J}_\pm\hat{J}_z$

NOTE: Don't just write m. It is m_j.

$$\hat{J}_{\pm} |j, m_j\rangle = [j(j+1) - m_j(m_j \pm 1)]^{1/2} \hbar \, |j, m_j \pm 1\rangle$$

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(e) Explain why there must be an eigenvector for which $\hat{J}_+ |j, m_j\rangle = 0$ and another eigenvector for which $\hat{J}_- |j, m_j\rangle = 0$. State (without proving) what the eigenvalues of $\hat{J}_z$ for these eigenvectors are.

$\hat{J}_+$ increases m_j but j remains the same.

$\hat{J}_-$ decreases m_j but j remains the same.

If there were a state where m_j exceeded j then $\hat{J}_z^2$ would be greater than $\hat{J}^2$ which is physically impossible. Therefore, there must be a state where $J_+ |j, m_j\rangle = 0$ to stop J_z^2 being greater than J^2.

The same applies in the opposite direction.

INCOMPLETE!

$m_j = 1$

(f) A system with $j=1$.

General state vector: $|\phi\rangle = a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle$

$$\hat{J}_y |\phi\rangle = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)|\phi\rangle$$

$\hat{J}_+ = \hat{J}_x + i\hat{J}_y$ — ①

$\hat{J}_- = \hat{J}_x - i\hat{J}_y$ — ②

① − ②: $\hat{J}_+ - \hat{J}_- = 2i\hat{J}_y$

$$\boxed{\hat{J}_y = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)}$$

$$\frac{1}{2i}\left\{\hat{J}_+|\phi\rangle - \hat{J}_-|\phi\rangle\right\}$$

$$\frac{1}{2i}\left\{\hat{J}_+[a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle] - \hat{J}_-[a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle]\right\}$$

$$\frac{1}{2i}\left\{[a(0) + b\sqrt{2}\hbar|1,1\rangle + c\sqrt{2}\hbar|1,0\rangle] - [a\sqrt{2}\hbar|1,0\rangle + b\sqrt{2}\hbar|1,-1\rangle + c(0)]\right\}$$

$$\frac{1}{2i}\left\{b\sqrt{2}\hbar|1,1\rangle + \sqrt{2}\hbar(c-a)|1,0\rangle - b\sqrt{2}\hbar|1,-1\rangle\right\}$$

$$\frac{\hbar}{\sqrt{2}i}(b|1,1\rangle + (c-a)|1,0\rangle - b|1,-1\rangle)$$

→ a − c in the answer!

$|\phi\rangle = a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle$

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(g) $\hat{J}_y|\phi\rangle = \frac{\hbar}{\sqrt{2}i}\left(b|1,1\rangle + (c-a)|1,0\rangle - b|1,-1\rangle\right)$

NOT VERY CLEAR!

Eigenvalues : $-\hbar, 0, \hbar$

For $-\hbar$: $\frac{b}{\sqrt{2}i} = c$ $\qquad \frac{a-c}{\sqrt{2}i} = b \qquad \frac{-b}{\sqrt{2}i} = c$ $\qquad$ let $a=1$, then $b = -\sqrt{2}i$

$b = \sqrt{2}ic \qquad\qquad c = 1$

$|\phi\rangle = |1,1\rangle - \sqrt{2}i|1,0\rangle + |1,-1\rangle$

Normalise

$|\phi\rangle = \frac{1}{2}|1,1\rangle - \frac{i}{\sqrt{2}}|1,0\rangle + \frac{1}{2}|1,-1\rangle$.

For $\hbar$: $\frac{b}{\sqrt{2}i} = a \qquad \frac{1}{\sqrt{2}i}(c-a) = b \qquad \frac{-b}{\sqrt{2}i} = c$ $\qquad$ let $a=1$, then $b = \sqrt{2}i$

$c = -1$

$a = \frac{b}{\sqrt{2}i}$

$b = \sqrt{2}ia \qquad\qquad c = \frac{-\sqrt{2}ia}{\sqrt{2}i}$

$c = -a$

$|\phi\rangle = a|1,1\rangle + b|1,0\rangle + c|1,-1\rangle$

$|\phi\rangle = |1,1\rangle + \sqrt{2}i|1,0\rangle - |1,-1\rangle$

Normalise

$|\phi\rangle = \frac{1}{2}|1,1\rangle + \frac{i}{\sqrt{2}}|1,0\rangle - \frac{1}{2}|1,-1\rangle$.

For 0 : $c - a = 0 \qquad b = 0$

$c = a$

let $a = 1$, then $c = 1$.

$|\phi\rangle = |1,1\rangle + |1,-1\rangle$

Normalise

$|\phi\rangle = \frac{1}{\sqrt{2}}|1,1\rangle + \frac{1}{\sqrt{2}}|1,-1\rangle$.

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