1 The rotation angle

The result of a Jacobi rotation $A \to A'$ is given by

$$A'_{ij} = \left[J(q,p)^T A J(q,p)\right]_{ij} = \sum_{nm} J(q,p)^T_{in} A_{nm} J(q,p)_{mj}$$

Assuming $p < q$ and using the definition of $J(p,q)$ we find

$$A'_{q,p-1} = \sum_{nm} A_{nm} J(q,p)^T_{qn} J(q,p)_{m,p-1} = \sum_{nm} A_{nm}(\delta_{nq}\cos\theta + \delta_{np}\sin\theta)(\delta_{m,p-1})$$

$$= \sum_{m} A_{n,p-1}(\delta_{nq}\cos\theta + \delta_{np}\sin\theta) = A_{q,p-1}\cos\theta + A_{p,p-1}\sin\theta$$

and then the Jacobi rotation $J(p,q)$ that zeros out element $A_{p-1,q}$ is defined by the angle

$$A'_{q,p-1} = 0 \implies \tan(\theta) = -\frac{A_{q,p-1}}{A_{p,p-1}} \implies \theta = \text{atan2}(-A_{q,p-1}, A_{p,p-1})$$

2 The algorithm

Consider a matrix

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \boldsymbol{*} & * & * & * & * \\ \boldsymbol{*} & \boldsymbol{*} & * & * & * \\ \boldsymbol{*} & \boldsymbol{*} & \boldsymbol{*} & * & * \end{bmatrix}$$

To get the Hessenberg decomposition of A the bold entries above has to be zeroed out. First transform A by applying the specific Jacobi rotation $J(2,3)$ that zeros out the element A_{31}.

$$A \to A' = \begin{bmatrix} * & {\color{red}*} & {\color{red}*} & * & * \\ {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} \\ {\color{red}\mathbf{0}} & {\color{red}*} & {\color{red}*} & {\color{red}*} & {\color{red}*} \\ \boldsymbol{*} & {\color{red}\boldsymbol{*}} & {\color{red}*} & * & * \\ \boldsymbol{*} & {\color{red}\boldsymbol{*}} & {\color{red}\boldsymbol{*}} & * & * \end{bmatrix}$$

On top of zeroing out that element, the transformation also acts on the rest of rows and columns 2 and 3 (the red elements above). Applying the Jacobi rotation $J(2,4)$ we get

$$A' \to A'' = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

from which it is clear that the previously obtained zero is preserved. Repeating the process clearly works for the whole column resulting in

$$A \to A''' = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \end{bmatrix}$$

The next transformation we wish to apply is $J(3,4)$. First consider the multiplication with $J(3,4)^T$ from the left which only modifies the rows 3 and 4 of A. However, notice also how the zeros are preserved since in that column the non-zero elements of $J(3,4)$ are multiplied by exactly those zeros.

$$J(3,4)^T A''' = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & * & * & * & * \end{bmatrix}$$

The same is true symmetrically for the multiplication by $J(3,4)$ from the right. Hence the zeros are preserved and we end up with

$$A''' \to A'''' = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & \mathbf{0} & * & * & * \\ \mathbf{0} & * & * & * & * \end{bmatrix}$$

The argument holds equally well for all the rows below, and together with the previous argument the zero we just created is also preserved. Following the pattern thus utimately results in the Hessenberg representation

$$A \to H = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ \mathbf{0} & * & * & * & * \\ \mathbf{0} & \mathbf{0} & * & * & * \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & * & * \end{bmatrix}$$

3 Determinant of Hessenberg matrix

First note that computing the determinant of a general matrix is trivially reduced to computing the determinant of its Hessian form since

$$\det A = \det(QHQ^T) = \det(QQ^T)\det H = \det H$$

Given a Hessenberg matrix one can expand along the first column to derive a simple recursive algorithm for computing the determinant.

$$H = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{bmatrix}$$

Let the two minors be denoted by h_1 and h_2 then

$$\det H = H_{11}\det h_1 - H_{21}\det h_2$$

Notice that the minors are themselves Hessenberg matrices, and so, the above formula holds for those as well.

For an $n \times n$ matrix, computing the determinant recursively in this fashion will take 2^n multiplications. Computing the Hessian decomposition itself takes $O(n^3)$ since there are $O(n^2)$ zeros, each taking n operations. The worse of the two is $O(2^n)$ which thus also is the scaling for computing the determinant through Hessenberg decomposition. This is significantly less than the $O(n!)$ one has from the Leibniz formula but significantly more than the $n^3 + n = O(n^3)$ it takes to compute the determinant through QR factorisation.