2. Normierung

$$\int_{-\infty}^{\infty} p(x)\,dx \overset{!}{=} 1$$

$$1 = C\int e^{-\frac{1}{2}ax^2+bx}\,dx$$

$$= C\int e^{-\frac{1}{2}(ax^2-2bx)}\,dx$$

$$= C\int e^{-\frac{1}{2}(ax^2-2bx+\frac{b^2}{a}-\frac{b^2}{a})}\,dx \qquad \Big|\; ax^2-2bx+\frac{b^2}{a} = \left(\sqrt{a}x-\frac{b}{\sqrt{a}}\right)^2$$

$$= C\int dx\, e^{-\frac{1}{2}(\sqrt{a}x-\frac{b}{\sqrt{a}})^2}\, e^{+\frac{b^2}{2a}}\,\cancel{dx}$$

$$= Ce^{+\frac{b^2}{2a}}\int dx\, e^{-\frac{1}{2}(\sqrt{a}x-\frac{b}{\sqrt{a}})^2} \qquad \Big|\; y=\sqrt{a}x-\frac{b}{\sqrt{a}}, \quad dx=\frac{1}{\sqrt{a}}dy$$

$$= Ce^{+\frac{b^2}{2a}}\frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}y^2} \qquad \Big|\; z^2=\frac{1}{2}y^2,\; z=\frac{y}{\sqrt{2}}, \quad \frac{dz}{dy}=\frac{1}{\sqrt{2}} \to dy=\sqrt{2}dz$$

$$= C e^{+\frac{b^2}{2a}}\frac{1}{\sqrt{a}}\cdot\sqrt{2}\,\underbrace{\int_{-\infty}^{\infty} dz\, e^{-z^2}}_{\sqrt{\pi}}$$

$$1 = C\cdot e^{+\frac{b^2}{2a}}\sqrt{\frac{2\pi}{a}} \quad \Leftrightarrow \quad \boxed{C = e^{-\frac{b^2}{2a}}\sqrt{\frac{a}{2\pi}}}$$

$$\int dx\, e^{-\frac{1}{2}x^2} = \sqrt{2\pi}$$

3. Mittelwert

$$\langle x\rangle = \int_{-\infty}^{\infty} x\, p(x)\, dx$$

$$= \int_{-\infty}^{\infty} x\, C e^{-\frac{1}{2}ax^2+bx} dx$$

$$= C \int_{-\infty}^{\infty} dx\; x\, e^{-\frac{1}{2}(\sqrt{a}x-\frac{b}{\sqrt{a}})^2} e^{\frac{b^2}{2a}}$$

$$= Ce^{\frac{b^2}{2a}} \int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}(\sqrt{a}x-\frac{b}{\sqrt{a}})^2} dx \quad \Big|\quad y=\sqrt{a}x-\frac{b}{\sqrt{a}} \Rightarrow x=\frac{y}{\sqrt{a}}+\frac{b}{a}, \quad \frac{dy}{dx}=\sqrt{a} \Rightarrow dx=\frac{dy}{\sqrt{a}}$$

$$= C e^{+\frac{b^2}{2a}} \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} dy \left(\frac{y}{\sqrt{a}}+\frac{b}{a}\right) e^{-\frac{1}{2}y^2}$$

$$= Ce^{\frac{b^2}{2a}} \frac{1}{\sqrt{a}} \left[ \int_{-\infty}^{\infty} dy\, \frac{y}{\sqrt{a}} e^{-\frac{1}{2}y^2} + \frac{b}{a} \underbrace{\int_{-\infty}^{\infty} dy\, e^{-\frac{1}{2}y^2}}_{\sqrt{2\pi}} \right]$$

$$= Ce^{\frac{b^2}{2a}} \frac{1}{\sqrt{a}} \left[ \frac{1}{\sqrt{a}} \underbrace{\int_{-\infty}^{\infty} dy\, y e^{-\frac{1}{2}y^2}}_{-\frac{1}{4}e^{-\frac{1}{2}y^2}\Big|_{-\infty}^{\infty} \Rightarrow \underline{\underline{0}}} + \sqrt{2\pi}\,\frac{b}{a} \right]$$

$$\langle x\rangle = Ce^{\frac{b^2}{2a}} \frac{\sqrt{2\pi}\, b}{\sqrt{a}\, a} \quad \Big|\quad C = e^{-\frac{b^2}{2a}} \sqrt{\frac{a}{2\pi}}$$

$$= \underline{\underline{\frac{b}{a}}}$$

## 3.3 Freies Wellenpaket

2. • Integral in $\psi(x,t)$ ausführen:

$$\psi(x,t) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{-(k-k_0)^2 d^2 - i\frac{\hbar k^2}{2m}t + ikx}$$

Abkürzungen: $a := d^2 + i\frac{\hbar}{2m}t$ , $b := k_0 d^2 + i\frac{x}{2}$ ,

$c := k_0^2 d^2$

$$\rightarrow \psi(x,t) = \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{-ak^2 + 2bk - c}$$

| Ergänze mit $+\frac{b^2}{a} - \frac{b^2}{a} = 0$, Binomische Formel...

$$= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \; e^{-a(k-\frac{b}{a})^2 + \frac{b^2}{a} - c}$$

$$= \frac{A}{\sqrt{2\pi}} e^{\frac{b^2}{a} - c} \underbrace{\int_{-\infty}^{\infty} dk \; e^{-a(k-\frac{b}{a})^2}}_{=\sqrt{\frac{\pi}{a}} \text{ s. Aufg. 3.2.2}}$$

$$= \frac{A}{\sqrt{2a}} e^{\frac{b^2}{a} - c}$$

• Dichte der Aufenthaltswahrscheinlichkeit:

$$|\psi(x,t)|^2 = \frac{A^2}{2a} e^{2\mathrm{Re}\left(\frac{b^2 - ac}{a}\right)}$$

| Die Imaginärteile des Exponenten subtrahieren sich weg nach der komplexen Konjugation!

3. $$|\psi(x,t)|^2 = \frac{A^2}{2|a|} e^{\frac{b^2}{a}-c} e^{\left(\frac{b^2}{a}-c\right)^*}$$

| · b, a, c einsetzen,
| · komplex konjugieren

⋮

$$= \frac{A^2}{2|a|} \exp(-2k_0^2 d) \exp\left(\frac{2k_0^2 d^6}{d^4+\left(\frac{\hbar t}{2m}\right)^2}\right)$$

$$\cdot \exp\left(\frac{2x k_0 d^2 \frac{\hbar t}{2m} - \frac{1}{2}x^2 d}{d^4+\left(\frac{\hbar t}{2m}\right)^2}\right)$$

$$\frac{\partial}{\partial x}|\psi(x,t)|^2 \overset{!}{=} 0 \qquad \text{(Maximalwert)}$$

$$\Leftrightarrow \frac{A^2}{2|a|} \exp(-2k_0^2 d) \exp\left(\frac{2k_0^2 d^6}{d^4+\left(\frac{\hbar t}{2m}\right)^2}\right) \cdot \exp\left(\frac{2x k_0 d^4 \frac{\hbar}{2m} t - \frac{1}{2}x^2 d}{d^4+\left(\frac{\hbar t}{2m}\right)^2}\right)$$

$$\cdot \left(\frac{2k_0 d^2 \frac{\hbar t}{2m} - x d^2}{d^4+\left(\frac{\hbar t}{2m}\right)^2}\right) \overset{!}{=} 0$$

$$\Leftrightarrow \quad 2k_0 d^2 \frac{\hbar t}{2m} = x d^2$$

$$\Rightarrow \quad x = \frac{k_0 \hbar}{m} t$$

$$v = \frac{\partial}{\partial t} x = \frac{k_0 \hbar}{2m}$$