6.3.1

(i) $\int u^*(x)\, A_1\, A_2\, v(x) dx$

$= \int dx\, \left((A_1 A_2)^+ u(x)\right)^* v(x)$

$= \left(\int dx\ v^*(x) (A_1 A_2)^+ u(x)\right)^*$

(ii) $\int u^*(x)\, A_1 A_2\, v(x)\, dx$

$= \int dx\, \left(A_1^+ u(x)\right)^* A_2 v(x)$

$= \int dx\ A_2^{+*} A_1^{+*} u^*(x)\, v(x)$

$= \left(\int dx\ v^*(x)\, A_2^+ A_1^+ u(x)\right)^*$

$\Rightarrow$ (i) und (ii) $\Rightarrow$ $(A_1 A_2)^+ = A_2^+ A_1^+$

6.3.2

$A^+ = A$

$(AB^+) = B^+ A^+ = BA$

(i) $[A, B] = 0 \Rightarrow AB = BA$

$\Rightarrow (AB)^+ = AB$

(ii) $[A, B] \neq 0 \Rightarrow AB \neq BA$

$(AB)^+ \neq AB$

6.4.1

$$\hat{a}_- u_n = \frac{1}{\sqrt{\hbar\omega_0 n}} \hat{a}_- \hat{a}_+ u_{n-1}$$

$$= \frac{1}{\sqrt{\hbar\omega_0 n}} \left(H + \frac{\hbar\omega_0}{2}\right) u_{n-1} \qquad H = \hbar\omega_0 \left[(n-1) + \frac{1}{2}\right]$$

$$= \frac{1}{\sqrt{\hbar\omega_0 n}} \left(\hbar\omega_0 (n - \frac{1}{2}) + \frac{\hbar\omega_0}{2}\right) u_{n-1}$$

$$= \sqrt{\hbar\omega_0 n}\; u_{n-1}(x)$$

6.4.2

$$\hat{n} u_n = \frac{\hat{a}_+ \hat{a}_-}{\hbar\omega_0} u_n$$

$$= \frac{1}{\hbar\omega_0} \hat{a}_+ \sqrt{\hbar\omega_0 n}\; u_{n-1}$$

$$= \frac{1}{\hbar\omega_0} \sqrt{\hbar\omega_0 n} \sqrt{\hbar\omega_0 n}\; u_n$$

$$= n\, u_n$$

6.4.3

Operatoren ohne ^

a) z.z.: $[a_-^m, a_+] = \hbar\omega_0 a_-^{m-1}$

Induktion:

- $m = 1$:

$$[a_-, a_+] = \hbar\omega_0$$

• $m \to m+1$:

$$[a_-^{m+1}, a_+] = a_-[a_-^m, a_+] + [a_-, a_+]\, a_-^m$$
$$= \hbar\omega_0 (a_- m a_-^{m-1} + a_-^m)$$
$$= \hbar\omega_0 (m+1)\, a_-^m$$

b) z.z.: $[a_-, a_+^m] = \hbar\omega_0 m a_+^{m-1}$

• $m = 1$ ✓

• $m \to m+1$

$$[a_-, a_+^{m+1}] = a_+[a_-, a_+^m] + [a_-, a_+]\, a_+^m$$
$$= \hbar\omega_0 (a_+ m a_+^{m-1} + a_+^m)$$
$$= \hbar\omega_0 (m+1)\, a_+^m$$

c) $$[\hat{n}, a_-^m] = [a_+ a_-, a_-^m] \frac{1}{\hbar\omega_0}$$
$$= (a_+[a_-, a_-^m] + [a_+, a_-^m]\, a_-) \frac{1}{\hbar\omega_0}$$
$$\underset{a)}{=} (0 - \hbar\omega_0 m a_-^{m-1} a_-) \frac{1}{\hbar\omega_0}$$
$$= -m a_-^m$$

d) $$[\hat{n}, a_+^m] = \left(a_+[a_-, a_+^m] + [a_+, a_+^m]a_-\right)\frac{1}{\hbar\omega}$$
$$= m a_+^m$$

6.4.4

$$\int dx\, u_n^+(x)\, u_m(x)$$
$$= \int \frac{1}{\sqrt{n!}} (u_0 a_+^n)^+ u_m(x)\, dx \qquad | a_+^+ = a_-$$
$$= \int \frac{1}{\sqrt{n!}} u_0^+ a_-^{n-m} \underbrace{a_-^m u_m(x)}_{=\sqrt{\hbar\omega_0}^m \sqrt{m!}\, u_0} dx \qquad | a_- u_m = \sqrt{\hbar\omega_0}\sqrt{m}\, u_{m-1}$$
$$| a_- u_{m-1} = \sqrt{\hbar\omega_0}\sqrt{m-1}\, u_{m-2}$$
$$= \int dx \sqrt{\hbar\omega_0}^m u_0^+ a_-^{n-m} u_0 \sqrt{\frac{m!}{n!}}$$

O.B.d.A: $n > m$

$\Rightarrow$ $n \neq m$: $a_- u_0 = 0$

$n = m$: $u_0^+ \mathbb{1} u_0 = 1$

$$\Rightarrow \int dx \sqrt{\hbar\omega_0}^m u_n^+ u_m = \delta_{nm} \sqrt{\hbar\omega_0}^m$$

6.4.5

a) $$\int dx\, |\phi_z(x)|^2 = \int dx\, e^{-|z|^2} \sum_{n,m=0}^{\infty} \frac{z^{*m}}{\sqrt{m!}} \frac{z^n}{\sqrt{n!}} u_m^+(x) u_n(x)$$
$$= e^{-|z|^2} \sum_{n=0}^{\infty} \frac{(|z|^2)^n}{n!} \sqrt{\hbar\omega_0}^n$$
$$= e^{-|z|^2} e^{+|z|^2} \sqrt{\hbar\omega_0}^n = 1 \sqrt{\hbar\omega_0}^n$$

b) $$a_- \phi_z = e^{-\frac{1}{2}|z|^2} \left(\underbrace{a_- u_0(x)}_{=0} + \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} a_- u_n(x) \right)$$

$$= e^{-\frac{1}{2}|z|^2} \sum_{n=1}^{\infty} \frac{z(z^{n-1})}{\sqrt{(n-1)!}} u_{n-1}(x) \sqrt{\hbar \omega_0}$$

$$\overset{n-1 \to n}{=} z \underbrace{e^{-\frac{1}{2}|z|^2} \sum_{n=1}^{\infty} \frac{z^n}{\sqrt{n!}} u_n(x)}_{\phi_z} \sqrt{\hbar \omega_0} = z \phi_z(x) \sqrt{\hbar \omega_0}$$