Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Modeling the Ideal Cipher in Linicrypt

Master Thesis

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Abstract

Todo

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Todo

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Preliminaries

2.1 Linicrypt

2.1.1 Definition of a Linicrypt program

The Linicrypt model for cryptographic constructions was introduced by Cramer & Rosulek in [?]. Summarizing the formalization from that paper, a pure Linicrypt program $\mathcal{P}$ is a straight line program whose intermediate variables are elements in a field $\mathbb{F}$. The only allowed operations to create an intermediate variable are:

- Retrieve an input, which is in $\mathbb{F}$
- Perform a linear combination of existing internal variables
- Call a random oracle $H : \{0,1\}^* \times \mathbb{F}^* \to \mathbb{F}^*$
- Sample from $\mathbb{F}$ uniformly

The program $\mathcal{P}$ can output one or more of its variables.

Below is an example of a Linicrypt program $\mathcal{P}^H$, written in conventional pseudocode on the left and in explicit Linicrypt on the right.

$\mathcal{P}^H(x, y)$		$\mathcal{P}^H(x, y)$
$r \leftarrow\$ \mathbb{F}$ **return** $H(x + r) + y$	$\rightsquigarrow$	$v_1 := x$ $v_2 := y$ $v_3 \leftarrow\$ \mathbb{F}$ $v_4 := v_1 + v_3$ $v_5 := H(v_4)$ $v_6 := v_5 + v_2$ **return** (v_6)

2.1.2 Type of Adversaries

The Linicrypt model only imposes computational restrictions on the constructions, not on the adversaries. Usually one considers arbitrary adversaries $\mathcal{A}$ that are computationally unbounded, but have bounded access to the random oracle H. Therefore the behaviour of an adversary is typically described in terms of the number of queries it makes.

2.1.3 Algebraic Representation

One of the advantages of restricting the computational model is that one can characterize Linicrypt programs with an algebraic representation. Let $\mathcal{P}$ be a linicrypt program with intermediate variables $v_1, \ldots, v_n$.

A **base variable** is an intermediate variable which was created by retrieving an input, calling the random oracle H or sampling from $\mathbb{F}$. Let base be the number of base variables and let $\boldsymbol{v}_{\mathsf{base}} \in \mathbb{F}^{\mathsf{base}}$ denote the vector of the base variables for an excecution of $\mathcal{P}$. A **derived variable** is one which is created by performing a linear combination of existing itermediate variables. Note, that derived variables are therefore linear combinations of base variables. As base variables are mostly independent of each other, it makes sense to *model them as independent vectors in* $\mathbb{F}^{\mathsf{base}}$. The derived variables are then modeled by linear combinations of these vectors.

Let v_i be an intermediate variable. We define the **associated vector** $\boldsymbol{v}_i$ to be the unique row vector such that $v_i = \boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}}$ for every excecution of $\mathcal{P}$. For example, if v_i is the j'th base variable, then $\boldsymbol{v}_i = \begin{bmatrix} 0, \ldots, 1, \ldots, 0 \end{bmatrix}$, where the 1 is in the j'th position. We follow the convention to write vectors in $\mathbb{F}^{\mathsf{base}}$ using a bold font.

The outputs of $\mathcal{P}$ can be described by a matrix with entries in $\mathbb{F}$. Let $o_1, \ldots, o_k$ be the output variables of $\mathcal{P}$. Then the **output matrix** $\boldsymbol{M}$ of $\mathcal{P}$ is defined by

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{o}_1 \\ \vdots \\ \boldsymbol{o}_k \end{bmatrix}.$$

By the definition of the associated vectors $\boldsymbol{o}_i$ we have $\boldsymbol{M} \times \boldsymbol{v}_{\mathsf{base}} = \begin{bmatrix} o_1, \ldots, o_k \end{bmatrix}^\top$. The output matrix describes the linear correlations in the output of $\mathcal{P}$.

But the output matrix doesn't describe all correlations in $\boldsymbol{v}_{\mathsf{base}}$. Namely, the relationship between the queries and answers to the random oracle H need to be captured algebraically. Let $v_i = H(t_i, (q_1, \ldots, q_n))$ be an operation in $\mathcal{P}$. The **associated oracle constraint** c to this operation is

$$c = \left(t_i, \begin{bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_n \end{bmatrix}, \boldsymbol{v}_i \right) = (t_i, \boldsymbol{Q}_i, \boldsymbol{v}_i).$$

This should be interpreted as the requirement that $\boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}} = H\left(t_i, \boldsymbol{Q}_i \times \boldsymbol{v}_{\mathsf{base}}\right)$. We denote the set of all (associated) oracle constraints of $\mathcal{P}$ by $\mathcal{C}$.

Wrapping up these definitions, we define the **algebraic representation** of $\mathcal{P}$ to be the tuple $(\boldsymbol{M}, \mathcal{C})$. A natural question that arises at this point is: Does the algebraic representation determine the behaviour of $\mathcal{P}$ completely?

2.1.4 Normalization and Indistinguishability

Todo: This leads to indistinguishability, normalization and theorem from first paper

2.1.5 Characterizing Collision Resistance in Linicrypt

1. Mention paper and results
2. Explain collision structure and degeneracy with the two examples

In a paper by I. McQuoid, T. Swope and M. Rosulek [**?**, Characterizing Collision and Second-Preimage Resistance in Linicrypt], the authors introduced a characterization of collision resistance and second-preimage resistance for a class of Linicrypt program based on the algebraic representation.

They identified two reasons why a *deterministic* Linicrypt $\mathcal{P}$ program can fail to be second-preimage resistant:

1. It is degenerate, meaning that it doesn't use all of its inputs independently
2. It has a collision structure, which means that one can change some intermediate value and compute what the input needs to be to counteract this change

Below are two examples, $\mathcal{P}^H_{\text{deg}}$ is degenerate and $\mathcal{P}^H_{\text{deg}}$ has a collision structure. Note, that you can choose $w' \neq w$ to be any value, then find a x' such that the output of $\mathcal{P}^H_{\text{cs}}$ stays the same, and finally find y' according to $w' = x' + y'$.

$\mathcal{P}^H_{\text{deg}}(x, y)$
$v := x + y$
return $H(v)$

$\mathcal{P}^H_{\text{cs}}(x, y)$
$w := x + y$
return $H(w) + x$

The precise definition of degenerate and collision structure will be discussed in chapter **??**, in a variation that is adapted to the goals of this thesis. The authors show that for any deterministic Linicrypt program which is degenerate or has a collision structure second-preimage resistance (and hence also collision resistance) is completely broken.

The main result of [**?**] is that they show that the converse of this is also true, for Linicrypt programs which use distinct nonces in each call to the random oracle. That is, if a Linicrypt program is not collision resistant, then it either has a collision structure, or it is degenerate.

Furthermore, checking for degeneracy and existence of a collision structure can be done efficiently.

Chapter 3

Extending Linicrypt to Ideal Ciphers

Let $\mathcal{P}$ be a Linicrypt program. For each query to E of the form $y = E(k, x)$ we define the associated constraint $(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{k} \in \mathbb{F}^{\mathsf{base}}$ is the row vector corresponding to $k \in \mathbb{F}$ and similarly for $\boldsymbol{x}$ and $\boldsymbol{y}$. Each query to D of the form $x = D(k, y)$, is associated with the constraint $(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x})$

To capture the fact that $E(k, x) = y$ should be associated to the same constraint as $D(k, y) = x$ for the same k, x and y, we introduce an equivalence relation on the constraints. For all $\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}' \in \mathbb{F}^{\mathsf{base}}$ we define

$$\begin{aligned}
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}') \\
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}')
\end{aligned}$$

The set of constraints $\mathcal{C}$ corresponding to $\mathcal{P}$ is then a subset of

$$\Big(\{E, D\} \times \mathbb{F}^{\mathsf{base}} \times \mathbb{F}^{\mathsf{base}} \times \mathbb{F}^{\mathsf{base}}\Big) \Big/ \sim$$

Todo: Include the idea that no constraint with the "same" input queries are used twice.

Each query should only be

Todo: Maybe scrap the idea of the equivalence relation, it seems to hinder more than it helps.

Todo: Instead of doing weird things with equivalence relation in Collision structure definition, explicitly add data of reverse or forward direction.

Definition 3.1 (Collision structure)**.** *Let $\mathcal{P} = (\boldsymbol{M}, \mathcal{C})$ be a Linicrypt program. A **collision structure** is an index i^* and a tuple $(c_1, \ldots, c_n)$ for $c_i = (O_i, \boldsymbol{k}_i, \boldsymbol{q}_i, \boldsymbol{a}_i)$ and $O_i \in \{E, D\}$, such that:*

1. $[c_1], \ldots, [c_n]$ *is an ordering of* $\mathcal{C}$
2. *The input or output corresponding to the query* c_{i^*} *can be fixed arbitrarily:*

$$\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{q}_{i^*}\}\big) \nsubseteq \mathsf{span}\big(\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{i^*-1}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{i^*-1}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i^*-1}\} \cup \mathsf{rows}\,(\boldsymbol{M})\,\big)$$

3. *For all* $j \geq i^*$ *the constraint* c_j *does not contradict previous constraints:*

$$\boldsymbol{a}_j \notin \mathsf{span}\big(\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1},\} \cup \{\boldsymbol{k}_j, \boldsymbol{q}_j\} \cup \mathsf{rows}\,(\boldsymbol{M})\,\big)$$

This is an mistake

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