Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Modeling the Ideal Cipher in Linicrypt

Master Thesis

Frederik Semmel

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Advisors: Fabio Banfi, Ueli Maurer

Institute of Theoretical Computer Science, ETH Zürich

Abstract

Todo

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Preliminaries

2.1 Linicrypt

2.1.1 Definition of a Linicrypt program

The Linicrypt model for cryptographic constructions was introduced by Cramer & Rosulek in [**?**]. Summarizing the formalization from that paper, a pure Linicrypt program $\mathcal{P}$ is a straight line program whose intermediate variables are elements in a field $\mathbb{F}$. The only allowed operations to create an intermediate variable are:

- Retrieve an input, which is in $\mathbb{F}$
- Perform a linear combination of existing internal variables
- Call a random oracle $H : \{0,1\}^* \times \mathbb{F}^* \to \mathbb{F}^*$
- Sample from $\mathbb{F}$ uniformly

The program $\mathcal{P}$ can output one or more of its variables.

Below is an example of a Linicrypt program $\mathcal{P}^H$, written in conventional pseudocode on the left and in explicit Linicrypt on the right.

$$
\boxed{\begin{array}{l}
\mathcal{P}^H(x,y) \\
\hline
\quad r \leftarrow\$\, \mathbb{F} \\
\quad \textbf{return } H(x+r)+y
\end{array}}
\quad \rightsquigarrow \quad
\boxed{\begin{array}{l}
\mathcal{P}^H(x,y) \\
\hline
\quad v_1 \coloneqq x \\
\quad v_2 \coloneqq y \\
\quad v_3 \leftarrow\$\, \mathbb{F} \\
\quad v_4 \coloneqq v_1 + v_3 \\
\quad v_5 \coloneqq H(v_4) \\
\quad v_6 \coloneqq v_5 + v_2 \\
\quad \textbf{return } (v_6)
\end{array}}
$$

2.1.2 Type of Adversaries

The Linicrypt model only imposes computational restrictions on the constructions, not on the adversaries. Usually one considers arbitrary adversaries $\mathcal{A}$ that are computationally unbounded, but have bounded access to the random oracle H. Therefore the behaviour of an adversary is typically described in terms of the number of queries it makes.

2.1.3 Algebraic Representation

One of the advantages of restricting the computational model is that one can characterize Linicrypt programs with an algebraic representation. Let $\mathcal{P}$ be a linicrypt program with intermediate variables $v_1, \ldots, v_n$.

A **base variable** is an intermediate variable which was created by retrieving an input, calling the random oracle H or sampling from $\mathbb{F}$. Let base be the number of base variables and let $\boldsymbol{v}_{\mathsf{base}} \in \mathbb{F}^{\mathsf{base}}$ denote the vector of the base variables for an excecution of $\mathcal{P}$. A **derived variable** is one which is created by performing a linear combination of existing itermediate variables. Note, that derived variables are therefore linear combinations of base variables. As base variables are mostly independent of each other, it makes sense to *model them as independent vectors in* $\mathbb{F}^{\mathsf{base}}$. The derived variables are then modeled by linear combinations of these vectors.

Let v_i be an intermediate variable. We define the **associated vector** $\boldsymbol{v}_i$ to be the unique row vector such that $v_i = \boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}}$ for every excecution of $\mathcal{P}$. For example, if v_i is the j'th base variable, then $\boldsymbol{v}_i = \begin{bmatrix}0, \ldots, 1, \ldots, 0\end{bmatrix}$, where the 1 is in the j'th position. We follow the convention to write vectors in $\mathbb{F}^{\mathsf{base}}$ using a bold font.

The outputs of $\mathcal{P}$ can be described by a matrix with entries in $\mathbb{F}$. Let $o_1, \ldots, o_k$ be the output variables of $\mathcal{P}$. Then the **output matrix** $\boldsymbol{M}$ of $\mathcal{P}$ is defined by

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{o}_1 \\ \vdots \\ \boldsymbol{o}_k \end{bmatrix}.$$

By the definition of the associated vectors $\boldsymbol{o}_i$ we have $\boldsymbol{M} \times \boldsymbol{v}_{\mathsf{base}} = \begin{bmatrix}o_1, \ldots, o_k\end{bmatrix}^\top$. The output matrix describes the linear correlations in the output of $\mathcal{P}$.

But the output matrix doesn't describe all correlations in $\boldsymbol{v}_{\mathsf{base}}$. Namely, the relationship between the queries and answers to the random oracle H need to be captured algebraically. Let $v_i = H(t_i, (q_1, \ldots, q_n))$ be an operation in $\mathcal{P}$. The **associated oracle constraint** c to this operation is

$$c = \left(t_i, \begin{bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_n \end{bmatrix}, \boldsymbol{v}_i \right) = (t_i, \boldsymbol{Q}_i, \boldsymbol{v}_i).$$

This should be interpreted as the requirement that $\boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}} = H\left(t_i, \boldsymbol{Q}_i \times \boldsymbol{v}_{\mathsf{base}}\right)$. We denote the set of all (associated) oracle constraints of $\mathcal{P}$ by $\mathcal{C}$.

As we want the base variables to be linearly indepentent from each other, we restrict ourselves to Linicrypt programs which don't make multiple calls to the random oracle with the same input. In the language of the algebraic reprensentation: We assume wlog that no two constraints in $\mathcal{C}$ share the same $(t, \boldsymbol{Q})$.

TODO: Base variables which are created by a call of the same vector have to be removed. This is what is done wlog in the first two papers. This case corresponds to calling the random orcacle twice on the same input. But all this seems to be closely related to the ability of the adversary to make queries collapse, therefore I leave this open until I have thought more about that.

Wrapping up these definitions, we define the **algebraic representation** of $\mathcal{P}$ to be the tuple $(\boldsymbol{M}, \mathcal{C})$. A natural question that arises at this point is: Does the algebraic representation determine the behaviour of $\mathcal{P}$ completely?

2.1.4 Normalization and Indistinguishability

Todo: This leads to indistinguishability, normalization and theorem from first paper

2.1.5 Characterizing Collision Resistance in Linicrypt

In a paper by I. McQuoid, T. Swope and M. Rosulek [**?**, Characterizing Collision and Second-Preimage Resistance in Linicrypt], the authors introduced a characterization of collision resistance and second-preimage resistance for a class of Linicrypt program based on the algebraic representation.

They identified two reasons why a *deterministic* Linicrypt $\mathcal{P}$ program can fail to be second-preimage resistant:

1. It is degenerate, meaning that it doesn't use all of its inputs independently
2. It has a collision structure, which means that one can change some intermediate value and compute what the input needs to be to counteract this change

Below are two example Linicrypt programs, $\mathcal{P}^H_{\text{deg}}$ is degenerate and $\mathcal{P}^H_{\text{deg}}$ has a collision structure. Note, that you can choose $w' \neq w$ to be any value, then find a x' such that the output of $\mathcal{P}^H_{\text{cs}}$ stays the same, and finally find y' according to $w' = x' + y'$.

$\mathcal{P}^H_{\text{deg}}(x, y)$
$v \coloneqq x + y$
return $H(v)$

$\mathcal{P}^H_{\text{cs}}(x, y)$
$w \coloneqq x + y$
return $H(w) + x$

The precise definition of degenerate and collision structure will be discussed in chapter **??**, in a variation that is adapted to the goals of this thesis. The authors show that for any deterministic Linicrypt program which is degenerate or has a collision structure second-preimage resistance (and hence also collision resistance) is completely broken.

The main result of [**?**] is that they show that the converse of this is also true, for Linicrypt programs which use distinct nonces in each call to the random oracle. That is, if a Linicrypt program is not collision resistant, then it either has a collision structure, or it is degenerate.

Furthermore, checking for degeneracy and existence of a collision structure can be done efficiently.

Chapter 3

Linicrypt with Ideal Ciphers

3.1 Adapting the Linicrypt model to use block ciphers

In this chapter we modify the Linicrypt model to make use of the ideal cipher model instead of the random oracle model. This means that a Linicrypt program gets access to a block cipher $\mathcal{E} = (E, D)$ where E and D are functions $\mathbb{F} \times \mathbb{F} \to \mathbb{F}$ instead of the hash function $H : \{0,1\}^* \times \mathbb{F}^* \to \mathbb{F}$. By the definition of a block cipher, $E_k := E(k, \cdot)$ is a permutation of $\mathbb{F}$ for all $k \in \mathbb{F}$ and $D_k := D(k, \cdot)$ is its inverse. In the ideal cipher model, we assume that the block cipher has no weakness. This is modelled by choosing each permutation E_k uniforly at random at the beginning of every security game.

The command $y = E(k, x)$ in a Linicrypt program has to be treated differently from the command $y = H(k, x)$ when considering collision resistance, because an attacker has access to the deterministic linicrypt program and both directions of the block cipher $\mathcal{E} = (E, D)$. Consider these two programs, $\mathcal{P}^H$ in standard Linicrypt and $\mathcal{P}^{\mathcal{E}}$ in ideal cipher Linicrypt.

$\mathcal{P}^H(k, x)$
return $H(k, x)$

$\mathcal{P}^{\mathcal{E}}(k, x)$
return $E(k, x)$

While $\mathcal{P}^H$ is collision resistant, it is trivial to find second preimages for $\mathcal{P}^{\mathcal{E}}$ For any $k' \in \mathbb{F}$ the pair $(k', D(k', E(k, x)))$ is a second preimage to (k, x).

The permutation and invertibility property of block ciphers have to be taken into account in both the algebraic representation and the characterization of collision resistance.

3.1.1 Algebraic representation for ideal cipher Linicrypt

Let $\mathcal{P}$ be a ideal cipher Linicrypt program. For each query to E of the form $y = E(k, x)$ we define the **associated ideal cipher constraint** $(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y})$. Each query to D of the form $x = D(k, y)$, is associated with the constraint $(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x})$.

As with standard Linicrypt, we want to exclude programs that make unnecessary queries to the block cipher. This way the base variables are linearly independent, except for the dependencies the adversary might introduce by carfully choosing the input. Hence we assume wlog that no two constraints have the same $(E, \boldsymbol{k}, \boldsymbol{x})$ or $(D, \boldsymbol{k}, \boldsymbol{y})$.

With ideal ciphers there is a second way to make an unnecessary query. That is by first computing $y = E(k, x)$ and then $x' = D(k, y)$. As D_k is the inverse of E_k we have $x = x'$ although $\boldsymbol{x}$ and $\boldsymbol{x}'$ are linearly independent.

Therfore for all $\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}' \in \mathbb{F}^{\mathsf{base}}$ we can assume there are no two constraints $(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y})$ and $(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}')$ in $\mathcal{C}$ for $\boldsymbol{x} \neq \boldsymbol{x}'$. Neither can there be $(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x})$ and $(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}')$ in $\mathcal{C}$ for $\boldsymbol{y} \neq \boldsymbol{y}'$.

TODO: Maybe it is simpler with equivalence relation called always colliding queries

$$
\begin{aligned}
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}') \\
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}')
\end{aligned}
$$

And saying that no two constraints in $\mathcal{C}$ are in the same equivalence class. This might be cleaner, if the equivalence relation used later to analyze repeated nonces case is defined similarly.

TODO: Instead of doing weird things with equivalence relation in Collision structure definition, explicitly add data of reverse or forward direction.

3.2 Collision Struture

outline:

- Describe it in words
- Definition

To explain the concept of a collision structure, we will make use of an example. Consider the following Linicrypt program:

$$
\boxed{
\begin{array}{l}
\mathcal{P}^{\mathcal{E}}_{\mathsf{col}}(a, b, c) \\
\hline
\quad w = E(c, b + c) + a \\
\quad \textbf{return } c + E(w, b)
\end{array}
}
$$

A second preimage to (a, b, c) can be found bythe following procedure: Choose some TODO: Finish this example of type 10,0,FF

TODO: Add the example of type 10,0,BB

Definition 3.1 (Collision structure). *Let $\mathcal{P} = (\boldsymbol{M}, \mathcal{C})$ be a Linicrypt program with $|\mathcal{C}| = n$. A* ***collision structure*** *for $\mathcal{P}$ is an index $1 \leq i^* \leq n$, an ordering $(c_1, \dots, c_n)$ of $\mathcal{C}$ for $c_i = (Op_i, \boldsymbol{k}_i, \boldsymbol{q}_i, \boldsymbol{a}_i)$ and a tuple $(dir_{i^*}, \dots, dir_n)$ for $dir_i \in \{\mathsf{Forward}, \mathsf{Backward}\}$, such that the following two conditions hold:*

1. *Let $\mathcal{F}_{i^*} = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{i^*-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{i^*-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{i^*-1}\}$. One of the following is true:*
   a) $dir_{i^*} = \mathsf{Forward}$ *and* $\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{q}_{i^*}\}\big) \not\subseteq \mathsf{span}\big(\mathcal{F}_{i^*} \cup \mathsf{rows}(\boldsymbol{M})\big)$
   b) $dir_{i^*} = \mathsf{Backward}$ *and* $\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{a}_{i^*}\}\big) \not\subseteq \mathsf{span}\big(\mathcal{F}_{i^*} \cup \mathsf{rows}(\boldsymbol{M})\big)$
2. *For all $j \geq i^*$ let $\mathcal{F}_j = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1}\}$. One of the following is true:*
   a) $dir_j = \mathsf{Forward}$ *and* $\boldsymbol{a}_j \notin \mathsf{span}\big(\mathcal{F}_j \cup \{\boldsymbol{k}_j, \boldsymbol{q}_j\} \cup \mathsf{rows}(\boldsymbol{M})\big)$
   b) $dir_j = \mathsf{Backward}$ *and* $\boldsymbol{q}_j \notin \mathsf{span}\big(\mathcal{F}_j \cup \{\boldsymbol{k}_j, \boldsymbol{a}_j\} \cup \mathsf{rows}(\boldsymbol{M})\big)$

TODO: Remove this wordy definition. All the info from here should be integrated into the example

Definition 3.2 (Collision structure). *Let $\mathcal{P} = (\boldsymbol{M}, \mathcal{C})$ be a Linicrypt program with $|\mathcal{C}| = n$. A* ***collision structure*** *for $\mathcal{P}$ is an index $1 \leq i^* \leq n$, an ordering $(c_1, \dots, c_n)$ of $\mathcal{C}$ for $c_i = (Op_i, \boldsymbol{k}_i, \boldsymbol{q}_i, \boldsymbol{a}_i)$ and a tuple $(dir_{i^*}, \dots, dir_n)$ for $dir_i \in \{\mathsf{Forward}, \mathsf{Backward}\}$, such that:*

1. *The i^*'th constraint is unconstrained by the output of $\mathcal{P}$ and previous fixed constraints. Let $\mathcal{F} = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{i^*-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{i^*-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{i^*-1}\}$ denote the vectors fixed by previous constraints in the ordering.*
   a) *if $dir_{i^*} = \mathsf{Forward}$, the input of the query associated to c_{i^*} is unconstrained:*
   $$\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{q}_{i^*}\}\big) \not\subseteq \mathsf{span}\big(\mathcal{F} \cup \mathsf{rows}(\boldsymbol{M})\big)$$
   b) *if $dir_{i^*} = \mathsf{Backward}$, the output of the query associated to c_{i^*} is unconstrained:*
   $$\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{a}_{i^*}\}\big) \not\subseteq \mathsf{span}\big(\mathcal{F} \cup \mathsf{rows}(\boldsymbol{M})\big)$$
2. *For all $j \geq i^*$ the constraint c_j does not contradict previous constraints. Let $\mathcal{F} = \{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1}\}$ denote the vectors fixed by previous constraints in the ordering.*
   a) *if $dir_j = \mathsf{Forward}$*
   $$\boldsymbol{a}_j \notin \mathsf{span}\big(\{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1},\} \cup \{\boldsymbol{k}_j, \boldsymbol{q}_j\} \cup \mathsf{rows}(\boldsymbol{M})\big)$$
   b) *if $dir_j = \mathsf{Backward}$*
   $$\boldsymbol{q}_j \notin \mathsf{span}\big(\{\boldsymbol{k}_1, \dots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \dots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \dots, \boldsymbol{a}_{j-1},\} \cup \{\boldsymbol{k}_j, \boldsymbol{a}_j\} \cup \mathsf{rows}(\boldsymbol{M})\big)$$

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