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Modeling the Ideal Cipher in Linicrypt

Master Thesis

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April 27, 2022

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Abstract

Todo

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Todo

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2.1 Linicrypt

2.1.1 Definition of a Linicrypt program

The Linicrypt model for cryptographic constructions was introduced by Cramer & Rosulek in [**?**]. Summarizing the formalization from that paper, a pure Linicrypt program $\mathcal{P}$ is a straight line program whose intermediate variables are elements in a field $\mathbb{F}$. The only allowed operations to create an intermediate variable are:

- Retrieve an input, which is in $\mathbb{F}$
- Perform a linear combination of existing internal variables
- Call a random oracle $H : \{0,1\}^* \times \mathbb{F}^* \to \mathbb{F}^*$
- Sample from $\mathbb{F}$ uniformly

The program $\mathcal{P}$ can output one or more of its variables.

2.1.2 Type of Adversary

The Linicrypt model only imposes computational restrictions on the constructions, not on the adversaries. Usually one considers arbitrary adversaries $\mathcal{A}$ that are computationally unbounded, but have bounded access to the random oracle H. Therefore the behaviour of an adversary is typically described in terms of the number of queries it makes.

2.1.3 Algebraic Representation

One of the advantages of restricting the computational model is that one can characterize Linicrypt programs with an algebraic representation. Let $\mathcal{P}$ be a linicrypt program with intermediate variables $v_1, \ldots, v_n$.

A **base variable** is an intermediate variable which was created by retrieving an input, calling the random oracle H or sampling from $\mathbb{F}$. Let base be the number of base variables and let $\boldsymbol{v}_{\mathsf{base}} \in \mathbb{F}^{\mathsf{base}}$ denote the vector of the base variables for an excecution of $\mathcal{P}$. A **derived variable** is one which is created by performing a linear combination of existing itermediate variables. Note, that derived variables are therefore linear combinations of base variables. As base variables are mostly independent of each other, it makes sense to *model them as independent vectors in* $\mathbb{F}^{\mathsf{base}}$. The derived variables are then modeled by linear combinations of these vectors.

Let v_i be an intermediate variable. We define the **associated vector** $\boldsymbol{v}_i$ to be the unique row vector such that $v_i = \boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}}$ for every excecution of $\mathcal{P}$. For example, if v_i is the j'th base variable, then $\boldsymbol{v}_i = \begin{bmatrix}0, \ldots, 1, \ldots, 0\end{bmatrix}$, where the 1 is in the j'th position. We follow the convention to write vectors in $\mathbb{F}^{\mathsf{base}}$ using a bold font.

The outputs of $\mathcal{P}$ can be described by a matrix with entries in $\mathbb{F}$. Let $o_1, \ldots, o_k$ be the output variables of $\mathcal{P}$. Then the **output matrix** $\boldsymbol{M}$ of $\mathcal{P}$ is defined by

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{o}_1 \\ \vdots \\ \boldsymbol{o}_k \end{bmatrix}.$$

By the definition of the associated vectors $\boldsymbol{o}_i$ we have $\boldsymbol{M} \times \boldsymbol{v}_{\mathsf{base}} = \begin{bmatrix}o_1, \ldots, o_k\end{bmatrix}^\top$. The output matrix describes the linear correlations in the output of $\mathcal{P}$.

But the output matrix doesn't describe all correlations in $\boldsymbol{v}_{\mathsf{base}}$. Namely, the relationship between the queries and answers to the random oracle H need to be captured algebraically. Let $v_i = H(t_i, (q_1, \ldots, q_n))$ be an operation in $\mathcal{P}$. The **associated oracle constraint** c to this operation is

$$c = \left(t_i, \begin{bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_n \end{bmatrix}, \boldsymbol{v}_i \right) = (t_i, \boldsymbol{Q}_i, \boldsymbol{v}_i).$$

This should be interpreted as the requirement that $\boldsymbol{v}_i \times \boldsymbol{v}_{\mathsf{base}} = H\left(t_i, \boldsymbol{Q}_i \times \boldsymbol{v}_{\mathsf{base}}\right)$. We denote the set of all (associated) oracle constraints of $\mathcal{P}$ by $\mathcal{C}$.

Wrapping up these definitions, we define the **algebraic representation** of $\mathcal{P}$ to be the tuple $(\boldsymbol{M}, \mathcal{C})$. A natural question that arises at this point is: Does the algebraic representation determine the behaviour of $\mathcal{P}$ completely?

Chapter 3

Extending Linicrypt to Ideal Ciphers

Let $\mathcal{P}$ be a Linicrypt program. For each query to E of the form $y = E(k, x)$ we define the associated constraint $(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y})$, where $\boldsymbol{k} \in \mathbb{F}^{\mathsf{base}}$ is the row vector corresponding to $k \in \mathbb{F}$ and similarly for $\boldsymbol{x}$ and $\boldsymbol{y}$. Each query to D of the form $x = D(k, y)$, is associated with the constraint $(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x})$

To capture the fact that $E(k, x) = y$ should be associated to the same constraint as $D(k, y) = x$ for the same k, x and y, we introduce an equivalence relation on the constraints. For all $\boldsymbol{k}, \boldsymbol{x}, \boldsymbol{x}', \boldsymbol{y}, \boldsymbol{y}' \in \mathbb{F}^{\mathsf{base}}$ we define

$$\begin{aligned}
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}') \\
(E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}') \\
(D, \boldsymbol{k}, \boldsymbol{y}, \boldsymbol{x}) &\sim (E, \boldsymbol{k}, \boldsymbol{x}, \boldsymbol{y}')
\end{aligned}$$

The set of constraints $\mathcal{C}$ corresponding to $\mathcal{P}$ is then a subset of

$$\Big(\{E, D\} \times \mathbb{F}^{\mathsf{base}} \times \mathbb{F}^{\mathsf{base}} \times \mathbb{F}^{\mathsf{base}}\Big) \Big/ \sim$$

Todo: Include the idea that no constraint with the "same" input queries are used twice.

Each query should only be

Todo: Maybe scrap the idea of the equivalence relation, it seems to hinder more than it helps.

Todo: Instead of doing weird things with equivalence relation in Collision structure definition, explicitly add data of reverse or forward direction.

Definition 3.1 (Collision structure)**.** *Let $\mathcal{P} = (\boldsymbol{M}, \mathcal{C})$ be a Linicrypt program. A* ***collision structure*** *is an index i^* and a tuple $(c_1, \ldots, c_n)$ for $c_i = (O_i, \boldsymbol{k}_i, \boldsymbol{q}_i, \boldsymbol{a}_i)$ and $O_i \in \{E, D\}$, such that:*

1. $[c_1], \ldots, [c_n]$ *is an ordering of* $\mathcal{C}$
2. *The input or output corresponding to the query* c_{i^*} *can be fixed arbitrarily:*

$$\mathsf{span}\big(\{\boldsymbol{k}_{i^*}, \boldsymbol{q}_{i^*}\}\big) \not\subseteq \mathsf{span}\big(\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{i^*-1}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{i^*-1}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{i^*-1}\} \cup \mathsf{rows}\,(\boldsymbol{M})\,\big)$$

3. *For all* $j \geq i^*$ *the constraint* c_j *does not contradict previous constraints:*

$$\boldsymbol{a}_j \notin \mathsf{span}\big(\{\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{j-1}, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{j-1}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_{j-1},\} \cup \{\boldsymbol{k}_j, \boldsymbol{q}_j\} \cup \mathsf{rows}\,(\boldsymbol{M})\,\big)$$

This is an mistake

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