TMA 4265 Stochastic Modelling

Week 37 solutions

Exercise 1:

a) The chain is reducible because it is not possible to go between any two states in the Markov chain. An equivalence class consists of those states that communicate with one another. For this Markov chain the equivalence classes are $\{1,2\},\ \{3,4\},\ \{5,6\}$.

b) Since the transition probability matrix diagonal has strictly positive values, all states have period 1, i.e. we have an aperiodic Markov chain.

c) State i is recurrent if beginning in state i, the process returns to state i at some later time with probability 1 and state i is transient if beginning in state i there is a positive probability that the process will never return to state i. Further, state i is absorbing if once the process has entered that state, it never leaves.
Transient states: $3, 4$
Recurrent states: $1, 2, 5, 6$
Absorbing states: none

Exercise 2:

Consider the Markov chain $\{X_n\}$ with transition probability matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0.05 & 0.8 & 0.15 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) This Markov chain is reducible because it is not possible to go between any two states. For example from state 1 to state 3.

b) Define $T = \min\{n \geq 0 \colon X_n = 1 \text{ or } X_n = 3\}$ and

$$P(X_T = 1 \mid X_1 = 1) = 1$$

$$P(X_T = 1 \mid X_1 = 2) = u$$

$$P(X_T = 1 \mid X_1 = 3) = 0$$

We use first step analysis to find u

$$
\begin{aligned}
u &= P(X_T = 1 \mid X_0 = 2) \\
&= \sum_{k=1}^{3} P(X_T = 1 \mid X_1 = k) P(X_1 = k \mid X_0 = 2) \\
&= 0.05 + 0.8u + 0
\end{aligned}
$$

This implies $u = 0.25$ and since we only have two possible states in the limiting distribution, we get $P(X_T = 3 \mid X_1 = 2) = 1 - u = 0.75$. Limiting distribution is

$$
\begin{aligned}
\lim_{n\to\infty} P(X_n = 1 \mid X_0 = 2) &= 0.25, \\
\lim_{n\to\infty} P(X_n = 2 \mid X_0 = 2) &= 0, \\
\lim_{n\to\infty} P(X_n = 3 \mid X_0 = 2) &= 0.75.
\end{aligned}
$$

Exercise 3:

a) Each throw is independent of other throws. Only the previous state (number of heads in a row) affects the probability of transitioning into a new state. Transition probability matrix is

$$
\mathbf{P} = \begin{pmatrix}
1/2 & 1/2 & 0 & 0 & 0 & \dots \\
1/2 & 0 & 1/2 & 0 & 0 & \dots \\
1/2 & 0 & 0 & 1/2 & 0 & \dots \\
1/2 & 0 & 0 & 0 & 1/2 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}
$$

State space is $\Omega = \mathbb{N}_0$.

b) Yes. Let i, j be any two states. We can always go from state i to 0, from there 0 to 1 and so on, until we have reached state j.

c) By the definition, the period of state i is

$$\gcd\{n \geq 0\colon P(X_n = i \mid X_0 = i) > 0\}.$$

Since all states communicate with each other, we only need to find the period of one of the states. All other states have the same period. For $i = 0$

$$P(X_1 = 0 \mid X_0 = 0) > 0,$$

hence $\gcd\{n \geq 0 \colon P(X_n = i \mid X_0 = i) > 0\} = 1$ and the period of state 0 is 1. This means that all states have period 1.

Since all states communicate with each other, we only need to prove that one of them is either recurrent or transient. All other states have the same property. For $i = 0$

$$\lim_{n\to\infty} P_{ii}^{(n)} = \frac{1}{\sum_{n\to\infty}^{\infty} n f_{ii}^{(n)}} = \frac{1}{\sum_{n\to\infty}^{\infty} n \left(\frac{1}{2}\right)^n} = \frac{1}{2} > 0 \tag{1}$$

where $f_{ii}^{(n)}$ is the first return probability. Also

$$\sum_{n=1}^{\infty} P_{ii}^{(n)} = \sum_{n=1}^{\infty} \frac{1}{2} = \infty.$$

Hence according to Theorem 4.2 state $i = 0$ is recurrent and all other states are also recurrent.

d) Equation implies that the chain is positive recurrent, which in combination with aperiodicity and irreducibility means it has a unique limiting distribution, see Theorem 4.4. Equation (1) gives us the limiting distribution probability for state 0

$$\pi_0 = \frac{1}{2}$$

and for states 1 and 2 we solve the equations

$$\pi_1 = \sum_{k=0}^{\infty} \pi_k P_{k1} = \frac{1}{4}$$

$$\pi_2 = \sum_{k=0}^{\infty} \pi_k P_{k2} = \frac{1}{2}\pi_1 = \frac{1}{8}.$$

We know that the answer is given to us by $\sum_{i=3}^{\infty} \pi_i$. Since $\sum_{i=0}^{\infty} \pi_i = 1$, the answer is

$$1 - \pi_0 - \pi_1 - \pi_2 = \frac{1}{8}.$$