TMA4265 Stochastic Modeling

Exercises week 41

Exercise 1. - Exercise 6.1.1 and 6.1.2. from the book

A pure birth process starting from $X(0) = 0$ has birth parameters $\lambda_0 = 1$, $\lambda_1 = 3$, $\lambda_2 = 2$, and $\lambda_3 = 5$. Let W_3 be the random time that it takes the process to reach state 3.

i) Determine the transition probability functions $P_n(t) = \Pr\{X(t) = n | X(0) = 0\}$ for $n = 0, 1, 2, 3$.

ii) Write W_3 as a sum of sojourn times and thereby deduce that the mean time is $\mathrm{E}[W_3] = \frac{11}{6}$.

iii) Determine the mean of $W_1 + W_2 + W_3$.

iv) What is the variance of W_3?

Exercise 2.

At a parking garage, customers arrive according to a Poisson process, with rate $\lambda = 0.5$ (time t is in minutes).

i) The garage opens at 08.00. What is the probability that no customer has arrived by 08.05?
What is the expected number of customers arriving during the first 15 minutes?
Given that two customers arrived during the first 10 minutes, what is the probability that no customer arrived during the first 5 minutes?

ii) Customers spend on average 100kr for parking, independently of each other, with standard deviation of 10kr.

Calculate the expected value of the total income at the garage during the day (08.00-16.00).
Calculate the variance of the total income at the garage during the day (08.00-16.00).

Next, consider a more realistic situation where the costumers in addition to arriving to the garage, also leave the garage. We assume costumers arrive independently of each other, according to the Poisson process described above. We further assume customers, independently of each other, spend an exponentially distributed time T in the parking garage, with expectation $1/\mu$, where $\mu = 1/30 = 0.0333$. The number of customers $N(t)$ in the garage, by time t can then be modeled by a birth-and-death process. The maximum capacity of the garage is $N_{\max} = 20$. If there are 20 customers in the garage, arriving customers will just drive by the garage, without forming a queue.

iii) Find the birth and death rates of the process.
Draw a transition diagram for the process.
The garage is closing. No new customers arrive, while the customers who are in the garage at the closing time, leave at the rates described above.
We consider a situation where there are two customers left in the garage at the closing time. Find the probability density, $f_T(t)$, of the waiting time until the parking garage is empty.

iv) Use long-run equality of process rates going in and out of states to find expressions for the long-run probabilities of the process.
Compute $P_{20} = \lim_{t\to\infty} P(N(t) = 20)$.
Compute the expected long-run number of customers in the garage; $E(N)$.

(Hint: You can look up Poisson probabilities in a table. If X

is Poisson distributed with parameter ν, we have

$$P(X = i) = \exp(-\nu) \frac{\nu^i}{i!}, \ i = 0, 1, 2, \ldots)$$

Denote the current parking fee per minute as c_{min}. For environmental reasons the authorities demand that garages must double the parking fee per minute. The owner of a garage thinks this will reduce the arrival rate of costumers. One assumes that the customers on average spend the same time T at the garage, as described above. One further assumes that the parking capacity will never be reached in this case.

v) Find the new arrival rate, λ^*, that keeps the long-term income of the garage the same as before the increase of the parking fee. (Take for granted the expected long-run number of customers $E(N)$ as the solution to D.)