TMA4265 Stochastic Processes
Week 36: Solutions.

Exercise 1

We derive the following transition structures:

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.1 & 0.7 & 0.2 \\ 0.5 & 0.1 & 0.4 \\ 0.3 & 0.6 & 0.1 \end{pmatrix} \end{array} \quad P^2 = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.42 & 0.26 & 0.32 \\ 0.22 & 0.6 & 0.18 \\ 0.36 & 0.33 & 0.31 \end{pmatrix} \end{array} \quad P^3 = \begin{array}{c} \\ A \\ B \\ C \end{array}\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 0.268 & 0.512 & 0.220 \\ 0.376 & 0.322 & 0.302 \\ 0.294 & 0.471 & 0.235 \end{pmatrix} \end{array}.$$

a)

$$\begin{aligned} P(X_3 = A) &= P(X_0 = A)P(X_3 = A|X_0 = A) + P(X_0 = B)P(X_3 = A|X_0 = B) \\ &\quad + P(X_0 = C)P(X_3 = A|X_0 = C) \\ &= 0.2 \cdot 0.268 + 0.5 \cdot 0.376 + 0.3 \cdot 0.294 = 0.3298. \end{aligned}$$

$$\begin{aligned} P(X_3 = B) &= P(X_0 = A)P(X_3 = B|X_0 = A) + P(X_0 = B)P(X_3 = B|X_0 = B) \\ &\quad + P(X_0 = C)P(X_3 = B|X_0 = C) \\ &= 0.2 \cdot 0.512 + 0.5 \cdot 0.322 + 0.3 \cdot 0.471 = 0.4047. \end{aligned}$$

$$P(X_3 = C) = 1 - P(X_3 = A) - P(X_3 = B) = 0.2655.$$

b)

$$P(X_3 = A|X_1 = B, X_0 = A) = P(X_3 = A|X_1 = B) = P(X_2 = A|X_0 = B) = 0.22.$$

c)

$$P(X_3 = A|X_2 = C, X_1 = B, X_0 = A) = P(X_3 = A|X_2 = C) = P(X_1 = A|X_0 = C) = 0.3.$$

d) $P(X_6 = A|X_3 = C) = P(X_3 = A|X_0 = C) = 0.294.$

e)

$$\begin{aligned} &P(X_3 = C|X_6 = A) \\ &= \frac{P(X_3 = C, X_6 = A)}{P(X_6 = A)} \\ &= \frac{P(X_6 = A|X_3 = C)P(X_3 = C)}{P(X_6 = A|X_3 = C)P(X_3 = C) + P(X_6 = A|X_3 = B)P(X_3 = B) + P(X_6 = A|X_3 = A)P(X_3 = A)} \\ &= \frac{P(X_3 = A|X_0 = C)P(X_3 = C)}{P(X_3 = A|X_0 = C)P(X_3 = C) + P(X_3 = A|X_0 = B)P(X_3 = B) + P(X_3 = A|X_0 = A)P(X_3 = A)} \\ &= \frac{0.294 \cdot 0.2655}{0.294 \cdot 0.2655 + 0.376 \cdot 0.4047 + 0.268 \cdot 0.3298} = 0.245. \end{aligned}$$

Exercise 2

a) We have
$P^4 = P \cdot P \cdot P \cdot P = \begin{pmatrix} 0.4769125 & 0.5230875 \\ 0.4359062 & 0.5640937 \end{pmatrix}$,
so the probability for rain four days from now is given by $P^4_{11} = 0.48$.

b) You can find the long-term probability for rain analytically from $\lim_{n\to\infty} P^n$. In Matlab/Python/R you can easily compute
$P^{100} = \begin{pmatrix} 0.4545455 & 0.5454545 \\ 0.4545455 & 0.5454545 \end{pmatrix}$
so the long-term probability of rain for a random day is 0.45.

Another solution is to solve the following system of equations to find the limiting distribution:

$$\begin{aligned} 0.7\pi_1 + 0.25\pi_2 &= \pi_1 \\ 0.3\pi_1 + 0.75\pi_2 &= \pi_2 \\ \pi_1 + \pi_2 &= 1. \end{aligned}$$

One of the equations is redundant (two unknowns and three equations), so we arbitrary strike ut equation number 2 in the system. Solving

$$\begin{aligned} 0.7\pi_1 + 0.25\pi_2 &= \pi_1 \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

gives $\pi_1 = 0.45$ and $\pi_2 = 0.55$.

c) See Blackboard for code.

Exercise 3

Let $\{X_n; n \geq 0\}$ be a Markov chain with the given transition probability and let $T = min\{n \geq 0 | X_n = 0 \text{ or } X_n = 3 \}$ denote the time of absorption.
a) We are interested in $P(X_T = 0|X_0 = 1)$ and use a first-step analysis to find this quantity. Let $u_i = P(X_T = 0|X_0 = i)$. Obviously $u_0 = 1$ and $u_3 = 0$. Further,

$$\begin{aligned} u_1 &= \sum_{j=0}^{3} u_j P_{1j} \\ &= u_0 P_{10} + u_1 P_{11} + u_2 P_{12} \\ &= 0.1 + 0.4u_1 + 0.1u_2 \end{aligned}$$

$$
\begin{aligned}
u_2 &= \sum_{j=0}^{3} u_j P_{2j} \\
&= u_0 P_{20} + u_1 P_{21} + u_2 P_{22} \\
&= 0.2 + 0.1 u_1 + 0.6 u_2
\end{aligned}
$$

Solving this linear system of two equations in two variables gives $u_1 = P(X_T = 0|X_0 = 1) \approx 0.261$.

b) We are interested in $E(T|X_0 = 1)$ and use a first-step analysis to find the desired quantity. Let $v_i = E(T|X_0 = i)$. Obviously $v_0 = v_3 = 0$. Further,

$$
\begin{aligned}
v_1 &= 1 + \sum_{j=0}^{3} v_j P_{1j} \\
&= 1 + v_1 P_{11} + v_2 P_{12} \\
&= 1 + 0.4 v_1 + 0.1 v_2
\end{aligned}
$$

$$
\begin{aligned}
v2 &= 1 + \sum_{j=0}^{3} v_j P_{2j} \\
&= 1 + v_1 P_{21} + v_2 P_{22} \\
&= 1 + 0.1 v_1 + 0.6 v_2.
\end{aligned}
$$

Solving this system gives the mean time of absorption starting in state 1, $v_1 = E(T|X_0 = 1) \approx 2.17$.