TMA4265 Stochastic Modeling

Week 35: Solutions

1: Coin Toss

a) Let B denote the event that we flipped the biased coin, and B' the event that we flipped the normal coin. Further, let H1 denote the event that the first toss results in a head and H2 denote the event that the second toss results in a head.

The probability that we flipped the biased coin is

$$P(B|H1) = \frac{P(B \cap H1)}{P(H1)} = \frac{P(B)P(H1|B)}{P(B)P(H1|B) + P(B')P(H1|B')} = \frac{\frac{1}{2} \cdot 1}{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}$$

by Bayes' theorem and the law of total probability.

b) We flip the same coin one more time, and the second outcome is also a head. Thus, we are given that both the first and the second outcome is a head. The probability that we are flipping the biased coin is then given by

$$\begin{aligned} P(B|H1, H2) &= \frac{P(H1, H2|B)P(B)}{P(H1, H2)} \\ &= \frac{P(B)P(H1|B)P(H2|H1, B)}{P(B)P(H1|B)P(H2|H1, B) + P(B')P(H1|B')P(H2|H1, B')} \\ &= \frac{\frac{1}{2} \cdot 1 \cdot 1}{\frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}} = \frac{4}{5}. \end{aligned}$$

Note that $P(H2|H1, B) = P(H2|B)$ and $P(H2|H1, B') = P(H2|B')$: The second outcome is independent of the first outcome given that we are throwing the same coin.

These results can be verified by simulations: A Python and R implementation can be found on the course web page.

2: Insurance claims

a) The moment generating function of the Normal distribution is given by $M_{Y_i}(t) = E(e^{Y_i t}) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ where $Y_i \sim \mathcal{N}(\mu, \sigma^2)$. We are going to use this to find the $k'th$ moments of the log-Gaussian distribution:

$$E(C_i^k) = E(e^{Y_i k}) = e^{\mu k + \frac{\sigma^2 k^2}{2}}.$$

Here we use that $C_i = e^{Y_i}$ and recognize $E(e^{Y_i k})$ as the moment generating function of the Normal distribution. Setting $k = 1$ we obtain an expression for the expected value of the log-Gaussian distribution:

$$E(C_i) = E(C_i^1) = e^{\mu \cdot 1 + \frac{\sigma^2 \cdot 1^2}{2}} = e^{\mu + \frac{\sigma^2}{2}}.$$

Now we can easily derive an expression for the variance of the log-Gaussian distribution as well:

$$Var(C_i) = E(C_i^2) - E(C_i)^2 = e^{\mu\cdot 2+\frac{\sigma^2\cdot 2^2}{2}} - (e^{\mu+\frac{\sigma^2}{2}})^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

b) Double expectation gives

$$E(\sum_{i=1}^{N} C_i) = E(E(\sum_{i=1}^{N} C_i|N)).$$

Since

$$E(\sum_{i=1}^{N} C_i|N) = \sum_{i=1}^{N} E(C_i) = \sum_{i=1}^{N} \exp(\mu + \frac{\sigma^2}{2}) = N\exp(\mu + \frac{\sigma^2}{2}),$$

we get

$$E(\sum_{i=1}^{N} C_i) = E(E(\sum_{i=1}^{N} C_i|N)) = E(N\exp(\mu + \frac{\sigma^2}{2})) = E(N)\exp(\mu + \frac{\sigma^2}{2}) = \lambda\exp(\mu + \frac{\sigma^2}{2}),$$

Inserting $\mu = 2$, $\sigma = 1$ and $\lambda = 6$ into the above expression we get that the expected total sum of claim amounts is $E(\sum_{i=1}^{N} C_i)$=1.339 mill. kr.

Double variance gives

$$\begin{aligned}
Var(\sum_{i=1}^{N} C_i|N) &= E(Var(\sum_{i=1}^{N} C_i|N)) + Var(E(\sum_{i=1}^{N} C_i|N)) \\
&= E(\sum_{i=1}^{N} Var(C_i)) + Var(\sum_{i=1}^{N} E(C_i)) \\
&= E(N[(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)]) + Var(N\exp(\mu + \frac{\sigma^2}{2})) \\
&= [(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)]E(N) + [\exp(\mu + \frac{\sigma^2}{2})]^2 Var(N) \\
&= (\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)\lambda + [\exp(\mu + \frac{\sigma^2}{2})]^2\lambda.
\end{aligned}$$

Inserting $\mu = 2$, $\sigma = 1$ and $\lambda = 6$ into the above expression we get that the variance of the total sum of claim amounts is $Var(\sum_{i=1}^{N} C_i|N) = 0.812$ mill. kr.

c) See Python and R implementations in separate files.