TMA4265 Stochastic Modeling

Week 38 Solutions

Exercise 1.

a) (a)

$$P(X(1) = 0) = e^{-2}$$

(b) Since the process has independent increments, the conditioning does not affect the answer e^{-2}.

b)

$$p_0 = \frac{e^{-\lambda}\lambda^0}{0!} = e^{-\lambda}$$
$$(p_k/p_{k-1}) = \frac{e^{-\lambda}\lambda^k(k-1)!}{e^{-\lambda}\lambda^{k-1}k!} = \frac{\lambda}{k}$$

c) By using Theorem 5.1 from the book, the definition of conditional probability and independence between X, Y, we get

$$\begin{aligned} P(X = k|N = n) &= \frac{P(X = k \cap Y = n-k)}{P(X+Y = n)} \\ &= \binom{n}{k} p^k (1-p)^{n-k}, \end{aligned}$$

where $p = \dfrac{\alpha}{\alpha + \beta}$.

d)

$$P(W_1 > 3) = 1 - P(W_1 \leq 3) = 1 - F_{W_1}(3) = e^{-6},$$

where F_{W_1} is the cumulative distribution function of the exponential distribution with parameter $\lambda = 2$.

e) (a)

$$P(3 < W_1 \leq 5) = P(W_1 \leq 5) - P(W_1 \leq 3) = e^{-6} - e^{-10}$$

(b)

$$P(X(3+2) - X(3) = 1) = \frac{(\lambda \cdot 2)^1 \exp(-\lambda \cdot 2)}{1!} = 4e^{-4}$$

f) If $\lambda = 6$ is the hourly rate, then $1/10$ is the rate per minute. Using the conditional probability and the independence of increments, we get

$$\begin{aligned} P(X(15) = 1 | X(60) = 1) &= \frac{P(X(15) = 1 \cap X(60) = 1)}{P(X(60) = 1)} \\ &= \frac{P(X(15) = 1) P(X(60) - X(15) = 0)}{P(X(60) = 1)} \\ &= \frac{1}{4}. \end{aligned}$$

Exercise 2. Let R_t be the number of goals scored by Rosenborg, let V_t be the number of goals scored by Vålerenga and let N_t be the total number of goals scored during a match where t is given in minutes. The number of goals scored by the teams in t minutes is then given by

$$P(R_t = n) = \frac{\exp\left(-\frac{\lambda_r}{90} t\right) \left(\frac{\lambda_r}{90} t\right)^n}{n!},$$

$$P(V_t = n) = \frac{\exp\left(-\frac{\lambda_v}{90} t\right) \left(\frac{\lambda_v}{90} t\right)^n}{n!}$$

with $\lambda_r = 2$ and $\lambda_v = 1.2$.

a) The total number of goals scored during one match is Poisson distributed with average $\lambda_{tot} = \lambda_r + \lambda_v$:

$$P(N_t = n) = \frac{\exp\left(-\left(\frac{\lambda_r}{90} + \frac{\lambda_v}{90}\right) t\right) \left(\left(\frac{\lambda_r}{90} + \frac{\lambda_v}{90}\right) t\right)^n}{n!}$$

because the distribution of a sum of Poisson distributions is Poisson distributed.

b)

$$P(N_{45} = 0) = \frac{\exp\left(-\left(\frac{\lambda_r}{90} + \frac{\lambda_v}{90}\right) \cdot 45\right) \left(\left(\frac{\lambda_r}{90} + \frac{\lambda_v}{90}\right) \cdot 45\right)^0}{0!} \approx 0.2019.$$

Alternatively, let X_{tot} denote the time until the first goal by either Rosenborg or Vålerenga in minutes. Then X_{tot} is exponentially distributed with expected value $\left(\frac{\lambda_r}{90}+\frac{\lambda_v}{90}\right)^{-1}$ and

$$\begin{aligned}
P(N_{45}=0) &= P(X_{tot}>45) \\
&= 1-\int_{t=0}^{t=45}\left(\frac{\lambda_r}{90}+\frac{\lambda_v}{90}\right)\exp\left(-\left(\frac{\lambda_r}{90}+\frac{\lambda_v}{90}\right)\cdot t\right)dt \\
&= 1-\left(1-\exp\left(-45\cdot\left(\frac{\lambda_r}{90}+\frac{\lambda_v}{90}\right)\right)\right) \\
&\approx 0.2019.
\end{aligned}$$

c) The probability that the final result is 2-2 is given by

$$P(R_{90}=2,V_{90}=2)=P(R_{90}=2)\cdot P(V_{90}=2)=0.0586,$$

where we use that the number of goals scored by Rosenberg is independent of the number of goals scored by Vålerenga.

d) The time until the first goal in the match is exponentially distributed with expected value $\left(\frac{\lambda_r}{90}+\frac{\lambda_v}{90}\right)^{-1}$. Inserting numbers, we get that the expected time until the first goal; is 28.125 minutes in this match.

e) Because of the memoryless property, the number of goals scored from $t=15$ to $t=45$ is independent of the number of goals scored from $t = 0$ to $t = 15$. Thus

$$\begin{aligned}
P(V_{45} > 1 | N_{15} = 0) &= P(V_{45} > 1 | V_{15} = 0) \\
&= \frac{P(V_{45} > 1, V_{15} = 0)}{P(V_{15} = 0)} \\
&= \frac{P(V_{45} - V_{15} > 1, V_{15} = 0)}{P(V_{15} = 0)} \\
&= \frac{P(V_{30} > 1) P(V_{15} = 0)}{P(V_{15} = 0)} \\
&= P(V_{30} > 1) \\
&= 1 - P(V_{30} = 0) \\
&= 1 - \frac{\exp\left(-\frac{\lambda_v}{90} \cdot 30\right) \left(\frac{\lambda_v}{90} \cdot 30\right)^0}{0!} \\
&\approx 0.33.
\end{aligned}$$

f) Example code can be found on Blackboard.