TMA4265 Stochastic Modeling
Week 42 Solutions

Exercise 1.

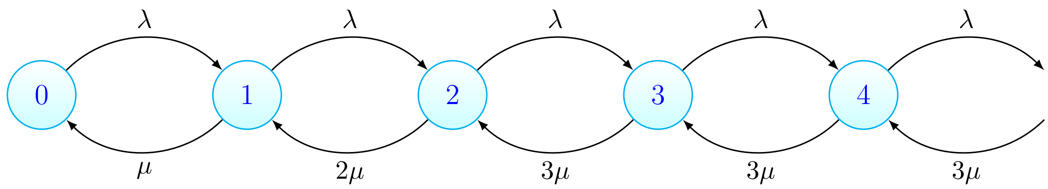
1. The given process is a birth and death process as the number of skiers in the stadium either increase with one (birth) or decrease with one (death) and all times until the next arrival (birth) and termination of shooting (death) are independent and exponentially distributed. The birth rates are given by

$$\lambda_n = \lambda, \quad n = 1, 2, \dots$$

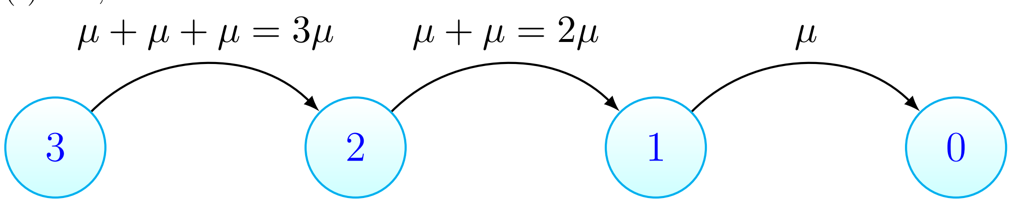
while the death rates are

$$\mu_1 = \mu, \quad \mu_2 = 2\mu, \quad \mu_3 = 3\mu, \dots .$$

This is a M/M/3 queue and the transition matrix is



2. The time until a skier is finished with shooting is exponentially distributed with rate $\mu = 1/30$. The time until the next skier is finished is then the minimum of independent and exponentially distributed variables which is exponential with rate equal to the sum of the individual rates. Meaning that either of the three completions, assuming $X(t) = 3$ will move the system to $X(t) = 2$, and so on



The expected time W until all the of the three skiers are finished is

$$E(W) = \frac{1}{3\mu} + \frac{1}{2\mu} + \frac{1}{\mu} \approx 55 \text{ seconds.}$$

3. Let T_i denote the time, starting from i, it takes for the process to enter state $i+1$, $i \geq 0$. Hence the expected time to reach state 3 (in seconds) is

$$
\begin{aligned}
E(T_0) &= 1/\lambda_0 = 12 \\
E(T_1) &= 1/\lambda_1 + \frac{\mu_1}{\lambda_1} E(T_0) = 16.8 \\
E(T_2) &= 1/\lambda_2 + \frac{\mu_2}{\lambda_2} E(T_1) = 25.44 \\
\Rightarrow \sum_{i=0}^{2} E(T_i) &= 54.24 \text{ seconds.}
\end{aligned}
$$

These expressions are obtained through first step analysis, in the book, Ch. 6.5, expressions 6.45 and 6.46). It is possible to solve this task using first step analysis as shown in the lectures. (Consider state 3 to be absorbing state, time it takes for going from state 3 to state 3 is zero.)

4. (For this task have a look at Ch. 6.4, expression 6.37 + but remember it is a queue, or look at Ch. 9.2.3, for example expression 9.17)
In general

$$
P_n = \frac{\theta_n}{\sum_{k=0} \theta_k},
$$

where $\theta_0 = 1$ and

$$
\theta_k = \frac{\lambda_0 \cdot \lambda_1 \cdot \ldots \cdot \lambda_{k-1}}{\mu_1 \cdot \ldots \cdot \mu_k}.
$$

Here,

$$
\begin{aligned}
\theta_1 &= \frac{\lambda}{\mu} \\
\theta_2 &= \frac{\lambda^2}{2\mu^2} \\
\theta_k &= \frac{\lambda^k}{2 \cdot 3^{k-2} \mu^k}, \quad k > 2.
\end{aligned}
$$

(If you follow the expressions from the book, you will be able to simplify it and get the expression above, this is for s=3, but due to the simplification, we get the k-2 in the exponent, hence k>2)

With $\lambda = 1/12$ and $\mu = 1/30$ we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \theta_k &= 1 + \frac{\lambda}{\mu} + \frac{1}{2} \sum_{k=2}^{\infty} \left(\frac{\lambda}{\mu}\right)^k \frac{1}{3^{k-2}} \\ &= 1 + \frac{\lambda}{\mu} + \frac{1}{2} \left(\frac{\lambda}{\mu}\right)^2 \sum_{k=0}^{\infty} \left(\frac{\lambda}{3\mu}\right)^k \\ &= 1 + \frac{\lambda}{\mu} + \frac{1}{2} \left(\frac{\lambda}{\mu}\right)^2 \frac{1}{1 - \frac{\lambda}{3\mu}} \\ &= 22.25.\end{aligned}$$

Hence, the limiting probabilities are

$$\begin{aligned}P_0 &= \frac{1}{\sum_{k=0}^{\infty} \theta_k} = \frac{4}{89} \\ P_1 &= \theta_1 P_0 = \frac{10}{89} \\ P_k &= \theta_k P_0 = \frac{25}{178} \left(\frac{5}{6}\right)^{k-2}, \quad k \geq 2.\end{aligned}$$

5. A skier can start shooting without waiting if either no, only one or only two shooting stands are busy. Hence the desired probability is given by

$$P_0 + P_1 + P_2 = \frac{53}{178}.$$

6. Use Equations for L_0 and L from page 458, expression 9.1 and below
L- average number of customers in the system
L_0 - average number of customers waiting in the system who are not yet served
s - fixed number of servers (in this case, shooting stands)

$$L_0 = \frac{\pi_0}{s!} (\frac{\lambda}{\mu})^s \frac{(\lambda/s\mu)}{(1 - \lambda/s\mu)^2}$$

$$W_0 = \frac{L_0}{\lambda}$$

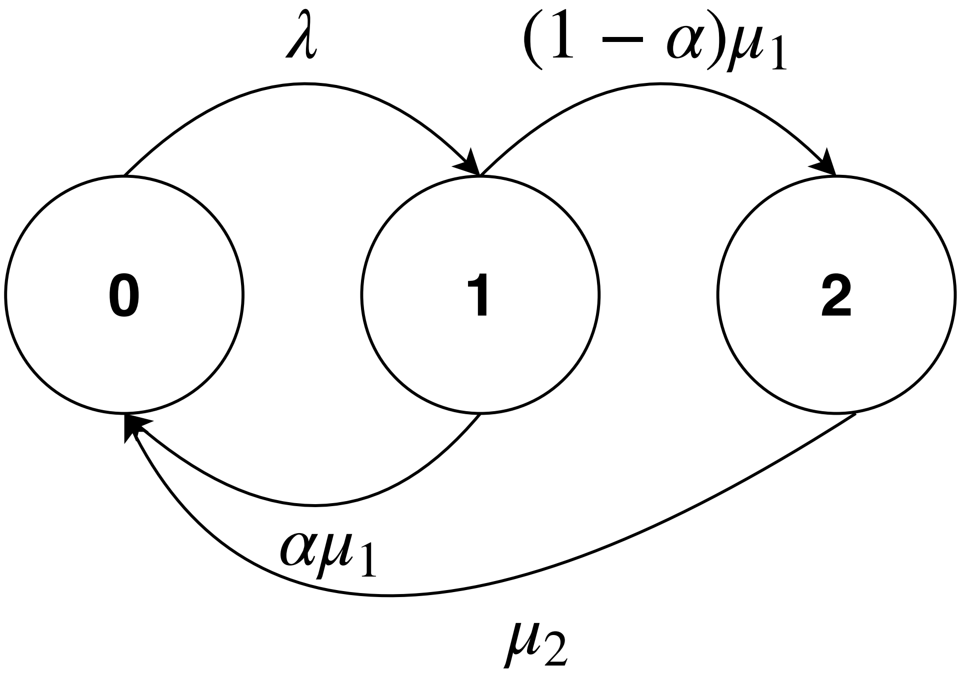
$$W = W_0 + \frac{1}{\mu}$$

$$L = \lambda W = \lambda(W_0 + \frac{1}{\mu}) = L_0 + \frac{\lambda}{\mu}$$

7. From Little's formula we get that the average amount of time W a skier spends in the system is given by

$$W = \frac{L}{\lambda} = \frac{6.011236}{1/12}.$$

Exercise 2. Let us assume we have three states: 0 (guide is free), 1 (guide is negotiating), 2 (guide is on tour). The diagram is



and the balance equations are:

$$\begin{aligned}
\text{leaving rate} &= \text{arrival rate} \\
\lambda P_0 &= \alpha\mu_1 P_1 + \mu_2 P_2 \\
\mu_1 P_1 &= \lambda P_0 \\
\mu_2 P_2 &= (1-\alpha)\mu_1 P_1
\end{aligned}$$

Then we can have P_0, P_1 and P_2

$$P_0 + P_1 + P_2 = 1$$

, then substituting the expressions above, we get:

$$P_0 + \frac{\lambda}{\mu_1} P_0 + \frac{(1-\alpha)\lambda}{\mu_2} P_0 = 1$$

Then finally, the proportion of time that the guide is free is:

$$P_0 = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \lambda\mu_2 + \lambda\mu_1(1-\alpha)}$$