TMA4265 Stochastic Modeling
Week 38: Solutions

Exercise 6.1.1. and 6.1.2. from the book

i) Probabilities through integration:

$$P_0(t) = e^{-\lambda_0 t},$$

$$P_1(t) = \lambda_0 e^{-\lambda_1 t} \int_0^t e^{\lambda_1 x} P_0(x) dx = \frac{1}{2} e^{-3t}(e^{2t} - 1),$$

$$P_2(t) = \lambda_1 e^{-\lambda_2 t} \int_0^t e^{\lambda_2 x} P_1(x) dx = \frac{3}{2}(e^{-t} + e^{-3t} - 2e^{-2t}), \quad \text{(see ch. 6.1 eq. 6.5)}$$

Correspondingly for $P_3(t)$.

ii) $W_3 = S_0 + S_1 + S_2$ and we know that $S_0 \sim \text{Exp}(\lambda_0)$, $S_1 \sim \text{Exp}(\lambda_1)$, $S_2 \sim \text{Exp}(\lambda_2)$.

$$EW_3 = \frac{1}{\lambda_0} + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = \frac{11}{6}. \quad \text{(Sojourn times mutually independent see p. 280)}$$

iii) $E(W_1 + W_2 + W_3) = E(W_1) + E(W_2) + E(W_3) = 3\dfrac{1}{\lambda_0} + 2\dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2}.$

iv) $\text{Var}(W_3) = \dfrac{1}{\lambda_0^2} + \dfrac{1}{\lambda_1^2} + \dfrac{1}{\lambda_2^2}.$

Exercise 2 (Problem 2 - exam 2016)

i) $\lambda = 0.5$.
$N(t)$ = the number of customers in time $(0, t)$. This is Poisson distributed with parameter λt.

$$P(N(15) = 0) = \exp(-\lambda 15),$$
$$E(N(15)) = \lambda 15,$$

$$\begin{aligned} P(N(5) = 0 | N(10) = 2) &= \frac{P(N(5) = 0, N(10) - N(5))}{P(N(10) = 2)} \\ &= \frac{\exp(-5\lambda)\frac{(\lambda 5)^2}{2}\exp(-5\lambda)}{\frac{(\lambda 10)^2}{2}\exp(-10\lambda)} \\ &= 0.25. \end{aligned}$$

The interpretation is that two independent events occur and since they are uniformly distributed within $(0, 10)$, the chance that both occur within $(5, 10)$ is 0.25.

ii) C_i is amount spent by customer i.

$$X = C_1 + C_2 + \ldots + C_{N(t)}.$$

By double expectation:

$$E(X) = E(E(X|N(t) = n)) = E(C)E(N(t)) = 100 \cdot \lambda 60 \cdot 8.$$

By double variance:

$$\begin{aligned}
\mathrm{Var}(X) &= E(\mathrm{Var}(X|N(t) = n)) + \mathrm{Var}(E(X|N(t) = n)) \\
&= \mathrm{Var}(C)E(N(t)) + E(C)^2\mathrm{Var}(N(t)) \\
&= 10^2 \cdot \lambda 60 \cdot 8 + 100^2 \cdot \lambda 60 \cdot 8.
\end{aligned}$$

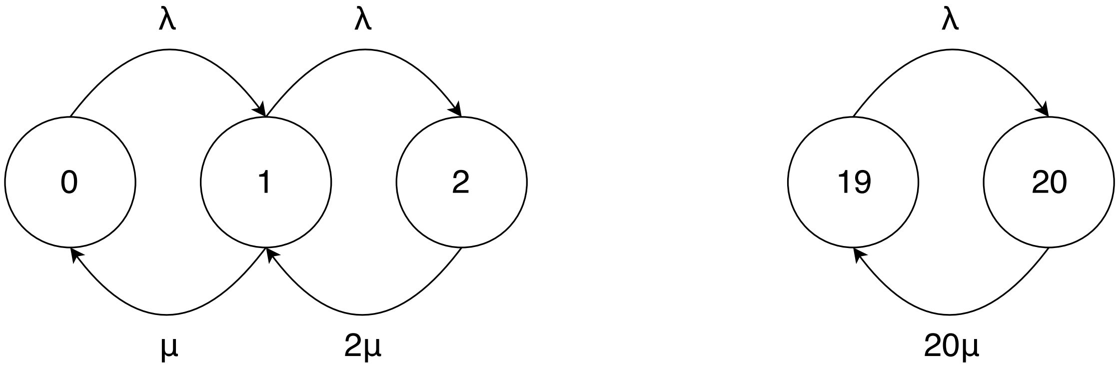
iii) Arrivals are run by the Poisson process

$$\lambda_i = \lambda.$$

Departures are run by the Poisson process, but there are i customers that can leave, so the rate is $i\mu$

$$\mu_i = i\mu.$$

The transition diagram is



Define waiting time $T = T_2 + T_1$. T_2 is exponentially distributed with parameter 2μ, while T_1 is exponentially distributed with parameter μ. The density of T is (see "1.2.5 Sums and convolutions" for this formula and when it is used)

$$f_T(t) = \int_0^t f_{T_2}(s) f_{T_1}(t - s) ds = 2\mu e^{-\mu t} - 2\mu e^{-2\mu t}.$$

iv) Long-run probabilities are $P_i = \lim_{t\to\infty} P(N(t) = i)$. By equating the rates out and rates in we get (see Ch. 6.4)

$$\begin{aligned} P_0\lambda &= P_1\mu \\ P_1(\lambda+\mu) &= P_0\lambda + P_2 2\mu \\ \ldots &= \ldots \\ P_{20}20\mu &= P_{19}\lambda \end{aligned}$$

(Remember expression for the power series of exp(x))

$$P_i = \frac{\nu^i}{i!}P_0, \quad \nu = \lambda/\mu$$

$$P_0 = \frac{1}{\sum_{i=0}^{20}\frac{\nu^i}{i!}} = \frac{1}{\exp(\nu)F_X(20)}$$

This gives us:

$$P_{20} = \frac{f_X(20)}{F_X(20)} = 0.045.$$

$$E(N) = \sum_{i=0}^{20} iP_i = P_0\sum_{i=0}^{20} i\frac{\nu^i}{i!} = P_0\nu\sum_{i=0}^{19}\frac{\nu^i}{i!}$$

$$\sum_{i=0}^{19}\frac{\nu^i}{i!} = \exp(\nu)\sum_{i=0}^{19}\exp(-\nu)\frac{\nu^i}{i!} = \exp(\nu)F_X(19)$$

$$E(N) = \frac{\nu\exp(\nu)F_X(19)}{\exp(\nu)F_X(20)} = \nu\frac{F_X(19)}{F_X(20)} = 15\frac{0.875}{0.917} = 14.31$$

v) Assume today's cost per minute of parking is c_{min}. The expected long-run pay per minute is
For the same income we get:

$$E(N)c_{\text{min}} = E_{\text{new}}(N)2c_{\text{min}}$$

$$E_{\text{new}}(N) = \frac{E(N)}{2}.$$

Here, we let λ^* be the new arrival rate, which is used when computing the left hand side. Since one assumes no capacity at the parking garage, the new probabilities are approximated by

$$P_i = \frac{x^i}{i!}P_0, \quad x = \lambda^*/\mu, \quad i = 0, 1, 2, \ldots$$

$$P_0 = e^{-x}.$$

These are Poisson probabilities with parameter x.

$$E_{\text{new}}(N) = \sum_{i=0}^{\infty} iP_i = \frac{\lambda^*}{\mu}$$

$$\lambda^* = \mu 7.155 = 0.0333 \cdot 7.155 = 0.238$$