$$\frac{}{\Gamma \vdash i : \mathsf{Int}}\ \textsc{T-Int} \qquad \frac{\Gamma \vdash e_1 : \mathsf{Int} \qquad \Gamma \vdash e_2 : \mathsf{Int} \qquad \Box \in \{+,-,*\}}{\Gamma \vdash e_1 \mathbin{\Box} e_2 : \mathsf{Int}}\ \textsc{T-Arith}$$

$$\frac{}{\Gamma \vdash \mathsf{true} : \mathsf{Bool}}\ \textsc{T-True} \qquad \frac{}{\Gamma \vdash \mathsf{false} : \mathsf{Bool}}\ \textsc{T-False}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Bool} \qquad \Gamma \vdash e_2 : \mathsf{T}_1 \qquad \Gamma \vdash e_3 : \mathsf{T}_2 \qquad (\mathsf{T}_1 = \mathsf{T}_2)}{\Gamma \vdash \mathsf{if}\, e_1\, \mathsf{then}\, e_2\, \mathsf{else}\, e_3 : \mathsf{T}_1}\ \textsc{T-If}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{Int} \qquad \Gamma \vdash e_2 : \mathsf{Int} \qquad \Box \in \{<,>,=\}}{\Gamma \vdash e_1 \mathbin{\Box} e_2 : \mathsf{Bool}}\ \textsc{T-Rel}$$

$$\frac{\Gamma(x) = \mathsf{T}}{\Gamma \vdash x : \mathsf{T}}\ \textsc{T-Var} \qquad \frac{\Gamma \vdash e_1 : \mathsf{T}_1 \qquad x : \mathsf{T}_1, \Gamma \vdash e_2 : \mathsf{T}_2}{\Gamma \vdash \mathsf{let}\, x = e_1\, \mathsf{in}\, e_2 : \mathsf{T}_2}\ \textsc{T-Let}$$

$$\frac{x : \mathsf{T}_1, \Gamma \vdash e : \mathsf{T}_2}{\Gamma \vdash (\mathsf{lambda}\, x : \mathsf{T}_1.\ e) : \mathsf{T}_1 \to \mathsf{T}_2}\ \textsc{T-Lambda}$$

$$\frac{\Gamma \vdash e_1 : \mathsf{T}_1 \to \mathsf{T}_2 \qquad \Gamma \vdash e_2 : \mathsf{T}_3 \qquad (\mathsf{T}_1 = \mathsf{T}_3)}{\Gamma \vdash (e_1\ e_2) : \mathsf{T}_2}\ \textsc{T-App}$$

$$\frac{f : \mathsf{T}, \Gamma \vdash e : \mathsf{T}}{\Gamma \vdash (\mathsf{fix}\, f : \mathsf{T}\ \mathsf{is}\ e) : \mathsf{T}}\ \textsc{T-Fix}$$

$$\frac{}{\Gamma \vdash \mathsf{Nil}[\mathsf{T}] : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Nil} \qquad \frac{\Gamma \vdash e_1 : \mathsf{T} \qquad \Gamma \vdash e_2 : \mathsf{List}[\mathsf{T}]}{\Gamma \vdash e_1 :: e_2 : \mathsf{List}[\mathsf{T}]}\ \textsc{T-Cons}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \mathsf{List}[\mathsf{T}_1] \qquad \Gamma \vdash e_2 : \mathsf{T}_2 \\ x : \mathsf{T}_1, y : \mathsf{List}[\mathsf{T}_1], \Gamma \vdash e_3 : \mathsf{T}_3 \qquad (\mathsf{T}_2 = \mathsf{T}_3)\end{array}}{\Gamma \vdash \mathsf{match}\ e_1\ \mathsf{with}\ \mathsf{Nil} \to e_2 \mid x :: y \to e_3\ \mathsf{end} : \mathsf{T}_2}\ \textsc{T-Match}$$

$$\frac{\Gamma \vdash e : \mathsf{T}_2 \qquad \mathsf{T}_1 = \mathsf{T}_2}{\Gamma \vdash (e\,@\,\mathsf{T}_1) : \mathsf{T}_1}\ \textsc{T-Annot}$$

Problem 5 (20 points)

Consider the following λ^+ expression:

```
(lambda n: Int.
    if n = 0 then (lambda k: Int. true) else (lambda m: Int. r (n-1) (m-1))) 7
```

Denote the above expression as e. We claim e is well-typed under the environment $(1) = r : \mathsf{Int} \to \mathsf{Int} \to \mathsf{Bool}$. Below is the skeleton of the derivation tree that shows $(1) \vdash e : \boxed{3}$ for some type $\boxed{3}$. The derivation starts from the bottom of the page with the double bar, and proceeds upwards. Complete the derivation tree by filling in the table on the next page. A $\boxed{\text{solid box}}$ should be filled with an λ^+ expression or type, a $\boxed{\text{dashed box}}$ with the name of a *typing rule*, and a circled number (n) with a typing environment. Note that type equalities of the form $\mathsf{T}_1 = \mathsf{T}_2$ are omitted from the tree, but you should mentally check that they hold.

$$\dfrac{(23)(r) = \mathsf{Int} \to \mathsf{Int} \to \mathsf{Bool}}{(23) \vdash r : \mathsf{Int} \to \mathsf{Int} \to \mathsf{Bool}}\ \text{T-VAR}$$

$$\dfrac{(23)(n) = \mathsf{Int}}{(23) \vdash n : \mathsf{Int}}\ \text{T-VAR} \qquad \dfrac{}{(23) \vdash 1 : \mathsf{Int}}\ \text{T-INT}$$

$$\dfrac{(23) \vdash n : \mathsf{Int} \qquad (23) \vdash 1 : \mathsf{Int}}{(23) \vdash n - 1 : \mathsf{Int}}\ \text{T-ARITH}$$

$$\dfrac{(23) \vdash r : \mathsf{Int} \to \mathsf{Int} \to \mathsf{Bool} \qquad (23) \vdash n - 1 : \mathsf{Int}}{(23) \vdash r\ (n-1) : \boxed{25}\ \text{int->bool}}\ \boxed{26}\ \text{app}$$

$$\dfrac{(23)(m) = \mathsf{Int}}{(23) \vdash m : \mathsf{Int}}\ \text{T-VAR} \qquad \dfrac{}{(23) \vdash 1 : \mathsf{Int}}\ \text{T-INT}$$

$$\dfrac{(23) \vdash m : \mathsf{Int} \qquad (23) \vdash 1 : \mathsf{Int}}{(23) \vdash m - 1 : \mathsf{Int}}\ \text{T-ARITH}$$

$$\dfrac{(23) \vdash r\ (n-1) : \boxed{25} \qquad (23) \vdash m - 1 : \mathsf{Int}}{\text{m,n,r}\ (23) \vdash r\ (n-1)\ (m-1) : \boxed{24}\ \text{bool}}\ \text{T-APP}$$

$$\dfrac{(23) \vdash r\ (n-1)\ (m-1) : \boxed{24}}{(7) \vdash \boxed{20} : \boxed{21}\ \text{int->bool}}\ \boxed{22}\ \text{Lambda}$$

(Resumed)

$\vdots$

$$\dfrac{(7)(n) = \mathsf{Int}}{(7) \vdash n : \mathsf{Int}}\ \text{T-VAR} \qquad \dfrac{}{(7) \vdash 0 : \mathsf{Int}}\ \text{T-INT}$$

$$\dfrac{(7) \vdash n : \mathsf{Int} \qquad (7) \vdash 0 : \mathsf{Int}}{(7) \vdash \underset{\text{n=0}}{\boxed{11}} : \boxed{12}\ \text{bool}}\ \boxed{13}\ \text{Rel}$$

$$\dfrac{\dfrac{}{(17) \vdash \overset{\text{true}}{\boxed{18}} : \mathsf{Bool}}\ \boxed{19}\ \text{true}}{(7) \vdash \boxed{14} : \boxed{15}\ \text{int->bool}}\ \boxed{16}\ \text{Lambda}$$

$\vdots$ (Continued above)

$$\dfrac{}{(7) \vdash \boxed{20} : \boxed{21}}\ \boxed{22}$$

$$\dfrac{(7) \vdash \boxed{11} : \boxed{12} \qquad (7) \vdash \boxed{14} : \boxed{15} \qquad (7) \vdash \boxed{20} : \boxed{21}}{\text{n:int, r:int->int->bool}\ (7) \vdash \boxed{8} : \boxed{9}\ \text{int->bool}}\ \boxed{10}\ \text{If}$$

$$\dfrac{(7) \vdash \boxed{8} : \boxed{9}}{(1) \vdash \boxed{2} : \boxed{5}\ \text{int->int->bool}}\ \boxed{6}\ \text{Lambda} \qquad \dfrac{}{(1) \vdash 7 : \mathsf{Int}}\ \text{T-INT}$$

$$\dfrac{(1) \vdash \boxed{2} : \boxed{5} \qquad (1) \vdash 7 : \mathsf{Int}}{\text{r:int->int->bool}\ (1) \vdash \left(\boxed{2}\ 7\right) : \boxed{3}\ \text{int->bool}}\ \boxed{4}\ \text{app}$$