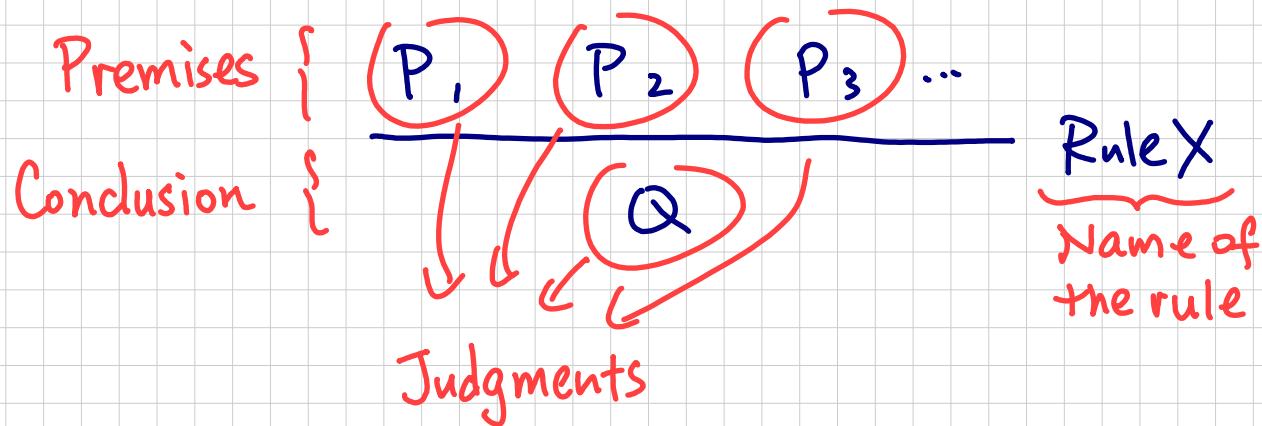


Inference Rules a.k.a. "Horizontal Bar Things"

A proof system may have multiple inference rules.

A rule is a ^(finite) sequence of premises and a conclusion.

Each premise/conclusion is a judgment.



Edge cases

1. A rule with 0 premise is ok.
2. > 1 conclusion is not ok.
3. Premises are usually unordered. I.e. $\frac{P_1 \ P_2}{Q} \cong \frac{P_2 \ P_1}{Q}$.
4. Each rule can be invoked as many times as you wish.
5. A proof system may have ≥ 1 forms of judgment.

Example 1 (Propositional logic)

Judgment: " \Box is true"

\square = A | B | C | ... \Rightarrow Propositional variables

| $\square \wedge \square$ ⇒ Conjunction

Rules :

$$\frac{\square_1 \quad \square_2}{\square_1 \wedge \square_2} \text{ /Intro}$$

$$\frac{\Box_1 \wedge \Box_2}{\Box_1} \wedge \text{Elim L}$$

$$\frac{\Box_1 \wedge \Box_2}{\Box_2} \wedge \text{ElimR}$$

Let's add a ad-hoc rule:

$$\text{No premise!} \quad \frac{}{P \wedge Q} \text{AH}$$

Let's show $Q \wedge P$

is derivable/provable.

Goal

work backward

$$\begin{array}{c}
 \text{work back} \\
 \uparrow \\
 \frac{\frac{P \wedge Q}{Q} \text{AH}}{P \wedge Q} \wedge\text{ElimR} \quad \frac{\frac{P \wedge Q}{P} \text{AH}}{P} \wedge\text{ElimL} \\
 \hline
 Q \wedge P \quad \wedge\text{Intro}
 \end{array}$$

Food for thought

What rules should we add if we introduce " \vee "?

Example 2 (Natural numbers)

Judgment : " \square is a Russian doll"

Abbrev. "□ ✓ "

$$\square = 8 \text{ (atomic)} \quad | \quad \text{a} \square \text{ (wrap)} \quad | \quad \text{a} \text{ (peanut)}$$

Rules :

B ✓ A

$$\frac{\square}{\text{---}} \quad \checkmark$$

W

Let's show : 

Food for thought:

Show " $\emptyset \vee$ " is not provable.

start here
→

The diagram consists of three horizontal blue lines spaced evenly apart. Below the top line is a small, irregular blue shape that looks like a blob or a small bump. To its right is a blue checkmark. Below the middle line is a larger, more complex blue shape that resembles a rounded peak or a small mountain. To its right is a blue checkmark. Below the bottom line is a very large, prominent blue shape that looks like a wide, shallow bowl or a large bump. To its right is a blue checkmark.

Example 3 (Big-step operational semantics for Peano arithmetic)

Judgment : " $\square \Rightarrow \square$ "

$$\square = 0$$

(zero)

$$\square \rightarrowtail \square$$

(successor)

$$\square \otimes \square$$

(addition)

" $\square_1 \Rightarrow \square_2$ provable
means \square_1 evaluates
to \square_2 .
"tensor"

Rules :

$$\frac{}{0 \otimes \square \Rightarrow \square} R\emptyset$$

$$\frac{\square_1 \otimes \square_2 \Rightarrow \square_3}{\square_1 \otimes \square_2 \Rightarrow \square_3} RS$$

Let's prove: $\exists ?$ such that $\square_0 \otimes \square_0 \Rightarrow ?$

Note: we'll discover what $?$ is as we do the proof.

$$\begin{cases} \square_1 = 0, \quad \square_2 = \square_0, \\ \square_3 = ???, \quad ? = \square_{??} \end{cases}$$

$$\begin{cases} \square_1 = \square_0, \quad \square_2 = \square_0, \\ \square_3 = ???, \quad ? = \square_{??} \end{cases}$$

$$\frac{}{0 \otimes \square_0 \Rightarrow ???} R\emptyset$$

$$\frac{\square_0 \otimes \square_0 \Rightarrow ??}{\square_0 \otimes \square_0 \Rightarrow ?} RS$$

} $R\emptyset$

$$\square = \square_0, \quad ??? = \square_0$$

$$\text{Thus, } ? = \square_{??} = \square_{\square_0}$$

$$\text{Or, } 0 + 1 \Rightarrow 3. \quad = \square_{\square_0}$$