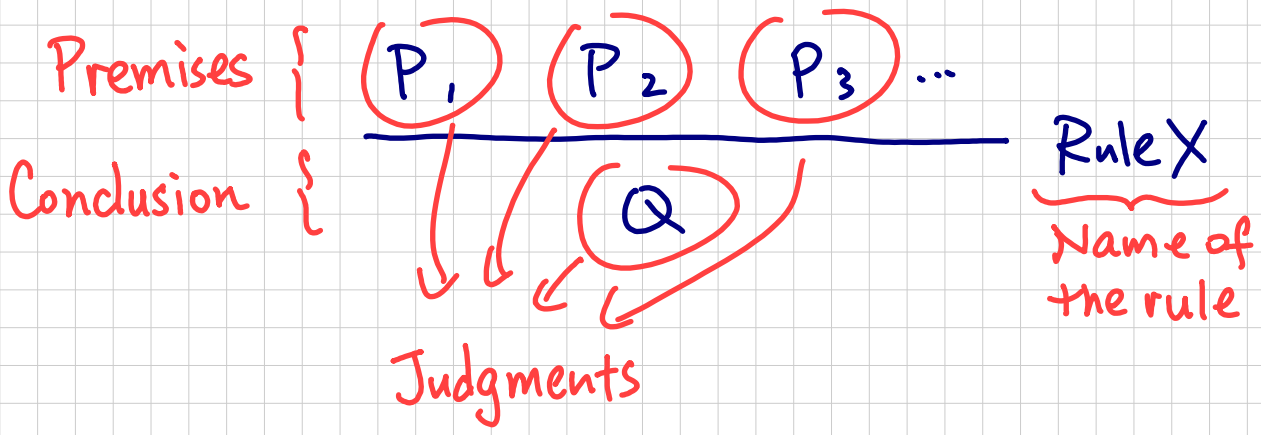
# Inference Rules a.k.a. "Horizontal Bar Things"

A proof system may have multiple inference rules.

A rule is a sequence (finite) of premises and a conclusion.

Each premise/conclusion is a judgment.




Edge cases

1. A rule with 0 premise is ok.
2. >1 conclusion is not ok.
3. Premises are usually unordered. I.e. $\frac{P_1\ P_2}{Q} \cong \frac{P_2\ P_1}{Q}$.
4. Each rule can be invoked as many times as you wish.
5. A proof system may have $\geq 1$ forms of judgment.

## Example 1 (Propositional logic)

Judgment: "$\square$ is true". Abbrev: "$\vdash \square$"

Provable means propositionally valid.

$\square$ = A | B | C | ⋯ ⇒ Propositional variables

| $\square \wedge \square$ ⇒ Conjuction

Rules:

$$\frac{\vdash \square_1 \quad \vdash \square_2}{\vdash \square_1 \wedge \square_2}\ \wedge\text{Intro} \qquad \frac{\vdash \square_1 \wedge \square_2}{\vdash \square_1}\ \wedge\text{ElimL} \qquad \frac{\vdash \square_1 \wedge \square_2}{\vdash \square_2}\ \wedge\text{ElimR}$$

Let's add an ad-hoc rule:

No premise! $\dfrac{}{\vdash P \wedge Q}$ AH

Let's show $\vdash Q \wedge P$ is derivable/provable.

work backward

$$\dfrac{\dfrac{\dfrac{}{\vdash P \wedge Q}\,\text{AH}}{\vdash Q}\,\wedge\text{ElimR} \qquad \dfrac{\dfrac{}{\vdash P \wedge Q}\,\text{AH}}{\vdash P}\,\wedge\text{ElimL}}{\vdash Q \wedge P}\,\wedge\text{Intro}$$

Goal →

### Food for thought

What rules should we add if we introduce "$\vee$"?

## Example 2 (Natural numbers)

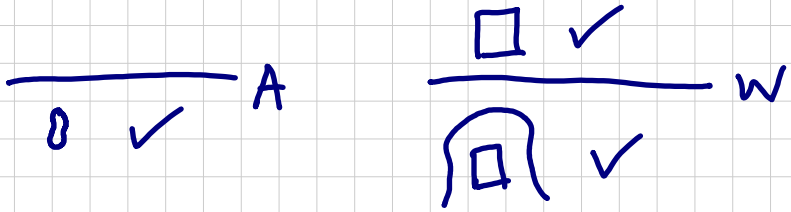
"□ ✓" provable = □ is a natural number

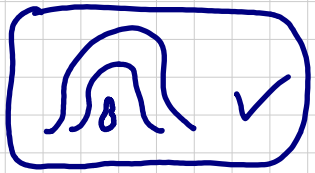
Judgment: "□ is a Russian doll"

Abbrev. "□ ✓"

□ = 


Rules:

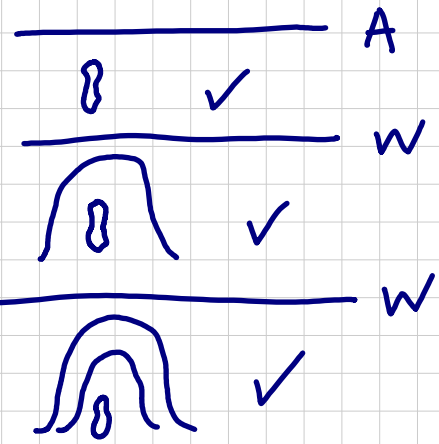



Let's show:  is provable

**Food for thought:**

Show "  " is not provable.

start here →



## Example 3 (Operational semantics for Peano arithmetic)

"$\square_1 \Rightarrow \square_2$" provable means $\square_1$ evaluates to $\square_2$.

Judgment: "$\square \Rightarrow \square$"

$\square$ = 0 (zero) | S$\square$ (successor) | $\square \otimes \square$ ("tensor") (addition)

Rules:

$$\frac{}{0 \otimes \square \Rightarrow \square}\ R0 \qquad \frac{\square_1 \otimes \square_2 \Rightarrow \square_3}{S\square_1 \otimes \square_2 \Rightarrow S\square_3}\ RS$$

Let's prove: $\exists ?$ such that $S0 \otimes S0 \Rightarrow ?$

Note: we'll discover what ? is as we do the proof.

$\square_1 = 0$, $\square_2 = S0$, $\square_3 = ???$, $?? = S???$

$$\frac{}{0 \otimes S0 \Rightarrow S???}\ R0 \qquad \frac{0 \otimes S0 \Rightarrow S???}{S0 \otimes S0 \Rightarrow SS???}\ RS$$

$\square = S0$, $??? = S0$

$\square_1 = S0$, $\square_2 = S0$, $\square_3 = ??$, $? = S??$

Thus, $? = S?? = SSS???$

$= SS0$

Or, $2 + 1 \Rightarrow 3$.