Inference Rules a.k.a. "Horizontal Bar Things"

A <u>proof system</u> may have multiple <u>inference rules</u>.

A rule is a (finite) sequence of <u>premises</u> and a <u>conclusion</u>.

Each premise/conclusion is a <u>judgment</u>.

Premises {

$$\frac{P_1 \quad P_2 \quad P_3 \quad \cdots}{Q}\ \underbrace{\text{Rule X}}_{\text{Name of the rule}}$$

Conclusion {

Judgments

<u>Edge cases</u>

1. A rule with <u>0 premise</u> is ok.
2. >1 conclusion is <u>not</u> ok.
3. Premises are usually <u>unordered</u>. I.e. $\dfrac{P_1 \; P_2}{Q} \cong \dfrac{P_2 \quad P_1}{Q}$.
4. Each rule can be invoked <u>as many times</u> as you wish.
5. A proof system may have <u>≥ 1 forms of judgment</u>.

Example 1 (Propositional logic)

Provable means propositionally valid.

Judgment: "$\square$ is true"

$\square = A \mid B \mid C \mid \cdots$ ⇒ Propositional variables

$\quad \mid \square \wedge \square$ ⇒ Conjuction

Rules:

$$\frac{\square_1 \qquad \square_2}{\square_1 \wedge \square_2} \wedge\text{Intro} \qquad \frac{\square_1 \wedge \square_2}{\square_1} \wedge\text{ElimL} \qquad \frac{\square_1 \wedge \square_2}{\square_2} \wedge\text{ElimR}$$

Food for thought

What rules should we add if we introduce "$\vee$"?

Let's add a ad-hoc rule:

No premise! $\dfrac{}{P \wedge Q}$ AH

Let's show $Q \wedge P$ is derivable/provable.

work backward

Goal →

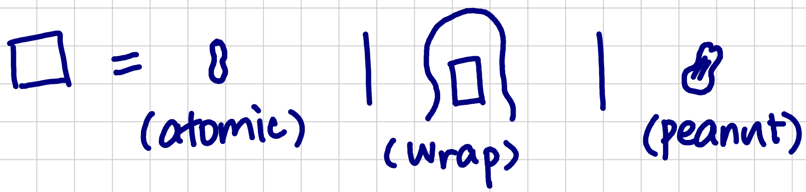
$$\dfrac{\dfrac{\dfrac{}{P \wedge Q}\text{AH}}{Q}\wedge\text{ElimR} \qquad \dfrac{\dfrac{}{P \wedge Q}\text{AH}}{P}\wedge\text{ElimL}}{Q \wedge P}\wedge\text{Intro}$$

Example 2 (Natural numbers)

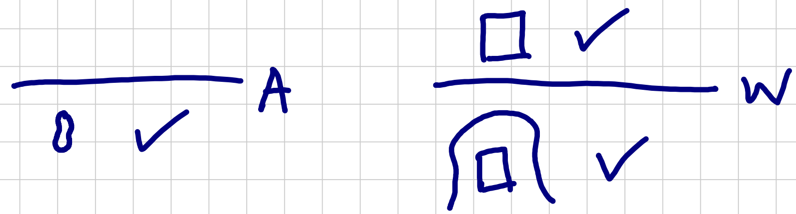
"$\square$ ✓" provable = $\square$ is a natural number

Judgment : "$\square$ is a Russian doll"

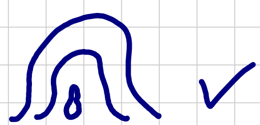
Abbrev. "$\square$ ✓"

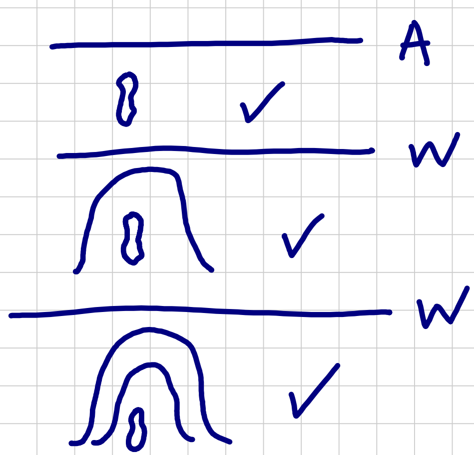


Rules :



Let's show :





start here →

Food for thought :

Show " ✓" is not provable.

"$\square_1 \Rightarrow \square_2$" provable means $\square_1$ evaluates to $\square_2$.

Example 3 (Big-step operational semantics for Peano arithmetic)

Judgment : "$\square \Rightarrow \square$"

$$\square \;=\; 0 \quad (\text{zero}) \quad\Big|\quad \overset{\frown}{\square} \quad (\text{successor}) \quad\Big|\quad \square \overset{\text{"tensor"}}{\otimes} \square \quad (\text{addition})$$

Rules :

$$\frac{}{0 \otimes \square \Rightarrow \square}\,R0 \qquad\qquad \frac{\square_1 \otimes \square_2 \Rightarrow \square_3}{\overset{\frown}{\square_1} \otimes \square_2 \Rightarrow \overset{\frown}{\square_3}}\,RS$$

Let's prove: $\exists\, ?$ such that $\overset{\frown}{\overset{\frown}{0}} \otimes \overset{\frown}{0} \Rightarrow ?$

Note: we'll discover what $?$ is as we do the proof.

$$\dfrac{\dfrac{\dfrac{}{0 \otimes \overset{\frown}{0} \Rightarrow ???}\,R0}{\overset{\frown}{0} \otimes \overset{\frown}{0} \Rightarrow ??}\,RS}{\overset{\frown}{\overset{\frown}{0}} \otimes \overset{\frown}{0} \Rightarrow ?}\,RS$$

R0: $\square = \overset{\frown}{0}$, $??? = \overset{\frown}{0}$

RS: $\square_1 = 0$, $\square_2 = \overset{\frown}{0}$, $\square_3 = ???$, $?? = \overset{\frown}{???}$

RS: $\square_1 = \overset{\frown}{0}$, $\square_2 = \overset{\frown}{0}$, $\square_3 = ??$, $? = \overset{\frown}{??}$

Thus, $? = \overset{\frown}{??} = \overset{\frown}{\overset{\frown}{???}} = \overset{\frown}{\overset{\frown}{\overset{\frown}{0}}}$

Or, $2 + 1 \Rightarrow 3$.