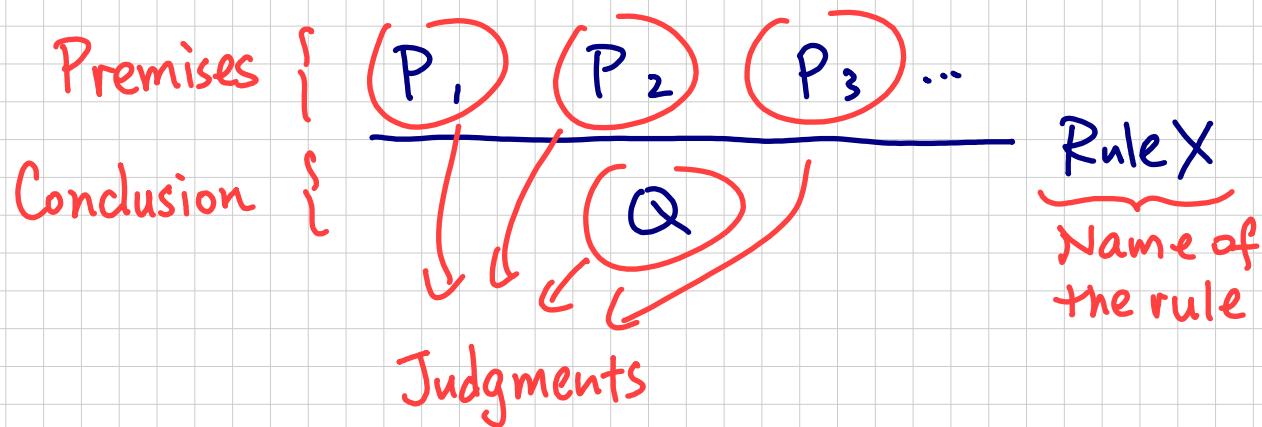


Inference Rules a.k.a. "Horizontal Bar Things"

A proof system may have multiple inference rules.

A rule is a ^(finite) sequence of premises and a conclusion.

Each premise/conclusion is a judgment.



Edge cases

1. A rule with 0 premise is ok.
2. > 1 conclusion is not ok.
3. Premises are usually unordered. I.e. $\frac{P_1 \ P_2}{Q} \cong \frac{P_2 \ P_1}{Q}$.
4. Each rule can be invoked as many times as you wish.
5. A proof system may have ≥ 1 forms of judgment.

Example 1 (Propositional logic)

Judgment: " \square is true". Abbrev: " $\vdash \square$ "

{ Provable means propositionally valid.

$\square = A \mid B \mid C \mid \dots \Rightarrow$ Propositional variables

$\mid \quad \square \wedge \square \Rightarrow$ Conjunction

Rules :

$$\frac{\vdash \square_1, \vdash \square_2}{\vdash \square_1 \wedge \square_2} \wedge \text{Intro}$$

$$\frac{\vdash \square_1, \wedge \square_2}{\vdash \square_1} \wedge \text{ElimL}$$

$$\frac{\vdash \square_1, \wedge \square_2}{\vdash \square_2} \wedge \text{ElimR}$$

Let's add an ad-hoc rule:

No premise!

$$\boxed{\frac{}{\vdash P \wedge Q}} \text{AH}$$

$$\boxed{\vdash Q \wedge P}$$

is derivable/provable.

Goal

Food for thought

What rules should we add if we introduce " \vee "?

work backward

$$\begin{array}{c} \vdash P \wedge Q \xrightarrow{\text{AH}} \vdash Q \\ \vdash P \wedge Q \xrightarrow{\wedge \text{ElimR}} \vdash Q \\ \vdash P \wedge Q \xrightarrow{\wedge \text{ElimL}} \vdash P \\ \vdash Q \wedge P \xrightarrow{\wedge \text{Intro}} \text{Goal} \end{array}$$

Example 2 (Natural numbers)

Judgment : " \square is a Russian doll "

Abbrev. "□ ✓ "

$$\square = 8 \text{ (atomic)} \quad | \quad \text{a} \square \text{ (wrap)} \quad | \quad \text{a} \text{ (peanut)}$$

Rules :

B ✓ A

$$\frac{\square}{\text{---}} \quad \checkmark$$

W

Let's show :  is provable

A handwritten note on lined paper. It features a large blue circle containing a question mark. To the left of the circle, there is a blue checkmark and the word "U:".

Food for thought :

Show " $\emptyset \checkmark$ " is not provable.

A

8 ✓

✓

A hand-drawn diagram on a grid background. A horizontal blue line at the top represents a reference level. Below it, a blue wavy line represents a wave packet. The wave packet has two main peaks. To the right of the diagram, there is a blue checkmark and the letter 'W'.

start here
→

$\{ \text{ " } \square \checkmark \text{ " provable}$
 $= \quad \square \text{ is a}$
 natural number

Example 3 (Operational semantics for Peano arithmetic)

Judgment : " $\square \Rightarrow \square$ "

$$\square = 0$$

(zero)

$$\square \rightarrowtail \square$$

(successor)

$$\square \otimes \square$$

(addition)

$\square_1 \Rightarrow \square_2$ provable
means \square_1 evaluates
to \square_2 .
"tensor"

Rules :

$$\frac{}{0 \otimes \square \Rightarrow \square} R\emptyset$$

$$\frac{\square_1 \otimes \square_2 \Rightarrow \square_3}{\square_1 \otimes \square_2 \Rightarrow \square_3} RS$$

Let's prove: $\exists ?$ such that $\square_1 \otimes \square_2 \Rightarrow ?$

Note: we'll discover what $?$ is as we do the proof.

$$\begin{cases} \square_1 = 0, \quad \square_2 = \square_1, \\ \square_3 = ??, \quad ?? = \square_3 \end{cases}$$

$$\frac{}{0 \otimes \square \Rightarrow ??} R\emptyset$$

$$\begin{cases} \square_1 = \square_1, \quad \square_2 = \square_1, \\ \square_3 = ??, \quad ?? = \square_3 \end{cases}$$

$$\frac{\square_1 \otimes \square_2 \Rightarrow ??}{\square_1 \otimes \square_2 \Rightarrow ?} RS$$

$\square = \square_1, \quad ?? = \square_1$
Thus, $? = \square_1 = \square$
Or, $2 + 1 \Rightarrow 3$.