The following is a review of the Quantitative Methods principles designed to address the learning outcome statements set forth by CFA Institute®. This topic is also covered in:

THE TIME VALUE OF MONEY

Study Session 2

EXAM FOCUS

This topic review covers time value of money concepts and applications. Procedures are presented for calculating the future value and present value of a single cash flow, an annuity, and a series of uneven cash flows. The impact of different compounding periods is examined, along with the procedures for solving for other variables in time value of money problems. Your main objective in this chapter is to master time value of money mechanics (i.e., learn how to crunch the numbers). There will be a lot of time value of money problems and applications on the exam, so be prepared to deal with them. Work all the questions and problems found at the end of this review. Make sure you know how to grind out all the time value of money problems on your calculator. The more rapidly you can do them (correctly), the more time you will have for the less predictable parts of the exam.

TIME VALUE OF MONEY CONCEPTS AND APPLICATIONS

In order for money to have time value, it must be possible to invest it at a positive rate of return. The rate of return (interest rate) that provides money with time value is composed of three components:

- **Risk-free rate.** This is the rate that is earned on a riskless investment, and it represents the compensation that investors require to defer current consumption. The rate on short-term U.S. Treasury securities is typically used to represent the risk-free rate.
- **Inflation premium.** This is the return that investors require to compensate them for the change of purchasing power over an investment horizon.
- **Risk premium.** This is the compensation that investors require for being exposed to various types of investment risk.

The concept of **compound interest** or **interest on interest** is deeply embedded in time value of money (TVM) procedures. When an investment is subjected to compound interest, the growth in the value of the investment from period to period reflects not only the interest earned on the original principal amount but also on the interest earned on the previous period's interest earnings—the interest on interest.

TVM applications frequently call for determining the **future value** (FV) of an investment's cash flows as a result of the effects of compound interest. Computing FV involves projecting the cash flows forward, on the basis of an appropriate compound interest rate, to the end of the investment's life. The computation of the **present value** (PV) works in the opposite direction—it brings the cash flows from an investment back to the beginning of the investment's life based on an appropriate compound rate of return.

Being able to measure the PV and/or FV of an investment's cash flows becomes useful when comparing investment alternatives because the value of the investment's cash flows must be measured at some common point in time, typically at the end of the investment horizon (FV) or at the beginning of the investment horizon (PV).

Using a Financial Calculator

It is very important that you be able to use a financial calculator when working TVM problems because the exam is constructed under the assumption that candidates have the ability to do so. There is simply no other way that you will have time to solve TVM problems. *CFA Institute allows only two types of calculators to be used for the exam—the TI BAII Plus® (including the BAII Plus Professional) and the HP 12C® (including the HP 12C Platinum). This topic review is written primarily with the TI BAII Plus in mind.* If you don't already own a calculator, go out and buy a *TI BAII Plus!* However, if you already own the HP 12C and are comfortable with it, by all means continue to use it.

The TI BAII Plus comes preloaded from the factory with the periods per year function (P/Y) set to 12. This automatically converts the annual interest rate (I/Y) into monthly rates. While appropriate for many loan-type problems, this feature is not suitable for the vast majority of the TVM applications we will be studying. So prior to using our Study Notes, please set your P/Y key to "1" using the following sequence of keystrokes:

[2nd] [P/Y] "1" [ENTER] [2nd] [QUIT]

As long as you do not change the P/Y setting, it will remain set at one period per year until the battery from your calculator is removed (it does not change when you turn the calculator on and off). If you want to check this setting at any time, press [2nd] [P/Y]. The display should read P/Y = 1.0. If it does, press [2nd] [QUIT] to get out of the "programming" mode. If it doesn't, repeat the procedure previously described to set the P/Y key. With P/Y set to equal 1, it is now possible to think of I/Y as the interest rate per compounding period and N as the number of compounding periods under analysis. Thinking of these keys in this way should help you keep things straight as we work through TVM problems.

Before we begin working with financial calculators, you should familiarize yourself with your TI by locating the TVM keys noted below. These are the only keys you need to know to work virtually all TVM problems.

- N = Number of compounding periods.
- I/Y = Interest rate per compounding period.
- PV = Present value.
- FV = Future value.
- PMT = Annuity payments, or constant periodic cash flow.
- CPT = Compute.

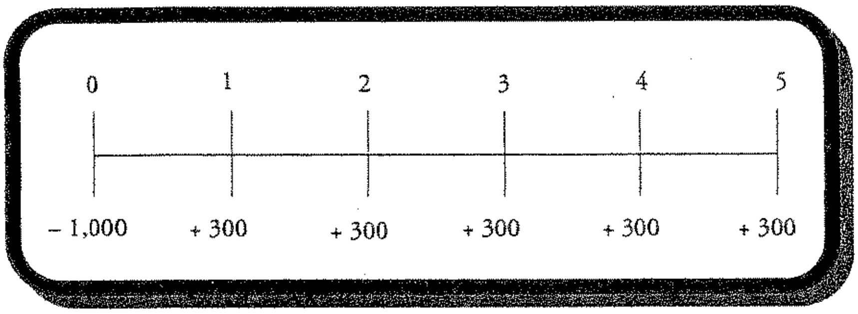
Professor's Note: We have provided a short online video in the Schweser Library on how to use your calculator. You can view it by logging in at www.schweser.com.

Time Lines

It is often a good idea to draw a time line before you start to solve a TVM problem. A **time line** is simply a diagram of the cash flows associated with a TVM problem. A cash flow that occurs in the present (today) is put at time 0. Cash outflows (payments) are given a negative sign, and cash inflows (receipts) are given a positive sign. Once the cash flows are assigned to a time line, they may be moved to the beginning of the investment period to calculate the PV through a process called **discounting** or to the end of the period to calculate the FV using a process called **compounding**.

Figure 1 illustrates a time line for an investment that costs $1,000 today (outflow) and will return a stream of cash payments (inflows) of $300 per year at the end of each of the next five years.

Figure 1: Time Line



Please recognize that the cash flows occur at the end of the period depicted on the time line. Furthermore, note that the end of one period is the same as the beginning of the next period. For example, the end of t = 2 is the same as the beginning of t = 3, but the beginning of year 3 cash flow appears at time t = 2 on the time line. Keeping this convention in mind will help you keep things straight when you are setting up TVM problems.

Professor's Note: Throughout the problems in this review, rounding differences may occur between the use of different calculators or techniques presented in this document. So don't panic if you are a few cents off in your calculations.

LOS 5.a: Explain an interest rate as the sum of a real risk-free rate, expected inflation, and premiums that compensate investors for distinct types of risk.

The **real risk-free rate** of interest is a theoretical rate on a single period loan that has no expectation of inflation in it. When we speak of a real rate of return, we are referring to an investor's increase in purchasing power (after adjusting for inflation). Since expected inflation in future periods is not zero, the rates we observe on U.S. Treasury bills (T-bills), for example, are risk-free rates but not *real* rates of return. T-bill rates are *nominal risk-free rates* because they contain an *inflation premium*. The approximate relation here is:

nominal risk-free rate = real risk-free rate + expected inflation rate

Securities may have one or more **types of risk**, and each added risk increases the required rate of return on the security. These types of risk are:

- **Default risk.** The risk that a borrower will not make the promised payments in a timely manner.
- **Liquidity risk.** The risk of receiving less than fair value for an investment if it must be sold for cash quickly.
- **Maturity risk.** As we will cover in detail in the section on debt securities, the prices of longer-term bonds are more volatile than those of shorter-term bonds. Longer maturity bonds have more maturity risk than shorter-term bonds and require a maturity risk premium.

Each of these risk factors is associated with a risk premium that we add to the nominal risk-free rate to adjust for greater default risk, less liquidity, and longer maturity relative to a very liquid, short-term, default risk-free rate such as that on T-bills. We can write:

required interest rate on a security = nominal risk-free rate
+ default risk premium
+ liquidity premium
+ maturity risk premium

LOS 5.b: Calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding, and solve time value of money problems when compounding periods are other than annual.

Financial institutions usually quote rates as **stated annual interest rates**, or **nominal rates**, along with a compounding frequency, as opposed to quoting rates as **periodic rates**—the rate of interest earned over a single compounding period. For example, a bank will quote a savings rate as 8%, compounded quarterly, rather than 2% per quarter. The rate of interest that investors actually realize as a result of compounding is known as the **effective annual rate** (EAR). EAR represents the annual rate of return actually being earned *after adjustments have been made for different compounding periods.*

EAR may be determined as follows:

$$\text{EAR} = (1 + \text{periodic rate})^m - 1$$

where:
periodic rate = nominal rate/m
m = the number of compounding periods per year

Obviously, the EAR for a stated rate of 8% *compounded annually* is not the same as the EAR for 8% *compounded semiannually,* or *quarterly.* Indeed, whenever compound interest is being used, the stated (nominal) rate and the actual (effective) rate of interest are equal only when interest is compounded annually. Otherwise, the greater the compounding frequency, the greater the EAR will be in comparison to the stated rate.

The computation of EAR is necessary when comparing investments that have different compounding periods. It allows for an apples-to-apples rate comparison.

Example: Computing EAR

Compute EAR if the nominal (stated) rate is 12%, compounded quarterly.

Answer:

Here m = 4, so the periodic rate is $\frac{12}{4} = 3\%$.

Thus, EAR = $(1 + 0.03)^4 - 1 = 1.1255 - 1 = 0.1255 = 12.55\%$.

This solution uses the [y^x] key on your financial calculator. The exact keystrokes on the TI for the above computation are 1.03 [y^x] 4 [=]. On the HP, the strokes are 1.03 [ENTER] 4 [y^x].

Example: Computing EARs for a range of compounding frequencies

Using a stated rate of 6%, compute EARs for semiannual, quarterly, monthly, and daily compounding.

Answer:

EAR with:

semiannual compounding	=	$(1 + 0.03)^2 - 1$	=	$1.06090 - 1 = 0.06090$	=	6.090%
quarterly compounding	=	$(1 + 0.015)^4 - 1$	=	$1.06136 - 1 = 0.06136$	=	6.136%
monthly compounding	=	$(1 + 0.005)^{12} - 1$	=	$1.06168 - 1 = 0.06168$	=	6.168%
daily compounding	=	$(1 + 0.00016438)^{365} - 1$	=	$1.06183 - 1 = 0.06183$	=	6.183%

Notice here that the EAR increases as the compounding frequency increases.

The limit of shorter and shorter compounding periods is called continuous compounding. To convert an annual stated rate to the EAR with continuous compounding, we use the formula $e^r - 1 = \text{EAR}$.

For 6%, we have $e^{0.06} - 1 = 6.1837\%$. The keystrokes are 0.06 [2nd] [e^x] [–] 1 [=] 0.061837.

LOS 5.c: Calculate and interpret the FV and PV of a single sum of money, ordinary annuity, a perpetuity (PV only), an annuity due, or a series of uneven cash flows.

Future Value of a Single Sum

Future value is the amount to which a current deposit will grow over time when it is placed in an account paying compound interest. The FV, also called the compound value, is simply an example of compound interest at work.

The formula for the FV of a *single* cash flow is:

$$FV = PV(1 + I/Y)^N$$

where:
PV = amount of money invested today (the present value)
I/Y = rate of return per compounding period
N = total number of compounding periods

In this expression, the investment involves a single cash outflow, PV, which occurs today, at $t = 0$ on the time line. The single sum FV formula will determine the value of an investment at the end of N compounding periods, given that it can earn a fully compounded rate of return, I/Y, over all of the periods.

The factor $(1 + I/Y)^N$ represents the compounding rate on an investment and is frequently referred to as the **future value factor**, or the **future value interest factor**, for a single cash flow at I/Y over N compounding periods. These are the values that appear in interest factor tables, which we will not be using.

Example: FV of a single sum

Calculate the FV of a $300 investment at the end of 10 years if it earns an annually compounded rate of return of 8%.

Answer:

To solve this problem with your calculator, input the relevant data and compute FV.

N = 10; I/Y = 8; PV = –300; CPT → FV = $647.68

Professor's Note: Note the negative sign on PV. This is not necessary, but it makes the FV come out as a positive number. If you enter PV as a positive number, ignore the negative sign that appears on the FV.

This relatively simple problem could also be solved using the following equation.

$$FV = 300(1 + 0.08)^{10} = \$647.68$$

On the TI calculator, enter 1.08 [y^x] 10 [×] 300 [=].

Present Value of a Single Sum

The PV of a single sum is today's value of a cash flow that is to be received at some point in the future. In other words, it is the amount of money that must be invested today, at a given rate of return over a given period of time, in order to end up with a specified FV. As previously mentioned, the process for finding the PV of a cash flow is known as *discounting* (i.e., future cash flows are "discounted" back to the present). The interest rate used in the discounting process is commonly referred to as the **discount rate** but may also be referred to as the **opportunity cost, required rate of return,** and the **cost of capital.** Whatever you want to call it, it represents the annual compound rate of return that can be earned on an investment.

The relationship between PV and FV can be seen by examining the FV expression stated earlier. Rewriting the FV equation in terms of PV, we get:

$$PV = FV \times \left[\frac{1}{(1+I/Y)^N}\right] = \frac{FV}{(1+I/Y)^N}$$

Note that for a single future cash flow, PV is always less than the FV whenever the discount rate is positive.

The quantity $1/(1 + I/Y)^N$ in the PV equation is frequently referred to as the **present value factor, present value interest factor,** or **discount factor** for a single cash flow at I/Y over N compounding periods.

Example: PV of a single sum

Given a discount rate of 9%, calculate the PV of a $1,000 cash flow that will be received in five years.

Answer:

To solve this problem, input the relevant data and compute PV.

N = 5; I/Y = 9; FV = 1,000; CPT → PV = –$649.93 (ignore the sign)

Professor's Note: With single sum PV problems, you can either enter FV as a positive number and ignore the negative sign on PV or enter FV as a negative number.

This relatively simple problem could also be solved using the following PV equation.

$$PV = \frac{1{,}000}{(1+0.09)^5} = \$649.93$$

On the TI, enter 1.09 [y^x] 5 [=] [1/x] [×] 1,000 [=].

The PV computed here implies that at a rate of 9%, an investor will be indifferent between $1,000 in five years and $649.93 today. Put another way, $649.93 is the amount that must be invested today at a 9% rate of return in order to generate a cash flow of $1,000 at the end of five years.

Annuities

An **annuity** is a stream of *equal cash flows* that occurs at *equal intervals* over a given period. Receiving $1,000 per year at the end of each of the next eight years is an example of an annuity. There are two types of annuities: **ordinary annuities** and **annuities due**. The *ordinary annuity* is the most common type of annuity. It is characterized by cash flows that occur at the *end* of each compounding period. This is a typical cash flow pattern for many investment and business finance applications. The other type of annuity is called an ***annuity due***, where payments or receipts occur at the beginning of each period (i.e., the first payment is today at $t = 0$).

Computing the FV or PV of an annuity with your calculator is no more difficult than it is for a single cash flow. You will know four of the five relevant variables and solve for the fifth (either PV or FV). The difference between single sum and annuity TVM problems is that instead of solving for the PV or FV of a single cash flow, we solve for the PV or FV of a stream of equal periodic cash flows, where the size of the periodic cash flow is defined by the payment (PMT) variable on your calculator.

Example: FV of an ordinary annuity

What is the future value of an ordinary annuity that pays $150 per year at the end of each of the next 15 years, given the investment is expected to earn a 7% rate of return?

Answer:

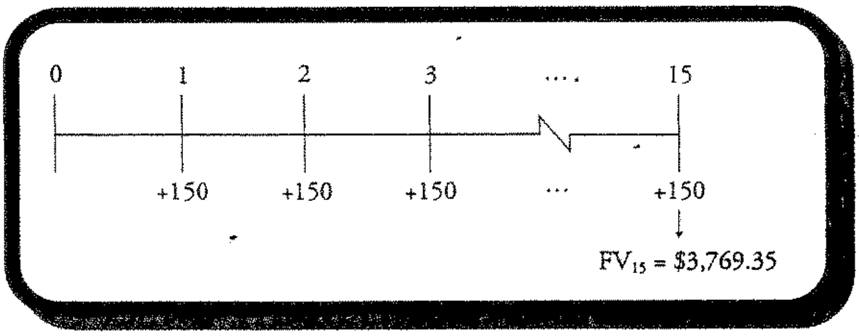
This problem can be solved by entering the relevant data and computing FV.

N = 15; I/Y = 7; PMT = –150; CPT → FV = $3,769.35

Implicit here is that PV = 0, clearing the TVM functions sets.

The time line for the cash flows in this problem is depicted in Figure 2.

Figure 2: FV of an Ordinary Annuity



As indicated here, the sum of the compounded values of the individual cash flows in this 15-year ordinary annuity is $3,769.35. Note that the annuity payments themselves amounted to $2,250 = 15 × $150, and the balance is the interest earned at the rate of 7% per year.

To find the PV of an ordinary annuity, we use the future cash flow stream, PMT, that we used with FV annuity problems, but we discount the cash flows back to the present (time = 0) rather than compounding them forward to the terminal date of the annuity.

Here again, the PMT variable is a *single* periodic payment, ***not*** the total of all the payments (or deposits) in the annuity. The PVA_O measures the collective PV of a stream of equal cash flows received at the end of each compounding period over a stated number of periods, N, given a specified rate of return, I/Y. The following examples illustrate how to determine the PV of an ordinary annuity using a financial calculator.

Example: PV of an ordinary annuity

What is the PV of an annuity that pays $200 per year at the end of each of the next 13 years given a 6% discount rate?

Answer:

The payments occur at the end of the year, so this annuity is an ordinary annuity. To solve this problem, enter the relevant information and compute PV.

N = 13; I/Y = 6; PMT = –200; CPT → PV = $1,770.54

The $1,770.54 computed here represents the amount of money that an investor would need to invest *today* at a 6% rate of return to generate 13 end-of-year cash flows of $200 each.

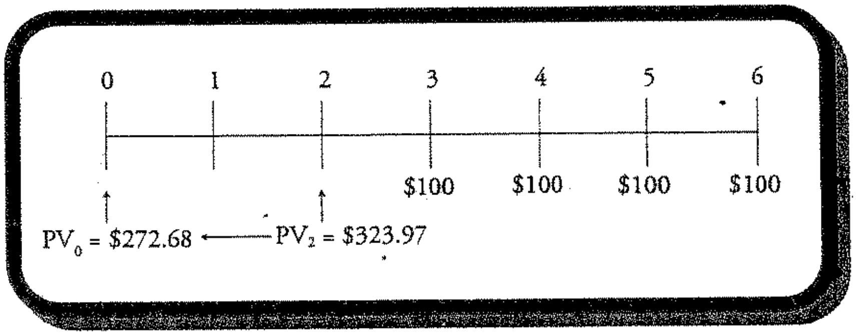
Example: PV of an ordinary annuity beginning later than t = 1

What is the present value of four $100 end-of-year payments if the first payment is to be received three years from today and the appropriate rate of return is 9%?

Answer:

The time line for this cash flow stream is shown in Figure 3.

Figure 3: PV of an Annuity Beginning at t = 3



Step 1: *Find the present value of the annuity as of the end of year 2 (PV_2).*

Input the relevant data and solve for PV_2.

N = 4; I/Y = 9; PMT = –100; CPT → PV = PV_2 = $323.97

Step 2: *Find the present value of PV_2.*

Input the relevant data and solve for PV_0.

N = 2; I/Y = 9; PMT = 0; FV = –323.97; CPT → PV = PV_0 = $272.68

In this solution, the annuity was treated as an ordinary annuity. The PV was computed one period before the first payment, and we discounted PV_2 = $323.97 over two years. We need to stress this important point. The PV annuity function on your calculator set in "END" mode gives you the value one period before the annuity begins. Although the annuity begins at t = 3, we discounted the result for only two periods to get the present (t = 0) value.

Future Value of an Annuity Due

Sometimes it is necessary to find the *FV of an annuity due* (FVA_D), an annuity where the annuity payments (or deposits) occur at the beginning of each compounding period. Fortunately, our financial calculators can be used to do this, but with one slight modification—the calculator must be set to the beginning-of-period (BGN) mode.

To switch between the BGN and END modes on the TI, press [2nd] [BGN] [2nd] [SET]. When this is done, "BGN" will appear in the upper right corner of the display window. If the display indicates the desired mode, press [2nd] [QUIT]. You will normally want your calculator to be in the ordinary annuity (END) mode, so remember to switch out of BGN mode after working annuity due problems. Note that nothing appears in the upper right corner of the display window when the TI is set to the END mode. It should be mentioned that while annuity due payments are made or received at the beginning of each period, the FV of an annuity due is calculated as of the end of the last period.

Another way to compute the FV of an annuity due is to calculate the FV of an ordinary annuity, and simply multiply the resulting FV by (1 + periodic compounding rate (I/Y)). Symbolically, this can be expressed as:

$$FVA_D = FVA_O \times (1 + I/Y)$$

The following examples illustrate how to compute the FV of an annuity due.

Example: FV of an annuity due

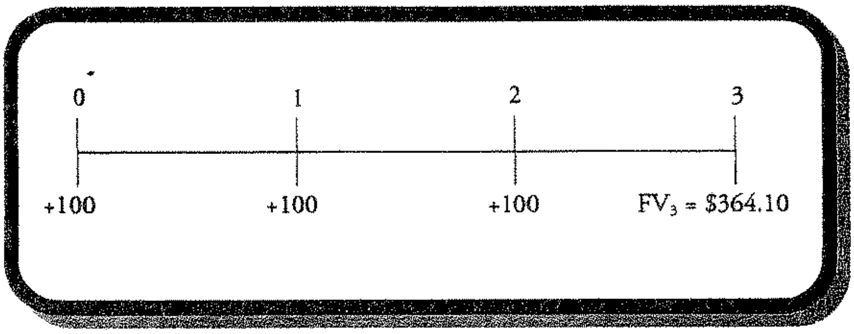
What is the future value of an annuity that pays $100 per year at the beginning of each of the next three years, commencing today, if the cash flows can be invested at an annual rate of 10%? Note in the time line in Figure 4 that the FV is computed as of the end of the last year in the life of the annuity, year 3, even though the final payment occurs at the beginning of year 3 (end of year 2).

Answer:

To solve this problem, put your calculator in the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), then input the relevant data and compute FV.

N = 3; I/Y = 10; PMT = –100; CPT → FV = $364.10

Figure 4: FV of an Annuity Due



Alternatively, we could calculate the FV for an ordinary annuity and multiply it by (1 + I/Y). Leaving your calculator in the END mode, enter the following inputs:

N = 3; I/Y = 10; PMT = –100; CPT → FVA_O = $331.00

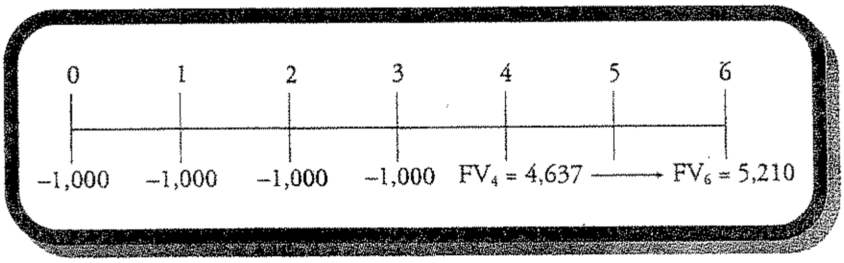
$$FVA_D = FVA_O \times (1 + I/Y) = 331.00 \times 1.10 = \$364.10$$

Example: FV of an annuity due

If you deposit \$1,000 in the bank today and at the beginning of each of the next *three* years, how much will you have six years from today at 6% interest? The time line for this problem is shown in Figure 5.

Answer:

Figure 5: FV for an Annuity Due



Step 1: *Compute the FV of the annuity due at the end of year 4 (FV_4).*

Set your calculator to the annuity due (BGN) mode, enter the relevant data, and compute FV_4.

N = 4; I/Y = 6; PMT = –1,000; CPT → FV = \$4,637.09

Step 2: *Find the future value of FV_4 two years from year 4.*

Enter the relevant data and compute FV_6.

N = 2; I/Y = 6; PV = –4,637.09; CPT → FV =\$5,210.23

Present Value of an Annuity Due

While less common than those for ordinary annuities, there may be problems on the exam where you have to find the *PV of an annuity due* (PVA_D). Using a financial calculator, this really shouldn't be much of a problem. With an annuity due, *there is one less discounting period* since the first cash flow occurs at t = 0 and thus is already its PV. This implies that, all else equal, the PV of an annuity due will be greater than the PV of an ordinary annuity.

As you will see in the next example, there are two ways to compute the PV of an annuity due. The first is to put the calculator in the BGN mode and then input all the relevant variables (PMT, I/Y, and N) as you normally would. The second, and far easier way, is to treat the cash flow stream as an ordinary annuity over N compounding periods, and simply multiply the resulting PV by (1 + periodic compounding rate (I/Y)). Symbolically, this can be stated as:

$$PVA_D = PVA_O \times (1 + I/Y)$$

The advantage of this second method is that you leave your calculator in the END mode and won't run the risk of forgetting to reset it. Regardless of the procedure used, the computed PV is given as of the beginning of the first period, t = 0.

Example: PV of an annuity due

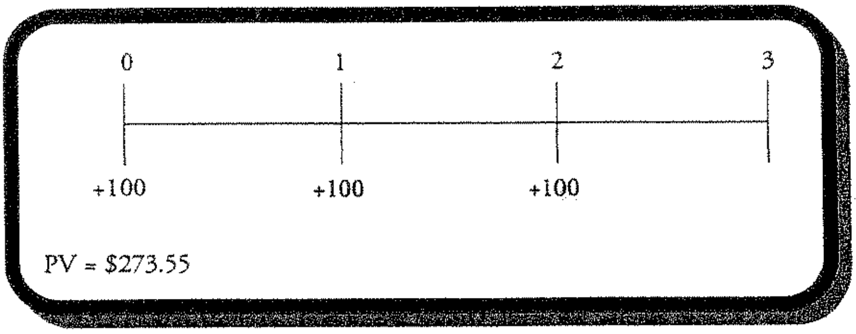
Given a discount rate of 10%, what is the present value of a 3-year annuity that makes a series of \$100 payments at the beginning of each of the next three years, starting today? The time line for this problem is shown in Figure 6.

Answer:

First, let's solve this problem using the calculator's BGN mode. Set your calculator to the BGN mode ([2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP), enter the relevant data, and compute PV.

$N = 3$; $I/Y = 10$; $PMT = -100$; $CPT \rightarrow PVA_D = \$273.55$

Figure 6: PV for an Annuity Due



Alternatively, this problem can be solved by leaving your calculator in the END mode. First, compute the PV of an ordinary 3-year annuity. Then multiply this PV by (1 + I/Y). To use this approach, enter the relevant inputs and compute PV.

$N = 3$; $I/Y = 10$; $PMT = -100$; $CPT \rightarrow PVA_O = \$248.69$

$PVA_D = PVA_O \times (1 + I/Y) = \$248.69 \times 1.10 = \$273.55$

Present Value of a Perpetuity

A perpetuity is a financial instrument that pays a fixed amount of money at set intervals over an *infinite* period of time. In essence, a perpetuity is a perpetual annuity. British consul bonds and most preferred stocks are examples of perpetuities since they promise fixed interest or dividend payments forever. Without going into all the excruciating mathematical details, the discount factor for a perpetuity is just one divided by the appropriate rate of return (i.e., 1/r). Given this, we can compute the PV of a perpetuity.

$$PV_{perpetuity} = \frac{PMT}{I/Y}$$

The PV of a perpetuity is the fixed periodic cash flow divided by the appropriate periodic rate of return.

As with other TVM applications, it is possible to solve for unknown variables in the $PV_{perpetuity}$ equation. In fact, you can solve for any one of the three relevant variables, given the values for the other two.

Example: PV of a perpetuity

Assume the preferred stock of Kodon Corporation pays $4.50 per year in annual dividends and plans to follow this dividend policy forever. Given an 8% rate of return, what is the value of Kodon's preferred stock?

Answer:

Given that the value of the stock is the PV of all future dividends, we have:

$$PV_{perpetuity} = \frac{4.50}{0.08} = \$56.25$$

Thus, if an investor requires an 8% rate of return, the investor should be willing to pay \$56.25 for each share of Kodon's preferred stock.

Example: Rate of return for a perpetuity

Using the Kodon preferred stock described in the preceding example, determine the rate of return that an investor would realize if she paid \$75.00 per share for the stock.

Answer:

Rearranging the equation for $PV_{perpetuity}$, we get:

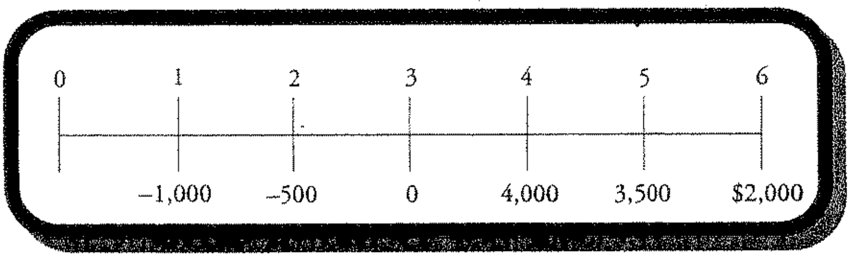
$$I/Y = \frac{PMT}{PV_{perpetuity}} = \frac{4.50}{75.00} = 0.06 = 6.0\%$$

This implies that the return (yield) on a \$75 preferred stock that pays a \$4.50 annual dividend is 6.0%.

PV and FV of Uneven Cash Flow Series

It is not uncommon to have applications in investments and corporate finance where it is necessary to evaluate a cash flow stream that is not equal from period to period. The time line in Figure 7 depicts such a cash flow stream.

Figure 7: Time Line for Uneven Cash Flows



This 6-year cash flow series is not an annuity since the cash flows are different every year. In fact, there is one year with zero cash flow and two others with negative cash flows. In essence, this series of uneven cash flows is nothing more than a stream of annual single sum cash flows. Thus, to find the PV or FV of this cash flow stream, all we need to do is sum the PVs or FVs of the individual cash flows.

Example: Computing the FV of an uneven cash flow series

Using a rate of return of 10%, compute the future value of the 6-year uneven cash flow stream described above at the end of the sixth year.

Answer:

The FV for the cash flow stream is determined by first computing the FV of each individual cash flow, then summing the FVs of the individual cash flows. Note that we need to preserve the signs of the cash flows.

FV_1:	PV = –1,000;	I/Y = 10;	N = 5;	CPT → FV = FV_1 =	–1,610.51
FV_2:	PV = –500;	I/Y = 10;	N = 4;	CPT → FV = FV_2 =	–732.05
FV_3:	PV = 0;	I/Y = 10;	N = 3;	CPT → FV = FV_3 =	0.00
FV_4:	PV = 4,000;	I/Y = 10;	N = 2;	CPT → FV = FV_4 =	4,840.00
FV_5:	PV = 3,500;	I/Y = 10;	N = 1;	CPT → FV = FV_5 =	3,850.00
FV_6:	PV = 2,000;	I/Y = 10;	N = 0;	CPT → FV = FV_6 =	2,000.00
				FV of cash flow stream = $\Sigma FV_{individual}$ =	**8,347.44**

Example: Computing PV of an uneven cash flow series

Compute the present value of this 6-year uneven cash flow stream described above using a 10% rate of return.

Answer:

This problem is solved by first computing the PV of each individual cash flow, then summing the PVs of the individual cash flows, which yields the PV of the cash flow stream. Again the signs of the cash flows are preserved.

PV_1:	FV = –1,000;	I/Y = 10;	N = 1;	CPT → PV = PV_1 =	–909.09
PV_2:	FV = –500;	I/Y = 10;	N = 2;	CPT → PV = PV_2 =	–413.22
PV_3:	FV = 0;	I/Y = 10;	N = 3;	CPT → PV = PV_3 =	0
PV_4:	FV = 4,000;	I/Y = 10;	N = 4;	CPT → PV = PV_4 =	2,732.05
PV_5:	FV = 3,500;	I/Y = 10;	N = 5;	CPT → PV = PV_5 =	2,173.22
PV_6:	FV = 2,000;	I/Y = 10;	N = 6;	CPT → PV = PV_6 =	1,128.95
				PV of cash flow stream = $\Sigma PV_{individual}$ =	**$4,711.91**

It is also possible to compute PV of an uneven cash flow stream by using the cash flow (CF) keys and the *net present value* (NPV) function on your calculator. This procedure is illustrated in the tables in Figures 8 and 9. In Figure 8, we have omitted the F01, F02, etc. values because they are all equal to 1. The F*n* variable indicates how many times a particular cash flow amount is repeated.

Figure 8: NPV Calculator Keystrokes—TI BAII Plus®

Key Strokes	*Explanation*	*Display*
[CF] [2nd] [CLR WORK]	Clear CF Memory Registers	CF0 = 0.00000
0 [ENTER]	Initial Cash Outlay	CF0 = 0.00000
[↓] 1,000 [+/–] [ENTER]	Period 1 Cash Flow	C01 = –1,000.00000
[↓] [↓] 500 [+/–] [ENTER]	Period 2 Cash Flow	C02 = –500.00000
[↓] [↓] 0 [ENTER]	Period 3 Cash Flow	C03 =0.00000
[↓] [↓] 4,000 [ENTER]	Period 4 Cash Flow	C04 =4,000.00000
[↓] [↓] 3,500 [ENTER]	Period 5 Cash Flow	C05 =3,500.00000
[↓] [↓] 2,000 [ENTER]	Period 6 Cash Flow	C06 =2,000.00000
[NPV] 10 [ENTER]	10% Discount Rate	I = 10.00000
[↓] [CPT]	Calculate NPV	NPV = 4,711.91226

Note that the BAII Plus Professional will give the NFV of 8,347.44 also if you press the ↓ key.

Figure 9: NPV Calculator Keystrokes—HP12C®

Key Strokes	*Explanation*	*Display*
[f] [FIN] [f] [REG]	Clear Memory Registers	0.00000
0 [g] [CF_0]	Initial Cash Outlay	0.00000
1,000 [CHS] [g] [CF_j]	Period 1 Cash Flow	–1,000.00000
500 [CHS] [g] [CF_j]	Period 2 Cash Flow	–500.00000
0 [g] [CF_j]	Period 3 Cash Flow	0.00000
4,000 [g] [CF_j]	Period 4 Cash Flow	4,000.00000
3,500 [g] [CF_j]	Period 5 Cash Flow	3,500.00000
2,000 [g] [CF_j]	Period 6 Cash Flow	2,000.00000
10 [i]	10% Discount Rate	10.00000
[f] [NPV]	Calculate NPV	4,711.91226

Solving Time Value of Money Problems When Compounding Periods Are Other Than Annual

While the conceptual foundations of TVM calculations are not affected by the compounding period, more frequent compounding does have an impact on FV and PV computations. Specifically, since an increase in the frequency of compounding increases the effective rate of interest, it also *increases* the FV of a given cash flow and *decreases* the PV of a given cash flow.

Example: The effect of compounding frequency on FV and PV

Compute the FV and PV of a $1,000 single sum for an investment horizon of one year using a stated annual interest rate of 6.0% with a range of compounding periods.

Answer:

Figure 10: Compounding Frequency Effect

Compounding Frequency	*Interest Rate per Period*	*Effective Rate of Interest*	*Future Value*	*Present Value*
Annual (m = 1)	6.000%	6.00%	$1,060.00	$943.396
Semiannual (m = 2)	3.000	6.090	1,060.90	942.596
Quarterly (m = 4)	1.500	6.136	1,061.36	942.184
Monthly (m = 12)	0.500	6.168	1,061.68	941.905
Daily (m = 365)	0.016438	6.183	1,061.83	941.769

There are two ways to use your financial calculator to compute PVs and FVs under different compounding frequencies:

1. Adjust the number of periods per year (P/Y) mode on your calculator to correspond to the compounding frequency (e.g., for quarterly, P/Y = 4). **WE DO NOT RECOMMEND THIS APPROACH!**

2. Keep the calculator in the annual compounding mode (P/Y = 1) and enter I/Y as the interest rate per compounding period, and N as the number of compounding periods in the investment horizon. Letting *m* equal the number of compounding periods per year, the basic formulas for the calculator input data are determined as follows:
   I/Y = the annual interest rate / m
   N = the number of years × m

The computations for the FV and PV amounts in Figure 10 are:

PV_A:	FV = –1,000;	I/Y = 6 / 1 = 6;	N = 1 × 1 = 1:	CPT → PV = PV_A = 943.396
PV_S:	FV = –1,000;	I/Y = 6 / 2 = 3;	N = 1 × 2 = 2:	CPT → PV = PV_S = 942.596
PV_Q:	FV = –1,000;	I/Y = 6/4 = 1.5;	N = 1 × 4 = 4:	CPT → PV = PV_Q = 942.184
PV_M:	FV = –1,000;	I/Y = 6/12 = 0.5;	N = 1 × 12 = 12:	CPT → PV = PV_M = 941.905
PV_D:	FV = –1,000;	I/Y = 6/365 = 0.016438;	N = 1 × 365 = 365:	CPT → PV = PV_D = 941.769
FV_A:	PV = –1,000;	I/Y = 6/1 = 6;	N = 1 × 1 = 1:	CPT → FV = FV_A = 1,060.00
FV_S:	PV = –1,000;	I/Y = 6/2 = 3;	N = 1 × 2 = 2:	CPT → FV = FV_S = 1,060.90
FV_Q:	PV = –1,000;	I/Y = 6/4 = 1.5;	N = 1 × 4 = 4:	CPT → FV = FV_Q = 1,061.36
FV_M:	PV = –1,000;	I/Y = 6/12 = 0.5;	N = 1 × 12 = 12:	CPT → FV = FV_M = 1,061.68
FV_D:	PV = –1,000;	I/Y = 6/365 = 0.016438;	N = 1 × 365 = 365:	CPT → FV = FV_D = 1,061.83

Example: FV of a single sum using quarterly compounding

Compute the FV of $2,000 today, five years from today using an interest rate of 12%, compounded quarterly.

Answer:

To solve this problem, enter the relevant data and compute FV:

$N = 5 \times 4 = 20$; $I/Y = 12 / 4 = 3$; $PV = -\$2{,}000$; CPT $\rightarrow$ FV = \$3,612.22

Example: FV using quarterly compounding and semiannual payments

What is the FV of four semiannual \$100 payments, given a nominal interest rate of 10%, compounded quarterly?

Answer:

The time line for this cash flow stream is shown in Figure 11.

Figure 11: FV Using Quarterly Compounding and Semiannual Payments

Time Line	0	1/4	1/2	3/4	1	1 1/4	1 1/2	1 3/4	2
			+100		+100		+100		+100

In order to solve an annuity problem using the TVM functions, the timing of the cash flows must correspond to the periodic interest rate. This problem may, however, be treated as an uneven cash flow stream using the quarterly rate of 2.5%.

The FV of the uneven cash flow stream described in this problem can be determined as follows:

PV = 100;	$N = 3 \times 2 = 6$;	$I/Y = 10 / 4 = 2.5\%$;	CPT $\rightarrow$ FV = FV_1 = 115.97
PV = 100;	$N = 2 \times 2 = 4$;	$I/Y = 10 / 4 = 2.5\%$;	CPT $\rightarrow$ FV = FV_2 = 110.38
PV = 100;	$N = 1 \times 2 = 2$;	$I/Y = 10 / 4 = 2.5\%$;	CPT $\rightarrow$ FV = FV_3 = 105.06
PV = 100;	$N = 0$;	$I/Y = 10 / 4 = 2.5\%$;	CPT $\rightarrow$ FV = FV_4 = 100.00
			The sum of these amounts (ΣFV) = \$431.41

Professor's Note: Ignore the minus signs on the solutions for FV in your calculator. \$100 today has a positive FV.

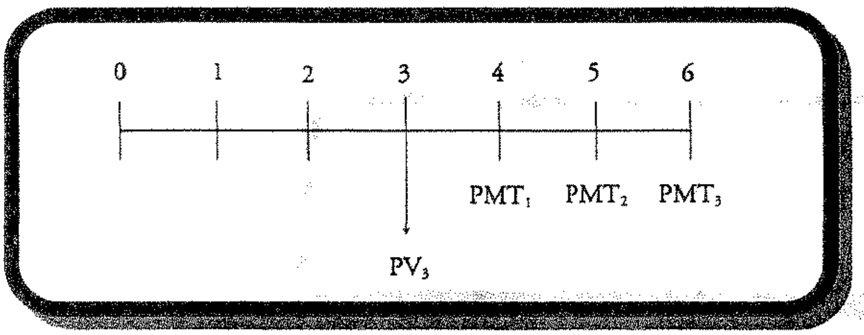
Note that in this example, the interest rate compounding period (quarterly) did not match the payment periods (semiannual) so we could not use our regular calculation functions for the future value of an annuity. We could, however, convert our quarterly rate to an effective semiannual rate and then calculate the future value of the annuity. A quarterly rate of 2.5% is equivalent to a semiannual rate of $1.025^2 - 1 = 5.0625\%$. Using this as the semiannual compounding rate for the annuity yields PMT = 100, N = 4, I/Y = 5.0625, CPT $\rightarrow$ FV = –431.41. In general, when there is a mismatch between the payment frequency and the compounding frequency, we must adjust the interest rate as above. If we just double the quarterly rate to 5%, we would get a future value (431.01) that is too low.

LOS 5.d: Draw a time line, specify a time index, and solve problems involving the time value of money as applied, for example, to mortgages and savings for college tuition or retirement.

The Time Index

In most of the PV problems we have discussed, cash flows were discounted back to the current period. In this case, the PV is said to be indexed to $t = 0$, or the time index is $t = 0$. For example, the PV of a 3-year ordinary annuity that is indexed to $t = 0$ is computed at the beginning of year 1 ($t = 0$). Contrast this situation with another 3-year ordinary annuity that doesn't start until year 4 and extends to year 6. It would not be uncommon to want to know the PV of this annuity at the beginning of year 4, in which case the time index is $t = 3$. The time line for this annuity is presented in Figure 12.

Figure 12: Indexing Time Line to Other Than t = 0



Loan Payments and Amortization

Loan amortization is the process of paying off a loan with a series of periodic loan payments, whereby a portion of the outstanding loan amount is paid off, or amortized, with each payment. When a company or individual enters into a long-term loan, the debt is usually paid off over time with a series of equal, periodic loan payments, and each payment includes the repayment of principal and an interest charge. The payments may be made monthly, quarterly, or even annually. Regardless of the payment frequency, the size of the payment remains fixed over the life of the loan. The amount of the principal and interest component of the loan payment, however, does not remain fixed over the term of the loan. Let's look at some examples to more fully develop the concept of amortization.

Example: Loan payment calculation: annual payments

A company plans to borrow $50,000 for five years. The company's bank will lend the money at a rate of 9% and requires that the loan be paid off in five equal end-of-year payments. Calculate the amount of the payment that the company must make in order to fully amortize this loan in five years.

Answer:

To determine the annual loan payment, input the relevant data and compute PMT.

N = 5; I/Y = 9; PV = –50,000; CPT → PMT = $12,854.62

Thus, the loan can be paid off in five equal annual payments of $12,854.62. Please note that FV = 0 in this computation; the loan will be fully paid off (amortized) after the five payments have been made.

Example: Loan payment calculation: quarterly payments

Using the loan described in the preceding example, determine the payment amount if the bank requires the company to make quarterly payments.

Answer:

The quarterly loan payment can be determined by inputting the relevant data and computing the payment (PMT):

N = 5 × 4 = 20; I/Y = 9 / 4 = 2.25; PV = –50,000; CPT → PMT = $3,132.10

Example: Constructing an amortization schedule

Construct an amortization schedule to show the interest and principal components of the end-of-year payments for a 10%, 5-year, $10,000 loan.

Answer:

The first step in solving this problem is to compute the amount of the loan payments. This is done by entering the relevant data and computing PMT:

N = 5; I/Y = 10%; PV = –$10,000; CPT → PMT = $2,637.97

Thus, the loan will be repaid via five equal $2,637.97 end-of-year payments. Each payment is made up of an interest component (profit to the lender) plus the partial recovery of loan principal, with principal recovery being scheduled so that the full amount of the loan is paid off by the end of year 5. The exact amount of the principal and interest components of each loan payment are presented and described in the amortization table shown in Figure 13.

Figure 13: Amortization Table

Period	*Beginning Balance*	*Payment*	*Interest Component (1)*	*Principal Component (2)*	*Ending Balance (3)*
1	$10,000.00	$2,637.97	$1,000.00	$1,637.97	$8,362.03
2	8,362.03	2,637.97	836.20	1,801.77	6,560.26
3	6,560.26	2,637.97	656.03	1,981.94	4,578.32
4	4,578.32	2,637.97	457.83	2,180.14	2,398.18
5	2,398.18	2,638.00*	239.82	2,398.18	0.00

*There is usually a slight amount of rounding error that must be recognized in the final period. The extra $0.03 associated with payment five reflects an adjustment for the rounding error and forces the ending balance to zero.

1. Interest component = beginning balance × periodic interest rate. In period 3, the interest component of the payment is $6,560.26 × 0.10 = $656.03.
2. Principal component = payment – interest. For example, the period 4 principal component is $2,637.97 – $457.83 = 2,180.14.
3. The ending balance in a given period, *t*, is the period's beginning balance minus the principal component of the payment, where the beginning balance for period *t* is the ending balance from period t – 1. For example, the period 2 ending balance equals $8,362.03 – $1,801.77 = $6,560.26, which becomes the period 3 beginning balance.

Professor's Note: Once you have solved for the payment, $2,637.97, the remaining principal on any payment date can be calculated by entering N = # of remaining payments and solving for the PV.

Example: Principal and interest component of a specific loan payment

Suppose you borrowed $10,000 at 10% interest to be paid semiannually over 10 years. Calculate the amount of the outstanding balance for the loan after the second payment is made.

Answer:

First the amount of the payment must be determined by entering the relevant information and computing the payment.

PV = –$10,000; I/Y = 10 / 2 = 5; N = 10 × 2 = 20; CPT → PMT = $802.43

The principal and interest component of the second payment can be determined using the following process:

Payment 1: Interest = ($10,000)(0.05) = $500

Principal = $802.43 – $500 = $302.43

Payment 2: Interest = ($10,000 – $302.43)(0.05) = $484.88

Principal = $802.43 – $484.88 = $317.55

Remaining balance = $10,000 – $302.43 – $317.55 = $9,380.02

The following examples will illustrate how to compute I/Y, N, or PMT in annuity problems.

Example: Computing an annuity payment needed to achieve a given FV

At an expected rate of return of 7%, how much must be deposited at the end of each year for the next 15 years to accumulate $3,000?

Answer:

To solve this problem, enter the three relevant known values and compute PMT.

N = 15; I/Y = 7; FV = +$3,000; CPT → PMT = –$119.38 (ignore sign)

Example: Computing a loan payment

Suppose you are considering applying for a $2,000 loan that will be repaid with equal end-of-year payments over the next 13 years. If the annual interest rate for the loan is 6%, how much will your payments be?

Answer:

The size of the end-of-year loan payment can be determined by inputting values for the three known variables and computing PMT.

N = 13; I/Y = 6; PV = –2,000; CPT → PMT = $225.92

Example: Computing the number of periods in an annuity

How many $100 end-of-year payments are required to accumulate $920 if the discount rate is 9%?

Answer:

The number of payments necessary can be determined by inputting the relevant data and computing N.

I/Y = 9%; FV = \$920; PMT = –\$100; CPT → N = 7 years

It will take seven annual \$100 payments, compounded at 9% annually, to accrue an investment value of \$920.

Professor's Note: Remember the sign convention. PMT and FV must have opposite signs or your calculator will issue an error message.

Example: Computing the number of years in an ordinary annuity

Suppose you have a \$1,000 ordinary annuity earning an 8% return. How many annual end-of-year \$150 withdrawals can be made?

Answer:

The number of years in the annuity can be determined by entering the three relevant variables and computing N.

I/Y = 8; PMT = 150; PV = –1,000; CPT → N = 9.9 years

Professor's Note: The HP calculator will round this to 10. This should not be a problem on the exam.

Example: Computing the rate of return for an annuity

Suppose you have the opportunity to invest \$100 at the end of each of the next five years in exchange for \$600 at the end of the fifth year. What is the annual rate of return on this investment?

Answer:

The rate of return on this investment can be determined by entering the relevant data and solving for I/Y.

N = 5; FV = \$600; PMT = –100; CPT → I/Y = 9.13%

Example: Computing the discount rate for an annuity

What rate of return will you earn on an ordinary annuity that requires a \$700 deposit today and promises to pay \$100 per year at the end of each of the next 10 years?

Answer:

The discount rate on this annuity is determined by entering the three known values and computing I/Y.

N = 10; PV = –700; PMT = 100; CPT → I/Y = 7.07%

Funding a Future Obligation

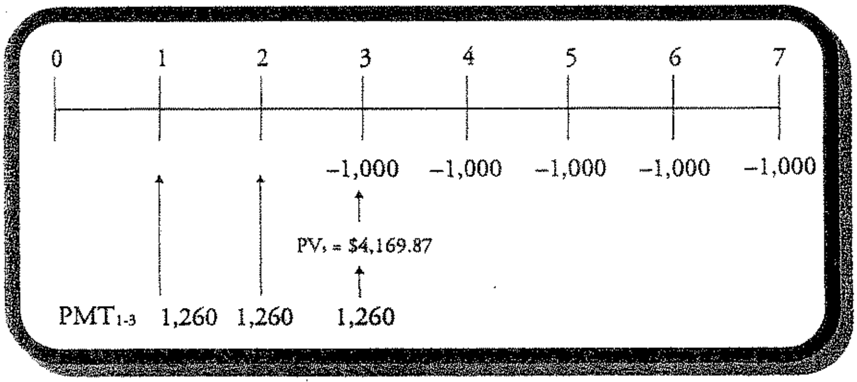
There are many TVM applications where it is necessary to determine the size of the deposit(s) that must be made over a specified period in order to meet a future liability. Two common examples of this type of application are (1) setting up a funding program for future college tuition, and (2) the funding of a retirement program. In most of these applications, the objective is to determine the size of the payment(s) or deposit(s) necessary to meet a particular monetary goal.

Example: Computing the required payment to fund an annuity due

Suppose you must make five annual $1,000 payments, the first one starting at the beginning of year 4 (end of year 3). To accumulate the money to make these payments you want to make three equal payments into an investment account, the first to be made one year from today. Assuming a 10% rate of return, what is the amount of these three payments?

The time line for this annuity problem is shown in Figure 14.

Figure 14: Funding an Annuity Due



Answer:

The first step in this type of problem is to determine the amount of money that must be available at the beginning of year 4 in order to satisfy the payment requirements. This amount is the PV of a 5-year annuity due at the beginning of year 4 (end of year 3). To determine this amount, set your calculator to the BGN mode, enter the relevant data, and compute PV.

N = 5; I/Y = 10; PMT = −1,000; CPT → PV = PV_3 = $4,169.87

Alternatively, you can leave your calculator in the END mode, compute the PV of a 5-year ordinary annuity, and multiply by 1.10.

N = 5; I/Y = 10; PMT = −1,000; CPT → PV = 3,790.79 × 1.1 = PV_3 = $4,169.87

A third alternative, with the calculator in END mode, is to calculate the t = 3 value of the last four annuity payments and then add $1,000.

N = 4; I/Y = 10; PMT = −1,000; CPT → PV = 3,169.87 + 1,000 = $4,169.87 = PV_3

PV_3 becomes the FV that you need three years from today from your three equal end-of-year deposits. To determine the amount of the three payments necessary to meet this funding requirement, be sure that your calculator is in the END mode, input the relevant data, and compute PMT.

N = 3; I/Y = 10; FV = −4,169.87; CPT → PMT = $1,259.78

The second part of this problem is an ordinary annuity. If you changed your calculator to BGN mode and failed to put it back in the END mode, you will get a PMT of $1,145, which is incorrect.

Example: Funding a retirement plan

Assume a 35-year-old investor wants to retire in 25 years at the age of 60. She expects to earn 12.5% on her investments prior to her retirement and 10% thereafter. How much must she deposit at the end of each year for the next 25 years in order to be able to withdraw $25,000 per year at the beginning of each year for the 30 years from age 60 to 90?

Answer:

This is a two-step problem. First determine the amount that must be on deposit in the retirement account at the end of year 25 in order to fund the 30-year, $25,000 annuity due. Second, compute the annuity payments that must be made to achieve the required amount.

Step 1: *Compute the amount required to meet the desired withdrawals.*

The required amount is the present value of the $25,000, 30-year annuity due at the beginning of year 26 (end of year 25). This can be determined by entering the relevant data, with the calculator in the END mode, and computing PV.

N = 29; I/Y = 10; PMT = –$25,000: CPT → PV = $234,240 (for 29 years)

Now add the first annuity payment to get $234,240 + $25,000 = $259,240. The investor will need $259,240 at the end of year 25.

Please note that we could have also performed this computation with our calculator in BGN mode as an annuity due. To do this, put your calculator in BGN mode [2nd] [BGN] [2nd] [SET] [2nd] [QUIT] on the TI or [g] [BEG] on the HP. Then enter:

N = 30; PMT = –25,000; I/Y = 10; CPT → PV = 259,240.14

If you do it this way, make certain you reset your calculator to the END mode.

Step 2: *The annuity payment that must be made to accumulate the required amount over 25 years can be determined by entering the relevant data and computing PMT.*

N = 25; I/Y = 12.5; FV = –259,240; CPT → PMT = $1,800.02

Thus, the investor must deposit $1,800 per year at the end of each of the next 25 years in order to accumulate $259,240. With this amount she will be able to withdraw $25,000 per year for the following 30 years.

Note that all these calculations assume that the investor will earn 12.5% on the payments prior to retirement and 10% on the funds held in the retirement account thereafter.

The Connection Between Present Values, Future Values, and Series of Cash Flows

As we have explained in the discussion of annuities and series of uneven cash flows, the sum of the present values of the cash flows is the present value of the series. The sum of the future values (at some future time = n) of a series of cash flows is the future value of that series of cash flows.

One interpretation of the present value of a series of cash flows is how much would have to be put in the bank today in order to make these future withdrawals and exhaust the account with the final withdrawal? Let's illustrate this with cash flows of $100 in year 1, $200 in year 2, $300 in year 3, and an assumed interest rate of 10%.

Calculate the present value of these three cash flows as:

$$\frac{100}{1.1}+\frac{200}{1.1^2}+\frac{300}{1.1^3}=\$481.59$$

If we put \$481.59 in an account yielding 10%, at the end of the year we would have 481.59 × 1.1 = \$529.75. Withdrawing \$100 would leave \$429.75.

Over the second year, the \$429.75 would grow to 429.75 × 1.1 = \$472.73. Withdrawing \$200 would leave \$272.73.

Over the third year, \$272.73 would grow to 272.73 × 1.1 = \$300, so that the last withdrawal of \$300 would empty the account.

The interpretation of the future value of a series of cash flows is straightforward as well. The FV answers the question, how much would be in an account when the last of a series of deposits is made? Using the same three cash flows—\$100, \$200, and \$300—and the same interest rate of 10%, we can calculate the future value of the series as:

$$100\ (1.1)^2 + 200\ (1.1) + 300 = \$641$$

This is simply the sum of the $t = 3$ value of each of the cash flows. Note that the $t = 3$ value and the $t = 0$ (present) value of the series are related by the interest rate, $481.59\ (1.1)^3 = 641$.

The \$100 cash flow (deposit) comes at $t = 1$, so it will earn interest of 10% compounded for two periods (until $t = 3$). The \$200 cash flow (deposit) will earn 10% between $t = 2$ and $t = 3$, and the final cash flow (deposit) of \$300 is made at $t = 3$, so \$300 is the future ($t = 3$) value of that cash flow.

We can also look at the future value in terms of how the account grows over time. At $t = 1$ we deposit \$100, so at time = 2 it has grown to \$110 and the \$200 deposit at $t = 2$ makes the account balance \$310. Over the next period the \$310 grows to 310 × 1.1 = \$341 at $t = 3$, and the addition of the final \$300 deposit puts the account balance at \$641. This is, of course, the future value we calculated initially.

Professor's Note: This last view of the future value of a series of cash flows suggests a quick way to calculate the future value of an uneven cash flow series. The process described previously for the future value of a series of end-of-period payments can be written mathematically as [(100 × 1.1) + 200] × 1.1 + 300 = 641, *and this might be a quick way to do some future value problems.*

Note that questions on the future value of an *annuity due* refer to the amount in the account one period after the last deposit is made. If the three deposits considered here were made at the beginning of each period (at $t = 0, 1, 2$) the amount in the account at the end of three years ($t = 3$) would be 10% higher (i.e., 641 × 1.1 = \$705.10).

The cash flow additivity principle refers to the fact that present value of any stream of cash flows equals the sum of the present values of the cash flows. There are different applications of this principle in time value of money problems. If we have two series of cash flows, the sum of the present values of the two series is the same as the present values of the two series taken together, adding cash flows that will be paid at the same point in time. We can also divide up a series of cash flows any way we like, and the present value of the "pieces" will equal the present value of the original series.

Example: Additivity principle

A security will make the following payments at the end of the next four years: \$100, \$100, \$400, and \$100. Calculate the present value of these cash flows using the concept of the present value of an annuity when the appropriate discount rate is 10%.

Answer:

We can divide the cash flows so that we have:

t = 1	t = 2	t = 3	t = 4	
100	100	100	100	cash flow series #1
0	0	300	0	cash flow series #2
\$100	\$100	\$400	\$100	

The additivity principle tells us that to get the present value of the original series, we can just add the present values of series #1 (a 4-period annuity) and series #2 (a single payment 3 periods from now).

For the annuity, N = 4, PMT = 100, FV = 0, I/Y = 10, CPT → PV = –\$316.99

For the single payment: N = 3, PMT = 0, FV = 300, I/Y = 10, CPT → PV = –\$225.39

The sum of these two values is 316.99 + 225.39 = \$542.38.

The sum of these two (present) values is identical (except for rounding) to the sum of the present values of the payments of the original series:

$$\frac{100}{1.1}+\frac{100}{1.1^2}+\frac{400}{1.1^3}+\frac{100}{1.1^4}=\$542.38$$

Key Concepts

1. The required rate of return on a security = real risk-free rate + expected inflation + default risk premium + liquidity premium + maturity risk premium.
2. Future value: $FV = PV(1 + I/Y)^N$; present value: $PV = FV / (1 + I/Y)^N$.
3. The effective annual rate when there are *m* compounding periods $=\left(1+\frac{\text{nominal rate}}{m}\right)^m-1$.
4. For non-annual time value of money problems, divide the stated annual interest rate by the number of compounding periods per year, *m*, and multiply the number of years by the number of compounding periods per year.
5. An annuity is a series of equal cash flows that occurs at evenly spaced intervals over time.
   - Ordinary annuity cash flows occur at the end of each time period.
   - Annuity due cash flows occur at the beginning of each time period.
6. Perpetuities are annuities with infinite lives (perpetual annuities):
   $$PV_{\text{perpetuity}}=\frac{PMT}{I/Y}$$
7. A mortgage is an amortizing loan, repaid in a series of equal payments (an annuity), where each payment consists of the periodic interest and a repayment of principal.
8. The present (future) value of any series of cash flows is equal to the sum of the present (future) values of the individual cash flows.

CONCEPT CHECKERS: THE TIME VALUE OF MONEY

1. The amount an investor will have in 15 years if $1,000 is invested today at an annual interest rate of 9% will be *closest* to:
   A. $1,350.
   B. $3,518.
   C. $3,642.
   D. $9,000.

2. Fifty years ago, an investor bought a share of stock for $10. The stock has paid no dividends during this period, yet it has returned 20%, compounded annually, over the past 50 years. If this is true, the share price is now *closest* to:
   A. $1,000.
   B. $4,550.
   C. $45,502.
   D. $91,004.

3. How much must be invested today at 0% to have $100 in three years?
   A. $77.75.
   B. $100.00.
   C. $126.30.
   D. $87.50.

4. How much must be invested today, at 8% interest, to accumulate enough to retire a $10,000 debt due seven years from today? The amount that must be invested today is *closest* to:
   A. $3,265.
   B. $5,835.
   C. $6,123.
   D. $8,794.

5. An analyst estimates that XYZ's earnings will grow from $3.00 a share to $4.50 per share over the next eight years. The rate of growth in XYZ's earnings is *closest* to:
   A. 4.9%.
   B. 5.2%.
   C. 6.7%.
   D. 7.0%.

6. If $5,000 is invested in a fund offering a rate of return of 12% per year, approximately how many years will it take for the investment to reach $10,000?
   A. 4 years.
   B. 5 years.
   C. 6 years.
   D. 7 years.

7. An investment is expected to produce the cash flows of $500, $200, and $800 at the end of the next three years. If the required rate of return is 12%, the present value of this investment is *closest* to:
   A. $835.
   B. $1,175.
   C. $1,235.
   D. $1,500.

8. Given an 8.5% discount rate, an asset that generates cash flows of $10 in year 1, –$20 in year 2, $10 in year 3, and is then sold for $150 at the end of year 4 has a present value of:
   A. $163.42.
   B. $150.00.
   C. $135.58.
   D. $108.29.

9. An investor has just won the lottery and will receive $50,000 per year at the end of each of the next 20 years. At a 10% interest rate, the present value of the winnings is *closest* to:
   A. $418,246.
   B. $425,678.
   C. $637,241.
   D. $2,863,750.

10. If $10,000 is invested today in an account that earns interest at a rate of 9.5%, what is the value of the equal withdrawals that can be taken out of the account at the end of each of the next five years if the investor plans to deplete the account at the end of the time period?
   A. $2,000.
   B. $2,453.
   C. $2,604.
   D. $2,750.

11. An investor is to receive a 15-year $8,000 annuity, the first payment to be received today. At an 11% discount rate, this annuity's worth today is *closest* to:
   A. $55,855.
   B. $57,527.
   C. $63,855.
   D. $120,000.

12. Given an 11% rate of return, the amount that must be put into an investment account at the end of each of the next ten years in order to accumulate $60,000 to pay for a child's education is *closest* to:
   A. $2,500.
   B. $4,432.
   C. $3,588.
   D. $6,000.

13. An investor will receive an annuity of $4,000 a year for ten years. The first payment is to be received five years from today. At a 9% discount rate, this annuity's worth today is *closest* to:
   A. $16,684.
   B. $18,186.
   C. $25,671.
   D. $40,000.

14. If $1,000 is invested today and $1,000 is invested at the beginning of each of the next three years at 12% interest (compounded annually), the amount an investor will have at the end of the fourth year will be *closest* to:
   A. $4,272.
   B. $4,779.
   C. $5,353.
   D. $6,792.

15. An investor is looking at a $150,000 home. If 20% must be put down and the balance is financed at 9% over the next 30 years, what is the monthly mortgage payment?
    A. $652.25.
    B. $799.33.
    C. $895.21.
    D. $965.55.

16. Given daily compounding, the growth of $5,000 invested for one year at 12% interest will be *closest* to:
    A. $5,600.
    B. $5,628.
    C. $5,637.
    D. $5,000.

17. Terry Corporation preferred stocks are expected to pay a $9 annual dividend forever. If the required rate of return on equivalent investments is 11%, a share of Terry preferred should be worth:
    A. $100.00.
    B. $81.82.
    C. $99.00.
    D. $122.22.

18. A share of George Co. preferred stock is selling for $65. It pays a dividend of $4.50 per year and has a perpetual life. The rate of return it is offering its investors is *closest* to:
    A. 4.5%.
    B. 6.5%.
    C. 6.9%.
    D. 14.4%.

19. If $10,000 is borrowed at 10% interest to be paid back over ten years, how much of the second year's payment is interest (assume annual loan payments)?
    A. $954.25.
    B. $937.26.
    C. $1,000.00.
    D. $1,037.26.

20. What is the effective annual rate for a credit card that charges 18% compounded monthly?
    A. 15.00%.
    B. 15.38%.
    C. 18.81%.
    D. 19.56%.

Comprehensive Problems: The Time Value of Money

1. The Parks plan to take three cruises, one each year. They will take their first cruise 9 years from today, the second cruise one year after that, and the third cruise 11 years from today. The type of cruise they will take currently costs $5,000, but they expect inflation will increase this cost by 3.5% per year on average. They will contribute to an account to save for these cruises that will earn 8% per year. What equal contributions must they make today and every year until their first cruise (10 contributions) in order to have saved enough for all three cruises at that time? They pay for cruises when taken.

2. A company's dividend in 1995 was $0.88. Over the next eight years, the dividends were $0.91, $0.99, $1.12, $0.95, $1.09, $1.25, $1.42, $1.26. Calculate the annually compounded growth rate of the dividend over the whole period.

3. An investment (a bond) will pay $1,500 at the end of each year for 25 years and on the date of the last payment will also make a separate payment of $40,000. If your required rate of return on this investment is 4%, how much would you be willing to pay for the bond today?

4. A bank quotes certificate of deposit (CD) yields both as annual percentage rates (APR) without compounding and as annual percentage yields (APY) that include the effects of monthly compounding. A $100,000 CD will pay $110,471.31 at the end of the year. Calculate the APR and APY the bank is quoting.

5. A client has $202,971.39 in an account that earns 8% per year, compounded monthly. The client's 35th birthday was yesterday and she will retire when the account value is $1 million.
   A. At what age can she retire if she puts no more money in the account?
   B. At what age can she retire if she puts $250/month into the account every month beginning one month from today?

6. At retirement nine years from now, a client will have the option of receiving a lump sum of £400,000 or 20 annual payments of £40,000 with the first payment made at retirement. What is the annual rate the client would need to earn on a retirement investment to be indifferent between the two choices?

Answers – Concept Checkers: The Time Value of Money

1. C N = 15; I/Y = 9; PV = –1,000; PMT = 0; CPT → FV = $3,642.48

2. D N = 50; I/Y = 20; PV = –10; PMT = 0; CPT → FV = $91,004.38

3. B Since no interest is earned, $100 is needed today to have $100 in three years.

4. B N = 7; I/Y = 8; FV = –10,000; PMT = 0; CPT → PV = $5,834.90

5. B N = 8; PV = –3; FV = 4.50; PMT = 0; CPT → I/Y = 5.1989

6. C PV = –5,000; I/Y = 12; FV = 10,000; PMT = 0; CPT → N = 6.12. Rule of 72 → 72/12 = six years.

   Note to HP12C users: One known problem with the HP12C is that it does not have the capability to round. In this particular question, you will come up with 7, although the correct answer is 6.1163. CFA Institute is aware of this problem, and hopefully you will not be faced with a situation like this on exam day (e.g., having to choose between two choices being so close together, like 6 and 7).

7. B Using your cash flow keys, CF_0 = 0; CF_1 = 500; CF_2 = 200; CF_3 = 800; I/Y = 12; NPV = $1,175.29.

   Or you can add up the present values of each single cash flow.

   PV_1 = N = 1; FV = –500; I/Y = 12; CPT → PV = 446.43
   PV_2 = N = 2; FV = –200; I/Y = 12; CPT → PV = 159.44
   PV_3 = N = 3; FV = –800; I/Y = 12; CPT → PV = 569.42

   Hence, 446.43 + 159.44 + 569.42 = $1,175.29.

8. D Using your cash flow keys, CF_0 = 0; CF_1 = 10; CF_2 = –20; CF_3 = 10; CF_4 = 150; I/Y = 8.5; NPV = $108.29.

9. B N = 20; I/Y = 10; PMT = –50,000; FV = 0; CPT → PV = $425,678.19

10. C PV = –10,000; I/Y = 9.5; N = 5; FV = 0; CPT → PMT = $2,604.36

11. C This is an annuity due. Switch to BGN mode.

    N = 15; PMT = –8,000; I/Y = 11; FV = 0; CPT → PV = 63,854.92. Switch back to END mode.

12. C N = 10; I/Y = 11; FV = –60,000; PV = 0; CPT → PMT = $3,588.08

13. B Two steps: (1) Find the PV of the 10-year annuity: N = 10; I/Y = 9; PMT = –4,000; FV = 0; CPT → PV = 25,670.63. This is the present value as of the end of year 4; (2) Discount PV of the annuity back four years: N = 4; PMT = 0; FV = –25,670.63; I/Y = 9; CPT → PV = 18,185.72.

14. C The key to this problem is to recognize that it is a 4-year annuity due, so switch to BGN mode: N = 4; PMT = –1,000; PV = 0; I/Y = 12; CPT → FV = 5,352.84. Switch back to END mode.

15. D N = 30 × 12 = 360; I/Y = 9 / 12 = 0.75; PV = –150,000(1 – 0.2) = –120,000; FV = 0; CPT → PMT = $965.55

16. C N = 1 × 365 = 365; I/Y = 12 / 365 = 0.0328767; PMT = 0; PV = –5,000; CPT → FV = $5,637.37

17. B 9 / 0.11 = $81.82

18. C 4.5 / 65 = 0.0692 or 6.92%

19\. B To get the annual payment, enter PV = –10,000; FV = 0; I/Y = 10; N = 10; CPT → PMT = 1,627.45. The first year's interest is $1,000 = 10,000 × 0.10, so the principal balance going into year 2 is 10,000 – 627.45 = $9,372.55. Year 2 interest = $937.26 = $9,372.55 × 0.10.

20\. D EAR = $(1 + 0.18/12)^{12} - 1$ = 19.56%

Answers – Comprehensive Problems: The Time Value of Money

1\. Our suggested solution method is:

$$\begin{aligned}\text{cost of first cruise} &= 5{,}000 \times 1.035^9\\ &= 6{,}814.49\\ \text{PV of first cruise cost} &= \frac{6{,}814.49}{(1.08)^9} = \$3{,}408.94\end{aligned}$$

$$\text{PV of second cruise cost} = \frac{(1.035)^{10}}{(1.08)^{10}} \times 5{,}000 = \$3{,}266.90$$

$$\text{PV of third cruise cost} = \left(\frac{1.035}{1.08}\right)^{11} \times 5{,}000 = \$3{,}130.78$$

PV of all three = 3,408.94 + 3,266.90 + 3,130.78 = $9,806.62. This is the amount needed in the account today so it's the PV of a 10-payment annuity due, solve for payment at 8% = $1,353.22.

2\. This problem is simpler than it may appear. The dividend grew from $0.88 to $1.26 in eight years. We know, then, that $0.88(1 + i)^8 = 1.26$, where i is the compound growth rate. Solving for i we get $\left(\frac{1.26}{0.88}\right)^{1/8} - 1 = 4.589\%$. You could also just enter $\frac{1.26}{0.88}$, press $[\sqrt{\ }]$ three times, get 1.045890 and see that the answer is 4.589%.

This technique for solving for a compound growth rate is a very useful one and you will see it often.

Calculator solution: PV = 0.88, N = 8, FV = –1.26, PMT = 0, CPT → I/Y = 4.589.

3\. We can take the present value of the payments (a regular annuity) and the present value of the $40,000 (lump sum) and add them together. N = 25, PMT = –1,500, i = 4, CPT → PV = 23,433.12 and 40,000 × $\left(\frac{1}{1.04}\right)^{25}$ = 15,004.67, for a total value of $38,437.79.

Alternatively, N = 25, PMT = –1,500, i = 4, FV = –40,000, CPT → PV = 38,437.79.

4\. For APR, PV = 100,000, FV = –110,471.31, PMT = 0, N = 12, CPT I/Y 0.8333, which is the monthly rate. The APR = 12 × 0.8333, or 10%.

APY = 110,471.31 / 100,000 – 1 = 0.10471 = 10.471% (equivalent to a compound monthly rate of 0.8333%)

5\. A. PV = –202,971.39, I/Y = 8/12 = 0.6667, PMT = 0, FV = 1,000,000, CPT → N = 240. 240 months is 20 years; she will be 55 years old.
B. Don't clear TVM functions. PMT = –250, CPT → N = 220, which is 18.335 years, so she will be 53. (N is actually 220.024, so the HP calculator displays 221.)

6. Setting the retirement date to t = 0 we have the following choices:

t = 0	t = 1	t = 2	...	t = 20
400,000				
40,000	40,000	40,000	...	40,000

One method: PV = 400,000 – 40,000 = 360,000; PMT = –40,000; N = 19; FV = 0; CPT → I/Y = 8.9196%

Or in *begin mode*: PV = 400,000; N = 20; FV = 0; PMT = –40,000; CPT → I/Y = 8.9196%

The following is a review of the Quantitative Methods principles designed to address the learning outcome statements set forth by CFA Institute®. This topic is also covered in:

DISCOUNTED CASH FLOW APPLICATIONS

Study Session 2

EXAM FOCUS

This topic review has a mix of topics, but all are important because of their usefulness and the certainty that some, if not all, of these topics will be on the exam. You must be able to use the cash flow functions on your calculator to calculate net present value and internal rate of return. We will use both of these in the Corporate Finance section and examine their strengths and weaknesses more closely there; but you must learn how to calculate them here. The time-weighted and money-weighted return calculations are standard tools for analysis. Calculating the various yield measures and the ability to calculate one from another are must-have skills. Don't hurry here, these concepts and techniques are foundation material and will turn up repeatedly at all three levels of the CFA® curriculum.

LOS 6.a: Calculate and interpret the net present value (NPV) and the internal rate of return (IRR) of an investment, contrast the NPV rule to the IRR rule, and identify any problems associated with the IRR rule.

The **net present value** (NPV) of an investment project is the present value of expected cash inflows associated with the project less the present value of the project's expected cash outflows, discounted at the appropriate cost of capital. The following procedure may be used to compute NPV.

- Identify all costs (outflows) and benefits (inflows) associated with an investment.
- Determine the appropriate discount rate or opportunity cost for the investment.
- Using the appropriate discount rate, find the PV of each cash flow. Inflows are positive and increase NPV. Outflows are negative and decrease NPV.
- Compute the NPV, the sum of the DCFs.

Mathematically, NPV is expressed as:

$$NPV = \sum_{t=0}^{N} \frac{CF_t}{(1+r)^t}$$

where:
CF_t = the expected net cash flow at time t
N = the estimated life of the investment
r = the discount rate = opportunity cost of capital

NPV is the PV of the cash flows less the initial (time = 0) outlay.

Example: Computing NPV

Calculate the NPV of an investment project with an initial cost of $5 million and positive cash flows of $1.6 million at the end of year 1, $2.4 million at the end of year 2, and $2.8 million at the end of year 3. Use 12% as the discount rate.

Answer:

The NPV for this project is the sum of the PVs of the project's individual cash flows and is determined as follows:

$$\text{NPV} = -\$5.0 + \frac{\$1.6}{1.12} + \frac{\$2.4}{(1.12)^2} + \frac{\$2.8}{(1.12)^3}$$

$$= -\$5.0 + \$1.42857 + \$1.91327 + \$1.99299$$

$$= \$0.3348 \text{ million, or } \$334{,}800$$

The procedures for calculating NPV with a TI BAII Plus® and an HP12C® hand-held financial calculator are presented in Figures 1 and 2.

Figure 1: Calculating NPV with the TI Business Analyst II Plus®

Key Strokes	*Explanation*	*Display*
[CF] [2nd] [CLR WORK]	Clear CF Memory Registers	CF0 = 0.00000
5 [+/–] [ENTER]	Initial Cash Outlay	CF0 = –5.00000
[↓]1.6 ENTER]	Period 1 Cash Flow	C01 = 1.60000
[↓] [↓] 2.4 [ENTER]	Period 2 Cash Flow	C02 = 2.40000
[↓] [↓] 2.8 [ENTER]	Period 3 Cash Flow	C03 = 2.80000
[NPV] 12 [ENTER]	12% discount rate	I =12.00000
[↓] [CPT]	Calculate NPV	NPV = 0.33482

Figure 2: Calculating NPV with the HP12C®

Key Strokes	*Explanation*	*Display*
[f] [FIN] [f] [REG]	Clear Memory Registers	0.00000
5 [CHS] [g] [CF_0]	Initial Cash Outlay	–5.00000
1.6 [g] [CF_j]	Period 1 Cash flow	1.60000
2.4 [g] [CF_j]	Period 2 Cash flow	2.40000
2.8 [g] [CF_j]	Period 3 Cash flow	2.80000
12 [i]	12% discount rate	12.00000
[f] [NPV]	Calculate NPV	0.33482

On the TI BAII Plus calculator, the sequence of [↓][↓] scrolls past the variables F01, F02, et cetera. The F here stands for frequency, and this is set to 1 by default. We did not enter anything because each cash flow amount occurred only once. If the period 2 cash flow, C02, were repeated three times (at t = 2, 3, and 4) we could input F02 = 3 to account for all three of these. The next input, C03, would then refer to the cash flow at t = 5.

On the HP 12C, you can also account for a repeated cash flow amount. Referring to the above example for the TI BAII Plus, the fact that the cash flow at period 2 is repeated for three periods is indicated by the sequence 3 [g] Nj immediately after the CF_j keystroke to input the amount of the period 2 cash flow.

Professor's Note: The NPV function can also be used to find the present value of any series of cash flows (positive or negative) over future periods. Just set $CF_0 = 0$ and input the cash flows CF_1 through CF_N as outlined above. The NPV is the present value of these cash flows since there is now no initial negative cash flow (initial cost).

The **internal rate of return** (IRR) is defined as the rate of return that equates the PV of an investment's expected benefits (inflows) with the PV of its costs (outflows). Equivalently, the IRR may be defined as the discount rate for which the NPV of an investment is zero.

Calculating IRR requires only that we identify the relevant cash flows for the investment opportunity being evaluated. Market-determined discount rates, or any other external (market-driven) data, are not necessary with the IRR procedure. The general formula for the IRR is:

$$0 = CF_0 + \frac{CF_1}{1+IRR} + \frac{CF_2}{(1+IRR)^2} + \cdots + \frac{CF_N}{(1+IRR)^N}$$

In the majority of IRR applications to capital budgeting, the initial cash flow, CF_0, represents the initial cost of the investment opportunity, and is therefore a negative value. As such, any discount rate less than the IRR will result in a positive NPV, and a discount rate greater than the IRR will result in a negative NPV. This implies that the NPV of an investment is zero when the discount rate used equals the IRR.

Example: Computing IRR

What is the IRR for the investment described in the preceding example?

Answer:

Substituting the investment's cash flows into the previous IRR equation results in the following equation:

$$0 = -5.0 + \frac{1.6}{1+IRR} + \frac{2.4}{(1+IRR)^2} + \frac{2.8}{(1+IRR)^3}$$

Solving this equation yields an IRR = 15.52%.

It is possible to solve IRR problems through a trial and error process. That is, keep guessing IRRs until you get the one that provides an NPV equal to zero. Practically speaking, a financial calculator or an electronic spreadsheet can and should be employed. The procedures for computing IRR with the TI BA II Plus and HP12C financial calculators are illustrated in Figures 3 and 4, respectively.

Figure 3: Calculating IRR with the TI Business Analyst II Plus®

Key Strokes	*Explanation*	*Display*
[CF] [2nd] [CLR WORK]	Clear Memory Registers	CF0 = 0.00000
5 [+/–] [ENTER]	Initial Cash Outlay	CF0 = –5.00000
[↓] 1.6 [ENTER]	Period 1 Cash Flow	C01 = 1.60000
[↓] [↓] 2.4 [ENTER]	Period 2 Cash Flow	C02 = 2.40000
[↓] [↓] 2.8 [ENTER]	Period 3 Cash Flow	C03 = 2.80000
[IRR] [CPT]	Calculate IRR	IRR = 15.51757

Figure 4: Calculating IRR with the HP12C®

Key Strokes	*Explanation*	*Display*
[f] [FIN] [f] [REG]	Clear Memory Registers	0.00000
5 [CHS] [g] [CF_0]	Initial Cash Outlay	–5.00000
1.6 [g] [CF_j]	Period 1 Cash flow	1.60000
2.4 [g] [CF_j]	Period 2 Cash flow	2.40000
2.8 [g] [CF_j]	Period 3 Cash flow	2.80000
[f] [IRR]	Calculate IRR	15.51757

The NPV Decision Rule Versus the IRR Rule

NPV decision rule. The basic idea behind NPV analysis is that if a project has a positive NPV, this amount goes to the firm's shareholders. As such, if a firm undertakes a project with a positive NPV, shareholder wealth is increased.

The NPV decision rules are summarized:

- Accept projects with a positive NPV. Positive NPV projects will increase shareholder wealth.
- Reject projects with a negative NPV. Negative NPV projects will decrease shareholder wealth.
- When two projects are mutually exclusive (only one can be accepted), the project with the higher positive NPV should be accepted.

IRR decision rule. Analyzing an investment (project) using the IRR method provides the analyst with a result in terms of a rate of return.

The following are decision rules of IRR analysis:

- Accept projects with an IRR that is greater than the firm's (investor's) required rate of return.
- Reject projects with an IRR that is less than the firm's (investor's) required rate of return.

Note that for a single project, the IRR and NPV rules lead to exactly the same accept/reject decision. If the IRR is greater than the required rate of return, the NPV is positive, and if the IRR is less than the required rate of return, the NPV is negative.

Problems Associated With the IRR Method

When the acceptance or rejection of one project has no effect on the acceptance or rejection of another, the two projects are considered to be independent projects. When only one of two projects may be accepted, the projects are considered to be mutually exclusive. For mutually exclusive projects, the NPV and IRR methods can give conflicting project rankings. This can happen when the projects' initial costs are of different sizes or when the timing of the cash flows is different. Let's look at an example that illustrates how NPV and IRR can yield conflicting results.

Example: Conflicting decisions between NPV and IRR

Assume NPV and IRR analysis of two mutually exclusive projects produced the results shown in Figure 5. As indicated, the IRR criteria recommends that Project A should be accepted. On the other hand, the NPV criteria indicates acceptance of Project B. Which project should be selected?

Figure 5: Ranking Reversals with NPV and IRR

Project	*Investment at t = 0*	*Cash Flow at t = 1*	*IRR*	*NPV at 10%*
A	–$5,000	$8,000	60%	$2,272.72
B	–$30,000	$40,000	33%	$6,363.64

Answer:

Investing in project A increases shareholder wealth by $2,272.72, while investing in project B increases shareholder wealth by $6,363.64. Since the overall goal of the firm is to maximize shareholder wealth, project B should be selected because it adds the most value to the firm.

Mathematically speaking, the NPV method assumes the reinvestment of a project's cash flows at the opportunity cost of capital, while the IRR method assumes that the reinvestment rate is the IRR. The discount rate used with the NPV approach represents the market-based opportunity cost of capital and is the required rate of return for the shareholders of the firm.

Given that shareholder wealth maximization is the ultimate goal of the firm, always *select the project with the greatest NPV when the IRR and NPV rules provide conflicting decisions.*

LOS 6.b: Define, calculate and interpret a holding period return (total return).

A holding period can be any period of time. The holding period of an investment may be a matter of days or as long as several years. The holding period return is simply the percentage increase in the value of an investment over the period it is held. If the asset has cash flows, such as dividend or interest payments, we refer to the return including the value of these interim cash flows as the total return.

As an example, consider a Treasury bill purchased for $980 and sold three months later for $992. The holding period return can be calculated as:

$$\text{HPR} = \frac{\text{ending value - beginning value}}{\text{beginning value}} = \frac{\text{ending value}}{\text{beginning value}} - 1 \text{ and we have } \frac{992}{980} - 1 = 0.0122 \text{ or } 1.22\%$$

We would say that the investor's 3-month holding period return was 1.22%.

As an example of a security that has interim cash payments, consider a stock that is purchased for $30 and is sold for $33 six months later, during which time it paid $0.50 in dividends. The holding period return (total return in this case) can be calculated as:

$$\text{HPR} = \frac{\text{ending value} - \text{beginning value} + \text{cash flow received}}{\text{beginning value}} = \frac{\text{ending value} + \text{cash flow received}}{\text{beginning value}} - 1$$

$$\text{and we have } \frac{33 + 0.50}{30} - 1 = 0.1167 \text{ or } 11.67\%,$$

which is the investor's total return over the holding period of six months.

LOS 6.c: Calculate, interpret, and distinguish between the money-weighted and time-weighted rates of return of a portfolio and appraise the performance of portfolios based on these measures.

The **money-weighted return** applies the concept of IRR to investment portfolios. The money-weighted rate of return is defined as the internal rate of return on a portfolio, taking into account all cash inflows and outflows. The beginning value of the account is an inflow as are all deposits into the account. All withdrawals from the account are outflows, as is the ending value.

Example: Money-weighted rate of return

Assume an investor buys a share of stock for $100 at t = 0 and at the end of the next year (t = 1), she buys an additional share for $120. At the end of year 2, the investor sells both shares for $130 each. At the end of each year in the holding period, the stock paid a $2.00 per share dividend. What is the money-weighted rate of return?

Step 1: Determine the timing of each cash flow and whether the cash flow is an inflow (+), into the account, or an outflow (–), available from the account.

t = 0:	purchase of first share	=	+$100.00
t = 1:	dividend from first share	=	–$2.00
	purchase of second share	=	+$120.00
	Subtotal, t = 1		+$118.00
t = 2:	dividend from two shares	=	–$4.00
	proceeds from selling shares	=	–$260.00
	Subtotal, t = 2		–$264.00

Step 2: Net the cash flows for each time period and set the PV of cash inflows equal to the present value of cash outflows.

$$PV_{inflows} = PV_{outflows}$$

$$\$100 + \frac{\$118}{(1+r)} = \frac{\$264}{(1+r)^2}$$

Step 3: Solve for *r* to find the money-weighted rate of return. This can be done using trial and error or by using the IRR function on a financial calculator or spreadsheet.

The intuition here is that we deposited $100 into the account at t = 0, then added $118 to the account at t = 1 (which, with the $2 dividend, funded the purchase of one more share at $120), and ended with a total value of $264.

To compute this value with a financial calculator, use these net cash flows and follow the procedure(s) described in Figures 6 or 7 to calculate the IRR.

Net cash flows: CF_0 = +100; CF_1 = +120 – 2 = +118; CF_2 = –260 + –4 = –264

Figure 6: Calculating Money-Weighted Return With the TI Business Analyst II Plus®

Key Strokes	*Explanation*	*Display*
[CF] [2nd][CLR WORK]	Clear Cash Flow Registers	CF0 = 0.00000
100 [ENTER]	Initial Cash Outlay	CF0 = +100.00000
[↓] 118 [ENTER]	Period 1 Cash Flow	C01 = +118.00000
[↓] [↓] 264 [+/–] [ENTER]	Period 2 Cash Flow	C02 = –264.00000
[IRR] [CPT]	Calculate IRR	IRR = 13.86122

Figure 7: Calculating Money-Weighted Return With the HP12C®

Key Strokes	*Explanation*	*Display*
[f] [FIN] [f] [REG]	Clear Memory Registers	0.00000
100 [g] [CF_0]	Initial Cash Outlay	+100.00000
118 [g] [CF_j]	Period 1 Cash Flow	+118.00000
264 [CHS] [g] [CF_j]	Period 2 Cash Flow	–264.00000
[f] [IRR]	Calculate IRR	13.86122

The money-weighted rate of return for this problem is 13.86%.

Time-weighted rate of return measures compound growth. It is the rate at which $1.00 compounds over a specified performance horizon. Time-weighting is the process of averaging a set of values over time. The *annual* time-weighted return for an investment may be computed by performing the following steps:

Step 1: Value the portfolio immediately preceding significant addition or withdrawals. Form subperiods over the evaluation period that correspond to the dates of deposits and withdrawals.

Step 2: Compute the holding period return (HPR) of the portfolio for each subperiod.

Step 3: Compute the product of (1 + HPR) for each subperiod to obtain a total return for the entire measurement period [i.e., $(1 + HPR_1) \times (1 + HPR_2) \ldots (1 + HPR_n)$]. If the total investment period is greater than one year, you must take the geometric mean of the measurement period return to find the annual time-weighted rate of return.

Example: Time-weighted rate of return

A share of stock is purchased at t = 0 for $100, and at the end of the next year, t = 1, another share is purchased for $120. At the end of year 2, both shares are sold for $130 each. At the end of both years 1 and 2, the stock paid a $2.00 per share dividend. What is the time-weighted rate of return for this investment? (This is the same investment as the preceding example.)

Answer:

Step 1: Break the evaluation period into two subperiods based on timing of cash flows.

Holding period 1: Beginning price = \$100.00
Dividends paid = \$2.00
Ending price = \$120.00

Holding period 2: Beginning price = \$240.00 (2 shares)
Dividends paid = \$4.00 (\$2 per share)
Ending price = \$260.00 (2 shares)

Step 2: Calculate the HPR for each holding period.

$HPR_1 = [(\$120 + 2) / \$100] - 1 = 22\%$
$HPR_2 = [(\$260 + 4) / \$240] - 1 = 10\%$

Step 3: Find the compound annual rate that would have produced a total return equal to the return on the account over the 2-year period.

$(1 + \text{time-weighted rate of return})^2 = (1.22)(1.10)$
$\text{time-weighted rate of return} = [(1.22)(1.10)]^{0.5} - 1 = 15.84\%$

This is the ***geometric mean*** return, which we examine in more detail later. In the investment management industry, ***the time-weighted rate of return is the preferred method of performance measurement, because it is not affected by the timing of cash inflows and outflows.***

In the preceding examples, the time-weighted rate of return for the portfolio was 15.84%, while the money-weighted rate of return for the same portfolio was 13.86%. The results are different because the money-weighted rate of return gave a larger weight to the year 2 HPR, which was 10%, versus the 22% HPR for year 1.

If funds are contributed to an investment portfolio just before a period of relatively poor portfolio performance, the money-weighted rate of return will tend to be lower than the time-weighted rate of return. On the other hand, if funds are contributed to a portfolio at a favorable time (just prior to a period of relatively high returns), the money-weighted rate of return will be higher than the time-weighted rate of return. The use of the time-weighted return removes these distortions and thus provides a better measure of a manager's ability to select investments over the period. If the manager has complete control over money flows into and out of an account, the money-weighted rate of return may be the more appropriate performance measure.

LOS 6.d: Calculate and interpret the bank discount yield, holding period yield, effective annual yield, and money market yield for a U.S. Treasury bill; and interpret and convert among holding period yields, money market yields, effective annual yields and the bond equivalent yields.

Pure discount instruments such as U.S. T-bills are quoted differently from U.S. government bonds. T-bills are quoted on a ***bank discount basis***, which is ***based on the face value*** of the instrument instead of the purchase price. The **bank discount yield** (BDY) is computed using the following formula:

$$r_{BD} = \frac{D}{F} \times \frac{360}{t}$$

where:
r_{BD} = the annualized yield on a bank discount basis
D = the dollar discount, which is equal to the difference between the face value of the bill and the purchase price
F = the face value (par value) of the bill
t = number of days remaining until maturity
360 = bank convention of number of days in a year

The *key distinction* of the bank discount yield is that it expresses the dollar discount from the face (par) value as a fraction of the face value, not the market price of the instrument. Another notable feature of the bank discount yield is that it is annualized by multiplying the discount-to-par by 360/t, where the market convention is to use a 360-day year versus a 365-day year. This type of annualizing method assumes no compounding (i.e., simple interest).

Example: Bank discount yield

Calculate the bank discount yield for a T-bill priced at $98,500 with a face value of $100,000 and 120 days until maturity.

Answer:

Substituting the relevant values into the bank discount yield equation in our example, we get:

$$r_{BD} = \frac{1{,}500}{100{,}000} \times \frac{360}{120} = 4.50\%$$

It is important for candidates to realize that a yield quoted on a bank discount basis is not representative of the return earned by an investor for the following reasons:

- Bank discount yield annualizes using simple interest and ignores the effects of compound interest.
- Bank discount yield is based on the face value of the bond, not its purchase price—investment returns should be evaluated relative to the amount invested.
- Bank discount yield is annualized based on a 360-day year rather than a 365-day year.

Holding period yield (HPY) or holding period return, is the total return an investor earns between the purchase date and the sale or maturity date. HPY is calculated using the following formula:

$$HPY = \frac{P_1 - P_0 + D_1}{P_0} = \frac{P_1 + D_1}{P_0} - 1$$

where:
P_0 = initial price of the instrument
P_1 = price received for instrument at maturity
D_1 = interest payment (distribution)

Example: HPY

What is the HPY for a T-bill priced at $98,500 with a face value of $100,000 and 120 days remaining until maturity?

Answer:

Using the HPY equation stated above, we have:

$$\begin{aligned} HPY &= (\$100{,}000 - \$98{,}500) / \$98{,}500 \\ &= \$1{,}500 / \$98{,}500 \\ &= 1.5228\% \end{aligned}$$

$D_1 = 0$ here because T-bills are a pure discount instrument (i.e., they make no interest payments).

The **effective annual yield** (EAY) is an annualized value, based on a 365-day year, that accounts for compound interest. It is calculated using the following equation:

$$EAY = (1 + HPY)^{365/t} - 1$$

Example: EAY

Compute the EAY using the HPY of 1.5228% from the previous example.

Answer:

The HPY is converted to an EAY as follows:

$$EAY = (1.015228)^{365/120} - 1 = 1.047042 - 1 = 4.7042\%$$

Note that we can convert from an EAY to HPY by using the reciprocal of the exponent and

$$(1.047042)^{\frac{120}{365}} - 1 = 1.5228\%.$$

The **money market yield** is equal to the annualized holding period yield, *assuming a 360-day year.* Using the money market yield makes the quoted yield on a T-bill comparable to yield quotes for interest-bearing money market instruments that pay interest on a 360-day basis. The money market yield is $\frac{360}{\#\,days} \times HPY$.

$$r_{MM} = HPY \times (360/t)$$

Given the bank discount yield, r_{BD}, the money market yield, r_{MM}, may be calculated using the equation:

$$r_{MM} = \frac{360 \times r_{BD}}{360 - (t \times r_{BD})}$$

Professor's Note: I couldn't remember this formula with a gun to my head, but I (and you) can easily convert a BDY to an HPY and the HPY to a money market yield using the previous formula.

Example: Money market yield, r_{MM}

What is the money market yield for a 120-day T-bill that has a bank discount yield equal to 4.50%?

Answer:

Given the r_{BD} for the T-bill, the first equation for r_{MM} is applied as follows:

$$r_{MM} = \frac{360 \times 0.045}{360 - (120 \times 0.045)}$$
$$= 16.2 / 354.6$$
$$= 4.569\%$$

Alternatively, we could first calculate the HPY for the T-bill and then annualize that. Actual discount is $0.045 \times \frac{120}{360} = 0.015$. Based on a \$1,000 face value, price = 1,000 (1 – 0.015) = 985 so that $HPY = \frac{1{,}000}{985} - 1$ and $r_{MM} = \left(\frac{1{,}000}{985} - 1\right) \times \frac{360}{120} = 0.04569 = 4.569\%$.

Converting Among Holding Period Yields, Money Market Yields, and Effective Annual Yields

Once we have established HPY, EAY, or r_{MM}, we can use one as a basis for calculating the other two. Remember:

- The HPY is the actual return an investor will receive if the money market instrument is held until maturity.
- The EAY is the annualized HPY on the basis of a 365-day year and incorporates the effects of compounding.
- The r_{MM} is the annualized yield that is based on price and a *360-day year* and does not account for the effects of compounding—it assumes simple interest.

Example: Converting among EAY, HPY and r_{MM}

Assume you purchased a T-bill that matures in 150 days for a price of \$98,000. The broker who sold you the T-bill quoted the money market yield at 4.898%. Compute the HPY and the EAY.

Answer:

Money market to holding period yield—r_{MM} is an annualized yield based on a 360-day year. To change the r_{MM} in this example into its HPY, we need to convert it to a 150-day holding period by multiplying it by (150 / 360). Thus:

$$HPY = r_{MM} \times (150 / 360)$$
$$= 0.04898 \times (150 / 360)$$
$$= 0.02041 = 2.041\%$$

Holding period yield to effective annual yield—the EAY is equal to the annualized HPY based on a 365-day year. Now that we have computed the HPY, simply annualize using a 365-day year to calculate the EAY as follows:

$$EAY = (1 + 0.02041)^{365/150} - 1$$
$$= 1.05039 - 1 = 5.039\%$$

Note that to convert the EAY back into the HPY, apply the reciprocal of the exponent to the EAY. This is the same as taking one plus the EAY to the power (t/365). For example, we can convert the EAY we just calculated back to the HPY as follows:

$$HPY = (1.05039)^{150/365} - 1 = 2.041\%$$

Professor's Note: On the Level 1 CFA® exam, you may be asked to convert from any one of these three yields into one of the others. You should note that the EAY and r_{MM} are merely annualized versions of the HPY. If you concentrate on converting back and forth between the HPY and the other yield figures, you will be well-prepared to answer these types of questions on the Level 1 exam.

Bond Equivalent Yield

The **bond-equivalent yield** refers to 2 × the semiannual discount rate. This convention stems from the fact that yields on U.S. bonds are quoted as twice the semiannual rate because the annual coupon interest is paid in two semiannual payments.

Example: Bond-equivalent yield calculation (1)

A 3-month loan has a holding period yield of 2%. What is the yield on a bond-equivalent basis?

Answer:

The first step is to convert the 3-month yield to a semiannual (6-month) yield:

$$1.02^2 - 1 = 4.04\%$$

The second step it to double it (2 × 4.04 = 8.08%) to get the bond-equivalent yield.

Example: Bond-equivalent yield calculation (2)

The effective annual yield on an investment is 8%. What is the yield on a bond-equivalent basis?

Answer:

The first step is to convert the effective annual yield to a semiannual yield.

$$1.08^{0.5} - 1 = 3.923\%$$

The second step is to double it: 2 × 3.923 = 7.846%.

Key Concepts

1. The NPV is the present value of future cash flows, discounted at the firm's cost of capital, less the project's cost. IRR is the discount rate that makes the NPV = 0 (equates the PV of the expected future cash flows to the project's initial cost).
2. The NPV rule is to accept a project if NPV > 0; the IRR rule is to accept a project if IRR > required rate of return. For an independent (single) project, these rules produce the exact same decision.
3. For mutually exclusive projects, IRR rankings and NPV rankings may differ due to differences in project size or in the timing of the cash flows. Choose the project with the higher NPV.
4. The money-weighted rate of return is the IRR calculated with end-of-period account values and is also the discount rate that makes the PV of cash inflows equal to the PV of cash outflows.
5. The time-weighted rate of return is calculated as the geometric mean of the compound holding period returns.
6. The bank discount yield is the percentage discount from face value, annualized by multiplying by $\frac{360}{\text{days to maturity}}$, $r_{BD} = \frac{D}{F} \times \frac{360}{\text{days}}$.

7. The holding period yield is calculated as:

$$HPY = \frac{P_1 - P_0 + D_1}{P_0} = \frac{P_1 + D_1}{P_0} - 1$$

8. The effective annual yield converts a t-day holding period yield to a compound annual yield based on a 365-day year:

$$EAY = (1 + HPY)^{365/t} - 1$$

9. A money market yield is annualized (without compounding) based on a 360-day year:

$$r_{MM} = HPY \times \frac{360}{t}$$

10. The bond equivalent yield is two times the effective semiannual rate of return.
11. To convert a bank discount yield to a money market yield, the calculation is:

$$r_{MM} = \frac{360 \times r_{BD}}{360 - (t \times r_{BD})}$$

CONCEPT CHECKERS: DISCOUNTED CASH FLOW APPLICATIONS

1. Which of the following statements does NOT accurately describe the IRR and NPV methods?
   A. The discount rate that gives an investment an NPV of zero is the investment's IRR.
   B. The IRR is the discount rate that equates the present value of cash inflows with the present value of cash outflows.
   C. If the NPV and IRR methods give conflicting decisions for mutually exclusive projects, the IRR decision should be used to select the project.
   D. The NPV method assumes that a project's cash flows will be reinvested at the cost of capital, while the IRR method assumes they will be reinvested at the IRR.

2. Which of the following statements does NOT accurately describe the IRR and NPV methods?
   A. A project's IRR can be positive even if the NPV is negative.
   B. A project with an IRR equal to the cost of capital will have an NPV of zero.
   C. A project's NPV may be positive even if the IRR is less than the cost of capital.
   D. Shareholder wealth will decrease if a company takes on a project with a negative NPV.

3. Which of the following statements does NOT accurately describe the IRR and/or NPV methods?
   A. The NPV tells how much the value of the firm has increased if you accept the project.
   B. When evaluating independent projects, the IRR and NPV methods always yield the same accept/reject decisions.
   C. When selecting between mutually exclusive projects, the NPV and IRR methods may give conflicting accept/reject decisions.
   D. When selecting between mutually exclusive projects, the project with the highest NPV should be accepted regardless of the sign of the NPV calculation.

4. A company is considering entering into a joint venture that will require an investment of $10 million. The project is expected to generate cash flows of $4 million, $3 million, and $4 million in each of the next three years, respectively. Assuming a discount rate of 10%, what is the project's NPV?
   A. –$879,000.
   B. –$309,000.
   C. +$243,000.
   D. –$1,523,000.

5. A company is considering entering into a joint venture that will require an investment of $10 million. The project is expected to generate cash flows of $4 million, $3 million, and $4 million in each of the next three years, respectively. Assuming a discount rate of 10%, what is the project's *approximate* IRR?
   A. 5%.
   B. 10%.
   C. 15%.
   D. 20%.

6. What should an analyst recommend based on the following information for two independent projects?

Project	*Investment at t = 0*	*Cash Flow at t = 1*	*IRR*	*NPV at 12%*
A	–$3,000	$5,000	66.67%	$1,464.29
B	–$10,000	$15,000	50.00%	$3,392.86

A. Reject A and reject B.
B. Accept A and reject B.
C. Reject A and accept B.
D. Accept A and accept B.

7. What should an analyst recommend based on the following information for two mutually exclusive projects?

Project	*Investment at t = 0*	*Cash Flow at t = 1*	*IRR*	*NPV at 12%*
A	–$3,000	$5,000	66.67%	$1,464.29
B	–$10,000	$15,000	50.00%	$3,392.86

A. Reject A and reject B.
B. Accept A and reject B.
C. Reject A and accept B.
D. Accept A and accept B.

8. Goodeal Inc. is considering the purchase of a new material handling system for a cost of $15 million. This system is expected to generate a positive cash flow of $1.8 million per year in perpetuity. What is the NPV of the proposed investment if the appropriate discount rate is 10.5%?
A. $2,142,857.
B. $13,200,000.
C. $16,575,000.
D. $17,142,857.

9. Goodeal Inc. is considering the purchase of a new material handling system for a cost of $15 million. This system is expected to generate a positive cash flow of $1.8 million per year in perpetuity. What is the IRR of the proposed investment if the appropriate hurdle rate is 10.5%?
A. 8.33%.
B. 10.5%.
C. 12.0%.
D. 17.1%.

10. Should a company accept a project that has an IRR of 14% and an NPV of $2.8 million if the cost of capital is 12%?
A. Yes, based only on the NPV.
B. Yes, based only on the IRR.
C. Yes, based on the NPV and the IRR.
D. No, based on both the NPV and IRR.

11. Which of the following statements does NOT represent a characteristic of the time-weighted rate of return? It is:
    A. not affected by the timing of cash flows.
    B. used to measure the compound rate of growth of $1 over a stated measurement period.
    C. the preferred method of performance measurement in the investment management industry.
    D. defined as the internal rate of return on an investment portfolio, taking into account all inflows and outflows.

Use the following data to answer Questions 12 and 13.

Assume an investor purchases a share of stock for $50 at time t = 0, and another share at $65 at time t = 1, and at the end of year 1 and year 2, the stock paid a $2.00 dividend. Also, at the end of year 2, the investor sold both shares for $70 each.

12. The money-weighted rate of return on the investment is:
    A. 15.45%.
    B. 18.02%.
    C. 16.73%.
    D. 15.79%.

13. The time-weighted rate of return on the investment is:
    A. 21.83%.
    B. 18.04%.
    C. 20.13%.
    D. 18.27%.

14. What is the bank discount yield for a T-bill that is selling for $99,000, with a face value of $100,000, and 95 days remaining until maturity?
    A. 1.51%.
    B. 4.50%.
    C. 3.79%.
    D. 6.00%.

15. What is the holding period yield for a T-bill that is selling for $99,000 if it has a face value of $100,000 and 95 days remaining until maturity?
    A. 2.03%.
    B. 3.48%.
    C. 3.79%.
    D. 1.01%.

16. What is the effective annual yield for a T-bill that is selling for $99,000 if it has a face value of $100,000 and 95 days remaining until maturity?
    A. 3.94%.
    B. 1.01%.
    C. 4.50%.
    D. 3.79%.

17. What is the money market yield for a T-bill that is selling for $99,000 if it has a face value of $100,000 and 95 days remaining until maturity?
    A. 3.90%.
    B. 3.83%.
    C. 3.79%.
    D. 7.52%.

18. Which of the following is FALSE regarding a bank discount yield?
    A. It is based on a 360 rather than a 365-day year.
    B. It ignores the opportunity to earn compound interest.
    C. It is based on the face value of the bond, not its purchase price.
    D. It reflects the nonannualized return an investor will earn over a security's life.

19. A 175-day T-bill has an effective annual yield of 3.80%. Its money-market yield is *closest* to:
    A. 3.80%.
    B. 3.71%.
    C. 3.65%.
    D. 1.80%.

Comprehensive Problems: Discounted Cash Flow Applications

1. Allison Rogers, CFA, makes the following statement, "The problems with bank discount yields quoted for T-bills is that they aren't yields, they ignore compounding, and they are based on a short year."

   A. Is she correct in all regards?

   B. Which of these problems is/are remedied by using the holding period yield?

   C. Which of these problems is/are remedied by using a money market yield?

   D. Which of these problems is/are remedied by using effective annual yields?

2. L. Adams buys 1,000 shares of Morris Tool stock for $35 per share. One year later the stock is $38 per share and has paid a dividend of $1.50 per share. Adams reinvests the dividends in additional shares at the time. At the end of the second year, the shares are trading for $37 and have paid $2 dividends over the period.

   L. Burns buys 500 shares of Morris Tool stock for $35 per share. One year later the stock is $38 per share and has paid a dividend of $1.50 per share. Burns reinvests the dividends in additional shares at that time and buys 500 additional shares. At the end of the second year, the shares are trading for $37 and have paid $2 in dividends over the period.

   A. Compare the annual time-weighted returns for the accounts of the two investors (no calculation required).

   B. Compare the annual money-weighted returns for the accounts of the two investors (no calculation required).

Answers – Concept Checkers: Discounted Cash Flow Applications

1. C If the NPV and IRR methods give conflicting decisions when selecting among mutually exclusive projects, always select the project with the greatest positive NPV.

2. C A project will have a negative NPV if its IRR is less than the firm's cost of capital.

3. D When selecting between two mutually exclusive projects, neither project should be accepted if they both have a negative NPV.

4. A $NPV = 4 / 1.10 + 3 / (1.10)^2 + 4 / (1.10)^3 - \$10 = -\$0.879038$ million, or –$879,038.

   Calculator approach: CF0 = –10; CF1 = 4; CF2 = 3; CF3 = 4; I = 10 → NPV = –$0.879038 (million).

5. A Use your test-taking skills here. You know from the previous question that the NPV is negative at 10%. Thus, the IRR must be less than 10%. This leaves only choice A to be the answer. Calculator solution: IRR = 4.9%.

6. D Both projects should be accepted since both projects have positive NPVs and will thus increase shareholder wealth.

7. C When the NPV and IRR rankings conflict, always select the project with the highest positive NPV in order to maximize shareholder wealth.

8. A $NPV = PV(\text{cash inflows}) - CF_0 = (1.8 \text{ million} / 0.105) - \$15 \text{ million} = \$2{,}142{,}857$. Accept the project.

9. C As a perpetuity, the following relationship applies: $1.8 million / IRR = $15 million. Thus, IRR = 1.8 / 15 = 12%. Since IRR > cost of capital (hurdle rate), accept the project.

10. C The project should be accepted on the basis of its positive NPV and its IRR, which exceeds the cost of capital.

11. D The money-weighted rate of return is the IRR of an investment's net cash flows.

12. B One way to do this problem is to set up the cash flows so that the PV of inflows = PV of outflows and plug in each of the multiple choices. $50 + 65 / (1 + r) = 2 / (1 + r) + 144 / (1 + r)^2 \rightarrow r = 18.02\%$. Or on your financial calculator, solve for IRR: $-50 - \frac{65-2}{1+IRR} + \frac{2(70+2)}{(1+IRR)^2} = 0$.

Calculating Money-Weighted Return With the TI Business Analyst II Plus®		
Key Strokes	*Explanation*	*Display*
[CF] [2nd] [CLR WORK]	Clear CF Memory Registers	CF0 = 0.00000
50 [+/–] [ENTER]	Initial cash inflow	CF0 =–50.00000
[↓] 63 [+/–][ENTER]	Period 1 cash inflow	C01 =–63.00000
[↓] [↓] 144 [ENTER]	Period 2 cash outflow	C02 =144.00000
[IRR] [CPT]	Calculate IRR	IRR = 18.02210

13. A $HPR_1 = (65 + 2) / 50 - 1 = 34\%$, $HPR_2 = (140 + 4) / 130 - 1 = 10.77\%$

    Time-weighted return $= [(1.34)(1.1077)]^{0.5} - 1 = 21.83\%$

14. C (1,000 / 100,000) × (360 / 95) = 3.79%

15. D (100,000 – 99,000) / 99,000 = 1.01%

16. A $(1 + 0.0101)^{365/95} - 1 = 3.94\%$.

17. B $(360 \times 0.0379) / (360 - (95 \times 0.0379)) = 3.83\%$, or $(1{,}000 / 99{,}000)(360 / 95) = 3.83\%$

18. D This is actually the definition of the holding period yield. All of the other answers are true statements regarding the bank discount yield.

19. B Since the effective yield is 3.8%, we know $\left[\frac{1{,}000}{\text{price}}\right]^{\frac{365}{175}} = 1.038$ and $\text{price} = \left[\frac{1{,}000}{1.038^{\frac{175}{365}}}\right] = \$982.28$ per \$1,000 face.

The money market yield is $\left(\frac{360}{175}\right) \times \text{HPY} = \left(\frac{360}{175}\right)\left(\frac{1{,}000}{982.28} - 1\right) = \frac{360}{175}(0.01804) = 3.711\%$. Alternatively, we can get the HPY from the EAY of 3.8% as $(1.038)^{\frac{175}{365}} - 1 = 1.804\%$.

Answers – Comprehensive Problems: Discounted Cash Flow Applications

1. A. She is correct in all regards. Bank discount yields are not true yields because they are based on a percentage of face (maturity) value instead of on the original amount invested. They are annualized without compounding since the actual discount from face value is simply multiplied by the number of periods in a "year." The "year" used is 360 days, so that is a shortcoming as well.

   B. The holding period yield uses the increase in value divided by the amount invested (purchase price), so it solves the problem that the BDY is not a true yield.

   C. The money market yield is also a true yield (a percentage of the initial investment), but does not solve the other two problems since it does not involve compounding and is based on a 360-day year.

   D. The effective annual yield solves all three shortcomings. It is based on the holding period yield (so it is a true yield), is a compound annual rate, and is based on a 365-day year.

2. A. Both investors have held the same single stock for both periods, so the time-weighted returns must be identical for both accounts.

   B. The performance of the stock (annual total return) was better in the first year than in the second. Since Burns increased his holdings for the second period by more than Adams, the Burns account has a greater weight on the poorer returns in a money-weighted returns calculation and will have a lower annual money-weighted rate of return over the two-year period than Adams.

The following is a review of the Quantitative Methods principles designed to address the learning outcome statements set forth by CFA Institute®. This topic is also covered in:

STATISTICAL CONCEPTS AND MARKET RETURNS

Study Session 2

EXAM FOCUS

This quantitative review is about the uses of descriptive statistics to summarize and portray important characteristics of large sets of data. The two key areas that you should concentrate on are (1) measures of central tendency and (2) measures of dispersion. Measures of central tendency include the arithmetic mean, geometric mean, weighted mean, median, and mode. Measures of dispersion include the range, mean absolute deviation, variance, and standard deviation. These measures quantify the variability of data around its "center." When describing investments, measures of central tendency provide an indication of an investment's expected reward. Measures of dispersion indicate the riskiness of an investment. For the Level 1 exam, you should know the properties of a normal distribution and be able to assess the effects of departures from normality, such as lack of symmetry (skewness) or the extent to which a distribution is peaked (kurtosis).

LOS 7.a: Differentiate between descriptive statistics and inferential statistics, and between a population and a sample, and explain the differences among the types of measurement scales.

The word **statistics** is used to refer to data (e.g., the average return on XYZ stock was 8% over the last ten years) and to the methods we use to analyze data. Statistical methods fall into one of two categories, **descriptive statistics** or **inferential statistics**.

Descriptive statistics are used to summarize the important characteristics of large data sets. The focus of this topic review is on the use of descriptive statistics to consolidate a mass of numerical data into useful information.

Inferential statistics, which will be discussed in subsequent topic reviews, pertain to the procedures used to make forecasts, estimates, or judgments about a large set of data on the basis of the statistical characteristics of a smaller set (a sample).

A **population** is defined as the set of all possible members of a stated group. A cross-section of the returns of all of the stocks traded on the New York Stock Exchange (NYSE) is an example of a population.

It is frequently too costly or time consuming to obtain measurements for every member of a population, if it is even possible. In this case, a sample may be used. A **sample** is defined as a subset of the population of interest. Once a population has been defined, a sample can be drawn from the population, and the sample's characteristics can be used to describe the population as a whole. For example, a sample of 30 stocks may be selected from among all of the stocks listed on the NYSE to represent the population of all NYSE-traded stocks.

Types of Measurement Scales

Different statistical methods use different levels of measurement, or measurement scales. **Measurement scales** may be classified into one of four major categories:

- **Nominal scales.** Nominal scales are the least accurate level of measurement. Observations are classified or counted with no particular order. An example would be assigning the number 1 to a municipal bond fund, the number 2 to a corporate bond fund, and so on for each fund style.

- **Ordinal scales.** Ordinal scales represent a higher level of measurement than nominal scales. When working with an ordinal scale, every observation is assigned to one of several categories. Then these categories are ordered with respect to a specified characteristic. For example, the ranking of 1,000 small cap growth stocks by performance may be done by assigning the number 1 to the 100 best performing stocks, the number 2 to the next 100 best performing stocks, and so on to the assignment of the number 10 to the 100 worst performing stocks. Based on this type of measurement, it can be concluded that a stock ranked 3 is better than a stock ranked 4, but the scale reveals nothing about performance differences or whether the difference between a 3 and a 4 is the same as the difference between a 4 and a 5.
- **Interval scale.** Interval scale measurements provide relative ranking, like ordinal scales, plus the assurance that differences between scale values are equal. Temperature measurement in degrees is a prime example. Certainly, 49°C is hotter than 32°C, and the temperature difference between 49°C and 32°C is the same as the difference between 67°C and 50°C. The weakness of the interval scale is that a measurement of zero does not necessarily indicate the total absence of what we are measuring. This means that interval-scale-based ratios are meaningless. For example, 30°F is not three times as hot as 10°F.
- **Ratio scales.** Ratio scales represent the most refined level of measurement. Ratio scales provide ranking and equal differences between scale values, and they also have a true zero point as the origin. The measurement of money is a good example. If you have zero dollars, you have no purchasing power, but if you have $4.00, you have twice as much purchasing power as a person with $2.00.

LOS 7.b: Explain the concepts of a parameter, a sample statistic, a frequency distribution, and relative versus cumulative relative frequency distributions.

A measure used to describe a characteristic of a population is referred to as a **parameter**. While many population parameters exist, investment analysis usually utilizes just a few, particularly the mean return and the standard deviation of returns.

In the same manner that a parameter may be used to describe a characteristic of a population, a **sample statistic** is used to measure a characteristic of a sample.

A **frequency distribution** is a tabular presentation of statistical data that aids the analysis of large data sets. Frequency distributions summarize statistical data by assigning it to specified groups, or intervals. Also, the data employed with a frequency distribution may be measured using any type of measurement scale.

Professor's Note: Intervals are also known as classes.

The following procedure describes how to **construct a frequency distribution.**

Step 1: *Define the intervals.* The first step in building a frequency distribution is to define the intervals to which data measurements (observations) will be assigned. An interval, also referred to as a class, is the set of values that an observation may take on. The range of values for each interval must have a lower and upper limit and be all-inclusive and nonoverlapping. Intervals must be *mutually exclusive* in a way that each observation can be placed in only one interval, and the total set of intervals should cover the total range of values for the entire population.

The number of intervals used is an important consideration. If too few intervals are used, the data may be too broadly summarized, and important characteristics may be lost. On the other hand, if too many intervals are used, the data may not be summarized enough.

Step 2: *Tally the observations.* After the intervals have been defined, the observations must be tallied, or assigned to their appropriate interval.

Step 3: *Count the observations.* Having tallied the data set, the number of observations that are assigned to each interval must be counted. The *absolute frequency*, or simply the frequency, is the actual number of observations that fall within a given interval.

Example: Constructing a frequency distribution

Use the data in Figure 1 to construct a frequency distribution for the returns on Intelco's common stock.

Figure 1: Annual Returns for Intelco, Inc. Common Stock

10.4%	22.5%	11.1%	–12.4%
9.8%	17.0%	2.8%	8.4%
34.6%	–28.6%	0.6%	5.0%
–17.6%	5.6%	8.9%	40.4%
–1.0%	–4.2%	–5.2%	21.0%

Answer:

Step 1: *Defining the interval.* For Intelco's stock, the range of returns is 69.0% (–28.6% → 40.4%). Using a return interval of 1% would result in 69 separate intervals, which in this case is too many. So let's use eight nonoverlapping intervals with a width of 10%. The lowest return intervals will be $-30\% \leq R_t < -20\%$, and the intervals will increase to $40\% \leq R_t \leq 50\%$.

Step 2: *Tally the observations and count the observations within each interval.* The tallying and counting of the observations is presented in Figure 2.

Figure 2: Tally and Interval Count for Returns Data

Interval	*Tallies*	*Absolute Frequency*
$-30\% \leq R_t < -20\%$	/	1
$-20\% \leq R_t < -10\%$	//	2
$-10\% \leq R_t < 0\%$	///	3
$0\% \leq R_t < 10\%$	///// //	7
$10\% \leq R_t < 20\%$	///	3
$20\% \leq R_t < 30\%$	//	2
$30\% \leq R_t < 40\%$	/	1
$40\% \leq R_t \leq 50\%$	/	1
Total		20

Tallying and counting the observations generates a frequency distribution that summarizes the pattern of annual returns on Intelco common stock. Notice that the interval with the greatest (absolute) frequency is the ($0\% \leq R_t < 10\%$) interval, which includes seven return observations. For any frequency distribution, the interval with the greatest frequency is referred to as the **modal interval**.

Relative Frequencies and Cumulative Relative Frequencies

The relative frequency is another useful way to present data. The relative frequency is calculated by dividing the absolute frequency of each return interval by the total number of observations. Simply stated, relative frequency is the percentage of total observations falling within each interval. Continuing with our example, the relative frequencies are presented in Figure 3.

Figure 3: Relative Frequencies

Interval	*Frequency*	*Relative Frequency*
$-30\% \leq R_t < -20\%$	1	1/20 = 0.05, or 5%
$-20\% \leq R_t < -10\%$	2	2/20 = 0.10, or 10%
$-10\% \leq R_t < 0\%$	3	3/20 = 0.15, or 15%
$0\% \leq R_t < 10\%$	7	7/20 = 0.35, or 35%
$10\% \leq R_t < 20\%$	3	3/20 = 0.15, or 15%
$20\% \leq R_t < 30\%$	2	2/20 = 0.10, or 10%
$30\% \leq R_t < 40\%$	1	1/20 = 0.05, or 5%
$40\% \leq R_t \leq 50\%$	1	1/20 = 0.05, or 5%
Total	20	100%

It is also possible to compute the **cumulative absolute frequency** and **cumulative relative frequency** by summing the absolute or relative frequencies starting at the lowest interval and progressing through the highest. The cumulative absolute frequencies and cumulative relative frequencies for the Intelco stock returns example are presented in Figure 4.

Figure 4: Cumulative Frequencies

Interval	*Absolute Frequency*	*Relative Frequency*	*Cumulative Absolute Frequency*	*Cumulative Relative Frequency*
$-30\% \leq R_t < -20\%$	1	5%	1	5%
$-20\% \leq R_t < -10\%$	2	10%	3	15%
$-10\% \leq R_t < 0\%$	3	15%	6	30%
$0\% \leq R_t < 10\%$	7	35%	13	65%
$10\% \leq R_t < 20\%$	3	15%	16	80%
$20\% \leq R_t < 30\%$	2	10%	18	90%
$30\% \leq R_t < 40\%$	1	5%	19	95%
$40\% \leq R_t \leq 50\%$	1	5%	20	100%
Total	20	100%		

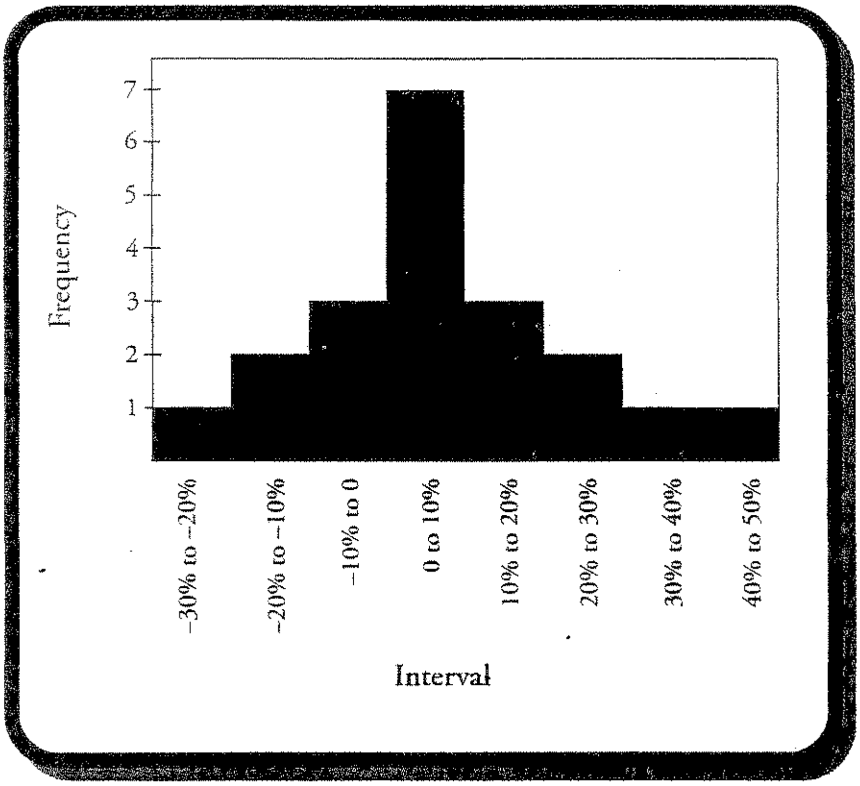
Notice that the cumulative absolute frequency or cumulative relative frequency for any given interval is the sum of the absolute or relative frequencies up to and including the given interval. For example, the cumulative absolute frequency for the ($0\% \leq R_t < 10\%$) interval is 13 = 1 + 2 + 3 + 7 and the cumulative relative frequency for this interval is 5% + 10% + 15% + 35% = 65%.

LOS 7.c: Analyze and interpret a histogram or a frequency polygon by describing the properties of a dataset.

A **histogram** is the graphical presentation of the absolute frequency distribution. A histogram is simply a bar chart of continuous data that has been classified into a frequency distribution. The attractive feature of a histogram is that it allows us to quickly see where most of the observations are concentrated.

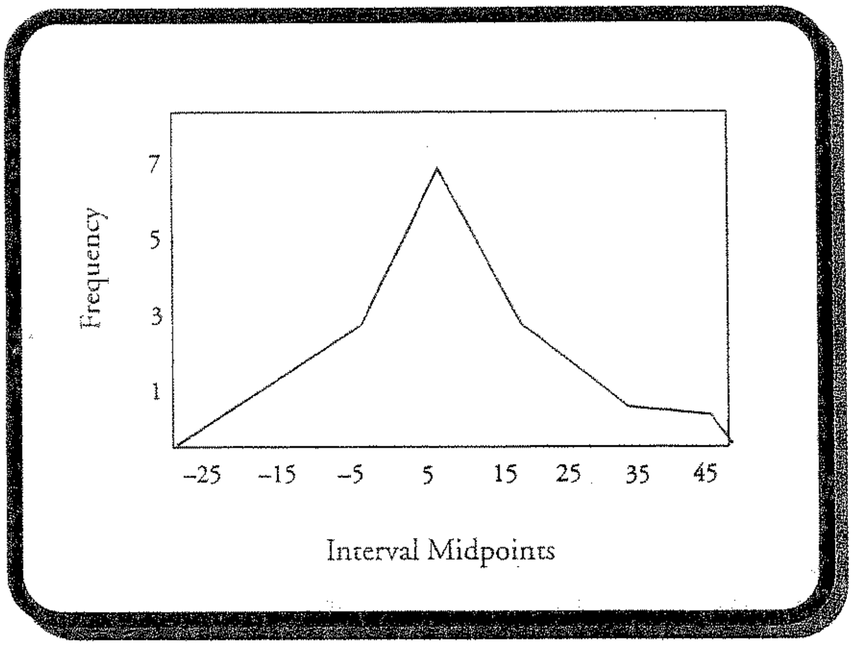
To construct a histogram, the intervals are scaled on the horizontal axis and the absolute frequencies are scaled on the vertical axis. The histogram for the relative frequency data in Figure 3 is provided in Figure 5.

Figure 5: Histogram of Stock Return Data



To construct a **frequency polygon**, the midpoint of each interval is plotted on the horizontal axis, and the absolute frequency for that interval is plotted on the vertical axis. Each point is then connected with a straight line. The frequency polygon for the returns data used in our example is in Figure 6.

Figure 6: Frequency Polygon of Stock Return Data



LOS 7.d: Define, calculate, and interpret measures of central tendency, including the population mean, sample mean, arithmetic mean, weighted average or mean (including a portfolio return viewed as a weighted mean), geometric mean, harmonic mean, median, and mode, quartiles, quintiles, deciles, and percentiles.

Measures of central tendency identify the center, or average, of a data set. This central point can then be used to represent the typical, or expected, value in the data set.

To compute the **population mean**, all the observed values in the population are summed (ΣX) and divided by the number of observations in the population, N. Note that the population mean is unique in that a given population only has one mean. The population mean is expressed as:

$$\mu = \frac{\sum_{i=1}^{N} X_i}{N}$$

The **sample mean** is the sum of all the values in a sample of a population, ΣX, divided by the number of observations in the sample, n. It is used to make *inferences* about the population mean. The sample mean is expressed as:

$$\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$$

Note the use of n, the sample size, versus N, the population size.

Example: Population mean and sample mean

Assume you and your research assistant are evaluating the stock of AXZ Corporation. You have calculated the stock returns for AXZ over the last 12 years to develop the data set shown below. Your research assistant has decided to conduct his analysis using only the returns for the five most recent years, which are displayed as the bold numbers in the data set. Given this information, calculate the population mean and the sample mean.

Data set: 12%, *25%*, *34%*, 15%, *19%*, 44%, *54%*, 33%, 22%, 28%, *17%*, 24%

Answer:

$$\mu = \text{population mean} = \frac{12+25+34+15+19+44+54+33+22+28+17+24}{12} = 27.25\%$$

$$\overline{X} = \text{sample mean} = \frac{25+34+19+54+17}{5} = 29.8\%$$

The population mean and sample mean are both examples of **arithmetic means**. The arithmetic mean is the sum of the observation values divided by the number of observations. It is the most widely used measure of central tendency and has the following properties:

- All interval and ratio data sets have an arithmetic mean.
- All data values are considered and included in the arithmetic mean computation.
- A data set has only one arithmetic mean (i.e., the arithmetic mean is unique).
- The sum of the deviations of each observation in the data set from the mean is always zero.

The arithmetic mean is the only measure of central tendency for which the sum of the deviations from the mean is zero. Mathematically, this property can be expressed as follows:

$$\text{sum of mean deviations} = \sum_{i=1}^{n}(X_i - \overline{X}) = 0$$

Example: Arithmetic mean and deviations from the mean

Compute the arithmetic mean for a data set described as:

Data set: [5, 9, 4, 10]

Answer:

The arithmetic mean of these numbers is:

$$\overline{X} = \frac{5+9+4+10}{4} = 7$$

The sum of the deviations from the mean is:

$$\sum_{i=1}^{n}(X_i - \overline{X}) = (5-7)+(9-7)+(4-7)+(10-7) = -2+2-3+3 = 0$$

Two weaknesses of the arithmetic mean are:

- *Extreme values.* Unusually large or small values can have a disproportionate effect on the computed value for the arithmetic mean.
- *Unknown number of observations.* The arithmetic mean cannot be determined for an open-ended data set (i.e., *n* is unknown).

The computation of a **weighted mean** recognizes that different observations may have a disproportionate influence on the mean. The weighted mean of a set of numbers is computed with the following equation:

$$\overline{X}_W = \sum_{i=1}^{n} w_i X_i = (w_1X_1 + w_2X_2 + \ldots + w_nX_n)$$

where:
$X_1, X_2, \ldots X_n$ = observed values
$w_1, w_2, \ldots w_n$ = corresponding weights associated with each of the observations such that $\Sigma w_i = 1$

Example: Weighted mean as a portfolio return

A portfolio consists of 50% common stocks, 40% bonds, and 10% cash. If the return on common stocks is 12%, the return on bonds is 7%, and the return on cash is 3%, what is the return to the portfolio?

Answer:

$$\overline{X}_W = w_{stock}R_{stock} + w_{bonds}R_{bonds} + w_{cash}R_{cash}$$

$$\overline{X}_W = (0.50 \times 0.12) + (0.40 \times 0.07) + (0.10 \times 0.03) = 0.091\text{, or } 9.1\%$$

The example illustrates an extremely important investments concept: *the return for a portfolio is the weighted average of the returns of the individual assets in the portfolio.* Asset weights are market weights, the market value of the asset relative to the market value of the entire portfolio.

The **median** is the midpoint of a data set when the data is arranged in ascending or descending order. Half the observations lie above the median and half are below. To determine the median, arrange the data from the highest to the lowest value, or lowest to highest value, and find the middle observation.

The median is important because the arithmetic mean can be affected by extremely large or small values (outliers). When this occurs, the median is a better measure of central tendency than the mean because it is not affected by extreme values.

Example: The median using an odd number of observations

What is the median return for five portfolio managers with 10-year annualized total returns record of: 30%, 15%, 25%, 21%, and 23%?

Answer:

First, arrange the returns in descending order.

30%, 25%, 23%, 21%, 15%

Then, select the observation that has an equal number of observations above and below it—the one in the middle. For the given data set, the third observation, 23%, is the median value.

Example: The median using an even number of observations

Suppose we add a sixth manager to the previous example with a return of 28%. What is the median return?

Answer:

Arranging the returns in descending order gives us:

30%, 28%, 25%, 23%, 21%, 15%

With an even number of observations, there is no single middle value. The median value in this case is the arithmetic mean of the two middle observations, 25% and 23%. Thus, the median return for the six managers is 24.0% = 0.5(25 + 23).

The **mode** is the value that occurs most frequently in a data set. A data set may have more than one mode or even no mode. When a distribution has one value that appears most frequently, it is said to be **unimodal**. When a set of data has two or three values that occur most frequently, it is said to be **bimodal** or **trimodal**, respectively.

Example: The mode

What is the mode of the following data set?

Data set: [30%, 28%, 25%, 23%, 28%, 15%, 5%]

Answer:

The mode is 28% because it is the value appearing most frequently.

The **geometric mean** is often used when calculating investment returns over multiple periods or when measuring compound growth rates. The general formula for the geometric mean, *G*, is as follows:

$$G = \sqrt[n]{X_1 \times X_2 \times \ldots \times X_n} = (X_1 \times X_2 \times \ldots \times X_n)^{1/n}$$

Note that this equation has a solution only if the product under the radical sign is non-negative.

When calculating the geometric mean for a returns data set, it is necessary to add 1 to each value under the radical and then subtract 1 from the result.

The geometric mean return (R_G) can be computed using the following equation:

$$1 + R_G = \sqrt[n]{(1 + R_1) \times (1 + R_2) \times \ldots \times (1 + R_n)}$$

where:
R_t = the return for period *t*

Example: Geometric mean return

For the last three years, the returns for Acme Corporation common stock have been −9.34%, 23.45%, and 8.92%. Compute the geometric mean return.

Answer:

$$1+R_G = \sqrt[3]{(-0.0934+1)\times(0.2345+1)\times(0.0892+1)}$$

$$1+R_G = \sqrt[3]{0.9066\times1.2345\times1.0892} = \sqrt[3]{1.21903} = (1.21903)^{1/3} = 1.06825$$

$$R_G = 1.06825 - 1 = 6.825\%$$

Solving this type of problem with your calculator is done as follows:

- On the TI, enter 1.21903; [y^x] 0.33333; [=]
- On the HP, enter 1.21903; [ENTER]; 0.33333; [y^x]

Note that the 0.33333 represents the one-third power.

Professor's Note: The geometric mean is always less than or equal to the arithmetic mean, and the difference increases as the dispersion of the observations increases from period to period. The only time the arithmetic and geometric means are equal is when there is no variability in the observations (i.e., all observations are equal).

A **harmonic mean** is used for certain computations, such as the average cost of shares purchased over time. The harmonic mean is calculated as $\frac{N}{\sum_{i=1}^{N}\frac{1}{X_i}}$, where there are N values of X_i.

Example: Calculating average cost with the harmonic mean

An investor purchases $1,000 of stock each month, and over the last three months the prices paid per share were $8, $9, and $10. What is the average cost per share for the shares acquired?

Answer:

$$\bar{X}_H = \frac{3}{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}} = \$8.926 \text{ per share}$$

To check this result, calculate the total shares purchased as $\frac{1{,}000}{8}+\frac{1{,}000}{9}+\frac{1{,}000}{10} = 336.11$ shares. The average price is $\frac{\$3{,}000}{336.11} = \$8.926$ per share.

The previous example illustrates the interpretation of the harmonic mean in its most common application. Note that the average price paid per share ($8.93) is less than the arithmetic average of the share prices, $\frac{8+9+10}{3} = 9$.

Also note that for values that are not all equal: harmonic mean < geometric mean < arithmetic mean.

Quartiles, Quintiles, Deciles, and Percentiles

Quantile is the general term for a value at or below which a stated proportion of the data in a distribution lies. Examples of quantiles include:

- *Quartiles*—the distribution is divided into quarters.
- *Quintile*—the distribution is divided into fifths.
- *Decile*—the distribution is divided into tenths.
- *Percentile*—the distribution is divided into hundredths (percents).

Note that any quantile may be expressed as a percentile. For example, the third quartile partitions the distribution at a value such that three-fourths, or 75%, of the observations fall below that value. Thus, the third quartile is the 75th percentile.

The formula for the position of the observation at a given percentile, *y*, with *n* data points sorted in ascending order is:

$$L_y = (n+1)\frac{y}{100}$$

Example: Quartiles

What is the third quartile for the following distribution of returns?

8%, 10%, 12%, 13%, 15%, 17%, 17%, 18%, 19%, 23%, 24%

Answer:

The third quartile is the point below which 75% of the observations lie. Recognizing that there are 11 observations in the data set, the third quartile can be identified as:

$$L_y = (11+1)\times\frac{75}{100} = 9$$

When the data is arranged in ascending order, the third quartile is the ninth data point from the left, or 19%. This means that 75% of all observations lie below 19%.

As you will see in the next example, if *L* is not a whole number, linear interpolation must be used to find the quantile.

Example: Quartiles

What is the third quartile for the following distribution of returns?

8%, 10%, 12%, 13%, 15%, 17%, 17%, 18%, 19%, 23%, 24%, 26%

Answer:

With 12 observations in this data set, the third quartile can be identified as:

$$L_y = (12+1)\times\frac{75}{100} = 9.75$$

This means that when the data is arranged in ascending order, the third quartile (75th percentile) is the ninth data point from the left, plus 0.75 × (distance between the 9th and 10th data values). Specifically, the third quartile is [19 + 0.75 × (23 – 19)] = 22%, indicating that 75% of all observations lie below 22%.

LOS 7.e: Define, calculate, and interpret 1) a range and mean absolute deviation, and 2) a sample and a population variance and standard deviation.

Dispersion is defined as the *variability around the central tendency*. The common theme in finance and investments is the tradeoff between reward and variability, where the central tendency is the measure of the reward and dispersion is a measure of risk.

The **range** is a relatively simple measure of variability, but when used with other measures it provides extremely useful information. The range is the distance between the largest and the smallest value in the data set, or:

range = maximum value – minimum value

Example: The range

What is the range for the 5-year annualized total returns for five investment managers if the managers' individual returns were 30%, 12%, 25%, 20%, and 23%?

Answer:

range = 30 – 12 = 18%

The **mean absolute deviation** (MAD) is the average of the absolute values of the deviations of individual observations from the arithmetic mean.

$$\text{MAD} = \frac{\sum_{i=1}^{n} \left| X_i - \overline{X} \right|}{n}$$

The computation of the MAD uses the absolute values of each deviation from the mean because the sum of the actual deviations from the arithmetic mean is zero.

Example: MAD

What is the MAD of the investment returns for the five managers discussed in the preceding example? How is it interpreted?

Answer:

annualized returns: [30%, 12%, 25%, 20%, 23%]

$$\overline{X} = \frac{[30 + 12 + 25 + 20 + 23]}{5} = 22\%$$

$$\text{MAD} = \frac{\left[|30-22| + |12-22| + |25-22| + |20-22| + |23-22|\right]}{5}$$

$$\text{MAD} = \frac{[8 + 10 + 3 + 2 + 1]}{5} = 4.8\%$$

This result can be interpreted to mean that, on average, an individual return will deviate ±4.8% from the mean return of 22%.

The **population variance** is defined as the average of the squared deviations from the mean. The population variance (σ^2) uses the values for all members of a population and is calculated using the following formula:

$$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}$$

Example: Population variance, σ^2

Assume the 5-year annualized total returns for the five investment managers used in the earlier example represent *all* of the managers at a small investment firm. What is the population variance of returns?

Answer:

$$\mu = \frac{[30 + 12 + 25 + 20 + 23]}{5} = 22\%$$

$$\sigma^2 = \frac{\left[(30-22)^2 + (12-22)^2 + (25-22)^2 + (20-22)^2 + (23-22)^2\right]}{5} = 35.60\left(\%^2\right)$$

Interpreting this result, we can say that the average variation from the mean return is 35.60% squared. Had we done the calculation using decimals instead of whole percents, the variance would be 0.00356. What is a percent squared? Yes, this is nonsense, but let's see what we can do so that it makes more sense.

As you have just seen, a major problem with using the variance is the difficulty of interpreting it. The computed variance, unlike the mean, is in terms of squared units of measurement. How does one interpret squared percents, squared dollars, or squared yen? This problem is mitigated through the use of the standard deviation. The **population standard deviation**, σ, is the square root of the population variance and is calculated as follows:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(X - \mu)^2}{N}}$$

Example: Population standard deviation, σ

Using the data from the preceding examples, compute the population standard deviation.

Answer:

$$\sigma = \sqrt{\frac{[(30-22)^2 + (12-22)^2 + (25-22)^2 + (20-22)^2 + (23-22)^2\ ]}{5}}$$

$$= \sqrt{35.60} = 5.97\%$$

Calculated with decimals instead of whole percents, we would get:

$$\sigma^2 = 0.00356 \text{ and } \sigma = \sqrt{0.00356} = 0.05966 = 5.97\%$$

Since the population standard deviation and population mean are both expressed in the same units (percent), these values are easy to relate. The outcome of this example indicates that the mean return is 22% and the standard deviation about the mean is 5.97%. Note that this is greater than the MAD of 4.8%, a result ($\sigma >$ MAD) that holds in general.

The **sample variance,** s^2, is the measure of dispersion that applies when we are evaluating a sample of n observations from a population. The sample variance is calculated using the following formula:

$$s^2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$$

The most noteworthy difference from the formula for population variance is that the denominator for s^2 is n – 1, one less than the sample size n, where σ^2 uses the entire population size N. Another difference is the use of the sample mean, $\overline{X}$, instead of the population mean, μ. Based on the mathematical theory behind statistical procedures, the use of the entire number of sample observations, n, instead of n – 1 as the divisor in the computation of s^2, will systematically *underestimate* the population parameter, σ^2, particularly for small sample sizes. This systematic underestimation causes the sample variance to be what is referred to as a **biased estimator** of the population variance. Using n – 1 instead of n in the denominator, however, improves the statistical properties of s^2 as an estimator of σ^2. Thus, s^2, as expressed in the equation above, is considered to be an unbiased estimator of σ^2.

Example: Sample variance

Assume that the 5-year annualized total returns for the five investment managers used in the preceding examples represent only a sample of the managers at a large investment firm. What is the sample variance of these returns?

Answer:

$$\overline{X} = \frac{[30+12+25+20+23]}{5} = 22\%$$

$$s^2 = \frac{\left[(30-22)^2 + (12-22)^2 + (25-22)^2 + (20-22)^2 + (23-22)^2\right]}{5-1} = 44.5\left(\%^2\right)$$

Thus, the sample variance of $44.5(\%^2)$ can be interpreted to be an unbiased estimator of the population variance.

As with the population standard deviation, the **sample standard deviation** can be calculated by taking the square root of the sample variance. The sample standard deviation, s, is defined as:

$$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}}$$

Example: Sample standard deviation

Compute the sample standard deviation based on the result of the preceding example.

Answer:

Since the sample variance for the preceding example was computed to be $44.5(\%^2)$, the sample standard deviation is:

$$s = [44.5(\%^2)]^{1/2} = 6.67\%$$

The results shown here mean that the sample standard deviation, $s = 6.67\%$, can be interpreted as an unbiased estimator of the population standard deviation, σ.

LOS 7.f: Contrast variance with semivariance and target semivariance.

Recall that variance measures the average squared deviation from the mean and that the average is based on $n - 1$ when calculating the variance of a sample.

Semivariance is calculated in the same manner, but only those observations that fall below the mean are included in the calculation. The formula is then:

$$\frac{\sum\limits_{\text{All } X_i < \bar{X}} (X_i - \bar{X})^2}{(\#\text{ of Xs less than } \bar{X}) - 1}$$

Semivariance is sometimes described as a measure of "downside risk" in an investments context. Note that for symmetric distributions, ranking portfolios by their semivariance will be equivalent to ranking them based on variance, since semivariance contains no additional information. For skewed distributions, the semivariance can provide additional information that variance does not. Even though semivariance has the appealing intuition of measuring downside risk, mathematically it does not have the attractive properties that variance does (e.g., we cannot sum semivariance for a portfolio of assets).

A related measure, **target semivariance**, is based on observations below a specific value. For example, we may be concerned with only negative returns values or values below an arbitrary returns "target," such as 4%. The calculation is the same as that of semivariance, with only observations below the target rate included. Target semivariance is a measure of "downside risk" relative to the target rate of return or target value.

LOS 7.g: Calculate and interpret the proportion of observations falling within a specified number of standard deviations of the mean, using Chebyshev's inequality.

Chebyshev's inequality states that for any set of observations, whether sample or population data and regardless of the shape of the distribution, the percentage of the observations that lie within *k* standard deviations of the mean is *at least* $1 - 1/k^2$ for all $k > 1$.

Example: Chebyshev's inequality

What is the minimum percentage of any distribution that will lie within ±2 standard deviations of the mean?

Answer:

Applying Chebyshev's inequality, we have:

$1 - 1/k^2 = 1 - 1/2^2 = 1 - 1/4 = 0.75$ or 75%

According to Chebyshev's inequality, the following relationships hold for any distribution. At least:

- 36% of observations lie within ±1.25 standard deviations of the mean.
- 56% of observations lie within ±1.50 standard deviations of the mean.
- 75% of observations lie within ±2 standard deviations of the mean.
- 89% of observations lie within ±3 standard deviations of the mean.
- 94% of observations lie within ±4 standard deviations of the mean.

The importance of Chebyshev's inequality is that it applies to any distribution. If we actually know the underlying distribution is normal, for example, we can be even more precise about the percentage of observations that will fall within 2 or 3 standard deviations of the mean.

LOS 7.h: Define, calculate, and interpret the coefficient of variation and the Sharpe ratio.

A direct comparison between two or more measures of dispersion may be difficult. For instance, suppose you are comparing the annual returns distribution for retail stocks with a mean of 8% and an annual returns distribution for a real estate portfolio with a mean of 16%. A direct comparison between the dispersion of the two distributions is not meaningful because of the relatively large difference in their means. To make a meaningful comparison, a relative measure of dispersion must be used. **Relative dispersion** is the amount of variability in a distribution relative to a reference point or benchmark. Relative dispersion is commonly measured with the **coefficient of variation** (CV), which is computed as:

$$CV = \frac{s_x}{\overline{X}} = \frac{\text{standard deviation of x}}{\text{average value of x}}$$

CV measures the amount of dispersion in a distribution relative to the distribution's mean. It is useful because it enables us to make a direct comparison of dispersion across different sets of data. In an investments setting, the CV is used to measure the risk (variability) per unit of expected return (mean).

Example: Coefficient of variation

You have just been presented with a report that indicates that the mean monthly return on T-bills is 0.25% with a standard deviation of 0.36%, and the mean monthly return for the S&P 500 is 1.09% with a standard deviation of 7.30%. Your unit manager has asked you to compute the CV for these two investments and to interpret your results.

Answer:

$$CV_{\text{T-bills}} = \frac{0.36}{0.25} = 1.44$$

$$CV_{\text{S\&P 500}} = \frac{7.30}{1.09} = 6.70$$

These results indicate that there is less dispersion (risk) per unit of monthly return for T-bills than there is for the S&P 500 (1.44 versus 6.70).

Professor's Note: To remember the formula for CV, remember that the coefficient of variation is a measure of variation, so standard deviation goes in the numerator. CV is variation per unit of return.

The Sharpe Ratio

The **Sharpe measure** (a.k.a., the *Sharpe ratio* or *reward-to-variability ratio*) is widely used for investment performance measurement to measure *excess* return per unit of risk. The Sharpe measure appears over and over throughout the CFA® curriculum. It is defined according to the following formula:

$$\text{Sharpe ratio} = \frac{\overline{r_p} - \overline{r_f}}{\sigma_p}$$

where:

$\overline{r_p}$ = portfolio return

$\overline{r_f}$ = risk-free return

σ_p = standard deviation of portfolio returns

Notice that the numerator of the Sharpe ratio uses a measure for a risk-free return. As such, the quantity $(\overline{r_p} - \overline{r_f})$, referred to as the **excess return** on Portfolio p, measures the extra reward that investors receive for exposing themselves to risk. Portfolios with large Sharpe ratios are preferred to portfolios with smaller ratios because it is assumed that rational investors prefer return and dislike risk.

Example: The Sharpe ratio

Assume that the mean monthly return on T-bills is 0.25% and that the mean monthly return and standard deviation for the S&P 500 are 1.30% and 7.30%, respectively. Using the T-bill return to represent the risk-free rate, as is common in practice, compute and interpret the Sharpe ratio.

Answer:

$$\text{Sharpe ratio} = \frac{1.30 - 0.25}{7.30} = 0.144$$

The Sharpe ratio of 0.144 indicates that the S&P 500 earned 0.144% of excess return per unit of risk, where risk is measured by standard deviation of portfolio returns.

LOS 7.i: Define and interpret skew, explain the meaning of a positively or negatively skewed return distribution, and describe the relative locations of the mean, median, and mode for a nonsymmetrical distribution.

A distribution is **symmetrical** if it is shaped identically on both sides of its mean. Distributional symmetry implies that intervals of losses and gains will exhibit the same frequency. For example, a symmetrical distribution with a mean return of zero will have losses in the –6% to –4% interval as frequently as it will have gains in the +4% to +6% interval. The extent to which a returns distribution is symmetrical is important because the degree of symmetry tells analysts if deviations from the mean are more likely to be positive or negative.

Skewness, or skew, refers to the extent to which a distribution is not symmetrical. Nonsymmetrical distributions may be either positively or negatively skewed and result from the occurrence of outliers in the data set. **Outliers** are observations with extraordinarily large values, either positive or negative.

- A *positively skewed* distribution is characterized by many outliers in the upper region, or right tail. A positively skewed distribution is said to be skewed right because of its relatively long upper (right) tail.
- A *negatively skewed* distribution has a disproportionately large amount of outliers that fall within its lower (left) tail. A negatively skewed distribution is said to be skewed left because of its long lower tail.

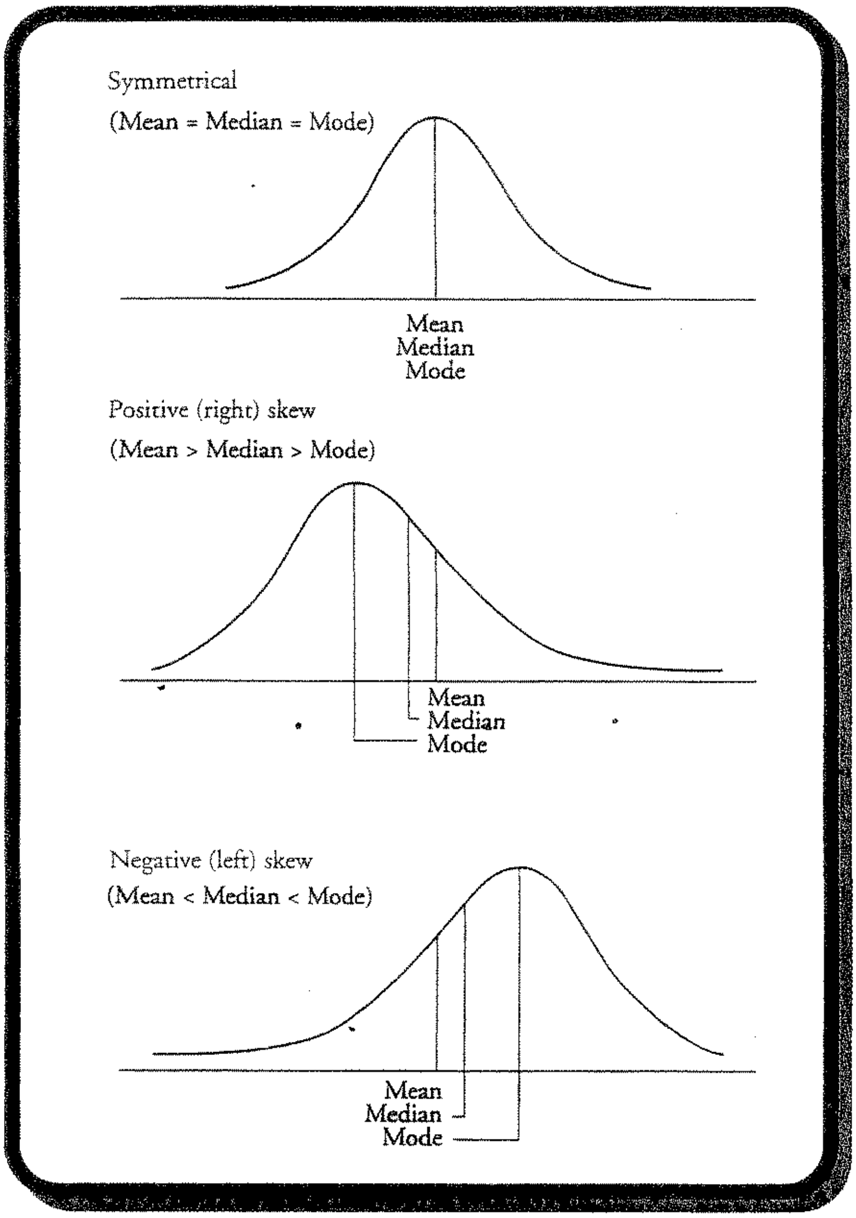
Mean, Median, and Mode for a Nonsymmetrical Distribution

Skewness affects the **location of the mean, median, and mode** of a distribution as summarized in the following bulleted list.

- For a symmetrical distribution, the mean, median, and mode are equal.
- For a positively skewed distribution, the mode is less than the median, which is less than the mean. The mean is affected by outliers; in a positively skewed distribution, there are large, positive outliers which will tend to "pull" the mean upward, or more positive. An example of a positively skewed distribution is that of housing prices. Suppose that you live in a neighborhood with 100 homes; 99 of them sell for $100,000, and one sells for $1,000,000. The median and the mode will be $100,000, but the mean will be $109,000. Hence, the mean has been "pulled" upward (to the right) by the existence of one home (outlier) in the neighborhood.
- For a negatively skewed distribution, the mean is less than the median, which is less than the mode. In this case, there are large, negative outliers which tend to "pull" the mean downward (to the left).

Professor's Note: The key to remembering how measures of central tendency are affected by skewed data is to recognize that skew affects the mean more than the median and mode, and the mean is "pulled" in the direction of the skew. The relative location of the mean, median, and mode for different distribution shapes is shown in Figure 7.

Figure 7: Effect of Skewness on Mean, Median, and Mode

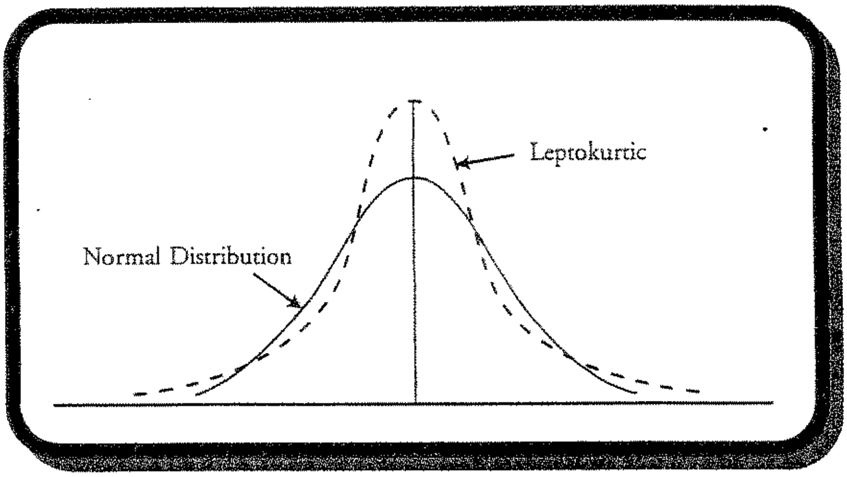


LOS 7.j: Define and interpret kurtosis, and measures of population and sample skew and kurtosis.

Kurtosis is a measure of the degree to which a distribution is more or less "peaked" than a normal distribution. **Leptokurtic** describes a distribution that is more peaked than a normal distribution, whereas **platykurtic** refers to a distribution that is less peaked, or flatter than a normal distribution.

As indicated in Figure 8, a leptokurtic return distribution will have more returns clustered around the mean and more returns with large deviations from the mean (fatter tails). Relative to a normal distribution, a leptokurtic distribution will have a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean. This means that there is a relatively greater probability of an observed value being either close to the mean or far from the mean. With regard to an investment returns distribution, a greater likelihood of a large deviation from the mean return is often perceived as an increase in risk.

Figure 8: Kurtosis



A distribution is said to exhibit **excess kurtosis** if it has either more or less kurtosis than the normal distribution. The computed kurtosis for all normal distributions is three. Statisticians, however, sometimes report excess kurtosis, which is defined as kurtosis minus three. Thus, a normal distribution has excess kurtosis equal to zero, a leptokurtic distribution has excess kurtosis greater than zero, and platykurtic distributions will have excess kurtosis less than zero.

Kurtosis is critical in a risk management setting. Most research of the distribution of securities returns has shown that returns are not normally distributed. Actual securities returns tend to exhibit both skewness and kurtosis. Skewness and kurtosis are critical concepts for risk management because when securities returns are modeled using an assumed normal distribution, the predictions from the models will not take into account the potential for extremely large, negative outcomes. In fact, most risk managers put very little emphasis on the mean and standard deviation of a distribution and focus more on the distribution of returns in the tails of the distribution—that is where the risk is. In general, greater positive kurtosis and more negative skew in returns distributions mean increased risk for an investor.

Measures of Sample Skew and Kurtosis

Sample skewness is equal to the sum of the cubed deviations from the mean divided by the cubed standard deviation and by the number of observations. Sample skewness for large samples is computed as:

$$\text{sample skewness}(S_K) = \frac{1}{n}\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^3}{s^3}$$

where:
s = sample standard deviation

Note that the denominator is always positive, but that the numerator can be positive or negative, depending on whether observations above the mean or observations below the mean tend to be further from the mean on average. When a distribution is right skewed, sample skewness is positive because the deviations above the mean are larger on average. A left-skewed distribution has a negative sample skewness.

Dividing by standard deviation cubed standardizes the statistic and allows **interpretation of the skewness measure**. If relative skewness is equal to zero, the data is not skewed. Positive levels of relative skewness imply a positively skewed distribution, whereas negative values of relative skewness imply a negatively skewed distribution. Values of S_K in excess of 0.5 in absolute value indicate significant levels of skewness.

Sample kurtosis is measured using deviations raised to the *fourth power.*

$$\text{sample kurtosis} = \frac{1}{n}\frac{\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^4}{s^4}$$

where:
s = sample standard deviation

To **interpret kurtosis**, note that it is measured relative to the kurtosis of a normal distribution, which is 3. Positive values of excess kurtosis indicate a distribution that is leptokurtic (more peaked, fat tails), whereas negative values indicate a platykurtic distribution (less peaked, thin tails). Excess kurtosis values that exceed 1.0 in absolute value are considered large. We can calculate kurtosis relative to that of a normal distribution as:

excess kurtosis = sample kurtosis – 3

Key Concepts

1. Descriptive statistics summarize the characteristics of a data set; inferential statistics are used to make probabilistic statements about a population based on a sample.
2. A population includes all members of a specified group, while a sample is a subset of the population used to draw inferences about the population.
3. Any measurable characteristic of a population is called a parameter; a characteristic of a sample is given by a sample statistic.
4. Data may be measured using different scales.
   - Nominal scale—data is put into a category with no particular order.
   - Ordinal scale—data is categorized and ordered with respect to some characteristic.
   - Interval scale—the difference in data values is meaningful, but zero does not represent the absence of what is being measured.
   - Ratio scale—the difference between observed values is meaningful, and a true zero point is the origin.
5. An interval is the set of return values, or range, that an observation falls within. A frequency distribution is a grouping of raw data into classes, or intervals.
6. Relative frequency is the percentage of total observations falling within each interval; cumulative relative frequency is the sum of the relative frequencies up to a point.
7. Histograms and frequency polygons are graphical tools used for portraying frequency distributions.
8. The arithmetic mean is $\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n}$. The geometric mean is $G = \sqrt[n]{X_1 \times X_2 \times \ldots \times X_n}$. The weighted mean is $\overline{X}_W = \sum_{i=1}^{n} w_i X_i$. The harmonic mean is $\overline{X}_H = \frac{N}{\sum_{i=1}^{N} \frac{1}{x_i}}$.
9. The median is the midpoint of a data set when the data is arranged from largest to smallest, and the mode of a data set is the value that appears most frequently.
10. Quantile is the general term for a value at or below which a stated proportion of the data in a distribution lies. Examples of quantiles include:
    - Quartiles—the distribution is divided into quarters.
    - Quintile—the distribution is divided into fifths.
    - Decile—the distribution is divided into tenths.
    - Percentile—the distribution is divided into hundredths (percents).

11. The range is the difference between the largest value and the smallest value in a data set.
12. Mean absolute deviation (MAD) is the average of the absolute values of the deviations from the arithmetic mean:

$$\text{MAD} = \frac{\sum_{i=1}^{n}\left|X_i - \overline{X}\right|}{n}$$

13. The variance is defined as the mean of the squared deviations from the arithmetic mean.
    - Population variance $= \sigma^2 = \dfrac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}$, where μ = population mean and N = size
    - Sample variance $= s^2 = \dfrac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{n-1}$, where $\overline{X}$ = sample mean and n = sample size
14. Standard deviation is the positive square root of the variance and is frequently used as a quantitative measure of risk.
15. Semivariance is a measure of downside risk that is calculated using only observations that are less than the mean, while target semivariance is calculated using only observations that are below the stated target return or value.
16. Chebyshev's inequality states that the proportion of the observations within *k* standard deviations of the mean is at least $1 - 1/k^2$ for all $k > 1$.
17. The coefficient of variation, $CV = \dfrac{s}{\overline{X}}$, expresses dispersion (risk) relative to the mean of a distribution.
18. The Sharpe measure (ratio) measures excess return per unit of risk:

$$\text{Sharpe ratio} = \frac{(\overline{r_p} - \overline{r_f})}{\sigma_p}$$

19. Skewness describes the degree to which a distribution is nonsymmetric about its mean.
    - A right-skewed distribution has positive sample skewness and a mean that is higher than the median that is higher than the mode.
    - A left-skewed distribution has negative skewness and a mean that is lower than the median that is lower than the mode.
20. Kurtosis measures the peakedness of a distribution and the probability of extreme outcomes.
    - Excess kurtosis is measured relative to a normal distribution, which has a kurtosis of 3.
    - Positive values of excess kurtosis indicate a distribution that is leptokurtic (fat tails, more peaked).
    - Negative values of excess kurtosis indicate a platykurtic distribution (thin tails, less peaked).
    - Excess kurtosis with an absolute value greater than 1 is considered large.

Concept Checkers: Statistical Concepts and Market Returns

1. The intervals in a frequency distribution should always have which of the following characteristics? The intervals should always:
   A. be truncated.
   B. be open ended.
   C. have a width of 10.
   D. be nonoverlapping.

Use the following frequency distribution for Questions 2 through 4.

Return, R	*Frequency*
−10% up to 0%	3
0% up to 10%	7
10% up to 20%	3
20% up to 30%	2
30% up to 40%	1

2. The number of intervals in this frequency table is:
   A. 1.
   B. 5.
   C. 16.
   D. 50.

3. The sample size is:
   A. 1.
   B. 5.
   C. 16.
   D. 50.

4. The relative frequency of the second class is:
   A. 0.0%.
   B. 10.0%.
   C. 16.0%.
   D. 43.8%.

Use the following data to answer Questions 5 through 13.

XYZ Corp. Annual Stock Prices

1995	*1996*	*1997*	*1998*	*1999*	*2000*
22%	5%	−7%	11%	2%	11%

5. What is the arithmetic mean return for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
   D. −7.0%.

6. What is the median return for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
   D. –7.0%.

7. What is the mode return for XYZ stock?
   A. 7.3%.
   B. 8.0%.
   C. 11.0%.
   D. –7.0%.

8. What is the range for XYZ stock returns?
   A. –7.0%.
   B. 11.0%.
   C. 22.0%.
   D. 29.0%.

9. What is the mean absolute deviation for XYZ stock returns?
   A. 0.00%.
   B. 5.20%.
   C. 7.33%.
   D. 29.0%.

10. Assuming that the distribution of XYZ stock returns is a population, what is the population variance?
    A. $5.0\%^2$.
    B. $6.8\%^2$.
    C. $7.7\%^2$.
    D. $80.2\%^2$.

11. Assuming that the distribution of XYZ stock returns is a population, what is the population standard deviation?
    A. 5.02%.
    B. 6.84%.
    C. 8.96%.
    D. 46.22%.

12. Assuming that the distribution of XYZ stock returns is a sample, the sample variance is *closest* to:
    A. $5.0\%^2$.
    B. $7.4\%^2$.
    C. $72.4\%^2$.
    D. $96.3\%^2$.

13. Assuming that the distribution of XYZ stock returns is a sample, what is the sample standard deviation?
    A. 7.4%.
    B. 9.8%.
    C. 72.4%.
    D. 96.3%.

14. For a skewed distribution, what is the minimum percentage of the observations that will lie between ±2.5 standard deviations of the mean based on Chebyshev's Inequality?
    A. 56%.
    B. 75%.
    C. 84%.
    D. Cannot be calculated for a skewed distribution.

Use the following data to answer Questions 15 and 16.

The annual returns for FJW's common stock over the years 2003, 2004, 2005, and 2006 were 15%, 19%, –8%, and 14%.

15. What is the arithmetic mean return for FJW's common stock?
    A. 8.62%.
    B. 10.00%.
    C. 14.00%.
    D. 15.25%.

16. What is the geometric mean return for FJW's common stock?
    A. 9.45%.
    B. 10.00%.
    C. 14.21%.
    D. It cannot be determined because the 2005 return is negative.

17. A distribution of returns that has a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean:
    A. is positively skewed.
    B. is a symmetric distribution.
    C. has positive excess kurtosis.
    D. has negative excess kurtosis.

18. Which of the following is *most accurate* regarding a distribution of returns that has a mean greater than its median?
    A. It is positively skewed.
    B. It is a symmetric distribution.
    C. It has positive excess kurtosis.
    D. It has negative excess skewness.

19. The harmonic mean of 3, 4, and 5 is:
    A. 3.74.
    B. 3.83.
    C. 4.
    D. 4.12.

Comprehensive Problems: Statistical Concepts and Market Returns

1. Year-end prices and dividends for Nopat Mutual Fund for each of six years are listed below along with the actual yield (return) on a money market fund called Emfund.

Year	*Nopat Fund Year-End Price*	*Nopat Fund Year-End Dividend*	*Nopat Annual Holding Period Return*	*Emfund Return for the Year*
1999	$28.50	$0.14		3.00%
2000	$26.80	$0.15	?	4.00%
2001	$29.60	$0.17	?	4.30%
2002	$31.40	$0.17	?	5.00%
2003	$34.50	$0.19	?	4.10%
2004	$37.25	$0.22	?	6.00%

Average risk-free rate over the five years 2000 – 2004 is 2.8%. Risk-free rate for 1999 is 2.8%

A. Calculate the annual holding period returns for a beginning-of-year investment in Nopat fund for each of the five years over the period 2000–2004 (% with two decimal places).

B. What is the arithmetic mean annual total return on an investment in Nopat fund shares (dividends reinvested) over the period 2000–2004?

C. What is the average compound annual rate of return on an investment in Nopat fund made at year end 1999 if it were held (dividends reinvested) until the end of 2004?

D. What is the median annual return on an Emfund investment over the 6-year period 1999–2004?

E. What is the sample standard deviation of the annual returns on money market funds over the 6-year period, using the Emfund returns as a sample?

F. What is the holding period return on a 6-year investment in Emfund made at the beginning of 1999?

G. If an investor bought $10,000 of Nopat Fund shares at the end of the year in each of the three years 2002–2004, what is the average price paid per share? What is the arithmetic mean of the three year-end prices?

H. What would have been the 1-year holding period return on a portfolio that had $60,000 invested in Nopat Fund and $40,000 invested in Emfund as of the beginning of 2004?

I. What is the coefficient of variation of the Nopat Fund annual total returns 2000–2004 and of the Emfund annual returns for the six years 1999–2004? Which is riskier?

J. What is the Sharpe ratio for an investment in the Nopat Fund over the five years from 2000–2004? What is the Sharpe ratio for an investment in the Emfund over the six years 1999–2004? Which Sharpe ratio is more preferred?

K. Calculate the range and mean absolute deviation of returns for an investment in the Emfund over the 6-year period 1999–2004.

L. Calculate the semivariance of returns on Emfund over the 6-year period.

M. What is the annual growth rate of dividends on Nopat Fund over the period from 1999–2004?

2. Identify the type of scale for each of the following:

A. Cars ranked as heavy, medium, or light.

B. Birds divided into categories of songbirds, birds of prey, scavengers, and game birds.

C. The height of each player on a baseball team.

D. The average temperature on 20 successive days in January in Chicago.

E. Interest rates on T-bills each year for 60 years.

3. Explain the difference between descriptive and inferential statistics.

4. An analyst has estimated the following parameters for the annual returns distributions for four portfolios:

Portfolio	*Mean Return E(R)*	*Variance of returns*	*Skewness*	*Kurtosis*
Portfolio A	10%	625	1.8	0
Portfolio B	14%	900	0.0	3
Portfolio C	16%	1250	–0.85	5
Portfolio D	19%	2000	1.4	2

She has been asked to evaluate the portfolios' risk and return characteristics. Assume that a risk-free investment will earn 5%.

A. Which portfolio would be preferred based on the Sharpe performance measure?

B. Which portfolio would be the most preferred based on the coefficient of variation?

C. Which portfolio(s) is/are symmetric?

D. Which portfolio(s) has/have fatter tails than a normal distribution?

E. Which portfolio is the riskiest based on its skewness?

F. Which portfolio is the riskiest based on its kurtosis?

G. Which portfolio will likely be considered more risky when judged by its semivariance rather than by its variance?

5. Which measure of central tendency is most affected by including rare but very large positive values?

6. A manager is responsible for managing part of an institutional portfolio to mimic the returns on the S&P 500 stock index. He is evaluated based on his ability to exactly match the returns on the index. His portfolio holds 200 stocks but has exactly the same dividend yield as the S&P 500 portfolio. Which of the statistical measures from this review would be an appropriate measure of his performance and how would you use it?

7. Below are the returns on 20 industry groups of stocks over the past year:

12%, –3%, 18%, 9%, –5%, 21%, 2%, 13%, 28%, –14%,

31%, 32%, 5%, 22%, –28%, 7%, 9%, 12%, –17%, 6%

A. What is the return on the industry group with the lowest rate of return in the top quartile?

B. What is the 40th percentile of this array of data?

C. What is the range of the data?

D. Based on a frequency distribution with 12 intervals, what is the relative frequency and cumulative relative frequency of the 10th interval (ascending order)?

ANSWERS – CONCEPT CHECKERS: STATISTICAL CONCEPTS AND MARKET RETURNS

1. D Intervals within a frequency distribution should always be nonoverlapping and closed ended so that each data value can be placed into only one interval. Intervals have no set width and should be set at a width so that data is adequately summarized without losing valuable characteristics.

2. B An interval is the set of return values that an observation falls within. Simply count the return intervals on the table—there are five of them.

3. C The sample size is the sum of all of the frequencies in the distribution, or 3 + 7 + 3 + 2 + 1 = 16.

4. D The relative frequency is found by dividing the frequency of the interval by the total number of frequencies.

$$\frac{7}{16} = 43.8\%$$

5. A [22% + 5% + –7% + 11% + 2% +11%] / 6 = 7.3%

6. B To find the median, rank the returns in order and take the middle value. –7%, 2%, 5%, 11%, 11%, 22%. In this case, because there is an even number of observations, the median is the average of the two middle values, or (5% + 11%) / 2 = 8.0%.

7. C The mode is the value that appears most often, or 11%.

8. D The range is calculated by taking the highest value minus the lowest value.

22% – (–7%) = 29.0%

9. C The mean absolute deviation is found by taking the mean of the absolute values of the deviations from the mean.

$$(|22 - 7.3| + |5 - 7.3| + |-7 - 7.3| + |11 - 7.3| + |2 - 7.3| + |11 - 7.3|) / 6 = 7.33\%$$

10. D The population variance, σ^2, is found by taking the mean of all squared deviations from the mean.

$$\sigma^2 = [(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / 6 = 80.2\%^2$$

11. C The population standard deviation, σ, is found by taking the square root of the population variance.

$$\sigma = \{[(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / 6\}^{½} = (80.2\%^2)^{0.5}$$
$$= 8.96\%$$

12. D The sample variance, s^2, uses n – 1 in the denominator.

$$s^2 = [(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / (6 - 1) = 96.3\%^2$$

13. B The sample standard deviation, s, is the square root of the sample variance.

$$s = \{[(22 - 7.3)^2 + (5 - 7.3)^2 + (-7 - 7.3)^2 + (11 - 7.3)^2 + (2 - 7.3)^2 + (11 - 7.3)^2] / (6 - 1)\}^{0.5}$$
$$= (96.3\%^2)^{0.5} = 9.8\%$$

14. C Applying Chebyshev's inequality, $1 - [1 / (2.5)^2] = 0.84$, or 84%.

15. B (15% + 19% + (–8%) + 14%) / 4 = 10%

16. A $(1.15 \times 1.19 \times 0.92 \times 1.14)^{0.25} - 1 = 9.45\%$

Professor's Note: This question could have been answered very quickly since the geometric mean must be less than the arithmetic mean computed in the preceding problem.

17. C A distribution that has a greater percentage of small deviations from the mean and a greater percentage of extremely large deviations from the mean will be leptokurtic and will exhibit excess kurtosis (positive). The distribution will be taller and have fatter tails than a normal distribution.

18. A A distribution with a mean greater than its median is positively skewed, or skewed to the right. The skew "pulls" the mean. *Note: Kurtosis deals with the height of the distribution and not the skewness.*

19. B $\bar{X}_H = \dfrac{3}{1/3 + 1/4 + 1/5} = 3.83$

ANSWERS – COMPREHENSIVE PROBLEMS: STATISTICAL CONCEPTS AND MARKET RETURNS

1. A. The annual holding period returns (total returns) are given in the table and are each calculated as (year-end price + year-end dividend)/previous year-end price – 1.

Year	*Nopat Fund Year-End Price*	*Nopat Fund Year-End Dividend*	*Nopat Annual Holding Period Return*	*Emfund Return for the Year*
1999	\$28.50	\$0.14		3.00%
2000	\$26.80	\$0.15	–5.44%	4.00%
2001	\$29.60	\$0.17	11.08%	4.30%
2002	\$31.40	\$0.17	6.66%	5.00%
2003	\$34.50	\$0.19	10.48%	4.10%
2004	\$37.25	\$0.22	8.61%	6.00%

B. The arithmetic mean of the holding period returns is 6.28%.

C. $((1 - 0.0544)(1.1108)(1.0666)(1.1048)(1.0861))^{1/5} - 1 = 6.10\%$

D. Median = (4.3 + 4.1) / 2 = 4.2%.

E. Sample standard deviation of Emfund returns over the six years is:

$$\{[(3-4.4)^2 + (4-4.4)^2 + (4.3-4.4)^2 + (5-4.4)^2 + (4.1-4.4)^2 + (6-4.4)^2] / 5\}^{1/2} = \left(\frac{5.14}{5}\right)^{1/2} = 1.01\%$$

F. $(1.03)(1.04)(1.043)(1.05)(1.041)(1.06) - 1 = 29.45\%$

G. The harmonic mean is 3/(1/31.4 + 1/34.5 + 1/37.25) = \$34.22 average purchase price per share. Arithmetic mean price = (31.4 + 34.5 + 37.25)/ 3 = \$34.38.

H. The portfolio return is a weighted average, 0.6 × 8.61% + 0.4 × 6% = 7.57%.

I. CV for Nopat = 6.77/6.28 = 1.08. CV for Emfund = 1.01/4.4 = 0.23. Emfund is less risky by this measure.

J. Sharpe ratio for Nopat is (6.28 – 2.8)/ 6.77 = 0.51. Sharpe measure for Emfund is (4.4 – 2.8) / 1.01 = 1.58. The Emfund is preferred using this criterion because it has higher excess returns per unit of risk.

K. Range is 6% – 3% = 3%. MAD is 0.73% = [(4.4% – 3%) + (4.4% – 4%) +(4.4% – 4.3%) +(5% – 4.4%) +(4.4% – 4.1%) +(6% – 4.4%)] / 6. Remember to use absolute values; we show all differences as positive to reflect that.

L. Semivariance = $[(3 - 4.4)^2 + (4 - 4.4)^2 + (4.3 - 4.4)^2 + (4.1 - 4.4)^2] / 3 = 0.74\%$.

M. Average annual growth rate of dividends is the geometric mean rate of growth: $(0.22/0.14)^{1/5} - 1 = 9.46\%$.

2. A. An ordinal scale.

B. A nominal scale.

C. A ratio scale.

D. An interval scale.

E. A ratio scale.

3. Descriptive statistics are used to summarize the important characteristics of large data sets to consolidate a mass of numerical data into useful information. Inferential statistics refers to using statistics to make forecasts, estimates, or judgments about a large set of data on the basis of the statistical characteristics of a smaller set (a sample).

4. A Portfolio D has the highest Sharpe ratio, $\frac{19-5}{\sqrt{2000}} = 0.313$ and is therefore the most preferred.

B. Portfolio B has the lowest coefficient of variation, $\frac{\sqrt{900}}{14} = 2.1429$ and is therefore the most preferred.

C. Portfolio B has no skew and is therefore a symmetric distribution (about its mean of 14%).

D. The kurtosis of a normal distribution is 3, so only portfolio C has positive excess kurtosis, indicating fatter tails (and more peakedness) relative to a normal distribution.

E. Negative skew indicates that returns below the mean are more extreme, so we would consider Portfolio C to be the most risky based on skew alone.

F. Larger kurtosis indicates greater likelihood of extreme outcomes and from a risk-management standpoint this indicates greater risk. Portfolio C has the greatest kurtosis.

G. Portfolio C has negative skew, indicating that it is not symmetric and has results below the mean that are more extreme. Thus its semivariance (or downside risk) has the potential to make it appear more risky than would be reflected in just its variance. Considering variance and semivariance as essentially average squared deviations from the mean, distributions with negative skew will have average squared deviations below the mean that are greater than the average squared deviations for the overall distribution.

5. The mean is most affected by large outliers in a distribution, compared to the median and mode, which may be unaffected.

6. Since the goal is to match the index returns we must focus on the differences between the returns on the manager's portfolio and those on the index he is attempting to mimic. These differences are referred to as "tracking error." The standard deviation or variance of the differences between his portfolio returns and the returns of the index over a number of periods would be a suitable measure of his performance. If you said mean absolute deviation, that is defensible as well as it is certainly one way to measure tracking error. It is, however, not the measure of tracking error we see used in practice.

7. A. With 20 datapoints, the top quartile (¼) is the top 5. Count down from the greatest value to find the 5th from the top is 21%.

   B. The location of the 40th percentile is $(20 + 1)(40/100) = 8.4$. The 8th and 9th lowest returns are 6% and 7%, so the 40th percentile is $6 + 0.4(7 - 6) = 6.4\%$.

   C. The range of the data is $32 - (-28) = 60$.

   D. Divide the range by 12 to get 5. The 10th interval from the bottom is the 3rd from the top. The top three intervals are $27 \le x \le 32$, $22 \le x < 27$, and $17 \le x < 22$. There are only two observations in the 10th interval, 18% and 21%. The relative frequency is $2/20 = 10\%$. Since there are four observations $\geq 22\%$, the cumulative relative frequency of the 10th interval is $(20 - 4)/20 = 80\%$.

The following is a review of the Quantitative Methods principles designed to address the learning outcome statements set forth by CFA Institute®. This topic is also covered in:

PROBABILITY CONCEPTS

Study Session 2

EXAM FOCUS

This topic review covers important terms and concepts associated with probability theory. Random variables, events, outcomes, conditional probability, and joint probability are described. Probability rules such as the addition rule and multiplication rule are introduced. These rules are frequently used by finance practitioners, so your understanding of and ability to apply probability rules is likely to be tested on the exam. Expected value, standard deviation, covariance, and correlation for individual asset and portfolio returns are discussed. A well-prepared candidate will be able to calculate and interpret these widely used measures.

This review also discusses counting rules, which lay the foundation for the binomial probability distribution that is covered in the next topic review.

LOS 8.a: Define a random variable, an outcome, an event, mutually exclusive events, and exhaustive events.

- A **random variable** is an uncertain quantity/number.
- An **outcome** is an observed value of a random variable.
- An **event** is a single outcome or a set of outcomes.
- **Mutually exclusive events** are events that cannot both happen at the same time.
- **Exhaustive events** are those that include all possible outcomes.

Consider rolling a six-sided die. The number that comes up is a *random variable*. If you roll a 4, that is an *outcome*. Rolling a 4 is an *event*, and rolling an even number is an *event*. Rolling a 4 and rolling a 6 are *mutually exclusive events*. Rolling an even number and rolling an odd number is a set of mutually exclusive and *exhaustive events*.

LOS 8.b: Explain the two defining properties of probability, and distinguish among empirical, subjective, and a priori probabilities.

There are **two defining properties of probability**.

- The probability of occurrence of any event (E_i) is between zero and 1 (i.e., $0 \leq P(E_i) \leq 1$).
- If a set of events, E_1, E_2, ... E_n, is mutually exclusive and exhaustive, the probabilities of those events sum to 1 (i.e., $\Sigma P(E_i) = 1$).

The first of the defining properties introduces the term $P(E_i)$, which is shorthand for the "probability of event i." If $P(E_i) = 0$, the event will never happen. If $P(E_i) = 1$, the event is certain to occur, and the outcome is not random.

The probability of rolling any one of the numbers 1–6 with a fair die is 1/6 = 0.1667 = 16.7%. The set of events—rolling a number equal to 1, 2, 3, 4, 5, or 6—is exhaustive, and the individual events are mutually exclusive, so the probability of this set of events is equal to 1. We are certain that one of the values in this set of events will occur.

An **empirical probability** is established by analyzing past data. An **a priori probability** is determined using a formal reasoning and inspection process. A **subjective probability** is the least formal method of developing probabilities and involves the use of personal judgment.

The following are examples of statements that use *empirical, a priori,* and *subjective probabilities* for developing probabilities.

- *Empirical probability.* "Historically, the Dow Jones Industrial Average (DJIA) has closed higher than the previous close two out of every three trading days. Therefore, the probability of the Dow going up tomorrow is two-thirds, or 66.7%."
- *A priori probability.* "Yesterday, 24 of the 30 DJIA stocks increased in value. Thus, if 1 of the 30 stocks is selected at random, there is an 80% (= 24/30) probability that its value increased yesterday."
- *Subjective probability.* "It is my personal feeling that the probability the DJIA will close higher tomorrow is 90%."

LOS 8.c: State the probability of an event in terms of odds for or against the event.

Stating the odds that an event will or will not occur is an alternative way of expressing probabilities. Consider an event that has a probability of occurrence of 0.125, which is one-eighth. The *odds* that the event will occur are

$\frac{0.125}{(1-0.125)} = \frac{1/8}{7/8} = \frac{1}{7}$, which we state as, "the odds for the event occurring are one-to-seven." The *odds against* the event occurring are the reciprocal of 1/7, which is seven-to-one.

We can also get the probability of an event from the odds by reversing these calculations. If we know that the odds for an event are one to six, we can compute the probability of occurrence as $\frac{1}{1+6} = \frac{1}{7} = 0.1429 = 14.29\%$.

Alternatively, the probability that the event will not occur is $\frac{6}{1+6} = \frac{6}{7} = 0.8571 = 85.71\%$.

Professor's Note: While I am quite familiar with the use of odds rather than probabilities at the horse track, I can't remember encountering odds for a stock or bond. The use of odds at the horse track lets you know how much you will win per $1 bet on a horse (less the track's percentage). If you bet on a 15-1 long shot and the horse wins, you will receive $15 and your $1 bet will be returned, so the profit is $15. Of course, if the horse loses, you would lose the $1 you bet and the "profit" is –$1.

One last point is that the expected return on the bet is zero, based on the probability of winning expressed in the odds. The probability of the horse winning when the odds are 15-to-1 is $\frac{1}{15+1} = \frac{1}{16}$ *and the probability of the horse losing is 15/16. The expected profit is* $\frac{1}{16} \times \$15 + \frac{15}{16} \times (-\$1) = 0$.

LOS 8.d: Distinguish between unconditional and conditional probabilities.

- **Unconditional probability** (a.k.a., *marginal probability*) refers to the probability of an event regardless of the past or future occurrence of other events. If we are concerned with the probability of an economic recession, regardless of the occurrence of changes in interest rates or inflation, we are concerned with the unconditional probability of a recession.

- A **conditional probability** is one where the occurrence of one event affects the probability of the occurrence of another event. For example, we might be concerned with the probability of a recession ***given*** that the monetary authority increases interest rates. This is a conditional probability. The key word to watch for here is "given." Using probability notation, "the probability of A ***given*** the occurrence of B" is expressed as $P(A \mid B)$, where the vertical bar (|) indicates "given," or "conditional upon." For our interest rate example above, the probability of a recession ***given*** an increase in interest rates is expressed as $P(\textit{recession} \mid \textit{increase in interest rates})$.

LOS 8.e: Define joint probability and calculate and interpret 1) the joint probability of two events, 2) the probability that at least one of two events will occur, given the probability of each and the joint probability of the two events, and 3) a joint probability of any number of independent events.

The **joint probability** of two events is the probability that they will both occur. We can calculate this from the conditional probability that A will occur given B occurs (a conditional probability) and the probability that B will occur (the unconditional probability of B). This calculation is sometimes referred to as the multiplication rule of probability. Using the notation for conditional and unconditional probabilities we can express this rule as:

$$P(AB) = P(A \mid B) \times P(B)$$

This expression is read as follows: "The joint probability of A and B, P(AB), is equal to the conditional probability of A ***given*** B, $P(A \mid B)$, times the unconditional probability of B, P(B)."

This relationship can be rearranged to define the conditional probability of A given B as follows:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

Example: Multiplication rule

Consider the following information:

- $P(I) = 0.4$, the probability of the monetary authority increasing interest rates (I) is 40%.
- $P(R \mid I) = 0.7$, the probability of a recession (R) given an increase in interest rates is 70%.

What is P(RI), the joint probability of a recession ***and*** an increase in interest rates?

Answer:

Applying the multiplication rule, we get the following result:

$$P(RI) = P(R \mid I) \times P(I)$$
$$P(RI) = 0.7 \times 0.4$$
$$P(RI) = 0.28$$

Don't let the cumbersome notation obscure the simple logic of this result. If an interest rate increase will occur 40% of the time and lead to a recession 70% of the time when it occurs, the joint probability of an interest rate increase and a resulting recession is (0.4)(0.7) = (0.28) = 28%.

Calculating the Probability That at Least One of Two Events Will Occur

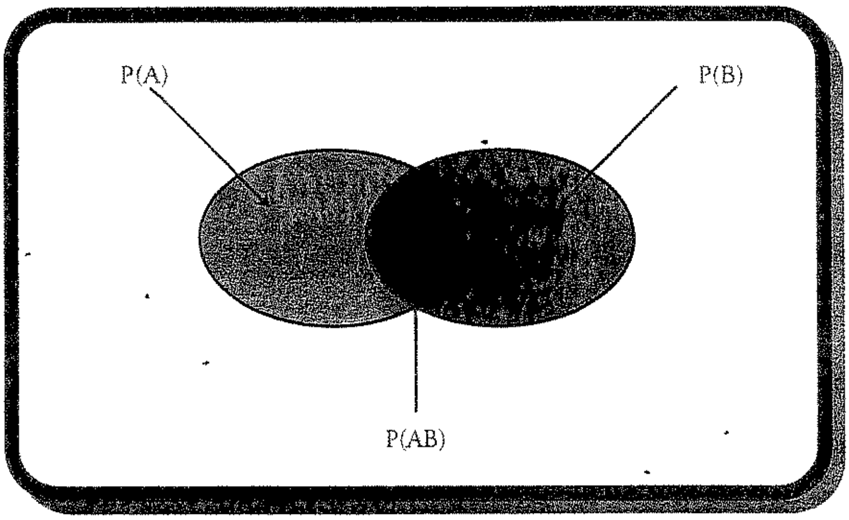
The addition rule for probabilities is used to determine the **probability that at least one of two events will occur.** For example, given two events, A and B, the addition rule can be used to determine the probability that either A or B will occur. If the events are ***not*** mutually exclusive, double counting must be avoided by subtracting the joint probability that *both* A and B will occur from the sum of the unconditional probabilities. This is reflected in the following general expression for the addition rule:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

For mutually exclusive events, where the joint probability, P(AB), is zero, the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event, P(A or B) = P(A) + P(B).

Figure 1 illustrates the addition rule with a Venn Diagram and highlights why the joint probability must be subtracted from the sum of the unconditional probabilities. Note that if the events are ***mutually exclusive*** the sets do not intersect, P(AB) = 0, and the joint probability is simply P(A) + P(B).

Figure 1: Venn Diagram



Example: Addition rule

Using the information in our previous interest rate and recession example and the fact that the unconditional probability of a recession, P(R), is 34%, determine the probability that either interest rates will increase *or a* recession will occur.

Answer:

Given that P(R) = 0.34, P(I) = 0.40, and P(RI) = 0.28, we can compute P(R or I) as follows:

$$P(R \text{ or } I) = P(R) + P(I) - P(RI)$$
$$P(R \text{ or } I) = 0.34 + 0.40 - 0.28$$
$$P(R \text{ or } I) = 0.46$$

Calculating a Joint Probability of any Number of Independent Events

On the roll of two dice, the **joint probability** of getting two 4s is calculated as:

P(4 on first die and 4 on second die) = P(4 on first die) × P(4 on second die) = $1/6 \times 1/6 = 1/36 = 0.0278$

On the flip of two coins, the probability of getting two heads is:

P(heads on first coin and heads on second coin) = $1/2 \times 1/2 = 1/4 = 0.25$

Hint: When dealing with *independent events*, the word *and* indicates multiplication, and the word *or* indicates addition. In probability notation:

$P(A \text{ or } B) = P(A) + P(B) - P(AB)$, and $P(A \text{ and } B) = P(A) \times P(B)$

The multiplication rule we used to calculate the joint probability of two independent events may be applied to any number of independent events as the following examples illustrate.

Example: Joint probability for more than two independent events

What is the probability of rolling three 4s in one simultaneous toss of three dice?

Answer:

Since the probability of rolling a 4 for each die is 1/6, the probability of rolling three 4s is:

P(three 4s on the roll of three dice) = $1/6 \times 1/6 \times 1/6 = 1/216 = 0.00463$

Similarly:

P(four heads on the flip of four coins) = $1/2 \times 1/2 \times 1/2 \times 1/2 = 1/16 = 0.0625$

Example: Joint probability for more than two independent events

Using empirical probabilities, suppose we observe that the DJIA has closed higher on two-thirds of all days in the past few decades. Furthermore, it has been determined that up and down days are independent. Based on this information, compute the probability of the DJIA closing higher for five consecutive days.

Answer:

P(DJIA up five days in a row) = $2/3 \times 2/3 \times 2/3 \times 2/3 \times 2/3 = (2/3)^5 = 0.132$

Similarly:

P(DJIA down five days in a row) = $1/3 \times 1/3 \times 1/3 \times 1/3 \times 1/3 = (1/3)^5 = 0.004$

LOS 8.f: Distinguish between dependent and independent events.

Independent events refer to events for which the occurrence of one has no influence on the occurrence of the others. The definition of independent events can be expressed in terms of conditional probabilities. Events A and B are independent if and only if:

$P(A \mid B) = P(A)$, or equivalently, $P(B \mid A) = P(B)$

If this condition is not satisfied, the events are **dependent events** (i.e., the occurrence of one is dependent on the occurrence of the other).

In our interest rate and recession example, recall that events I and R are not independent: the occurrence of I affects the probability of the occurrence of R. In this example, the independence conditions for I and R are violated because:

$P(R) = 0.34$, but $P(R \mid I) = 0.7$, the probability of a recession is greater when there is an increase in interest rates.

The best examples of independent events are found with the a priori probabilities of dice tosses or coin flips. A die has "no memory." Therefore, the event of rolling a 4 on the second toss is independent of rolling a 4 on the first toss. This idea may be expressed as:

$$P(\text{4 on second toss} \mid \text{4 on first toss}) = P(\text{4 on second toss}) = 1/6 \text{ or } 0.167$$

The idea of independent events also applies to flips of a coin:

$$P(\text{heads on first coin} \mid \text{heads on second coin}) = P(\text{heads on first coin}) = 1/2 \text{ or } 0.50$$

LOS 8.g: Calculate, using the total probability rule, an unconditional probability.

The **total probability rule** highlights the relationship between unconditional and conditional probabilities of mutually exclusive and exhaustive events. It is used to explain the unconditional probability of an event in terms of probabilities that are conditional upon other events.

In general, the unconditional probability of event R, $P(R) = P(R \mid S_1) \times P(S_1) + P(R \mid S_2) \times P(S_2) + \ldots + P(R \mid S_N) \times P(S_N)$, where the set of events $\{S_1, S_2, \ldots S_N\}$ is mutually exclusive and exhaustive.

Example: An investment application of unconditional probability

Building upon our ongoing example about interest rates and economic recession, we can assume that a recession can only occur with either of the two events—interest rates increase (I) or interest rates do not increase (I^C)—since these events are mutually exclusive and exhaustive. I^C is read "the complement of I," which means "not I." Therefore, the probability of I^C is $1 - P(I)$. It is logical, therefore, that the sum of the two joint probabilities must be the unconditional probability of a recession. This can be expressed as follows:

$$P(R) = P(RI) + P(RI^C)$$

Applying the multiplication rule, we may restate this expression as:

$$P(R) = P(R \mid I) \times P(I) + P(R \mid I^C) \times P(I^C)$$

Assume that $P(R \mid I) = 0.70$, $P(R \mid I^C)$, the probability of recession if interest rates do not rise, is 10% and that $P(I) = 0.40$ so that $P(I^C) = 0.60$. The unconditional probability of a recession can be calculated as follows:

$$\begin{aligned} P(R) &= P(R \mid I) \times P(I) + P(R \mid I^C) \times P(I^C) \\ &= (0.70)(0.40) + (0.10)(0.60) \\ &= 0.28 + 0.06 = 0.34 \end{aligned}$$

Expected Value

Now that we have developed some probability concepts and tools for working with probabilities, we can apply this knowledge to determine the average value for a random variable that results from multiple experiments. This average is called an expected value. In any given experiment, the observed value for a random variable may not equal its expected value, and even if it does, the outcome from experiment to experiment will be different. The degree of dispersion of outcomes around the expected value of a random variable is measured using the variance and standard deviation. When pairs of random variables are being observed, the covariance and correlation are used to measure the extent of the relationship between the observed values for the two variables from one experiment to another.

The **expected value** is the weighted average of the possible outcomes of a random variable, where the weights for each outcome is the probability that the outcome will occur. The mathematical representation for the expected value of random variable X is:

$$E(X) = \Sigma P(x_i)x_i = P(x_1)x_1 + P(x_2)x_2 + \ldots + P(x_n)x_n$$

Here, E is referred to as the expectations operator and is used to indicate the computation of a probability-weighted average. The symbol x_1 represents the first observed value (observation) for random variable X; x_2 is the second observation, and so on through the nth observation. The concept of expected value may be demonstrated using the a priori probabilities associated with a coin toss. On the flip of one coin, the occurrence of the event "heads" may be used to assign the value of one to a random variable. Alternatively, the event "tails" means the random variable equals zero. Statistically, we would formally write:

if heads, then X = 1
if tails, then X = 0

For a fair coin, P(heads) = P(X = 1) = 0.5, and P(tails) = P(X = 0) = 0.5. The expected value can be computed as follows:

$$E(X) = \Sigma P(x_i)x_i = P(X = 0)(0) + P(X = 1)(1) = (0.5)(0) + (0.5)(1) = 0.5$$

In any individual flip of a coin, X cannot assume a value of 0.5. Over the long term, however, the average of all the outcomes is expected to be 0.5. Similarly, the expected value of the roll of a fair die, where X = number that faces up on the die, is determined to be:

$$E(X) = \Sigma P(x_i)x_i = (1/6)(1) + (1/6)(2) + (1/6)(3) + (1/6)(4) + (1/6)(5) + (1/6)(6)$$
$$E(X) = 3.5$$

We can never roll a 3.5 on a die, but over the long term, 3.5 should be the average value of all outcomes.

The expected value is, statistically speaking, our "best guess" of the outcome of a random variable. While a 3.5 will never appear when a die is rolled, the average amount by which our guess differs from the actual outcomes is minimized when we use the expected value calculated this way.

Professor's Note: When we had historical data in an earlier topic review, we calculated the mean or simple arithmetic average and used deviations from the mean to calculate the variance and standard deviation. The calculations given here for the expected value (or weighted mean) is based on a probability model, whereas our earlier calculations were based on a sample or population of outcomes. Note that when the probabilities are equal, the simple mean is the expected value. For the roll of a die, all six outcomes are equally likely, so $\frac{1+2+3+4+5+6}{6} = 3.5$ *gives us the same value as the probability model. However, with a probability model, the probabilities of the possible outcomes need not be equal and the simple mean is not necessarily the expected outcome as the following example illustrates.*

Example: Expected value for stock returns

In the past, Tillard Corporation has fallen short, met, or exceeded analysts' earnings forecasts with the relative frequencies of 20%, 45%, and 35%, respectively. Historically, on the day Tillard announces earnings, the stock has fallen 3%, increased 1%, or increased 4%, depending on whether the earnings announcement fell short of, met, or exceeded the forecast earnings, respectively. What is the expected value of Tillard's stock return?

Answer:

The computation of the expected value of Tillard's stock return is demonstrated in Figure 2. As indicated in Figure 2, the expected value is the sum of the values in the column of the $P(x_i)x_i$ calculations.

Figure 2: Expected Value Computation

Event	*Probability of Event* $P(x_i)$	*Stock Return* x_i	$P(x_i)x_i$
Fall short of forecast	0.20	–0.03	–0.0060
Meet forecast	0.45	0.01	0.0045
Exceed forecast	0.35	0.04	0.0140
The expected value of the return on Tillard's stock = $\Sigma P(x_i)x_i$ = 0.0125			

LOS 8.h: Explain the use of conditional expectation in investment applications.

Conditional expected values are calculated using conditional probabilities. In investments, forecasts are frequently made using the expected value for a stock's return, earnings, and dividends. After the initial forecast, new and relevant information may surface that can affect the forecasted value(s). When this happens, the original forecast must be refined, and it is done using conditional expected values. As the name implies, ***conditional expected values*** are expected values that are contingent upon the occurrence of some other event. Let's look at an example to more fully develop this idea.

Example: Conditional expected value

Continuing with the preceding example, suppose the probability of the company's earnings announcement actually falling short of, meeting, or exceeding the earnings forecasts depends upon (is *conditional* upon) situations such as the general state of the economy when the announcement is made. Given a "good" economy at the time of the earnings announcement, the conditional probabilities for the level of announced earnings relative to forecasted earnings are as follows:

P(fall short of forecasted earnings | good economy) = 0.10
P(meet forecasted earnings | good economy) = 0.50
P(exceed forecasted earnings | good economy) = 0.40

Given a "poor" economy at the time of the announcement, the corresponding probabilities are:

P(fall short of forecasted earnings | poor) = 0.30
P(meet forecasted earnings | poor) = 0.40
P(exceed forecasted earnings | poor) = 0.30

Using these conditional probabilities, it is now possible to compute the conditional expected value for each possible state of the economy.

Answer:

The expected value of Tillard's stock returns given a "good" economy at the time earnings are announced is:

$$E(X \mid \text{good economy}) = (0.10)(-0.03) + (0.50)(0.01) + (0.40)(0.04) = 0.018$$

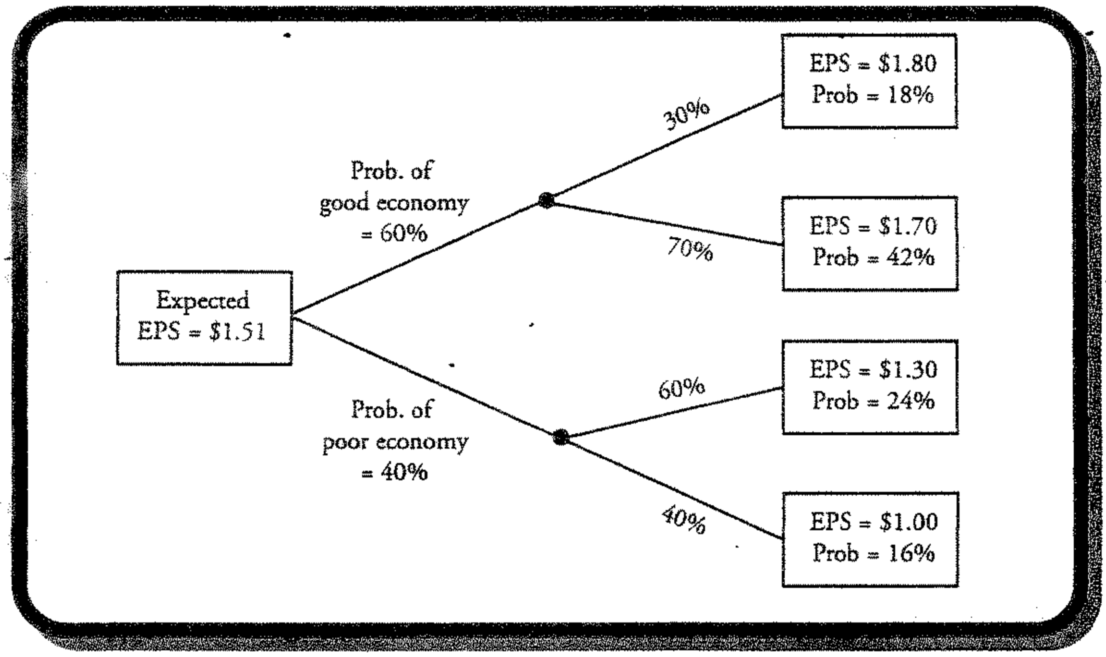
Similarly, the expected value of Tillard's stock return, given a "poor" economy at the time earnings are announced is:

$$E(X \mid \text{poor economy}) = (0.30)(-0.03) + (0.40)(0.01) + (0.30)(0.04) = 0.007$$

LOS 8.i: Diagram an investment problem, using a tree diagram.

You might well wonder where the returns and probabilities used in calculating expected values come from. A general framework called a tree diagram is used to show the probabilities of various outcomes. In Figure 3, we have shown estimates of EPS for four different outcomes: a good economy and relatively good results at the company, a good economy and relatively poor results at the company, a poor economy and relatively good results at the company, a poor economy and relatively poor results at the company. Using the rules of probability we can calculate the probabilities of each of the four EPS outcomes shown in the boxes on the right-hand side of the "tree."

Figure 3: A Tree Diagram



The expected EPS of $1.51 is simply calculated as:

$$0.18 \times 1.80 + 0.42 \times 1.70 + 0.24 \times 1.30 + 0.16 \times 1.00 = \$1.51$$

Note that the probabilities of the four possible outcomes sum to 1.

Covariance and Correlation

The variance and standard deviation measure the dispersion, or volatility, of only one variable. In many finance situations, however, we are interested in how two random variables move in relation to each other. For investment applications, one of the most frequently analyzed pairs of random variables is the returns of two assets. Investors and managers frequently ask questions such as, "what is the relationship between the return for Stock A and Stock B?" or "what is the relationship between the performance of the S&P 500 and that of the automotive industry?" As you will soon see, the covariance and correlation are measures that provide useful information about how two random variables, such as asset returns, are related.

LOS 8.j: Define, calculate and interpret covariance and correlation.

Covariance is a measure of how two assets move together. It is the expected value of the product of the deviations of the two random variables from their respective expected values. A common symbol for the covariance between random variables X and Y is Cov(X,Y). Since we will be mostly concerned with the covariance of asset returns, the following formula has been written in terms of the covariance of the return of asset i, R_i, and the return of asset j, R_j:

$$\text{Cov}(R_i,R_j) = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$$

The following are *properties of the covariance:*

- The covariance is a general representation of the same concept as the variance. That is, the variance measures how a random variable moves with itself, and the covariance measures how one random variable moves with another random variable.
- The covariance of R_A with itself is equal to the variance of R_A; that is, $\text{Cov}(R_A,R_A) = \text{Var}(R_A)$.
- The covariance may range from negative infinity to positive infinity.

To aid in the interpretation of covariance, consider the returns of a stock and of a put option on the stock. These two returns will have a negative covariance because they move in opposite directions. The returns of two automotive stocks would likely have a positive covariance, and the returns of a stock and a riskless asset would have a zero covariance because the riskless asset's returns never move, regardless of movements in the stock's return. While the formula for covariance given above is correct, the method of computing the covariance of returns from a joint probability model uses a probability-weighted average of the products of the random variable's deviations from their means for each possible outcome. The following example illustrates this calculation.

Example: Covariance

Assume that the economy can be in three possible states (S) next year: boom, normal, or slow economic growth. An expert source has calculated that P(boom) = 0.30, P(normal) = 0.50, and P(slow) = 0.20. The returns for Stock A, R_A, and Stock B, R_B, under each of the economic states are provided in Figure 4. What is the covariance of the returns for Stock A and Stock B?

Answer:

First, the expected returns for each of the stocks must be determined.

$$E(R_A) = (0.3)(0.20) + (0.5)(0.12) + (0.2)(0.05) = 0.13$$
$$E(R_B) = (0.3)(0.30) + (0.5)(0.10) + (0.2)(0.00) = 0.14$$

The covariance can now be computed using the procedure described in Figure 4.

Figure 4: Covariance Computation

Event	*P(S)*	R_A	R_B	$P(S) \times [R_A - E(R_A)] \times [R_B - E(R_B)]$
Boom	0.3	0.20	0.30	(0.3)(0.2 – 0.13)(0.3 – 0.14) = 0.00336
Normal	0.5	0.12	0.10	(0.5)(0.12 – 0.13)(0.1 – 0.14) = 0.00020
Slow	0.2	0.05	0.00	(0.2)(0.05 – 0.13)(0 – 0.14) = 0.00224
				$Cov(R_A, R_B) = \Sigma P(S) \times [R_A - E(R_A)] \times [R_B - E(R_B)]$ = **0.00580**

The preceding example illustrates the use of a joint probability function. A *joint probability function* for two random variables gives the probability of the joint occurrence of specified outcomes. In this case, we only had three joint probabilities:

$P(R_A = 0.2 \text{ and } R_B = 0.3) = 0.30$
$P(R_A = 0.12 \text{ and } R_B = 0.1) = 0.50$
$P(R_A = 0.05 \text{ and } R_B = 0.0) = 0.20$

Joint probabilities are often presented in a table such as the one shown in Figure 5. According to Figure 5, $P(R_A = 0.12 \text{ and } R_B = 0.10) = 0.50$. This is the boldfaced probability represented in the cell at the intersection of the column labeled $R_B = 0.10$ and the row labeled $R_A = 0.12$. Similarly, $P(R_A = 0.20 \text{ and } R_B = 0.10) = 0$.

Figure 5: Joint Probability Table

Joint Probabilities	$R_B = 0.30$	$R_B = 0.10$	$R_B = 0.00$
$R_A = 0.20$	0.30	0	0
$R_A = 0.12$	0	**0.50**	0
$R_A = 0.05$	0	0	0.20

In more complex applications, there would likely be positive values where the zeros appear in Figure 5. In any case, the sum of all the probabilities in the cells on the table must equal 1.

In practice, the covariance is difficult to interpret. This is mostly because it can take on extremely large values, ranging from negative to positive infinity, and, like the variance, these values are expressed in terms of square units.

To make the covariance of two random variables easier to interpret, it may be divided by the product of the random variable's standard deviations. The resulting value is called the **correlation coefficient**, or simply, correlation. The relationship between covariances, standard deviations, and correlations can be seen in the following expression for the correlation of the returns for asset *i* and *j*:

$$\text{Corr}(R_i, R_j) = \frac{\text{Cov}(R_i, R_j)}{\sigma(R_i)\sigma(R_j)}, \text{ which implies } \text{Cov}(R_i, R_j) = \text{Corr}(R_i, R_j)\sigma(R_i)\sigma(R_j)$$

The correlation between two random return variables may also be expressed as $\rho(R_i, R_j)$, or $\rho_{i,j}$.

Properties of correlation of two random variables R_i and R_j are summarized here:

- Correlation measures the strength of the linear relationship between two random variables.
- Correlation has no units.
- The correlation ranges from −1 to +1.
- That is, $-1 \le \text{Corr}(R_i, R_j) \le +1$
- If $\text{Corr}(R_i, R_j) = 1.0$, the random variables have perfect positive correlation. This means that a movement in one random variable results in a proportional positive movement in the other relative to its mean.
- If $\text{Corr}(R_i, R_j) = -1.0$, the random variables have perfect negative correlation. This means that a movement in one random variable results in an exact opposite proportional movement in the other relative to its mean.
- If $\text{Corr}(R_i, R_j) = 0$, there is no linear relationship between the variables, indicating that prediction of R_i cannot be made on the basis of R_j using linear methods.

Example: Correlation

Using our previous example, **compute** and **interpret** the correlation of the returns for stocks A and B given that $\sigma^2(R_A) = 0.0028$ and $\sigma^2(R_B) = 0.0124$ and recalling that $\text{Cov}(R_A, R_B) = 0.0058$.

Answer:

First, it is necessary to convert the variances to standard deviations.

$$\sigma(R_A) = (0.0028)^{1/2} = 0.0529$$
$$\sigma(R_B) = (0.0124)^{1/2} = 0.1114$$

Now, the correlation between the returns of Stock A and Stock B can be computed as:

$$\text{Corr}(R_A, R_B) = \frac{0.0058}{(0.0529)(0.1114)} = 0.9842$$

The closeness of this value to +1 indicates that the strength of the linear relationship is positive and very strong.

LOS 8.k: Calculate and interpret the expected value, variance, and standard deviation particularly for return on a portfolio.

The **expected value** and **variance for a portfolio of assets** can be determined using the properties of the individual assets in the portfolio. To do this, it is necessary to establish the portfolio weight for each asset. As indicated in the formula below, the weight, *w*, of portfolio asset *i* is simply the market value currently invested in the asset divided by the current market value of the entire portfolio.

$$w_i = \frac{\text{market value of investment in asset i}}{\text{market value of the portfolio}}$$

Portfolio expected value. The expected value of a portfolio composed of *n* assets with weights, w_i, and expected values, R_i, can be determined using the following formula:

$$E(R_p) = \sum_{i=1}^{N} w_i E(R_i) = w_1E(R_1) + w_2E(R_2) + \ldots + w_nE(R_n)$$

More often, we have expected returns (rather than expected prices). When the R_i are returns, the expected return for a portfolio, $E(R_P)$, is calculated using the asset weights and the same formula as above.

Portfolio variance. The variance of the portfolio return uses the portfolio weights also, but in a more complicated way:

$$\text{Var}(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \text{Cov}(R_i, R_j)$$

The way this formula works, particularly in its use of the double summation operator, $\sum\sum$, is best explained using two-asset and three-asset portfolio examples.

Example: Variance of a two-asset portfolio

Symbolically express the variance of a portfolio composed of risky asset A and risky asset B.

Answer:

Application of the variance formula provides the following:

$$\text{Var}(R_p) = w_A w_A \text{Cov}(R_A,R_A) + w_A w_B \text{Cov}(R_A,R_B) + w_B w_A \text{Cov}(R_B,R_A) + w_B w_B \text{Cov}(R_B,R_B)$$

Now, since $\text{Cov}(R_A,R_B) = \text{Cov}(R_B,R_A)$, and $\text{Cov}(R_A,R_A) = \sigma^2(R_A)$, this expression reduces to the following:

$$\text{Var}(R_P) = w_A^2\sigma^2(R_A) + w_B^2\sigma^2(R_B) + 2w_A w_B \text{Cov}(R_A,R_B)$$

Professor's Note: Know this formula! The calculation is required by an LOS in Quantitative Methods.

Since $\text{Cov}(R_A,R_B) = \sigma(R_B)\sigma(R_A)\rho(R_A,R_B)$, another way to present this formula is:

$$\text{Var}(R_P) = w_A^2\sigma^2(R_A) + w_B^2\sigma^2(R_B) + 2w_A w_B \sigma(R_A)\sigma(R_B)\rho(R_A,R_B)$$

Example: Variance of a three-asset portfolio

A portfolio composed of risky assets A, B, and C will have a variance of return determined as:

$$\begin{aligned}\text{Var}(R_p) = {} & w_A w_A \text{Cov}(R_A,R_A) + w_A w_B \text{Cov}(R_A,R_B) + w_A w_C \text{Cov}(R_A,R_C) \\ & + w_B w_A \text{Cov}(R_B,R_A) + w_B w_B \text{Cov}(R_B,R_B) + w_B w_C \text{Cov}(R_B,R_C) \\ & + w_C w_A \text{Cov}(R_C,R_A) + w_C w_B \text{Cov}(R_C,R_B) + w_C w_C \text{Cov}(R_C,R_C)\end{aligned}$$

which can be reduced to the following expression:

$$\begin{aligned}\text{Var}(R_p) = {} & w_A^2\sigma^2(R_A) + w_B^2\sigma^2(R_B) + w_C^2\sigma^2(R_C) \\ & + 2w_A w_B \text{Cov}(R_A,R_B) + 2w_A w_C \text{Cov}(R_A,R_C) + 2w_B w_C \text{Cov}(R_B,R_C)\end{aligned}$$

A portfolio composed of four assets will have four $w_i^2\sigma^2(R_i)$ terms and six $2w_i w_j \text{Cov}(R_i,R_j)$ terms. A portfolio with five assets will have five $w_i^2\sigma^2(R_i)$ terms and ten $2w_i w_j \text{Cov}(R_i,R_j)$ terms. In fact, the expression for the variance of an *n*-asset portfolio will have $n(n-1)/2$ unique $\text{Cov}(R_i,R_j)$ terms since $\text{Cov}(R_i,R_j) = \text{Cov}(R_j,R_i)$.

Professor's Note: I would expect that if there is a problem on the exam that requires the calculation of the variance (standard deviation) of a portfolio of risky assets, it would involve only two risky assets.

The following formula is useful when we want to compute covariances, given correlations and variances.

$$\text{Cov}(R_i,R_j) = \sigma(R_i)\sigma(R_j)\rho(R_i,R_j)$$

LOS 8.1: Calculate covariance given a joint probability function.

Example: Expected value, variance, and covariance

What is the expected value, variance, and covariance(s) for a portfolio that consists of $400 in Asset A and $600 in Asset B? The joint probabilities of the returns of the two assets are in Figure 6.

Figure 6: Probability Table

Joint Probabilities	$R_B = 0.40$	$R_B = 0.20$	$R_B = 0.00$
$R_A = 0.20$	0.15	0	0
$R_A = 0.15$	0	0.60	0
$R_A = 0.04$	0	0	0.25

Answer:

The asset weights are:

$$w_A = \$400 \;/\; (\$400 + \$600) = 0.40$$
$$w_B = \$600 \;/\; (\$400 + \$600) = 0.60$$

The expected return of the individual assets is determined as:

$$E(R_A) = P(R_{A1},R_{B1})R_{A1} + P(R_{A2},R_{B2})R_{A2} + P(R_{A3},R_{B3})R_{A3}$$
$$E(R_A) = (0.15)(0.20) + (0.60)(0.15) + (0.25)(0.04) = 0.13$$
$$E(R_B) = P(R_{B1},R_{A1})R_{B1} + P(R_{B2},R_{A2})R_{B2} + P(R_{B3},R_{A3})R_{B3}$$
$$E(R_B) = (0.15)(0.40) + (0.60)(0.20) + (0.25)(0.00) = 0.18$$

The variance for the individual asset returns is determined as:

$$\text{Var}(R_A) = P(R_{A1},R_{B1})[(R_{A1} - E(R_A)]^2 + P(R_{A2},R_{B2})[(R_{A2} - E(R_A)]^2 + P(R_{A3},R_{B3})[(R_{A3} - E(R_A)]^2$$
$$\text{Var}(R_A) = (0.15)(0.20 - 0.13)^2 + (0.6)(0.15 - 0.13)^2 + (0.25)(0.04 - 0.13)^2 = 0.0030$$
$$\text{Var}(R_B) = P(R_{B1},R_{A1})[(R_{B1} - E(R_B)]^2 + P(R_{A2},R_{B2})[(R_{B2} - E(R_B)]^2 + P(R_{B3},R_{A3})[(R_{B3} - E(R_B)]^2$$
$$\text{Var}(R_B) = (0.15)(0.40 - 0.18)^2 + (0.6)(0.20 - 0.18)^2 + (0.25)(0.00 - 0.18)^2 = 0.0156$$

The covariance of the individual asset returns is determined as:

$$\begin{aligned}\text{Cov}(R_A, R_B) = \;& P(R_{A1},R_{B1})[R_{A1} - E(R_A)][(R_{B1} - E(R_B)] \\ &+ P(R_{A2},R_{B2})[R_{A2} - E(R_A)][(R_{B2} - E(R_B)] \\ &+ P(R_{A3},R_{B3})[R_{A3} - E(R_A)][(R_{B3} - E(R_B)]\end{aligned}$$

$$\begin{aligned}Cov(R_A, R_B) &= 0.15(0.20 - 0.13)(0.40 - 0.18)\\ &+ 0.60(0.15 - 0.13)(0.20 - 0.18)\\ &+ 0.25(0.04 - 0.13)(0.00 - 0.18)\\ &= 0.0066\end{aligned}$$

Using the weights $w_A = 0.40$ and $w_B = 0.60$, the expected return and variance of the portfolio are computed as:

$$E(R_P) = w_A E(R_A) + w_B E(R_B) = (0.4)(0.13) + (0.6)(0.18) = 0.16$$

$$\begin{aligned}Var(R_p) &= (0.40)^2(0.003) + (0.60)^2(0.0156) + 2(0.4)(0.60)(0.0066)\\ &= 0.009264\end{aligned}$$

Please note that as tedious as this example was, if more of the cells in the joint probability matrix were not zero, it could have been even more tedious.

Example: Correlation and covariance

Consider a portfolio of three assets, X, Y, and Z, where the individual market value of these assets is \$600, \$900, and \$1,500, respectively. The market weight, expected return, and variance for the individual assets are presented below. The correlation matrix for the asset returns are shown in Figure 7. Using this information, compute the variance of the portfolio return.

$E(R_X) = 0.10$ $\quad Var(R_X) = 0.0016$ $\quad w_X = 0.2$
$E(R_Y) = 0.12$ $\quad Var(R_Y) = 0.0036$ $\quad w_Y = 0.3$
$E(R_Z) = 0.16$ $\quad Var(R_Z) = 0.0100$ $\quad w_Z = 0.5$

Figure 7: Stock X, Y, and Z Returns Correlation

Correlation Matrix			
Returns	R_X	R_Y	R_Z
R_X	1.00	0.46	0.22
R_Y	0.46	1.00	0.64
R_Z	0.22	0.64	1.00

Answer:

The expected return for the portfolio may be determined as:

$$E(R_p) = \sum w_i E(R_i) = w_1 E(R_1) + w_2 E(R_2) + w_3 E(R_3)$$
$$E(R_P) = (0.20)(0.10) + (0.30)(0.12) + (0.50)(0.16)$$
$$E(R_P) = 0.136$$

The variance of a three-asset portfolio return is determined using the formula:

$$\begin{aligned}Var(R_p) = &\ w_X^2\sigma^2(R_X) + w_Y^2\sigma^2(R_Y) + w_Z^2\sigma^2(R_Z) + 2w_Xw_YCov(R_X,R_Y) + 2w_Xw_ZCov(R_X,R_Z)\\ &+ 2w_Yw_ZCov(R_Y,R_Z)\end{aligned}$$

Here we must make use of the relationship $Cov(R_i,R_j) = \sigma(R_i)\sigma(R_j)\rho(R_i,R_j)$, since we are not provided with the covariances.

Let's solve for the covariances, then substitute the resulting values into the portfolio return variance equation.

$$\text{Cov}(R_X,R_Y) = (0.0016)^{½}(0.0036)^{½}(0.46) = 0.001104$$
$$\text{Cov}(R_X,R_Z) = (0.0016)^{½}(0.0100)^{½}(0.22) = 0.000880$$
$$\text{Cov}(R_Y,R_Z) = (0.0036)^{½}(0.0100)^{½}(0.64) = 0.003840$$

Now we can solve for the variance of the portfolio returns as:

$$\text{Var}(R_P) = (0.20)^2(0.0016) + (0.30)^2(0.0036) + (0.50)^2(0.01) + (2)(0.2)(0.3)(0.001104) + (2)(0.2)(0.5)(0.00088) + (2)(0.3)(0.5)(0.00384)$$

$$\text{Var}(R_P) = 0.004348$$

Example: Covariance matrix

Assume you have a portfolio that consists of Stock S and a put option, O, on Stock S. The corresponding weights of these portfolio assets are $w_S = 0.90$ and $w_O = 0.10$. Using the covariance matrix provided in Figure 8, calculate the variance of the return for the portfolio.

Figure 8: Returns Covariance for Stock S and Put O

Covariance Matrix		
Returns	R_S	R_O
R_S	0.0011	–0.0036
R_O	–0.0036	0.016

Answer:

This is the simplest type of example because the most tedious calculations have already been performed. Simply extract the appropriate values from the covariance matrix and insert them into the variance formula.

Recall that the covariance of an asset with itself is the variance. Thus, the terms along the diagonal in the covariance matrix are return variances.

The portfolio return variance can be computed as:

$$\text{Var}(R_P) = (0.90)^2(0.0011) + (0.10)^2(0.016) + 2(0.90)(0.10)(-0.0036) = 0.000403$$

LOS 8.m: Calculate and interpret an updated probability, using Bayes' formula.

Bayes' formula is used to update a given set of prior probabilities for a given event in response to the arrival of new information. The rule for updating prior probability of an event is:

$$\text{updated probability} = \frac{\text{probability of new information for a given event}}{\text{unconditional probability of new information}} \times \text{prior probability of event}$$

Note in the following example of the application of Bayes' formula that we can essentially reverse a given set of conditional probabilities. This means that, given P(B), P(A | B), and P(A | B^C), it is possible to use Bayes' formula to compute P(B | A).

Example: Bayes' formula

Electcomp Corporation manufactures electronic components for computers and other devices. There is speculation that Electcomp is about to announce a major expansion into overseas markets. The expansion will occur, however, only if Electcomp's managers estimate overseas demand to be sufficient to support the necessary sales. Furthermore, if demand is sufficient and overseas expansion occurs, Electcomp is likely to raise its prices.

Using *O* to represent the event of overseas expansion, *I* to represent a price increase, and I^C to represent no price increase, an industry analyst has estimated the unconditional and conditional probabilities shown as follows:

$$P(I) = 0.3$$
$$P(I^C) = 0.7$$
$$P(O \mid I) = 0.6$$
$$P(O \mid I^C) = 0.4$$

The analyst's estimates for P(I) and $P(I^C)$ are called the *priors* because they reflect what is already known. They do not reflect the current information about the possible overseas expansion.

Application of Bayes' formula allows us to compute P(I | O), the probability that prices will increase given that Electcomp announces that it will expand overseas (the new information). Using the multiplication rule, we can express the joint probability of *I* and *O*:

$$P(O \mid I) = P(IO) / P(I), \text{ and } P(IO) = P(I \mid O) \times P(O)$$

Based on these relationships, Bayes' formula can be expressed using the information from this example as indicated below [i.e., substitute P(IO) from the second equation into the first and solve for P(O | I)]:

$$P(I \mid O) = \frac{P(O \mid I)}{P(O)} \times P(I)$$

In order to solve this equation, P(O) must be determined. This can be done using the total probability rule:

$$P(O) = P(O \mid I) \times P(I) + P(O \mid I^C) \times P(I^C)$$
$$P(O) = (0.6 \times 0.3) + (0.4 \times 0.7)$$
$$P(O) = 0.46$$

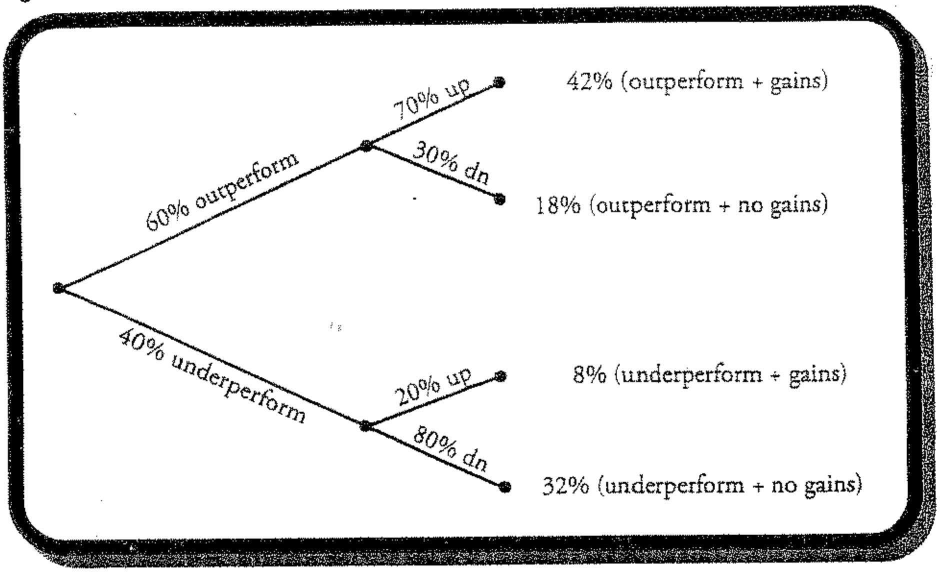
Now the updated probability of the increase in prices given that Electcomp expands overseas can be computed:

$$P(I \mid O) = \frac{0.60}{0.46} \times 0.30 = 0.3913$$

This means that if the new information of "expand overseas" is announced, the prior probability estimate of P(I) = 0.30 must be increased to 0.3913.

Another illustration of the use of Bayes' formula may make it easier to remember and apply. Consider the following possibilities: There is a 60% probability the economy will outperform, and if it does, there is a 70% chance a stock will be up and a 30% chance the stock will go down. There is a 40% chance the economy will underperform, and if it does, there is a 20% chance the stock in question will increase in value (have gains) and an 80% chance it will not. Let's diagram this situation.

Figure 9: A Probability Model



In Figure 9, we have multiplied the probabilities to calculate the probabilities of each of the four outcome pairs. Note that these sum to 1. Given that the stock has gains, what is our updated probability of an outperforming economy? We sum the probability of stock gains in both states (outperform and underperform) to get 42% + 8% = 50%. Given that the stock has gains, the probability that the economy has outperformed is $\frac{42\%}{50\%} = 84\%$. In the previous notation the priors are as follows:
probability of economic outperformance = P(O) = 60%, the probability of stock gains given economic outperformance is P(G | O) = 70%, and the (unconditional) probability of a gain in stock price is 50%.

We are seeking P(O | G), the probability of outperformance given gains. Bayes' formula says:

$$P(O|G) = \frac{P(G|O) \times P(O)}{P(G)}, \text{ which is our } \frac{42\%}{50\%} = 84\%$$

LOS 8.n: Solve counting problems using the factorial, combination, and permutation notations, and identify which counting method is appropriate to solve a particular counting problem.

Labeling refers to the situation where there are n items that can each receive one of k different labels. The number of items that receives label 1 is n_1 and the number that receive label 2 is n_2, and so on such that $n_1 + n_2 + n_3 + \ldots + n_k = n$. The total number of ways that the labels can be assigned is:

$$\frac{n!}{(n_1!) \times (n_2!) \times \ldots \times (n_k!)}$$

where:
the symbol "!" stands for **factorial**. For example, $4! = 4 \times 3 \times 2 \times 1 = 24$, and $2! = 2 \times 1 = 2$.

The general expression for n factorial is:

$n! = n \times (n - 1) \times (n - 2) \times (n - 3) \times \ldots \times 1$, where by definition, $0! = 1$

Calculator help: On the TI, factorial is [2nd] [x!] (above the multiplication sign). On the HP, factorial is [g] [n!]. To compute 4! on the TI, enter [4][2nd][x!] = 24. On the HP, press [4][ENTER][g][n!].

Example: Labeling

Consider a portfolio consisting of eight stocks. Your goal is to designate four of the stocks as "long-term holds," three of the stocks as "short-term holds," and one stock as "sell." How many ways can these eight stocks be labeled?

Answer:

There are 8! = 40,320 total possible sequences that can be followed to assign the three labels to the eight stocks. However, the order that each stock is assigned a label does not matter. For example, it does not matter which of the first three stocks labeled "long-term" is the first to be labeled. Thus, there are 4! ways to assign the long-term label. Continuing this reasoning to the other categories, there are $4! \times 3! \times 1!$ equivalent sequences for assigning the labels. To eliminate the counting of these redundant sequences, the total number of possible sequences (8!) must be divided by the number of redundant sequences ($4! \times 3! \times 1!$).

Thus, the number of ***different*** ways to label the eight stocks is:

$$\frac{8!}{4! \times 3! \times 1!} = \frac{40{,}320}{24 \times 6 \times 1} = 280$$

A special case of labeling arises when the number of labels equals 2 (k = 2). That is, the *n* items can only be in one of two groups, and $n_1 + n_2 = n$. In this case, we can let $r = n_1$ and $n_2 = n - r$. Since there are only two categories, we usually talk about choosing *r* items. Then $(n - r)$ are not chosen. The general formula for labeling when k = 2 is called the **combination formula** (or *binomial formula*) and is expressed as:

$$_nC_r = \frac{n!}{(n-r)!r!},$$

where $_nC_r$ is the number of possible ways (combinations) of selecting *r* items from a set of *n* items when the order of selection is not important. This is also written $\binom{n}{r}$ and read "n choose r."

Another useful formula is the **permutation formula.** A permutation is a specific ordering of a group of objects. The question of how many different groups of size *r* in specific order can be chosen from *n* objects is answered by the permutation formula. The number of permutations of *r* objects from *n* objects = $\frac{n!}{(n-r)!}$. We will give an example using this formula shortly.

Professor's Note: The combination formula $_nC_r$ and the permutation formula $_nP_r$ are both available on the TI calculator. To calculate the number of different groups of three stocks from a list of eight stocks (i.e., $_8C_3$) the sequence is 8 [2nd] [$_nC_r$] 3 [=] which yields 56. If we want to know the number of differently ordered groups of three that can be selected from a list of eight, we enter 8 [2nd] [$_nP_r$] 3 [=] to get 336 which is the number of permutations, $\frac{8!}{(8-3)!}$. This function is not available on the HP calculator. Remember, current policy permits you to bring both calculators to the exam if you choose.

Example: Number of choices in any order

How many ways can three stocks be sold from an eight-stock portfolio?

Answer:

This is similar to the preceding labeling example. Since order does not matter, we take the total number of possible ways to select three of the eight stocks and divide by the number of possible redundant selections. Thus, the answer is:

$$\frac{8!}{5! \times 3!} = 56$$

In the preceding two examples, ordering did not matter. The order of selection could, however, be important. For example, suppose we want to liquidate only one stock position per week over the next three weeks. Once we choose three particular stocks to sell, the order in which they are sold must be determined. In this case, the concept of permutation comes into play. The *permutation formula* is:

$$_{n}P_{r} = \frac{n!}{(n-r)!},$$

where $_{n}P_{r}$ is the number of possible ways (permutations) to select *r* items from a set of *n* items when the order of selection is important. The permutation formula implies that there are r! *more* ways to choose *r* items if the *order* of selection *is important* than if order is not important.

Example: Permutation

How many ways are there to sell three stocks out of eight if the order of the sales is important?

Answer:

$$_{n}P_{r} = {}_{8}P_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 336$$

This is 3! times the 56 possible combinations computed in the preceding example for selecting the three stocks when the order was not important.

There are five guidelines that may be used to determine which counting method to employ when dealing with counting problems:

- The *multiplication rule* of counting is used when there are *two or more groups*. The key is that only *one* item may be selected from each group.
- *Factorial* is used by itself when there are *no groups*—we are only arranging a given set of *n* items. Given *n* items, there are n! ways of arranging them.
- The *labeling formula* applies to *three or more sub-groups* of predetermined size. Each element of the entire group must be assigned a place, or label, in one of the three or more sub-groups.
- The *combination formula* applies to *only two groups* of predetermined size. Look for the word "choose" or "combination."
- The *permutation formula* applies to *only two groups* of predetermined size. Look for a specific reference to "order" being important.

KEY CONCEPTS

1. A random variable is an uncertain outcome determined by chance.
2. The two properties of probability are:
   - The sum of the probabilities of all possible mutually exclusive events is 1.
   - The probability of any event is not larger than 1 or smaller than zero.
3. A priori probability measures probabilities based on well-defined inputs; empirical probability measures probability from observations or experiments; and subjective probability is an informed guess.
4. Unconditional probability (marginal probability) is the probability of an event occurring; conditional probability, $P(A|B)$, is the probability of an event (A) occurring given that another event (B) has occurred.
5. The general rule of multiplication is used to find the probability that two events will occur when one event is conditional on the other is $P(A \text{ and } B) = P(A|B) \times P(B)$ which is $P(A \text{ and } B) = P(A) \times P(B)$ for independent events.
6. The general rule of addition is that $P(A \text{ or } B) = P(A) + P(B) - P(AB)$.
7. The probability of an independent event is unaffected by the occurrence of other events, but the probability of a dependent event is changed by the occurrence of another event.
8. The probability that any one of a set of independent events will occur is the sum of their probabilities, and the probability that they will all occur is the product of their probabilities.
9. Using the total probability rule, the expected value is the probability-weighted average of the conditional expected values: $E(X) = \sum_{i=1}^{n} [P_i(S_i)] \times E(X_i | S_i)$, where S_i is a set of mutually exclusive and exhaustive events.
10. The expected value of a random variable, E(X), equals $\sum_{i=1}^{n} P(x_i)X_i$, and the variance of a random variable, Var(X), equals $\sum_{i=1}^{n} P(x_i)\,[X_i - E(X)]^2 = \sigma_X^2$.
11. Conditional expectations are used in investments to update expectations when a conditioning event has occurred.
12. Correlation is a standardized measure of association between two random variables; it ranges in value from −1 to +1 and is equal to $\dfrac{\text{Cov}(A,B)}{\sigma_A \sigma_B}$.
13. The expected returns and variance of a two-asset portfolio are given by:

$$E(R_p) = w_1E(R_1) + w_2E(R_2)$$

$$\begin{aligned} \text{Var}(R_p) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\text{Cov}_{12} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_1\sigma_2\rho_{12} \end{aligned}$$

14. Bayes' formula for updating probabilities based on the occurrence of an event *O* is:

$$P(I \mid O) = \frac{P(O \mid I)}{P(O)} \times P(I)$$

15. The number of ways to order *n* objects is *n* factorial, $n! = n \times (n-1) \times (n-2) \times \ldots \times 1$.
16. The number of ways to choose a subset of *r* from a set of *n* when order doesn't matter is $\dfrac{n!}{(n-r)!r!}$; when order matters, there are $\dfrac{n!}{(n-r)!}$ permutations.

CONCEPT CHECKERS: PROBABILITY CONCEPTS

1. Given the conditional probabilities in the table below and the unconditional probabilities P(Y = 1) = 0.3 and P(Y = 2) = 0.7, what is the expected value of X?

x_i	$P(x_i / Y = 1)$	$P(x_i / Y = 2)$
0	0.2	0.1
5	0.4	0.8
10	0.4	0.1

A. 4.3.
B. 5.0.
C. 5.3.
D. 5.7.

Use the following data to answer Questions 2 through 6.

Joint Probabilities			
Returns	$R_B = 0.5$	$R_B = 0.0$	$R_B = 0.0$
$R_A = -0.1$	0.4	0	0
$R_A = 0.1$	0	0.3	0
$R_A = 0.3$	0	0	0.3

2. Given the joint probability table, the expected return of Stock A is *closest* to:
A. 0.12.
B. 0.08.
C. 0.20.
D. 0.15.

3. Given the joint probability table, the standard deviation of Stock B is *closest* to:
A. 0.060.
B. 0.212.
C. 0.045.
D. 0.245.

4. Given the joint probability table, the variance of Stock A is *closest* to:
A. 0.0276.
B. 0.1661.
C. 0.0450.
D. 0.0129.

5. Given the joint probability table, the covariance between A and B is *closest* to:
A. 0.03690.
B. 0.00129.
C. –0.03600.
D. –0.00129.

6. Given the joint probability table, the correlation between R_A and R_B is *closest* to:
   A. −0.88.
   B. −0.33.
   C. +0.33.
   D. +0.50.

7. The probability that the DJIA will increase tomorrow is 2/3. The probability of an increase in the DJIA stated as odds is:
   A. two-to-one.
   B. one-to-three.
   C. one-to-two.
   D. two-to-three.

8. A discrete uniform distribution (each event has an equal probability of occurrence) has the following possible outcomes for X [1, 2, 3, 4]. The variance of this distribution is *closest* to:
   A. 0.00.
   B. 1.00.
   C. 1.25.
   D. 2.00.

9. If events A and B are mutually exclusive, then:
   A. $P(A \mid B) = P(A)$.
   B. $P(A \mid B) = P(B)$.
   C. $P(AB) = P(A) \times P(B)$.
   D. $P(A \text{ or } B) = P(A) + P(B)$.

10. At a charity ball, 800 names were put into a hat. Four of the names are identical. On a random draw, what is the probability that one of these four names will be drawn?
    A. 0.004.
    B. 0.005.
    C. 0.010.
    D. 0.025.

11. Among 900 taxpayers with incomes below $100,000, 35 were audited by the IRS. The probability that a randomly chosen individual with an income below $100,000 was audited is *closest* to:
    A. 0.039.
    B. 0.125.
    C. 0.350.
    D. 1.000.

12. Which of the following values *cannot* be the probability of an event?
    A. 0.00.
    B. 0.78.
    C. 1.25.
    D. 1.00.

13. Two mutually exclusive events:
    A. always occur together.
    B. cannot occur together.
    C. can sometimes occur together.
    D. occur together based only upon mutual consent.

14. Two events are said to be independent if the occurrence of one event:
    A. means that the second event cannot occur.
    B. means that the second event is certain to occur.
    C. affects the probability of the occurrence of the other event.
    D. does not affect the probability of the occurrence of the other event.

Use the following conditional probabilities to answer Questions 15 through 18.

State of the Economy	*Probability of the Economic State*	*Stock Performance*	*Conditional Probability of Stock Performance*
Good	0.30	Good Neutral Poor	0.60 0.30 0.10
Neutral	0.50	Good Neutral Poor	0.30 0.40 0.30
Poor	0.20	Good Neutral Poor	0.10 0.60 0.30

15. What is the conditional probability of having good stock performance in a poor economic environment?
    A. 0.02.
    B. 0.03.
    C. 0.10.
    D. 0.30.

16. What is the joint probability of having a good economy and a neutral stock performance?
    A. 0.09.
    B. 0.20.
    C. 0.30.
    D. 1.30.

17. What is the total probability of having a good performance in the stock?
    A. 0.20.
    B. 0.35.
    C. 0.65.
    D. 1.00.

18. Given that the stock had good performance, the probability the state of the economy was good is *closest* to:
    A. 0.35.
    B. 0.46.
    C. 0.51.
    D. 1.00.

19. Consider a universe of ten bonds from which an investor will ultimately purchase six bonds for his portfolio. If the order in which he buys these bonds is not important, how many potential six-bond combinations are there?
    A. 7.
    B. 210.
    C. 5,040.
    D. Cannot be determined.

20. The correlation of returns between Stocks A and B is 0.50. The covariance between these two securities is 0.0043, and the standard deviation of the return of Stock B is 26%. The variance of returns for Stock A is:
    A. 0.0331.
    B. 0.0011.
    C. 0.2656.
    D. 0.0112.

21. There are ten sprinters in the Olympic finals. How many different ways can the gold, silver, and bronze medals be awarded?
    A. 120.
    B. 720.
    C. 1,440.
    D. 604,800.

22. Which of the following is **NOT** a probability distribution?

    A. $X = [1,2,3,4]$; Prob $[X_i] = \frac{X_i}{10}$.

    B. $X = [1,2,3,4]$; Prob $[X_i] = \frac{X_i^2}{30}$.

    C. $X = [5,10]$; Prob $[X_i] = \frac{8 - X_i}{5}$.

    D. $X = [5,10]$; Prob $[X_i] = \frac{X_i - 3}{9}$.

Comprehensive Problems: Probability Concepts

1. Given the following probability data for the return on the market and the return on Best Oil, calculate the covariance of returns between Best Oil and the market.

Joint Probability Table

	R_{Best} = 20%	R_{Best} = 10%	R_{Best} = 5%
R_{Mkt} = 15%	0.4	0	0
R_{Mkt} = 10%	0	0.2	0
R_{Mkt} = 0%	0	0	0.4

2. The correlation of returns between the returns on Cape Products and Dogger Industries is 0.6. The standard deviation of returns for Cape is 15% and the standard deviation of returns for Dogger is 20%. The expected return for Dogger is 18% and the expected return for Cape is 12%. Calculate the expected returns and standard deviation of returns on a portfolio that has $300,000 invested in Dogger and $200,000 invested in Cape.

3. M. Atwood, an analyst, has developed a scoring system for bonds and found that if the score from a bond is less than 20, there is a probability of 85% that it will default within five years. If a bond's score is greater than or equal to 20, there is only a 40% chance that it will default within five years. Given that a randomly chosen bond currently has a 25% probability of a score less than 20, what is the probability that a bond that defaults within the next five years had a score of 20 or higher?

4. A bond that matures in one year is priced at $950 today. You estimate that it has a 10% probability of default. If the bond defaults, you expect to recover $600. If it does not default, it will pay $1,080 at maturity. The nominal 1-year risk-free rate is 7.5%.

   A. What are the odds against this bond defaulting?

   B. What is the expected payoff on the bond in one year?

   C. What is the expected return on the bond?

   D. What would be the price of the bond if its expected return were equal to the risk-free rate?

5. You are considering a portfolio of three stocks:
   - Stock A (55% of the portfolio) has an expected return of 8% with a standard deviation of 24%.
   - Stock B (25% of the portfolio) has an expected return of 4% with a standard deviation of 18%.
   - Stock C (20% of the portfolio) has an expected return of 3% with a standard deviation of 15%.

   The correlations between these stocks' returns are:
   - Stock A with Stock B: 0.85.
   - Stock A with Stock C: 0.30.
   - Stock B with Stock C: –0.15.

   A. Based on these data, construct a covariance matrix for the returns on the three stocks.

   B. Calculate the expected return and standard deviation of the portfolio.

   C. Provide a set of three mutually exclusive and exhaustive events with respect to the relation between this portfolio's realized return and its expected return.

   D. If you add three more stocks to the portfolio, how many variances and how many unique covariances will you need to calculate the portfolio variance?

6. You are forecasting the sales of a building materials supplier by assessing the expansion plans of its largest customer, a homebuilder. You estimate the probability that the customer will increase its orders for building materials to 25%. If the customer does increase its orders, you estimate the probability that the homebuilder will start a new development at 70%. If the customer does not increase its orders from this supplier, you estimate only a 20% chance that it will start the new development. Later you find out that the homebuilder will start the new development. In light of this new information, what is your new (updated) probability that the builder will increase its orders from this supplier?

ANSWERS – CONCEPT CHECKERS: PROBABILITY CONCEPTS

1. C $E(X|Y = 1) = (0.2)(0) + (0.4)(5) + (0.4)(10) = 6$ and $E(X|Y = 2) = (0.1)(0) + (0.8)(5) + (0.1)(10) = 5$

   $E(X) = (0.3)(6) + (0.7)(5) = 5.30$

2. B $E(R_A) = (0.4)(-0.1) + (0.3)(0.1) + (0.3)(0.3) = 0.08$

3. D Expected return of Stock B = $(0.4)(0.5) = 0.20$

   $VAR(R_B) = 0.4(0.5 - 0.2)^2 + 0.3(0 - 0.2)^2 + 0.3(0 - 0.2)^2 = 0.06$

   Standard deviation = $(0.06)^{½} = 0.2449$

4. A $E(R_A) = (0.4)(-0.1) + (0.3)(0.1) + (0.3)(0.3) = 0.08$

   $VAR(R_A) = 0.4(-0.1 - 0.08)^2 + 0.3(0.1 - 0.08)^2 + 0.3(0.3 - 0.08)^2 = 0.0276$

5. C $COV(R_A,R_B) = 0.4(-0.1 - 0.08)(0.5 - 0.2) + 0.3(0.1 - 0.08)(0 - 0.2) + 0.3(0.3 - 0.08)(0 - 0.2) = -0.03600$

6. A $CORR(R_A,R_B) = COV(R_A,R_B) / \sigma(R_A)\sigma(R_B) = -0.036 / (0.1661 \times 0.24494) = -0.036 / 0.0406845 = -0.8849$

7. A Odds for E $= P(E) / [1 - P(E)] = \dfrac{2/3}{1/3} = 2/1 =$ two-to-one

8. C Expected value = $(1/4)(1 + 2 + 3 + 4) = 2.5$

   Variance = $1.25 = (1/4)[(1 - 2.5)^2 + (2 - 2.5)^2 + (3 - 2.5)^2 + (4 - 2.5)^2]$

   Note that since each observation is equally likely, each has 25% (1/4) chance of occurrence.

9. D There is no intersection of events when events are mutually exclusive. $P(A|B) = P(A) \times P(B)$ is only true for independent events. Note that since A and B are mutually exclusive (cannot both happen), $P(A|B)$ and $P(AB)$ must both be equal to zero, making answers A, B, and C incorrect.

10. B P(name 1 or name 2 or name 3 or name 4) = 1/800 + 1/800 + 1/800 + 1/800 = 4/800 = 0.005

11. A 35 / 900 = 0.0389

12. C Probabilities may range from zero (meaning no chance of occurrence) through 1 (which means a sure thing).

13. B One or the other may occur, not both.

14. D Two events are said to be independent if the occurrence of one event does not affect the probability of the occurrence of the other event.

15. C Go to the poor state and read off the probability of good performance [i.e., P(poor performance | good economy) = 0.10].

16. A P(good economy and neutral performance) = P(good economy)P(neutral performance | good economy) = (0.3)(0.3) = 0.09

17. B (0.3)(0.6) + (0.5)(0.3) + (0.2)(0.1) = 0.35. This is the sum of all the joint probabilities for good performance over all states [i.e., $\sum P(\text{economic state}_i)$ P(good performance | economic state$_i$)].

18. C This is an application of Bayes' formula. P(good economy | good performance) = Prob(good stock performance | good economy) × Prob(good economy)/P(good stock performance).

$$= \frac{(0.6)(0.3)}{(0.3)(0.6)+(0.5)(0.3)+(0.2)(0.1)} = \frac{0.18}{0.35} = 0.5143$$

19. B $${}_{n}C_{r} = \frac{n!}{(n-r)!r!} = {}_{10}C_{6} = \frac{10!}{(10-6)!6!} = \frac{10!}{4!6!} = 210$$

20. B $$\text{Corr}(R_A,R_B) = \frac{\text{Cov}(R_A,R_B)}{[\sigma(R_A)][\sigma(R_B)]}$$

$$\sigma^2(R_A) = \left[\frac{\text{Cov}(R_A,R_B)}{\sigma(R_B)\text{Corr}(R_A,R_B)}\right]^2 = \left[\frac{0.0043}{(0.26)(0.5)}\right]^2 = 0.0331^2 = 0.0011$$

21. B Since the order of the top three finishers matters, we need to use the permutation formula.

$${}_{10}P_{3} = \frac{10!}{(10-3)!} = 720$$

22. C $\frac{8-5}{5} + \frac{8-10}{5} = \frac{1}{5}$, and $\frac{8-10}{5}$ is negative, so this satisfies neither of the requirements for a probability distribution. The others have prob[X_i] between zero and one and $\sum P[X_i] = 1$, and thus satisfy both requirements for a probability distribution.

Answers – Comprehensive Problems: Probability Concepts

1. $E(R_{Best}) = 0.4(20\%) + 0.2(10\%) + 0.4(5\%) = 12\%$

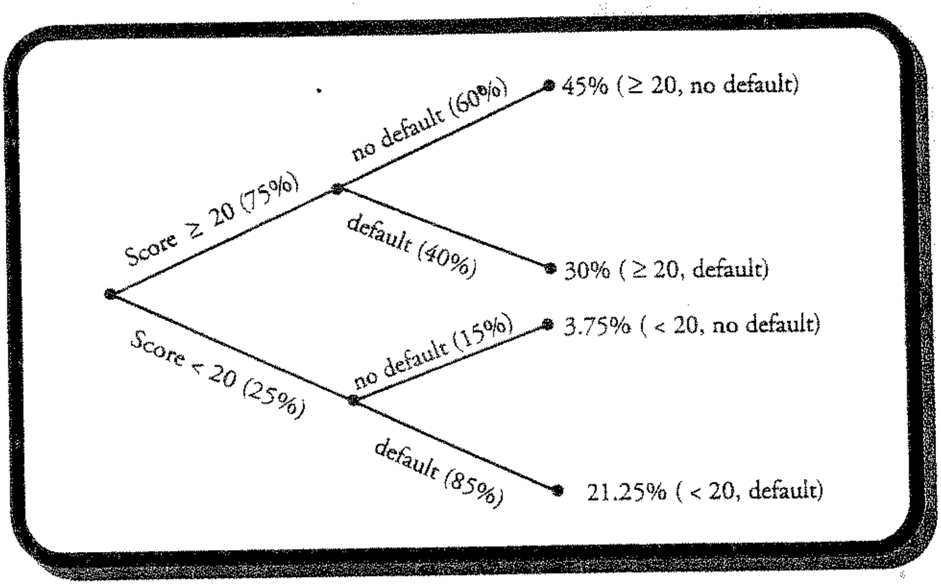
$E(R_{Mkt}) = 0.4(15\%) + 0.2(10\%) + 0.4(0\%) = 8\%$

$$\begin{aligned} \text{Cov}(R_{Best}, R_{Mkt}) &= 0.4(20\% - 12\%)(15\% - 8\%) \\ &+ 0.2(10\% - 12\%)(10\% - 8\%) \\ &+ 0.4(5\% - 12\%)(0\% - 8\%) \\ &= 0.4(8)(7) + 0.2(-2)(2) + 0.4(-7)(-8) = 44 \end{aligned}$$

Remember the units of covariance (like variance) are percent squared here. We used whole number percents in the calculations and got 44; if we had used decimals, we would have gotten 0.0044.

2. The portfolio weight for Dogger (W_D) is $\frac{300}{500} = 60\%$ and for Cape, the portfolio weight is $\frac{200}{500} = 40\%$. The expected return on the portfolio is 0.6(18%) + 0.4(12%) = 15.6%. The variance is $(0.6)^2(0.2)^2 + (0.4)^2(0.15)^2 + 2(0.6)(0.4)(0.6)(0.2)(0.15) = 0.02664$. The standard deviation of portfolio returns is $\sqrt{0.02664} = 16.32\%$.

3. Construct the following tree:



Total probability of default = 30% + 21.25% = 51.25%. Percent of defaulting bonds with score $\geq 20 = \frac{30\%}{51.25\%} = 58.5\%$. A bond that defaults in the next five years has a 58.5% probability of having a current score greater than or equal to 20. Note that we have employed Bayes' theorem here to update the score expectation based on the additional information that a bond has defaulted.

4. A. The odds for the bond defaulting are $\frac{0.10}{1-0.10} = \frac{1}{9}$, or 1 to 9. The odds against the bond defaulting are the reciprocal, or 9 to 1.

B. The expected payoff on the bond at maturity is:

P(default) × bond value if it defaults + P(no default) × bond value if it does not default = 0.1(600) + 0.9(1,080) = 60 + 972 = $1,032

C. The expected return is 1,032 / 950 – 1 = 0.0863, or 8.63%.

D. $\frac{1{,}032}{\text{price}} - 1 = 0.075$ so price would need to be $\frac{1{,}032}{1.075} = \$960$.

5. A. First calculate the variances on each of the three stocks:

$\text{Var(A)} = (0.24)^2 = 0.0576$

$\text{Var(B)} = (0.18)^2 = 0.0324$

$\text{Var(C)} = (0.15)^2 = 0.0225$

These will be the diagonal entries in the covariance matrix.

Covariance Matrix of Returns for Stocks A, B, and C

	Stock A	Stock B	Stock C
Stock A	0.0576		
Stock B		0.0324	
Stock C			0.0225

Next calculate the covariance for each pair of stocks. The correlation (ρ_{xy}) = Cov(x,y)/$\sigma_x\sigma_y$. Rearranging that, we get Cov(x,y) = $\rho_{xy}\ \sigma_x\sigma_y$. So:

Cov(A,B) = 0.85 × 0.24 × 0.18 = 0.0367

Cov(A,C) = 0.30 × 0.24 × 0.15 = 0.0108

Cov(B,C) = –0.15 × 0.18 × 0.15 = –0.0041

These results complete the covariance matrix:.

Covariance Matrix of Returns for Stocks A, B, and C

	Stock A	Stock B	Stock C
Stock A	0.0576	0.0367	0.0108
Stock B	0.0367	0.0324	–0.0041
Stock C	0.0108	–0.0041	0.0225

B. The expected return on the portfolio is a weighted average of the individual stock returns:

$E[R_P] = 0.55(0.08) + 0.25(0.04) + 0.20(0.03) = 0.06$ or 6%

For a 3-asset portfolio, the portfolio variance is calculated as:

$$Var(R_P) = W_A{}^2\sigma^2(R_A) + W_B{}^2\sigma^2(R_B) + W_C{}^2\sigma^2(R_C) + 2W_AW_BCov(R_A,R_B) + 2W_AW_CCov(R_A,R_C) + 2W_BW_CCov(R_B,R_C)$$

Substituting, we get:

$$Var(R_P) = 0.55^2(0.0576) + 0.25^2(0.0324) + 0.20^2(0.0225) + 2(0.55)(0.25)(0.0367) + 2(0.55)(0.20)(0.0108) + 2(0.25)(0.20)(-0.0041)$$

$$= 0.0174 + 0.0020 + 0.0009 + 0.0101 + 0.0024 - 0.0004$$

$$= 0.0324$$

The portfolio standard deviation is $\sqrt{0.0324} = 0.1800$, or 18%.

C. You can answer this question any number of different ways, but here is the most obvious:

Event 1: The realized return is greater than the expected return.

Event 2: The realized return is equal to the expected return.

Event 3: The realized return is less than the expected return.

D. With six assets in the portfolio, there will be 6 variance terms and 15 unique covariance terms. The covariance matrix will have 6 × 6 = 36 cells. The covariance of a stock return with itself is its variance; those will be the six entries on the diagonal. The other 30 cells are the covariance pairs, but since each pair appears twice in the matrix—Cov(A,B) is the same as Cov(B,A)—the number of unique covariance pairs is half of that, or 15. For any portfolio of n assets, the portfolio variance calculation would involve n variance terms and $n(n-1)/2$ unique covariance terms.

6. The prior probability that the builder will increase its orders is 25%.

P(increase) = 0.25

P(no increase) = 0.75

There are four possible outcomes:

- Builder increases its orders and starts new development.
- Builder increases its orders and does not start new development.
- Builder does not increase its orders and starts new development.
- Builder does not increase its orders and does not start new development.

The probabilities of each outcome are as follows:

- P(increase and development) = (0.25)(0.70) = **0.175**.
- P(increase and no development) = (0.25)(0.30) = 0.075.
- P(no increase and development) = (0.75)(0.20) = **0.15**.
- P(no increase and no development) = (0.75)(0.80) = 0.60.

We want to update the probability of an increase in orders, given the new information that the builder is starting the development. We can apply Bayes' formula:

$$P(\text{increase} \mid \text{development}) = \frac{P(\text{development} \mid \text{increase}) \times P(\text{increase})}{P(\text{development})}$$

From our assumptions, P(development | increase) = 0.70, and P(increase) = 0.25, so the numerator is (0.70)(0.25) = 0.175.

P(development) is the sum of P(development and increase) and P(development and no increase).

P(development)= 0.175 + 0.15 = 0.325

$$\text{Thus, } P(\text{increase} \mid \text{development}) = \frac{(0.7)\times(0.25)}{0.175+0.15} = \frac{0.175}{0.325} = 0.5385\text{, or } 53.85\%$$

Professors Note: I can never remember this formula, so I set these problems up like the probability model (tree) in the notes and focus on the probabilities of the new information—development in this case—which I have put in bold. Total probability of development is 17.5 + 15 = 32.5. Of that probability, 17.5 / 32.5, or 53.85% of the time, development is paired with an increase in sales!

The following is a review of the Quantitative Methods principles designed to address the learning outcome statements set forth by CFA Institute®. This topic is also covered in:

COMMON PROBABILITY DISTRIBUTIONS

Study Session 3

EXAM FOCUS

This topic review contains a lot of very testable material. Learn the difference between discrete and continuous probability distributions. The binomial and normal distributions are the most important here. You must learn the properties of both distributions and memorize the formulas for the mean and variance of the binomial distribution and for the probability of a particular value when given a binomial probability distribution. Learn what shortfall risk is and how to calculate and use Roy's safety-first criterion. Know how to standardize a normally distributed random variable, use a *z*-table, and construct confidence intervals. These skills will be used repeatedly in the topic reviews that follow. Additionally, understand the basic features of the lognormal distribution, Monte Carlo simulation, and historical simulation. Finally, it would be a good idea to know how to get continuously compounded rates of return from holding period returns. Other than that, no problem.

LOS 9.a: Define and explain a probability distribution and distinguish between discrete and continuous random variables.

LOS 9.b: Describe the set of possible outcomes of a specified discrete random variable.

A **probability distribution** describes the probabilities of all the possible outcomes for a random variable. The probabilities of all possible outcomes must sum to 1. A simple probability distribution is that for the roll of one fair die; there are six possible outcomes and each one has a probability of 1/6, so they sum to 1. The probability distribution of all the possible returns on the S&P 500 index for the next year is a more complex version of the same idea.

A **discrete random variable** is one for which the number of possible outcomes can be counted, and for each possible outcome, there is a measurable and positive probability. An example of a discrete random variable is the number of days it rains in a given month, because there is a finite number of possible outcomes—the number of days it can rain in a month is defined by the number of days in the month.

A **continuous random variable** is one for which the number of possible outcomes is infinite, even if lower and upper bounds exist. The actual amount of daily rainfall between zero and 100 inches is an example of a continuous random variable because the actual amount of rainfall can take on an infinite number of values. Daily rainfall can be measured in inches, half inches, quarter inches, thousandths of inches, or in even smaller increments. Thus, the number of possible daily rainfall amounts between zero and 100 inches is essentially infinite.

The assignment of probabilities to the possible outcomes for discrete and continuous random variables provides us with discrete probability distributions and continuous probability distributions. The difference between these types of distributions is most apparent for the following properties:

- For a *discrete distribution*, $p(x) = 0$ when x cannot occur, or $p(x) > 0$ if it can. Recall that $p(x)$ is read: "the probability that random variable $X = x$." For example, the probability of it raining on 33 days in June is zero because this cannot occur, but the probability of it raining 25 days in June has some positive value.