The following is a review of the Analysis of Fixed Income Investments principles designed to address the learning outcome statements set forth by CFA Institute. This topic is also covered in:

INTRODUCTION TO THE VALUATION OF DEBT SECURITIES

Study Session 16

EXAM FOCUS

Bond valuation is all about calculating the present value of the promised cash flows. If your time-value-of-money skills are not up to speed, take the time now to revisit the Study Session 2 review of TVM concepts. The material in this topic review is very important. Calculating the value of a bond by discounting expected cash flows should become an easy exercise. The final material, on discounting a bond's expected cash flows using spot rates and the idea of "arbitrage-free" bond valuation, is quite important as well. A good understanding here will just make what follows easier to understand.

LOS 70.a: Explain the steps in the bond valuation process (i.e., estimate expected cash flows, determine an appropriate discount rate or rates, and compute the present value of the cash flows).

The general procedure for valuing fixed-income securities (or any security) is to take the present values of all the expected cash flows and add them up to get the value of the security.

There are three steps in the bond valuation process:

1. **Estimate the cash flows** over the life of the security. For a bond, there are two types of cash flows: (1) the coupon payments and (2) the return of principal.

2. **Determine the appropriate discount rate** based on the risk of (uncertainty about) the receipt of the estimated cash flows.

3. **Calculate the present value of the estimated cash flows** by multiplying the bond's expected cash flows by the appropriate discount factors.

LOS 70.b: Identify the types of bonds for which estimating the expected cash flows is difficult, and explain the problems encountered when estimating the cash flows for these bonds.

Certainly, one problem in estimating future cash flows for bonds is predicting defaults and any potential credit problems that make the receipt of future cash flows uncertain. Aside from credit risk, however, we can identify three situations where estimating future cash flows poses additional difficulties.

1. **The principal repayment stream is not known with certainty.** This category includes bonds with embedded options (puts, calls, prepayment options, and accelerated sinking fund provisions). For these bonds, the future stream of principal payments is uncertain and will depend to a large extent on the future path of interest rates. For example, lower rates will increase prepayments of mortgage passthrough securities, and principal will be repaid earlier.

2. **The coupon payments are not known with certainty.** With floating-rate securities, future coupon payments depend on the path of interest rates. With some floating-rate securities, the coupon payments may depend on the price of a commodity or the rate of inflation over some future period.

3. **The bond is convertible or exchangeable into another security.** Without information about future stock prices and interest rates, we don't know when the cash flows will come or how large they will be.

LOS 70.c: Compute the value of a bond, given the expected annual or semiannual cash flows and the appropriate single (constant) discount rate, explain how the value of a bond changes if the discount rate increases or decreases, and compute the change in value that is attributable to the rate change.

For a Treasury bond, the appropriate rate used to value the promised cash flows is the risk-free rate. This may be a single rate, used to discount all of the cash flows, or a series of discount rates that correspond to the times until each cash flow arrives.

For non-Treasury securities, we must add a risk premium to the risk-free (Treasury) rate to determine the appropriate discount rate. This risk premium is one of the yield spread measures covered in the previous review and is the added yield to compensate for greater credit risk, liquidity risk, call risk, prepayment risk, etc. When using a single discount rate to value bonds, the risk premium is added to the risk-free rate to get the appropriate discount rate for all of the expected cash flows.

yield on a risky bond = yield on a default-free bond + risk premium

Other things being equal, the riskier the security, the higher the yield differential (or risk premium) we need to add to the on-the-run Treasury yields.

Computing the Value of a Bond

Valuation with a single yield (discount rate). Recall that we valued an annuity using the time value of money keys on the calculator. For an option-free coupon bond, the coupon payments can be valued as an annuity. In order to take into account the payment of the par value at maturity, we will enter this final payment as the future value. This is the basic difference between valuing a coupon bond and valuing an annuity.

For simplicity, consider a security that will pay \$100 per year for 10 years and make a single \$1,000 payment at maturity (in 10 years). If the appropriate discount rate is 8% for all the cash flows, the value is:

$$\frac{100}{1.08}+\frac{100}{1.08^2}+\frac{100}{1.08^3}+\frac{100}{1.08^4}+\ldots\ldots+\frac{100}{1.08^{10}}+\frac{1{,}000}{1.08^{10}}=\$1{,}134.20=\text{present value of expected cash flows}$$

This is simply the sum of the present values of the future cash flows, \$100 per year for 10 years and \$1,000 (the principal repayment) to be received at the end of the tenth year, at the same time as the final coupon payment.

The calculator solution is:

N = 10; PMT = 100; FV = 1,000; I/Y = 8; CPT → PV = –\$1,134.20

where:
N = number of years
PMT = the *annual* coupon payment
I/Y = the *annual* discount rate
FV = the par value or selling price at the end of an assumed holding period

Professor's Note: Take note of a couple of points here. The discount rate is entered as a whole number in percent, 8, not 0.08. The ten coupon payments of \$100 each are taken care of in the N = 10 entry, the principal repayment is in the FV = 1,000 entry. Lastly, note that the PV is negative; it will be the opposite sign to the sign of PMT and FV. The calculator is just "thinking" that if you receive the payments and future value (you own the bond), you must pay the present value of the bond today (you must buy the bond). That's why the PV amount is negative; it is a cash outflow to *a bond buyer. Just make sure that you give the payments and future value the same sign, and then you can ignore the sign on the answer (PV).*

The Change in Value When Interest Rates Change

Bond values and bond yields are inversely related. An *increase* in the discount rate will *decrease* the present value of a bond's expected cash flows; a decrease in the discount rate will increase the present value of a bond's expected cash flows. The change in bond value in response to a change in the discount rate can be calculated as the difference between the present values of the cash flows at the two different discount rates.

Example: Changes in required yield

A bond has a par value of $1,000, a 6% semiannual coupon, and three years to maturity. Compute the bond values when the yield to maturity is 3, 6, and 12%.

Answer:

$$\text{At I/Y} = \frac{3}{2};\ N = 3\times 2;\ FV = 1{,}000;\ PMT = \frac{60}{2};\ CPT \rightarrow PV = -1{,}085.458$$

$$\text{At I/Y} = \frac{6}{2};\ N = 3\times 2;\ FV = 1{,}000;\ PMT = \frac{60}{2};\ CPT \rightarrow PV = -1{,}000.000$$

$$\text{At I/Y} = \frac{12}{2};\ N = 3\times 2;\ FV = 1{,}000;\ PMT = \frac{60}{2};\ CPT \rightarrow PV = -852.480$$

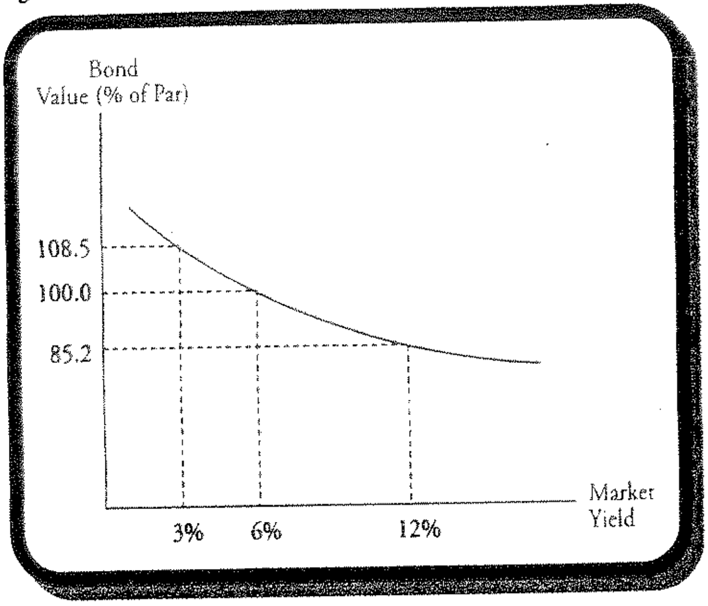
We have illustrated here a point covered earlier; if the yield to maturity equals the coupon rate, the bond value is equal to par. If the yield to maturity is higher (lower) than the coupon rate, the bond is trading at a discount (premium) to par.

We can now calculate the percentage change in price for changes in yield. If the required yield decreases from 6% to 3%, the value of the bond increases by $\frac{1{,}085.46}{1{,}000.00} - 1 = 8.546\%$. If the yield increases from 6% to 12%, the bond value decreases by $\frac{852.48}{1{,}000.00} - 1 = -14.752\%$.

Professor's Note: Notice that in these calculations, you only need to change the interest rate (I/Y) and then compute PV once the values of N, PMT, and FV have been entered. The TVM keys "remember" the values for these inputs even after the calculator has been turned off!

Price-yield profile. If you plot a bond's yield to its corresponding value, you'll get a graph like the one shown in Figure 1. Here we see that higher prices are associated with lower yields. This graph is called the *price-yield curve*. Note that it is not a straight line but is curved. For option-free bonds, the price-yield curve is convex toward the origin, meaning it looks like half of a smile.

Figure 1: The Price-Yield Profile



LOS 70.d: Explain how the price of a bond changes as the bond approaches its maturity date, and compute the change in value that is attributable to the passage of time.

Prior to maturity, a bond can be selling at a significant discount or premium to par value. However, regardless of its required yield, the price will converge to par value as maturity approaches. Consider the bond in the previous example ($1,000 par value, 3-year life, paying 6% semiannual coupons). The bond values corresponding to required yields of 3, 6, and 12% as the bond approaches maturity are presented in Figure 2.

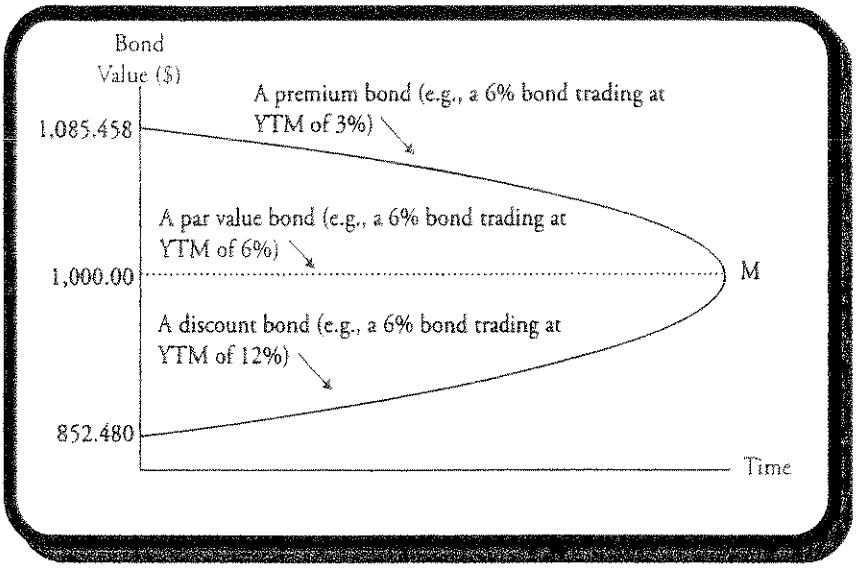
Figure 2: Bond Values and the Passage of Time

Time to Maturity	*YTM = 3%*	*YTM = 6%*	*YTM = 12%*
3.0 years	$1,085.40	$1,000.00	$852.48
2.5	1,071.74	1,000.00	873.63
2.0	1,057.82	1,000.00	896.05
1.5	1,043.68	1,000.00	919.81
1.0	1,029.34	1,000.00	945.00
0.5	1,014.78	1,000.00	971.69
0.0	1,000.00	1,000.00	1,000.00

To compute the change in bond value due to the passage of time, just revalue the bond with the number of periods (remaining until maturity) reduced. Note that in the preceding example, the value of a 6% bond with three years until maturity and a yield to maturity of 3% is FV = 1,000; PMT = 30; N = 6; I/Y = 1.5; CPT → PV = $1,085.46. To see the effect of the passage of time (with the yield to maturity held constant) just enter N = 5 CPT → PV to get the value one period (six months) from now of $1,071.74, or N = 4 CPT → PV to get the value two periods (one year) from now of $1,057.82.

The change in value associated with the passage of time for the three bonds represented in Figure 2 is presented graphically in Figure 3.

Figure 3: Premium, Par, and Discount Bonds



LOS 70.e: Compute the value of a zero-coupon bond.

Since a zero-coupon bond has only a single payment at maturity, the value of a zero is simply the present value of the par or face value. Given the yield to maturity, the calculation is:

$$\text{bond value} = \frac{\text{maturity value}}{(1+i)^{\text{number of years} \times 2}}$$

Note that this valuation model requires just three pieces of information:

- The bond's maturity value, assumed to be $1,000.
- The semiannual discount rate, *i*.
- The life of the bond, *N* years.

Alternatively, using the TVM keys, we enter:

PMT = 0; FV = par; N = # years × 2; I/Y = YTM/2 = semiannual discount rate; CPT → PV

Although zero-coupon bonds do not pay coupons, it is customary to value zero-coupon bonds using semiannual discount rates. Note that *N* is now two times the number of years to maturity and that the semiannual discount rate is one-half the yield to maturity expressed as a BEY.

Example: Valuing a zero-coupon bond

Compute the value of a 10-year, $1,000 face value zero-coupon bond with a yield to maturity of 8%.

Answer:

To find the value of this bond given its yield to maturity of 8% (a 4% semiannual rate), we can calculate:

$$\text{bond value} = \frac{1{,}000}{\left(1+\frac{0.08}{2}\right)^{10\times 2}} = \frac{1{,}000}{(1.04)^{20}} = \$456.39$$

Or, using a calculator, use the following inputs:

$$N = 10 \times 2 = 20;\ FV = 1{,}000;\ I/Y = \tfrac{8}{2} = 4;\ PMT = 0;\ CPT \rightarrow PV = -\$456.39$$

The difference between the current price of the bond (\$456.39) and its par value (\$1,000) is the amount of compound interest that will be earned over the 10-year life of the issue.

Professor's Note: Exam questions will likely specify whether annual or semiannual discounting should be used. Just be prepared to value a zero-coupon bond either way.

LOS 70.f: Explain the arbitrage-free valuation approach and the market process that forces the price of a bond toward its arbitrage-free value, and explain how a dealer could generate an arbitrage profit if a bond is mispriced.

Yield to maturity is a summary measure and is essentially an internal rate of return based on a bond's cash flows and its market price. In the traditional valuation approach, we get the yield to maturity of bonds with maturity and risk characteristics similar to those of the bond we wish to value. Then we use this rate to discount the cash flows of the bond to be valued.

With the **arbitrage-free valuation approach**, we *discount each cash flow using a discount rate that is specific to the maturity of each cash flow.* Again, these discount rates are called **spot rates** and can be thought of as the required rates of return on zero-coupon bonds maturing at various times in the future.

The arbitrage-free valuation approach simply says that the value of a Treasury bond based on (Treasury) spot rates must be equal to the value of the "parts" (i.e., the sum of the present values of all of the expected cash flows). If this is not the case, there must be an arbitrage opportunity. If a bond is selling for less than the sum of the present values of its expected cash flows, an arbitrageur will buy the bond and sell the "pieces." If the bond is selling for more than the sum of the values of the pieces (individual cash flows), one could buy the pieces, package them to "make" a bond, and then sell the bond "package" to earn an arbitrage profit.

The first step in checking for arbitrage-free valuation is to value a coupon bond using the appropriate spot rates. The second step is to compare this value to the market price of the bond. If the computed value is not equal to the market price, there is an arbitrage profit to be earned by buying the lower-priced alternative (either the bond or the individual cash flows) and selling the higher-priced alternative. Of course, this assumes that there are zero-coupon bonds available that correspond to the coupon bond's cash flows.

Example: Arbitrage-free valuation

Consider a 6% Treasury note with 1.5 years to maturity. Spot rates (expressed as semiannual yields to maturity) are: 6 months = 5%, 1 year = 6%, 1.5 years = 7%. If the note is selling for \$992, compute the arbitrage profit, and explain how a dealer would perform the arbitrage.

Answer:

To value the note, note that the cash flows (per \$1,000 par value) will be \$30, \$30, and \$1,030 and that the semiannual discount rates are half the stated yield to maturity.

Using the semiannual spot rates, the present value of the expected cash flows is:

$$\text{present value using spot rates} = \frac{30}{1.025} + \frac{30}{1.03^2} + \frac{1{,}030}{1.035^3} = \$986.55$$

This value is less than the market price of the note, so we will buy the individual cash flows (zero-coupon bonds), combine them into a 1.5-year note "package," and sell the package for the market price of the note. This will result in an immediate and riskless profit of 992.00 – 986.55 = $5.45 per bond.

Determining whether a bond is over- or undervalued is a 2-step process. First, compute the value of the bond using either the spot rates or yield to maturity, remembering that both are often given as two times the semiannual discount rate(s). Second, compare this value to the market price given in the problem to see which is higher.

How a Dealer Can Generate an Arbitrage Profit

Recall that the Treasury STRIPS program allows dealers to divide Treasury bonds into their coupon payments (by date) and their maturity payments in order to create zero-coupon securities. The program also allows reconstitution of Treasury bonds/notes by putting the individual cash flows back together to create Treasury securities. Ignoring any costs of performing these transformations, the ability to separate and reconstitute Treasury securities will insure that the arbitrage-free valuation condition is met.

The STRIPS program allows for just the arbitrage we outlined previously. If the price of the bond is greater than its arbitrage-free value, a dealer could buy the individual cash flows and sell the package for the market price of the bond. If the price of the bond is less than its arbitrage-free value, an arbitrageur can make an immediate and riskless profit by purchasing the bond and selling the parts for more than the cost of the bond.

Such arbitrage opportunities and the related buying of bonds priced "too low" and sales of bonds priced "too high" will force the bond prices toward equality with their arbitrage-free values, eliminating further arbitrage opportunities.

Key Concepts

1. To value a bond, one must estimate the amount and timing of the cash flows from coupons and par value, determine the appropriate discount rate(s), and calculate the sum of the present values of the cash flows.
2. Certain bond features, including embedded options, convertibility, or floating rates, can make the estimation of future cash flows uncertain, which adds complexity to the estimation of bond values.
3. In bond valuation, the approximate discount rate is a function of a risk-free rate (e.g., U.S. Treasury yields) and a risk premium (the incremental yield spread required to compensate the investor for incurring any additional risks relative to Treasury securities).
4. When discounting a bond's expected cash flows, the period of the discount rate must match the frequency of the payments.
5. The required yield on a bond can change during the life of the bond, but the cash flows generally will not. Therefore, decreases (increases) in required yield will increase (decrease) the bond's price.
6. When interest rates (yields) do not change, a bond's price will move toward its par value as time passes and the maturity date approaches.
7. The value of a zero-coupon bond calculated using a semiannual discount rate (i) is:

   $$\text{bond value} = \frac{\text{maturity value}}{(1+i)^{\text{number of years} \times 2}}$$

8. If the required yield equals the coupon rate, the bond will trade at par. If the required yield exceeds the coupon rate, the bond will trade at a discount. If the required yield is less than the coupon rate, the bond will trade at a premium.
9. YTM is really an average of the required rates for the individual cash flows of a bond; the required rate for an individual cash flow is called the spot rate.
10. Bond prices are considered to be arbitrage-free if the sum of the present values of the individual cash flows, discounted using spot rates, is equal to the bond price.

11. If bonds prices are not equal to their arbitrage-free values, dealers can generate arbitrage profits by buying the lower-priced alternative (either the bond or the individual cash flows) and selling the higher-priced alternative (either the individual cash flows or a package of the individual cash flows equivalent to the bond).

Concept Checkers: Introduction to the Valuation of Debt Securities

1. An analyst observes a 5-year, 10% coupon bond with semiannual payments. The face value is £1,000. How much is each coupon payment?
   A. £50.
   B. £25.
   C. £500.
   D. £100.

2. A 20-year, 10% annual-pay bond has a par value of $1,000. What would this bond be trading for if it were being priced to yield 15% as an annual rate?
   A. $685.14.
   B. $687.03.
   C. $828.39.
   D. $832.40.

3. An analyst observes a 5-year, 10% semiannual-pay bond. The face amount is £1,000. The analyst believes that the yield to maturity for this bond should be 15%. Based on this yield estimate, the price of this bond would be:
   A. £828.40.
   B. £832.39.
   C. £1,189.53.
   D. £1,193.04.

4. Two bonds have par values of $1,000. Bond A is a 5% annual-pay, 15-year bond priced to yield 8% as an annual rate; the other (Bond B) is a 7.5% annual-pay, 20-year bond priced to yield 6% as an annual rate. The values of these two bonds would be:

	Bond A	Bond B
A.	$740.61	$847.08
B.	$740.61	$1,172.04
C.	$743.22	$1,172.04
D.	$743.22	$847.08

5. Bond A is a 15-year, 10.5% semiannual-pay bond priced with a yield to maturity of 8%, while Bond B is a 15-year, 7% semiannual-pay bond priced with the same yield to maturity. Given that both bonds have par values of $1,000, the prices of these two bonds would be:

	Bond A	Bond B
A.	$1,216.15	$913.54
B.	$1,216.15	$944.41
C.	$746.61	$944.41
D.	$746.61	$913.54

Use the following data to answer Questions 6 through 8.

An analyst observes a 20-year, 8% option-free bond with semiannual coupons. The required semiannual-pay yield to maturity on this bond was 8%, but suddenly it drops to 7.25%.

6. As a result of the drop, the price of this bond:
   A. will increase.
   B. will decrease.
   C. will stay the same.
   D. cannot be determined without additional information.

7. Prior to the change in the required yield, what was the price of the bond?
   A. 92.64.
   B. 100.00.
   C. 107.85.
   D. Cannot be determined without additional information.

8. The percentage change in the price of this bond when the rate decreased is *closest* to:
   A. 7.86%.
   B. 7.79%.
   C. 8.00%.
   D. 8.15%.

9. Treasury spot rates (expressed as semiannual-pay yields to maturity) are as follows: 6 months = 4%, 1 year = 5%, 1.5 years = 6%. A 1.5-year, 4% Treasury note is trading at $965. The arbitrage trade and arbitrage profit are:
   A. buy the bond, sell the pieces, earn $7.09 per bond.
   B. sell the bond, buy the pieces, earn $7.09 per bond.
   C. buy the bond, sell the pieces, earn $7.91 per bond.
   D. sell the bond, buy the pieces, earn $7.91 per bond.

10. A $1,000, 5%, 20-year annual-pay bond has a yield of 6.5%. If the yield remains unchanged, how much will the bond value increase over the next three years?
   A. $13.58.
   B. $13.62.
   C. $13.78.
   D. $13.96.

11. The value of a 17-year, zero-coupon bond with a maturity value of $100,000 and a semiannual-pay yield of 8.22% is *closest* to:
   A. $24,618.
   B. $25,425.
   C. $26,108.
   D. $91,780.

ANSWERS – CONCEPT CHECKERS: INTRODUCTION TO THE VALUATION OF DEBT SECURITIES

1. A $\text{CPN} = 1{,}000 \times \frac{0.10}{2} = £50$

2. B bond value $= \sum_{t=1}^{20} \frac{100}{(1+0.15)^t} + \frac{1{,}000}{(1+0.15)^{20}} = \$687.03$

   N = 20; I/Y = 15; FV = 1,000; PMT = 100; CPT → PV = –$687.03

3. A N = 10; I/Y = 7.5; FV = 1,000; PMT = 50; CPT → PV = –$828.40

4. C Bond A: N = 15; I/Y = 8; FV = 1,000; PMT = 50; CPT → PV = –$743.22

   Bond B: N = 20; I/Y = 6; FV = 1,000; PMT = 75; CPT → PV = –$1,172.04

   Because the coupon on Bond A is less than its required yield, the bond will sell at a discount; conversely, because the coupon on Bond B is greater than its required yield, the bond will sell at a premium.

5. A Bond A: N = 15 × 2 = 30; I/Y = $\frac{8}{2}$ = 4; FV = 1,000; PMT = $\frac{105}{2}$ = 52.50; CPT → PV = –$1,216.15

   Bond B: N = 15 × 2 = 30; I/Y = $\frac{8}{2}$ = 4; FV = 1,000; PMT = $\frac{70}{2}$ = 35; CPT → PV = –$913.54

6. A The price-yield relationship is inverse. If the required yield falls, the bond's price will rise, and vice versa.

7. B If YTM = stated coupon rate ⇒ bond price = 100 or par value.

8. A The new value is $40 = \text{N}$, $\frac{7.25}{2} = \text{I/Y}$, $40 = \text{PMT}$, $1{,}000 = \text{FV}$

   CPT → PV = –1,078.55, an increase of 7.855%

9. A arbitrage-free value $= \frac{20}{1.02} + \frac{20}{1.025^2} + \frac{1020}{1.03^3} = \$972.09$

   Since the bond price ($965) is less, buy the bond and sell the pieces for an arbitrage profit of $7.09 per bond.

10. B With 20 years to maturity, the value of the bond with an annual-pay yield of 6.5% is 20 = N, 50 = PMT, 1,000 = FV, 6.5 = I/Y, CPT – PV = –834.72. With 17 = N, CPT → PV = –848.34, so the value will increase $13.62.

11. B $\text{PMT} = 0$, $\text{N} = 2 \times 17 = 34$, $\text{I/Y} = \frac{8.22}{2} = 4.11$, $\text{FV} = 100{,}000$

    CPT → PV = –25,424.75, or

    $$\frac{100{,}000}{(1.0411)^{34}} = \$25{,}424.76$$

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YIELD MEASURES, SPOT RATES, AND FORWARD RATES

Study Session 16

EXAM FOCUS

This topic review gets a little more specific about yield measures and introduces some yield measures that you will (almost certainly) need to know for the exam: current yield, yield to maturity, and yield to call. Please pay particular attention to the concept of a bond equivalent yield and how to convert various yields to a bond equivalent basis. The other important thing about the yield measures here is to understand what they are telling you so that you understand their limitations. The Level 1 exam may place as much emphasis on these issues as on actual yield calculations.

The final section of this review introduces forward rates. The relationship between forward rates and spot rates is an important one. At a minimum, you should be prepared to solve for spot rates given forward rates and to solve for an unknown forward rate given two spot rates. You should also get a firm grip on the concept of an option-adjusted spread, when it is used and how to interpret it, as well as how and when it differs from a zero-volatility spread.

LOS 71.a: Explain the sources of return from investing in a bond (i.e., coupon interest payments, capital gain/loss, reinvestment income).

Debt securities that make explicit interest payments have three sources of return:

1. The periodic *coupon interest payments* made by the issuer.
2. The *recovery of principal, along with any capital gain or loss* that occurs when the bond matures, is called, or is sold.
3. *Reinvestment income*, or the income earned from reinvesting the periodic coupon payments (i.e., the compound interest on reinvested coupon payments).

The interest earned on reinvested income is an important source of return to bond investors. The uncertainty about how much reinvestment income a bondholder will realize is what we have previously addressed as *reinvestment risk*.

LOS 71.b: Compute and interpret the traditional yield measures for fixed-rate bonds (e.g., current yield, yield to maturity, yield to first call, yield to first par call date, yield to refunding, yield to put, yield to worst, cash flow yield) and explain the assumptions underlying traditional yield measures and the limitations of the traditional yield measures.

Current yield is the simplest of all return measures, but it offers limited information. This measure looks at just one source of return: *a bond's annual interest income*—it does not consider capital gains/losses or reinvestment income. The formula for the current yield is:

$$\text{current yield} = \frac{\text{annual cash coupon payment}}{\text{bond price}}$$

Example: Computing current yield

Consider a 20-year, \$1,000 par value, 6% *semiannual-pay* bond that is currently trading at \$802.07. Calculate the current yield.

Answer:

The *annual* cash coupon payments total:

annual cash coupon payment = par value × stated coupon rate = \$1,000 × 0.06 = \$60

Since the bond is trading at \$802.07, the current yield is:

$$\text{current yield} = \frac{60}{802.07} = 0.0748\text{, or } 7.48\%$$

Note that current yield is based on *annual* coupon interest so that it is the same for a semiannual-pay and annual-pay bond with the same coupon rate and price.

Yield to maturity (YTM) is an annualized internal rate of return, based on a bond's price and its promised cash flows. For a bond with semiannual coupon payments, the yield to maturity is stated as two times the semiannual internal rate of return implied by the bond's price. The formula that relates bond price and YTM for a semiannual coupon bond is:

$$\text{bond price} = \frac{CPN_1}{\left(1 + \frac{YTM}{2}\right)} + \frac{CPN_2}{\left(1 + \frac{YTM}{2}\right)^2} + \cdots + \frac{CPN_{2N} + \text{Par}}{\left(1 + \frac{YTM}{2}\right)^{2N}}$$

where:
CPN_t = the (semiannual) coupon payment received after t semiannual periods
N = number of years to maturity
YTM = yield to maturity

YTM and price contain the same information. That is, given the YTM, you can calculate the price and given the price, you can calculate the YTM.

We cannot easily solve for YTM from the bond price. Given a bond price and the coupon payment amount, we could solve it by trial and error, trying different values of YTM until the present value of the expected cash flows is equal to price. Fortunately, your calculator will do exactly the same thing, only faster. It uses a trial and error algorithm to find the discount rate that makes the two sides of the pricing formula equal.

Example: Computing YTM

Consider a 20-year, \$1,000 par value bond, with a 6% coupon rate (semiannual payments) that is currently trading at \$802.07. Calculate the YTM.

Answer:

Using a financial calculator, you'd find the YTM on this bond as follows:

PV = –802.07; N = 20 × 2 = 40; FV = 1,000; PMT = 60/2 = 30; CPT → I/Y = 4.00

4% is the semiannual discount rate, $\frac{\text{YTM}}{2}$ in the formula, so the YTM = 2 × 4% = 8%.

Note that the signs of PMT and FV are positive and the sign of PV is negative; you must do this to avoid the dreaded "Error 5" message on the TI calculator. If you get the "Error 5" message, you can assume you have not assigned a negative value to the price (PV) of the bond and a positive sign to the cash flows to be received from the bond.

There are certain relationships that exist between different yield measures, depending on whether a bond is trading at par, at a discount, or at a premium. These relationships are shown in Figure 1.

Figure 1: Par, Discount, and Premium Bonds

Bond Selling at:	*Relationship*
Par	coupon rate = current yield = yield to maturity
Discount	coupon rate < current yield < yield to maturity
Premium	coupon rate > current yield > yield to maturity

These conditions will hold in all cases; every discount bond will have a nominal yield (coupon rate) that is less than its current yield and a current yield that is less than its YTM.

The yield to maturity calculated in the previous example (2 × the semiannual discount rate) is referred to as a **bond equivalent yield** (BEY) and we will also refer to it as a semiannual YTM or semiannual-pay YTM. If you are given yields that are identified as BEY, you will know that you must divide by two to get the semiannual discount rate. With bonds that make annual coupon payments, we can calculate an **annual-pay yield to maturity,** which is simply the internal rate of return for the expected annual cash flows.

Example: Calculating YTM for annual coupon bonds

Consider an annual pay 20-year, $1,000 par value, with a 6% coupon rate that is trading at $802.07. Calculate the *annual-pay YTM.*

Answer:

The relation of price and annual-pay YTM on this bond is:

$$802.07 = \sum_{t=1}^{20} \frac{60}{(1+\text{YTM})^t} + \frac{1{,}000}{(1+\text{YTM})^{20}} \Rightarrow \text{YTM} = 8.019\%$$

Here we have separated the coupon cash flows and the principal repayment.

The calculator solution is:

PV = –802.07; N = 20; FV = 1,000; PMT = 60; CPT → I/Y = 8.019; 8.019% is the annual-pay YTM.

Use a discount rate of 8.019%, and you'll find the present value of the bond's future cash flows (annual coupon payments and the recovery of principal) will equal the current market price of the bond. *The discount rate is the bond's YTM.*

For zero-coupon Treasury bonds, the convention is to quote the yields as BEYs (semiannual-pay YTMs).

Example: Calculating YTM for zero-coupon bonds

A 5-year Treasury STRIP is priced at $768. Calculate the semiannual-pay YTM and annual-pay YTM.

Answer:

The direct calculation method, based on the geometric mean covered in Quantitative Methods, is:

$$\text{the semiannual-pay YTM or BEY} = \left[\left(\frac{1{,}000}{768}\right)^{\frac{1}{10}} - 1\right] \times 2 = 5.35\%$$

$$\text{the annual-pay YTM} = \left(\frac{1{,}000}{768}\right)^{\frac{1}{5}} - 1 = 5.42\%$$

Using the TVM calculator functions:

- PV = –768; FV =1,000; PMT = 0; N = 10; CPT → I/Y = 2.675% × 2 = 5.35% for the semiannual-pay YTM, and PV = –768; FV =1,000; PMT = 0; N = 5; CPT → I/Y = 5.42% for the annual-pay YTM.

The annual-pay YTM of 5.42% means that $768 earning compound interest of 5.42%/yr. would grow to $1,000 in five years.

The **yield to call** is used to calculate the yield on callable bonds that are selling at a premium to par. For bonds trading at a premium to par, the *yield to call* may be less than the yield to maturity. This can be the case when the call price is below the current market price.

The calculation of the yield to call is the same as the calculation of yield to maturity, except that the *call price is substituted* for the par value in FV and the *number of semiannual periods until the call date is substituted* for periods to maturity, N. When a bond has a period of call protection, we calculate the **yield to first call** over the period until the bond may first be called, and use the first call price in the calculation as FV. In a similar manner, we can calculate the yield to any subsequent call date using the appropriate call price.

If the bond contains a provision for a call at *par* at some time in the future, we can calculate the *yield to first par call* using the number of years until the par call date and par for the maturity payment. If you have a good understanding of the yield to maturity measure, the YTC is not a difficult calculation; just be very careful about the number of years to the call and the call price for that date. An example will illustrate the calculation of these yield measures.

Example: Computing the YTM, YTC, and yield to first par call

Consider a 20-year, 10% semiannual-pay bond priced at 112 that can be called in five years at 102 and called at par in seven years. Calculate the YTM, YTC, and yield to first par call.

Professor's Note: Bond prices are often expressed as a percent of par (e.g., 100 = par).

Answer:

The YTM can be calculated as:
N = 40; PV = –112; PMT = 5; FV = 100; CPT → I/Y = 4.361% × 2 = 8.72% = YTM.

To compute the yield to first call (YTFC), we substitute the number of semiannual periods until the first call date (10) for N, and the first call price (102) for FV, as follows:

N = 10; PV = –112; PMT = 5; FV = 102;

CPT → I/Y = 3.71% and 2 × 3.71 = 7.42% = YTFC

To calculate the yield to first par call (YTFPC) we will substitute the number of semiannual periods until the first par call date (14) for N and par (100) for FV as follows:

N = 14; PV = –112; PMT = 5; FV = 100; CPT → I/Y = 3.873% × 2 = 7.746% = YTFPC

Note that the yield to call, 7.42%, is significantly lower than the yield to maturity, 8.72%. If the bond were trading at a discount to par value, there would be no reason to calculate the yield to call. For a discount bond, the YTC will be higher than the YTM since the bond will appreciate more rapidly with the call to at least par and, perhaps, an even greater call price. Bond yields are quoted on a yield to call basis when the YTC is less than the YTM, which can only be the case for bonds trading at a premium to the call price.

The **yield to worst** is the worst yield outcome of any that are possible given the call provisions of the bond. In the above example the yield to first call is less than the YTM and less than the yield to first par call. So the worst possible outcome is a yield of 7.42%; the yield to first call is the *yield to worst*.

The **yield to refunding** refers to a specific situation where a bond is currently callable and current rates make calling the issue attractive to the issuer, but where the bond covenants contain provisions giving protection from refunding until some future date. The calculation of the yield to refunding is just like that of YTM or YTC. The difference here is that the yield to refunding would use the call price, but the date (and therefore the number of periods used in the calculation) is the date when refunding protection ends. Recall that bonds that are callable but not currently refundable can be called using funds from sources other than the issuance of a lower coupon bond.

The **yield to put** (YTP) is used if a bond has a put feature and is selling at a discount. The yield to put will likely be higher than the yield to maturity. The yield to put calculation is just like the yield to maturity with the number of semiannual periods until the put date as N, and the put price as FV.

Example: Computing YTM and YTP

Consider a 3-year, 6%, $1,000 *semiannual-pay* bond. The bond is selling for $925.40. The first put opportunity is at par in two years. Calculate the YTM and the YTP.

Answer:

Yield to maturity is calculated as:

N = 6; FV = 1,000; PMT = 30; PV = –925.40; CPT → I/Y = 4.44 × 2 = 8.88% = YTM

Yield to put is calculated as:

N = 4; FV = 1,000; PMT = 30; PV = –925.40; CPT → I/Y = 5.11 × 2 = 10.22% = YTP

In this example, the yield to put is higher than the YTM and, therefore, would be the appropriate yield to look at for this bond.

The **cash flow yield** (CFY) is used for mortgage-backed securities and other amortizing asset-backed securities that have monthly cash flows. In many cases, the amount of the principal repayment can be greater than the amount required to amortize the loan over its original life. Cash flow yield (CFY) incorporates an assumed schedule of monthly cash flows based on assumptions as to how prepayments are likely to occur. Once we have projected the monthly cash flows, we can calculate CFY as a *monthly* internal rate of return based on the market price of the security.

Professor's Note: It is unlikely that you will be required to actually calculate a CFY on the exam and more likely that you could be required to interpret one. If you need to calculate a CFY, just use the cash flow keys, put the price of the security as a negative value as CF_0, enter the monthly cash flows sequentially as CFn's, and solve for IRR, which will be a monthly rate.

The following formula is used to convert a (monthly) CFY into bond equivalent form:

$$\text{bond equivalent yield} = \left[(1 + \text{monthly CFY})^6 - 1\right] \times 2$$

Here, we have converted the monthly yield into a semiannual yield and then doubled it to make it equivalent to a semiannual-pay YTM or bond equivalent yield.

A limitation of the CFY measure is that actual prepayment rates may differ from those assumed in the calculation of CFY.

The Assumptions and Limitations of Traditional Yield Measures

The primary *limitation of the yield to maturity measure* is that it does not tell us the compound rate of return that we will realize on a fixed-income investment over its life. This is because we do not know the rate of interest we will realize on the reinvested coupon payments (the reinvestment rate). Reinvestment income can be a significant part of the overall return on a bond. As noted earlier, the uncertainty about the return on reinvested cash flows is referred to as *reinvestment risk*. It is higher for bonds with higher coupon rates, other things equal, and potentially higher for callable bonds as well.

The realized yield on a bond is the actual compound return that was earned on the initial investment. It is usually computed at the end of the investment horizon. For a bond to have a ***realized yield*** equal to its YTM, all cash flows prior to maturity must be reinvested at the YTM, and the bond must be held until maturity. If the "average" reinvestment rate is below the YTM, the realized yield will be below the YTM. For this reason it is often stated that: ***The yield to maturity assumes cash flows will be reinvested at the YTM and assumes that the bond will be held until maturity.*** This is the point of LOS here.

The other internal rate of return measures, YTC and YTP, suffer from the same shortcomings since they are calculated like YTMs and do not account for reinvestment income. The CFY measure is also an internal rate of return measure and can differ greatly from the realized yield if reinvestment rates are low, since scheduled principal payments and prepayments must be reinvested along with the interest payments.

LOS 71.c: Explain the importance of reinvestment income in generating the yield computed at the time of purchase, calculate the amount of income required to generate that yield, and discuss the factors that affect reinvestment risk.

Reinvestment income is important because if the reinvestment rate is less than the YTM, the realized yield on the bond will be less than the YTM. The realized yield will always be between the YTM and the assumed reinvestment rate.

If a bondholder holds a bond until maturity and reinvests all coupon interest payments, the total amount generated by the bond over its life has three components:

- Bond principal.
- Coupon interest.
- Interest on reinvested coupons.

Once we calculate the total amount needed for a particular level of compound return over a bond's life, we can subtract the principal and coupon payments to determine the amount of reinvestment income necessary to achieve the target yield. An example will illustrate this calculation.

Example: Calculating required reinvestment income for a bond

If you purchase a 6%, 10-year Treasury bond at par, how much reinvestment income must be generated over its life to provide the investor with a compound return of 6% on a semiannual basis?

Answer:

Assuming the bond has a par value of $100, we first calculate the total value that must be generated 10 years (20 semiannual periods) from now as:

$$100(1.03)^{20} = \$180.61$$

There are 20 bond coupons of $3 each, totaling $60, and a payment of $100 of principal at maturity.

Therefore, the required reinvestment income over the life of the bond is:

$$180.61 - 100 - 60 = \$20.61$$

Professor's Note: If we had purchased the bond at a premium or discount, we would still use the purchase price (which would not equal 100) and the required compound return to calculate the total future dollars required, and then subtract the maturity value and the total coupon payments to get the required reinvestment income.

Factors That Affect Reinvestment Risk

Other things being equal, a coupon bond's **reinvestment risk** will *increase* with:

- *Higher coupons*—because there's more cash flow to reinvest.
- *Longer maturities*—because more of the total value of the investment is in the coupon cash flows (and interest on coupon cash flows).

In both cases, the amount of reinvested income will play a bigger role in determining the bond's total return and, therefore, introduce more reinvestment risk. A non-callable zero-coupon bond has no reinvestment risk over its life because there are no cash flows to reinvest prior to maturity.

LOS 71.d: Compute and interpret the bond equivalent yield of an annual-pay bond, and the annual-pay yield of a semiannual-pay bond.

This LOS requires that you be able to turn a semiannual return into an annual return, and an annual return into a semiannual return.

Example: Comparing bonds with different coupon frequencies

Suppose that Daimler-Chrysler has a semiannual coupon bond trading in the U.S. with a YTM of 6.25%, and an annual coupon bond trading in Europe with a YTM = 6.30%. Which bond has the greater yield?

Answer:

To determine the answer, we can convert the yield on *the annual-pay bond* to a (semiannual-pay) bond equivalent yield. That is:

$$\text{BEY of an annual-pay bond} = \left[(1 + \text{annual YTM})^{\frac{1}{2}} - 1\right] \times 2$$

Thus, the BEY of the 6.30% annual-pay bond is:

$$[(1+0.0630)^{0.5} - 1]\times 2 = [1.031 - 1]\times 2 = 0.031\times 2 = 0.062 = 6.2\%$$

The 6.25% semiannual-pay bond provides the better (bond equivalent) yield.

Alternatively, we could convert the YTM of the semiannual-pay bond (which is a bond equivalent yield) to an equivalent annual-pay basis. The equivalent annual yield (EAY—*sometimes known as the effective annual yield*) to the 6.25% semiannual-pay YTM is:

$$\text{equivalent annual yield} = \left(1+\frac{0.0625}{2}\right)^2 - 1 = 0.0635 \rightarrow 6.35\%$$

The EAY of the semiannual-pay bond is 6.35%, which is greater than the 6.3% for the annual-pay bond. Therefore, the semiannual-pay bond has a greater yield as long as we put the yields on an equivalent basis, calculating both as annual yields or calculating both as bond equivalent yields (semiannual yields × 2).

LOS 71.e: Describe the methodology for computing the theoretical Treasury spot rate curve and compute the value of a bond using spot rates.

The par yield curve gives the YTMs of bonds currently trading near their par values (YTM ≈ coupon rate) for various maturities. Here, we need to use these yields to get the theoretical Treasury spot rate curve by a process called bootstrapping.

The method of bootstrapping can be a little confusing, so let's first get the main idea and then go through a more realistic and detailed example. The general idea is that we will solve for spot rates by knowing the prices of coupon bonds. We always know one spot rate to begin with and then calculate the spot rate for the next longer period. When we know two spot rates, we can get the third based on the market price of a bond with three cash flows by using the spot rates to get the present values of the first two cash flows.

As an example of this method, consider that we know the prices and yields of three annual-pay bonds as shown in Figure 2. All three bonds are trading at par or $1,000.

Figure 2: Prices and Yield for Three Annual-Pay Bonds

Maturity	*Coupon*	*Yield*	*Price*
1 year	3%	3%	$1,000
2 years	4%	4%	$1,000
3 years	5%	5%	$1,000

Since the one-year bond makes only one payment (it's an annual-pay bond) of $1,030 at maturity, the one-year spot rate is 3%, the yield on this single payment. The two-year bond makes two payments, a $40 coupon in one year and a $1,040 payment at maturity in two years. Since the spot rate to discount the 2-year bond's first cash flow is 3%, and since we know that the sum of the present values of the bond's cash flows must equal its (no arbitrage) price of $1,000, we can write:

$$\frac{40}{1.03}+\frac{1{,}040}{(1+\text{2-year spot rate})^2}=\$1{,}000$$

Based on this we can solve for the two-year spot rate as follows:

1. $$\frac{1{,}040}{(1+\text{2-year spot})^2}=1{,}000-\frac{40}{1.03}=1{,}000-38.83=961.17$$

2. $$\frac{1{,}040}{961.17}=(1+\text{2-year spot})^2=1.082$$

3. $$\text{2-year spot}=(1.082)^{\frac{1}{2}}-1=0.04019=4.019\%$$

Now that we have both the 1-year and 2-year spot rates, we can use the cash flows and price of the 3-year bond to write:

$$\frac{50}{1.03}+\frac{50}{(1.0419)^2}+\frac{1{,}050}{(1+\text{3-year spot})^3}=1{,}000$$

And solve for the 3-year spot rate:

$$1{,}000-\frac{50}{1.03}-\frac{50}{(1.0419)^2}=\frac{1{,}050}{(1+\text{3-year spot})^3}$$

$$1{,}000-48.54-46.06=\frac{1{,}050}{(1+\text{3-year spot})^3}$$

$$905.40=\frac{1{,}050}{(1+\text{3-year spot})^3}$$

$$\left(\frac{1{,}050}{905.40}\right)^{1/3}-1=\text{3-year spot}=0.05063=5.063\%$$

So we can state that:

$$\frac{50}{1.03}+\frac{50}{(1.0419)^2}+\frac{1,050}{(1.05063)^3}=\$1,000$$

We have just solved for the 2-year and 3-year spot rates by the method of bootstrapping.

In practice, Treasury bonds pay semiannually and their YTMs are semiannual-pay YTMs. The next example illustrates the method of bootstrapping when coupons are paid semiannually.

Consider the yields on coupon Treasury bonds trading at par given in Figure 3. YTM for the bonds is expressed as a bond equivalent yield (semiannual-pay YTM).

Figure 3: Par Yields for Three Semiannual-Pay Bonds

Maturity	*YTM*	*Coupon*	*Price*
6 months	5%	5%	100
1 year	6%	6%	100
18 months	7%	7%	100

The bond with six months left to maturity has a semiannual discount rate of $0.05/2 = 0.025 = 2.5\%$ or 5% on an annual BEY basis. Since this bond will only make one payment of 102.5 in six months, the YTM is the spot rate for cash flows to be received six months from now.

The bootstrapping process proceeds from this point using the fact that the 6-month annualized spot rate is 5% together with the price/YTM information on the 1-year bond. We will use the formula for valuing a bond using spot rates that we covered earlier.

Noting that the 1-year bond will make two payments, one in six months of 3.0 and one in one year of 103.0, and that the appropriate spot rate to discount the coupon payment (which comes six months from now), we can write:

$$\frac{3}{1.025}+\frac{103}{\left(1+S_{1.0}/2\right)^2}=100,\ \text{where } S_{1.0} \text{ is the annualized 1-year spot rate, and solve for } S_{1.0}/2 \text{ as:}$$

$$\frac{103}{\left(1+S_{1.0}/2\right)^2}=100-\frac{3}{1.025}=100-2.927=97.073$$

$$\frac{103}{97.073}=\left(1+S_{1.0}/2\right)^2 \text{ so } \sqrt{\frac{103}{97.073}}-1=S_{1.0}/2=0.030076 \text{ and } S_{1.0}=2\times0.030076=0.060152=6.0152\%$$

Now that we have the 6-month and 1-year spot rates, we can use this information and the price of the 18-month bond to set the bond price equal to the value of the bond's cash flows as:

$$\frac{3.5}{1.025}+\frac{3.5}{(1.030076)^2}+\frac{103.5}{\left(1+\frac{S_{1.5}}{2}\right)^3}=100,$$ where $S_{1.5}$ is the annualized 1.5-year spot rate, and solve for $\frac{S_{1.5}}{2}$

$$\frac{103.5}{\left(1+\frac{S_{1.5}}{2}\right)^3}=100-\frac{3.5}{1.025}-\frac{3.5}{(1.030076)^2}=100-3.415-3.30=93.285$$

$$\frac{103.5}{93.285}=\left(1+\frac{S_{1.5}}{2}\right)^3 \text{ so } \left(\frac{103.5}{93.285}\right)^{\frac{1}{3}}-1=\frac{S_{1.5}}{2}=0.035244 \text{ and } S_{1.5}=2\times0.035244=0.070488=7.0488\%$$

To summarize the method of bootstrapping spot rates from the par yield curve:

1. Begin with the 6-month spot rate.
2. Set the value of the 1-year bond equal to the present value of the cash flows with the 1-year spot rate divided by two as the only unknown.
3. Solve for the 1-year spot rate.
4. Use the 6-month and 1-year spot rates and equate the present value of the cash flows of the 1.5 year bond equal to its price, with the 1.5 year spot rate as the only unknown.
5. Solve for the 1.5-year spot rate.

Professor's Note: You are responsible for "describing" this methodology, not for "computing" theoretical spot rates.

Example: Valuing a bond using spot rates

Given the following spot rates (in BEY form):

0.5 years = 4%
1.0 years = 5%
1.5 years = 6%

Calculate the value of a 1.5 year, 8% Treasury bond.

Answer:

Simply lay out the cash flows and discount by the spot rates, which are one-half the quoted rates since they are quoted in BEY form.

$$\frac{4}{\left(1+\frac{0.04}{2}\right)^1}+\frac{4}{\left(1+\frac{0.05}{2}\right)^2}+\frac{104}{\left(1+\frac{0.06}{2}\right)^3}=102.9 \text{ or, with the TVM calculator function,}$$

N = 1; PMT = 0; I/Y = 2; FV = 4; CPT → PV = –3.92
N = 2; PMT = 0; I/Y = 2.5; FV = 4; CPT → PV = –3.81
N = 3; PMT = 0; I/Y = 3; FV = 104; CPT → PV = –95.17

Add these values together to get 102.9.

LOS 71.f: Distinguish between the nominal spread and the zero-volatility spread and explain the limitations of the nominal spread.

The **nominal spread** is the simplest of the spread measures to use and to understand. It is simply an issue's YTM minus the YTM of a Treasury security of similar maturity. Therefore, the use of the nominal spread suffers from the same limitations as the YTM. YTM uses a single discount rate to value the cash flows, so it *ignores the shape of the spot yield curve*. In fact, YTM for a coupon bond is theoretically correct only to the extent that the spot rate curve is flat. When the yield curve is steep, nominal yield spreads are affected and nominal spreads for amortizing securities are the most affected.

The Zero-Volatility Spread

One way to get a bond's nominal spread to Treasuries would be to add different amounts to the yield of a comparable Treasury bond, and value the bond with those YTM's. The amount added to the Treasury yield that produces a bond value equal to the market price of the bond must be the nominal yield spread.

This may seem like an odd way to get the spread, but it makes sense when you see how the **zero-volatility spread**, or static spread, is calculated. The zero-volatility spread (Z-spread) is the equal amount that we must add to each rate on the Treasury spot yield curve in order to make the present value of the risky bond's cash flows equal to its market price. Instead of measuring the spread to YTM, the *zero-volatility spread* measures the spread to Treasury spot rates necessary to produce a spot rate curve that "correctly" prices a risky bond (i.e., produces its market price).

For a risky bond, the value obtained from discounting the expected cash flows at Treasury spot rates will be too high because the Treasury spot rates are lower than those appropriate for a risky bond. In order to value it correctly, we have to increase each of the Treasury spot rates by some equal amount so that the present value of the risky bond's cash flows discounted at the (increased) spot rates equals the market value of the bond. The following example will illustrate the process for calculating the Z-spread.

Example: Zero-volatility spread

1-, 2-, and 3-year spot rates on Treasuries are 4%, 8.167%, and 12.377%, respectively. Consider a 3-year, 9% annual coupon corporate bond trading at 89.464. The YTM = 13.50% and the YTM of a 3-year Treasury is 12%. Compute the nominal spread and the zero-volatility spread of the corporate bond.

Answer:

The *nominal spread* is:

$$\text{nominal spread} = \text{YTM}_{\text{Bond}} - \text{YTM}_{\text{Treasury}} = 13.50 - 12.00 = 1.50\%$$

To compute the *Z-spread*, set the present value of the bond's cash flows equal to today's market price. Discount each cash flow at the appropriate zero-coupon bond spot rate *plus* a fixed spread = ZS. Solve for ZS in the following equation and you have the Z-spread:

$$89.464 = \frac{9}{(1.04+\text{ZS})^1} + \frac{9}{(1.08167+\text{ZS})^2} + \frac{109}{(1.12377+\text{ZS})^3} \Rightarrow \text{ZS} = 1.67\% \text{ or } 167 \text{ basis points}$$

Note that this spread is found by trial-and-error. In other words, pick a number "ZS," plug it into the right-hand side of the equation, and see if the result equals 89.464. If the right-hand side equals the left, then you have found the Z-spread. If not, pick another "ZS" and start over.

Professor's Note: This is not a calculation you are expected to make; this example is to help you understand how a Z-spread differs from a nominal spread.

LOS 71.g: Explain an option-adjusted spread for a bond with an embedded option and explain the option cost.

The **option-adjusted spread** (OAS) measure is used when a bond has embedded options. A callable bond, for example, must have a greater yield than an identical option-free bond, and a greater nominal spread or Z-spread. Without accounting for the value of the options, these spread measures will suggest the bond is a great value when, in fact, the additional yield is compensation for call risk. Loosely speaking, the *option-adjusted spread* takes the option yield component out of the Z-spread measure; the option-adjusted spread is the spread to the Treasury spot rate curve that the bond would have if it were option-free. The OAS is the spread for non-option characteristics like credit risk, liquidity risk, and interest rate risk.

Professor's Note: The actual method of calculation is reserved for Level 2; for our purposes, however, an understanding of what the OAS is will be sufficient.

Option Cost in Percent

If we calculate an option-adjusted spread for a callable bond, it will be less than the bond's Z-spread. The difference is the extra yield required to compensate for the call option. Calling that extra yield the **option cost**, we can write:

Z-spread – OAS = option cost in percent

Example: Cost of an embedded option

Suppose you learn that a bond is callable and that it has an OAS of 135bp. You also know that similar bonds have a Z-spread of 167 basis points. Compute the cost of the embedded option.

Answer:

The option cost = Z-spread – OAS = 167 – 135 = 32 basis points.

For embedded short calls (e.g., callable bonds): option cost > 0 (you receive compensation for writing the option to the issuer) → OAS < Z-spread. In other words, you *require more yield on the callable bond* than for an option-free bond.

For embedded puts (e.g., putable bonds), option cost < 0 (i.e., you must pay for the option) → OAS > Z-spread. In other words, you *require less yield on the putable bond* than for an option-free bond.

LOS 71.h: Define a forward rate, and compute spot rates from forward rates and forward rates from spot rates.

A **forward rate** is a borrowing/lending rate for a loan to be made at some future date. The notation used must identify both the length of the lending/borrowing period and when in the future the money will be loaned/borrowed. Thus ${}_1f_1$ is the rate for a 1-year loan one year from now and ${}_1f_2$ is the rate for a 1-year loan to be made two years from now, etc. Rather than introduce a separate notation, we can represent the current 1-year rate as ${}_1f_0$. To get the present values of a bond's expected cash flows, we need to discount each cash flow by the forward rates for each of the periods until it is received.

The Relationship Between Short-Term Forward Rates and Spot Rates

The idea here is that *borrowing for three years at the 3-year rate or borrowing for 1-year periods, three years in succession, should have the same cost.*

This relation is illustrated as $(1+S_3)^3 = (1 + {}_1f_0)(1 + {}_1f_1)(1 + {}_1f_2)$ and the reverse as

$S_3 = \left[(1 + {}_1f_0)(1 + {}_1f_1)(1 + {}_1f_2)\right]^{\frac{1}{3}} - 1$, which is the geometric mean we covered in Study Session 2.

Example: Computing spot rates from forward rates

If the current 1-year rate is 2%, the 1-year forward rate (${}_1f_1$) is 3% and the 2-year forward rate (${}_1f_2$) is 4%, what is the 3-year spot rate?

Answer:

$$S_3 = \left[(1.02)(1.03)(1.04)\right]^{\frac{1}{3}} - 1 = 2.997\%$$

This can be interpreted to mean that a dollar compounded at 2.997% for three years would produce the same ending value as a dollar that earns compound interest of 2% the first year, 3% the next year, and 4% for the third year.

Professor's Note: You can get a very good approximation of the 3-year spot rate with the simple average of the forward rates. In the previous example we got 2.997% and the simple average of the three annual rates is $\frac{2+3+4}{3} = 3\%$*!*

Forward Rates Given Spot Rates

We can use the same relationship we used to calculate spot rates from forward rates to calculate forward rates from spot rates.

Our basic relation between forward rates and spot rates (for two periods) is:

$$(1+S_2)^2 = (1+{}_1f_0)(1+{}_1f_1)$$

Which, again, tells us that an investment has the same expected yield (borrowing has the same expected cost) whether we invest (borrow) for two periods at the two-period spot rate, S_2, or for one period at the current rate, S_1, and for the next period at the expected forward rate, ${}_1f_1$. Clearly, given two of these rates, we can solve for the other.

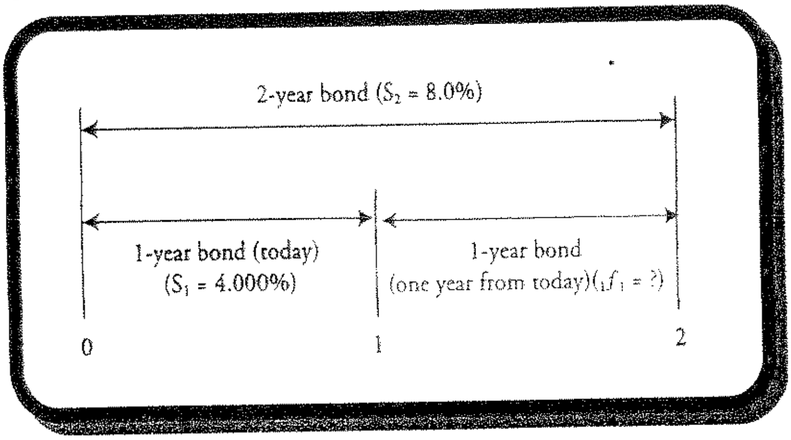
Example: Computing a forward rate from spot rates

The two-period spot rate, S_2, is 8% and the current one-period (spot) rate is 4% (this is both S_1 and ${}_1f_0$). Calculate the forward rate for one period, one period from now, ${}_1f_1$.

Answer:

Figure 4 illustrates the problem.

Figure 4: Finding a Forward Rate



From our original equality, $(1+S_2)^2 = (1+S_1)(1+{}_1f_1)$, we can get $\frac{(1+S_2)^2}{(1+S_1)} - 1 = {}_1f_1$ or, since we know that both choices have the same payoff in two years:

$$(1.08)^2 = (1.04)(1+{}_1f_1)$$

$$(1+{}_1f_1) = \frac{(1.08)^2}{(1.04)}$$

$${}_1f_1 = \frac{(1.08)^2}{(1.04)} - 1 = \frac{1.1664}{1.04} - 1 = 12.154\%$$

In other words, investors are willing to accept 4.0% on the 1-year bond today (when they could get 8.0% on the 2-year bond today) only because they can get 12.154% on a 1-year bond one year from today. This future rate that can be locked in today is a *forward rate*.

Similarly, we can back other forward rates out of the spot rates. We know that:

$$(1+S_3)^3 = (1+S_1)(1+{}_1f_1)(1+{}_1f_2)$$

And that:

$$(1+S_2)^2 = (1+S_1)(1+{}_1f_1), \text{ so we can write } (1+S_3)^3 = (1+S_2)^2(1+{}_1f_2)$$

This last equation says that investing for three years at the 3-year spot rate should produce the same ending value as investing for two years at the 2-year spot rate and then for a third year at ${}_1f_2$, the 1-year forward rate, two years from now.

Solving for the forward rate, ${}_1f_2$, we get:

$$\frac{(1+S_3)^3}{(1+S_2)^2} - 1 = {}_1f_2$$

Example: Forward rates from spot rates

Let's extend the previous example to three periods. The current 1-year spot rate is 4.0%, the current 2-year spot rate is 8.0%, and the current 3-year spot rate is 12.0%. Calculate the 1-year forward rates one and two years from now.

Answer:

We know the following relation must hold:

$$(1+S_2)^2 = (1+S_1)(1+{}_1f_1)$$

We can use it to solve for the 1-year forward rate one year from now:

$$(1.08)^2 = (1.04)(1+{}_1f_1), \text{ so } {}_1f_1 = \frac{(1.08)^2}{(1.04)} - 1 = 12.154\%$$

We also know that the relations:

$$(1+S_3)^3 = (1+S_1)(1+{}_1f_1)(1+{}_1f_2)$$

and, equivalently $(1+S_3)^3 = (1+S_2)^2(1+{}_1f_2)$ must hold.

Substituting values for S_3 and S_2, we have:

$$(1.12)^3 = (1.08)^2 \times (1+{}_1f_2)$$

so that the 1-year forward rate two years from now is:

$$ {}_1f_2 = \frac{(1.12)^3}{(1.08)^2} - 1 = 20.45\% $$

To verify these results, we can check our relations by calculating:

$$ S_3 = \left[(1.04)(1.12154)(1.2045)\right]^{\frac{1}{3}} - 1 = 12.00\% $$

This may all seem a bit complicated, but the basic relation, that borrowing for successive periods at one-period rates should have the same cost as borrowing at multiperiod spot rates, can be summed up as:

$(1+S_2)^2 = (1+S_1)(1+{}_1f_1)$ for two periods, and $(1+S_3)^3 = (1+S_2)^2(1+{}_1f_2)$ for three periods.

Professor's Note: Simple averages also give decent approximations for calculating forward rates from spot rates. In the above example, we had spot rates of 4% for one year and 8% for two years. Two years at 8% is 16%, so if the first year rate is 4%, the second year rate is close to 16 – 4 = 12% (actual is 12.154). Given a 2-year spot rate of 8% and a 3-year spot rate of 12%, we could approximate the 1-year forward rate from time two to time three as (3 × 12) – (2 × 8) = 20. That may be close enough (actual is 20.45) to answer a multiple choice question and, in any case, serves as a good check to make sure the exact rate you calculate is reasonable.

LOS 71.i. Compute the value of a bond using forward rates.

Example: Computing a bond value using forward rates

The current 1-year rate (${}_1f_0$) is 4% and the 1-year forward rate for lending from time = 1 to time = 2 is ${}_1f_1$ = 5%, and the 1-year forward rate for lending from time = 2 to time = 3 is ${}_1f_2$ = 6%. Value a 3-year annual-pay bond with a 5% coupon and a par value of $1,000.

Answer:

$$ \text{bond value} = \frac{50}{1+{}_1f_0} + \frac{50}{(1+{}_1f_0)(1+{}_1f_1)} + \frac{1{,}050}{(1+{}_1f_0)(1+{}_1f_1)(1+{}_1f_2)} = $$

$$ \frac{50}{1.04} + \frac{50}{(1.04)(1.05)} + \frac{1{,}050}{(1.04)(1.05)(1.06)} = \$1{,}000.98 $$

Professor's Note: If you think this looks a little like valuing a bond using spot rates, as we did for arbitrage-free valuation, you are right. The discount factors are equivalent to spot rate discount factors.

Key Concepts

1. There are three sources of return to a coupon bond: coupon interest payments, reinvestment income on the coupons, and capital gain or loss on the principal value.
2. Yield to maturity (YTM) can be calculated on a semiannual or annual basis and when calculated on a semiannual basis is termed a bond equivalent yield (BEY).
3. YTM is not the realized yield on an investment unless the reinvestment rate is equal to the YTM.
4. Other important yield measures are the current yield, yield to call, yield to put, and (monthly) cash flow yield.
5. Reinvestment risk is higher when the coupon rate is greater (maturity held constant) and when the bond has longer maturity (coupon rate held constant).
6. The spot rate curve gives the rates to discount each separate cash flow based on the time when it will be received. YTM is an IRR measure that does not account for the shape of the yield curve.
7. There are three commonly used yield spread measures:
   - Nominal spread: bond YTM – Treasury YTM.
   - Zero-volatility spread (*Z*-spread or static spread): spread to the spot yield curve.
   - Option-adjusted spread (OAS): spread to the spot yield curve after adjusting for the effects of embedded options, reflects the spread for credit risk and illiquidity.
8. The *Z*-spread – OAS = option cost in percent.
   - For callable bonds: Z-spread > OAS and option cost > 0.
   - For putable bonds: Z-spread < OAS and option cost < 0.
9. Forward rates are current lending/borrowing rates for loans to be made in future periods.
10. Spot rates for a maturity of *N* periods are the geometric mean of forward rates over the *N* periods.

Concept Checkers: Yield Measures, Spot Rates, and Forward Rates

Use the following data to answer Questions 1 through 4.

An analyst observes a Widget & Co. 7.125%, 4-year, semiannual-pay bond trading at 102.347% of par (where par = $1,000). The bond is callable at 101 in two years, and putable at 100 in two years.

1. What is the bond's current yield?
   A. 6.962%.
   B. 7.500%.
   C. 7.426%.
   D. 7.328%.

2. What is the bond's yield to maturity?
   A. 3.225%.
   B. 6.450%.
   C. 6.334%.
   D. 5.864%.

3. What is the bond's yield to call?
   A. 3.167%.
   B. 5.664%.
   C. 6.334%.
   D. 5.864%.

4. What is the bond's yield to put?
   A. 2.932%.
   B. 6.450%.
   C. 4.225%.
   D. 5.864%.

5. Based on semiannual compounding, what would the YTM be on a 15-year, zero-coupon, $1,000 par value bond that's currently trading at $331.40?
   A. 3.750%.
   B. 5.151%.
   C. 7.500%.
   D. 7.640%.

6. An analyst observes a bond with an *annual* coupon that's being priced to yield 6.350%. What is this issue's bond equivalent yield?
   A. 3.175%.
   B. 3.126%.
   C. 6.252%.
   D. 6.172%.

7. An analyst determines that the cash flow yield of GNMA Pool 3856 is 0.382% *per month*. What is the bond equivalent yield?
   A. 9.582%.
   B. 9.363%.
   C. 4.682%.
   D. 4.628%.

8. For the YTM to equal the actual compound return an investor realizes on an investment in a coupon bond, we must assume all of the following EXCEPT:
   A. cash flows will be paid as promised.
   B. the bond will not be sold at a capital loss.
   C. cash flows will be reinvested at the YTM rate.
   D. The bond will be held until maturity.

9. The 4-year spot rate is 9.45%, and the 3-year spot rate is 9.85%. What is the 1-year forward rate three years from today?
   A. 0.400%.
   B. 9.850%.
   C. 8.258%.
   D. 11.059%.

10. An investor purchases a bond that is putable at the option of the holder. The option has value. He has calculated the *Z*-spread as 223 basis points. The option-adjusted spread will be:
    A. equal to 223 basis points.
    B. less than 223 basis points.
    C. greater than 223 basis points.
    D. It is not possible to determine from the data given.

Use the following data to answer Questions 11 and 12.

Given:
- Current 1-year rate = 5.5%
- 1f1 = 7.63%
- 1f2 = 12.18%
- 1f3 = 15.5%

11. The value of a 4-year, 10% annual-pay, $1,000 par value bond would be *closest* to:
    A. $844.55.
    B. $995.89.
    C. $1,009.16.
    D. $1,085.62.

12. Using annual compounding, the value of a 3-year, zero-coupon, $1,000 par value bond would be:
    A. $708.
    B. $785.
    C. $852.
    D. $948.

13. A bond's nominal spread, zero-volatility spread, and option-adjusted spread will all be equal for a coupon bond if:
    A. the coupon is low and the yield curve is flat.
    B. the yield curve is flat and the bond is not callable.
    C. the bond is option free.
    D. the coupon is high, the yield curve is flat, and the bond has no embedded options.

14. The zero-volatility spread will be zero:
    A. for any bond that is option-free.
    B. if the yield curve is flat.
    C. for a zero-coupon bond.
    D. for an on-the-run Treasury bond.

Comprehensive Problems: Yield Measures, Spot Rates, and Forward Rates

1. An investor buys a 10-year, 7% coupon, semiannual-pay bond for 92.80. He sells it three years later, just after receiving the sixth coupon payment, when its yield to maturity is 6.9%. Coupon interest has been placed in an account that yields 5% (BEY). State the sources of return on this bond, and calculate the dollar return from each source based on a $100,000 bond.

2. What is the yield on a bond equivalent basis of an annual-pay 7% coupon bond priced at par?

3. What is the annual-pay yield to maturity of a 7% coupon semi-annual pay bond?

4. The yield to maturity on a bond equivalent basis on 6-month and 1-year T-bills are 2.8% and 3.2% respectively. A 1.5-year, 4% Treasury note is selling at par.
   A. What is the 18-month Treasury spot rate?
   B. If a 1.5-year corporate bond with a 7% coupon is selling for 102.395, what is the nominal spread for this bond? Is the zero-volatility spread (in basis points) 127, 130, or 133?

5. Assume the following spot rates (as BEYs).

Years to maturity	*Spot rates*
0.5	4.0%
1.0	4.4%
1.5	5.0%
2.0	5.4%

   A. What is the 6-month forward rate one year from now?
   B. What is the 1-year forward rate one year from now?
   C. What is the value of a 2-year, 4.5% coupon Treasury note?

6. Assume the current 6-month rate is 3.5% and the 6-month forward rates (all as BEYs) are those in the following table.

Periods From Now	*Forward Rates*
1	3.8%
2	4.0%
3	4.4%
4	4.8%

   A. Calculate the corresponding spot rates.
   B. What is the value of a 1.5-year, 4% Treasury note?

7. Consider the following three bonds that all have par values of $100,000.
   I. A 10-year zero coupon bond priced at 48.20.
   II. A 5-year 8% semiannual-pay bond priced with a YTM of 8%.
   III. A 5-year 9% semiannual-pay bond priced with a YTM of 8%.

   A. What is the dollar amount of reinvestment income that must be earned on each bond if it is held to maturity and the investor is to realize the current YTM?
   B. Rank the three bonds in terms of how important reinvestment income is to an investor who wishes to realize the stated YTM of the bond at purchase by holding it to maturity.

ANSWERS – CONCEPT CHECKERS: YIELD MEASURES, SPOT RATES, AND FORWARD RATES

1. A current yield $= \dfrac{71.25}{1{,}023.47} = 0.06962$, or 6.962%

2. B $1{,}023.47 = \displaystyle\sum_{t=1}^{8} \frac{35.625}{(1+\text{YTM}/2)^t} + \frac{1{,}000}{(1+\text{YTM}/2)^8} \Rightarrow \text{YTM} = 6.450\%$

   N = 8; FV = 1,000; PMT = 35.625; PV = –1,023.47 → CPT I/Y = 3.225 × 2 = 6.45%

3. C $1{,}023.47 = \displaystyle\sum_{t=1}^{4} \frac{35.625}{(1+\text{YTC}/2)^t} + \frac{1{,}010}{(1+\text{YTC}/2)^4} \Rightarrow \text{YTC} = 6.334\%$

   N = 4; FV = 1,010; PMT = 35.625; PV = –1,023.47; CPT → I/Y = 3.167 × 2 = 6.334%

4. D $1{,}023.47 = \displaystyle\sum_{t=1}^{4} \frac{35.625}{(1+\text{YTP}/2)^t} + \frac{1{,}000}{(1+\text{YTP}/2)^4} \Rightarrow \text{YTP} = 5.864\%$

   N = 4; FV = 1,000; PMT = 35.625; PV = –1,023.47; CPT → I/Y = 2.932 × 2 = 5.864%

5. C $\left[\left(\dfrac{1{,}000}{331.40}\right)^{\frac{1}{30}} - 1\right] \times 2 = 7.5\%$ or,

   Solving with a financial calculator:

   N = 30; FV = 1,000; PMT = 0; PV = –331.40; CPT → I/Y = 3.750 × 2 = 7.500%

6. C bond equivalent yield $= \left([1+\text{EAY}]^{\frac{1}{2}} - 1\right) \times 2 = \left([1.0635]^{\frac{1}{2}} - 1\right) \times 2 = 6.252\%$

7. D bond equivalent yield $= \left([1+\text{CFY}]^{6} - 1\right) \times 2 = \left([1.00382]^{6} - 1\right) \times 2 = 4.628\%$

8. B For a bond purchased at a premium to par value, a decrease in the premium over time (a capital loss) is already factored into the calculation of YTM.

9. C $(1.0945)^4 = (1.0985)^3 \times (1 + {}_1f_3)$

   $\dfrac{(1.0945)^4}{(1.0985)^3} - 1 = {}_1f_3 = 8.258\%$

10. C For embedded puts (e.g., putable bonds): option cost < 0, ⇒ OAS > Z-spread.

11. C Spot rates: $S_1 = 5.5\%$.

$$S_2 = [(1.055)(1.0763)]^{\frac{1}{2}} - 1 = 6.56\%$$

$$S_3 = [(1.055)(1.0763)(1.1218)]^{\frac{1}{3}} - 1 = 8.39\%$$

$$S_4 = [(1.055)(1.0763)(1.1218)(1.155)]^{\frac{1}{4}} - 1 = 10.13\%$$

Bond value:
N = 1; FV = 100; I/Y = 5.5; CPT → PV = 94.79
N = 2; FV = 100; I/Y = 6.56; CPT → PV = 88.07
N = 3; FV = 100; I/Y = 8.39; CPT → PV = 78.53
N = 4; FV = 1,100; I/Y = 10.13; CPT → PV = 747.77
Total: $1,009.16

12. B Find the spot rate for 3-year lending:

$$S_3 = [(1.055)(1.0763)(1.1218)]^{\frac{1}{3}} - 1 = 8.39\%$$

Value of the bond: N = 3; FV = 1,000; I/Y = 8.39; CPT→PV = 785.29

or

$$\frac{\$1{,}000}{(1.055)(1.0763)(1.1218)} = \$785.05$$

13. D If the yield curve is flat, the nominal spread and the Z-V spread are equal. If the bond is option-free, the Z-V spread and OAS are equal. The coupon rate is not relevant.

14. D A Treasury bond is the best answer. The Treasury spot yield curve will correctly price an on-the-run Treasury bond at its arbitrage-free price, so the Z-V spread is zero.

Answers – Comprehensive Problems: Yield Measures, Spot Rates, and Forward Rates

1. The three sources of return are coupon interest payments, recovery of principal/capital gain or loss, and reinvestment income.

Coupon interest payments:
0.07 / 2 × $100,000 × 6 = $21,000

Recovery of principal/capital gain or loss:
Calculate the sale price of the bond: N = (10 – 3) × 2 = 14; I/Y = 6.9 / 2 = 3.45; PMT = 0.07 / 2 × 100,000 = 3,500; FV = 100,000; CPT → PV = –100,548

Capital gain = 100,548 – 92,800 = $7,748

Reinvestment income:
We can solve this by treating the coupon payments as a 6-period annuity, calculating the future value based on the semiannual interest rate, and subtracting the coupon payments. The difference must be the interest earned by reinvesting the coupon payments.

N = 3 × 2 = 6; I/Y = 5 / 2 = 2.5; PV = 0; PMT = –3,500; CPT → FV = $22,357

Reinvestment income = 22,357 – (6 × 3,500) = $1,357

2. BEY = 2 × semiannual discount rate

semiannual discount rate = $(1.07)^{1/2} - 1 = 0.344 = 3.44\%$

BEY = 2 × 3.44% = 6.88%

3. annual-pay YTM = $(1+\frac{0.07}{2})^2 - 1 = 0.0712 = 7.12\%$

4. A. Since the T-bills are zero coupon instruments, their YTMs are the 6-month and 1-year spot rates. To solve for the 1.5-year spot rate we set the bond's market price equal to the present value of its (discounted) cash flows:

$$100 = \frac{2}{1+\frac{0.028}{2}} + \frac{2}{\left(1+\frac{0.032}{2}\right)^2} + \frac{102}{\left(1+\frac{S_{1.5}}{2}\right)^3}$$

$$100 = 1.9724 + 1.9375 + \frac{102}{\left(1+\frac{S_{1.5}}{2}\right)^3}$$

$$\left(1+\frac{S_{1.5}}{2}\right)^3 = \frac{102}{100 - 1.9724 - 1.9375} = 1.0615$$

$$1+\frac{S_{1.5}}{2} = 1.0615^{1/3} = 1.0201$$

$S_{1.5} = 0.0201 \times 2 = 0.0402 = 4.02\%$

B. Compute the YTM on the corporate bond:

N = 1.5 × 2 = 3; PV = –102.395; PMT = 7 / 2 = 3.5; FV = 100; CPT → I/Y = 2.6588 × 2 = 5.32%

nominal spread = $YTM_{Bond} - YTM_{Treasury}$ = 5.32% – 4.0% = 1.32%, or 132 bp

Solve for the zero-volatility spread by setting the present value of the bond's cash flows equal to the bond's price, discounting each cash flow by the Treasury spot rate plus a fixed Z-spread.

$$102.4 = \frac{3.5}{1+\frac{0.028+ZS}{2}} + \frac{3.5}{\left(1+\frac{0.032+ZS}{2}\right)^2} + \frac{103.5}{\left(1+\frac{0.0402+ZS}{2}\right)^3}$$

Substituting each of the choices into this equation gives the following bond values:

Z-spread	*Bond value*
127 bp	102.4821
130 bp	102.4387
133 bp	102.3953

Since the price of the bond is 102.395, a Z-spread of 133 bp is the correct one.

Note that, assuming one of the three zero-volatility spreads given is correct, you could calculate the bond value using the middle spread (130) basis points, get a bond value (102.4387) that is too high, and know that the higher zero-volatility spread is the only one that could generate a present value equal to the bond's market price.

Also note that according to the LOS you are not responsible for this calculation. Working through this example, however, should ensure that you understand the concept of a zero-volatility spread well.

5. A. $\left(1+\frac{S_{1.5}}{2}\right)^3=\left(1+\frac{S_{1.0}}{2}\right)^2\left(1+\frac{{}_{0.5}f_{1.0}}{2}\right)$

$$\left(1+\frac{{}_{0.5}f_{1.0}}{2}\right)=\frac{\left(1+\frac{S_{1.5}}{2}\right)^3}{\left(1+\frac{S_{1.0}}{2}\right)^2}=\frac{1.025^3}{1.022^2}=1.03103$$

$${}_{0.5}f_{1.0}=0.03103\times 2=0.0621=6.21\%$$

B. ${}_1f_1$ here, refers to the 1-year rate, one year from today, expressed as a BEY.

$$\left(1+\frac{S_2}{2}\right)^4=\left(1+\frac{S_1}{2}\right)^2\left(1+\frac{{}_1f_1}{2}\right)^2$$

$$\left(1+\frac{{}_1f_1}{2}\right)^2=\frac{\left(1+\frac{S_2}{2}\right)^4}{\left(1+\frac{S_1}{2}\right)^2}$$

$$\frac{{}_1f_1}{2}=\sqrt{\frac{\left(1+\frac{S_2}{2}\right)^4}{\left(1+\frac{S_1}{2}\right)^2}}-1$$

$$\frac{{}_1f_1}{2}=\sqrt{\frac{\left(1+\frac{0.054}{2}\right)^4}{\left(1+\frac{0.044}{2}\right)^2}}-1=0.0320$$

$${}_1f_1=2\times 0.0320=6.40\%$$

Note that the approximation 2 × 5.4 – 4.4 = 6.4 works very well here and is quite a bit less work.

C. Discount each of the bond's cash flows (as a percent of par) by the appropriate spot rate:

$$\text{bond value} = \frac{2.25}{1+\frac{0.040}{2}} + \frac{2.25}{\left(1+\frac{0.044}{2}\right)^2} + \frac{2.25}{\left(1+\frac{0.050}{2}\right)^3} + \frac{102.25}{\left(1+\frac{0.054}{2}\right)^4} = \frac{2.25}{1.02} + \frac{2.25}{1.0445} + \frac{2.25}{1.0769} + \frac{102.25}{1.1125} = 98.36$$

6. A. $\left(1+\frac{S_{1.0}}{2}\right)^2 = \left(1+\frac{S_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{0.5}}{2}\right) = \left(1+\frac{0.035}{2}\right)\left(1+\frac{0.038}{2}\right) = 1.0368$

$$\frac{S_{1.0}}{2} = 1.0368^{1/2} - 1 = 0.0182$$

$S_{1.0} = 0.0182 \times 2 = 0.0364 = 3.64\%$

$$\left(1+\frac{S_{1.5}}{2}\right)^3 = \left(1+\frac{S_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{1.0}}{2}\right) = \left(1+\frac{0.035}{2}\right)\left(1+\frac{0.038}{2}\right)\left(1+\frac{0.040}{2}\right) = 1.0576$$

$$\frac{S_{1.5}}{2} = 1.0576^{1/3} - 1 = 0.0188$$

$S_{1.5} = 0.0188 \times 2 = 0.0376 = 3.76\%$

$$\left(1+\frac{S_{2.0}}{2}\right)^4 = \left(1+\frac{S_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{1.0}}{2}\right)\left(1+\frac{{}_{0.5}f_{1.5}}{2}\right) = \left(1+\frac{0.035}{2}\right)\left(1+\frac{0.038}{2}\right)\left(1+\frac{0.040}{2}\right)\left(1+\frac{0.044}{2}\right) = 1.0809$$

$$\frac{S_{2.0}}{2} = 1.0809^{1/4} - 1 = 0.0196$$

$S_{2.0} = 0.0196 \times 2 = 0.0392 = 3.92\%$

B. $\frac{2}{1+\frac{0.035}{2}} + \frac{2}{\left(1+\frac{0.0364}{2}\right)^2} + \frac{102}{\left(1+\frac{0.0376}{2}\right)^3} = 100.35$

7. A. Bond (I) has no reinvestment income and will realize its current YTM at maturity unless it defaults. For the coupon bonds to realize their current YTM, their coupon income would have to be reinvested at the YTM.

Bond (II): $(1.04)^{10}$ (100,000) – 100,000 – 10(4,000) = \$8,024.43

Bond (III): First, we must calculate the current bond value. N = 5 × 2 = 10; I/Y = 8 / 2 = 4; FV = 100,000; PMT = 4,500; CPT → PV = –104,055.45

$(1.04)^{10}$ (104,055.45) – 100,000 – 10(4,500) = \$9,027.49

B. Reinvestment income is most important to the investor with the 9% coupon bond, followed by the 8% coupon bond and the zero-coupon bond. In general, reinvestment risk increases with the coupon rate on a bond.

The following is a review of the Analysis of Fixed Income Investments principles designed to address the learning outcome statements set forth by CFA Institute. This topic is also covered in:

INTRODUCTION TO THE MEASUREMENT OF INTEREST RATE RISK

Study Session 16

EXAM FOCUS

This topic review is about the relation of yield changes and bond price changes, primarily based on the concepts of duration and convexity. There is really nothing in this study session that can be safely ignored; the calculation of duration, the use of duration, and the limitations of duration as a measure of bond price risk are all important. You should work to understand what convexity is and its relation to the interest rate risk of fixed-income securities. There are two important formulas: the formula for effective duration and the formula for estimating the price effect of a yield change based on both duration and convexity. Finally, you should get comfortable with how and why the convexity of a bond is affected by the presence of embedded options.

LOS 72.a: Distinguish between the full valuation approach (the scenario analysis approach) and the duration/convexity approach for measuring interest rate risk, and explain the advantage of using the full valuation approach.

The **full valuation** or **scenario analysis approach** to measuring interest rate risk is based on applying the valuation techniques we have learned for a given change in the yield curve (i.e., for a given *interest rate scenario*). For a single option-free bond this could be simply, "if the YTM increases by 50 bp or 100 bp, what is the impact on the value of the bond?" More complicated scenarios can be used as well, such as the effect on the bond value of a steepening of the yield curve (long-term rates increase more than short-term rates). If our valuation model is good, the exercise is straightforward: plug in the rates described in the interest rate scenario(s), and see what happens to the values of the bonds. If the valuation model used is sufficiently good, this is the theoretically preferred approach. Applied to a portfolio of bonds, one bond at a time, we can get a very good idea of how different changes in interest rates will affect the value of the portfolio.

The **duration/convexity approach** provides an approximation of the actual interest rate sensitivity of a bond or bond portfolio. Its main advantage is its simplicity compared to the full valuation approach. The full valuation approach can get quite complex and time consuming for a portfolio of more than a few bonds, especially if some of the bonds have more complex structures, such as call provisions. As we will see shortly, limiting our scenarios to parallel yield curve shifts and "settling" for an estimate of interest rate risk allows us to use the summary measures, duration, and convexity. This greatly simplifies the process of estimating the value impact of overall changes in yield.

Compared to the duration/convexity approach, the full valuation approach is more precise and can be used to evaluate the price effects of more complex interest rate scenarios. Strictly speaking, the duration-convexity approach is appropriate only for estimating the effects of parallel yield curve shifts.

Example: The full valuation approach

Consider two option-free bonds. Bond X is an 8% annual-pay bond with five years to maturity, priced at 108.4247 to yield 6% (N = 5; PMT = 8.00; FV = 100; I/Y = 6.00%; CPT → PV = −108.4247).

Bond Y is a 5% annual-pay bond with 15 years to maturity, priced at 81.7842 to yield 7%.

Assume a $10 million face-value position in each bond and two scenarios. The first scenario is a parallel shift in the yield curve of +50 basis points and the second scenario is a parallel shift of +100 basis points. Note that the bond price of 108.4247 is the price per $100 of par value. With $10 million of par value bonds, the market value will be $10.84247 million.

Answer:

The full valuation approach for the two simple scenarios is illustrated in Figure 1.

Figure 1: The Full Valuation Approach

		Market Value of:			
Scenario	*Yield Δ*	*Bond X (in millions)*	*Bond Y (in millions)*	*Portfolio*	*Portfolio Value Δ%*
Current	+0 bp	$10.84247	$8.17842	$19.02089	
1	+50 bp	$10.62335	$7.79322	$18.41657	–3.18%
2	+100 bp	$10.41002	$7.43216	$17.84218	–6.20%

N = 5; PMT = 8; FV = 100; I/Y = 6% + 0.5%; CPT → PV = –106.2335
N = 5; PMT = 8; FV = 100; I/Y = 6% + 1%; CPT → PV = –104.1002
N = 15; PMT = 5; FV = 100; I/Y = 7% + 0.5%; CPT → PV = –77.9322
N = 15; PMT = 5; FV = 100; I/Y = 7% + 1%; CPT → PV = –74.3216
Portfolio value change 50 bp: (18.41657 – 19.02089) / 19.02089 = –0.03177 = –3.18%
Portfolio value change 100 bp: (17.84218 – 19.02089) / 19.02089 = –0.06197 = –6.20%

It's worth noting that, on an individual bond basis, the effect of an increase in yield on the bonds' values is less for Bond X than for Bond Y (i.e., with a 50 bp increase in yields, the value of Bond X falls by 2.02%, while the value of Bond Y falls by 4.71%; and with a 100 bp increase, X falls by 3.99%, while Y drops by 9.12%). This, of course, is totally predictable since Bond Y is a longer-term bond and has a lower coupon—both of which mean more interest rate risk.

Professor's Note: Let's review the effects of bond characteristics on duration (price sensitivity). Holding other characteristics the same, we can state:

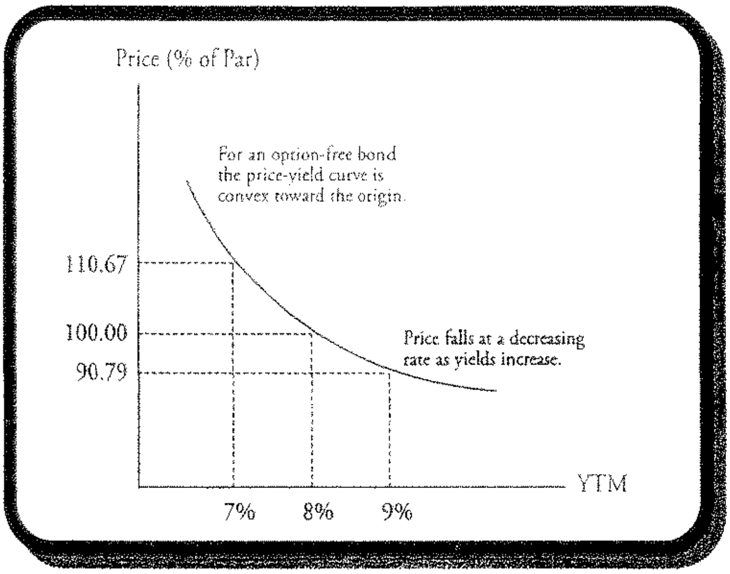
- *Higher (lower) coupon means lower (higher) duration.*
- *Longer (shorter) maturity means higher (lower) duration.*
- *Higher (lower) market yield means lower (higher) duration.*

Finance professors love to test these relations.

LOS 72.b: Describe the price volatility characteristics for option-free, callable, prepayable, and putable bonds when interest rates change (including the concepts of "positive convexity" and "negative convexity").

We established earlier that the relation between price and yield for a straight coupon bond is negative. An increase in yield (discount rate) leads to a decrease in the value of a bond. The precise nature of this relationship for an option-free, 8%, 20-year bond is illustrated in Figure 2.

Figure 2: Price-Yield Curve for an Option-Free, 8%, 20-Year Bond



First, note that the price-yield relationship is negatively sloped, so the price falls as the yield rises. Second, note that the relation follows a curve, not a straight line. Since the curve is convex (toward the origin) we say that an option-free bond has positive convexity. Because of convexity, the price of an option-free bond *increases more when yields fall than it decreases when yields rise.* In Figure 2 we have illustrated that, for an 8%, 20-year option-free bond, a 1% decrease in the YTM will increase the price to 110.67, a *10.67% increase* in price. A 1% increase in YTM will cause the bond value to decrease to 90.79, a *9.22% decrease* in value.

If the price-yield relation were a straight line, there would be no difference between the price increase and the price decline in response to equal decreases and increases in yields. Convexity is a good thing for a bond owner; for a given volatility of yields, price increases are larger than price decreases. The convexity property is often expressed by saying, "a bond's price falls at a decreasing rate as yields rise." For the price-yield relationship to be convex, the slope (rate of decrease) of the curve must be decreasing as we move from left to right (i.e., towards higher yields).

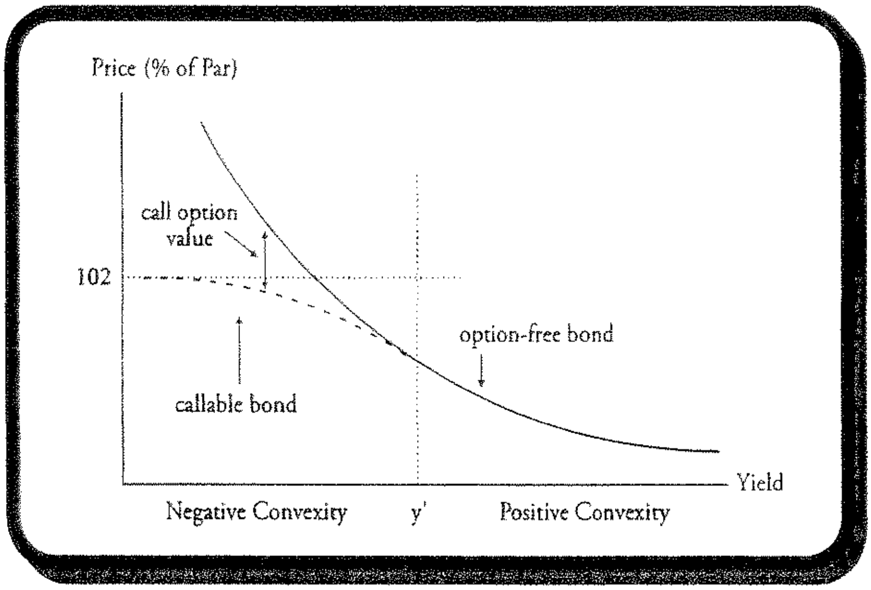
Note that the duration (interest rate sensitivity) of a bond at any yield is the (absolute value of) the slope of the price-yield function at that yield. The convexity of the price-yield relation for an option-free bond can help you remember a result presented earlier, that the duration of a bond is less at higher market yields.

Callable Bonds, Prepayable Securities, and Negative Convexity

With a **callable** or **prepayable** debt, the upside price appreciation in response to decreasing yields is limited (sometimes called price compression). Consider the case of a bond that is currently callable at 102. The fact that the issuer can call the bond at any time for $1,020 per $1,000 of face value puts an effective upper limit on the value of the bond. As Figure 3 illustrates, as yields fall and the price approaches $1,020, the price-yield curve rises more slowly than that of an identical but noncallable bond. When the price begins to *rise at a decreasing rate* in response to further decreases in yield, the price-yield curve "bends over" to the left and exhibits negative convexity.

Thus, in Figure 3, so long as yields remain *below level y'*, callable bonds will exhibit *negative convexity;* however, at yields *above level y'*, those same callable bonds will exhibit *positive convexity*. In other words, at higher yields the value of the call options becomes very small so that a callable bond will act very much like a noncallable bond. It is only at lower yields that the callable bond will exhibit negative convexity.

Figure 3: Price-Yield Function of a Callable Bond



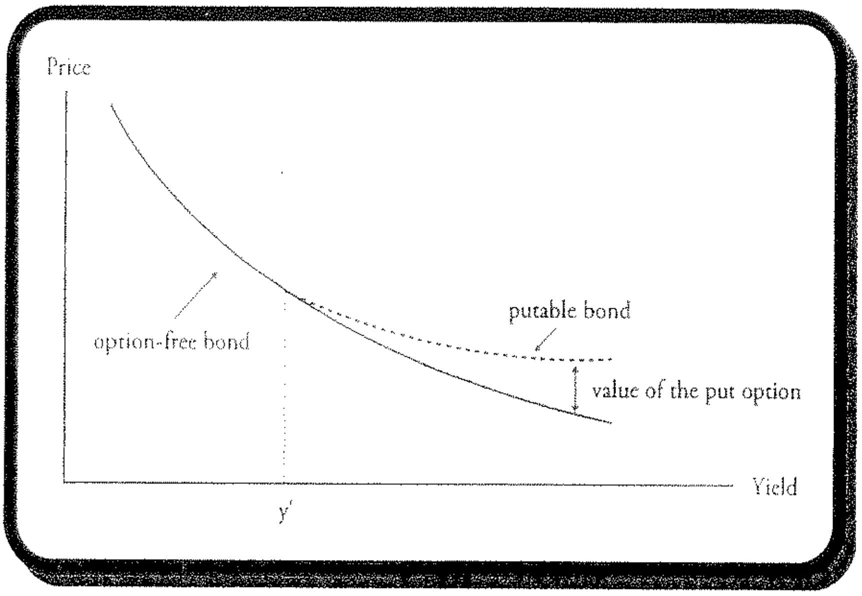
In terms of price sensitivity to interest rate changes, the slope of the price-yield curve at any particular yield tells the story. Note that as yields fall, the slope of the price-yield curve for the callable bond decreases, becoming almost zero (flat) at very low yields. This tells us how a call feature affects price sensitivity to changes in yield. At higher yields, the interest rate risk of a callable bond is very close or identical to that of a similar option-free bond. At lower yields, the price volatility of the callable bond will be much lower than that of an identical but noncallable bond.

The effect of a prepayment option is quite similar to that of a call; at low yields it will lead to negative convexity and reduce the price volatility (interest rate risk) of the security. Note that when yields are low and callable and prepayable securities exhibit less interest rate risk, reinvestment risk rises. At lower yields the probability of a call and the prepayment rate both rise, increasing the risk of having to reinvest principal repayments at the lower rates.

The Price Volatility Characteristics of Putable Bonds

The value of a put increases at higher yields and decreases at lower yields opposite to the value of a call option. Compared to an option-free bond, a **putable bond** will have *less* price volatility at higher yields. This comparison is illustrated in Figure 4.

Figure 4: Comparing the Price-Yield Curves for Option-Free and Putable Bonds



In Figure 4, the price of the putable bond falls more slowly in response to increases in yield above y' because the value of the embedded put rises at higher yields. The slope of the price-yield relation is flatter, indicating less price sensitivity to yield changes (lower duration) for the putable bond at higher yields. At yields below y', the value of the put is quite small, and a putable bond's price acts like that of an option-free bond in response to yield changes.

LOS 72.c: Compute and interpret the effective duration of a bond, given information about how the bond's price will increase and decrease for given changes in interest rates, and compute the approximate percentage price change for a bond, given the bond's effective duration and a specified change in yield.

In our introduction to the concept of duration, we described it as the ratio of the percentage change in price to change in yield. Now that we understand convexity, we know that the price change in response to rising rates is smaller than the price change in response to falling rates for option-free bonds. The formula we will use for calculating the **effective duration** of a bond uses the average of the price changes in response to equal increases and decreases in yield to account for this fact. If we have a callable bond that is trading in the area of negative convexity, the price increase is smaller than the price decrease, but using the average still makes sense.

The formula for calculating the effective duration of a bond is:

$$\text{effective duration} = \frac{(\text{bond price when yields fall} - \text{bond price when yields rise})}{2 \times (\text{initial price}) \times (\text{change in yield in decimal form})}$$

which we will sometimes write as $\text{duration} = \dfrac{V_- - V_+}{2V_0(\Delta y)}$

where:
V_- = bond value if the yield decreases by Δy
V_+ = bond value if the yield increases by Δy
V_0 = initial bond price
Δy = change in yield used to get V_- and V_+, *expressed in decimal form*

Consider the following example of this calculation.

Example: Calculating effective duration

Consider a 20-year, semiannual-pay bond with an 8% coupon that is currently priced at $908.00 to yield 9%. If the yield declines by 50 basis points (to 8.5%), the price will increase to $952.30, and if the yield increases by 50 basis points (to 9.5%), the price will decline to $866.80. Based on these price and yield changes, calculate the effective duration of this bond.

Answer:

Let's approach this intuitively to gain a better understanding of the formula. We begin by computing the average of the percentage change in the bond's price for the yield increase and the percentage change in price for a yield decrease. We can calculate this as:

$$\text{average } \textit{percentage} \text{ price change } = \frac{(\$952.30 - \$866.80)}{2 \times \$908.00} = 0.0471\%, \text{ or } 4.71\%$$

The 2 in the denominator is to obtain the average price change, and the $908 in the denominator is to obtain this average change as a percentage of the current price.

To get the duration (to scale our result for a 1% change in yield), the final step is to divide this average percentage price change by the change in interest rates that caused it. In the example, the yield change was 0.5%, which we need to write in decimal form as 0.005. Our estimate of the duration is:

$$\frac{0.0471}{0.005} = \frac{4.71\%}{0.50\%} = 9.42 = \text{duration}$$

Using the formula previously given, we have:

$$\text{effective duration} = \frac{(\$952.3 - \$866.8)}{2 \times \$908 \times 0.005} = 9.416$$

The interpretation of this result, as you should be convinced by now, is that a 1% change in yield produces an approximate change in the price of this bond of 9.42%. Note, however, that this estimate of duration was based on a change in yield of 0.5% and will perform best for yield changes close to this magnitude. Had we used a yield change of 0.25% or 1%, we would have obtained a slightly different estimate of effective duration.

This is an important concept and you are required to learn the formula for the calculation. To further help you understand this formula and remember it, consider the following.

The price increase in response to a 0.5% decrease in rates was $\frac{\$44.30}{\$908} = 4.879\%$. The price decrease in response to a 0.5% increase in rates was $\frac{\$41.20}{\$908} = 4.537\%$. The average of the percentage price increase and the percentage price decrease is 4.71%. Since we used a 0.5% change in yield to get the price changes, we need to double this and get a 9.42% change in price for a 1% change in yield. The duration is 9.42.

For bonds with no embedded options, modified duration and effective duration will be equal or very nearly equal. In order to calculate effective duration for a bond with an embedded option, we need a pricing model that takes account of how the cash flows change when interest rates change.

Approximate Percentage Price Change for a Bond Based on Effective Duration

Multiply effective duration by the change in yield to get the magnitude of the price change and then change the sign to get the direction of the price change right (yield up, price down).

percentage change in bond price = –effective duration × change in yield in percent

Example: Using effective duration

What is the expected percentage price change for a bond with an effective duration of nine in response to an increase in yield of 30 basis points?

Answer:

$-9 \times 0.3\% = -2.7\%$

We expect the bond's price to decrease by 2.7% in response to the yield change. If the bond were priced at $980, the new price is 980 × (1 – 0.027) = $953.54. Don't make this hard; it's not.

LOS 72.d: Distinguish among the alternative definitions of duration (modified, effective or option-adjusted, and Macaulay), and explain why effective duration is the most appropriate measure of interest rate risk for bonds with embedded options.

The formula we used to calculate duration based on price changes in response to equal increases and decreases in YTM, duration $= \frac{V_- - V_+}{2V_0(\Delta y)}$, is the formula for effective (option-adjusted) duration. This is the preferred measure because it gives a good approximation of interest rate sensitivity for both option-free bonds and *bonds with embedded options*.

Macaulay duration is an estimate of a bond's interest rate sensitivity based on the time, in years, until promised cash flows will arrive. Since a 5-year zero-coupon bond has only one cash flow five years from today, its Macaulay duration is five. The change in value in response to a 1% change in yield for a 5-year zero-coupon bond is approximately 5%. A 5-year coupon bond has some cash flows that arrive earlier than five years from today (the coupons), so its Macaulay duration is less than five. This is consistent with what we learned earlier: the higher the coupon, the less the price sensitivity (duration) of a bond.

Macaulay duration is the earliest measure of duration, and because it was based on the time, duration is often stated as years. Because Macaulay duration is based on the expected cash flows for an option-free bond, it is not an appropriate estimate of the price sensitivity of bonds with embedded options.

Modified duration is derived from Macaulay duration and offers a slight improvement over Macaulay duration in that it takes the current YTM into account. Like Macaulay duration, and for the same reasons, modified duration is not an appropriate measure of interest rate sensitivity for bonds with embedded options. For option-free bonds, however, effective duration (based on small changes in YTM) and modified duration will be very similar.

Professor's Note: The LOS here do not require that you calculate either Macaulay duration or modified duration, only effective duration. For your own understanding, however, note that the relation is

$$\text{modified duration} = \frac{\text{Macaulay duration}}{1 + \text{periodic market yield}}$$

. This accounts for the fact we learned earlier that duration decreases as YTM increases. Graphically, the slope of the price-yield curve is less steep at higher yields.

Effective Duration for Bonds With Embedded Options

As noted earlier, in comparing the various duration measures, both Macaulay and modified duration are calculated directly from the promised cash flows for a bond with no adjustment for the effect of any embedded options on cash flows. Effective duration is calculated from expected price changes in response to changes in yield that explicitly take into account a bond's option provisions (i.e., they are in the price-yield function used).

Interpreting Duration

We can interpret duration in three different ways.

First, duration is the slope of the price-yield curve at the bond's current YTM. Mathematically, the slope of the price-yield curve is the first derivative of the price-yield curve with respect to yield.

A second interpretation of duration, as originally developed by Macaulay, is a weighted average of the time (in years) until each cash flow will be received. The weights are the proportions of the total bond value that each cash flow represents. The answer, again, comes in years.

A third interpretation of duration is the approximate percentage change in price for a 1% change in yield. This interpretation, price sensitivity in response to a change in yield, is the preferred, and most intuitive, interpretation of duration.

Professor's Note: The fact that duration was originally calculated and expressed in years has been a source of confusion for many candidates and finance students. Practitioners regularly speak of "longer duration securities." This confusion is the reason for this part of the LOS. The most straightforward interpretation of duration is the one that we have used up to this point: "It is the approximate percentage change in a bond's price for a 1% change in YTM." I have seen duration expressed in years in CFA exam questions; just ignore the years and use the number. I have also seen questions asking whether duration becomes longer or shorter in response to a change; longer means higher or more interest rate sensitivity. A duration of 6.82 years means that for a 1% change in YTM, a bond's value will change approximately 6.82%. This is the best way to "interpret" duration.

LOS 72.e: Compute the duration of a portfolio, given the duration of the bonds comprising the portfolio, and identify the limitations of portfolio duration.

The concept of duration can also be applied to portfolios. In fact, one of the benefits of duration as a measure of interest rate risk is that the duration of a portfolio is simply the weighted average of the durations of the individual securities in the portfolio. Mathematically, the duration of a portfolio is:

$$\text{portfolio duration} = w_1 D_1 + w_2 D_2 + \ldots + w_N D_N$$

where:
w_i = market value of bond i divided by the market value of the portfolio
D_i = the duration of bond i
N = the number of bonds in the portfolio

Example: Calculating portfolio duration

Suppose you have a two-security portfolio containing Bonds A and B. The market value of Bond A is $6,000, and the market value of Bond B is $4,000. The duration of Bond A is 8.5, and the duration of Bond B is 4.0. Calculate the duration of the portfolio.

Answer:

First, find the weights of each bond. Since the market value of the portfolio is $10,000 = $6,000 + $4,000, the weight of each security is as follows:

$$\text{weight in Bond A} = \frac{\$6{,}000}{\$10{,}000} = 60\%$$

$$\text{weight in Bond B} = \frac{\$4{,}000}{\$10{,}000} = 40\%$$

Using the formula for the duration of a portfolio, we get:

$$\text{portfolio duration} = (0.6 \times 8.5) + (0.4 \times 4.0) = 6.7$$

Limitations of Portfolio Duration

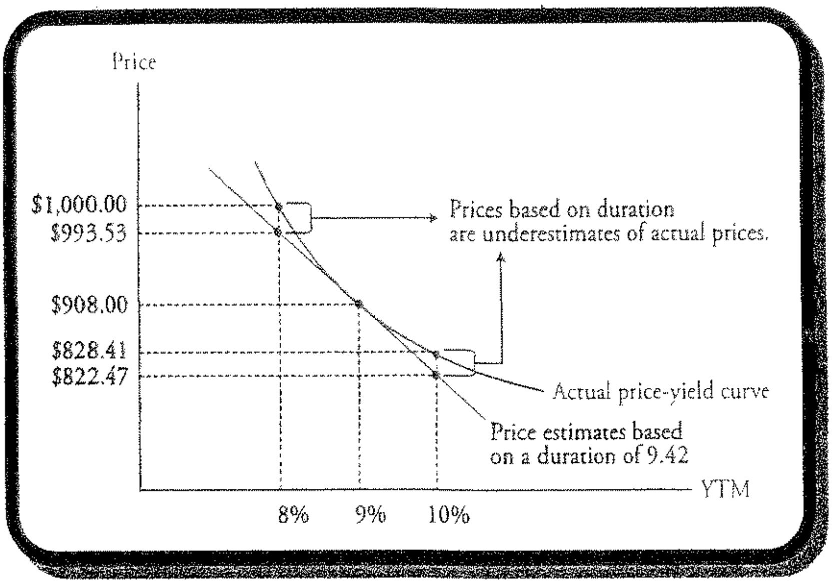
The limitations of portfolio duration as a measure of interest rate sensitivity stem from the fact that yields may not change equally on all the bonds in the portfolio. With a portfolio that includes bonds with different maturities, credit risks, and embedded options, there is no reason to suspect that the yields on individual bonds will change by equal amounts when the yield curve changes. As an example, a steepening of the yield curve can increase yields on long-term bonds and leave the yield on short-term bonds unchanged. It is for this reason that we say that duration is a good measure of the sensitivity of portfolio value to *parallel* changes in the yield curve.

LOS 72.f: Describe the convexity measure of a bond and estimate a bond's percentage price change, given the bond's duration and convexity and a specified change in interest rates.

Convexity is a measure of the curvature of the price-yield curve. The more curved the price-yield relation is, the greater the convexity. A straight line has a convexity of zero. If the price-yield "curve" were, in fact, a straight line, the convexity would be zero. The reason we care about convexity is that the more curved the price-yield relation is, the worse our duration-based estimates of bond price changes in response to changes in yield are.

As an example, consider again an 8%, 20-year Treasury bond priced at $908 so that it has a yield to maturity of 9%. We previously calculated the effective duration of this bond as 9.42. Figure 5 illustrates the differences between actual bond price changes and duration-based estimates of price changes at different yield levels.

Figure 5: Duration-Based Price Estimates vs. Actual Bond Prices



Based on a value of 9.42 for duration, we would estimate the new prices after 1% changes in yield (to 8% and to 10%) as $1.0942 \times 908 = \$993.53$ and $(1 - 0.0942) \times 908 = \$822.47$, respectively. These price estimates are shown in Figure 5 along the straight line tangent to the actual price-yield curve.

The actual price of the 8% bond at a YTM of 8% is, of course, par value ($1,000). Based on a YTM of 10%, the actual price of the bond is $828.41, about $6 higher than our duration based estimate of $822.47. Note that price estimates based on duration are less than the actual prices for both a 1% increase and a 1% decrease in yield.

Figure 5 illustrates why convexity is important and why estimates of price changes based solely on duration are inaccurate. If the price-yield relation were a straight line (i.e., if convexity were zero), duration alone would provide good estimates of bond price changes for changes in yield of any magnitude. The greater the convexity, the greater the error in price estimates based solely on duration. A method of incorporating convexity into our estimates of bond price changes in response to yield changes is the subject of the next LOS.

A Bond's Approximate Percentage Price Change Based on Duration and Convexity

By combining duration and convexity we can obtain a more accurate estimate of the percentage change in price of a bond, especially for relatively large changes in yield. The formula for estimating a bond's percentage price change based on its convexity and duration is:

percentage change in price = duration effect + convexity effect

$$= \left\{\left[-\text{duration} \times (\Delta y)\right] + \left[\text{convexity} \times (\Delta y)^2\right]\right\} \times 100$$

With Δy entered as a decimal, the "× 100" is necessary to get an answer in percent.

Example: Estimating price changes with duration and convexity

Consider an 8% Treasury bond with a current price of $908 and a YTM of 9%. Calculate the percentage change in price of both a 1% increase and a 1% decrease in YTM based on a duration of 9.42 and a convexity of 68.33.

Answer:

The duration effect, as we calculated earlier, is $9.42 \times 0.01 = 0.0942 = 9.42\%$. The convexity effect is $68.33 \times 0.01^2 \times 100 = 0.00683 \times 100 = 0.683\%$. The total effect for a *decrease in yield of 1%* (from 9% to 8%) is $9.42\% + 0.683\% = +10.103\%$ and the estimate of the new price of the bond is $1.10103 \times 908 = 999.74$. This is much closer to the actual price of $1,000 than our estimate using only duration.

The total effect for an *increase in yield of 1%* (from 9% to 10%) is $-9.42\% + 0.683\% = -8.737\%$ and the estimate of the bond price is $(1 - 0.08737)(908) = \$828.67$. Again, this is much closer to the actual price ($828.40) than the estimate based solely on duration.

There are a few points worth noting here. First, the convexity adjustment is always positive when convexity is positive because $(\Delta y)^2$ is always positive. This goes along with the illustration in Figure 5, which shows that the duration-only based estimate of a bond's price change suffered from being an underestimate of the percentage increase in the bond price when yields fell, and an overestimate of the percentage decrease in the bond price when yields rose. Recall, that for a callable bond, convexity can be negative at low yields. When convexity is negative, the convexity adjustment to the duration-only based estimate of the percentage price change will be negative for both yield increases and yield decreases.

Professor's Note: Different dealers may calculate the convexity measure differently. Often the measure is calculated in a way that requires us to divide the measure by two in order to get the correct convexity adjustment. For exam purposes, the formula we've shown here is the one you need to know. However, you should also know that there can be some variation in how different dealers calculate convexity.

LOS 72.g: Distinguish between modified convexity and effective convexity.

Effective convexity takes into account changes in cash flows due to embedded options, while modified convexity does not. The difference between modified convexity and effective convexity mirrors the difference between modified duration and effective duration. Recall that modified duration is calculated without any adjustment to a bond's cash flows for embedded options. Also recall that effective duration was appropriate for bonds with embedded options because the inputs (prices) were calculated under the assumption that the cash flows could vary at different yields because of the embedded options in the securities. Clearly, effective convexity is the appropriate measure to use for bonds with embedded options, since it is based on bond values that incorporate the effect of embedded options on the bond's cash flows.

LOS 72.h: Compute the price value of a basis point (PVBP), and explain its relationship to duration.

The price value of a basis point (PVBP) is the dollar change in the price/value of a bond or a portfolio when the yield changes by one basis point, or 0.01%. We can calculate the PVBP directly for a bond by changing the YTM by one basis point and computing the change in value. As a practical matter, we can use duration to calculate the price value of a basis point as:

duration $\times$ 0.0001 $\times$ bond value = price value of a basis point

The following example demonstrates this calculation.

Example: Calculating the price value of a basis point

A bond has a market value of \$100,000 and a duration of 9.42. What is the price value of a basis point?

Answer:

Using the duration formula, the percentage change in the bond's price for a change in yield of 0.01% is: $0.01\% \times 9.42 = 0.0942\%$. We can calculate 0.0942% of the original \$100,000 portfolio value as $0.000942 \times 100{,}000 = \$94.20$. If the bond's yield increases (decreases) by one basis point, the portfolio value will fall (rise) by \$94.20. \$94.20 is the (duration-based) price value of a basis point for this bond.

We could also directly calculate the price value of a basis point for this bond by increasing the YTM by 0.01% (0.0001) and calculating the change in bond value. This would give us the PVBP for an increase in yield. This would be very close to our duration-based estimate because duration is a very good estimate of interest rate risk for small changes in yield. We can ignore the convexity adjustment here because it is of very small magnitude: $(\Delta y)^2 = (0.0001)^2 = 0.00000001$, which is very small indeed!

KEY CONCEPTS

1. The full valuation approach to measuring interest rate risk involves using a pricing model to value individual bonds and can be used to find the price impact of any scenario of interest rate/yield curve changes. Its advantages are its flexibility and precision.
2. The duration/convexity approach is based on summary measures of interest rate risk and, while simpler to use for a portfolio of bonds than the full valuation approach, is theoretically correct only for parallel shifts of the yield curve.
3. Callable bonds and prepayable securities will have less interest rate risk (lower duration) at lower yields and putable bonds will have less interest rate risk at higher yields, compared to option-free bonds.
4. Option-free bonds have a price-yield relationship that is curved (convex toward the origin) and are, therefore, said to exhibit positive convexity. Bond prices fall less rapidly in response to yield increases than they rise in response to lower yields.
5. Callable bonds exhibit negative convexity at low yield levels; bond prices rise less rapidly in response to yield decreases than they fall in response to yield increases.
6. Effective duration is calculated as the ratio of the average percentage price change for equal increases and decreases in yield to the change in yield, effective duration $= \dfrac{V_- - V_+}{2V_0(\Delta y)}$.
7. Percentage change in bond price = –duration × change in yield in percent.
8. Macaulay duration and modified duration are based on a bond's promised cash flows, while effective duration takes into account the effect of embedded options on a bond's cash flows.
9. Effective duration is appropriate for estimating price changes in bonds with embedded options in response to yield changes; Macaulay and modified duration are not.
10. The most intuitive interpretation of duration is as the percentage change in a bond's price for a 1% change in yield.
11. The duration of a portfolio of bonds is equal to a weighted average of the individual bond durations, where the weights are the proportions of total portfolio value in each bond position.
12. Portfolio duration is limited because it gives the sensitivity of portfolio value only to yield changes that are equal for all bonds in the portfolio, an unlikely scenario for most portfolios.
13. Because of convexity, the duration measure is a poor approximation of price sensitivity for yield changes that are not absolutely small. The convexity adjustment accounts for the curvature of the price-yield relationship.

14. Incorporating both duration and convexity, we can estimate the percentage change in price in response to a change in yield of (Δy) as $\left\{\left[-\text{duration}\times(\Delta y)\right]+\left[\text{convexity}\times(\Delta y)^2\right]\right\}\times 100$.
15. Effective convexity considers expected changes in cash flows that may occur for bonds with embedded options, while modified convexity does not.
16. PVBP measures the price impact, in dollars, of a one basis point change in yield on a bond or bond portfolio.

Concept Checkers: Introduction to the Measurement of Interest Rate Risk

1. Why is the price/yield profile of a callable bond less convex than that of an otherwise identical option-free bond? The price:
   A. increase is capped from above, at or near the call price as the required yield decreases.
   B. increase is capped from above, at or near the call price as the required yield increases.
   C. decrease is limited from below, at or near the call price as the required yield decreases.
   D. decrease is limited from below, at or near the call price as the required yield increases.

2. You own $15 million face of the 4.65% semiannual-pay Portage Health Authority bonds. The bonds have exactly 17 years to maturity and are currently priced to yield 4.39%. Using the full valuation approach, the interest rate exposure (in percent of value) for this bond position given a 75 basis point increase in required yield is *closest* to:
   A. –9.104%.
   B. –9.031%.
   C. –8.344%.
   D. –8.283%.

3. You are estimating the interest rate risk of a 14% semiannual-pay coupon with six years to maturity. The bond is currently trading at par. Using a 25 basis point change in yield, the effective duration of the bond is *closest* to:
   A. 0.389.
   B. 0.397.
   C. 3.889.
   D. 3.970.

4. Suppose that the bond in Question 3 is callable at par today. Using a 25 basis point change in yield, the bond's effective duration assuming that its price cannot exceed 100 is *closest* to:
   A. 1.972.
   B. 1.998.
   C. 19.72.
   D. 19.98.

5. Suppose that you determine that the modified duration of a bond is 7.87. The percentage change in price using duration for a yield decrease of 110 basis points is *closest* to:
   A. –8.657%.
   B. –7.155%.
   C. +7.155%.
   D. +8.657%.

6. A bond has a convexity of 57.3. The convexity effect if the yield decreases by 110 basis points is *closest* to:
   A. –1.673%.
   B. –0.693%.
   C. +0.693%.
   D. +1.673%.

7. Assume you're looking at a bond that has an effective duration of 10.5 and a convexity of 97.3. Using both of these measures, the estimated percentage change in price for this bond, in response to a decline in yield of 200 basis points, is *closest* to:
   A. 22.95%.
   B. 19.05%.
   C. 17.11%.
   D. 24.89%.

8. An analyst has determined that if market yields rise by 100 basis points, a certain high-grade corporate bond will have a convexity effect of 1.75%. Further, she's found that the total estimated percentage change in price for this bond should be –13.35%. Given this information, it follows that the bond's percentage change in price due to duration is:
   A. –15.10%.
   B. –11.60%.
   C. +15.10%.
   D. +16.85%.

9. The total price volatility of a typical noncallable bond can be found by:
   A. adding the bond's convexity effect to its effective duration.
   B. adding the bond's negative convexity to its modified duration.
   C. subtracting the bond's negative convexity from its positive convexity.
   D. subtracting the bond's modified duration from its effective duration, then add any positive convexity.

10. The current price of a $1,000 7-year 5.5% semiannual coupon bond is $1,029.23. The bond's PVBP is *closest* to:
   A. $5.93.
   B. $0.60.
   C. $0.05.
   D. $5.74.

11. The effect on a bond portfolio's value of a decrease in yield would be *most accurately* estimated by using:
   A. the price value of a basis point.
   B. the portfolio duration.
   C. the full valuation approach.
   D. both the portfolio's duration and convexity.

12. An analyst has noticed lately that the price of a particular bond has risen less when the yield falls by 0.1% than the price falls when rates increase by 0.1%. She could conclude that the bond:
   A. is an option-free bond.
   B. has an embedded put option.
   C. a zero-coupon bond.
   D. has negative convexity.

13. Which of the following measures is *lowest* for a currently callable bond?
   A. Macaulay duration.
   B. Effective duration.
   C. Modified duration.
   D. Cannot be determined.

Comprehensive Problems: Introduction to the Measurement of Interest Rate Risk

Use the following information to answer Questions 1 through 6.

A bond dealer provides the following selected information on a portfolio of fixed-income securities.

Par Value	*Mkt. Price*	*Coupon*	*Modified Duration*	*Effective Duration*	*Effective Convexity*
$2 million	100	6.5%	8	8	154
$3 million	93	5.5%	6	1	50
$1 million	95	7%	8.5	8.5	130
$4 million	103	8%	9	5	–70

1. What is the effective duration for the portfolio?
2. Calculate the price value of a basis point for this portfolio.
3. Which bond(s) likely has/have no embedded options? (identify bonds by coupon)
4. Which bond(s) is/are likely callable?
5. Which bond(s) is/are likely putable?
6. What is the approximate price change for the 7% bond if its yield to maturity increases by 25 basis points?
7. Why might two bond dealers differ in their estimates of a portfolio's effective duration?
8. Why might portfolio effective duration be an inadequate measure of interest rate risk for a bond portfolio even if we assume the bond effective durations are correct?

Answers – Concept Checkers: Introduction to the Measurement of Interest Rate Risk

1. A As the required yield decreases on a callable bond, the rate of increase in the price of the bond begins to slow down and eventually level off as it approaches the call price, a characteristic known as "negative convexity."

2. C We need to compare the value of the bond today to the value if the YTM increases by 0.75%.

   Price today = 103.092

   $$N = 34;\ PMT = \frac{4.65}{2} = 2.325;\ FV = 100;\ I/Y = \frac{4.39}{2} = 2.195\%;\ CPT \rightarrow PV = -103.092$$

   Price after a 75 basis point increase in the YTM is 94.490

   $$N = 34;\ PMT = \frac{4.65}{2} = 2.325;\ FV = 100;\ I/Y = \frac{5.14}{2} = 2.57\%;\ CPT \rightarrow PV = -94.490$$

   $$\text{Interest rate exposure} = \frac{94.490 - 103.092}{103.092} = -8.344\%$$

3. D $V_- = 100.999$

   $$N = 12;\ PMT = \frac{14.00}{2} = 7.00;\ FV = 100;\ I/Y = \frac{13.75}{2} = 6.875\%;\ CPT \rightarrow PV = -100.999$$

   $V_+ = 99.014$

   $$N = 12;\ PMT = \frac{14.00}{2} = 7.00;\ FV = 100;\ I/Y = \frac{14.25}{2} = 7.125\%;\ CPT \rightarrow PV = -99.014$$

   $V_0 = 100.000$

   $\Delta y = 0.0025$

   $$\text{duration} = \frac{V_- - V_+}{2V_0(\Delta y)} = \frac{100.999 - 99.014}{2(100)0.0025} = 3.970$$

4. A $V_- = 100.000$

   $V_+ = 99.014$

   $V_0 = 100.000$

   $\Delta y = 0.0025$

   $$\text{duration} = \frac{V_- - V_+}{2V_0(\Delta y)} = \frac{100.000 - 99.014}{2(100)0.0025} = 1.972$$

5. D $\text{Est.}\left[\Delta V\_\%\right] = -7.87 \times (-1.10\%) = 8.657\%$

6. C $\text{convexity effect} = \text{convexity} \times (\Delta y)^2 = \left[57.3(0.011)^2\right] \times 100 = 0.693\%$

7. D Total estimated price change = (duration effect + convexity effect)

$$\left\{\left[-10.5\times(-0.02)\right]+\left[97.3\times(-0.02)^2\right]\right\}\times 100 = 21.0\% + 3.89\% = 24.89\%$$

8. A Total percentage change in price = duration effect + convexity effect. Thus:

$$-13.35 = \text{duration effect} + 1.75 \Rightarrow \text{duration effect} = -15.10\%$$

(Note the duration effect must be negative because yields are *rising*.)

9. A Total percentage change in price = duration effect + convexity effect. Thus:

Total percentage change in price = effective duration + convexity effect

(Note that since this is a noncallable bond, you can use either effective or modified duration in the above equation.)

10. B PVBP = initial price – price if yield is changed by 1 bp. First, we need to calculate the yield so that we can calculate the price of the bond with a 1 basis point change in yield. Using a financial calculator: PV = –1,029.23; FV = 1,000; PMT = 27.5 = (0.055 × 1,000) / 2; N =14 = 2 × 7 years; CPT → I/Y = 2.49998, multiplied by 2 = 4.99995, or 5.00%. Next, compute the price of the bond at a yield of 5.00% + 0.01%, or 5.01%. Using the calculator: FV = 1,000; PMT = 27.5; N = 14; I/Y = 2.505 (5.01 / 2); CPT → PV = $1,028.63. Finally, PVBP = $1,029.23 – $1,028.63 = $0.60.

11. C The full valuation approach is the most complex method, but also the most accurate.

12. D A bond with negative convexity will rise less in price in response to a decrease in yield than it will fall in response to an equal-sized increase in rates.

13. B The interest rate sensitivity of a bond with an embedded call option will be less than that of an option-free bond. Effective duration takes the effect of the call option into account and will, therefore, be less than Macaulay or modified duration.

Answers – Comprehensive Problems: Introduction to the Measurement of Interest Rate Risk

1. Portfolio effective duration is the weighted average of the effective durations of the portfolio bonds.

   Numerators in weights are market values (par value × price as percent of par). Denominator is total market value of the portfolio.

$$\frac{2}{9.86}(8)+\frac{2.79}{9.86}(1)+\frac{0.95}{9.86}(8.5)+\frac{4.12}{9.86}(5)=4.81 \quad \text{(weights are in millions)}$$

2. Price value of a basis point can be calculated using effective duration for the portfolio and the portfolio's market value, together with a yield change of 0.01%. Convexity can be ignored for such a small change in yield.

   4.81 × 0.0001 × 9,860,000 = $4,742.66

3. The 6.5% and 7% coupon bonds likely have no embedded options. For both of these bonds, modified duration and effective duration are identical, which would be the case if they had no embedded options. (It is possible that these bonds have options that are so far out of the money that the bond prices act as if there is no embedded option. One example might be a conversion option to common stock at $40 per share when the market value of the shares is $2.)

4. The 8% bond is likely callable. It is trading at a premium, its effective duration is less than modified duration, and it exhibits negative convexity. Remember, call price can be above par.

5. The 5.5% bond is likely putable. It is trading at a significant discount, its effective duration is much lower than its modified duration (close to zero in fact), and its convexity is positive but low. Note that a putable bond may trade below par when the put price is below par (also if there is risk that the issuer cannot honor the put). If it were callable, we would expect its modified and effective durations to be closer in value because the market price is significantly below likely call prices.

6. Based on the effective duration and effective convexity of the 7% bond, the approximate price change is:

$$[-8.5 \times 0.0025] + [130 \times 0.0025^2] \times 950{,}000 = -\$19{,}415.63$$

7. In order to estimate effective duration, the dealers must use a pricing model for the bonds and choose a specific yield change. Differences in models or the yield change used can lead to differences in their estimates of effective duration.

8. Effective duration is based on small changes in yield and is appropriate for parallel changes in the yield curve (or equal changes in the yields to maturity for all portfolio bonds). Other types of yield changes will make portfolio duration an inadequate measure of portfolio interest rate risk.

The following is a review of the Derivative Investments principles designed to address the learning outcome statements set forth by CFA Institute. This topic is also covered in:

DERIVATIVE MARKETS AND INSTRUMENTS

Study Session 17

EXAM FOCUS

This topic review contains introductory material for the upcoming reviews of specific types of derivatives. Derivatives-specific definitions and terminology are presented along with information about derivatives markets. Upon completion of this review, candidates should be familiar with the basic concepts that underlie derivatives and the general arbitrage framework. There is little contained in this review that will not be elaborated upon in the five reviews that follow.

LOS 73.a: Define a derivative and differentiate between exchange-traded and over-the-counter derivatives.

A **derivative** is a security that *derives* its value from the value or return of another asset or security.

A physical exchange exists for many options contracts and futures contracts. **Exchange-traded derivatives** are standardized and backed by a clearinghouse.

Forwards and *swaps* are custom instruments and are traded/created by dealers in a market with no central location. A dealer market with no central location is referred to as an **over-the-counter** market. They are largely unregulated markets and each contract is with a counterparty, which may expose the owner of a derivative to default risk (when the counterparty does not honor their commitment).

Some *options* trade in the over-the-counter market, notably bond options.

LOS 73.b: Define a forward commitment, identify the types of forward commitments, and describe the basic characteristics of forward contracts, futures contracts, options (calls and puts), and swaps.

A **forward commitment** is a legally binding promise to perform some action in the future. Forward commitments include forward contracts, futures contracts, and swaps.

Forward contracts and futures contracts can be written on equities, indexes, bonds, physical assets, or interest rates.

In a **forward contract** one party agrees to buy, and the counterparty to sell, a physical asset or a security at a specific price on a specific date in the future. If the future price of the asset increases, the buyer (at the older, lower price) has a gain, and the seller a loss.

A **futures contract** is a forward contract that is standardized and exchange-traded. The main differences with forwards are that futures are traded in an active secondary market, are regulated, backed by the clearinghouse, and require a daily settlement of gains and losses.

A **swap** is a series of forward contracts. In the simplest swap, one party agrees to pay the short-term (floating) rate of interest on some principal amount, and the counterparty agrees to pay a certain (fixed) rate of interest in return. Swaps of different currencies and equity returns are also common.