Project 2: Shift Register Sequences in Rust

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Contents

Home Exercise 1

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1. $p(x) = x^4 + x^2 + 1$ over $\mathbb{F}_2$

Since $x^4 + x^2 + 1 = (x^2 + x + 1)^2$ it is not irreducible, and therefore not primitive.

2. $p(x) = x^3 + x + 1$ over $\mathbb{F}_3$

Since $x^3 + x + 1 = x^3 + 3x^2 + 4x + 4 = (x + 2)(x^2 + x + 2)$, it clearly has factors and is therefore not irreducible. Since it is not irreducible, it is not primitive.

3. $p(x) = x^2 + \alpha^5 x + 1$ over $\mathbb{F}_{2^4}$, where $\alpha^4 + \alpha + 1 = 0$

If there is an i such that $p(a^i) = 0$, then $p(x)$ has a root, and is therefore not primitive nor irreducible.
If $i = 6$, then $p(\alpha^6) = \alpha^{12} + \alpha^{11} + 1 = (\alpha^3 + \alpha^2 + \alpha + 1) + (\alpha^3 + \alpha^2 + \alpha) + 1 = 2\alpha^3 + 2\alpha^2 + 2\alpha + 2 = 0$ As shown, $p(x)$ is reducible and therefore not primitive.

Lab Exercise 1

1. $p(x) = x^{23} + x^5 + 1$ over $\mathbb{F}_2$

> Primitive($x^{23} + x^5 + 1$) **mod** 2
> True

Therefore $p(x)$ is primitive, and therefore irreducible.

2. $p(x) = x^{23} + x^6 + 1$ over $\mathbb{F}_2$

> Primitive($x^{23} + x^6 + 1$) **mod** 2
> False
> Irreduc($x^{23} + x^6 + 1$) **mod** 2
> False

Therefore $p(x)$ is neither a primitive nor irreducible.

3. $p(x) = x^{18} + x^3 + 1$ over $\mathbb{F}_2$

> Primitive($x^{18} + x^3 + 1$) **mod** 2
> False
> Irreduc($x^{18} + x^3 + 1$) **mod** 2
> True

Therefore $p(x)$ is not a primitive, but it is irreducible.

4. $p(x) = x^8 + x^6 + 1$ over $\mathbb{F}_7$

> Primitive($x^8 + x^6 + 1$) **mod** 7
> False
> Irreduc($x^8 + x^6 + 1$) **mod** 7
> False

Therefore $p(x)$ is neither a primitive nor irreducible.

5. $p(x) = x^6 + \alpha^5 x + 1$ over $\mathbb{F}_{2^4}$

> G4 := GF(2, 4, $\alpha^4 + \alpha + 1$)
> G4 := $\mathbb{F}_{16}$
> x := α
> a := G4:-ConvertIn($x^6 + \alpha^5 * x + 1$)
> a := $1 \mod 2$
> x := α^2
> a := G4:-ConvertIn($x^6 + \alpha^5 * x + 1$)
> a := $\alpha^2 + 1 \mod 2$
> x := α^3

...

> x := α^{14}
> a := G4:-ConvertIn($x^6 + \alpha^5 * x + 1$)
> a := $\alpha^3 \mod 2$
> G4:-isPrimitiveElement(a)
> false

Since the function wasn't evaluated to 0 for any of the α^i, $p(x)$ is irreducible. But is not primitive.

Home Exercise 2

$|\mathbb{F}_{2^4}| = 16 \implies \alpha^{15} \equiv 1$, therefore the possible orders for a polynomial consisting of one α are all possible factors of 15, that is 1, 3, 5 and 15.

1. α

$ord(\alpha) = n \implies \alpha^n \equiv \alpha^{15} \implies n = 15.$ The order of α is 15.

2. α^2

$ord(\alpha) = n \implies \alpha^{2n} \equiv \alpha^{15} \implies n = 15.$ The order of α is 15.

3. α^3

$ord(\alpha) = n \implies \alpha^{3n} \equiv \alpha^{15} \implies n = 5.$ The order of α is 5.

4. α^5

$ord(\alpha) = n \implies \alpha^{5n} \equiv \alpha^{15} \implies n = 3.$ The order of α is 3.

Lab Exercise 2

```
> G18 := GF(2, 18, α^18 + α^3 + 1)
> G18 := 𝔽_{2^18}
```

1. α

```
> a := G18:-ConvertIn(α);
> ...
> G18:-order(a)
> 189
```

2. α^2

```
> a := G18:-ConvertIn(α^2);
> ...
```

```
> G18:-order(a)
> 189
```

3. α^3

```
> a := G18:-ConvertIn(α^3);
> ...
> G18:-order(a)
> 63
```

4. $\alpha + \alpha^3$

```
> a := G18:-ConvertIn(α+α^3);
> ...
> G18:-order(a)
> 262143
```

Home Exercise 3

1. $p(x) = x^4 + x^2 + 1$ over $\mathbb{F}_2$

Since $x^4 + x^2 + 1 = (x^2 + x + 1)^2$ we can describe $C(D) = C_1(D)^n$, where $C(D) = (1 + D + D^2)^2$ and $C_1(D) = 1 + D + D^2$ with $n = 2$. Therefore we can calculate $L_1 = \deg C_1(D) = 2$, with the period $T_1 = 3$ of C_1. We can then calculate $T_2 = p^m T_1 = 2^1 2 = 4$, with $m = 1$ since $2^0 < 2 \leq 2^1$. We can then plug these numbers into the formula $1(1) \oplus \frac{q^{L_1}-1}{T_1}(T_1) \oplus \frac{q^{L_1}(q^{L_1}-1)}{T_2}(T_2)$. Which results in the cycle set $1(1) \oplus 1(3) \oplus 2(6)$.

2. $p(x) = x^3 + x + 1$ over $\mathbb{F}_3$

Since $x^3+x+1 = (x+2)(x^2+x+2)$ we can describe $C(D) = C_1(D) * C_2(D)$, where $C(D) = D^3 + D + 1$, $C_1(D) = D + 2$ and $D^2 + D + 2$. We can see that the order of C_1, $T_{1_1} = 1$ since

$$1 = 2(D + 2) + (-2D)$$

We can also see that the order of C_2, $T_{2_1} = 8$ since

$$
\begin{aligned}
1 &= 2(D^2 + D + 2) + (-2D^2 - 2D) \\
1 &= (2 + 2D)(D^2 + D + 2) + (-2D^3 - D^2) \\
1 &= (2 + 2D + D^2)(D^2 + D + 2) + (-D^4) \\
1 &= (2 + 2D + D^2 + D^4)(D^2 + D + 2) + (-D^6 - D^5) \\
1 &= (2 + 2D + D^2 + D^4)(D^2 + D + 2) + (-D^6 - D^5) \\
1 &= (2 + 2D + D^2 + D^4 + D^5)(D^2 + D + 2) + (-D^7 - 2D^6) \\
1 &= (2 + 2D + D^2 + D^4 + D^5 + 2D^6)(D^2 + D + 2) + (-2D^8)
\end{aligned}
$$

We can then plug these numbers into the formula $1(1) \oplus \frac{q^{L_1} - 1}{T_1}(T_1) \oplus \frac{q^{L_1}(q^{L_1} - 1)}{T_2}(T_2)$. Which results in the cycle set $1(1) \oplus 2(1)$ for C_1, and the cycle set $1(1) \oplus 1(8)$ for C_2. Resulting in the cycle set $(1(1) \oplus 2(1)) \times (1(1) \oplus 1(8)) = 1(1) \oplus 1(8) \oplus 2(1) \oplus 2(8) = 3(1) \oplus 3(8)$ for $C(D)$.

Lab Exercise 3

1. $p(x) = x^{23} + x^5 + 1$ over $\mathbb{F}_2$

Calculate the order through: `> G23 := GF(2, 23,` $\alpha^{23} + \alpha^5 + 1$`)`

`> G23 :=` $\mathbb{F}_{2^{23}}$

`> a := G23:-ConvertIn(`α`)`

`> a :=` $\alpha \mod 2$

`> G23:-order(a)`

`> 8388607`

With the order $T_1 = 8388607$, we can calculate the cycle set through the formula $1(1) \oplus \frac{q^{L_1} - 1}{T_1}(T_1) \oplus \frac{q^{L_1}(q^{L_1} - 1)}{T_2}(T_2)$. Resulting in a cycle set of $1(1) \oplus 1(8388607)$.

2. $p(x) = x^{23} + x^6 + 1$ over $\mathbb{F}_2$

`> Factor(`$x^{23} + x^6 + 1$`) mod 2;`

`>` $(x^4 + x^3 + 1) \times (x^{16} + x^{15} + x^{13} + x^{12} + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1) \times (x^3 + x + 1)$

Which gives us the three polynomials $C_1(D) = D^4 + D^3 + 1$, $C_2(D) = D^{16} + D^{15} + D^{13} + D^{12} + D^8 + D^6 + D^4 + D^3 + D^2 + D + 1$ and $C_3(D) = D^3 + D + 1$.

We can then calculate the order of C_1, $T_1 = 15$ through:

> G4 := GF(2, 4, $\alpha^4 + \alpha^3 + 1$)
> G4 := $\mathbb{F}_{16}$
> a := G4:-ConvertIn(α)
> a := $\alpha \mod 2$
> G4:-order(a)
> 15

We can then calculate the order of C_2, $T_2 = 21845$ through:

> G16 := GF(2, 16, $\alpha^{16} + \alpha^{15} + \alpha^{13} + \alpha^{12} + \alpha^8 + \alpha^6 + \alpha^4 + \alpha^3 + \alpha^2 + \alpha + 1$)
> G16 := $\mathbb{F}_{2^{16}}$
> a := G16:-ConvertIn(α)
> a := $\alpha \mod 2$
> G16:-order(a)
> 21845

And lastly we can calculate the order of C_3, $T_3 = 7$ through:

> G3 := GF(2, 3, $\alpha^3 + \alpha + 1$)
> G3 := $\mathbb{F}_8$
> a := G3:-ConvertIn(α)
> a := $\alpha \mod 2$
> G3:-order(a)
> 7

With these number we can calculate the cycle sets through the formula $1(1) \oplus \frac{q^{L_1}-1}{T_1}(T_1) \oplus \frac{q^{L_1}(q^{L_1}-1)}{T_2}(T_2)$. Resulting in the cycle set $1(1) \oplus 1(15)$ for C_1, $1(1) \oplus 3(21845)$ for C_2 and $1(1) \oplus 1(7)$ for C_3. Meaning we can calculate the cycle set of $C(D)$ through $(1(1) \oplus 1(15)) \times (1(1) \oplus 3(21845)) \times (1(1) \oplus 1(7)) = 1(1) \oplus 1(7) \oplus 3(21845) \oplus 3(152915) \oplus 1(15) \oplus 1(105) \oplus 3(327675) \oplus 3(2293725)$

Home Exercise 4

One primitive of degree 4 over $\mathbb{F}_2$ is $x^4 + x^3 + 1$. As it satisfies $P(0) = P(1) = 1$ and it doesn't have any factors.

Lab Exercise 4

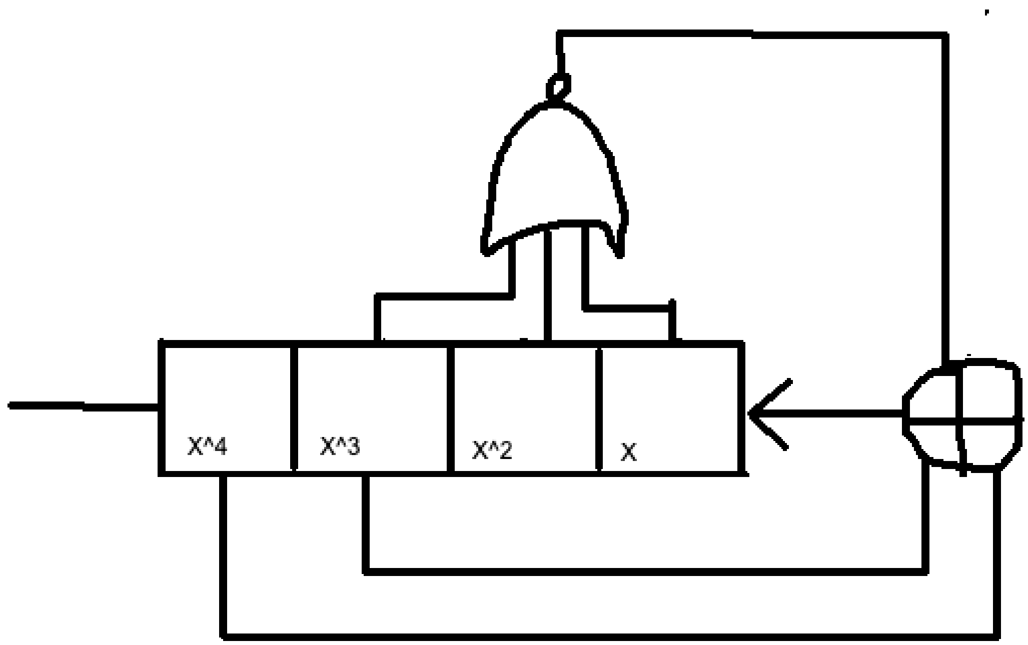
One way to find primtive polynomials is to simply iterate through all possible polynomials and filter them based on if they are primitive. This can be achieved through the following code:

```
> for i from 0 to 4 do
    for j from 0 to 4 do
      for k from 0 to 4 do
        for l from 0 to 4 do
          if (Primitive(l*x^4+k*x^3+j*x^2+i*x+1) mod 2)
            then print(l, k, j, i, 1) break
          end if
        end do
      end do
    end do
  end do
> 2, 2, 1, 0, 1
```

So one example of a primitive polynomial of degree 4 over $\mathbb{F}_5$ is $2x^4 + 2x^3 + x^2 + 1$.

Home Exercise 5

In order to generate a basic de Bruijne sequence of the format $...s \to s'...$ we can use just a simple XOR operation between over the x^4 and x^3 slots, resulting in the 16-length de Bruijn 1100010011010111. But it doesn't include the 0 state, so in order to insert it we add a simple NOR check between the x^3, x^2 and x slots, changing the sequence of states from $... \to 1000 \to 0001 \to ...$ to $... \to 1000 \to 0000 \to 0001 \to ...$, making the sequence into a $...s \to 0 \to s'...$ sequence. The resulting de Bruijn sequence of length 16 starting from $x^4 + x^3 + 1$ is 1100001001101011.



Lab Exercise 5

The program is a fairly simple implementation in rust, where it simply generates two different sequences, one for $\mathbb{Z}_2$ based on the the home assignment, and one for $\mathbb{Z}_5$. It then combines them into a single sequence, creating a de Bruijn sequence of $\mathbb{Z}_{10}$ and prints it out.

So it simply generetes two start vectors, both with the values [0,0,0,1] at line 8 and 9. These are then iterated through a loop 10003 times in order to generate their sequences. For the $\mathbb{Z}_2$ sequence, it simply adds the slots together based on the primitive $x^4 + x^3 + 1$ and reduces them to modulo 2, and then appends the result to the sequence, except in the special cases of [1,0,0,0] and [0,0,0,0], where we have special conditions for in order to add the 0 element, that is $...s \to 0 \to s'....$ The code for this can be seen on line 13-21.
For the $\mathbb{Z}_5$ sequence, it simply adds the slots together based on the primitive $2x^4 + 2x^3 + x^2 + 1$ and reduces them to modulo 5, and then appends the result to the sequence, except in the special case of [2,0,0,0] and [0,0,0,0], the reasoning for choicing these specific numbers is as the linear implemen-

tation would go through $[2,0,0,0] \rightarrow [0,0,0,1]$, resetting the cycle, therefore we add the 0 element, [0,0,0,0], inbetween these, creating the cycle ... $\rightarrow$ 2000 $\rightarrow$ 0000 $\rightarrow$ 0001 $\rightarrow$ .... The code for this can be seen on line 23-30.

The $\mathbb{Z}_{10}$ sequence is then created by adding these two sequences together and then printing them out. The code for this can be seen on line 35-44.

Source code

```
1  use std::io::*;
2
3  fn main() {
4      exercise5();
5  }
6
7  fn exercise5() {
8      let mut z2 = vec![0, 0, 0, 1];
9      let mut z5 = vec![0, 0, 0, 1];
10     // create file sequence.txt
11     for i in 0..10003 {
12         z2.push(
13             match z2.clone()[i..] {
14                 // Special case of 1000->0000->0001
15                 [0, 0, 0, 0] => 1,
16                 [1, 0, 0, 0] => 0,
17                 // General case x^4 + x^3 + 1
18                 [a,b,_,_] => (-(a+b) as i32).rem_euclid(2),
19                 _ => unreachable!()
20             }
21         );
22         z5.push(
23             match z5.clone()[i..] {
24                 // Special case of 2000->0000->0001
25                 [0, 0, 0, 0] => 1,
26                 [2, 0, 0, 0] => 0,
27                 // General case 2x^4 + 2x^3 + x^2 + 1
```

```
                [a,b,c, _] => (-(2*a+2*b+c) as i32).rem_euclid(5),
                _ => unreachable!()
            }
        );
    }
    // print to file
    let mut file = std::fs::File::create("output.txt").unwrap();
    file.write_all(
        z2.iter().zip(z5.iter()).skip(4)
        .map(|(a,b)| {
            return (5*a + b).to_string();
        })
        .reduce(|a,b|
            a + "\n" + &b
        ).unwrap().as_bytes()
    ).unwrap();
}
```

de Bruijn sequence $\mathbb{Z}_{10}$

048947276564243902650936958044084457452858514347246827087872334744
990547657412282486052888701008326648279880331621980717579424291476
371599930118235707195552031834992528986304250367351588513309309606
078684432644564649668433380089254879914317009616457551142731993615
667242360479093898932347147538365771410904780509567113384096362658
900129107516076653103702591718869233073288271866610018408648378674
003722854507789412381275274989620348036636369684234511670947875303
460058074776613037129706355793204522592706756404082476173966541416
447627275571301731691748556444153059463879612445448818295681012712
673718968022290389190656820005109849277651424840760548695304453495
290785801439229637258782233924994509765241273298150788820105337619
377983033612693526757442474197182659943016328525269550203633994707
898130470086280658801335435915157863443714951919966343383058470987
941436205916195750114723694816566224281097454889843239219708386572
141540973505956211383459181765840017415706157660310820754626886423
352378372686611006345819387862400822780905778441283177072998912039

3086181869634239016625497870303910553529776113082179251855743209027547256751404532971628966041461497172775521306236646298551444603554918879112490498363795631017217628268963022740884645656320050159394777601429345715098690304903990745785301484279182758732238429949059760241723793605788320150387164877933038117648076752442924692637659443061378070769500208138949257893130920581735658301380485460657813448219906469961343833553925987441481255461695700119228649366561224731592909889343284269253886522146045928055951211833954636765340062465251657610315325709176881423802873827686111051395364887812405327735455773441733672527998412084358163686913428406617099782030841550807977111353267470685524325407709275670140903792617896104191199262777502135128619179850144910850946887411294099381879513106226717376891302724583919565132050065484977710147439526059864030940894529578030193477463775823228347949455971024622874815578332060088261987743308316719357670244742964718765444351187352576900025318849475784313542553628565330183098091565731349326945196991134833850847598244193175091669520016427819486651122923654745988434373476470888602219109547355590121633890918676034051296070665711036037525467683142830782837768111150189081988731245037728095572344623862707799341253485361868641347345616259973203534650535797211180376292568502437045725477562014540874716789110464169471777700218017816467980014941580549688241174459483687901315127626287684130722953846956013250056093997721019248907155981403544584907957303064397291877532327339749495592102912782936557333251058371698724335336626485762024924791926876044480168280757640007036839497573431804750817856033063359354656523139437649069694113933880539759324464367054666902006147736498660112742860929598343482397192588810226415909285554012613884546867103450179152566521108108707096763314733573738776311160068453698823129008727359552234912881725779434170398081686814139239516175992320803960508579221163087174756800248209527097751201904582926678411091466492677720026306736196793001944653509968324162495493868740136017717178763413522790839695101370055154899722106429845265593140804953945795230351489274687703237238929499554210741773748655233370155382669822438038617198571202942974647687104493066373575714005208638499752343630970536785103351385480965602318448719456964411843883508975432491486250966640205119728199861011724781547959334393289264758831027146545478550401711883909686210390067460756602115315825259671331923852828877131161056390869832317405827285950223941783627577443462089353668631

141842890616759423253089150585742216135826297563002932590725977012
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282986502338206508376693224830881626985212074479291976821049435618
285752140502581839997023481359250867801038018809359651023634982649
569144163488380589704329419817059661402501692736998110162297360979
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851736696243051499153559913329117945498863240631562717678213463077
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279682203732783929999004260296273798600238325605387664322933583617
698021252497474697632109448516378570214500753638999202393185470586
730108306835485960102813993719956414461398383558920437446936255961
140700664728699311061279281597904339348739269753331522691545978000
451266383959631215345517465751102610860825759121381478352878822136
116506395864332812950827785400273496283677522448417539358663131913
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866412480069491585544138246629459983132901860627676232184185272958
841406018705056593442721564748957100481458094589902407308519397791
322532872839799440092157462787931007338706058871143724880836676430
262079474796921326049935168780202640552536889442073486354755812306
038518359854101523684937699014194168483885534209329919367554111902
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0931953549589402452358064897741327037827389794440542652917787431005
2388251558821148229835386671430712574929796421371099480668730207714
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2897642038315726465954230504562618889041362158450679602016026608815
8752041713864738578233271366160568903148328629058772304502784917788
6220274489170898531136319639289561120942870808965003074276163587762
4206305748259572042701751909587421173466091966942127225526356781114
1743551949658004463379162995943318245636067762123713963527795334419
0156370555604349227606479890210543690809958440290285351989724137720
8737288974444504760746778243150288370655832119327938088662143521775
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