

Extending some categories to categories with families

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1 Categories with families

Definition 1. A *category with families* is given by

- A category \mathbb{C} with a terminal object $\mathbf{1}$,
- A functor $F = (\text{Ty}, \text{Tm}) : \mathbb{C}^{\text{op}} \rightarrow \text{Fam}(\text{Set})$. For the morphism part, we introduce the notation $_ \{ \cdot \}$ for both types and terms, i.e. if $f : \Delta \rightarrow \Gamma$ then $_ \{ f \} : \text{Ty}(\Gamma) \rightarrow \text{Ty}(\Delta)$ and for every $\sigma \in \text{Ty}(\Delta)$ we have $_ \{ f \} : \text{Tm}(\Delta, \sigma) \rightarrow \text{Tm}(\Gamma, \sigma \{ f \})$.
- For each object Γ in \mathbb{C} and $\sigma \in \text{Ty}(\Gamma)$ an object $\Gamma.\sigma$ together with a morphism $\mathbf{p}(\sigma) : \Gamma.\sigma \rightarrow \Gamma$ (the *first projection*) and a term $\mathbf{v}_\sigma \in \text{Tm}(\Gamma.\sigma, \sigma \{ \mathbf{p}(\sigma) \})$ (the *second projection*) with the following universal property: for each $f : \Delta \rightarrow \Gamma$ and $M \in \text{Tm}(\Delta, \sigma \{ f \})$ there exists a unique morphism $\theta = \langle f, M \rangle_\sigma : \Delta \rightarrow \Gamma.\sigma$ such that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma \{ \theta \} = M$.

2 Set

Directly from Dybjer [Dyb96], Hofmann [Hof97], Buisse and Dybjer [BD08],

...

Choose $\mathbb{C} = \text{Set}$ (with $\mathbf{1} = \mathbf{1}$ any singleton), and define

$$\begin{aligned}\text{Ty}(\Gamma) &= \{ \sigma \mid \sigma : \Gamma \rightarrow \text{Set} \} \\ \text{Tm}(\Gamma, \sigma) &= \prod_{\gamma \in \Gamma} \sigma(\gamma)\end{aligned}$$

(this should really be $\sigma : \Gamma \rightarrow U$ for some universe (U, T) for size considerations, and accordingly $\text{Tm}(\Gamma, \sigma) = \prod_{\gamma \in \Gamma} T(\sigma(\gamma))$). For $f : \Delta \rightarrow \Gamma$, $\sigma : \text{Ty}(\Gamma)$, $h :$

$\text{Tm}(\Gamma, \sigma)$, define

$$\begin{aligned}\sigma\{f\} &: \text{Ty}(\Delta) = \{\sigma \mid \sigma : \Delta \rightarrow \text{Set}\} \\ \sigma\{f\} &= \sigma \circ f \\ h\{f\} &: \text{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta)) \\ h\{f\} &= h \circ f\end{aligned}$$

For the context comprehension, define

$$\begin{aligned}\Gamma.\sigma &= \sum_{\gamma \in \Gamma} \sigma(\gamma) \\ \mathbf{p}(\sigma) &: \sum_{\gamma \in \Gamma} \sigma(\gamma) \rightarrow \Gamma \\ \mathbf{p}(\sigma)(\langle \gamma, s \rangle) &= \gamma \\ \mathbf{v}_\sigma \in \text{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\}) &= \prod_{\langle \gamma, s \rangle \in \Gamma.\sigma} \sigma(\gamma) \\ \mathbf{v}_\sigma(\langle \gamma, s \rangle) &= s\end{aligned}$$

Given $f : \Delta \rightarrow \Gamma$ and $M \in \text{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta))$, we define

$$\theta = \langle f, M \rangle_\sigma : \Delta \rightarrow \underbrace{\Gamma.\sigma}_{\sum_{\gamma \in \Gamma} \sigma(\gamma)}$$

by $\theta(\delta) = \langle f(\delta), M(\delta) \rangle$. We then have $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$, and any other function satisfying these equations must be extensionally equal to θ , hence θ is unique.

3 Fam(Set)

$\text{Fam}(\text{Set})$ can also be extended to a category with families. We start with $\mathbb{C} = \text{Fam}(\text{Set})$ (and $\mathbb{1} = (\mathbf{1}, \lambda x. \mathbf{1})$), and define

$$\text{Ty}(X, Y) = \{(A, B) \mid A : X \rightarrow \text{Set}, B : (x : X) \rightarrow Y(x) \rightarrow A(x) \rightarrow \text{Set}\}$$

$$\text{Tm}((X, Y), (A, B)) = \{(h, k) \mid h : \prod_{x \in X} A(x), k : \prod_{x \in X, y \in Y(x)} B(x, y, h(x))\}$$

(similar size considerations apply as for Set). For $(f, g) : (X, Y) \rightarrow (X', Y')$, $(A, B) : \text{Ty}(X', Y')$, $(h, k) : \text{Tm}((X', Y'), (A, B))$, define

$$\begin{aligned}(A, B)\{f, g\} &: \text{Ty}(X, Y) = \{(A, B) \mid A : X \rightarrow \text{Set}, B : (x : X) \rightarrow Y(x) \rightarrow A(x) \rightarrow \text{Set}\} \\ (A, B)\{f, g\} &= (A, B) \circ (f, g) = (A \circ f, \lambda x, y. B(f(x), g(x, y))) \\ (h, k)\{f, g\} &: \text{Tm}(\Delta, \sigma\{f\}) \\ (h, k)\{f, g\} &= (h, k) \circ (f, g) = (h \circ f, \lambda x, y. k(f(x), g(x, y)))\end{aligned}$$

For the context comprehension, define

$$\begin{aligned}(X, Y).(A, B) &= (\sum_{x \in X} A(x), \lambda \langle x, a \rangle . \sum_{y \in Y(x)} B(x, y, a)) \\ \mathbf{p}(A, B) &= (\text{fst}, \lambda x. \text{fst}) \\ \mathbf{v}_{A, B} &= (\text{snd}, \lambda x. \text{snd})\end{aligned}$$

Given $(f, g) : (X', Y') \rightarrow (X, Y)$ and $(h, k) \in \text{Tm}((X', Y'), (A, B)\{f, g\})$, we define

$$(\theta, \psi) = \langle (f, g), (h, k) \rangle_{(A, B)} : (X', Y') \rightarrow (X, Y).(A, B)$$

by $\theta(x) = \langle f(x), h(x) \rangle$, $\psi(x, y) = \langle g(x, y), k(x, y) \rangle$. It is clear that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$ and that these conditions force (θ, ψ) to be unique.

4 BiAlg(F, G) for $F, G : \mathbb{C} \rightarrow \mathbb{D}$

Lemma 2. *BiAlg(F, G) has a terminal object if \mathbb{C} and \mathbb{D} does, and G preserves terminal objects (i.e. $G(\mathbf{1}_{\mathbb{C}}) \cong \mathbf{1}_{\mathbb{D}}$).*

Proof. Define $\mathbf{1}_{\text{BiAlg}(F, G)} := (\mathbf{1}_{\mathbb{C}}, !_{F(\mathbf{1}_{\mathbb{C}})})$ where $!_{F(\mathbf{1}_{\mathbb{C}})}$ is the unique map $F(\mathbf{1}_{\mathbb{C}}) \rightarrow \mathbf{1}_{\mathbb{D}}$. For any object (X, f) , the unique morphism $(X, f) \rightarrow (\mathbf{1}_{\mathbb{C}}, !_{F(\mathbf{1}_{\mathbb{C}})})$ is given by the unique arrow $!_X$ from X to $\mathbf{1}_{\mathbb{C}}$ in \mathbb{C} , and the diagram

$$\begin{array}{ccc} FX & \xrightarrow{f} & GX \\ \downarrow F(!_X) & & \downarrow G(!_X) \\ F(\mathbf{1}_{\mathbb{C}}) & \xrightarrow{!_{F(\mathbf{1}_{\mathbb{C}})}} & G(\mathbf{1}_{\mathbb{C}}) = \mathbf{1}_{\mathbb{D}} \end{array}$$

commutes since both paths are arrows into $\mathbf{1}_{\mathbb{D}}$, hence equal. \square

4.1 Some CwF preliminaries

Clairambault [Cla06, 4.1] defines a category $\text{Type}_{\mathbb{C}}(\Gamma)$ of types in context Γ from the base category \mathbb{C} . The morphisms between $A, B \in \text{Ty}_{\mathbb{C}}(\Gamma)$ are defined to be the terms $f \in \text{Tm}_{\mathbb{C}}(\Gamma.A, B\{\mathbf{p}(A)\})$, with identity given by \mathbf{v}_A . We will be mostly interested in the composition of two terms $f \in \text{Tm}_{\mathbb{C}}(\Gamma.A, B\{\mathbf{p}(A)\})$ and $g \in \text{Tm}_{\mathbb{C}}(\Gamma.B, C\{\mathbf{p}(B)\})$, which is defined to be

$$g \bullet f := g\{\langle \mathbf{p}(A), f \rangle_B\}.$$

The following proposition says that comprehension is a functor from “families in \mathbb{C} ” to \mathbb{C} , which is quite convenient.

Lemma 3. *Given $g : \Gamma' \rightarrow \Gamma$ and $M \in \text{Tm}(\Gamma'.\sigma', \sigma\{g \circ \mathbf{p}(\sigma')\})$, one can construct $\psi : \Gamma'.\sigma' \rightarrow \Gamma.\sigma$.*

Proof. Take $\psi := \langle g \circ \mathbf{p}(\sigma'), M \rangle_\sigma$. \square

Lemma 4. Let $f : \Delta \rightarrow \Gamma$, $M \in \text{Tm}(\Delta, \sigma\{f\})$, $h : \Theta \rightarrow \Delta$. Then $\langle f, M \rangle_\sigma \circ h = \langle f \circ h, M\{h\} \rangle_{\sigma\{f\}}$.

Proof. $\langle f, M \rangle_\sigma \circ h$ satisfies the universal property for $f \circ h$ and $M\{h\}$. \square

Lemma 5. For every $M \in \text{Tm}(\Gamma, \sigma)$, there is $\overline{M} : \Gamma \rightarrow \Gamma.\sigma$ such that $\mathbf{p}(\sigma) \circ \overline{M} = \text{id}$ and $\mathbf{v}_\sigma\{\overline{M}\} = M$.

Proof. There is no choice but to define $\overline{M} := \langle \text{id}, M \rangle_\sigma$. \square

4.2 Some box preliminaries

In order to extend $\text{BiAlg}(F, G)$ to a CwF (with $F, G : \mathbb{C} \rightarrow \mathbb{D}$), we assume that \mathbb{C} and \mathbb{D} are CwFs , and that \square_F and \square_G exist and satisfy certain requirements. We collect these here.

Definition 6. Assume that \mathbb{C} and \mathbb{D} are CwFs . They *has boxes*, if for each $F : \mathbb{C} \rightarrow \mathbb{D}$, $\Gamma \in \mathbb{C}$ and $\sigma \in \text{Ty}(\Gamma)$, there exists $\square_F(\Gamma, \sigma) \in \text{Ty}(F(\Gamma))$ and an isomorphism $\varphi_F : F(\Gamma.\sigma) \rightarrow F(\Gamma).\square_F(\Gamma, \sigma)$ such that $\mathbf{p} \circ \varphi = F(\mathbf{p})$.

For every $f \in \text{Tm}(\Gamma, \sigma)$, we also demand the existence of $\text{dmap}_F(\Gamma, \sigma, f) \in \text{Tm}(F(\Gamma), \square_F(\Gamma, \sigma))$ such that $\varphi_F \circ F(f) = \text{dmap}_F(\Gamma, \sigma, f)$.

Finally, we also require a morphism part of \square_F , namely that for $f : \Delta \rightarrow \Gamma$, $M \in \text{Tm}(\Delta, \sigma\{f\})$ we have $\square_F(f, M) \in \text{Tm}(F(\Delta), \square_F(\Gamma, \sigma)\{F(f)\})$ with “naturality condition” $\varphi_F \circ F(\langle f, M \rangle) = \langle F(f), \square_F(f, M) \rangle$.

Assume that \mathbb{C} and \mathbb{D} are CwFs with boxes. We assume that for $f : \Delta \rightarrow \Gamma$, we have

$$\square_G(\Gamma, \sigma)\{G(f)\} = \square_G(\Delta, \sigma\{f\}) \quad (1)$$

and similarly $\square_G(\text{id}_\Gamma, N)\{G(f)\} = \square_G(\text{id}_\Delta, N\{f\})$ for all $N \in \text{Tm}(\Gamma, \sigma)$ and $f : \Delta \rightarrow \Gamma$. Here, $\square_G(\text{id}_\Gamma, N)\{G(f)\} \in \text{Tm}(G(\Delta), \square_G(\Gamma, \sigma)\{G(f)\})$ and $\square_G(\text{id}_\Delta, N\{f\}) \in \text{Tm}(G(\Delta), \square_G(\Delta, \sigma\{f\}))$, so both sides of the equation have the same type by (1).

Remark 7. Demanding equality on the nose instead of isomorphism simplifies matters – we are spared transporting terms hidden inside substitutions along the isomorphisms. I guess it should be possible in principle though.

However, with the usual definition of \square_G , one (almost) never has equality. (In Set , for example, the left hand side is $\{y : G(\Sigma \Gamma \sigma) \mid \dots\}$ and the right hand side $\{y : G(\Sigma \Delta (\sigma \circ f)) \mid \dots\}$.) If $G = U$ is a forgetful functor, though, then the usual definition of $\square_U(\Gamma, \sigma)$ is isomorphic to a $X(\Gamma, \sigma)$ such that $X(\Gamma, \sigma)\{U(f)\} = X(\Delta, \sigma\{f\})$. I see no harm in replacing $\square_U(\Gamma, \sigma)$ with $X(\Gamma, \sigma)$ for U , so that we get an actual equality? The properties we need \square_U to have are of course preserved by isomorphism anyway.

For F , we only require the existence of

$$\phi_F(f) \in \text{Tm}(F(\Delta).\square_F(\Delta, \sigma\{f\}), \square_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}(\square_F(\Delta, \sigma\{f\}))\})$$

which should be functorial in f , i.e. $\phi_F(\text{id}) = \mathbf{v}_\sigma$ and

$$\phi_F(f \circ g) = \phi_F(f)\{\langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle\}.$$

Remark 8. $\phi_F(f)$ and $\phi_F(g)$ are not composable in $\text{Type}(F(\Gamma))$, as their types depend on f and g , but the “composition” above should be composition in some more liberal category (where \mathbf{v}_σ still is the identity)? It is in any case exactly what we need, and holds e.g. in Set (I have not checked $\text{Fam}(\text{Set})$, but would be very surprised if it did not hold).

We assume that \square_F , $\phi_F(f)$ and substitution relate in the following ways: for every $f : \Delta \rightarrow \Gamma$, $M \in \text{Tm}(\Gamma, \sigma\{f\})$ and $g : \Theta \rightarrow \Delta$, we have

- $\square_F(f, M)\{F(g)\} = \square_F(f \circ g, M\{g\})$ (“ $\square_F(f \circ g) = \square_F(f) \circ \square_F(g)$ ”),
- $\square_F(\mathbf{p}, \mathbf{v}_\sigma) = \mathbf{v}_{\square_F(\Gamma, \sigma)}\{\varphi_F\}$ (“ $\square_F(\text{id}) = \text{id}$ ”),
- $\phi_F(f)\{\overline{\square_F(\text{id}, M)}\} = \square_F(f, M)$.

4.3 The construction

4.3.1 Types

Define

$$\text{Ty}_{\text{BiAlg}(F, G)}(\Gamma, h) = \{(\sigma, M) \mid \sigma \in \text{Ty}_{\mathbb{C}}(\Gamma), M \in \text{Tm}_{\mathbb{D}}(F(\Gamma). \square_F(\Gamma, \sigma), \square_G(\Gamma, \sigma)\{h \circ \mathbf{p}\})\}$$

For substitutions, assume $f : (\Delta, h') \rightarrow (\Gamma, h)$, i.e. $f : \Delta \rightarrow \Gamma$ and $G(f) \circ h' = h \circ F(f)$. Define for $(\sigma, M) \in \text{Ty}_{\text{BiAlg}(F, G)}(\Gamma, h)$

$$(\sigma, M)\{f\} = (\sigma\{f\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}) \in \text{Ty}_{\text{BiAlg}(F, G)}(\Delta, h')$$

We should check that this makes sense. Since $\sigma \in \text{Ty}_{\mathbb{C}}(\Gamma)$, we have $\sigma\{f\} \in \text{Ty}_{\mathbb{C}}(\Delta)$. We now need a term in $\text{Tm}_{\mathbb{D}}(F(\Delta). \square_F(\Delta, \sigma\{f\}), \square_G(\Delta, \sigma\{f\})\{h' \circ \mathbf{p}\})$. Since $F(f) : F(\Delta) \rightarrow F(\Gamma)$ and

$$\phi_F \in \text{Tm}(F(\Delta). \square_F(\Delta, \sigma\{f\}), \square_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}\}),$$

Lemma 3 applies and we get $g := \langle F(f) \circ \mathbf{p}, \phi_F \rangle : F(\Delta). \square_F(\Delta, \sigma\{f\}) \rightarrow F(\Gamma). \square_F(\Gamma, \sigma)$, so that

$$M\{g\} \in \text{Tm}_{\mathbb{D}}(F(\Delta). \square_F(\Delta, \sigma\{f\}), \square_G(\Gamma, \sigma)\{h \circ \mathbf{p} \circ g\})$$

and since

$$h \circ \mathbf{p} \circ g = h \circ \mathbf{p} \circ \langle F(f) \circ \mathbf{p}, \phi_F \rangle = h \circ F(f) \circ \mathbf{p} = G(f) \circ h' \circ \mathbf{p}$$

and $\square_G(\Gamma, \sigma)\{G(f)\} = \square_G(\Delta, \sigma\{f\})$, we in fact have

$$M\{g\} \in \text{Tm}_{\mathbb{D}}(F(\Delta). \square_F(\Delta, \sigma\{f\}), \square_G(\Delta, \sigma\{f\})\{h' \circ \mathbf{p}\})$$

as needed. Functoriality follows from functoriality of $\phi_F(f)$ and functoriality one level down:

$$\begin{aligned} (\sigma, M)\{\text{id}\} &= (\sigma\{\text{id}\}, M\{\langle F(\text{id}) \circ \mathbf{p}, \phi_F(\text{id}) \rangle\}) \\ &= (\sigma, M\{\langle \mathbf{p}, \mathbf{v}_\sigma \rangle\}) = (\sigma, M\{\text{id}\}) = (\sigma, M) \end{aligned}$$

$$\begin{aligned}
(\sigma, M)\{f\}\{g\} &= (\sigma\{f\}\{g\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}\{\langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle\}) \\
&= (\sigma\{f \circ g\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle \circ \langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle\}) \\
&= (\sigma\{f \circ g\}, M\{\langle F(f) \circ \mathbf{p} \circ \langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle, \phi_F(f) \rangle\{\langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle\}\}) \\
&= (\sigma\{f \circ g\}, M\{\langle F(f) \circ F(g) \circ \mathbf{p}, \phi_F(f \circ g) \rangle\}) \\
&= (\sigma\{f \circ g\}, M\{\langle F(f \circ g) \circ \mathbf{p}, \phi_F(f \circ g) \rangle\}) \\
&= (\sigma, M)\{f \circ g\}
\end{aligned}$$

4.3.2 Terms

Define

$$\text{Tm}((\Gamma, h), (\sigma, M)) = \{N \in \text{Tm}_{\mathbb{C}}(\Gamma, \sigma) \mid \square_G(\text{id}_{\Gamma}, N)\{h\} = M\{\overline{\square_F(\text{id}_{\Gamma}, N)}\}\}$$

If $f : (\Delta, h') \rightarrow (\Gamma, h)$, we define $N\{f\}$ for $N \in \text{Tm}((\Gamma, h), (\sigma, M))$ to be $N\{f\}$ inherited from \mathbb{C} . We have to check that $N\{f\} \in \text{Tm}((\Delta, h'), (\sigma, M)\{f\}) = \text{Tm}((\Delta, h'), (\sigma\{f\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}))$, i.e. that

$$\square_G(\text{id}_{\Delta}, N\{f\})\{h'\} = M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}\{\overline{\square_F(\text{id}_{\Delta}, N\{f\})}\}$$

given that $\square_G(\text{id}_{\Gamma}, N)\{h\} = M\{\overline{\square_F(\text{id}_{\Gamma}, N)}\}$. We calculate

$$\begin{aligned}
\square_G(\text{id}_{\Delta}, N\{f\})\{h'\} &= \square_G(\text{id}_{\Gamma}, N)\{G(f)\}\{h'\} \\
&= \square_G(\text{id}_{\Gamma}, N)\{h\}\{F(f)\} \\
&= M\{\langle \text{id}_{F(\Gamma)}, \square_F(\text{id}_{\Gamma}, N) \rangle\}\{F(f)\} \\
&= M\{\langle F(f), \square_F(\text{id}_{\Gamma}, N)\{F(f)\} \rangle\} \\
&= M\{\langle F(f), \square_F(f, N\{f\}) \rangle\} \\
&= M\{\langle F(f), \phi_F(f)\{\overline{\square_F(\text{id}, N\{f\})}\} \rangle\} \\
&= M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}\{\overline{\square_F(\text{id}, N\{f\})}\}.
\end{aligned}$$

4.3.3 Context comprehension

Given $(\Gamma, h) \in \text{BiAlg}(F, G)$ and $(\sigma, M) \in \text{Ty}(\Gamma, h)$, we define

$$\begin{aligned}
(\Gamma, h).(\sigma, M) &:= (\Gamma, \sigma, \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F) \\
\mathbf{p}((\sigma, M)) &:= \mathbf{p}(\sigma) \\
\mathbf{v}_{(\sigma, M)} &:= \mathbf{v}_{\sigma} \\
\langle f, N \rangle_{(\sigma, M)} &:= \langle f, N \rangle_{\sigma}
\end{aligned}$$

We have to check that (i) $\mathbf{p}((\sigma, M)) : (\Gamma, h).(\sigma, M) \rightarrow (\Gamma, h)$, (ii) $\mathbf{v}_{(\sigma, M)} \in \text{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{\mathbf{p}((\sigma, M))\})$ and that (iii) $\langle f, N \rangle_{\sigma} : (\Delta, h') \rightarrow (\Gamma, h).(\sigma, M)$ when $f : (\Delta, h') \rightarrow (\Gamma, h)$ and $N \in \text{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{p\})$.

For (i), we need that $G(\mathbf{p}) \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F = h \circ F(\mathbf{p})$. But $G(\mathbf{p}) = \mathbf{p} \circ \varphi_G$, so

$$\begin{aligned} G(\mathbf{p}) \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F &= \mathbf{p} \circ \varphi_G \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \\ &= \mathbf{p} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \\ &= h \circ \mathbf{p} \circ \varphi_F \\ &= h \circ F(\mathbf{p}). \end{aligned}$$

Hence $\mathbf{p}((\sigma, M)) : (\Gamma, h).(\sigma, M) \rightarrow (\Gamma, h)$. For (ii), we need to show that $\mathbf{v}_\sigma \in \text{Tm}((\Gamma, \sigma, \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F), (\sigma\{\mathbf{p}\}, M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p}) \rangle\}))$, i.e. that $\square_G(\text{id}_\Gamma, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} = M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p}) \rangle\}\{\square_F(\text{id}_\Gamma, \mathbf{v}_\sigma)\}$. First, note that $\square_G(\text{id}_\Gamma, \mathbf{v}_\sigma) = \square_G(\mathbf{p}(\sigma), \mathbf{v}_\sigma)$ because

$$\begin{aligned} \square_G(\text{id}, \mathbf{v}) &= \square_G(\mathbf{p} \circ \bar{\mathbf{v}}, \mathbf{v}\{\bar{\mathbf{v}}\}) = \square_G(\mathbf{p}, \mathbf{v})\{G(\bar{\mathbf{v}})\} \\ &= \square_G(\mathbf{p}, \mathbf{v}\{\bar{\mathbf{v}}\}) = \square_G(\mathbf{p}, \mathbf{v}) \end{aligned}$$

Now

$$\begin{aligned} \square_G(\text{id}_\Gamma, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} &= \square_G(\mathbf{p}, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= \mathbf{v}_\sigma\{\varphi_G \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= \mathbf{v}_\sigma\{\langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= M\{\varphi_F\} \end{aligned}$$

but also

$$\begin{aligned} M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p}) \rangle\}\{\overline{\square_F(\text{id}_\Gamma, \mathbf{v}_\sigma)}\} &= M\{\langle F(\mathbf{p}), \phi_F(\mathbf{p})\{\overline{\square_F(\text{id}_\Gamma, \mathbf{v}_\sigma)}\}\rangle\} \\ &= M\{\langle F(\mathbf{p}), \square_F(\mathbf{p}, \mathbf{v}_\sigma)\rangle\} \\ &= M\{\varphi_F \circ F(\langle \mathbf{p}, \mathbf{v}_\sigma \rangle)\} \\ &= M\{\varphi_F \circ F(\text{id})\} \\ &= M\{\varphi_F\}. \end{aligned}$$

Finally, we have to check that $\langle f, N \rangle$ really is a morphism. (Uniqueness of $\langle f, N \rangle$ is of course inherited from \mathbb{C} .) We are given $f : (\Delta, h') \rightarrow (\Gamma, h)$ and $N \in \text{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{p\})$, that is we have $G(f) \circ h' = h \circ F(f)$ and

$$\square_G(\text{id}_\Delta, N)\{h'\} = M\{\langle F(f) \circ \mathbf{p}, \phi_F(f) \rangle\}\{\overline{\square_F(\text{id}_\Delta, N)}\}$$

which simplifies to

$$\square_G(\text{id}_\Delta, N)\{h'\} = M\{\langle F(f), \square_F(f, N) \rangle\}$$

We also need that $\square_G(\text{id}_\Delta, N) = \square_G(f, N)$ (they have the same type thanks to our assumption $\square_G(\Gamma, \sigma)\{G(f)\} = \square_G(\Delta, \sigma\{f\})$) – can this be proven from the facts we have or should it be added to our global list of assumptions? Thus, we have

$$\square_G(f, N)\{h'\} = \square_G(\text{id}, N)\{h'\} = M\{\langle F(f), \square_F(f, N) \rangle\}$$

We have to show that

$$G(\langle f, N \rangle) \circ h' = \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \circ F(\langle f, N \rangle)$$

or equivalently

$$\varphi_G \circ G(\langle f, N \rangle) \circ h' = \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \circ F(\langle f, N \rangle)$$

We have

$$\begin{aligned} \varphi_G \circ G(\langle f, N \rangle) \circ h' &= \langle G(f), \square_G(f, N) \rangle \circ h' \\ &= \langle G(f) \circ h', \square_G(f, N)\{h'\} \rangle \\ &= \langle h \circ F(f), M\{\langle F(f), \square_F(f, N) \rangle\} \rangle \end{aligned}$$

and also

$$\begin{aligned} \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \circ F(\langle f, N \rangle) &= \langle h \circ \mathbf{p}, M \rangle \circ \langle F(f), \square_F(f, N) \rangle \\ &= \langle h \circ F(f), M\{\langle F(f), \square_F(f, N) \rangle\} \rangle \end{aligned}$$

and we are done.

References

- [BD08] Alexandre Buisse and Peter Dybjer. The interpretation of intuitionistic type theory in locally cartesian closed categories – an intuitionistic perspective. *Electronic Notes in Theoretical Computer Science*, 218:21–32, 2008.
- [Cla06] Pierre Clairambault. From categories with families to locally cartesian closed categories. Technical report, ENS Lyon, 2006.
- [Dyb96] Peter Dybjer. Internal type theory. *Lecture Notes in Computer Science*, 1158:120–134, 1996.
- [Hof97] Martin Hofmann. Syntax and semantics of dependent types. In Andrew Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 79 – 130. Cambridge University Press, 1997.