

# Generalising $\square_F$ and $\text{dmap}_F$ for non-endofunctors

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## 1 The standard situation

In the ordinary situation, we have an endofunctor  $F : \text{Set} \rightarrow \text{Set}$  representing an inductive definition, i.e. we have a set  $\mu F$  and a constructor  $c : F(\mu F) \rightarrow \mu F$ . The eliminator has type

$$\begin{aligned} \text{elim}_F : (P : \mu F \rightarrow \text{Set}) \rightarrow (g : (x : F(\mu F)) \rightarrow \square_F(\mu F, P, x) \rightarrow P(c(x))) \\ \rightarrow (x : \mu F) \rightarrow P(x) \end{aligned}$$

and computation rule

$$\text{elim}_F(P, g, c(x)) = g(x, \text{dmap}_F(\mu F, P, \text{elim}_F(P, g), x)).$$

Here,

$$\square_F : (A : \text{Set}) \rightarrow (P : A \rightarrow \text{Set}) \rightarrow F(A) \rightarrow \text{Set}$$

gives the induction hypothesis (it is called ‘IH’ in the induction-recursion papers) and

$$\begin{aligned} \text{dmap}_F : (A : \text{Set}) \rightarrow (P : A \rightarrow \text{Set}) \rightarrow ((x : A) \rightarrow P(x)) \\ \rightarrow (x : F(A)) \rightarrow \square_F(A, P, x) \end{aligned}$$

takes care of the recursive calls (in the IR papers, it is called  $\text{mapIH}$ ). Both  $\square_F$  and  $\text{dmap}_F$  can be explicitly defined:

$$\begin{aligned} \square_F(A, P, x) &= \{y : F(\Sigma A P) \mid F(\pi_0)(y) = x\} \\ \text{dmap}_F(A, P, g, x) &= F(\widehat{g})(x) \end{aligned}$$

where  $\widehat{g} : A \rightarrow \Sigma A P$  is defined by  $\widehat{g}(z) = \langle z, g(z) \rangle$ . Note that  $\pi_0 \circ \widehat{g} = \text{id}$ , so that

$$F(\pi_0)(F(\widehat{g})(x)) = F(\pi_0 \circ \widehat{g})(x) = F(\text{id})(x) = \text{id}(x) = x$$

which shows that indeed  $\text{dmap}_F(A, P, g, x) : \square_F(A, P, x)$ . One can show that this definition of  $\square_F$  is naturally isomorphic to the definition in the IR papers (see Problem 2 in [problems.pdf](#)).

However, we prefer to just use some abstract properties of  $\square_F$  and  $\text{dmap}_F$ , namely

$$F(\Sigma A B) \cong \Sigma F(A) \square_F(A, B) \quad (*\square)$$

and this isomorphism  $\varphi$  satisfies

$$\begin{array}{ccc} F(\Sigma A B) & \xrightarrow{\varphi} & \Sigma F(A) \square_F(A, B) \\ F(\pi_0) \downarrow & \swarrow \pi_0 & \\ F(A) & & \end{array} \quad (**\square)$$

i.e.  $\pi_0 \circ \varphi = F(\pi_0)$ . For  $\text{dmap}_F$ , we have for all  $f : (x : A) \rightarrow B(x)$

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\widehat{f})} & F(\Sigma A B) \\ \widehat{\text{dmap}_F(f)} \downarrow & \swarrow \varphi & \\ \Sigma F(A) \square_F(A, B) & & \end{array} \quad (*\text{dmap})$$

i.e.  $\widehat{\text{dmap}_F(f)} = \varphi \circ F(\widehat{f})$ .

## 2 Generalising $\square_F$ and $\text{dmap}_F$ for non-endofunctors

Let us now try to generalise things, first by moving away from the category  $\text{Set}$  to an arbitrary category  $\mathbb{C}$  (with some structure, of course), and then by replacing the endofunctor  $F : \mathbb{C} \rightarrow \mathbb{C}$  with just a functor  $F : \mathbb{C} \rightarrow \mathbb{D}$ .

### 2.1 The category $\llbracket \text{Fam} \rrbracket(\mathbb{C})$ of interpretations of families in $\mathbb{C}$

Let us for the moment assume that the category  $\mathbb{C}$  is a model of type theory. I want to define a category  $\llbracket \text{Fam} \rrbracket(\mathbb{C})$  of “interpretations of families in  $\mathbb{C}$ ”. It has as objects an interpretation of ‘ $A : \text{Set}, B : A \rightarrow \text{Set}$ ’ in  $\mathbb{C}$ , and a morphism from  $\llbracket A, B \rrbracket$  to  $\llbracket A', B' \rrbracket$  is an interpretation of ‘ $f : A \rightarrow A', g : (x : A) \rightarrow B(x) \rightarrow B'(f(x))$ ’.

Assuming that the interpretation of ‘ $A : \text{Set}$ ’ in  $\mathbb{C}$  is  $\llbracket A \rrbracket : \mathbb{C}$ , there should also be a forgetful functor  $U : \llbracket \text{Fam} \rrbracket(\mathbb{C}) \rightarrow \mathbb{C}$  defined by  $U(\llbracket A \rrbracket, \llbracket B \rrbracket) = \llbracket A \rrbracket$  and  $U(\llbracket f \rrbracket, \llbracket g \rrbracket) = \llbracket f \rrbracket$ .

**Example 1** ( $\mathbb{C} = \text{Set}$ ). For  $\mathbb{C} = \text{Set}$ , we recover the usual category  $\text{Fam}(\text{Set})$  of families of sets, i.e. we have objects pairs  $(A, B)$  where  $A : \text{Set}$  and  $B : A \rightarrow \text{Set}$ , and a morphism  $(A, B) \rightarrow (A', B')$  is a pair  $f : A \rightarrow A', g : (x : A) \rightarrow B(x) \rightarrow B'(f(x))$ .

The forgetful functor  $U$  is the usual forgetful functor  $U : \text{Fam}(\text{Set}) \rightarrow \text{Set}$ ;  $U(A, B) = A$ ,  $U(f, g) = f$ .

**Example 2** ( $\mathbb{C} = \mathbf{Fam}(\mathbf{Set})$ ). Unless I am mistaken, the objects in  $\llbracket \mathbf{Fam} \rrbracket(\mathbf{Fam}(\mathbf{Set}))$  are tuples  $((A_0, A_1), (B_0, B_1))$  where

$$\begin{aligned} A_0 &: \mathbf{Set}, \\ A_1 &: A_0 \rightarrow \mathbf{Set}, \\ B_0 &: A_0 \rightarrow \mathbf{Set}, \\ B_1 &: (x : A_0) \rightarrow A_1(x) \rightarrow B_0(x) \rightarrow \mathbf{Set}. \end{aligned}$$

A morphism from  $((A_0, A_1), (B_0, B_1))$  to  $((A'_0, A'_1), (B'_0, B'_1))$  consists of

$$\begin{aligned} f_0 &: A_0 \rightarrow A'_0, \\ f_1 &: (x : A_0) \rightarrow A_1(x) \rightarrow A'_1(f_0(x)), \\ g_0 &: (x : A_0) \rightarrow B_0(x) \rightarrow B'_0(f_0(x)) \\ g_1 &: (x : A_0) \rightarrow (y : A_1(x)) \rightarrow (z : B_0(x)) \rightarrow B_1(x, y, z) \\ &\quad \rightarrow B'_1(f_0(x), f_1(x, y), g_0(x, z)). \end{aligned}$$

$$U((A_0, A_1), (B_0, B_1)) = (A_0, A_1), \quad U((f_0, f_1), (g_0, g_1)) = (f_0, f_1).$$

**Question 1.** Is there a better way to present  $\llbracket \mathbf{Fam} \rrbracket(\mathbb{C})$ ?

## 2.2 $\Sigma$ as a functor from $\llbracket \mathbf{Fam} \rrbracket(\mathbb{C})$ to $\mathbb{C}$

Now, I notice that the sigma constructor takes a family  $(A, B)$  of sets and gives me a set  $\Sigma A B$  back. It is also functorial, since if I have a morphism  $(f, g) : (A, B) \rightarrow (A', B')$ , then I can construct a morphism  $[f, g] : \Sigma A B \rightarrow \Sigma A' B'$  by defining

$$[f, g]\langle a, b \rangle = \langle f(a), g(a, b) \rangle.$$

The projection  $\pi_0$  takes me from  $\Sigma A B$  to  $A = U(A, B)$ , and  $\pi_0 \circ [f, g] = f \circ \pi_0$ . It is thus a natural transformation  $\pi_0 : \Sigma \rightarrow U$ .

In the general case, I would thus like to generalise  $\Sigma$  to a functor

$$\Sigma_{\mathbb{C}} : \llbracket \mathbf{Fam} \rrbracket(\mathbb{C}) \rightarrow \mathbb{C}$$

together with a natural transformation  $\pi_0 : \Sigma_{\mathbb{C}} \rightarrow U$ .

**Question 2.** Do we need  $\pi_1 : (x : \Sigma_{\mathbb{C}}(A, B)) \rightarrow B(\pi_0(x))$  as well?

**Question 3.** I probably would like  $\Sigma_{\mathbb{C}}$  to be the interpretation of  $\Sigma$  in  $\mathbb{C}$ . For morphisms, this would mean that I want  $\Sigma_{\mathbb{C}}(f, g)$  to be the interpretation of  $[f, g]$ ?

## 2.3 $\square$ is a functor $\square' : \mathbf{Set}^{\mathbf{Set}} \rightarrow \mathbf{Fam}(\mathbf{Set})^{\mathbf{Fam}(\mathbf{Set})}$

Looking at the type

$$\square_F : (A : \mathbf{Set}) \rightarrow (P : A \rightarrow \mathbf{Set}) \rightarrow F(A) \rightarrow \mathbf{Set}$$

of  $\square_F$  again, we see that we can write this as

$$\square' : \text{Set}^{\text{Set}} \rightarrow \text{Fam}(\text{Set})^{\text{Fam}(\text{Set})}$$

if we define  $\square'(F) = \lambda(A, B). (F(A), \square_F(A, B))$ . For  $\square'(F)$  to be a functor  $\text{Fam}(\text{Set}) \rightarrow \text{Fam}(\text{Set})$ , it must also act on morphisms. Suppose  $(f, g) : (A, B) \rightarrow (A', B')$ . We must find a morphism from  $\square'(F)(A, B)$  to  $\square'(F)(A', B')$ , i.e. from

$$(F(A), \{y : F(\Sigma A B) \mid F(\pi_0)(y) = x\})$$

to

$$(F(A'), \{y : F(\Sigma A' B') \mid F(\pi_0)(y) = x\}).$$

But  $f : A \rightarrow A'$ , so  $m = F(f) : F(A) \rightarrow F(A')$  can be our first component. Now we need to define

$$\begin{aligned} n : (x : F(A)) &\rightarrow \{y : F(\Sigma A B) \mid F(\pi_0)(y) = x\} \\ &\rightarrow \{z : F(\Sigma A' B') \mid F(\pi_0)(z) = F(f)(x)\} \end{aligned}$$

Let  $n(x) = F([f, g])$ . We need to prove that  $F(\pi_0)(F([f, g])(y)) = F(f)(x)$  for  $y : F(\Sigma A B)$  such that  $F(\pi_0)(y) = x$ . But  $\pi_0 \circ [f, g] = f \circ \pi_0$ , so

$$F(\pi_0)(F([f, g])(y)) = F(\pi_0 \circ [f, g])(y) = F(f \circ \pi_0)(y) = F(f)(F(\pi_0)(y)) = F(f)(x).$$

Thus we can take  $\square'(F)(f, g) = (n, m) = (F(f), \lambda x. F([f, g]))$ . This shows that  $\square'(F) : \text{Fam}(\text{Set})^{\text{Fam}(\text{Set})}$  if  $F : \text{Set} \rightarrow \text{Set}$  is a functor.

However, we want more! We want  $\square'$  to be a functor  $\square' : \text{Set}^{\text{Set}} \rightarrow \text{Fam}(\text{Set})^{\text{Fam}(\text{Set})}$ . We have defined the object part of  $\square'$ , what is the morphism part? Given a natural transformation  $\eta : F \rightarrow G$ , we have to construct a natural transformation  $\square'(\eta) : \square'(F) \rightarrow \square'(G)$ . The component of  $\square'(\eta)$  at  $(A, B)$  consists of a function  $f : F(A) \rightarrow G(A)$  and a function  $g : (x : F(A)) \rightarrow \square_F(A, B, x) \rightarrow \square_G(A, B, f(x))$ . For  $f$ , we can choose  $f = \eta_A$ . Choose  $g(x) = \eta_{\Sigma A B}$ . Certainly  $\eta_{\Sigma A B} : F(\Sigma A B) \rightarrow G(\Sigma A B)$ , but do we have  $G(\pi_0)(\eta_{\Sigma A B}(y)) = \eta_a(x)$ , given that  $F(\pi_0)(y) = x$ ? Yes, because substituting  $x = F(\pi_0)(y)$ , we end up with the equation  $G(\pi_0)(\eta_{\Sigma A B}(y)) = \eta_a(F(\pi_0)(y))$ , which is exactly the naturality condition!

$$\begin{array}{ccc} F(\Sigma A B) & \xrightarrow{\eta_{\Sigma A B}} & G(\Sigma A B) \\ \downarrow F(\pi_0) & & \downarrow G(\pi_0) \\ F(A) & \xrightarrow{\eta_A} & G(A) \end{array}$$

Hence we can define  $\square'(\eta)_{(A, B)} = (\eta_A, \lambda x. \eta_{\Sigma A B})$ . The naturality of  $\square'(\eta)$  follows directly from the naturality of  $\eta$ .

Now that  $\Sigma$  and  $\square$  can be seen as functors, we can replace the isomorphism  $\varphi$  in  $(*\square)$  with a natural isomorphism  $\eta : F \circ \Sigma \rightarrow \Sigma \circ \square$ . One can easily check

that the isomorphism given in Problem 1 in fact is natural:

$$\begin{array}{ccc}
F(\Sigma A B) & \xrightarrow{\eta_{(A,B)}} & \Sigma F(A) \square_F(A, B) \\
F([f,g]) \downarrow & & \downarrow [F(f), [F([f,g])]] \\
F(\Sigma A' B') & \xrightarrow{\eta_{(A',B')}} & \Sigma F(A') \square_F(A', B')
\end{array}$$

## 2.4 A generalised $\square_F$ for $F$ not an endofunctor

Assume that  $\mathbb{C}$  and  $\mathbb{D}$  are categories such that  $\llbracket \text{Fam} \rrbracket(\mathbb{C})$  and  $\llbracket \text{Fam} \rrbracket(\mathbb{D})$  make sense. (Then  $\Sigma_{\mathbb{C}}$  and  $\Sigma_{\mathbb{D}}$  should make sense as well.) We can then generalise  $\square_F$  to a functor

$$\square : \mathbb{D}^{\mathbb{C}} \rightarrow \llbracket \text{Fam} \rrbracket(\mathbb{C}) \rightarrow \llbracket \text{Fam} \rrbracket(\mathbb{C})$$

such that there is a natural isomorphism

$$\eta : F \circ \Sigma_{\mathbb{C}} \xrightarrow{\sim} \Sigma_{\mathbb{D}} \circ \square_F \quad (*\square)$$

satisfying

$$\begin{array}{ccc}
F(\Sigma_{\mathbb{C}} A B) & \xrightarrow{\eta_{(A,B)}} & \Sigma_{\mathbb{D}} F(A) \square_F(A, B) \\
F(\pi_0) \downarrow & \swarrow \pi_0 & \\
F(A) & & 
\end{array} \quad (**\square)$$

i.e.  $\pi_0 \circ \eta_{(A,B)} = F(\pi_0)$ .

## 2.5 The Inverse image

If  $\mathbb{C}$  also has an inverse image operation which maps  $f : X \rightarrow Y$  to  $f^* : \llbracket \text{Fam} \rrbracket(\mathbb{C})$  with  $U(f^*) = Y$ , we can define  $\square_F(X) = (F(U(X)), (F(\pi_{0_X}))^*)$ .

We have to check that  $(*\square)$  and  $(**\square)$  hold. ... more conditions on  $\cdot^*$  needed (corresponding to being left adjoint to reindexing) ...

## 3 $\Sigma$ in the relevant categories

Let us write  $\text{Pow}_{\mathbb{C}}(X)$  for the interpretation of  $X \rightarrow \text{Set}$  in  $\mathbb{C}$ .

**Definition 3.** A category  $\mathbb{C}$  has *sigma types* if  $\llbracket \text{Fam} \rrbracket(\mathbb{C})$  and a forgetful functor  $U : \llbracket \text{Fam} \rrbracket(\mathbb{C}) \rightarrow \mathbb{C}$  exists, together with a functor  $\Sigma_{\mathbb{C}} : \llbracket \text{Fam} \rrbracket(\mathbb{C}) \rightarrow \mathbb{C}$  and a natural transformation  $\pi_0 : \Sigma_{\mathbb{C}} \rightarrow U$ .

**Definition 4.** Let  $\mathbb{C}, \mathbb{D}$  be categories having sigma types. We say that  $\mathbb{C}$  and  $\mathbb{D}$  have *boxes* if there is a functor  $\square : \mathbb{D}^{\mathbb{C}} \rightarrow \llbracket \text{Fam} \rrbracket(\mathbb{D})^{\llbracket \text{Fam} \rrbracket(\mathbb{C})}$  satisfying  $(*\square)$  and  $(**\square)$ .

Section 2.5 starts exploring the idea that if  $\mathbb{C}$  and  $\mathbb{D}$  have sigma types, and  $\mathbb{D}$  in addition an inverse image operation, then  $\mathbb{C}$  and  $\mathbb{D}$  have boxes.

### 3.1 $\Sigma$ and $\square$ in Set

This has been covered already.

### 3.2 $\Sigma$ and $\square$ in Fam(Set)

$$\begin{aligned} \Sigma_{\text{Fam}(\text{Set})} : (A : \text{Set})(B : A \rightarrow \text{Set}) \rightarrow \\ (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(a) \rightarrow P(a) \rightarrow \text{Set}) \rightarrow \text{Fam}(\text{Set}) \end{aligned}$$

$$\Sigma_{\text{Fam}(\text{Set})}(A, B, P, Q) = (\Sigma A P, \lambda \langle a, p \rangle. \Sigma b : B(a). Q(a, b, p))$$

$$(\pi_0, \overline{\pi_0})_{(A, B), (P, Q)} : \Sigma_{\text{Fam}(\text{Set})}(A, B, P, Q) \rightarrow (A, B)$$

$$(\pi_0, \overline{\pi_0})_{(A, B), (P, Q)}(\langle a, p \rangle, \langle b, q \rangle) = (a, b)$$

Let  $F : \text{Fam}(\text{Set}) \rightarrow \text{Set}$ . Then

$$\square_F : \llbracket \text{Fam} \rrbracket(\text{Fam}(\text{Set})) \rightarrow \text{Fam}(\text{Set})$$

$$\begin{aligned} \square_F((A, B), (P, Q), x) &= \{y : F(\Sigma_{\text{Fam}(\text{Set})}(A, B)(P, Q)) \mid F(\pi_0, \overline{\pi_0})(y) = x\} \\ &= \{y : F(\Sigma A P, \lambda \langle a, p \rangle. \Sigma b : B(a). Q(a, b, p)) \mid F(\pi_0, \overline{\pi_0})(y) = x\} \end{aligned}$$

We can also turn

$$\begin{aligned} f : (x : A) \rightarrow P(x) \\ g : (x : A) \rightarrow (y : B(x)) \rightarrow Q(x, y, f(x)) \end{aligned}$$

(the interpretation of  $f : (x : A) \rightarrow P(x)$  into

$$\widehat{(f, g)} : (A, B) \rightarrow \Sigma_{\text{Fam}(\text{Set})}(A, B)(P, Q)$$

by defining  $\widehat{(f, g)} = (\widehat{f}, \widehat{g})$  where  $\widehat{f}(x) = \langle x, f(x) \rangle$  and  $\widehat{g}(x, y) = \langle y, g(x, y) \rangle$ .

### 3.3 Inverse image in Fam(Set)

Given  $(f, g) : (A, B) \rightarrow (A', B')$ , we can define  $(f, g)^{-1} : (A', B') \rightarrow \text{Fam}(\text{Set})$  by

$$(f, g)^{-1} = (f^{-1}, g^{-1})$$

where  $f^{-1} : A' \rightarrow \text{Set}$ ,  $f^{-1}(x) = \{y : A \mid f(y) = x\}$ , and

$$g^{-1} : (x : A') \rightarrow B'(x) \rightarrow f^{-1}(x) \rightarrow \text{Set},$$

$g^{-1}(x, b, y) = \{z : B(y) \mid g(y, z) = b\}$ . The equality  $g(y, z) = b$  typechecks since  $f(y) = x$ , hence  $g(y, z) : B'(f(y)) = B'(x)$ .

### 3.4 $\Sigma$ and $\square$ in $\mathbf{BiAlg}(F, G)$ for $F, G : \mathbb{C} \rightarrow \mathbb{D}$

Assume that  $\mathbb{C}$  and  $\mathbb{D}$  have boxes. Then  $\mathbf{BiAlg}(F, G)$  have sigma types, and  $\llbracket \mathbf{Fam} \rrbracket(\mathbf{BiAlg}(F, G)) = \mathbf{BiAlg}(\square_F, \square_G)$ .

**Remark 5.** If one notices that  $\square$  is the morphism part of the (weak) functor  $\llbracket \mathbf{Fam} \rrbracket : \mathbf{Cat}_{\text{models of TT}} \rightarrow \mathbf{Cat}$ , this becomes

$$\llbracket \mathbf{Fam} \rrbracket(\mathbf{BiAlg}(F, G)) = \mathbf{BiAlg}(\llbracket \mathbf{Fam} \rrbracket(F), \llbracket \mathbf{Fam} \rrbracket(G)),$$

which looks rather natural.

#### 3.4.1 $\Sigma$ in $\mathbf{BiAlg}(F, G)$

Let  $\varphi_F : F(\Sigma(A, P)) \rightarrow \Sigma(\square_F(A, P))$  and similarly  $\varphi_G$  be the natural isomorphisms for the boxes in  $\mathbb{C}, \mathbb{D}$ . Given  $(A, P) : \llbracket \mathbf{Fam} \rrbracket(\mathbb{C})$  and  $(c, d) : \square_F(A, P) \rightarrow \square_G(A, P)$ , i.e. an object in  $\llbracket \mathbf{Fam} \rrbracket(\mathbf{BiAlg}(F, G))$ , we construct

$$\Sigma_{\mathbf{BiAlg}(F, G)}(A, c)(P, d) = (\Sigma_{\mathbb{C}} A P, \varphi_G^{-1} \circ [c, d] \circ \varphi_F)$$

where  $[c, d] = \Sigma_{\mathbb{C}}(c, d) : \Sigma_{\mathbb{C}} \square_F(A, P) \rightarrow \Sigma_{\mathbb{C}} \square_G(A, P)$  is the morphism part of  $\Sigma_{\mathbb{C}}$  applied to  $(c, d)$ . For morphisms, given  $(f, g) : ((A, P), (c, d)) \rightarrow \llbracket \mathbf{Fam} \rrbracket(\mathbf{BiAlg}(F, G))((A', P'), (c', d'))$ , i.e.  $(f, g) : (A, P) \rightarrow \llbracket \mathbf{Fam} \rrbracket(\mathbb{C})(A', P')$  such that

$$\begin{array}{ccc} \square_F(A, P) & \xrightarrow{(c, d)} & \square_G(A, P) \\ \square_F(f, g) \downarrow & & \downarrow \square_G(f, g) \\ \square_F(A', P') & \xrightarrow{(c', d')} & \square_G(A', P') \end{array} \quad (1)$$

commutes, we define  $\Sigma_{\mathbf{BiAlg}(F, G)}(f, g) = \Sigma_{\mathbb{C}}(f, g)$ . We have to check that this really is a morphism in  $\mathbf{BiAlg}(F, G)$ , i.e. that the following commutes:

$$\begin{array}{ccc} F(\Sigma_{\mathbb{C}} A P) & \xrightarrow{\varphi_G^{-1} \circ [c, d] \circ \varphi_F} & G(\Sigma_{\mathbb{C}} A P) \\ F[f, g] \downarrow & & \downarrow G[f, g] \\ F(\Sigma_{\mathbb{C}} A' P') & \xrightarrow{\varphi_G^{-1} \circ [c', d'] \circ \varphi_F} & G(\Sigma_{\mathbb{C}} A' P') \end{array}$$

By the naturality of  $\varphi$ , we have  $\Sigma \square_F(f, g) \circ \varphi_F = \varphi_F \circ F \Sigma(f, g)$  or equivalently  $F \Sigma(f, g) = \varphi_F^{-1} \circ \Sigma \square_F(f, g) \circ \varphi_F$ . This shows that the left and rightmost squares in the following diagram commutes. The outermost square commutes by (1), and the top and bottom squares by definition, so that the square in the middle

commutes by diagram chasing, and we are done:

$$\begin{array}{ccc}
\Sigma_{\mathbb{C}}\Box_F(A, P) & \xrightarrow{[c, d]} & \Sigma_{\mathbb{C}}\Box_G(A, P) \\
\uparrow \varphi_F & & \uparrow \varphi_G \downarrow \varphi_G^{-1} \\
F(\Sigma_{\mathbb{C}} A P) & \xrightarrow{\varphi_G^{-1} \circ [c, d] \circ \varphi_F} & G(\Sigma_{\mathbb{C}} A P) \\
\downarrow F[f, g] & & \downarrow G[f, g] \\
F(\Sigma_{\mathbb{C}} A' P') & \xrightarrow{\varphi_G^{-1} \circ [c', d'] \circ \varphi_F} & G(\Sigma_{\mathbb{C}} A' P') \\
\uparrow \varphi_F \downarrow \varphi_F^{-1} & & \uparrow \varphi_G^{-1} \\
\Sigma_{\mathbb{C}}\Box_F(A', P') & \xrightarrow{[c', d']} & \Sigma_{\mathbb{C}}\Box_G(A', P')
\end{array}$$

$\Sigma\Box_F(f, g)$  (left curved arrow)       $\Sigma\Box_G(f, g)$  (right curved arrow)

### 3.4.2 $\pi_0$ in $\mathbf{BiAlg}(F, G)$

For  $\pi_0 : \Sigma_{\mathbf{BiAlg}(F, G)}(A, c)(P, d) \rightarrow (A, c)$ , we of course define  $(\pi_0)_{\mathbf{BiAlg}(F, G)} = (\pi_0)_{\mathbb{C}}$ . We have to check that the diagram

$$\begin{array}{ccc}
F(\Sigma_{\mathbb{C}} A P) & \xrightarrow{\varphi_G^{-1} \circ [c, d] \circ \varphi_F} & G(\Sigma_{\mathbb{C}} A P) \\
F(\pi_0) \downarrow & & \downarrow G(\pi_0) \\
F(A) & \xrightarrow{c} & G(A)
\end{array}$$

commutes. But by  $(**\Box)$ ,  $G(\pi_0) = \pi_0 \circ \varphi_G$  so that

$$G(\pi_0) \circ \varphi_G^{-1} \circ [c, d] \circ \varphi_F = \pi_0 \circ [c, d] \circ \varphi_F = c \circ \pi_0 \circ \varphi_F = c \circ F(\pi_0).$$

### 3.4.3 $\Box_{\hat{G}}$ for $\hat{G} : \mathbb{Fam}(\mathbf{BiAlg}(F, U)) \rightarrow \mathbf{Fam}(\mathbf{Set})$

First, notice that  $\Box_U(A, B, P, Q, x) \cong P(x)$  for the forgetful functor  $U$ :

$$\begin{aligned}
\Box_U(A, B, P, Q, x) &= \{z : U(\Sigma_{\mathbf{Fam}(\mathbf{Set})}(A, B)(P, Q) \mid U(\pi_0, \bar{\pi}_0)(z) = x\} \\
&= \{z : \Sigma A P \mid \pi_0(z) = x\} \\
&= \{\langle x, y \rangle : \Sigma A P\} \cong P(x)
\end{aligned}$$

where the isomorphism sends  $\langle x, y \rangle : \Sigma A P$  to  $y : P(x)$  and the inverse sends  $y : P(x)$  to  $\langle x, y \rangle$ . Thus, we can replace the morphism  $d : (x : F(A, B)) \rightarrow \Box_F(A, B, P, Q, x) \rightarrow \Box_U(A, B, P, Q, c(x))$  with a morphism  $d' : (x : F(A, B)) \rightarrow \Box_F(A, B, P, Q, x) \rightarrow P(c(x))$  in the definition of  $\mathbb{Fam}(\mathbf{BiAlg}(F, U))$ . This also simplifies the definition of the morphism of  $\Sigma_{\mathbf{BiAlg}(F, U)}$  to be  $[c, d'] \circ \varphi_F$  instead of  $\varphi_U^{-1} \circ [c, d] \circ \varphi_F$ .

We now define

$$\Box_{\hat{G}} : \mathbb{Fam}(\mathbf{BiAlg}(F, U)) \rightarrow \mathbb{Fam}(\mathbf{Fam}(\mathbf{Set})).$$



by

$$\square_{\widehat{G}}((A, B), (P, Q), (c, d)) = (\widehat{G}(A, B, c), (\widehat{G}(\pi_0, \overline{\pi_0}))^*).$$

Explicitly, we have

$$\begin{aligned} \square_{\widehat{G}}((A, B), (P, Q), (c, d)) = \\ (F(A, B), G(A, B, c), \square_F(A, B, P, Q), \square_G(A, B, P, Q, c, d)) \end{aligned}$$

where  $\square_F$  is exactly the  $\square_F$  from Section 3.2, i.e.

$$\square_F(A, B, P, Q, x) = \{y : F(\Sigma_{\text{Fam}(\text{Set})} (A, B) (P, Q)) \mid F(\pi_0, \overline{\pi_0})(y) = x\},$$

and  $\square_G(A, B, P, Q, c, d) : (x : F(A, B)) \rightarrow G(A, B, c, x) \rightarrow \square_F(A, B, P, Q, x) \rightarrow \text{Set}$  is defined by

$$\begin{aligned} \square_G(A, B, P, Q, c, d, x, b, y) = \\ \{z : G(\Sigma_{\text{Fam}(\text{Set})} (A, B) (P, Q), [c, d] \circ \varphi_F, y) \mid G(\pi_0, \overline{\pi_0}, y, z) = b\} \end{aligned}$$

## 4 Example

Let us try to calculate  $\square_F$  and  $\square_G$  for the contexts and types.

$$\begin{aligned} F(A, B) &= 1 + \Sigma A B \\ G(A, B, c, x) &= 1 + \Sigma a : A, b : B(a), B(c(\text{inr}(\langle a, b \rangle))) . c(x) = a. \end{aligned}$$

For  $\square_F$ , we get

$$\square_F(A, B, P, Q, x) = \{y : 1 + \Sigma \langle a, p \rangle : (\Sigma A P) . \Sigma b : B(a) . Q(a, b, p) \mid (1 + [\pi_0, \overline{\pi_0}])(y) = x\}$$

so that

$$\begin{aligned} \square_F(A, B, P, Q, \text{inl}(\star)) &\cong \mathbf{1} \\ \square_F(A, B, P, Q, \text{inr}(\langle a, b \rangle)) &\cong (p : P(a)) \times Q(a, b, p) \end{aligned}$$

For  $\square_G$ , we have in all its glory

$$\begin{aligned} \square_G(A, B, P, Q, c, \bar{c}, x, y, \bar{x}) = \\ \{z : 1 + \Sigma \langle a, p \rangle : (\Sigma A P) . \langle b, q \rangle : \Sigma b : B(a) . Q(a, b, p) . \Sigma b' : B(c(\text{inr}(\langle a, b \rangle))) . \\ Q(c(\text{inr}(\langle a, b \rangle)), b', \bar{c}(\text{inr}(\langle a, b \rangle)), \text{inr}(\langle \langle a, b \rangle, \langle a, b \rangle \rangle)) . [c, \bar{c}](\varphi_F(\bar{x}) = (a, p) \\ \mid (1 + [\pi_0, [\pi_0, \overline{\pi_0}])(\bar{x}, z) = y\} \end{aligned}$$

which, ignoring indicies, should be

$$\begin{aligned} \square_G(A, B, P, Q, x, \text{inl}(\star), \bar{x}) &\cong \mathbf{1} \\ \square_F(A, B, P, Q, x, \text{inr}(\langle a, b, b' \rangle), \bar{x}) &\cong \\ (p : P(a)) \times (q : Q(a, b, p)) \times Q(c(\text{inr}(\langle a, b \rangle)), b', \bar{c}(\text{inr}(\langle a, b \rangle)), \text{inr}(\langle p, q \rangle))) . \end{aligned}$$

## 5 $\text{dmap}_F$ and $\text{dmap}_{\widehat{G}}$

We want to define

$$\begin{aligned} \text{dmap}_F : (A : \text{Set})(B : A \rightarrow \text{Set}) \rightarrow \\ (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\ (f : (x : A) \rightarrow P(x)) \rightarrow \\ (g : (x : A) \rightarrow (y : B(x)) \rightarrow Q(x, y, f(x))) \rightarrow \\ (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \end{aligned}$$

and

$$\begin{aligned} \text{dmap}_G : (A : \text{Set})(B : A \rightarrow \text{Set}) \rightarrow \\ (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\ (\text{in}_F : F(A, B) \rightarrow A) \\ (\overline{\text{in}_F} : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow P(\text{in}_F(x))) \rightarrow \\ (f : (x : A) \rightarrow P(x)) \rightarrow \\ (g : (x : A) \rightarrow (y : B(x)) \rightarrow Q(x, y, f(x))) \rightarrow \\ (x : F(A, B)) \rightarrow (y : G(A, B, \text{in}_F, x)) \\ \rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, x, y, \text{dmap}_F(A, B, P, Q, f, g, x)) \end{aligned}$$

which of course are the components of  $\text{dmap}_{\widehat{G}}$  which is the interpretation of  $\text{dmap}$  in  $\text{BiAlg}(F, U)$ . Thus, we should define

$$\text{dmap}_F(A, B, P, Q, f, g, x) = F(\widehat{f}, \widehat{g})(x)$$

which works since  $F(\widehat{f}, \widehat{g}) : F(A, B) \rightarrow F(\Sigma_{\text{Fam}(\text{Set})}(A, B)(P, Q))$ ,  $(\pi_0, \overline{\pi_0}) \circ (\widehat{f}, \widehat{g}) = \text{id}$  and

$$\square_F(A, B, P, Q, x) = \{y : F(\Sigma_{\text{Fam}(\text{Set})}(A, B)(P, Q)) \mid F(\pi_0, \overline{\pi_0})(y) = x\}.$$

Similarly, we should define

$$\text{dmap}_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, f, g, x, y) = G(\widehat{(f, g)})(x, y)$$

but this requires  $\overline{\text{in}_F}(x, F(\widehat{f}, \widehat{g})(x)) = f(\text{in}_F(x))$  – why is this true? The fact that  $(f, g)$  is a dependent morphism probably means that this should hold? It will hold for  $f = \text{elim}_F(\dots)$ , right, since  $\text{elim}_F(\dots, \text{in}_F(x)) = \overline{\text{in}_F}(x, \text{dmap}_F(\dots))$ , but this condition should be recorded and explained somehow...

## 6 The eliminator

The eliminator should have type the interpretation of

$$\begin{aligned} \text{elim}_F : (A : \text{Set})(\text{in}_F : F(A) \rightarrow A)(P : A \rightarrow \text{Set}) \rightarrow \\ ((x : F(A)) \rightarrow \square_F(A, P, x) \rightarrow P(\text{in}_F(x))) \rightarrow (x : A) \rightarrow P(x) \end{aligned}$$

in our current category, which becomes

$$\begin{aligned}
& \text{elim}_F : (A : \text{Set})(B : A \rightarrow \text{Set}) \rightarrow \\
& (\text{in}_F : F(A, B) \rightarrow A)(\text{in}_G : (x : F(A, B)) \rightarrow G(A, B, \text{in}_F, x) \rightarrow B(\text{in}_F(x))) \rightarrow \\
& (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\
& (\overline{\text{in}_F} : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow P(\text{in}_F(x))) \\
& (\overline{\text{in}_G} : (x : F(A, B)) \rightarrow (y : G(A, B, \text{in}_F, x)) \rightarrow (\bar{x} : \square_F(A, B, P, Q, x)) \\
& \rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, x, y, \bar{x}) \rightarrow Q(\text{in}_F(x), \text{in}_G(x, y), \overline{\text{in}_F}(x, \bar{x}))) \\
& \rightarrow (x : A) \rightarrow P(x)
\end{aligned}$$

$$\begin{aligned}
& \text{elim}_G : (A : \text{Set})(B : A \rightarrow \text{Set}) \rightarrow \\
& (\text{in}_F : F(A, B) \rightarrow A)(\text{in}_G : (x : F(A, B)) \rightarrow G(A, B, \text{in}_F, x) \rightarrow B(\text{in}_F(x))) \rightarrow \\
& (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\
& (\overline{\text{in}_F} : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow P(\text{in}_F(x))) \\
& (\overline{\text{in}_G} : (x : F(A, B)) \rightarrow (y : G(A, B, \text{in}_F, x)) \rightarrow (\bar{x} : \square_F(A, B, P, Q, x)) \\
& \rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, x, y, \bar{x}) \rightarrow Q(\text{in}_F(x), \text{in}_G(x, y), \overline{\text{in}_F}(x, \bar{x}))) \\
& \rightarrow (x : A) \rightarrow (y : B(x)) \rightarrow Q(x, y, \text{elim}_F(A, B, \text{in}_F, \text{in}_G, P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, x))
\end{aligned}$$

The computation rules should be

$$\begin{aligned}
& \text{elim}_F(A, B, \text{in}_F, \text{in}_G, P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x)) = \\
& \overline{\text{in}_F}(x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x))
\end{aligned}$$

$$\begin{aligned}
& \text{elim}_G(A, B, \text{in}_F, \text{in}_G, P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x), \text{in}_G(x, y)) = \\
& \overline{\text{in}_G}(x, y, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x), \\
& \text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), \overline{\text{in}_F}, x, y))
\end{aligned}$$

## 6.1 Elim $\implies$ init

The proof seems to be the interpretation of the standard proof.

Assume that we have the eliminator for  $(A, B)$ ,  $(\text{in}_F, \text{in}_G)$ . Need to define

$$\begin{aligned}
& \text{fold}_F : (X : \text{Set})(Y : X \rightarrow \text{Set}) \rightarrow \\
& (c : F(X, Y) \rightarrow X)(d : (x : F(X, Y)) \rightarrow G(X, Y, c, x) \rightarrow Y(c(x))) \rightarrow \\
& A \rightarrow X
\end{aligned}$$

$$\begin{aligned}
& \text{fold}_G : (X : \text{Set})(Y : X \rightarrow \text{Set}) \rightarrow \\
& (c : F(X, Y) \rightarrow X)(d : (x : F(X, Y)) \rightarrow G(X, Y, c, x) \rightarrow Y(c(x))) \rightarrow \\
& (x : A) \rightarrow B(x) \rightarrow Y(\text{fold}_F(X, Y, c, d, x))
\end{aligned}$$

Choose  $P(x) := X$ ,  $Q(x, y, \bar{x}) := Y(\bar{x})$ . If we can define

$$\overline{\text{in}}_F : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow X$$

and

$$\begin{aligned} \overline{\text{in}}_G : (x : F(A, B)) &\rightarrow (y : G(A, B, \text{in}_F, x)) \rightarrow (\bar{x} : \square_F(A, B, P, Q, x)) \\ &\rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}}_F, x, y, \bar{x}) \rightarrow Y(\overline{\text{in}}_F(x, \bar{x})), \end{aligned}$$

we are in business, as we then can define  $\text{fold}_F(X, Y, c, d) := \text{elim}_F(\dots, \overline{\text{in}}_F, \overline{\text{in}}_G) : A \rightarrow X$  and  $\text{fold}_G(X, Y, c, d) := \text{elim}_G(\dots, \text{in}_F, \text{in}_G) : (x : A) \rightarrow B(x) \rightarrow Y(\text{fold}_F(X, Y, c, d, x))$ .

But now, notice that the non-dependence of  $P$  and  $Q$  turns  $\Sigma A P$  and  $\Sigma b : B(a). Q(a, b, p)$  into non-dependent products  $A \times X$  and  $B(a) \times Y(x)$  so that  $\pi_1 : A \times X \rightarrow X$  and  $\bar{\pi}_1 : (\langle a, x \rangle : A \times X) \rightarrow B(a) \times Y(x) \rightarrow Y(x)$  make up an ordinary morphism in  $\text{Fam}(\text{Set})$ . Thus, we get

$$F(\pi_1, \bar{\pi}_1) : F(A \times X, B \times Y) \rightarrow F(X, Y)$$

and

$$\begin{aligned} G(\pi_1, \bar{\pi}_1) : (x : F(A \times X, B \times Y)) &\rightarrow G(A \times X, B \times Y, [\text{in}_F, c] \circ \varphi_F, x) \\ &\rightarrow G(X, Y, c, F(\pi_1, \bar{\pi}_1)(x)) \end{aligned}$$

and we can define  $\overline{\text{in}}_F(x, y) = c(F(\pi_1, \bar{\pi}_1)(\varphi_F^{-1}(\langle x, y \rangle)))$  and  $\overline{\text{in}}_G(x, y, \bar{x}, \bar{y}) = d(F(\pi_1, \bar{\pi}_1)\varphi_F^{-1}(\langle x, \bar{x} \rangle), G(\pi_1, \bar{\pi}_1)(\varphi_F^{-1}(\langle x, \bar{x} \rangle), \varphi_G^{-1}(\langle y, \bar{y} \rangle)))$ . Then

$$\text{elim}_F(A, B, X, Y, \overline{\text{in}}_F, \overline{\text{in}}_G) : A \rightarrow X$$

$$\text{elim}_G(A, B, X, Y, \text{in}_F, \text{in}_G) : (x : A) \rightarrow B(x) \rightarrow Y(\text{elim}_F(A, B, X, Y, \overline{\text{in}}_F, \overline{\text{in}}_G, x))$$

as required, and we can define  $\text{fold}_F(X, Y, c, d) = \text{elim}_F(A, B, X, Y, \overline{\text{in}}_F, \overline{\text{in}}_G)$  and  $\text{fold}_G(X, Y, c, d) = \text{elim}_G(A, B, X, Y, \text{in}_F, \text{in}_G)$ .

Let us write simply  $\text{fold}_F$  and  $\text{fold}_G$  for  $\text{fold}_F(X, Y, c, d)$  and  $\text{fold}_G(X, Y, c, d)$ . We have to check that  $\text{fold}_F \circ \text{in}_F = c \circ F(\text{fold}_F, \text{fold}_G)$  and  $\text{fold}_G(\text{in}_F(x)) \circ \text{in}_G(x) = d(F(\text{fold}_F, \text{fold}_G)) \circ G(\text{fold}_F, \text{fold}_G)$ .

We have

$$\begin{aligned} \text{fold}_F(\text{in}_F(x)) &= \text{elim}_F(X, Y, \overline{\text{in}}_F, \overline{\text{in}}_G, \text{in}_F(x)) \\ &= \overline{\text{in}}_F(x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x)) \\ &= c(F(\pi_1, \bar{\pi}_1)(\varphi_F^{-1}(\langle x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x) \rangle))) \\ &= c(F(\pi_1, \bar{\pi}_1)(\varphi_F^{-1}(\text{dmap}_F(\widehat{\text{fold}}_F, \widehat{\text{fold}}_G)(x)))) \\ &= c(F(\pi_1, \bar{\pi}_1)(\varphi_F^{-1}(\varphi_F(F(\widehat{\text{fold}}_F, \widehat{\text{fold}}_G)(x))))) \\ &= c(F((\pi_1, \bar{\pi}_1) \circ (\widehat{\text{fold}}_F, \widehat{\text{fold}}_G))(x)) \\ &= c(F(\text{fold}_F, \text{fold}_G)(x)) \end{aligned}$$

and similarly

$$\begin{aligned}
\text{fold}_G(\text{in}_F(x), \text{in}_G(x, y)) &= \text{elim}_G(X, Y, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x), \text{in}_G(x, y)) \\
&= \overline{\text{in}_G}(x, y, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x), \text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), x, y)) \\
&= d(F(\text{fold}_F, \text{fold}_G)x, G(\pi_1, \overline{\pi_1})(\varphi_G^{-1}(\langle y, \text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), x, y) \rangle))) \\
&= d(F(\text{fold}_F, \text{fold}_G)x, G(\pi_1, \overline{\pi_1})(\varphi_G^{-1}(\text{dmap}_G(\widehat{\text{elim}_F(\dots)}, \text{elim}_G(\dots))(x, y)))) \\
&= d(F(\text{fold}_F, \text{fold}_G)x, G(\pi_1, \overline{\pi_1})(\varphi_G^{-1}(\varphi_G(G(\widehat{\text{fold}_F}, \widehat{\text{fold}_G})(x, y)))) \\
&= d(F(\text{fold}_F, \text{fold}_G)x, G((\pi_1, \overline{\pi_1}) \circ (\widehat{\text{fold}_F}, \widehat{\text{fold}_G}))(x, y)) \\
&= d(F(\text{fold}_F, \text{fold}_G)x, G(\text{fold}_F, \text{fold}_G)(x, y))
\end{aligned}$$

**Question 4.** Uniqueness? How for arbitrary  $F, G$ ? (For functors arising from axiomatisation of ind.-ind. by induction on the code.)

## 6.2 Init $\implies$ elim

Yet another proof to interpret. Assume that  $(A, B)$ ,  $(\text{in}_F, \text{in}_G)$  is the initial object in our category  $E$ . In other words, we have

$$\begin{aligned}
\text{fold}_F : (X : \text{Set})(Y : X \rightarrow \text{Set}) \rightarrow \\
(c : F(X, Y) \rightarrow X)(d : (x : F(X, Y)) \rightarrow G(X, Y, c, x) \rightarrow Y(c(x))) \rightarrow \\
A \rightarrow X
\end{aligned}$$

$$\begin{aligned}
\text{fold}_G : (X : \text{Set})(Y : X \rightarrow \text{Set}) \rightarrow \\
(c : F(X, Y) \rightarrow X)(d : (x : F(X, Y)) \rightarrow G(X, Y, c, x) \rightarrow Y(c(x))) \rightarrow \\
(x : A) \rightarrow B(x) \rightarrow Y(\text{fold}_F(X, Y, c, d, x))
\end{aligned}$$

such that

$$\begin{array}{ccc}
F(A, B) & \xrightarrow{\text{in}_F} & A \\
F(\text{fold}_F, \text{fold}_G) \downarrow & (p) & \downarrow \text{fold}_F(X, Y, c, d) \\
F(X, Y) & \xrightarrow{c} & X
\end{array}$$

and

$$\begin{array}{ccc}
G(A, B, \text{in}_F, x) & \xrightarrow{\text{in}_G(x)} & B(\text{in}_F(x)) \\
G(\text{fold}_F, \text{fold}_G, (p), x) \downarrow & & \downarrow \text{fold}_G(X, Y, c, d, \text{in}_F(x)) \\
G(X, Y, c, F(\text{fold}_F, \text{fold}_G)(x)) & \xrightarrow{d(F(\text{fold}_F, \text{fold}_G)(x))} & Y(\text{fold}_F(\text{in}_F(x))) \\
& & \parallel (p) \\
& & Y(c(F(\text{fold}_F, \text{fold}_G)(x)))
\end{array}$$

We must construct

$$\begin{aligned} \text{elim}_F : (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\ (\overline{\text{in}_F} : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow P(\text{in}_F(x))) \\ (\overline{\text{in}_G} : (x : F(A, B)) \rightarrow (y : G(A, B, \text{in}_F, x)) \rightarrow (\bar{x} : \square_F(A, B, P, Q, x)) \\ \rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, x, y, \bar{x}) \rightarrow Q(\text{in}_F(x), \text{in}_G(x, y), \overline{\text{in}_F}(x, \bar{x}))) \\ \rightarrow (x : A) \rightarrow P(x) \end{aligned}$$

$$\begin{aligned} \text{elim}_G : (P : A \rightarrow \text{Set})(Q : (x : A) \rightarrow B(x) \rightarrow P(x) \rightarrow \text{Set}) \rightarrow \\ (\overline{\text{in}_F} : (x : F(A, B)) \rightarrow \square_F(A, B, P, Q, x) \rightarrow P(\text{in}_F(x))) \\ (\overline{\text{in}_G} : (x : F(A, B)) \rightarrow (y : G(A, B, \text{in}_F, x)) \rightarrow (\bar{x} : \square_F(A, B, P, Q, x)) \\ \rightarrow \square_G(A, B, P, Q, \text{in}_F, \overline{\text{in}_F}, x, y, \bar{x}) \rightarrow Q(\text{in}_F(x), \text{in}_G(x, y), \overline{\text{in}_F}(x, \bar{x}))) \\ \rightarrow (x : A) \rightarrow (y : B(x)) \rightarrow Q(x, y, \text{elim}_F(A, B, \text{in}_F, \text{in}_G, P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, x)) \end{aligned}$$

such that

$$\text{elim}_F(P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x)) = \overline{\text{in}_F}(x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x))$$

$$\begin{aligned} \text{elim}_G(P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x), \text{in}_G(x, y)) = \\ \overline{\text{in}_G}(x, y, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x), \\ \text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), \overline{\text{in}_F}(x, y))). \end{aligned}$$

Recall that  $\Sigma_{\text{Fam}(\text{Set})}(A, B) (P, Q) = (\langle a, p \rangle : \Sigma A P, \Sigma b : B(a). Q(a, b, p))$ . Define  $h_F : F(\Sigma_{\text{Fam}(\text{Set})}(A, B) (P, Q)) \rightarrow \Sigma A P$  and

$$\begin{aligned} h_G : (x : F(\Sigma_{\text{Fam}(\text{Set})}(A, B) (P, Q))) \rightarrow \\ G(\Sigma_{\text{Fam}(\text{Set})}(A, B) (P, Q), h_F, x) \rightarrow \Sigma b : B(a). Q(a, b, p) \end{aligned}$$

(where  $a = \pi_0(h_F(x))$ ,  $p = \pi_1(h_F(x))$ ) by

$$h_F(x) = [\text{in}_F, \overline{\text{in}_F}](\varphi_F(x))$$

$$h_G(x, y) = [\text{in}_G(\pi_0(\varphi_F(x))), \overline{\text{in}_G}(\pi_0(\varphi_F(x))), \pi_1(\varphi_F(x))](\varphi_G(x, y)).$$

**Remark 6.** The order of  $\overline{\text{in}_G}$ 's arguments is wrong, so that the above technically is a lie – you need to move the  $\square_G$  argument earlier with a lambda.

This is nothing but the two parts of a composition  $\Sigma((\text{in}_F, \text{in}_G), (\overline{\text{in}_F}, \overline{\text{in}_G})) \circ (\varphi_F, \varphi_G)$  in  $\text{Fam}(\text{Set})$ , of course.

Consider the diagrams

$$\begin{array}{ccc}
F(A, B) & \xrightarrow{\text{in}_F} & A \\
\downarrow F(\text{fold}(h)) & & \downarrow \text{fold}_F(h) \\
F(\Sigma(A, B)(P, Q)) & \xrightarrow{h_F} & \Sigma A P \\
\downarrow F(\pi_0, \overline{\pi_0}) & & \downarrow \pi_0 \\
F(A, B) & \xrightarrow{\text{in}_F} & A
\end{array}$$
  

$$\begin{array}{ccc}
G(A, B, \text{in}_F, x) & \xrightarrow{\text{in}_G(x)} & B(\text{in}_F(x)) \\
\downarrow G(\text{fold}(h), x) & & \downarrow \text{fold}_G(h)(\text{in}_F(x)) \\
G(\Sigma(A, B)(P, Q), h_F, F(\text{fold}(h))(x)) & \xrightarrow{h_G(F(\text{fold}(h))(x))} & Y(\text{fold}_F(\text{in}_F(x))) \\
\downarrow G(\pi_0, \overline{\pi_0}, F(\text{fold}(h))(x)) & & \downarrow \text{fold}_G(h)(\text{in}_F(x)) \\
G(A, B, \text{in}_F, x) & \xrightarrow{\text{in}_G(x)} & B(\text{in}_F(x))
\end{array}$$

The topmost squares commutes because  $(\text{fold}_F, \text{fold}_G)$  is a morphism in  $E$ , and the lower squares commute by calculation, using that  $\pi_0 \circ \varphi_F = F(\pi_0, \overline{\pi_0})$  and  $\overline{\pi_0}(-) \circ \varphi_G(x) = G(\pi_0, \overline{\pi_0}, x)$ . Thus  $(\pi_0, \overline{\pi_0}) \circ (\text{fold}_F(h), \text{fold}_G(h)) : (A, B) \rightarrow (A, B)$  is a morphism, but so is  $(\text{id}, \overline{\text{id}})$  so that  $\pi_0 \circ \text{fold}_F = \text{id}$  and  $\overline{\pi_0} \circ \text{fold}_G = \overline{\text{id}}$  by initiality. Hence we can define  $\text{elim}_F(\dots) : (x : A) \rightarrow P(x)$  and  $\text{elim}_G(\dots) : (x : A) \rightarrow B(x) \rightarrow Q(x, y, \text{elim}_F(x))$  by  $\text{elim}_F(\dots, x) = \pi_1(\text{fold}_F(h)(x))$  and  $\text{elim}_G(\dots, x, y) = \pi_1(\text{fold}_G(h)(x, y))$ .

Finally, we must check that the computation rules hold:

$$\text{elim}_F(P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x)) = \overline{\text{in}_F}(x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x))$$

$$\begin{aligned}
\text{elim}_G(P, Q, \overline{\text{in}_F}, \overline{\text{in}_G}, \text{in}_F(x), \text{in}_G(x, y)) = \\
\overline{\text{in}_G}(x, y, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x), \\
\text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), \overline{\text{in}_F}, x, y)).
\end{aligned}$$

For  $\text{elim}_F$ , this is exactly like the usual initial algebra case. We use

**Lemma 7.** (i)  $\pi_0 \circ \varphi_F \circ F(\text{fold}(h)) = \text{id}$

$$(ii) \pi_1 \circ \varphi_F \circ F(\text{fold}(h)) = \text{dmap}_F(\pi_1 \circ \text{fold}_F(h), \overline{\pi_1} \circ \text{fold}_G(h)).$$

*Proof.* (i) has already been taken care of. For (ii), notice that  $\text{fold}.(h) = \pi_1 \circ \text{fold}.(h)$  since  $\text{fold}.(h)(x) = \langle \pi_0(\text{fold}.(h)(x)), \pi_1(\text{fold}.(h)(x)) \rangle = \langle x, \pi_1(\text{fold}.(h)(x)) \rangle =$

$\pi_1(\widehat{\text{fold}}(h)(x))$ . Thus we have

$$\begin{aligned}\pi_1 \circ \varphi \circ F(\text{fold}_F(h), \text{fold}_G(h)) &= \pi_1 \circ \varphi \circ F(\pi_1 \circ \widehat{\text{fold}}_F(h), \overline{\pi_1} \circ \widehat{\text{fold}}_G(h)) \\ &= \pi_1 \circ \text{dmap}_F(\pi_1 \circ \widehat{\text{fold}}_F(h), \overline{\pi_1} \circ \widehat{\text{fold}}_G(h)) \\ &= \text{dmap}(\pi_1 \circ \text{fold}_F(h), \overline{\pi_1} \circ \text{fold}_G(h))\end{aligned}$$

□

$$\begin{aligned}\text{elim}_F(P, Q, \overline{\text{in}}_F, \overline{\text{in}}_G, \text{in}_F(x)) &= \pi_1(\text{fold}_F(h)(\text{in}_F(x))) \\ &= \pi_1(h_F(F(\text{fold}(h))(x))) \\ &= \overline{\text{in}}_F(\pi_0(\varphi_F(F(\text{fold}(h))(x))), \pi_1(\varphi_F(F(\text{fold}(h))(x)))) \\ &= \overline{\text{in}}_F(x, \text{dmap}_F(\pi_1 \circ \text{fold}_F(h), \overline{\pi_1} \circ \text{fold}_G(h), x)) \\ &= \overline{\text{in}}_F(x, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x))\end{aligned}$$

For  $\text{elim}_G$ , we analogously need that

$$(iii) \quad \overline{\pi_0} \circ \varphi_G \circ G(\text{fold}(h)) = \overline{\text{id}}$$

$$(iv) \quad \overline{\pi_1} \circ \varphi_G \circ G(\text{fold}(h)) = \text{dmap}_G(\pi_1 \circ \text{fold}_F(h), \overline{\pi_1} \circ \text{fold}_G(h)).$$

and then the proof goes through in the same way:

$$\begin{aligned}\text{elim}_G(P, Q, \overline{\text{in}}_F, \overline{\text{in}}_G, \text{in}_F(x), \text{in}_G(x, y)) &= \pi_1(\text{fold}_G(h)(\text{in}_F(x), \text{in}_G(x, y))) \\ &= \pi_1(h_G(F(\text{fold}(h))(x), G(\text{fold}(h))(x, y))) \\ &= \overline{\text{in}}_G(\pi_0(\odot_F), \pi_0(\odot_G), \pi_1(\odot_F), \pi_1(\odot_G)) \\ &= \overline{\text{in}}_G(x, y, \text{dmap}_F(\text{elim}_F(\dots), \text{elim}_G(\dots), x), \\ &\quad \text{dmap}_G(\text{elim}_F(\dots), \text{elim}_G(\dots), x, y))\end{aligned}$$

(where  $\odot_F = \varphi_F(F(\text{fold}(h))(x))$  and  $\odot_G = \varphi_G(F(\text{fold}(h))(x), G(\text{fold}(h))(x, y))$ .)