Extending some categories to categories with families

Fredrik Nordvall Forsberg

January 11, 2011

1 Categories with families

Definition 1. A *category with families* is given by

- A category $\mathbb{C}$ with a terminal object $[]$,
- A functor $F = (\mathrm{Ty}, \mathrm{Tm}) : \mathbb{C}^{\mathrm{op}} \to \mathrm{Fam}(\mathrm{Set})$. For the morphism part, we introduce the notation $\_\{\cdot\}$ for both types and terms, i.e. if $f : \Delta \to \Gamma$ then $\_\{f\} : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$ and for every $\sigma \in \mathrm{Ty}(\Delta)$ we have $\_\{f\} : \mathrm{Tm}(\Delta, \sigma) \to \mathrm{Tm}(\Gamma, \sigma\{f\})$.
- For each object Γ in $\mathbb{C}$ and $\sigma \in \mathrm{Ty}(\Gamma)$ an object $\Gamma.\sigma$ together with a morphism $\mathbf{p}(\sigma) : \Gamma.\sigma \to \Gamma$ (the *first projection*) and a term $\mathbf{v}_\sigma \in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\})$ (the *second projection*) with the following universal property: for each $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\})$ there exists a unique morphism $\theta = \langle f, M\rangle_\sigma : \Delta \to \Gamma.\sigma$ such that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$.

2 Set

Directly from Dybjer [Dyb96] , Hofmann [Hof97], Buisse and Dybjer [BD08], ...

Choose $\mathbb{C} = \mathrm{Set}$ (with $[] = \mathbf{1}$ any singleton), and define

$$\mathrm{Ty}(\Gamma) = \{\sigma \mid \sigma : \Gamma \to \mathrm{Set}\}$$
$$\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma \in \Gamma} \sigma(\gamma)$$

(this should really be $\sigma : \Gamma \to U$ for some universe (U, T) for size considerations, and accordingly $\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma\in\Gamma} T(\sigma(\gamma))$). For $f : \Delta \to \Gamma$, $\sigma : \mathrm{Ty}(\Gamma)$, $h :$

$\mathrm{Tm}(\Gamma, \sigma)$, define

$$
\begin{aligned}
&\sigma\{f\} : \mathrm{Ty}(\Delta) = \{\sigma \mid \sigma : \Delta \to \mathrm{Set}\} \\
&\sigma\{f\} = \sigma \circ f \\
&h\{f\} : \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta)) \\
&h\{f\} = h \circ f
\end{aligned}
$$

For the context comprehension, define

$$
\begin{aligned}
\Gamma.\sigma &= \sum_{\gamma \in \Gamma} \sigma(\gamma) \\
\mathbf{p}(\sigma) &: \sum_{\gamma \in \Gamma} \sigma(\gamma) \to \Gamma \\
\mathbf{p}(\sigma)(\langle \gamma, s \rangle) &= \gamma \\
\mathbf{v}_\sigma &\in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\}) = \prod_{\langle \gamma, s \rangle \in \Gamma.\sigma} \sigma(\gamma) \\
\mathbf{v}_\sigma(\langle \gamma, s \rangle) &= s
\end{aligned}
$$

Given $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta))$, we define

$$
\theta = \langle f, M \rangle_\sigma : \Delta \to \underbrace{\Gamma.\sigma}_{\sum_{\gamma \in \Gamma} \sigma(\gamma)}
$$

by $\theta(\delta) = \langle f(\delta), M(\delta) \rangle$. We then have $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$, and any other function satisfying these equations must be extensionally equal to θ, hence θ is unique.

3 Fam(Set)

Fam(Set) can also be extended to a category with families. We start with $\mathbb{C} = \mathrm{Fam}(\mathrm{Set})$ (and $[] = (\mathbf{1}, \lambda x.\mathbf{1})$), and define

$$
\begin{aligned}
\mathrm{Ty}(X, Y) &= \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
\mathrm{Tm}((X, Y), (A, B)) &= \{(h, k) \mid h : \prod_{x \in X} A(x), k : \prod_{x \in X, y \in Y(x)} B(x, y, h(x))\}
\end{aligned}
$$

(similar size considerations apply as for Set). For $(f, g) : (X, Y) \to (X', Y')$, $(A, B) : \mathrm{Ty}(X', Y')$, $(h, k) : \mathrm{Tm}((X', Y'), (A, B))$, define

$$
\begin{aligned}
&(A, B)\{f, g\} : \mathrm{Ty}(X, Y) = \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
&(A, B)\{f, g\} = (A, B) \circ (f, g) = (A \circ f, \lambda x, y \ .\ B(f(x), g(x, y)) \\
&(h, k)\{f, g\} : \mathrm{Tm}(\Delta, \sigma\{f\}) \\
&(h, k)\{f, g\} = (h, k) \circ (f, g) = (h \circ f, \lambda x, y \ .\ k(f(x), g(x, y)))
\end{aligned}
$$

For the context comprehension, define

$$
\begin{aligned}
(X, Y).(A, B) &= (\sum_{x \in X} A(x), \lambda \langle x, a \rangle \ . \sum_{y \in Y(x)} B(x, y, a)) \\
\mathbf{p}(A, B) &= (\mathrm{fst}, \lambda x.\mathrm{fst}) \\
\mathbf{v}_{A,B} &= (\mathrm{snd}, \lambda x.\mathrm{snd})
\end{aligned}
$$

Given $(f, g) : (X', Y') \to (X, Y)$ and $(h, k) \in \mathrm{Tm}((X', Y'), (A, B)\{f, g\})$, we define

$$(\theta, \psi) = \langle (f, g), (h, k) \rangle_{(A,B)} : (X', Y') \to (X, Y).(A, B)$$

by $\theta(x) = \langle f(x), h(x) \rangle$, $\psi(x, y) = \langle g(x, y), k(x, y) \rangle$. It is clear that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$ and that these conditions force (θ, ψ) to be unique.

4 $\mathbf{BiAlg}(F, G)$ for $F, G : \mathbb{C} \to \mathbb{D}$

Lemma 2. *BiAlg(F, G) has a terminal object if $\mathbb{C}$ and $\mathbb{D}$ does, and G preserves terminal objects (i.e. $G(\mathbf{1}_\mathbb{C}) \cong \mathbf{1}_\mathbb{D}$).*

Proof. Define $\mathbf{1}_{\mathrm{BiAlg}(F,G)} \coloneqq (\mathbf{1}_\mathbb{C}, !_{F(\mathbf{1}_\mathbb{C})})$ where $!_{F(\mathbf{1}_\mathbb{C})}$ is the unique map $F(\mathbf{1}_\mathbb{C}) \to \mathbf{1}_D$. For any object (X, f), the unique morphism $(X, f) \to (\mathbf{1}_\mathbb{C}, !_{F(\mathbf{1}_\mathbb{C})})$ is given by the unique arrow $!_X$ from X to $\mathbf{1}_C$ in $\mathbb{C}$, and the diagram

$$
\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "F(!_X)"'] & GX \arrow[d, "G(!_X)"] \\
F(\mathbf{1}_C) \arrow[r, "!_{F(\mathbf{1}_C)}"'] & G(\mathbf{1}_\mathbb{C}) = \mathbf{1}_\mathbb{D}
\end{tikzcd}
$$

commutes since both paths are arrows into $\mathbf{1}_\mathbb{D}$, hence equal. □

4.1 Some CwF preliminaries

Clairambault [Cla06, 4.1] defines a category $\mathrm{Type}_\mathbb{C}(\Gamma)$ of types in context Γ from the base category $\mathbb{C}$. The morphisms between $A, B \in \mathrm{Ty}_\mathbb{C}(\Gamma)$ are defined to be the terms $f \in \mathrm{Tm}_\mathbb{C}(\Gamma.A, B\{\mathbf{p}(A)\})$, with identity given by $\mathbf{v}_A$. We will be mostly interested in the composition of two terms $f \in \mathrm{Tm}_\mathbb{C}(\Gamma.A, B\{\mathbf{p}(A)\})$ and $g \in \mathrm{Tm}_\mathbb{C}(\Gamma.B, C\{\mathbf{p}(B)\})$, which is defined to be

$$g \bullet f \coloneqq g\{\langle \mathbf{p}(A), f \rangle_B\}.$$

The following proposition says that comprehension is a functor from "families in $\mathbb{C}$" to $\mathbb{C}$, which is quite convenient.

Lemma 3. *Given $g : \Gamma' \to \Gamma$ and $M \in Tm(\Gamma'.\sigma', \sigma\{g \circ \mathbf{p}(\sigma')\})$, one can construct $\psi : \Gamma'.\sigma' \to \Gamma.\sigma$.*

Proof. Take $\psi \coloneqq \langle g \circ \mathbf{p}(\sigma'), M \rangle_\sigma$. □

Lemma 4. *Let $f : \Delta \to \Gamma$, $M \in Tm(\Delta, \sigma\{f\})$, $h : \Theta \to \Delta$. Then $\langle f, M\rangle_\sigma \circ h = \langle f \circ h, M\{h\}\rangle_{\sigma\{f\}}$.*

Proof. $\langle f, M\rangle_\sigma \circ h$ satisfies the universal property for $f \circ h$ and $M\{h\}$. $\square$

4.2 The construction

Assume that $\mathbb{C}$ and $\mathbb{D}$ are CwFs (with boxes, or can this be defined for all CwFs?).

Assume further that for $f : \Delta \to \Gamma$, we have

$$\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$$

Remark 5. Demanding equality on the nose instead of isomorphism simplifies matters – we are spared transporting terms hidden inside substitutions along the isomorphisms. I guess it should be possible in principle though.

However, with the ususal definition of $\Box_G$, one (almost) never has equality. (In Set, for example, the left hand side is $\{y : G(\Sigma\ \Gamma\ \sigma) \mid \ldots\}$ and the right hand side $\{y : G(\Sigma\ \Delta\ (\sigma \circ f)) \mid \ldots\}$.) If $G = U$ is a forgetful functor, though, then the usual definition of $\Box_U(\Gamma, \sigma)$ is isomorphic to a $X(\Gamma, \sigma)$ such that $X(\Gamma, \sigma)\{U(f)\} = X(\Delta, \sigma\{f\})$. I see no harm in replacing $\Box_U(\Gamma, \sigma)$ with $X(\Gamma, \sigma)$ for U, so that we get an actual equality? The properties we need $\Box_U$ to have are of course preserved by isomorphism anyway.

For F, we only require the existence of

$$\phi_F(f) \in \mathrm{Tm}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}(\Box_F(\Delta, \sigma\{f\}))\})$$

which should be functorial in f, i.e. $\phi_F(\mathrm{id}) = \mathbf{v}_\sigma$ and

$$\phi_F(f \circ g) = \phi_F(f)\{\langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle\}.$$

Remark 6. $\phi_F(f)$ and $\phi_F(g)$ are not composable in Type($F(\Gamma)$), as their types depend on f and g, but the "composition" above should be composition in some more liberal category (where $\mathbf{v}_\sigma$ still is the identity)? It is in any case exactly what we need, and holds e.g. in Set (I have not checked Fam(Set), but would be very surprised if it did not hold).

4.2.1 Types

Define

$$\mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Gamma, h) = \{(\sigma, M) \mid \sigma \in \mathrm{Ty}_{\mathbb{C}}(\Gamma), M \in \mathrm{Tm}_{\mathbb{D}}(F(\Gamma).\Box_F(\Gamma, \sigma), \Box_G(\Gamma, \sigma)\{h \circ \mathbf{p}\})\}$$

For substitutions, assume $f : (\Delta, h') \to (\Gamma, h)$, i.e. $f : \Delta \to \Gamma$ and $G(f) \circ h' = h \circ F(f)$. Define for $(\sigma, M) \in \mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Gamma, h)$

$$(\sigma, M)\{f\} = (\sigma\{f\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f)\rangle\}) \in \mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Delta, h')$$

We should check that this makes sense. Since $\sigma \in \mathrm{Ty}_{\mathbb{C}}(\Gamma)$, we have $\sigma\{f\} \in \mathrm{Ty}_{\mathbb{C}}(\Delta)$. We now need a term in $\mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Delta, \sigma\{f\})\{h \circ \mathbf{p}\})$. Since $F(f) : F(\Delta) \to F(\Gamma)$ and

$$\phi_F \in \mathrm{Tm}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}\}),$$

Lemma 3 applies and we get $g := \langle F(f) \circ \mathbf{p}, \phi_F \rangle : F(\Delta).\Box_F(\Delta, \sigma\{f\}) \to F(\Gamma).\Box_F(\Gamma, \sigma)$, so that

$$M\{g\} \in \mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Gamma, \sigma)\{h \circ \mathbf{p} \circ g\})$$

and since

$$h \circ \mathbf{p} \circ g = h \circ \mathbf{p} \circ \langle F(f) \circ \mathbf{p}, \phi_F \rangle = h \circ F(f) \circ \mathbf{p} = G(f) \circ h' \circ \mathbf{p}$$

and $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$, we in fact have

$$M\{g\} \in \mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Delta, \sigma\{f\})\{h' \circ \mathbf{p}\})$$

as needed. Functoriality follows from functoriality of $\phi_F(f)$ and functoriality one level down:

$$\begin{aligned}(\sigma, M)\{\mathrm{id}\} &= (\sigma\{\mathrm{id}\}, M\{\langle F(\mathrm{id}) \circ \mathbf{p}, \phi_F(\mathrm{id})\rangle\}) \\ &= (\sigma, M\{\langle p, \mathbf{v}_\sigma\rangle\}) = (\sigma, M\{\mathrm{id}\}) = (\sigma, M)\end{aligned}$$

$$\begin{aligned}(\sigma, M)\{f\}\{g\} &= (\sigma\{f\}\{g\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f)\rangle\}\{\langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle\}) \\ &= (\sigma\{f \circ g\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f)\rangle \circ \langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle\}) \\ &= (\sigma\{f \circ g\}, M\{\langle F(f) \circ \mathbf{p} \circ \langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle, \phi_F(f)\{\langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle\}\}) \\ &= (\sigma\{f \circ g\}, M\{\langle F(f) \circ F(g) \circ \mathbf{p}, \phi_F(f \circ g)) \\ &= (\sigma\{f \circ g\}, M\{\langle F(f \circ g) \circ \mathbf{p}, \phi_F(f \circ g)) \\ &= (\sigma, M)\{f \circ g\}\end{aligned}$$

4.2.2 Terms

4.2.3 Context comprehension

References

[BD08] Alexandre Buisse and Peter Dybjer. The interpretation of intuitionistic type theory in locally cartesian closed categories – an intuitionistic perspective. *Electronic Notes in Theoretical Computer Science*, 218:21–32, 2008.

[Cla06] Pierre Clairambault. From categories with families to locally cartesian closed categories. Technical report, ENS Lyon, 2006.

[Dyb96] Peter Dybjer. Internal type theory. *Lecture Notes in Computer Science*, 1158:120–134, 1996.

[Hof97] Martin Hofmann. Syntax and semantics of dependent types. In Andrew Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 79 – 130. Cambridge University Press, 1997.