Generalising $\Box_F$ and dmap_F for non-endofunctors

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December 14, 2010

1 The standard situation

In the ordinary situation, we have an endofunctor $F : \mathrm{Set} \to \mathrm{Set}$ representing an inductive definition, i.e. we have a set μF and a constructor $c : F(\mu F) \to \mu F$. The eliminator has type

$$\begin{aligned}\mathrm{elim}_F : (P : \mu F \to \mathrm{Set}) \to (g : (x : F(\mu F)) \to \Box_F(\mu F, P, x) \to P(c(x))) \\ \to (x : \mu F) \to P(x)\end{aligned}$$

and computation rule

$$\mathrm{elim}_F(P, g, c(x)) = g(x, \mathrm{dmap}_F(\mu F, P, \mathrm{elim}_F(P, g), x)).$$

Here,

$$\Box_F : (A : \mathrm{Set}) \to (P : A \to \mathrm{Set}) \to F(A) \to \mathrm{Set}$$

gives the induction hypothesis (it is called 'IH' in the induction-recursion papers) and

$$\begin{aligned}\mathrm{dmap}_F : (A : \mathrm{Set}) \to (P : A \to \mathrm{Set}) \to ((x : A) \to P(x)) \\ \to (x : F(A)) \to \Box_F(A, P, x)\end{aligned}$$

takes care of the recursive calls (in the IR papers, it is called mapIH). Both $\Box_F$ and dmap_F can be explicitly defined:

$$\begin{aligned}\Box_F(A, P, x) &= \{y : F(\Sigma\ A\ P) \mid F(\pi_0)(y) = x\} \\ \mathrm{dmap}_F(A, P, g, x) &= F(\widehat{g})(x)\end{aligned}$$

where $\widehat{g} : A \to \Sigma\ A\ P$ is defined by $\widehat{g}(z) = \langle z, g(z)\rangle$. Note that $\pi_0 \circ \widehat{g} = \mathrm{id}$, so that

$$F(\pi_0)(F(\widehat{g})(x)) = F(\pi_0 \circ \widehat{g})(x)) = F(\mathrm{id})(x) = \mathrm{id}(x) = x$$

which shows that indeed $\mathrm{dmap}_F(A, P, g, x) : \Box_F(A, P, x)$. One can show that this definition of $\Box_F$ is naturally isomorphic to the definition in the IR papers (see Problem 2 in `problems.pdf`).

However, we prefer to just use some abstract properties of $\Box_F$ and dmap_F, namely

$$F(\Sigma\ A\ B) \cong \Sigma\ F(A)\ \Box_F(A,B) \qquad (*_\Box)$$

and this isomorphism φ satisfies

$$\begin{array}{ccc} F(\Sigma\ A\ B) & \xrightarrow{\varphi} & \Sigma\ F(A)\ \Box_F(A,B) \\ {\scriptstyle F(\pi_0)}\downarrow & \swarrow_{\pi_0} & \\ F(A) & & \end{array} \qquad (**_\Box)$$

i.e. $\pi_0 \circ \varphi = F(\pi_0)$. For dmap_F, we have for all $f : (x : A) \to B(x)$

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\widehat{f})} & F(\Sigma\ A\ B) \\ {\scriptstyle \widehat{\mathrm{dmap}_F(f)}}\downarrow & \swarrow_{\varphi} & \\ \Sigma\ F(A)\ \Box_F(A,B) & & \end{array} \qquad (*_{\mathrm{dmap}})$$

i.e. $\widehat{\mathrm{dmap}_F(f)} = \varphi \circ F(\widehat{f})$.

2 Generalising $\Box_F$ and dmap_F for non-endofunctors

Let us now try to generalise things, first by moving away from the category Set to an arbitrary category $\mathbb{C}$ (with some structure, of course), and then by replacing the endofunctor $F : \mathbb{C} \to \mathbb{C}$ with just a functor $F : \mathbb{C} \to \mathbb{D}$.

2.1 The category $[\![\mathbf{Fam}]\!](\mathbb{C})$ of interpretations of families in $\mathbb{C}$

Let us for the moment assume that the category $\mathbb{C}$ is a model of type theory. I want to define a category $[\![\mathrm{Fam}]\!](\mathbb{C})$ of "interpretations of families in $\mathbb{C}$". It has as objects an interpretation of '$A : \mathrm{Set}, B : A \to \mathrm{Set}$' in $\mathbb{C}$, and a morphism from $[\![A, B]\!]$ to $[\![A', B']\!]$ is an interpretation of '$f : A \to A', g : (x : A) \to B(x) \to B'(f(x))$'.

Assuming that the interpretation of '$A : \mathrm{Set}$' in $\mathbb{C}$ is $[\![A]\!] : \mathbb{C}$, there should also be a forgetful functor $U : [\![\mathrm{Fam}]\!](\mathbb{C}) \to \mathbb{C}$ defined by $U([\![A]\!], [\![B]\!]) = [\![A]\!]$ and $U([\![f]\!], [\![g]\!]) = [\![f]\!]$.

Example 1 ($\mathbb{C} = \mathrm{Set}$)**.** For $\mathbb{C} = \mathrm{Set}$, we recover the usual category Fam(Set) of families of sets, i.e. we have objects pairs (A, B) where $A : \mathrm{Set}$ and $B : A \to \mathrm{Set}$, and a morphism $(A, B) \to (A', B')$ is a pair $f : A \to A'$, $g : (x : A) \to B(x) \to B'(f(x))$.

The forgetful functor U is the usual forgetful functor $U : \mathrm{Fam(Set)} \to \mathrm{Set}$; $U(A, B) = A$, $U(f, g) = f$.

Example 2 ($\mathbb{C} = \mathrm{Fam}(\mathrm{Set})$)**.** Unless I am mistaken, the objects in $[\![\mathrm{Fam}]\!](\mathrm{Fam}(\mathrm{Set}))$ are tuples $((A_0, A_1), (B_0, B_1))$ where

$$\begin{aligned}
&A_0 : \mathrm{Set},\\
&A_1 : A_0 \to \mathrm{Set},\\
&B_0 : A_0 \to \mathrm{Set},\\
&B_1 : (x : A_0) \to A_1(x) \to B_0(x) \to \mathrm{Set}.
\end{aligned}$$

A morphism from $((A_0, A_1), (B_0, B_1))$ to $((A'_0, A'_1), (B'_0, B'_1))$ consists of

$$\begin{aligned}
&f_0 : A_0 \to A'_0,\\
&f_1 : (x : A_0) \to A_1(x) \to A'_1(f_0(x)),\\
&g_0 : (x : A_0) \to B_0(x) \to B'_0(f_0(x))\\
&g_1 : (x : A_0) \to (y : A_1(x)) \to (z : B_0(x)) \to B_1(x, y, z)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \to B'_1(f_0(x), f_1(x, y), g_0(x, z)).
\end{aligned}$$

$U((A_0, A_1), (B_0, B_1)) = (A_0, A_1)$, $U((f_0, f_1), (g_0, g_1)) = (f_0, f_1)$.

Question 1. Is there a better way to present $[\![\mathrm{Fam}]\!](\mathbb{C})$?

2.2 Σ as a functor from $[\![\mathbf{Fam}]\!](\mathbb{C})$ to $\mathbb{C}$

Now, I notice that the sigma constructor takes a family (A, B) of sets and gives me a set $\Sigma\ A\ B$ back. It is also functorial, since if I have a morphism $(f, g) : (A, B) \to (A', B')$, then I can construct a morphism $[f, g] : \Sigma\ A\ B \to \Sigma\ A'\ B'$ by defining

$$[f, g]\langle a, b\rangle = \langle f(a), g(a, b)\rangle.$$

The projection π_0 takes me from $\Sigma\ A\ B$ to $A = U(A, B)$, and $\pi_0 \circ [f, g] = f \circ \pi_0$. It is thus a natural transformation $\pi_0 : \Sigma \dot{\to} U$.

In the general case, I would thus like to generalise Σ to a functor

$$\Sigma_{\mathbb{C}} : [\![\mathrm{Fam}]\!](\mathbb{C}) \to \mathbb{C}$$

together with a natural transformation $\pi_0 : \Sigma_{\mathbb{C}} \dot{\to} U$.

Question 2. Do we need $\pi_1 : (x : \Sigma_{\mathbb{C}}(A, B)) \to B(\pi_0(x))$ as well?

Question 3. I probably would like $\Sigma_{\mathbb{C}}$ to be the interpretation of Σ in $\mathbb{C}$. For morphisms, this would mean that I want $\Sigma_{\mathbb{C}}(f, g)$ to be the interpretation of $[f, g]$?

2.3 $\Box$ is a functor $\Box' : \mathbf{Set}^{\mathbf{Set}} \to \mathbf{Fam}(\mathbf{Set})^{\mathbf{Fam(Set)}}$

Looking at the type

$$\Box_F : (A : \mathrm{Set}) \to (P : A \to \mathrm{Set}) \to F(A) \to \mathrm{Set}$$

of $\Box_F$ again, we see that we can write this as

$$\Box' : \mathrm{Set}^{\mathrm{Set}} \to \mathrm{Fam}(\mathrm{Set})^{\mathrm{Fam}(\mathrm{Set})}$$

if we define $\Box'(F) = \lambda(A, B).\ (F(A), \Box_F(A, B))$. For $\Box'(F)$ to be a functor Fam(Set) to Fam(Set), it must also act on morphisms. Suppose $(f, g) : (A, B) \to (A', B')$. We must find a morphism from $\Box'(F)(A, B)$ to $\Box'(F)(A', B')$, i.e. from

$$(F(A), \{y : F(\Sigma\ A\ B) \mid F(\pi_0)(y) = x\})$$

to

$$(F(A'), \{y : F(\Sigma\ A'\ B') \mid F(\pi_0)(y) = x\}).$$

But $f : A \to A'$, so $m = F(f) : F(A) \to F(A')$ can be our first component. Now we need to define

$$\begin{aligned} n : (x : F(A)) \to \{y : F(\Sigma\ A\ B) \mid F(\pi_0)(y) = x\} \\ \to \{z : F(\Sigma\ A'\ B') \mid F(\pi_0)(z) = F(f)(x)\} \end{aligned}$$

Let $n(x) = F([f, g])$. We need to prove that $F(\pi_0)(F([f, g])(y)) = F(f)(x)$ for $y : F(\Sigma\ A\ B)$ such that $F(\pi_0)(y) = x$. But $\pi_0 \circ [f, g] = f \circ \pi_0$, so

$$F(\pi_0)(F([f, g])(y)) = F(\pi_0 \circ [f, g])(y) = F(f \circ \pi_0)(y) = F(f)(F(\pi_0)(y)) = F(f)(x).$$

Thus we can take $\Box'(F)(f, g) = (n, m) = (F(f), \lambda x.\ F([f, g]))$. This shows that $\Box'(F) : \mathrm{Fam}(\mathrm{Set})^{\mathrm{Fam}(\mathrm{Set})}$ if $F : \mathrm{Set} \to \mathrm{Set}$ is a functor.

However, we want more! We want $\Box'$ to be a functor $\Box' : \mathrm{Set}^{\mathrm{Set}} \to \mathrm{Fam}(\mathrm{Set})^{\mathrm{Fam}(\mathrm{Set})}$. We have defined the object part of $\Box'$, what is the morphism part? Given a natural transformation $\eta : F \mathrel{\dot{\to}} G$, we have to construct a natural transformation $\Box'(\eta) : \Box'(F) \mathrel{\dot{\to}} \Box'(G)$. The component of $\Box'(\eta)$ at (A, B) consists of a function $f : F(A) \to G(A)$ and a function $g : (x : F(A)) \to \Box_F(A, B, x) \to \Box_G(A, B, f(x))$. For f, we can choose $f = \eta_A$. Choose $g(x) = \eta_{\Sigma\ A\ B}$. Certainly $\eta_{\Sigma AB} : F(\Sigma\ A\ B) \to G(\Sigma\ A\ B)$, but do we have $G(\pi_0)(\eta_{\Sigma AB}(y)) = \eta_a(x)$, given that $F(\pi_0)(y) = x$? Yes, because substituting $x = F(\pi_0)(y)$, we end up with the equation $G(\pi_0)(\eta_{\Sigma AB}(y)) = \eta_a(F(\pi_0)(y))$, which is exactly the naturality condition!

$$\begin{array}{ccc} F(\Sigma\ A\ B) & \xrightarrow{\eta_{\Sigma AB}} & G(\Sigma\ A\ B) \\ {\scriptstyle F(\pi_0)}\downarrow & & \downarrow{\scriptstyle G(\pi_0)} \\ F(A) & \xrightarrow[\eta_A]{} & G(A) \end{array}$$

Hence we can define $\Box'(\eta)_{(A,B)} = (\eta_A, \lambda x.\ \eta_{\Sigma AB})$. The naturality of $\Box'(\eta)$ follows directly from the naturality of η.

Now that Σ and $\Box$ can be seen as functors, we can replace the isomorphism φ in $(*_\Box)$ with a natural isomorphism $\eta : F \circ \Sigma \mathrel{\dot{\to}} \Sigma\ \circ \Box$. One can easily check

that the isomorphism given in Problem 1 in fact is natural:

$$\begin{array}{ccc} F(\Sigma\ A\ B) & \xrightarrow{\eta_{(A,B)}} & \Sigma\ F(A)\ \Box_F(A,B) \\ {\scriptstyle F([f,g])}\downarrow & & \downarrow{\scriptstyle [F(f),[F([f,g])]]} \\ F(\Sigma\ A'\ B') & \xrightarrow[\eta_{(A',B')}]{} & \Sigma\ F(A')\ \Box_F(A',B') \end{array}$$

2.4 A generalised $\Box_F$ for F not an endofunctor

Assume that $\mathbb{C}$ and $\mathbb{D}$ are categories such that $[\![\mathrm{Fam}]\!](\mathbb{C})$ and $[\![\mathrm{Fam}]\!](\mathbb{D})$ make sense. (Then $\Sigma_{\mathbb{C}}$ and $\Sigma_{\mathbb{D}}$ should make sense as well.) We can then generalise $\Box_F$ to a functor

$$\Box : \mathbb{D}^{\mathbb{C}} \to [\![\mathrm{Fam}]\!](\mathbb{C}) \to [\![\mathrm{Fam}]\!](\mathbb{C})$$

such that there is a natural isomorphism

$$\eta : F \circ \Sigma_{\mathbb{C}} \dot{\to} \Sigma_{\mathbb{D}}\ \circ \Box_F \qquad (*_\Box)$$

satisfying

$$\begin{array}{ccc} F(\Sigma_{\mathbb{C}}\ A\ B) & \xrightarrow{\eta_{(A,B)}} & \Sigma_{\mathbb{D}}\ F(A)\ \Box_F(A,B) \\ {\scriptstyle F(\pi_0)}\downarrow & \swarrow_{\pi_0} & \\ F(A) & & \end{array} \qquad (**_\Box)$$

i.e. $\pi_0 \circ \eta_{(A,B)} = F(\pi_0)$.

2.5 The Inverse image

If $\mathbb{C}$ also has an inverse image operation which maps $f : X \to Y$ to $f^* : [\![\mathrm{Fam}]\!](\mathbb{C})$ with $U(f^*) = Y$, we can define $\Box_F(X) = (F(U(X)), (F(\pi_{0_X}))^*)$.

We have to check that $(*_\Box)$ and $(**_\Box)$ hold. ... more conditions on $\cdot^*$ needed (corresponding to being left adjoint to reindexing) ...

3 Σ in the relevant categories

Let us write $\mathrm{Pow}_{\mathbb{C}}(X)$ for the interpretation of $X \to \mathrm{Set}$ in $\mathbb{C}$.

Definition 3. A category $\mathbb{C}$ *has sigma types* if $[\![\mathrm{Fam}]\!](\mathbb{C})$ and a forgetful functor $U : [\![\mathrm{Fam}]\!](\mathbb{C}) \to \mathbb{C}$ exists, together with a functor $\Sigma_{\mathbb{C}} : [\![\mathrm{Fam}]\!](\mathbb{C}) \to \mathbb{C}$ and a natural transformation $\pi_0 : \Sigma_{\mathbb{C}} \to U$.

Definition 4. Let $\mathbb{C}$, $\mathbb{D}$ be categories having sigma types. We say that $\mathbb{C}$ *and* $\mathbb{D}$ *have boxes* if there is a functor $\Box : \mathbb{D}^{\mathbb{C}} \to [\![\mathrm{Fam}]\!](\mathbb{D})^{[\![}\mathrm{Fam}]\!](C)$ satisfying $(*_\Box)$ and $(**_\Box)$.

Section 2.5 starts exploring the idea that if $\mathbb{C}$ and $\mathbb{D}$ have sigma types, and $\mathbb{D}$ in addition an inverse image operation, then $\mathbb{C}$ and $\mathbb{D}$ have boxes.

3.1 Σ and $\Box$ in Set

This has been covered already.

3.2 Σ and $\Box$ in Fam(Set)

$$\Sigma_{\mathrm{Fam(Set)}} : (A : \mathrm{Set})(B : A \to \mathrm{Set}) \to (P : A \to \mathrm{Set})(Q : (x : A) \to B(a) \to P(a) \to \mathrm{Set}) \to \mathrm{Fam(Set)}$$

$$\Sigma_{\mathrm{Fam(Set)}}(A, B, P, Q) = (\Sigma\ A\ P, \lambda\langle a, p\rangle.\Sigma b{:}B(a).\ Q(a, b, p))$$

$$(\pi_0, \overline{\pi_0})_{(A,B),(P,Q)} : \Sigma_{\mathrm{Fam(Set)}}(A, B, P, Q) \to (A, B)$$

$$(\pi_0, \overline{\pi_0})_{(A,B),(P,Q)}(\langle a, p\rangle, \langle b, q\rangle) = (a, b)$$

Let $F : \mathrm{Fam(Set)} \to \mathrm{Set}$. Then

$$\Box_F : [\![\mathrm{Fam}]\!](\mathrm{Fam(Set)}) \to \mathrm{Fam(Set)}$$

$$\begin{aligned}\Box_F((A, B), (P, Q), x) &= \{y : F(\Sigma_{\mathrm{Fam(Set)}}\ (A, B)\ (P, Q))|F(\pi_0, \overline{\pi_0})(y) = x\} \\ &= \{y : F(\Sigma\ A\ P, \lambda\langle a, p\rangle.\Sigma b{:}B(a).\ Q(a, b, p))|F(\pi_0, \overline{\pi_0})(y) = x\}\end{aligned}$$

3.3 Inverse image in Fam(Set)

Given $(f, g) : (A, B) \to (A', B')$, we can define $(f, g)^{-1} : (A', B') \to \mathrm{Fam(Set)}$ by

$$(f, g)^{-1} = (f^{-1}, g^{-1})$$

where $f^{-1} : A' \to \mathrm{Set}$, $f^{-1}(x) = \{y : A \mid f(y) = x\}$, and

$$g^{-1} : (x : A') \to B'(x) \to f^{-1}(x) \to \mathrm{Set},$$

$g^{-1}(x, b, y) = \{z : B(y) \mid g(y, z) = b\}$. The equality $g(y, z) = b$ typechecks since $f(y) = x$, hence $g(y, z) : B'(f(y)) = B'(x)$.

3.4 Σ and $\Box$ in BiAlg(F, G) for $F, G : \mathbb{C} \to \mathbb{D}$

Assume that $\mathbb{C}$ and $\mathbb{D}$ have boxes. Then $\mathrm{BiAlg}(F, G)$ have sigma types, and $[\![\mathrm{Fam}]\!](\mathrm{BiAlg}(F, G)) = \mathrm{BiAlg}(\Box_F, \Box_G)$.

Remark 5. If one notices that $\Box$ is the morphism part of the (weak) functor $[\![\mathrm{Fam}]\!] : \mathrm{Cat}_{\text{models of TT}} \to \mathrm{Cat}$, this becomes

$$[\![\mathrm{Fam}]\!](\mathrm{BiAlg}(F, G)) = \mathrm{BiAlg}([\![\mathrm{Fam}]\!](F), [\![\mathrm{Fam}]\!](G)),$$

which looks rather natural.

3.4.1 Σ in BiAlg(F, G)

Let $\varphi_F : F(\Sigma(A, P)) \to \Sigma(\Box_F(A, P))$ and similarly φ_G be the natural isomorphisms for the boxes in $\mathbb{C}$, $\mathbb{D}$. Given $(A, P) : [\![\text{Fam}]\!](\mathbb{C})$ and $(c, d) : \Box_F(A, P) \to \Box_G(A, P)$, i.e. an object in $[\![\text{Fam}]\!](\text{BiAlg}(F, G))$, we construct

$$\Sigma_{\text{BiAlg}(F,G)}\ (A, c)\ (P, d) = (\Sigma_{\mathbb{C}}\ A\ P, \varphi_G^{-1} \circ [c, d] \circ \varphi_F)$$

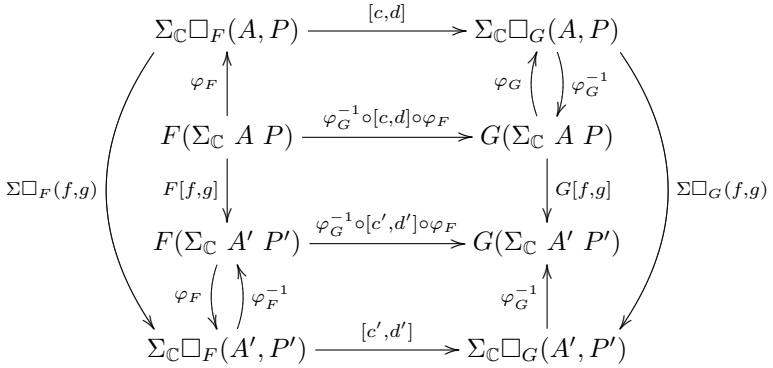
where $[c, d] = \Sigma_{\mathbb{C}}(c, d) : \Sigma_{\mathbb{C}}\Box_F(A, P) \to \Sigma_{\mathbb{C}}\Box_G(A, P)$ is the morphism part of $\Sigma_{\mathbb{C}}$ applied to (c, d). For morphisms, given $(f, g) : ((A, P), (c, d)) \to_{[\![\text{Fam}]\!](\text{BiAlg}(F,G))} ((A', P'), (c', d'))$, i.e. $(f, g) : (A, P) \to_{[\![\text{Fam}]\!](\mathbb{C})} (A', P')$ such that

$$\begin{array}{ccc} \Box_F(A, P) & \xrightarrow{(c,d)} & \Box_G(A, P) \\ {\scriptstyle \Box_F(f,g)}\downarrow & & \downarrow{\scriptstyle \Box_G(f,g)} \\ \Box_F(A', P') & \xrightarrow{(c',d')} & \Box_G(A', P') \end{array} \tag{1}$$

commutes, we define $\Sigma_{\text{BiAlg}(F,G)}(f, g) = \Sigma_{\mathbb{C}}(f, g)$. We have to check that this really is a morphism in BiAlg(F, G), i.e. that the following commutes:

$$\begin{array}{ccc} F(\Sigma_{\mathbb{C}}\ A\ P) & \xrightarrow{\varphi_G^{-1} \circ [c,d] \circ \varphi_F} & G(\Sigma_{\mathbb{C}}\ A\ P) \\ {\scriptstyle F[f,g]}\downarrow & & \downarrow{\scriptstyle G[f,g]} \\ F(\Sigma_{\mathbb{C}}\ A'\ P') & \xrightarrow{\varphi_G^{-1} \circ [c',d'] \circ \varphi_F} & G(\Sigma_{\mathbb{C}}\ A'\ P') \end{array}$$

By the naturality of φ, we have $\Sigma\Box_F(f, g) \circ \varphi_F = \varphi_F \circ F\Sigma(f, g)$ or equivalently $F\Sigma(f, g) = \varphi^{-1} \circ \Sigma\Box_F(f, g) \circ \varphi_F$. This shows that the left and rightmost squares in the following diagram commutes. The outermost square commutes by (1), and the top and bottom squares by definition, so that the square in the middle commutes by diagram chasing, and we are done:



3.4.2 π_0 in $\mathbf{BiAlg}(F, G)$

For $\pi_0 : \Sigma_{\mathrm{BiAlg}(F,G)}\ (A, c)\ (P, d) \to (A, c)$, we of course define $(\pi_0)_{\mathrm{BiAlg}(F,G)} = (\pi_0)_{\mathbb{C}}$. We have to check that the diagram

$$\begin{array}{ccc} F(\Sigma_{\mathbb{C}}\ A\ P) & \xrightarrow{\varphi_G^{-1} \circ [c,d] \circ \varphi_F} & G(\Sigma_{\mathbb{C}}\ A\ P) \\ {\scriptstyle F(\pi_0)}\downarrow & & \downarrow{\scriptstyle G(\pi_0)} \\ F(A) & \xrightarrow{\quad c \quad} & G(A) \end{array}$$

commutes. But by $(**_{\Box})$, $G(\pi_0) = \pi_0 \circ \varphi_G$ so that

$$G(\pi_0) \circ \varphi_G^{-1} \circ [c, d] \circ \varphi_F = \pi_0 \circ [c, d] \circ \varphi_F = c \circ \pi_0 \circ \varphi_F = c \circ F(\pi_0).$$

3.4.3 $\Box_{\widehat{G}}$ for $\widehat{G} : [\![\mathbf{Fam}]\!](\mathbf{BiAlg}(F, U)) \to \mathbf{Fam}(\mathbf{Set})$

First, notice that $\Box_U(A, B, P, Q, x) \cong P(x)$ for the forgetful functor U:

$$\begin{aligned} \Box_U(A, B, P, Q, x) &= \{z : U(\Sigma_{\mathrm{Fam(Set)}}\ (A, B)\ (P, Q) \mid U(\pi_0, \overline{\pi_0})(z) = x\} \\ &= \{z : \Sigma\ A\ P \mid \pi_0(z) = x\} \\ &= \{\langle x, y\rangle : \Sigma\ A\ P\ \} \cong P(x) \end{aligned}$$

where the isomorphism sends $\langle x, y\rangle : \Sigma\ A\ P$ to $y : P(x)$ and the inverse sends $y : P(x)$ to $\langle x, y\rangle$. Thus, we can replace the morphism $d : (x : F(A, B)) \to \Box_F(A, B, P, Q, x) \to \Box_U(A, B, P, Q, c(x))$ with a morphism $d' : (x : F(A, B)) \to \Box_F(A, B, P, Q, x) \to P(c(x))$ in the definition of $[\![\mathrm{Fam}]\!](\mathrm{BiAlg}(F, U))$. This also simplifies the definition of the morphism of $\Sigma_{\mathrm{BiAlg}(F,U)}$ to be $[c, d'] \circ \varphi_F$ instead of $\varphi_U^{-1}[c, d] \circ \varphi_F$.

We now define

$$\Box_{\widehat{G}} : [\![\mathrm{Fam}]\!](\mathrm{BiAlg}(F, U)) \to [\![\mathrm{Fam}]\!](\mathrm{Fam}(\mathrm{Set})).$$

by

$$\Box_{\widehat{G}}((A, B), (P, Q), (c, d)) = (\widehat{G}(A, B, c), (\widehat{G}(\pi_0, \overline{\pi_0}))^*).$$

Explicitly, we have

$$\begin{aligned} &\Box_{\widehat{G}}((A, B), (P, Q), (c, d)) = \\ &\qquad (F(A, B), G(A, B, c), \Box_F(A, B, P, Q), \Box_G(A, B, P, Q, c, d)) \end{aligned}$$

where $\Box_F$ is exactly the $\Box_F$ from Section 3.2, i.e.

$$\Box_F(A, B, P, Q, x) = \{y : F(\Sigma_{\mathrm{Fam(Set)}}\ (A, B)\ (P, Q)) \mid F(\pi_0, \overline{\pi_0})(y) = x\},$$

and $\Box_G(A, B, P, Q, c, d) : (x : F(A, B) \to G(A, B, c, x) \to \Box_F(A, B, P, Q, x) \to$ Set is defined by

$$\begin{aligned} &\Box_G(A, B, P, Q, c, d, x, b, y) = \\ &\qquad \{z : G(\Sigma_{\mathrm{Fam(Set)}}\ (A, B)\ (P, Q), [c, d] \circ \varphi_F, y) \mid G(\pi_0, \overline{\pi_0}, y, z) = b\} \end{aligned}$$

4 Example

Let us try to calculate $\Box_F$ and $\Box_G$ for the contexts and types.

$$\begin{aligned} F(A,B) &= 1 + \Sigma\ A\ B \\ G(A,B,c,x) &= 1 + \Sigma a : A, b : B(a), B(c(\mathrm{inr}(\langle a,b\rangle))).\ c(x) = a. \end{aligned}$$

For $\Box_F$, we get

$$\Box_F(A,B,P,Q,x) = \{y : 1+\Sigma\langle a,p\rangle : (\Sigma\ A\ P)\ .\ \Sigma b\colon B(a)\ .\ Q(a,b,p) \mid (1+[\pi_0,\overline{\pi_0}])(y) = x\}$$

so that

$$\begin{aligned} \Box_F(A,B,P,Q,\mathrm{inl}(\star)) &\cong \mathbf{1} \\ \Box_F(A,B,P,Q,\mathrm{inr}(\langle a,b\rangle)) &\cong (p : P(a)) \times Q(a,b,p) \end{aligned}$$

For $\Box_G$, we have in all its glory

$$\begin{aligned} &\Box_G(A,B,P,Q,c,\overline{c},x,y,\overline{x}) = \\ &\{z : 1 + \Sigma\langle a,p\rangle : (\Sigma\ A\ P)\ .\langle b,q\rangle : \Sigma b\colon B(a)\ .\ Q(a,b,p)\ .\Sigma b' : B(c(\mathrm{inr}(\langle a,b\rangle)))\ . \\ &\quad Q(c(\mathrm{inr}(\langle a,b\rangle)),b',\overline{c}(\mathrm{inr}(\langle a,b\rangle),\mathrm{inr}\langle\langle a,b\rangle,\langle a,b\rangle\rangle)).[c,\overline{c}](\varphi_F(\overline{x}) = (a,p) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mid (1 + [\pi_0,[\pi_0,\overline{\pi_0}]])(\overline{x},z) = y\} \end{aligned}$$

which, ignoring indicies, should be

$$\begin{aligned} &\Box_G(A,B,P,Q,x,\mathrm{inl}(\star),\overline{x}) \cong \mathbf{1} \\ &\Box_F(A,B,P,Q,x,\mathrm{inr}(\langle a,b,b'\rangle),\overline{x}) \cong \\ &\quad (p : P(a)) \times (q : Q(a,b,p)) \times Q(c(\mathrm{inr}(\langle a,b\rangle)),b',\overline{c}(\mathrm{inr}(\langle a,b\rangle),\mathrm{inr}(\langle p,q\rangle))). \end{aligned}$$