## All is well if G preserves pullbacks

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February 1, 2011

## Abstract

Using too many diagrams, which makes the text a lot longer than it could be, we derive the special properties needed of G from the assumption that G preserves pullbacks.

## 1 General assumptions

For every  $\Gamma \in \mathbb{C}$  and  $\sigma \in \mathrm{Ty}(\Gamma)$ , there exists a unique  $\square_F(\Gamma, \sigma) \in \mathrm{Ty}(F(\Gamma))$  and an isomorphism  $\varphi_F : F(\Gamma, \sigma) \to F(\Gamma)$ .  $\square_F(\Gamma, \sigma)$  such that  $\mathbf{p} \circ \varphi = F(\mathbf{p})$ . [Up to iso etc.]

We have a morphism part: for  $f: \Delta \to \Gamma$ ,  $M \in \text{Tm}(\Delta, \sigma\{f\})$  there exists  $\Box_F(f, M) \in \text{Tm}(F(\Delta), \Box_F(\Gamma, \sigma)\{F(f)\})$ .

Finally, we have for every  $f: \Delta \to \Gamma$ ,  $\sigma \in \mathrm{Ty}(\Gamma)$  a term

$$\phi_F(f) \in \operatorname{Tm}(F(\Delta), \square_F(\Delta, \sigma\{f\}), \square_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}\}).$$

We use the following global assumptions, which we expect to hold for every  $F:\mathbb{C}\to\mathbb{D}$ :

- $\varphi_F \circ F(\langle f, M \rangle) = \langle F(f), \square_F(f, M) \rangle.$
- $\phi_F(f \circ g) = \phi_F(f) \{ \langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle \}.$
- $\phi_F(f)\{\overline{\Box_F(\mathrm{id},M)}\}=\Box_F(f,M).$

The only assumption on G is that it preserves pullbacks.

$$2 \quad \Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$$

**Theorem 1.** if G preserves pullbacks, then  $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$ .

*Proof.* We show that there is an isomorphism  $\gamma: G(\Delta.\sigma\{f\}) \to G(\Delta)$ .  $\square_G(\Gamma,\sigma)\{G(f)\}$  satisfying  $\mathbf{p} \circ \gamma = G(\mathbf{p})$ . By the uniqueness of  $\square_G(\Delta,\sigma\{f\})$ , we must thus have  $\square_G(\Gamma,\sigma)\{G(f)\} = \square_G(\Delta,\sigma\{f\})$ .

The following diagram is always a pullback in every CwF:

$$\begin{array}{ccc} \Delta.\sigma\{f\} & \xrightarrow{f \circ \mathbf{p}, \mathbf{v} \rangle} \Gamma.\sigma \\ \mathbf{p} & & \downarrow \mathbf{p} \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

Since G preserves pullbacks, also

$$G(\Delta.\sigma\{f\}) \xrightarrow{G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle)} G(\Gamma.\sigma)$$

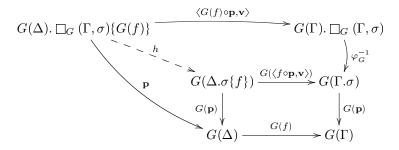
$$G(\mathbf{p}) \downarrow \qquad \qquad \downarrow G(\mathbf{p})$$

$$G(\Delta) \xrightarrow{G(f)} G(\Gamma)$$

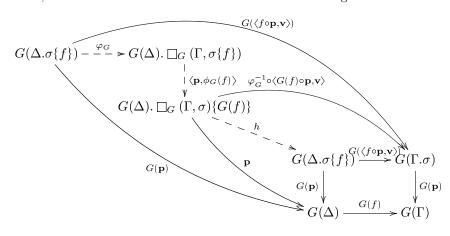
is a pullback square. But now note that

$$G(\mathbf{p}) \circ \varphi_G \circ \langle G(f) \circ \mathbf{p}, \mathbf{v} \rangle = \mathbf{p} \circ \langle G(f) \circ \mathbf{p}, \mathbf{v} \rangle = G(f) \circ \mathbf{p}$$

so that we get a mediating arrow h in the following pullback diagram:



This h will be our  $\gamma^{-1}$ . We can construct  $\gamma$  explicitly by  $\gamma := \langle \mathbf{p}, \phi_G(f) \rangle \circ \varphi_G$ . By the uniqueness of the mediating arrow in the following diagram, we get  $h \circ \gamma = \mathrm{id}$  for free, at least after we have checked that the outer triangles commute.



The bottom one is straightforward; it can be split up into two commuting triangles:

$$G(\Delta.\sigma\{f\}) - \overset{\varphi_G}{-} \Rightarrow G(\Delta). \ \Box_G \ (\Gamma, \sigma\{f\}) \overset{\langle \mathbf{p}, \phi_G(f) \rangle}{-} \overset{\langle \mathbf{p}, \phi_G(f) \rangle}{\Rightarrow} G(\Delta). \ \Box_G \ (\Gamma, \sigma)\{G(f)\}$$

For the top triangle, we need the following identity:

**Lemma 2.** 
$$\phi_G(f)\{\varphi_G\} = \Box_G(f \circ p, \mathbf{v})$$

*Proof.* First, write  $\varphi_G$  in a complicated way:

$$\varphi_{G} = \varphi_{G} \circ id$$

$$= \varphi_{G} \circ G(\langle \mathbf{p}, \mathbf{v} \rangle)$$

$$= \langle G(\mathbf{p}), \square_{G}(\mathbf{p}, \mathbf{v}) \rangle$$

$$= \langle G(\mathbf{p}), \phi_{G}(\mathbf{p}) \{ \overline{\square_{G}(id, \mathbf{v})} \} \rangle$$

$$= \langle G(\mathbf{p}) \circ \mathbf{p}, \phi_{G}(\mathbf{p}) \rangle \circ \overline{\square_{G}(id, \mathbf{v})}$$

Then note that  $\phi_G(f)\{\langle G(\mathbf{p}) \circ \mathbf{p}, \phi_G(\mathbf{p}) \rangle\} = \phi_G(f \circ \mathbf{p})$ . Thus, we have

$$\phi_G(f)\{\varphi_G\} = \phi_G(f)\{\langle G(\mathbf{p}) \circ \mathbf{p}, \phi_G(\mathbf{p}) \rangle \circ \overline{\square_G(\mathrm{id}, \mathbf{v})}\}$$
$$= \phi_G(f \circ \mathbf{p})\{\overline{\square_G(\mathrm{id}, \mathbf{v})}\}$$
$$= \square_G(f \circ \mathbf{p}, \mathbf{v}).$$

Now, we can just calculate: we want

$$\varphi_G^{-1} \circ \langle G(f) \circ \mathbf{p}, \phi_G(f) \rangle \circ \varphi_G = G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle)$$

or equivalently

$$\langle G(f) \circ \mathbf{p}, \phi_G(f) \rangle \circ \varphi_G = \varphi_G \circ G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle).$$

We have

$$\langle G(f) \circ \mathbf{p}, \phi_{G}(f) \rangle \circ \varphi_{G} = \langle \mathbf{p} \circ \varphi_{G}, \phi_{G}(f) \{ \varphi_{G} \} \rangle$$

$$= \langle G(f) \circ \mathbf{p} \circ \varphi_{G}, \phi_{G}(f) \{ \varphi_{G} \} \rangle$$

$$= \langle G(f) \circ G(\mathbf{p}), \phi_{G}(f) \{ \varphi_{G} \} \rangle$$

$$= \langle G(f) \circ G(\mathbf{p}), \square_{G}(f \circ \mathbf{p}, \mathbf{v}) \rangle$$

$$= \langle G(f \circ \mathbf{p}), \square_{G}(f \circ \mathbf{p}, \mathbf{v}) \rangle$$

$$= \varphi_{G} \circ G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle).$$

Hence, we have  $h\circ \gamma=\mathrm{id}.$  To prove  $\gamma\circ h=\mathrm{id},$  consider the commuting square

$$G(\Delta). \square_{G} (\Gamma, \sigma) \{G(f)\} \xrightarrow{\langle G(f) \circ \mathbf{p}, \mathbf{v} \rangle} G(\Gamma). \square_{G} (\Gamma, \sigma)$$

$$\downarrow \varphi_{G}^{-1} \downarrow \varphi_{G}^{-1}$$

$$G(\Delta. \sigma\{f\}) \xrightarrow{G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle)} G(\Gamma. \sigma)$$

Since

$$G(\langle f \circ \mathbf{p}, \mathbf{v} \rangle) \circ h = \varphi^{-1} \circ \langle G(f \circ \mathbf{p}), \square_G(f \circ \mathbf{p}, \mathbf{v}) \rangle \circ h$$
$$= \varphi^{-1} \circ \langle G(f) \circ G(\mathbf{p}) \circ h, \square_G(f \circ \mathbf{p}, \mathbf{v}) \{h\} \rangle$$
$$= \varphi^{-1} \circ \langle G(f) \circ \mathbf{p}, \square_G(f \circ \mathbf{p}, \mathbf{v}) \{h\} \rangle,$$

the square tells us that  $\square_G(f \circ \mathbf{p}, \mathbf{v})\{h\} = \mathbf{v}$ . Now we can show that  $\gamma \circ h = \mathrm{id}$ :

$$\gamma \circ h = \langle \mathbf{p}, \phi_G(f) \rangle \circ \varphi_G \circ h 
= \langle \mathbf{p} \circ \varphi_G \circ h, \phi_G(f) \{ \varphi_G \circ h \} \rangle 
= \langle G(\mathbf{p}) \circ h, \square_G(f \circ \mathbf{p}, \mathbf{v}) \{ h \} \rangle 
= \langle \mathbf{p}, \mathbf{v} \rangle = \mathrm{id}$$

Thus we have found our isomorphism and have  $\gamma^{-1} = h$ . Furthermore, we see that  $G(\mathbf{p}) \circ \gamma^{-1} = \mathbf{p}$  or equivalently  $\mathbf{p} \circ \gamma = G(\mathbf{p})$ , so by the uniqueness of  $\square_G$ , we must have  $\square_G(\Gamma, \sigma)\{G(f)\} = \square_G(\Delta, \sigma\{f\})$ .