Extending some categories to categories with families

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1 Categories with families

Definition 1. A *category with families* is given by

- A category $\mathbb{C}$ with a terminal object $[]$,
- A functor $F = (\mathrm{Ty}, \mathrm{Tm}) : \mathbb{C}^{\mathrm{op}} \to \mathrm{Fam}(\mathrm{Set})$. For the morphism part, we introduce the notation $\_\{\cdot\}$ for both types and terms, i.e. if $f : \Delta \to \Gamma$ then $\_\{f\} : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$ and for every $\sigma \in \mathrm{Ty}(\Delta)$ we have $\_\{f\} : \mathrm{Tm}(\Delta, \sigma) \to \mathrm{Tm}(\Gamma, \sigma\{f\})$.
- For each object Γ in $\mathbb{C}$ and $\sigma \in \mathrm{Ty}(\Gamma)$ an object $\Gamma.\sigma$ together with a morphism $\mathbf{p}(\sigma) : \Gamma.\sigma \to \Gamma$ (the *first projection*) and a term $\mathbf{v}_\sigma \in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\})$ (the *second projection*) with the following universal property: for each $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\})$ there exists a unique morphism $\theta = \langle f, M \rangle_\sigma : \Delta \to \Gamma.\sigma$ such that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$.

2 Set

Directly from Dybjer [Dyb96] , Hofmann [Hof97], Buisse and Dybjer [BD08], ...

Choose $\mathbb{C} = \mathrm{Set}$ (with $[] = \mathbf{1}$ any singleton), and define

$$\mathrm{Ty}(\Gamma) = \{\sigma \mid \sigma : \Gamma \to \mathrm{Set}\}$$
$$\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma \in \Gamma} \sigma(\gamma)$$

(this should really be $\sigma : \Gamma \to U$ for some universe (U, T) for size considerations, and accordingly $\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma\in\Gamma} T(\sigma(\gamma))$). For $f : \Delta \to \Gamma$, $\sigma : \mathrm{Ty}(\Gamma)$, h :

$\mathrm{Tm}(\Gamma, \sigma)$, define

$$
\begin{aligned}
\sigma\{f\} &: \mathrm{Ty}(\Delta) = \{\sigma \mid \sigma : \Delta \to \mathrm{Set}\} \\
\sigma\{f\} &= \sigma \circ f \\
h\{f\} &: \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta)) \\
h\{f\} &= h \circ f
\end{aligned}
$$

For the context comprehension, define

$$
\begin{aligned}
\Gamma.\sigma &= \sum_{\gamma \in \Gamma} \sigma(\gamma) \\
\mathbf{p}(\sigma) &: \sum_{\gamma \in \Gamma} \sigma(\gamma) \to \Gamma \\
\mathbf{p}(\sigma)(\langle \gamma, s \rangle) &= \gamma \\
\mathbf{v}_\sigma &\in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\}) = \prod_{\langle \gamma, s \rangle \in \Gamma.\sigma} \sigma(\gamma) \\
\mathbf{v}_\sigma(\langle \gamma, s \rangle) &= s
\end{aligned}
$$

Given $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta))$, we define

$$
\theta = \langle f, M \rangle_\sigma : \Delta \to \underbrace{\Gamma.\sigma}_{\sum_{\gamma \in \Gamma} \sigma(\gamma)}
$$

by $\theta(\delta) = \langle f(\delta), M(\delta) \rangle$. We then have $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$, and any other function satisfying these equations must be extensionally equal to θ, hence θ is unique.

3 Fam(Set)

Fam(Set) can also be extended to a category with families. We start with $\mathbb{C} = \mathrm{Fam}(\mathrm{Set})$ (and $[] = (\mathbf{1}, \lambda x.\mathbf{1})$), and define

$$
\begin{aligned}
\mathrm{Ty}(X, Y) &= \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
\mathrm{Tm}((X, Y), (A, B)) &= \{(h, k) \mid h : \prod_{x \in X} A(x), k : \prod_{x \in X, y \in Y(x)} B(x, y, h(x))\}
\end{aligned}
$$

(similar size considerations apply as for Set). For $(f, g) : (X, Y) \to (X', Y')$, $(A, B) : \mathrm{Ty}(X', Y')$, $(h, k) : \mathrm{Tm}((X', Y'), (A, B))$, define

$$
\begin{aligned}
&(A, B)\{f, g\} : \mathrm{Ty}(X, Y) = \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
&(A, B)\{f, g\} = (A, B) \circ (f, g) = (A \circ f, \lambda x, y \,.\, B(f(x), g(x, y)) \\
&(h, k)\{f, g\} : \mathrm{Tm}(\Delta, \sigma\{f\}) \\
&(h, k)\{f, g\} = (h, k) \circ (f, g) = (h \circ f, \lambda x, y \,.\, k(f(x), g(x, y)))
\end{aligned}
$$

For the context comprehension, define

$$\begin{aligned}
(X,Y).(A,B) &= (\sum_{x\in X} A(x), \lambda\langle x,a\rangle \,.\, \sum_{y\in Y(x)} B(x,y,a)) \\
\mathbf{p}(A,B) &= (\mathrm{fst}, \lambda x.\mathrm{fst}) \\
\mathbf{v}_{A,B} &= (\mathrm{snd}, \lambda x.\mathrm{snd})
\end{aligned}$$

Given $(f,g) : (X',Y') \to (X,Y)$ and $(h,k) \in \mathrm{Tm}((X',Y'),(A,B)\{f,g\})$, we define

$$(\theta,\psi) = \langle (f,g),(h,k)\rangle_{(A,B)} : (X',Y') \to (X,Y).(A,B)$$

by $\theta(x) = \langle f(x), h(x)\rangle$, $\psi(x,y) = \langle g(x,y), k(x,y)\rangle$. It is clear that $\mathbf{p}(\sigma)\circ\theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$ and that these conditions force (θ,ψ) to be unique.

4 BiAlg(F,G) for $F,G : \mathbb{C} \to \mathbb{D}$

Lemma 2. *BiAlg(F,G) has a terminal object if $\mathbb{C}$ and $\mathbb{D}$ does, and G preserves terminal objects (i.e. $G(\mathbf{1}_\mathbb{C}) \cong \mathbf{1}_\mathbb{D}$).*

Proof. Define $\mathbf{1}_{\mathrm{BiAlg}(F,G)} \coloneqq (\mathbf{1}_\mathbb{C}, !_{F(\mathbf{1}_\mathbb{C})})$ where $!_{F(\mathbf{1}_\mathbb{C})}$ is the unique map $F(\mathbf{1}_\mathbb{C}) \to \mathbf{1}_D$. For any object (X,f), the unique morphism $(X,f) \to (\mathbf{1}_\mathbb{C}, !_{F(\mathbf{1}_\mathbb{C})})$ is given by the unique arrow $!_X$ from X to $\mathbf{1}_C$ in $\mathbb{C}$, and the diagram

$$\begin{tikzcd}
FX \arrow[r, "f"] \arrow[d, "F(!_X)"'] & GX \arrow[d, "G(!_X)"] \\
F(\mathbf{1}_C) \arrow[r, "!_{F(\mathbf{1}_C)}"'] & G(\mathbf{1}_\mathbb{C}) = \mathbf{1}_\mathbb{D}
\end{tikzcd}$$

commutes since both paths are arrows into $\mathbf{1}_\mathbb{D}$, hence equal. ◻

4.1 Some CwF preliminaries

Clairambault [Cla06, 4.1] defines a category $\mathrm{Type}_\mathbb{C}(\Gamma)$ of types in context Γ from the base category $\mathbb{C}$. The morphisms between $A,B \in \mathrm{Ty}_\mathbb{C}(\Gamma)$ are defined to be the terms $f \in \mathrm{Tm}_\mathbb{C}(\Gamma.A, B\{\mathbf{p}(A)\})$, with identity given by $\mathbf{v}_A$. We will be mostly interested in the composition of two terms $f \in \mathrm{Tm}_\mathbb{C}(\Gamma.A, B\{\mathbf{p}(A)\})$ and $g \in \mathrm{Tm}_\mathbb{C}(\Gamma.B, C\{\mathbf{p}(B)\})$, which is defined to be

$$g \bullet f \coloneqq g\{\langle \mathbf{p}(A), f\rangle_B\}.$$

The following proposition says that comprehension is a functor from "families in $\mathbb{C}$" to $\mathbb{C}$, which is quite convenient.

Lemma 3. *Given $g : \Gamma' \to \Gamma$ and $M \in Tm(\Gamma'.\sigma', \sigma\{g\circ\boldsymbol{p}(\sigma')\})$, one can construct $\psi : \Gamma'.\sigma' \to \Gamma.\sigma$.*

Proof. Take $\psi \coloneqq \langle g \circ \mathbf{p}(\sigma'), M\rangle_\sigma$. ◻

Lemma 4. *Let $f : \Delta \to \Gamma$, $M \in Tm(\Delta, \sigma\{f\})$, $h : \Theta \to \Delta$. Then $\langle f, M\rangle_\sigma \circ h = \langle f \circ h, M\{h\}\rangle_{\sigma\{f\}}$.*

Proof. $\langle f, M\rangle_\sigma \circ h$ satisfies the universal property for $f \circ h$ and $M\{h\}$. □

Lemma 5. *For every $M \in Tm(\Gamma, \sigma)$, there is $\overline{M} : \Gamma \to \Gamma.\sigma$ such that $\boldsymbol{p}(\sigma) \circ \overline{M} = id$ and $\boldsymbol{v}_\sigma\{\overline{M}\} = M$.*

Proof. There is no choice but to define $\overline{M} := \langle \mathrm{id}, M\rangle_\sigma$. □

4.2 Some box preliminaries

In order to extend $\mathrm{BiAlg}(F, G)$ to a CwF (with $F, G : \mathbb{C} \to \mathbb{D}$), we assume that $\mathbb{C}$ and $\mathbb{D}$ are CwFs, and that $\Box_F$ and $\Box_G$ exist and satisfy certain requirements. We collect these here.

Definition 6. Assume that $\mathbb{C}$ and $\mathbb{D}$ are CwFs. They *has boxes*, if for each $F : \mathbb{C} \to \mathbb{D}$, $\Gamma \in \mathbb{C}$ and $\sigma \in \mathrm{Ty}(\Gamma)$, there exists $\Box_F(\Gamma, \sigma) \in \mathrm{Ty}(F(\Gamma))$ and an isomorphism $\varphi_F : F(\Gamma.\sigma) \to F(\Gamma).\Box_F\,(\Gamma, \sigma)$ such that $\mathbf{p} \circ \varphi = F(\mathbf{p})$.

We also require a morphism part of $\Box_F$, namely that for $f : \Delta \to \Gamma$, $M \in \mathrm{Tm}(\Delta, \sigma\{f\})$ we have $\Box_F(f, M) \in \mathrm{Tm}(F(\Delta), \Box_F(\Gamma, \sigma)\{F(f)\})$ with "naturality condition" $\varphi_F \circ F(\langle f, M\rangle) = \langle F(f), \Box_F(f, M)\rangle$.

Assume that $\mathbb{C}$ and $\mathbb{D}$ are CwFs with boxes. We assume that for $f : \Delta \to \Gamma$, we have

$$\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\}) \tag{1}$$

and similarly $\Box_G(\mathrm{id}_\Gamma, N)\{G(f)\} = \Box_G(\mathrm{id}_\Delta, N\{f\})$ for all $N \in \mathrm{Tm}(\Gamma, \sigma)$ and $f : \Delta \to \Gamma$. Here, $\Box_G(\mathrm{id}_\Gamma, N)\{G(f)\} \in \mathrm{Tm}(G(\Delta), \Box_G(\Gamma, \sigma)\{G(f)\})$ and $\Box_G(\mathrm{id}_\Delta, N\{f\}) \in \mathrm{Tm}(G(\Delta), \Box_G(\Delta, \sigma\{f\}))$, so both sides of the equation have the same type by (1).

Remark 7. Demanding equality on the nose instead of isomorphism simplifies matters – we are spared transporting terms hidden inside substitutions along the isomorphisms. I guess it should be possible in principle though.

However, with the usual definition of $\Box_G$, one (almost) never has equality. (In Set, for example, the left hand side is $\{y : G(\Sigma\ \Gamma\ \sigma) \mid \ldots\}$ and the right hand side $\{y : G(\Sigma\ \Delta\ (\sigma \circ f)) \mid \ldots\}$.) If $G = U$ is a forgetful functor, though, then the usual definition of $\Box_U(\Gamma, \sigma)$ is isomorphic to a $X(\Gamma, \sigma)$ such that $X(\Gamma, \sigma)\{U(f)\} = X(\Delta, \sigma\{f\})$. I see no harm in replacing $\Box_U(\Gamma, \sigma)$ with $X(\Gamma, \sigma)$ for U, so that we get an actual equality? The properties we need $\Box_U$ to have are of course preserved by isomorphism anyway.

For F, we only require the existence of

$$\phi_F(f) \in \mathrm{Tm}(F(\Delta).\Box_F\,(\Delta, \sigma\{f\}), \Box_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}(\Box_F(\Delta, \sigma\{f\}))\})$$

which should be functorial in f, i.e. $\phi_F(\mathrm{id}) = \mathbf{v}_\sigma$ and

$$\phi_F(f \circ g) = \phi_F(f)\{\langle F(g) \circ \mathbf{p}, \phi_F(g)\rangle\}.$$

Remark 8. $\phi_F(f)$ and $\phi_F(g)$ are not composable in $\mathrm{Type}(F(\Gamma))$, as their types depend on f and g, but the "composition" above should be composition in some more liberal category (where $\mathbf{v}_\sigma$ still is the identity)? It is in any case exactly what we need, and holds e.g. in Set (I have not checked Fam(Set), but would be very surprised if it did not hold).

We assume that $\Box_F$, $\phi_F(f)$ and substitution relate in the following ways: for every $f : \Delta \to \Gamma$, $M \in \mathrm{Tm}(\Gamma, \sigma\{f\})$ and $g : \Theta \to \Delta$, we have

- $\Box_F(f, M)\{F(g)\} = \Box_F(f \circ g, M\{g\})$ ("$\Box_F(f \circ g) = \Box_F(f) \circ \Box_F(g)$"),
- $\Box_F(\mathbf{p}, \mathbf{v}_\sigma) = \mathbf{v}_{\Box_F(\Gamma,\sigma)}\{\varphi_F\}$ ("$\Box_F(\mathrm{id}) = \mathrm{id}$"),
- $\phi_F(f)\{\overline{\Box_F(\mathrm{id}, M)}\} = \Box_F(f, M)$.

4.3 The construction

4.3.1 Types

Define

$$\mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Gamma, h) = \{(\sigma, M) \mid \sigma \in \mathrm{Ty}_{\mathbb{C}}(\Gamma), M \in \mathrm{Tm}_{\mathbb{D}}(F(\Gamma).\Box_F(\Gamma, \sigma), \Box_G(\Gamma, \sigma)\{h \circ \mathbf{p}\})\}$$

For substitutions, assume $f : (\Delta, h') \to (\Gamma, h)$, i.e. $f : \Delta \to \Gamma$ and $G(f) \circ h' = h \circ F(f)$. Define for $(\sigma, M) \in \mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Gamma, h)$

$$(\sigma, M)\{f\} = (\sigma\{f\}, M\{\langle F(f) \circ \mathbf{p}, \phi_F(f)\rangle\}) \in \mathrm{Ty}_{\mathrm{BiAlg}(F,G)}(\Delta, h')$$

We should check that this makes sense. Since $\sigma \in \mathrm{Ty}_{\mathbb{C}}(\Gamma)$, we have $\sigma\{f\} \in \mathrm{Ty}_{\mathbb{C}}(\Delta)$. We now need a term in $\mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Delta, \sigma\{f\})\{h \circ \mathbf{p}\})$. Since $F(f) : F(\Delta) \to F(\Gamma)$ and

$$\phi_F \in \mathrm{Tm}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}\}),$$

Lemma 3 applies and we get $g := \langle F(f) \circ \mathbf{p}, \phi_F\rangle : F(\Delta).\Box_F(\Delta, \sigma\{f\}) \to F(\Gamma).\Box_F(\Gamma, \sigma)$, so that

$$M\{g\} \in \mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Gamma, \sigma)\{h \circ \mathbf{p} \circ g\})$$

and since

$$h \circ \mathbf{p} \circ g = h \circ \mathbf{p} \circ \langle F(f) \circ \mathbf{p}, \phi_F\rangle = h \circ F(f) \circ \mathbf{p} = G(f) \circ h' \circ \mathbf{p}$$

and $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$, we in fact have

$$M\{g\} \in \mathrm{Tm}_{\mathbb{D}}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_G(\Delta, \sigma\{f\})\{h' \circ \mathbf{p}\})$$

as needed. Functoriality follows from functoriality of $\phi_F(f)$ and functoriality one level down:

$$\begin{aligned}(\sigma, M)\{\mathrm{id}\} &= (\sigma\{\mathrm{id}\}, M\{\langle F(\mathrm{id}) \circ \mathbf{p}, \phi_F(\mathrm{id})\rangle\}) \\ &= (\sigma, M\{\langle p, \mathbf{v}_\sigma\rangle\}) = (\sigma, M\{\mathrm{id}\}) = (\sigma, M)\end{aligned}$$

$$
\begin{aligned}
(\sigma, M)\{f\}\{g\} &= (\sigma\{f\}\{g\}, M\{\langle F(f)\circ \mathbf{p}, \phi_F(f)\rangle\}\{\langle F(g)\circ \mathbf{p}, \phi_F(g)\rangle\}) \\
&= (\sigma\{f\circ g\}, M\{\langle F(f)\circ \mathbf{p}, \phi_F(f)\rangle \circ \langle F(g)\circ \mathbf{p}, \phi_F(g)\rangle\}) \\
&= (\sigma\{f\circ g\}, M\{\langle F(f)\circ \mathbf{p}\circ \langle F(g)\circ \mathbf{p}, \phi_F(g)\rangle, \phi_F(f)\{\langle F(g)\circ \mathbf{p}, \phi_F(g)\rangle\}) \\
&= (\sigma\{f\circ g\}, M\{\langle F(f)\circ F(g)\circ \mathbf{p}, \phi_F(f\circ g)) \\
&= (\sigma\{f\circ g\}, M\{\langle F(f\circ g)\circ \mathbf{p}, \phi_F(f\circ g)) \\
&= (\sigma, M)\{f\circ g\}
\end{aligned}
$$

4.3.2 Terms

Define

$$\mathrm{Tm}((\Gamma, h), (\sigma, M)) = \{N \in \mathrm{Tm}_{\mathbb{C}}(\Gamma, \sigma) \mid \Box_G\,(\mathrm{id}_\Gamma, N)\{h\} = M\{\overline{\Box_F(\mathrm{id}_\Gamma, N)}\}\}$$

If $f : (\Delta, h') \to (\Gamma, h)$, we define $N\{f\}$ for $N \in \mathrm{Tm}((\Gamma, h), (\sigma, M))$ to be $N\{f\}$ inherited from $\mathbb{C}$. We have to check that $N\{f\} \in \mathrm{Tm}((\Delta, h'), (\sigma, M)\{f\})$ $= \mathrm{Tm}((\Delta, h'), (\sigma\{f\}, M\{\langle F(f)\circ \mathbf{p}, \phi_F(f)\rangle\}))$, i.e. that

$$\Box_G(\mathrm{id}_\Delta, N\{f\})\{h'\} = M\{\langle F(f)\circ \mathbf{p}, \phi_F(f)\rangle\}\{\overline{\Box_F(\mathrm{id}_\Delta, N\{f\})}\}$$

given that $\Box_G(\mathrm{id}_\Gamma, N)\{h\} = M\{\overline{\Box_F(\mathrm{id}_\Gamma, N)}\}$. We calculate

$$
\begin{aligned}
\Box_G(\mathrm{id}_\Delta, N\{f\})\{h'\} &= \Box_G(\mathrm{id}_\Gamma, N)\{G(f)\}\{h'\} \\
&= \Box_G(\mathrm{id}_\Gamma, N)\{h\}\{F(f)\} \\
&= M\{\langle \mathrm{id}_{F(\Gamma)}, \Box_F(\mathrm{id}_\Gamma, N)\rangle\}\{F(f)\} \\
&= M\{\langle F(f), \Box_F(\mathrm{id}_\Gamma, N)\{F(f)\}\rangle\} \\
&= M\{\langle F(f), \Box_F(f, N\{f\})\rangle\} \\
&= M\{\langle F(f), \phi_F(f)\{\overline{\Box_F(\mathrm{id}, N\{f\})}\rangle\} \\
&= M\{\langle F(f)\circ \mathbf{p}, \phi_F(f)\rangle\}\{\overline{\Box_F(\mathrm{id}, N\{f\})}\}.
\end{aligned}
$$

4.3.3 Context comprehension

Given $(\Gamma, h) \in \mathrm{BiAlg}(F, G)$ and $(\sigma, M) \in \mathrm{Ty}(\Gamma, h)$, we define

$$
\begin{aligned}
(\Gamma, h).(\sigma, M) &:= (\Gamma.\sigma, \varphi_G^{-1} \circ \langle h\circ \mathbf{p}, M\rangle \circ \varphi_F) \\
\mathbf{p}((\sigma, M)) &:= \mathbf{p}(\sigma) \\
\mathbf{v}_{(\sigma, M)} &:= \mathbf{v}_\sigma \\
\langle f, N\rangle_{(\sigma, M)} &:= \langle f, N\rangle_\sigma
\end{aligned}
$$

We have to check that (i) $\mathbf{p}((\sigma, M)) : (\Gamma, h).(\sigma, M) \to (\Gamma, h)$, (ii) $\mathbf{v}_{(\sigma,M)} \in \mathrm{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{\mathbf{p}((\sigma, M))\})$ and that (iii) $\langle f, N\rangle_\sigma : (\Delta, h') \to (\Gamma, h).(\sigma, M)$ when $f : (\Delta, h') \to (\Gamma, h)$ and $N \in \mathrm{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{p\})$.

For (i), we need that $G(\mathbf{p}) \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F = h \circ F(\mathbf{p})$. But $G(\mathbf{p}) = \mathbf{p} \circ \varphi_G$, so

$$\begin{aligned} G(\mathbf{p}) \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F &= \mathbf{p} \circ \varphi_G \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \\ &= \mathbf{p} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F \\ &= h \circ \mathbf{p} \circ \varphi_F \\ &= h \circ F(\mathbf{p}). \end{aligned}$$

Hence $\mathbf{p}((\sigma, M)) : (\Gamma, h).(\sigma, M) \to (\Gamma, h)$. For (ii), we need to show that $\mathbf{v}_\sigma \in \mathrm{Tm}((\Gamma.\sigma, \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F), (\sigma\{\mathbf{p}\}, M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p})\}))$, i.e. that $\Box_G(\mathrm{id}_\Gamma, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} = M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p})\rangle\}\{\overline{\Box_F(\mathrm{id}_\Gamma, \mathbf{v}_\sigma)}\}$. First, note that $\Box_G(\mathrm{id}_\Gamma, \mathbf{v}_\sigma) = \Box_G(\mathbf{p}(\sigma), \mathbf{v}_\sigma)$ because

$$\begin{aligned} \Box_G(\mathrm{id}, \mathbf{v}) &= \Box_G(\mathbf{p} \circ \overline{\mathbf{v}}, \mathbf{v}\{\overline{\mathbf{v}}\}) = \Box_G(\mathbf{p}, \mathbf{v})\{G(\overline{\mathbf{v}})\} \\ &= \Box_G(\mathbf{p}, \mathbf{v}\{\overline{\mathbf{v}}\}) = \Box_G(\mathbf{p}, \mathbf{v}) \end{aligned}$$

Now

$$\begin{aligned} \Box_G(\mathrm{id}_\Gamma, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} &= \Box_G(\mathbf{p}, \mathbf{v}_\sigma)\{\varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= \mathbf{v}_\sigma\{\varphi_G \circ \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= \mathbf{v}_\sigma\{\langle h \circ \mathbf{p}, M \rangle \circ \varphi_F\} \\ &= M\{\varphi_F\} \end{aligned}$$

but also

$$\begin{aligned} M\{\langle F(\mathbf{p}) \circ \mathbf{p}, \phi_F(\mathbf{p})\rangle\}\{\overline{\Box_F(\mathrm{id}_\Gamma, \mathbf{v}_\sigma)}\} &= M\{\langle F(\mathbf{p}), \phi_F(\mathbf{p})\{\overline{\Box_F(\mathrm{id}_\Gamma, \mathbf{v}_\sigma)}\}\rangle\} \\ &= M\{\langle F(\mathbf{p}), \Box_F(\mathbf{p}, \mathbf{v}_\sigma)\rangle\} \\ &= M\{\varphi_F \circ F(\langle \mathbf{p}, \mathbf{v}_\sigma \rangle)\} \\ &= M\{\varphi_F \circ F(\mathrm{id})\} \\ &= M\{\varphi_F\}. \end{aligned}$$

Finally, we have to check that $\langle f, N \rangle$ really is a morphism. (Uniqueness of $\langle f, N \rangle$ is of course inherited from $\mathbb{C}$.) We are given $f : (\Delta, h') \to (\Gamma, h)$ and $N \in \mathrm{Tm}((\Gamma, h).(\sigma, M), (\sigma, M)\{p\})$, that is we have $G(f) \circ h' = h \circ F(f)$ and

$$\Box_G(\mathrm{id}_\Delta, N)\{h'\} = M\{\langle F(f) \circ \mathbf{p}, \phi_F(f)\rangle\}\{\overline{\Box_F(\mathrm{id}_\Delta, N)}\}$$

which simplifies to

$$\Box_G(\mathrm{id}_\Delta, N)\{h'\} = M\{\langle F(f), \Box_F(f, N)\rangle\}$$

We also need that $\Box_G(\mathrm{id}_\Delta, N) = \Box_G(f, N)$ (they have the same type thanks to our assumption $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$) – can this be proven from the facts we have or should it be added to our global list of assumptions? Thus, we have

$$\Box_G(f, N)\{h'\} = \Box_G(\mathrm{id}, N)\{h'\} = M\{\langle F(f), \Box_F(f, N)\rangle\}$$

We have to show that

$$G(\langle f, N\rangle) \circ h' = \varphi_G^{-1} \circ \langle h \circ \mathbf{p}, M\rangle \circ \varphi_F \circ F(\langle f, N\rangle)$$

or equivalently

$$\varphi_G \circ G(\langle f, N\rangle) \circ h' = \langle h \circ \mathbf{p}, M\rangle \circ \varphi_F \circ F(\langle f, N\rangle)$$

We have

$$\begin{aligned}
\varphi_G \circ G(\langle f, N\rangle) \circ h' &= \langle G(f), \Box_G(f, N)\rangle \circ h' \\
&= \langle G(f) \circ h', \Box_G(f, N)\{h'\}\rangle \\
&= \langle h \circ F(f), M\{\langle F(f), \Box_F(f, N)\rangle\}\rangle
\end{aligned}$$

and also

$$\begin{aligned}
\langle h \circ \mathbf{p}, M\rangle \circ \varphi_F \circ F(\langle f, N\rangle) &= \langle h \circ \mathbf{p}, M\rangle \circ \langle F(f), \Box_F(f, N)\rangle \\
&= \langle h \circ F(f), M\{\langle F(f), \Box_F(f, N)\rangle\}\rangle
\end{aligned}$$

and we are done.

Question 1. Can we recover the inverse image type (and hence $\Box_F$) from $\mathbb{C}$ as well?

5 The equivalence of elim and init in the CwF formulation

Since our constructor now has type in : $F(A) \to G(A)$, we need to change the type of elim accordingly. This means that the type and specification of dmap_F has to be changed as well; in particular, it depends on both F and G, and should more accurately be called $\mathrm{dmap}_{F,G}$.

Definition 9. For every $g \in \mathrm{Tm}(G(\Gamma), \Box_G(\Gamma, \sigma))$, we demand the existence of $\mathrm{dmap}_{F,G}(\Gamma, \sigma, g) \in \mathrm{Tm}(F(\Gamma), \Box_F(\Gamma, \sigma))$ such that if $f \in \mathrm{Tm}(\Gamma, \sigma)$ then

$$\varphi_F \circ F(\overline{f}) = \overline{\mathrm{dmap}_F(\Gamma, \sigma, \mathbf{v}\{\varphi_G \circ G(\overline{f})\})}.$$

Note that if $G = \mathrm{ID}$, then $\Box_G(A, P) = P$ and $\varphi_G = \mathrm{id}$, and the above equation collapses to $\varphi_F \circ F(\overline{f}) = \overline{\mathrm{dmap}_F(\Gamma, \sigma, f)}$, since $\mathbf{v}\{\overline{f}\} = f$.

Let $F, G : \mathbb{C} \to \mathbb{D}$ with $\mathbb{C}$ and $\mathbb{D}$ CwFs. The elimination principle for $A \in \mathbb{C}$, in : $F(A) \to G(A)$ says that if $P \in \mathrm{Ty}(A)$ and $g \in \mathrm{Tm}(F(A). \Box_F (A, P), \Box_G(A, P)\{\mathrm{in} \circ \mathbf{p}\})$ then there exist $\mathrm{elim}(P, g) \in \mathrm{Tm}(G(A), \Box_G(A, P))$. The computation rule says

$$\mathrm{elim}(P, g)\{\mathrm{in}\} = g\{\overline{\mathrm{dmap}(A, P, \mathrm{elim}(P, g))}\}.$$

For $G = \mathrm{ID}$ the identity functor, we can choose $\Box_{\mathrm{ID}}(A, P) = P$, and this reduces to $\mathrm{elim}(P, g) \in \mathrm{Tm}(A, P)$ if $g \in \mathrm{Tm}(F(A).\Box_F(A, P), P\{\mathrm{in} \circ \mathbf{p}\})$. In Set, this means that $\mathrm{elim}(P, g) : (x : A) \to P(x)$ if $g : (x : F(A)) \to \Box_F(A, P, x) \to P(\mathrm{in}(x))$.

5.1 Init $\Longrightarrow$ elim

Theorem 10. *Let $F, G : \mathbb{C} \to \mathbb{D}$ with $\mathbb{C}$ and $\mathbb{D}$ CwFs. If (A, in) is initial in BiAlg(F, G) then the elimination principle holds for (A, in).*

Proof. Let $P \in \mathrm{Ty}(A)$ and $g \in \mathrm{Tm}(F(A).\Box_F(A, P), \Box_G(A, P)\{\mathrm{in} \circ \mathbf{p}\})$ be given. Then $h := \varphi_G^{-1} \circ \langle \mathrm{in} \circ p, g\rangle \circ \varphi_F : F(A.P) \to G(A.P)$, so by initiality, we get $\mathrm{fold}(h) : A \to A.P$ such that $h \circ F(\mathrm{fold}(h)) = G(\mathrm{fold}(h)) \circ \mathrm{in}$. Hence the following diagram commutes:

$$\begin{tikzcd}
F(A) \arrow[rrr, "\mathrm{in}"] \arrow[d, "F(\mathrm{fold}(h))"'] & & & G(A) \arrow[d, "G(\mathrm{fold}(h))"] \\
F(A.P) \arrow[dr, "\varphi_F"] \arrow[dd, "F(\mathbf{p})"'] & & & G(A.P) \arrow[dl, "\varphi_G"', shift right] \arrow[dd, "G(\mathbf{p})"] \\
& F(A).\Box_F(A, P) \arrow[r, "\langle \mathrm{in}\circ p, g\rangle"] \arrow[dl, "\mathbf{p}"] & G(A).\Box_G(A, P) \arrow[ur, "\varphi_G^{-1}"', shift right] \arrow[dr, "\mathbf{p}"] & \\
F(A) \arrow[rrr, "\mathrm{in}"'] & & & G(A)
\end{tikzcd}$$

This means that $\mathbf{p} \circ \mathrm{fold}(h)$ is a morphism in $\mathrm{BiAlg}(F, G)$, so by initiality, we must have $\mathbf{p} \circ \mathrm{fold}(h) = \mathrm{id}$. We now define $\mathrm{elim}(P, g) := \mathbf{v}\{\varphi_G \circ G(\mathrm{fold}(h))\}$. We then have

$$\begin{aligned}
\mathrm{elim}(p, G) \in \mathrm{Tm}(G(A), &\Box_G(A, P)\{\mathbf{p} \circ \varphi_G \circ U(\mathrm{fold}(h))\}) \\
= \mathrm{Tm}(G(A), &\Box_G(A, P)\{U(\mathbf{p}) \circ U(\mathrm{fold}(h))\}) \\
= \mathrm{Tm}(G(A), &\Box_G(A, P)\{U(\mathbf{p} \circ \mathrm{fold}(h))\}) \\
= \mathrm{Tm}(G(A), &\Box_G(A, P)\{U(\mathrm{id})\}) \\
= \mathrm{Tm}(G(A), &\Box_G(A, P))
\end{aligned}$$

as required.

We must check that the computation rule $\mathrm{elim}(P, g)\{\mathrm{in}\} = g\{\overline{\mathrm{dmap}(A, P, \mathrm{elim}(P, g))}\}$ holds. Note first that since $\mathbf{p} \circ \mathrm{fold}(h) = \mathrm{id}$, we have

$$\mathrm{fold}(h) = \langle \mathbf{p} \circ \mathrm{fold}(h), \mathbf{v}\{\mathrm{fold}(h)\}\rangle = \langle \mathrm{id}, \mathbf{v}\{\mathrm{fold}(h)\}\rangle = \overline{\mathbf{v}\{\mathrm{fold}(h)\}}$$

Using this, we have

$$
\begin{aligned}
\mathrm{elim}(P, g)\{\mathrm{in}\} &= \mathbf{v}\{\varphi_G \circ G(\mathrm{fold}(h)) \circ \mathrm{in}\} \\
&= \mathbf{v}\{\varphi_G \circ \varphi_G^{-1} \circ \langle \mathrm{in} \circ \mathbf{p}, g\rangle \circ \varphi_F \circ F(\mathrm{fold}(h))\} \\
&= g\{\varphi_F \circ F(\mathrm{fold}(h))\} \\
&= g\{\varphi_F \circ F(\overline{\mathbf{v}\{\mathrm{fold}(h)\}})\} \\
&= g\{\overline{\mathrm{dmap}(A, P, \mathbf{v}\{\varphi_G \circ G(\overline{\mathbf{v}\{\mathrm{fold}(h)\}})\})}\} \\
&= g\{\overline{\mathrm{dmap}(A, P, \mathbf{v}\{\varphi_G \circ G(\mathrm{fold}(h))\})}\} \\
&= g\{\overline{\mathrm{dmap}(A, P, \mathrm{elim}(P, g))}\}
\end{aligned}
$$

as required. □

Let $\mathbb{E}$ be the equaliser category which we intend to interpret inductive-inductive definitions in. It inherits a CwF structure from $\mathrm{BiAlg}(\widehat{G}, U)$.

Lemma 11. *$\mathbb{E}$ is preserved under $\Gamma.\sigma$, i.e. if $\Gamma \in \mathbb{E}$ then $\Gamma.\sigma \in \mathbb{E}$.*

Corollary 12. *Let $F, G : \mathbb{C} \to \mathbb{D}$ with $\mathbb{C}$ and $\mathbb{D}$ CwFs. If (A, in) is initial in $\mathbb{E}$ then the elimination principle holds for (A, in).*

5.2 Elim $\Longrightarrow$ weak init

This is the easy direction in the concrete case, but in the abstract setting, I cannot see how to avoid postulating some more properties of the CwFs $\mathbb{C}$ and $\mathbb{D}$, namely that they have "constant families".

Definition 13. A CwF $\mathbb{C}$ has *constant families* if there for each $\Gamma, \Delta \in \mathbb{C}$ exists $\check{\Delta} \in \mathrm{Ty}(\Gamma)$ such that $\mathrm{Tm}(\Theta, \check{\Delta}\{g\}) \cong \mathrm{Hom}(\Theta, \Delta)$ for all $g : \Theta \to \Gamma$. In other words:

- If $f \in \mathrm{Tm}(\Theta, \check{\Delta}\{g\})$ then $\check{f} : \Theta \to \Delta$.
- If $f : \Theta \to \Delta$ then $\hat{f} \in \mathrm{Tm}(\Theta, \check{\Delta}\{g\})$.
- $\hat{\check{f}} = \check{\hat{f}} = f$.

We also require $\check{f} \circ g = \widetilde{f\{g\}}$.

As usual, we have $F, G : \mathbb{C} \to \mathbb{D}$, and we insist that both $\mathbb{C}$ and $\mathbb{D}$ have constant families. We also insist that $\widetilde{G(\Delta)} = \Box_G(\Gamma, \check{\Delta})$.

Here is something which I don't know how to state, but which is quite important for the argument to go through: Given $f : G(X) \to G(Y)$, we want to "lift" this to $\hat{f} : X \to Y$ such that $G(\hat{f}) = f$, but this will not be possible for all such f. To be more concrete: the functor G we are interested in is a forgetful functor $G : \mathrm{BiAlg}(F, V) \to \mathbb{C}$, $G(X, h) = X$, $G(f, p) = f$. Thus, lifting $f : G(X, h) \to G(Y, h')$ would mean that $V(f) \circ h = h' \circ F(f)$ which of course is not true for all f.

However, the f we are interested in will satisfy its own equation $G(f) \circ \overline{h} = \overline{h}' \circ H(f)$ which will contain the equation $V(f) \circ h = h' \circ F(f)$ because we have applied the equaliser (which means that $\overline{h} = (h, k)$ etc). Thus, this particular f *can* be lifted, but how to state this property abstractly? (I suppose a formulation mentioning forgetful functors $\mathrm{BiAlg}(F, V) \to \mathbb{C}$ would be acceptable, but it would be nicer to keep it more abstract.) For now, I will just assume that I have this lifting and that it satisfies

$$M = \mathbf{v}\{\varphi_G \circ G(\overline{\widehat{M'}})\}$$

where $M \in \mathrm{Tm}(G(\Gamma), \square_G(\Gamma, \check{\Delta}))$ and $M' = \widehat{\widehat{M}}$.

Theorem 14. *Let $F, G : \mathbb{C} \to \mathbb{D}$ with $\mathbb{C}$ and $\mathbb{D}$ CwFs with constant families. Assume the assumptions mentioned above. If the elimination principle holds for (Γ, in), then (Γ, in) is weakly initial in $\mathbb{E}$.*

Proof. Let $(\Delta, h) \in \mathbb{E}$. We have to construct $\mathrm{fold}(h) : \Gamma \to \Delta$ such that $G(\mathrm{fold}(h)) \circ \mathrm{in} = h \circ F(\mathrm{fold}(h))$.

Notice that $\mathbf{v}_{\check{\Delta}} \in \mathrm{Tm}(\Gamma.\check{\Delta}, \check{\Delta}\{\mathbf{p}\})$ so that $\check{\mathbf{v}} : \Gamma.\check{\Delta} \to \Delta$. Hence we have $\psi := h \circ F(\check{\mathbf{v}}) \circ \varphi_F^{-1} : F(\Gamma).\square_F\, (\Gamma, \check{\Delta}) \to G(\Delta)$. Since $\widecheck{G(\Delta)} = \square_G(\Gamma, \check{\Delta})$, we then have $\widehat{\psi} \in \mathrm{Tm}(F(\Gamma).\square_F\, (\Gamma, \check{\Delta}), \square_G(\Gamma, \check{\Delta})\{\mathrm{in} \circ p\})$ so that $\mathrm{elim}(\check{\Delta}, \widehat{\psi}) \in \mathrm{Tm}(G(\Gamma), \square_G(\Gamma, \check{\Delta})$. Hence $\zeta := \mathrm{elim}(\check{\Delta}, \widehat{\psi}) : G(\Gamma) \to G(\Delta)$.

Now consider $\widehat{\zeta} : \Gamma \to \Delta$. <Argument why $\widehat{\zeta}$ exists here.> Note that $\mathrm{elim}(\check{\Delta}, \widehat{\psi}) = \mathbf{v}\{\varphi_G \circ G(\overline{\widehat{\widehat{\zeta}}})\}$ and calculate:

$$\begin{aligned}
U(\widehat{\zeta}) \circ \mathrm{in} = \zeta \circ \mathrm{in} &= \widecheck{\mathrm{elim}(\check{\Delta}, \widehat{\psi})} \circ \mathrm{in} \\
&= \widecheck{\mathrm{elim}(\check{\Delta}, \widehat{\psi})}\{\mathrm{in}\} \\
&= \widehat{\psi}\{\overline{\mathrm{dmap}(\Gamma, \check{\Delta}, \widecheck{\mathrm{elim}(\check{\Delta}, \widehat{\psi})})}\} \\
&= \psi \circ \overline{\mathrm{dmap}(\Gamma, \check{\Delta}, \mathrm{elim}(\check{\Delta}, \widehat{\psi}))} \\
&= h \circ F(\check{\mathbf{v}}) \circ \varphi_F^{-1} \circ \overline{\mathrm{dmap}(\Gamma, \check{\Delta}, \mathbf{v}\{\varphi_G \circ G(\overline{\widehat{\widehat{\zeta}}})\})} \\
&= h \circ F(\check{\mathbf{v}}) \circ \varphi_F^{-1} \circ \varphi_F \circ F(\overline{\widehat{\widehat{\zeta}}}) \\
&= h \circ F(\check{\mathbf{v}} \circ \overline{\widehat{\widehat{\zeta}}}) = h \circ F(\widecheck{\mathbf{v}\{\overline{\widehat{\widehat{\zeta}}}\}}) \\
&= h \circ F(\check{\widehat{\widehat{\zeta}}}) = h \circ F(\widehat{\zeta})
\end{aligned}$$

Thus we can define $\mathrm{fold}(h) = \widehat{\zeta}$. ∎

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