All is well if G preserves pullbacks

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Abstract

Using too many diagrams, which makes the text a lot longer than it could be, we derive the special properties needed of G from the assumption that G preserves pullbacks.

1 General assumptions

For every $\Gamma \in \mathbb{C}$ and $\sigma \in \mathrm{Ty}(\Gamma)$, there exists a unique $\Box_F(\Gamma, \sigma) \in \mathrm{Ty}(F(\Gamma))$ and an isomorphism $\varphi_F : F(\Gamma.\sigma) \to F(\Gamma).\Box_F(\Gamma, \sigma)$ such that $\mathbf{p} \circ \varphi = F(\mathbf{p})$. [Up to iso etc.]

We have a morphism part: for $f : \Delta \to \Gamma$, $M \in \mathrm{Tm}(\Delta, \sigma\{f\})$ there exists $\Box_F(f, M) \in \mathrm{Tm}(F(\Delta), \Box_F(\Gamma, \sigma)\{F(f)\})$.

Finally, we have for every $f : \Delta \to \Gamma$, $\sigma \in \mathrm{Ty}(\Gamma)$ a term

$$\phi_F(f) \in \mathrm{Tm}(F(\Delta).\Box_F(\Delta, \sigma\{f\}), \Box_F(\Gamma, \sigma)\{F(f) \circ \mathbf{p}\}).$$

We use the following global assumptions, which we expect to hold for every $F : \mathbb{C} \to \mathbb{D}$:

- $\varphi_F \circ F(\langle f, M \rangle) = \langle F(f), \Box_F(f, M) \rangle$.
- $\phi_F(f \circ g) = \phi_F(f)\{\langle F(g) \circ \mathbf{p}, \phi_F(g) \rangle\}$.
- $\phi_F(f)\{\overline{\Box_F(\mathrm{id}, M)}\} = \Box_F(f, M)$.

The only assumption on G is that it preserves pullbacks.

2 $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$

Theorem 1. *if G preserves pullbacks, then $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$.*

Proof. We show that there is an isomorphism $\gamma : G(\Delta.\sigma\{f\}) \to G(\Delta).\Box_G(\Gamma, \sigma)\{G(f)\}$ satisfying $\mathbf{p} \circ \gamma = G(\mathbf{p})$. By the uniqueness of $\Box_G(\Delta, \sigma\{f\})$, we must thus have $\Box_G(\Gamma, \sigma)\{G(f)\} = \Box_G(\Delta, \sigma\{f\})$.

The following diagram is always a pullback in every CwF:

$$\begin{tikzcd}
\Delta.\sigma\{f\} \arrow[r, "\langle f\circ\mathbf{p},\mathbf{v}\rangle"] \arrow[d, "\mathbf{p}"'] & \Gamma.\sigma \arrow[d, "\mathbf{p}"] \\
\Delta \arrow[r, "f"] & \Gamma
\end{tikzcd}$$

Since G preserves pullbacks, also

$$\begin{tikzcd}
G(\Delta.\sigma\{f\}) \arrow[r, "G(\langle f\circ\mathbf{p},\mathbf{v}\rangle)"] \arrow[d, "G(\mathbf{p})"'] & G(\Gamma.\sigma) \arrow[d, "G(\mathbf{p})"] \\
G(\Delta) \arrow[r, "G(f)"] & G(\Gamma)
\end{tikzcd}$$

is a pullback square. But now note that

$$G(\mathbf{p}) \circ \varphi_G \circ \langle G(f) \circ \mathbf{p}, \mathbf{v}\rangle = \mathbf{p} \circ \langle G(f) \circ \mathbf{p}, \mathbf{v}\rangle = G(f) \circ \mathbf{p}$$

so that we get a mediating arrow h in the following pullback diagram:

$$\begin{tikzcd}
G(\Delta).\,\Box_G\,(\Gamma,\sigma)\{G(f)\} \arrow[rr, "\langle G(f)\circ\mathbf{p},\mathbf{v}\rangle"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "\mathbf{p}"'] & & G(\Gamma).\,\Box_G\,(\Gamma,\sigma) \arrow[d, bend left, "\varphi_G^{-1}"] \\
& G(\Delta.\sigma\{f\}) \arrow[r, "G(\langle f\circ\mathbf{p},\mathbf{v}\rangle)"] \arrow[d, "G(\mathbf{p})"'] & G(\Gamma.\sigma) \arrow[d, "G(\mathbf{p})"] \\
& G(\Delta) \arrow[r, "G(f)"] & G(\Gamma)
\end{tikzcd}$$

This h will be our γ^{-1}. We can construct γ explicitly by $\gamma \coloneqq \langle \mathbf{p}, \phi_G(f)\rangle \circ \varphi_G$. By the uniqueness of the mediating arrow in the following diagram, we get $h \circ \gamma = \mathrm{id}$ for free, at least after we have checked that the outer triangles commute.

$$\begin{tikzcd}
G(\Delta.\sigma\{f\}) \arrow[r, dashed, "\varphi_G"] \arrow[ddrrr, bend left, "G(\langle f\circ\mathbf{p},\mathbf{v}\rangle)"] \arrow[dddrr, bend right, "G(\mathbf{p})"'] & G(\Delta).\,\Box_G\,(\Gamma,\sigma\{f\}) \arrow[d, dashed, "\langle\mathbf{p},\phi_G(f)\rangle"] & & \\
& G(\Delta).\,\Box_G\,(\Gamma,\sigma)\{G(f)\} \arrow[drr, bend left, "\varphi_G^{-1}\circ\langle G(f)\circ\mathbf{p},\mathbf{v}\rangle"] \arrow[dr, dashed, "h"] \arrow[ddr, bend right, "\mathbf{p}"] & & \\
& & G(\Delta.\sigma\{f\}) \arrow[r, "G(\langle f\circ\mathbf{p},\mathbf{v}\rangle)"] \arrow[d, "G(\mathbf{p})"'] & G(\Gamma.\sigma) \arrow[d, "G(\mathbf{p})"] \\
& & G(\Delta) \arrow[r, "G(f)"] & G(\Gamma)
\end{tikzcd}$$

The bottom one is straightforward; it can be split up into two commuting triangles:

$$\begin{tikzcd}
G(\Delta.\sigma\{f\}) \arrow[r, dashed, "\varphi_G"] \arrow[dr, "G(\mathbf{p})"'] & G(\Delta).\Box_G\,(\Gamma,\sigma\{f\}) \arrow[r, dashed, "\langle \mathbf{p}, \phi_G(f)\rangle"] \arrow[d, "\mathbf{p}"] & G(\Delta).\Box_G\,(\Gamma,\sigma)\{G(f)\} \arrow[dl, "\mathbf{p}"] \\
& G(\Delta) &
\end{tikzcd}$$

For the top triangle, we need the following identity:

Lemma 2. $\phi_G(f)\{\varphi_G\} = \Box_G(f \circ p, \boldsymbol{v})$

Proof. First, write φ_G in a complicated way:

$$\begin{aligned}
\varphi_G &= \varphi_G \circ \mathrm{id} \\
&= \varphi_G \circ G(\langle \mathbf{p}, \mathbf{v}\rangle) \\
&= \langle G(\mathbf{p}), \Box_G(\mathbf{p}, \mathbf{v})\rangle \\
&= \langle G(\mathbf{p}), \phi_G(\mathbf{p})\{\overline{\Box_G(\mathrm{id}, \mathbf{v})}\}\rangle \\
&= \langle G(\mathbf{p}) \circ \mathbf{p}, \phi_G(\mathbf{p})\rangle \circ \overline{\Box_G(\mathrm{id}, \mathbf{v})}
\end{aligned}$$

Then note that $\phi_G(f)\{\langle G(\mathbf{p}) \circ \mathbf{p}, \phi_G(\mathbf{p})\rangle\} = \phi_G(f \circ \mathbf{p})$. Thus, we have

$$\begin{aligned}
\phi_G(f)\{\varphi_G\} &= \phi_G(f)\{\langle G(\mathbf{p}) \circ \mathbf{p}, \phi_G(\mathbf{p})\rangle \circ \overline{\Box_G(\mathrm{id}, \mathbf{v})}\} \\
&= \phi_G(f \circ \mathbf{p})\{\overline{\Box_G(\mathrm{id}, \mathbf{v})}\} \\
&= \Box_G(f \circ \mathbf{p}, \mathbf{v}).
\end{aligned}$$

□

Now, we can just calculate: we want

$$\varphi_G^{-1} \circ \langle G(f) \circ \mathbf{p}, \phi_G(f)\rangle \circ \varphi_G = G(\langle f \circ \mathbf{p}, \mathbf{v}\rangle)$$

or equivalently

$$\langle G(f) \circ \mathbf{p}, \phi_G(f)\rangle \circ \varphi_G = \varphi_G \circ G(\langle f \circ \mathbf{p}, \mathbf{v}\rangle).$$

We have

$$\begin{aligned}
\langle G(f) \circ \mathbf{p}, \phi_G(f)\rangle \circ \varphi_G &= \langle \mathbf{p} \circ \varphi_G, \phi_G(f)\{\varphi_G\}\rangle \\
&= \langle G(f) \circ \mathbf{p} \circ \varphi_G, \phi_G(f)\{\varphi_G\}\rangle \\
&= \langle G(f) \circ G(\mathbf{p}), \phi_G(f)\{\varphi_G\}\rangle \\
&= \langle G(f) \circ G(\mathbf{p}), \Box_G(f \circ \mathbf{p}, \mathbf{v})\rangle \\
&= \langle G(f \circ \mathbf{p}), \Box_G(f \circ \mathbf{p}, \mathbf{v})\rangle \\
&= \varphi_G \circ G(\langle f \circ \mathbf{p}, \mathbf{v}\rangle).
\end{aligned}$$

Hence, we have $h \circ \gamma = \mathrm{id}$. To prove $\gamma \circ h = \mathrm{id}$, consider the commuting square

$$\begin{tikzcd}
G(\Delta).\,\square_G\,(\Gamma,\sigma)\{G(f)\} \arrow[rr, "\langle G(f)\circ \mathbf{p}, \mathbf{v}\rangle"] \arrow[dr, dashed, "h"] & & G(\Gamma).\,\square_G\,(\Gamma,\sigma) \arrow[d, bend left, "\varphi_G^{-1}"] \\
& G(\Delta.\sigma\{f\}) \arrow[r, "G(\langle f\circ \mathbf{p}, \mathbf{v}\rangle)"] & G(\Gamma.\sigma)
\end{tikzcd}$$

Since

$$\begin{aligned}
G(\langle f \circ \mathbf{p}, \mathbf{v}\rangle) \circ h &= \varphi^{-1} \circ \langle G(f \circ \mathbf{p}), \square_G(f \circ \mathbf{p}, \mathbf{v})\rangle \circ h \\
&= \varphi^{-1} \circ \langle G(f) \circ G(\mathbf{p}) \circ h, \square_G(f \circ \mathbf{p}, \mathbf{v})\{h\}\rangle \\
&= \varphi^{-1} \circ \langle G(f) \circ \mathbf{p}, \square_G(f \circ \mathbf{p}, \mathbf{v})\{h\}\rangle,
\end{aligned}$$

the square tells us that $\square_G(f \circ \mathbf{p}, \mathbf{v})\{h\} = \mathbf{v}$. Now we can show that $\gamma \circ h = \mathrm{id}$:

$$\begin{aligned}
\gamma \circ h &= \langle \mathbf{p}, \phi_G(f)\rangle \circ \varphi_G \circ h \\
&= \langle \mathbf{p} \circ \varphi_G \circ h, \phi_G(f)\{\varphi_G \circ h\}\rangle \\
&= \langle G(\mathbf{p}) \circ h, \square_G(f \circ \mathbf{p}, \mathbf{v})\{h\}\rangle \\
&= \langle \mathbf{p}, \mathbf{v}\rangle = \mathrm{id}
\end{aligned}$$

Thus we have found our isomorphism and have $\gamma^{-1} = h$. Furthermore, we see that $G(\mathbf{p}) \circ \gamma^{-1} = \mathbf{p}$ or equivalently $\mathbf{p} \circ \gamma = G(\mathbf{p})$, so by the uniqueness of $\square_G$, we must have $\square_G(\Gamma, \sigma)\{G(f)\} = \square_G(\Delta, \sigma\{f\})$. $\square$