Extending some categories to categories with families

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1 Categories with families

Definition 1. A *category with families* is given by

- A category $\mathbb{C}$ with a terminal object $[]$,
- A functor $F = (\mathrm{Ty}, \mathrm{Tm}) : \mathbb{C}^{\mathrm{op}} \to \mathrm{Fam}(\mathrm{Set})$. For the morphism part, we introduce the notation $\_\{\cdot\}$ for both types and terms, i.e. if $f : \Delta \to \Gamma$ then $\_\{f\} : \mathrm{Ty}(\Gamma) \to \mathrm{Ty}(\Delta)$ and for every $\sigma \in \mathrm{Ty}(\Delta)$ we have $\_\{f\} : \mathrm{Tm}(\Delta, \sigma) \to \mathrm{Tm}(\Gamma, \sigma\{f\})$.
- For each object Γ in $\mathbb{C}$ and $\sigma \in \mathrm{Ty}(\Gamma)$ an object $\Gamma.\sigma$ together with a morphism $\mathbf{p}(\sigma) : \Gamma.\sigma \to \Gamma$ (the *first projection*) and a term $\mathbf{v}_\sigma \in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\})$ (the *second projection*) with the following universal property: for each $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\})$ there exists a unique morphism $\theta = \langle f, M\rangle_\sigma : \Delta \to \Gamma.\sigma$ such that $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$.

2 Set

Directly from Dybjer [Dyb96] , Hofmann [Hof97], Buisse and Dybjer [BD08], ...

Choose $\mathbb{C} = \mathrm{Set}$ (with $[] = \mathbf{1}$ any singleton), and define

$$\mathrm{Ty}(\Gamma) = \{\sigma \mid \sigma : \Gamma \to \mathrm{Set}\}$$
$$\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma \in \Gamma} \sigma(\gamma)$$

(this should really be $\sigma : \Gamma \to U$ for some universe (U, T) for size considerations, and accordingly $\mathrm{Tm}(\Gamma, \sigma) = \prod_{\gamma\in\Gamma} T(\sigma(\gamma))$). For $f : \Delta \to \Gamma$, $\sigma : \mathrm{Ty}(\Gamma)$, h :

$\mathrm{Tm}(\Gamma, \sigma)$, define

$$\begin{aligned}
&\sigma\{f\} : \mathrm{Ty}(\Delta) = \{\sigma \mid \sigma : \Delta \to \mathrm{Set}\} \\
&\sigma\{f\} = \sigma \circ f \\
&h\{f\} : \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta)) \\
&h\{f\} = h \circ f
\end{aligned}$$

For the context comprehension, define

$$\begin{aligned}
\Gamma.\sigma &= \sum_{\gamma \in \Gamma} \sigma(\gamma) \\
\mathbf{p}(\sigma) &: \sum_{\gamma \in \Gamma} \sigma(\gamma) \to \Gamma \\
\mathbf{p}(\sigma)(\langle \gamma, s \rangle) &= \gamma \\
\mathbf{v}_\sigma &\in \mathrm{Tm}(\Gamma.\sigma, \sigma\{\mathbf{p}(\sigma)\}) = \prod_{\langle \gamma, s \rangle \in \Gamma.\sigma} \sigma(\gamma) \\
\mathbf{v}_\sigma(\langle \gamma, s \rangle) &= s
\end{aligned}$$

Given $f : \Delta \to \Gamma$ and $M \in \mathrm{Tm}(\Delta, \sigma\{f\}) = \prod_{\delta \in \Delta} \sigma(f(\delta))$, we define

$$\theta = \langle f, M \rangle_\sigma : \Delta \to \underbrace{\Gamma.\sigma}_{\sum_{\gamma \in \Gamma} \sigma(\gamma)}$$

by $\theta(\delta) = \langle f(\delta), M(\delta) \rangle$. We then have $\mathbf{p}(\sigma) \circ \theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$, and any other function satisfying these equations must be extensionally equal to θ, hence θ is unique.

3 Fam(Set)

Fam(Set) can also be extended to a category with families. We start with $\mathbb{C} = \mathrm{Fam}(\mathrm{Set})$ (and $[] = (\mathbf{1}, \lambda x.\mathbf{1})$), and define

$$\begin{aligned}
\mathrm{Ty}(X, Y) &= \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
\mathrm{Tm}((X, Y), (A, B)) &= \{(h, k) \mid h : \prod_{x \in X} A(x), k : \prod_{x \in X, y \in Y(x)} B(x, y, h(x))\}
\end{aligned}$$

(similar size considerations apply as for Set). For $(f, g) : (X, Y) \to (X', Y')$, $(A, B) : \mathrm{Ty}(X', Y')$, $(h, k) : \mathrm{Tm}((X', Y'), (A, B))$, define

$$\begin{aligned}
&(A, B)\{f, g\} : \mathrm{Ty}(X, Y) = \{(A, B) \mid A : X \to \mathrm{Set}, B : (x : X) \to Y(x) \to A(x) \to \mathrm{Set}\} \\
&(A, B)\{f, g\} = (A, B) \circ (f, g) = (A \circ f, \lambda x, y\ .\ B(f(x), g(x, y))) \\
&(h, k)\{f, g\} : \mathrm{Tm}(\Delta, \sigma\{f\}) \\
&(h, k)\{f, g\} = (h, k) \circ (f, g) = (h \circ f, \lambda x, y\ .\ k(f(x), g(x, y)))
\end{aligned}$$

For the context comprehension, define

$$
\begin{aligned}
(X,Y).(A,B) &= (\sum_{x\in X} A(x), \lambda\langle x,a\rangle \,.\, \sum_{y\in Y(x)} B(x,y,a)) \\
\mathbf{p}(A,B) &= (\mathrm{fst}, \lambda x.\mathrm{fst}) \\
\mathbf{v}_{A,B} &= (\mathrm{snd}, \lambda x.\mathrm{snd})
\end{aligned}
$$

Given $(f,g) : (X',Y') \to (X,Y)$ and $(h,k) \in \mathrm{Tm}((X',Y'),(A,B)\{f,g\})$, we define

$$(\theta,\psi) = \langle (f,g),(h,k)\rangle_{(A,B)} : (X',Y') \to (X,Y).(A,B)$$

by $\theta(x) = \langle f(x), h(x)\rangle$, $\psi(x,y) = \langle g(x,y), k(x,y)\rangle$. It is clear that $\mathbf{p}(\sigma)\circ\theta = f$ and $\mathbf{v}_\sigma\{\theta\} = M$ and that these conditions force (θ,ψ) to be unique.

4 BiAlg(F,G) for $F,G : \mathbb{C} \to \mathbb{D}$

References

[BD08] Alexandre Buisse and Peter Dybjer. The interpretation of intuitionistic type theory in locally cartesian closed categories – an intuitionistic perspective. *Electronic Notes in Theoretical Computer Science*, 218:21–32, 2008.

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[Hof97] Martin Hofmann. Syntax and semantics of dependent types. In Andrew Pitts and Peter Dybjer, editors, *Semantics and Logics of Computation*, pages 79 – 130. Cambridge University Press, 1997.