Problem 1: $F(\Sigma\ A\ B) \cong \Sigma\ (FA)\ (\square_F B))$

Here is a useful little construction: let $\widehat{\cdot} : ((x : A) \to B(a)) \to A \to \Sigma\ A\ B$ be defined by $\widehat{g}(x) = \langle x, g(x) \rangle$. Notice that $\pi_0 \circ \widehat{g} = \mathrm{id}$ and $\pi_1 \circ \widehat{g} = g$ for any g.

Definition 1. Given functor $F : \mathrm{Set} \to \mathrm{Set}$, we define

$$\square_F : \{A : \mathrm{Set}\} \to (P : A \to \mathrm{Set}) \to F(A) \to \mathrm{Set}$$
$$\mathrm{dmap}_F : \{A : \mathrm{Set}\} \to \{P : A \to \mathrm{Set}\} \to ((x : A) \to P(x)) \to (x : F(A)) \to \square_F(P, x)$$

by

$$\square_F(B, x) = \{y : F\ (\Sigma\ A\ B)\ |\ F(\pi_0)(y) = x\}$$
$$\mathrm{dmap}_F(g, x) = \langle F(\widehat{g})(x), \mathrm{refl} \rangle$$

Notice that we are using the fact that $\pi_0 \circ \widehat{g} = \mathrm{id}$, so that $F(\pi_0)(F(\widehat{g})(x)) = F(\pi_0 \circ \widehat{g})(x) = F(\mathrm{id})(x) = x$, and refl has the right type.

Problem 1. (i) There is $\varphi : F(\Sigma\ A\ B) \xrightarrow{\cong} \Sigma\ (FA)\ (\square_F B))$.

(ii) $\pi_0 \circ \varphi = F(\pi_0)$.

(iii) For $g : (x : A) \to B(x)$, we have $\widehat{\mathrm{dmap}_F(g)} = \varphi \circ F(\widehat{g})$.

Proof. (i) Define
$$\varphi : F(\Sigma\ A\ B) \to \Sigma\ (FA)\ (\square_F B))$$

and
$$\psi : \Sigma\ (FA)\ (\square_F B)) \to F(\Sigma\ A\ B)$$

by
$$\varphi(y) = \langle F(\pi_0)(y), \langle y, \mathrm{refl} \rangle \rangle$$
$$\psi(\langle x, \langle y, p \rangle \rangle) = y.$$

Then $\psi(\varphi(y)) = \psi(\langle F(\pi_0)\ y, \langle y, \mathrm{refl} \rangle \rangle) = y$ and for every $\langle x, \langle y, p \rangle \rangle : \Sigma\ (FA)\ (\square_F B))$, we have $x = F(\pi_0)(y)$ by p and $p = \mathrm{refl}$ by proof irrelevance, so that $\varphi(\psi(\langle x, \langle y, p \rangle \rangle)) = \varphi(y) = \langle F(\pi_0)(y), \langle y, \mathrm{refl} \rangle \rangle = \langle x, \langle y, p \rangle \rangle$. Hence $F(\Sigma\ A\ B) \cong \Sigma\ (FA)\ (\square_F B))$.

(ii) $\pi_0(\varphi(y)) = \pi_0(\langle F(\pi_0)(y), \langle y, \mathrm{refl} \rangle \rangle) = F(\pi_0)(y)$.

(iii) $\widehat{\mathrm{dmap}_F(g)}(x) = \langle x, \mathrm{dmap}_F(g, x) \rangle = \langle x, \langle F(\widehat{g})(x), \mathrm{refl} \rangle \rangle$, and also $\varphi(F(\widehat{g})(x)) = \langle F(\pi_0)(F(\widehat{g})(x)), \langle (F(\widehat{g})(x), \mathrm{refl}) \rangle \rangle = \langle F(\pi_0 \circ \widehat{g})(x)), \langle (F(\widehat{g})(x), \mathrm{refl}) \rangle \rangle = \langle x, \langle (F(\widehat{g})(x), \mathrm{refl}) \rangle \rangle$ since $\pi_0 \circ \widehat{g} = \mathrm{id}$. Hence $\widehat{\mathrm{dmap}_F(g)} = \varphi \circ F(\widehat{g})$. $\square$

Problem 2: $\Box$ for containers $\cong$ $\Box$ for functors

Definition 2. For a container $S \triangleleft P$, we define

$$\Box_{S \triangleleft P} : \{A : \text{Set}\} \to (B : A \to \text{Set}) \to [\![S \triangleleft P]\!] \, A \to \text{Set}$$

by $\Box_{S \triangleleft P}(B, \langle s, f \rangle) = (p : P(s)) \to B(f(p))$.

Problem 2. For all $A : \text{Set}$, $B : A \to \text{Set}$ and $\langle s, f \rangle : [\![S \triangleleft P]\!](A)$,

$$\Box_{S \triangleleft P}(B, \langle s, f \rangle) \cong \Box_{[\![S \triangleleft P]\!]}(B, \langle s, f \rangle).$$

Proof. Define

$$\varphi : ((p : P(s)) \to B(f(p))) \to \{\langle s', f' \rangle : \Sigma s' : S \,.(P(s') \to \Sigma \, A \, B) \mid \langle s', \pi_0 \circ f' \rangle = \langle s, f \rangle\}$$

and

$$\psi : \{\langle s', f' \rangle : \Sigma s' : S \,.(P(s') \to \Sigma \, A \, B) \mid \langle s', \pi_0 \circ f' \rangle = \langle s, f \rangle\} \to ((p : P(s)) \to B(f(p)))$$

by

$$\varphi(g) = \langle \langle s, (\lambda p \,.\, \langle f(p), g(p) \rangle) \rangle, \langle \text{refl}, \text{ext}(\text{refl}) \rangle \rangle$$

$$\psi(\langle \langle s', f' \rangle, r \rangle) = \pi_1 \circ f'$$

(there's lots of invented notation for equality proofs and hidden substitutions going on here). Now $\psi(\varphi(g)) = \psi(\langle \langle s, (\lambda p \,.\, \langle f(p), g(p) \rangle) \rangle, \langle \text{refl}, \text{ext}(\text{refl}) \rangle \rangle) = \pi_1 \circ (\lambda p \,.\, \langle f(p), g(p) \rangle) = g$ and

$$\begin{aligned}
\varphi(\psi(\langle \langle s', f' \rangle, r \rangle)) &= \varphi(\pi_1 \circ f') \\
&= \langle \langle s, (\lambda p \,.\, \langle f(p), \pi_1(f'(p)) \rangle) \rangle, \langle \text{refl}, \text{ext}(\text{refl}) \rangle \rangle \\
&= \langle \langle s', (\lambda p \,.\, \langle \pi_0(f'(p)), \pi_1(f'(p)) \rangle) \rangle, r \rangle \\
&= \langle \langle s', f' \rangle, r \rangle
\end{aligned}$$

where we have used eta for Σ and Π in the last equality.

$\Box$

We can make $\Box_F$ into a functor $\boxdot_F : \text{Fam}(\text{Set}) \to \text{Fam}(\text{Set})$ [in general $\text{Fam}(\mathbb{C}) \to \text{Fam}(\mathbb{D})$?] by defining $\boxdot_F(A, B) = (F(A), \Box_F(A, B))$ for objects and $\boxdot(f, g) = (F(f), \lambda \, x \, \langle y, \text{refl} \rangle \,.\langle F([f, g])(y), \text{refl} \rangle)$ where $[f, g] : \Sigma \, A \, B \to \Sigma \, A' \, B'$ is the obvious map, and the last refl is of the right type since $\pi_0 \circ [f, g] = f \circ \pi_0$.

Similarly, we can define $\boxdot_{S \triangleleft P}(A, B) = ([\![S \triangleleft P]\!](A), \Box_{S \triangleleft P})$ for objects and and $\boxdot_{S \triangleleft P}(f, g) = ([\![S \triangleleft P]\!](f), \lambda(s, h) \, j \, p \,.\, g(h(p), j(p)))$ for morphisms $(f, g) : (A, B) \to (A', B')$. The result can be strengthened to:

Proposition 3. *There is a natural isomorphism* $\eta : \Box_{S \triangleleft P} \Rightarrow \Box_{[\![S \triangleleft P]\!]}$.

Proof. We have already constructed the components at each (A, B) and shown them to be isomorphisms in the proof of Problem 2. All that is left to do is to check the naturality condition, but this follows from a straightforward verification:

$$
\begin{array}{ccc}
j & \longmapsto & \langle\langle s, (\lambda\ p\ .\ \langle h(p), j(p)\rangle)\rangle, \mathrm{refl}\rangle \\
\uparrow & & \uparrow \\
\downarrow & & \downarrow \\
\lambda\ p\ .\ g(h(p), j(p)) & \longmapsto & \langle\langle s, (\lambda\ p\ .\ \langle f(h(p)), g(h(p), j(p))\rangle)\rangle, \mathrm{refl}\rangle
\end{array}
$$

$\square$

Problem 3: Init $\implies$ elim (in Set)

Let $F : \mathrm{Set} \to \mathrm{Set}$ be a functor and $(\mu F, \mathrm{in}_F)$ a F-algebra, i.e.

$$\mu F : \mathrm{Set} \qquad\qquad \mathrm{in}_F : F(\mu F) \to \mu F.$$

$\square_F$ and $\mathbf{dmap}_F$

Let us first record what we need and expect from $\square_F$ and dmap_F. They should have types

$$\square_F : \{A : \mathrm{Set}\} \to (P : A \to \mathrm{Set}) \to F(A) \to \mathrm{Set}$$
$$\mathrm{dmap}_F : \{A : \mathrm{Set}\} \to \{P : A \to \mathrm{Set}\} \to ((x : A) \to P(x)) \to (x : F(A)) \to \square_F(P, x)$$

and satisfy

$$F(\Sigma\ A\ P) \cong \Sigma\ (FA)\ (\square_F P)) \tag{$*_\square$}$$

and this isomorphism φ must satisfy

$$
\begin{array}{ccc}
F(\Sigma\ A\ P) & \xrightarrow{\ \varphi\ } & \Sigma\ (F\ A)\ (\square_F\ P) \\
{\scriptstyle F(\pi_0)}\Big\downarrow & \swarrow{\scriptstyle \pi_0} & \\
F\ A & &
\end{array}
\tag{$**_\square$}
$$

i.e. $\pi_0 \circ \varphi = F(\pi_0)$. For dmap_F, we must for all $f : (x : A) \to B(x)$ have

$$
\begin{array}{ccc}
F\ A & \xrightarrow{\ F(\widehat{f})\ } & F(\Sigma\ A\ B) \\
{\scriptstyle \widehat{\mathrm{dmap}_F}(f)}\Big\downarrow & \swarrow{\scriptstyle \varphi} & \\
\Sigma\ (F\ A)\ (\square_F\ B) & &
\end{array}
\tag{$*_{\mathrm{dmap}}$}
$$

i.e. $\widehat{\mathrm{dmap}_F}(f) = \varphi \circ F(\widehat{f})$.

For $\square_F(P, x) = \{y : F\ (\Sigma\ A\ P) \mid F(\pi_0)(y) = x\}$ and $\mathrm{dmap}_F(g, x) = \langle F(\widehat{g})(x), \mathrm{refl}\rangle$, this holds, as proved in Problem 1.

The equivalence of elim and init

Principle (Elim). The elimination principle for F says that we have

$$\mathrm{elim}_F : (P : \mu F \to \mathrm{Set}) \to \big((x : F(\mu F)) \to \Box_F(P, x) \to P(\mathrm{in}_F(x))\big)$$
$$\to (x : \mu F) \to P(x)$$

and computation rule

$$\mathrm{elim}_F(P, g, \mathrm{in}_F(x)) = g(x, \mathrm{dmap}_F(\mathrm{elim}_F(P, g), x))$$

Principle (Init). The initial algebra principle for F says that $(\mu F, \mathrm{in}_F)$ is the *initial* algebra for F, i.e. for any other F-algebra (X, f), there is $\mathrm{fold}(f) : \mu F \to X$ such that

$$
\begin{array}{ccc}
F(\mu F) & \xrightarrow{\ \mathrm{in}_F\ } & \mu F \\
{\scriptstyle F(\mathrm{fold}(f))}\downarrow & & \downarrow{\scriptstyle \mathrm{fold}(f)} \\
F(X) & \xrightarrow{\ f\ } & X
\end{array}
$$

commutes.

Problem 3. Init $\implies$ elim.

Proof. Assume that $(\mu F, \mathrm{in}_F)$ is initial. We must construct

$$\mathrm{elim}_F : (P : \mu F \to \mathrm{Set}) \to \big((x : F(\mu F)) \to \Box_F(P, x) \to P(\mathrm{in}_F(x))\big)$$
$$\to (x : \mu F) \to P(x)$$

such that
$$\mathrm{elim}_F(P, g, \mathrm{in}_F(x)) = g(x, \mathrm{dmap}_F(\mathrm{elim}_F(P, g), x)).$$

Let $P : \mu F \to \mathrm{Set}$ and $g : (x : F(\mu F)) \to \Box_F(P, x) \to P(\mathrm{in}_F(x))$ be given. The plan is to make $\Sigma\ \mu F\ P$ into a F-algebra $(\Sigma\ \mu F\ P, h)$ and then show that $\pi_0 \circ \mathrm{fold}(h) = \mathrm{id}$, so that $\pi_1 \circ \mathrm{fold}(h) : (x : \mu F) \to P(x)$. We can then define $\mathrm{elim}_F(P, g, x) = \pi_1(\mathrm{fold}(f)(x))$ and must show that the computation rule holds.

First, we need to define $h : F(\Sigma\ \mu F\ P) \to \Sigma\ \mu F\ P$. Let $\varphi : F(\Sigma\ \mu F\ P) \xrightarrow{\cong} \Sigma\ (F\ A)\ (\Box_F\ P)$ be the witness of $(*_\Box)$. We can then define

$$h(x) = \langle \mathrm{in}_F(\pi_0(\varphi(x))), g(\pi_0(\varphi(x)), \pi_1(\varphi(x))) \rangle,$$

so that $\mathrm{fold}(h) : \mu F \to \Sigma\ \mu F\ P$. We also know that

$$
\begin{array}{ccc}
F(\mu F) & \xrightarrow{\ \mathrm{in}_F\ } & \mu F \\
{\scriptstyle F(\mathrm{fold}(h))}\downarrow & & \downarrow{\scriptstyle \mathrm{fold}(h)} \\
F(\Sigma\ \mu F\ P) & \xrightarrow{\ h\ } & \Sigma\ \mu F\ P
\end{array}
\tag{1}
$$

4

commutes. Now $\pi_0(h(x)) = \operatorname{in}_F(\pi_0(\varphi(x))) \stackrel{(**_\square)}{=} \operatorname{in}_F(F(\pi_0)(x))$, so that the diagram

$$
\begin{array}{ccc}
F(\Sigma\ \mu F\ P) & \xrightarrow{\ h\ } & \Sigma\ \mu F\ P \\
{\scriptstyle F(\pi_0)}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
F(\mu F) & \xrightarrow{\ \operatorname{in}_F\ } & \mu F
\end{array}
\tag{2}
$$

commutes. Pasting Diagram (1) and (2) together, we get the commuting diagram

$$
\begin{array}{ccc}
F(\mu F) & \xrightarrow{\ \operatorname{in}_F\ } & \mu F \\
{\scriptstyle F(\mathrm{fold}(h))}\downarrow & & \downarrow{\scriptstyle \mathrm{fold}(h)} \\
F(\Sigma\ \mu F\ P) & \xrightarrow{\ h\ } & \Sigma\ \mu F\ P \\
{\scriptstyle F(\pi_0)}\downarrow & & \downarrow{\scriptstyle \pi_0} \\
F(\mu F) & \xrightarrow{\ \operatorname{in}_F\ } & \mu F
\end{array}
$$

which shows that $\pi_0 \circ \mathrm{fold}(h) : (\mu F, \operatorname{in}_F) \to (\mu F, \operatorname{in}_F)$ is a morphism in the category of F-algebras. But also $\mathrm{id}_{\mu F}$ is such a morphism, and since $(\mu F, \operatorname{in}_F)$ is initial, we must have $\pi_0 \circ \mathrm{fold}(h) = \mathrm{id}$. Hence we can define $\mathrm{elim}_F(P, g) :=$ $\pi_1 \circ \mathrm{fold}(h) : (x : \mu F) \to P(x)$.

It remains to be shown that $\mathrm{elim}_F(P, g, \operatorname{in}_F(x)) = g(x, \mathrm{dmap}_F(\mathrm{elim}_F(P, g), x))$. Unfolding the definition of elim_F, we see that we must prove $\pi_1(\mathrm{fold}(h)(\operatorname{in}_F(x))) = g(x, \mathrm{dmap}_F(\pi_1 \circ \mathrm{fold}(h), x))$. Since $\mathrm{fold}(h) \circ \operatorname{in}_F = h \circ F(\mathrm{fold}(h))$ by (1), the left hand side reduces to $\pi_1(h(F(\mathrm{fold}(h)(x))))$ which is equal to $g(\pi_0(\varphi(F(\mathrm{fold}(h)(x)))), \pi_1(\varphi(F(\mathrm{fold}(h)(x)))))$ by the definition of h. It is thus enough to show that

(i) $\pi_0 \circ \varphi \circ F(\mathrm{fold}(h)) = \mathrm{id}$

(ii) $\pi_1 \circ \varphi \circ F(\mathrm{fold}(h)) = \mathrm{dmap}_F(\pi_1 \circ \mathrm{fold}(h))$.

Identity (i) is easily taken care of: by $(**_\square)$, $\pi_0 \circ \varphi = F(\pi_0)$, so that we have $\pi_0(\varphi(F(\mathrm{fold}(h)(x)))) = F(\pi_0)(F(\mathrm{fold}(h)(x))) = F(\pi_0 \circ \mathrm{fold}(h))(x) = F(\mathrm{id})(x) = x$.

For (ii), note that eta for Σ implies $\mathrm{fold}(h)(x) = \langle \pi_0(\mathrm{fold}(h)(x)), \pi_1(\mathrm{fold}(h)(x)) \rangle$ $= \langle x, \pi_1(\mathrm{fold}(h)(x)) \rangle$ so that $\mathrm{fold}(h) = \widehat{f}$ for $f := \pi_1 \circ \mathrm{fold}(h)$. Hence, using $\pi_1 \circ \widehat{g} = g$ several times for different functions g, we have

$$
\begin{aligned}
\mathrm{dmap}_F(\pi_1 \circ \mathrm{fold}(h)) &= \mathrm{dmap}_F(\pi_1 \circ \widehat{f}) \\
&= \mathrm{dmap}_F(f) \\
&= \pi_1 \circ \widehat{\mathrm{dmap}_F}(f) \\
&\stackrel{(*_{\mathrm{dmap}})}{=} \pi_1 \circ \varphi \circ F(\widehat{f}) = \pi_1 \circ \varphi \circ F(\mathrm{fold}(h))
\end{aligned}
$$

which takes care of (ii), and we are done. $\qquad\square$

Problem 4: Elim $\implies$ init (in Set)

Problem 4. Elim $\implies$ init.

Proof. Let $(X.f)$ be a F-algebra. We must construct $\text{fold}(f) : \mu F \to X$ such that $\text{fold}(f) \circ \text{in}_F = f \circ F(\text{fold}(f))$. We have

$$\text{elim}_F : (P : \mu F \to \text{Set}) \to \big(g : (x : F(\mu F)) \to \square_F(P, x) \to P(\text{in}_F(x))\big)$$
$$\to (x : \mu F) \to P(x)$$

such that $\text{elim}_F(P, g, \text{in}_F(x)) = g(x, \text{dmap}_F(\text{elim}_F(P, g), x))$, so let us choose $P = K\ X$ to be constantly X and $g(x, y) = f(F(\pi_1)(\varphi^{-1}(\langle x, y \rangle)))$, and define $\text{fold}(f) = \text{elim}_F(K\ X, g)$. Then $\text{fold}(f) : \mu F \to X$, and

$$\begin{aligned}
\text{fold}(f)(\text{in}_F(x)) &= \text{elim}_F(P, g, \text{in}_F(x)) \\
&= g(x, \text{dmap}_F(\text{elim}_F(P, g), x)) \\
&= f(F(\pi_1)(\varphi^{-1}(\langle x, \text{dmap}_F(\text{elim}_F(P, g), x) \rangle))) \\
&= f \circ F(\pi_1) \circ \varphi^{-1} \circ (\lambda x\ .\ \langle x, \text{dmap}_F(\text{fold}(f), x) \rangle)(x) \\
&= f \circ F(\pi_1) \circ \varphi^{-1} \circ \widehat{\text{dmap}_F(\text{fold}(f))}(x) \\
&\overset{(*_{\text{dmap}})}{=} f \circ F(\pi_1) \circ \varphi^{-1} \circ \varphi \circ F(\widehat{\text{fold}(f)})(x) \\
&= f \circ F(\pi_1 \circ \widehat{\text{fold}(f)})(x) \\
&= f \circ F(\text{fold}(f))(x),
\end{aligned}$$

so $\text{fold}(f) \circ \text{in}_F = f \circ F(\text{fold}(f))$ as required. $\qquad\square$