Projectile motion with the Coriolis effect

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Abstract

This exercise deals with the trajectory of a projectile in 3D with the adiabatic air resistance model and the Coriolis effect using the Runge-Kutta algorithm.

1. Setting up the cannon in 3 dimensions

In three dimensions Runge-Kutta takes the form

$$\boldsymbol{w(t)} = \begin{bmatrix} s(t) \\ e(t) \\ u(t) \\ v_s(t) \\ v_e(t) \\ v_u(t) \end{bmatrix}, \ \boldsymbol{w'(t)} = \begin{bmatrix} v_s(t) \\ v_e(t) \\ v_u(t) \\ a_s(t) \\ a_e(t) \\ a_u(t) \end{bmatrix},$$

$$\boldsymbol{w(t+\Delta t)} = \boldsymbol{w(t)} + \boldsymbol{w'(t)}\Delta t = \begin{bmatrix} s(t) + v_s(t)\Delta t \\ e(t) + v_e(t)\Delta t \\ u(t) + v_u(t)\Delta t \\ v_s(t) + a_s(t)\Delta t \\ v_e(t) + a_e(t)\Delta t \\ v_u(t) + a_u(t)\Delta t \end{bmatrix}. \tag{1}$$

s, e and u are the positions in the directions south, east and up. v_i is the velocity in the given directions, and a_i is the acceleration in the given directions. At (1) a total system is set up to hand out data for each iteration.

The acceleration due to adiabatic air resistance is dependent on the velocities and the altitude

$$a_{L,i} = -B_2 v_i v (1 - \frac{au}{T_0})^{\alpha}$$

with the constants, $a = 6.5 \cdot 10^{-3}\mathrm{K\,m^{-1}}$, $\alpha = 2.5$ for air, and T_0 is the temperature at sea level which here is set equal 290 K. The air resistance constant B_2 was initially set to $4 \cdot 10^{-5}\mathrm{m^{-1}}$, but then the maximum range were only 80 km. Since this was not enough to reach Paris from Crépy the constant, B_2 was multiplied with 0.4. Furthermore any altitude, u, above 40 km is set to zero in order to avoid complex numbers. Next the acceleration due to the Coriolis effect depends on the cross product of the velocity vector v and the rotation vector Ω from figure 1

$$\boldsymbol{v} = \begin{bmatrix} v_s \\ v_e \\ v_u \end{bmatrix}, \ \boldsymbol{\Omega} = \omega \begin{bmatrix} -\cos\phi \\ 0 \\ \sin\phi \end{bmatrix},$$

$$a_C = -2\Omega \times v = 2\omega \begin{bmatrix} v_e \sin\phi \\ -v_u \cos\phi - v_s \sin\phi \\ v_e \cos\phi \end{bmatrix}.$$

ω is earth's angular velocity at $7.29 \cdot 10^{-5} \mathrm{rad\,s^{-1}}$ and ϕ is the latitude angle as seen in figure 1.

Coordinate System Of The Earth

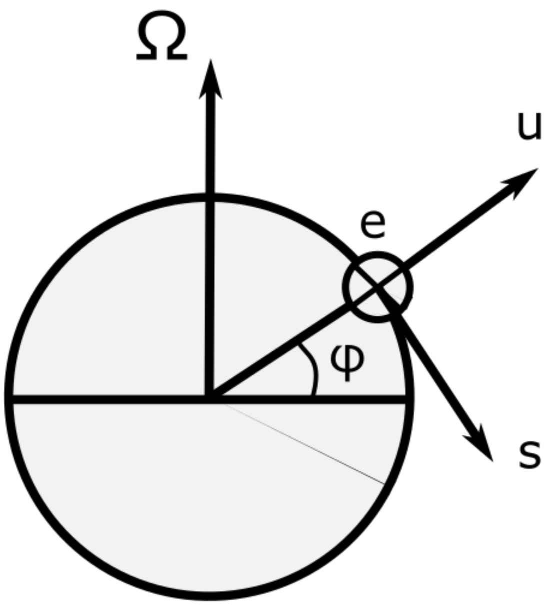

Figure 1: The model represents a cross section of earth with north and south being respectively up and down, and east and west being respectively in and out. ϕ is the angle between the equator and the position with the centre of the earth in the corner.

The final Runge-Kutta then takes the form of

$$\boldsymbol{w(t+\Delta t)} = \boldsymbol{w(t)}+\boldsymbol{w'(t)}\Delta t = \begin{bmatrix} s(t)+v_s(t)\Delta t \\ e(t)+v_e(t)\Delta t \\ u(t)+v_u(t)\Delta t \\ v_s(t)+(2\omega v_e \sin\phi - \frac{B_2}{m}v_s v(1-\frac{ay}{T_0})^\alpha)\Delta t \\ v_e(t)-(2\omega(v_u\cos\phi + v_s\sin\phi)+\frac{B_2}{m}v_e v(1-\frac{ay}{T_0})^\alpha)\Delta t \\ v_u(t)+(g+2\omega v_e\cos\phi - \frac{B_2}{m}v_u v(1-\frac{aa(t)}{T_0})^\alpha)\Delta t \end{bmatrix}. \tag{2}$$

The gravitational acceleration, g, is equal to $-9.81\,\mathrm{m\,s^{-2}}$. Furthermore the initial velocity has to be determined. In three dimensions the angle ζ, as seen in figure 2, will decide which direction on the ground the cannon points. In this task this is done by excluding the Coriolis effect and simply calculating which direction to fire in order to move in a straight line from the start coordinates to the end coordinates, which made ζ to be -0.9994 rad.

Coordinate System On The Ground

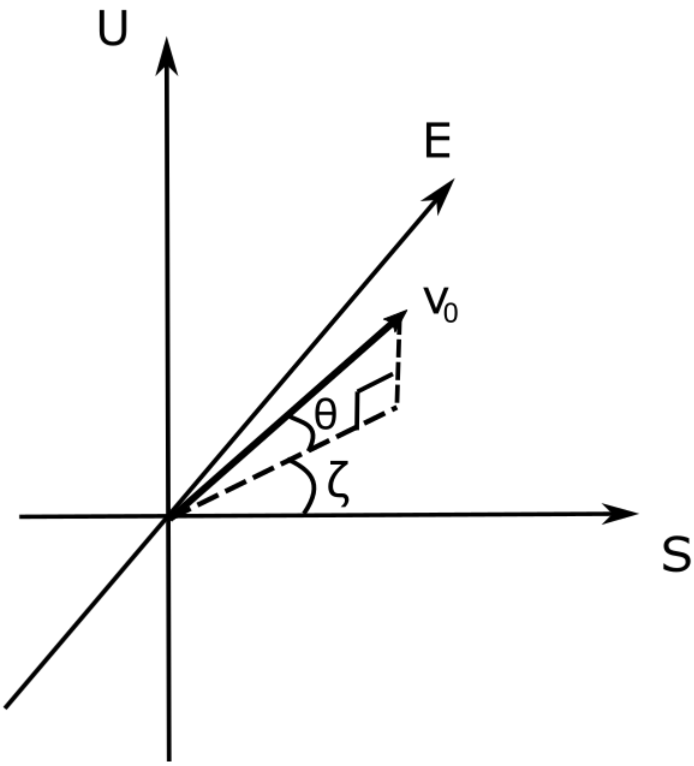

Figure 2: The angles ζ and ϕ stand perpendicular on each other. ζ decides the direction between east and south, while ϕ decides the direction upwards. v_0 is the initial speed.

The step length, Δt is set to 0.1 s, and six arrays are made, one for each element in the Runge-Kutta. In this task the curvature of the globe is also accounted for. The algorithm stops when u reaches below the surface, which is calculated by the equation for spheres

$$(s - s_0)^2 + (e - e_0)^2 + (u - u_0)^2 \leq R^2,$$

where s_0, e_0 and u_0 are the initial start positions. The distance covered on the ground is then interpolated between the last two values for both south and east, and then the total distance covered is found by using the Pythagorean theorem. The total time elapsed is equal to the amount of elements in the u array times the time step Δt. The maximum height is found as the largest value in the u array. The angle ϕ is set to $45°$ and the initial speed to 1640 $\mathrm{m\,s^{-1}}$. With these values the total ground distance covered was 159.3 km and the highest altitude achieved was 42.4 km. The total time elapsed was 188.9 s. Figure 3, 4 and 5 present the projectile trajectory in different directions.

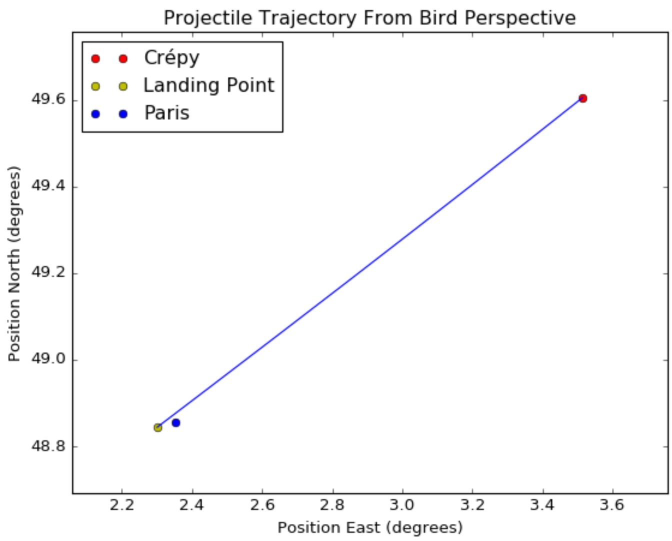

Figure 3: The plot presents the projectile trajectory as seen from above, with position east and north on the axes measured in degrees. The projectile starts in the upper right corner and travels downwards towards left in an almost straight line. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris.

2. Reaching the target

As figure 3 shows the Coriolis effect deviates the path to the right of the target. In fact by 0.0115 rad, and the distance from the landing point to Paris is 5.5 km. In order to reach the desired target a while loop is used to improve the trajectory for each shot until it reaches errors small enough to accept. If the angle between the two lines from Crépy to Paris, and Crépy to the landing point is larger than the acceptable difference the ground angle, ζ, from figure 2 is adjusted in the right way by a constant angle step. The same is done for the upwards angle, θ, if the distance between the landing point and

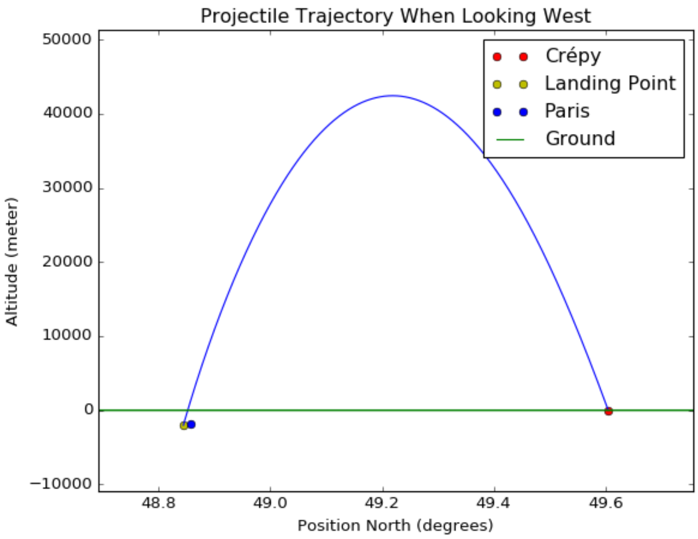

Figure 4: The plot presents the projectile trajectory as seen from looking straight west, with position north on the x-axis measured in degrees and altitude on the y-axes measured in metres. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris. The green line is the ground.

the target is larger than the acceptable error. The new error distance will be calculated for both θ + the angle step in upwards direction, and for - the angle step. The desired new angle will then be chosen depending on if the previous shot was too short or too long. Here the acceptable errors were 100 m and 0.0005 rad, and the angle steps used were 0.001 rad in the ζ-direction and 0.1 degree in the θ-direction. Final error were then 73.5 m away and 0.0003 rad away. The final θ angle was 42.3 degrees and the final θ angle was shifted a total of 0.11 rad becoming 0.9884 rad. As figure 6, 7 and 8 show the projectile now hit rather spot on!

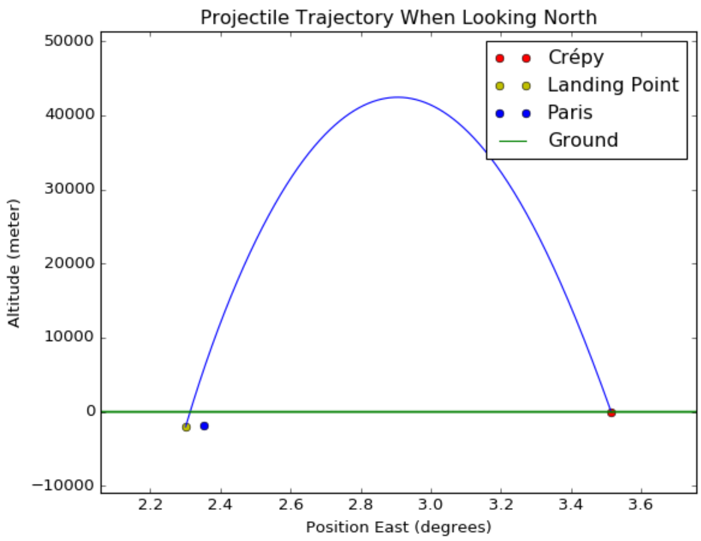

Figure 5: The plot presents the projectile trajectory as seen from looking straight north, with position east on the x-axis measured in degrees and altitude on the y-axes measured in metres. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris. The green line is the ground.

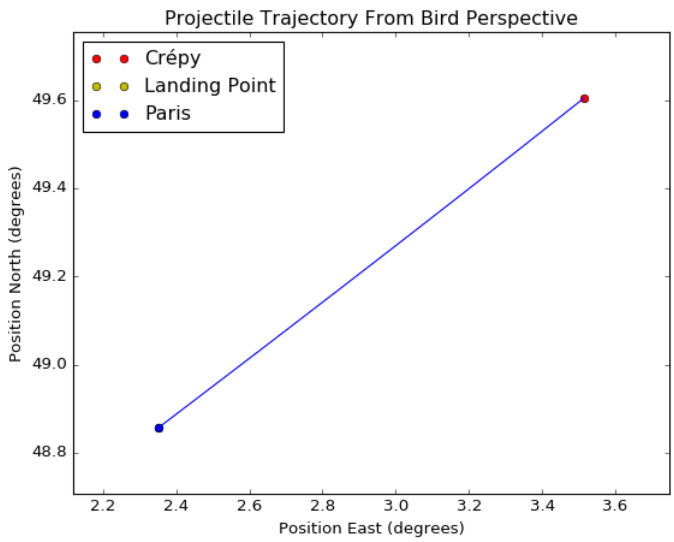

Figure 6: The plot presents the projectile trajectory as seen from above, with position east and north on the axes measured in degrees. The projectile starts in the upper right corner and travels downwards towards left in an almost straight line. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris.

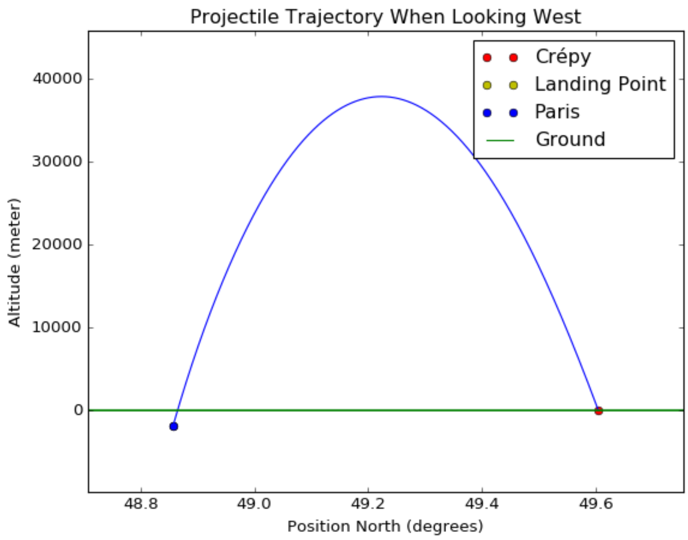

Figure 7: The plot presents the projectile trajectory as seen from looking straight west, with position north on the x-axis measured in degrees and altitude on the y-axes measured in metres. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris. The green line is the ground.

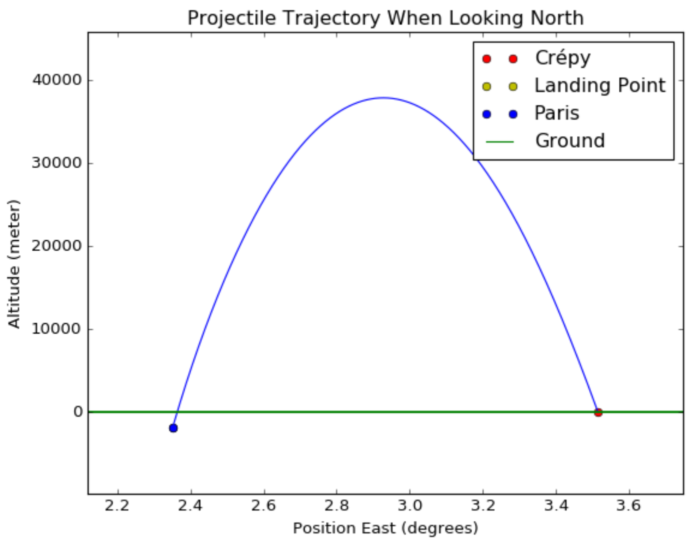

Figure 8: The plot presents the projectile trajectory as seen from looking straight north, with position east on the x-axis measured in degrees and altitude on the y-axes measured in metres. The red dot is the starting point, Crépy, the yellow dot is the landing point and the blue dot is the desired target, Paris. The green line is the ground.