Motion of charged particles in earth's magnetic field

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Abstract

This exercise deals with the three dimensional trajectory of charged particles in earth's magnetic field using a simple magnetic dipole model and the Runge-Kutta algorithm.

1. Earth's magnetic field

The earth's magnetic field can be described as a simple dipole. The approximation is somewhat decent within a certain distance from earth but as the distance increases the model breaks down. [1] The dipole field has only two components in polar coordinates, B_r and B_θ

$$B_r = -2B_0 \frac{R_E}{r}^3 \cos\theta \qquad\qquad B_\theta = -B_0 \frac{R_E}{r}^3 \sin\theta, \tag{1}$$

where $B_0 = 3{,}1 \cdot 10^{-5}\,\mathrm{T}$, $R_E = 6{,}371 \cdot 10^6\,\mathrm{m}$ is the radius of earth, r is the radial coordinate with origin in earth's centre, and θ is the angular coordinate measured from the magnetic south pole. If $r < R_E$ the magnetic field is set to be zero. In Cartesian coordinates (1) becomes

$$B_x = -3B_0 R_E^3 \frac{xz}{r^5}, \qquad B_y = -3B_0 R_E^3 \frac{yz}{r^5} \qquad B_z = -B_0 R_E^3 \frac{3z^2 - r^2}{r^5}. \tag{2}$$

The coordinate system is aligned with the ecliptic plane such that the x-axis is the line from the sun to the earth and the z-axis is perpendicular to the ecliptic plane as in figure 1. Earth's magnetic dipole is tilted an angle ϕ with respect to the ecliptic plane. Here ϕ is set to 13.5°, which is the tilt between earth's spin axis and the ecliptic plane minus the tilt between earth's dipole axis and the spin axis. In order to simplify the calculations the tilt is set to the xz-plane only such that in order to calculate the new magnetic dipole field a rotation matrix around the y-axis can be applied

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{3}$$

The magnetic field vector is calculated from (2) with the new coordinates u, v and w before being rotated back again with the same rotation matrix (3), but with the opposite angle, $-\phi$.

Ecliptic coordinate system

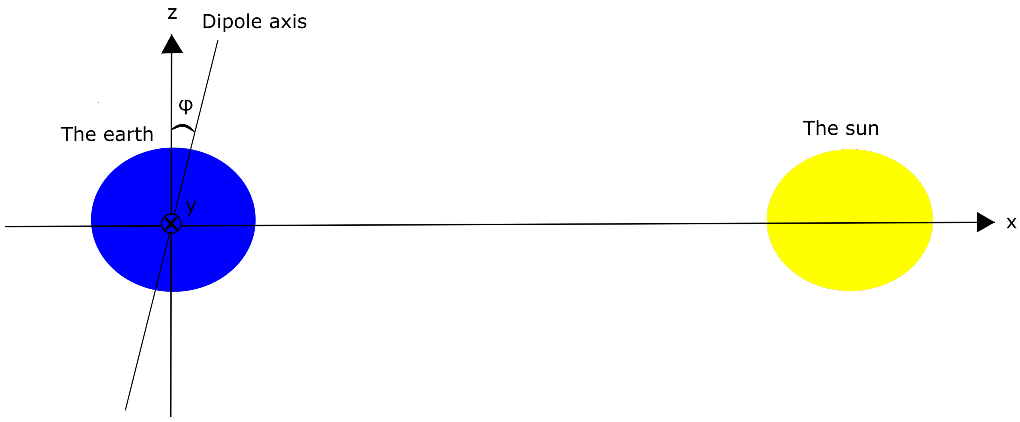
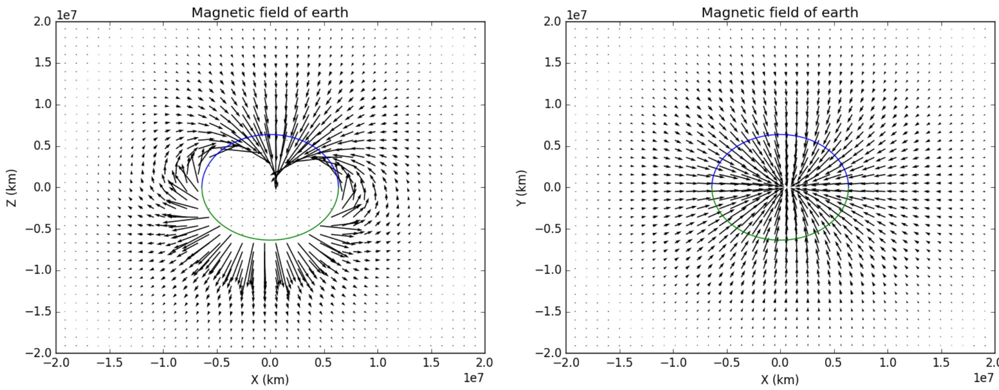


Figure 1: The model presents the coordinate system of earth's magnetic field. The blue sphere represents earth and the yellow sphere presents the sun. The magnetic dipole axis is tilted from the z-axis by an angle ϕ. Scales and lengths are not to scale.

Earths magnetic dipole field



(a) Magnetic field in the plane $y = 0$.

(b) Magnetic field in the plane $z = 1.4R_E$.

Figure 2: The figures presents earth's magnetic field in the planes $y = 0$ and $z = 1.4R_E$ according to (2). The arrows presents the field lines with relative strengths and the circle presents earth.

2. Charged particles in a magnetic field

A magnetic force acts on charged particles moving in a magnetic field

$$\vec{F_m} = q\vec{v} \times \vec{B}, \tag{4}$$

where q is the charge, $\vec{v}$ is the velocity and $\vec{B}$ is the magnetic field as in (2). From (4) Newton's second law gives the acceleration vector

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \frac{q}{m} \begin{bmatrix} v_y B_z - v_z B_y \\ v_z B_x - v_x B_z \\ v_x B_y - v_y B_x \end{bmatrix},$$

with m as the particle mass. In three dimensions Runge-Kutta uses the matrix w

$$\boldsymbol{w(t)} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}, \qquad \boldsymbol{w'(t)} = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \\ a_x(t) \\ a_y(t) \\ a_z(t) \end{bmatrix}.$$

The fourth order Runge-Kutta uses four derivatives

$$w_1'(t) = w'(t), \qquad w_2'(t) = (w(t) + w_1'(t)\frac{h}{2})',$$

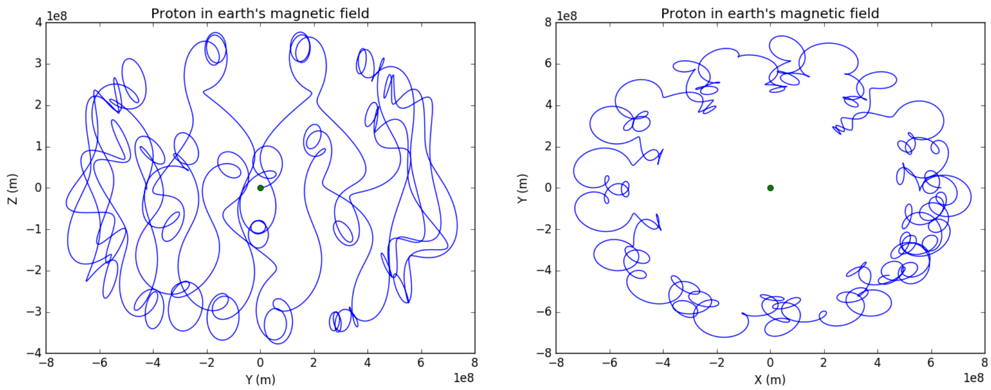
$$w_3'(t) = (w(t) + w_2'(t)\frac{h}{2})', \qquad w_4'(t) = (w(t) + w_3'(t)h)'.$$

The total iteration over a time step h is then

$$w(t + ht) = w(t) + \frac{h}{6}[w_1'(t) + 2w_2'(t) + 2w_3'(t) + w_4'(t)].$$

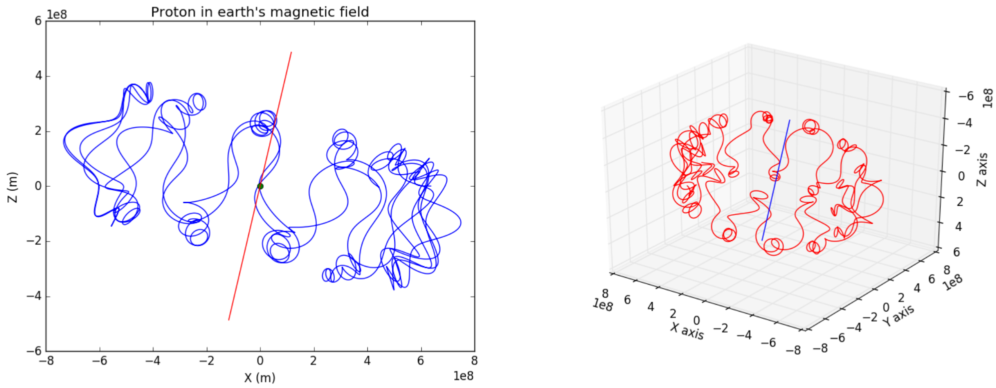
3. Trajectory of a proton in earth's magnetic field

A proton with mass $m = 1{,}67 \cdot 10^{-27}\,\mathrm{kg}$ and charge $q = 1{,}6 \cdot 10^{-19}\,\mathrm{C}$ is released from the sun starting here at $x = 100R_E$ with initial velocity $v_x = -200\,\mathrm{km\,s^{-1}}$. 15000 iterations are executed with time steps of $h = 10\,\mathrm{s}$. As figure 3 and 4 shows, the proton travels in helices towards the magnetic poles before being pushed out again as if it hit a spring. It then moves onto the opposite pole along the magnetic field lines from figure 1 and repeats the elastic helix collision. The trips from pole to pole form a ring around the dipole axis where the dipole vector and ring vector are parallel. Protons with this initial condition seem to be trapped in magnetic orbit and never reaches earth. Aurora Borealis comes to life only when particles reaches the atmosphere. For that perhaps other initial conditions are required or however the simple dipole model is too simplified in order to bring these charged particles all the way to the atmosphere.



(a) Proton trajectory in the yz plane. **(b)** Proton trajectory in the xy plane.

Figure 3: The plots presents the proton trajectory in earth's magnetic field with initial position $x = 100R_E$ and velocity $v_x = -200\,\mathrm{km\,s^{-1}}$. The blue line is the projection and the green dot is the centre of earth.



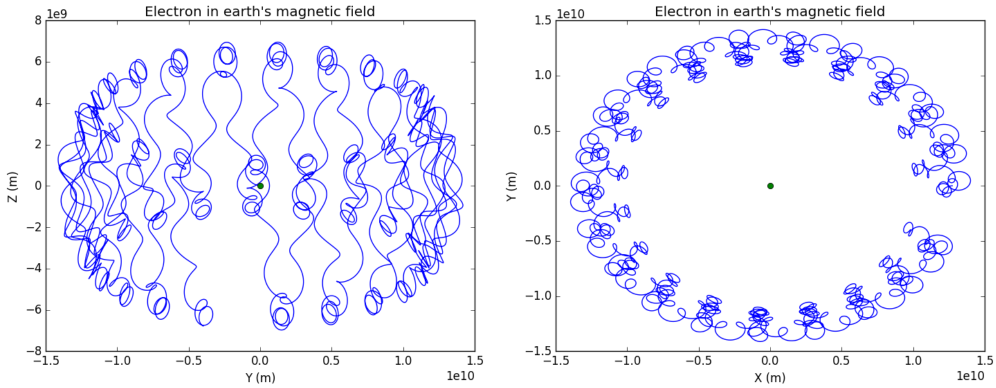
(a) Proton trajectory in the xz plane. The blue line, red line and green dot presents respectively the proton trajectory, the magnetic dipole axis and earth's centre.

(b) Proton trajectory in three dimensions. The red line and the green line presents respectively the proton trajectory and the magnetic dipole axis.

Figure 4: The plots presents the proton trajectory in earth's magnetic field with initial position $x = 100R_E$ and velocity $v_x = -200\,\mathrm{km\,s^{-1}}$.

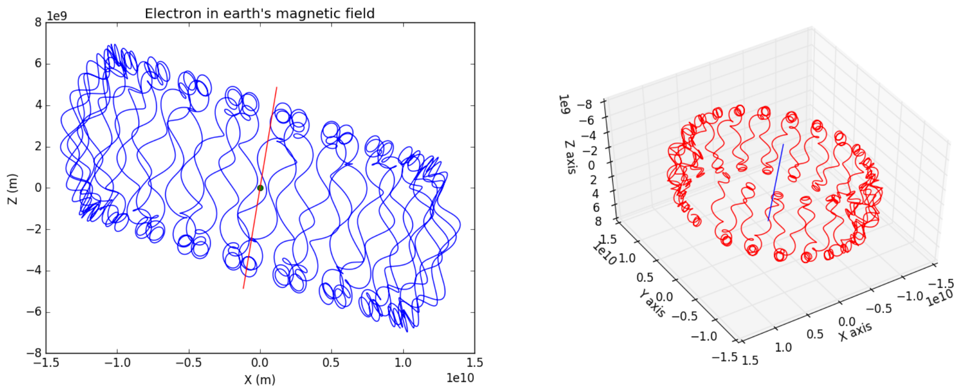
4. Trajectory of an electron in earth's magnetic field

This time an electron with mass $m = 9{,}11 \cdot 10^{-31}$ kg and charge $q = -1{,}6 \cdot 10^{-19}$ C is released from the sun with an initial velocity $v_x = -500\,\mathrm{km\,s^{-1}}$ starting at the position $x = 2000R_E$. The time step $h = 100\,\mathrm{s}$ and a total of 18000 iterations were executed. As figure 5 and 6 shows the particle motion is similar to the proton's, but it has more trips between the poles per cycle around the magnetic dipole axis. The distance from earth is also a lot larger, and the ring vector and the dipole vector are not parallel as for the proton.



(a) Electron trajectory in the yz plane. **(b)** Electron trajectory in the xy plane.

Figure 5: The plots presents the electron trajectory in earth's magnetic field with initial position $x = 2000R_E$ and velocity $v_x = -500\,\mathrm{km\,s^{-1}}$. The blue line is the projection and the green dot is the centre of earth.



(a) Electron trajectory in the xz plane. The blue line, red line and green dot presents respectively the electron trajectory, the magnetic dipole axis and earth's centre.

(b) Electron trajectory in three dimensions. The red line and the green line presents respectively the electron trajectory and the magnetic dipole axis.

Figure 6: The plots presents the electron trajectory in earth's magnetic field with initial position $x = 2000R_E$ and velocity $v_x = -500\,\mathrm{km\,s^{-1}}$.

[1] `https:ccmc.gsfc.nasa.gov/RoR_WWW/presentations/Dipole.pdf`