Creating 'Nice' Cubics

Meta mathematical problems study mathematics using mathematics. One branch of this, popularised by Cuoco in 2000, concerns itself with the creation of 'nice' mathematical problems.

These 'nice' problems come in all kinds of different flavours that depend on the situation. One anecdote I often draw to describe 'nice problems' is the feeling you have when, after a lengthy exam question, your answer perfectly simplifies to something beautiful and you think 'well that *must* be right!'[1] How do the exam writers make this magic happen?

Here, we focus on the creation of cubic equations that have integer roots and *rational* turning points. These are appropriate for problems at A-level, for example.

Let us begin by assuming our cubic equation has integer roots. Note that we can recover roots and turning points under horizontal translation, so without loss of generality consider these roots to be at zero and integers a, b such that

$$f(x) = x(x-a)(x-b) = x^3 - (a+b)x^2 + abx.$$

Its derivative is found

$$f'(x) = 3x^2 - 2(a+b)x + ab$$

and thus, with the quadratic formula, the turning points of f are determined by

$$x = \frac{2(a+b) \pm \sqrt{4(a+b)^2 - 12ab}}{6} = \frac{a+b \pm \sqrt{(a+b)^2 - 3ab}}{3}.$$

This formulation of the turning points will be our focus moving forwards. Indeed, if we can eliminate the surd such that

$$(a+b)^2 - 3ab = a^2 - ab + b^2 = c^2 \tag{†}$$

for some integer c, then the turning points will be rational.

1. A simple case where $a = b$**.** With repeated roots, (†) is easily solved by forcing $a^2 = c^2$ for any integer c. In this case the turning points are found at $x = a$ and $x = a/3$, meaning f will have at least one integer turning point. If $a = 3n$, it will have two integer turning points. For example,

$$f(x) = x(x-3)^2 = x^3 - 6x^2 + 9x \tag{1}$$

has turning points at $x = 1$ and $x = 3$. By means of translation, we can construct an entire family of cubics with integer turning points from (1) alone, say

$$g(x) := f(x-2) = x^3 - 12x^2 + 45x - 50$$

which has turning points at $x = 3$ and $x = 5$. Table 1 gives a few examples of cubics constructed in this way.

[1] Conversely, if you have an answer that is lengthy, irrational, or just *looks wrong*, you may immediately look back on your workings trying to spot a mistake.

Construction	Expansion	Roots	Turning Points
$f_a(x) = x(x-3)^2$	$x^3 - 6x^2 + 9x$	$x = 0, x = 3$	$(1, 4), (3, 0)$
$g_{1a}(x) = f_a(x-2)$	$x^3 - 12x^2 + 45x - 50$	$x = 2, x = 5$	$(3, 4), (5, 0)$
$g_{2a}(x) = f_a(x+3)$	$x^3 + 9x^2 + 24x + 20$	$x = -5, x = -2$	$(-4, 4), (-2, 0)$
$f_b(x) = x(x+2)^2$	$x^3 + 4x^2 + 4x$	$x = 0, x = -2$	$(-2, 0), \big(-\frac{2}{3}, -\frac{32}{27}\big)$
$g_{1b}(x) = f_b(x-4)$	$x^3 - 8x^2 + 20x - 16$	$x = 2, x = 4$	$(2, 0), \big(\frac{10}{3}, -\frac{32}{27}\big)$
$g_{2b}(x) = f_b(x+1)$	$x^3 + 7x^2 + 15x + 9$	$x = -3, x = -1$	$(-3, 0), \big(-\frac{5}{3}, -\frac{32}{27}\big)$

Table 1: Cubics with integer roots and rational turning points constructed by (a) forcing repeated roots ($a = b$) in $f(x) = x(x-a)(x-b)$ and (b) defining $a^2 = c^2$ for any $c \in \mathbb{Z}$. Two additional cubics from each family of translations are also given.

2. A more general case. Consider (†) once more. This condition on integers a, b, and c precisely defines the relationship needed to be an 'Eisenstein triple' (a, b, c). These are very similar in nature to Pythagorean triples (with this connection being explored in more detail later), and can be explicitly generated by integers p and q with the construction

$$(a = p^2 - q^2, \quad b = 2pq - q^2, \quad c = p^2 - pq + q^2).$$

For example, $(5, 8, 7)$ is an Eisenstein triple generated by $p = 3$ and $q = 2$. Thus the cubic equation

$$f(x) = x(x-5)(x-8)$$

will have rational turning points; indeed, they are at $x = 2$ and $x = 20/3$. If either criteria $a + b \pm c = 3n$ are satisfied, the cubic will have at least one integer turning point.

Table 2 highlights a few more examples of cubics generated with the Eisenstein integers.

Construction	Expansion	Turning Points
$f_a(x) = x(x+8)(x+3)$	$x^3 + 11x^2 + 24x$	$(-6, 36)$ and $(-4/3, -400/27)$
$g_{1a}(x) = f_a(x+2)$	$x^3 + 17x^2 + 80x + 100$	$(-8, 36)$ and $(-10/3, -400/27)$
$g_{2a}(x) = f_a(x-1)$	$x^3 + 8x^2 + 5x - 14$	$(-5, 36)$ and $(-1/3, -400/27)$
$f_b(x) = x(x-10)(x-16)$	$x^3 - 26x^2 + 160x$	$(4, 288)$ and $(40/3, -3200/7)$
$g_{1b}(x) = f_b(x+8)$	$x^3 - 2x^2 - 64x + 128$	$(-4, 288)$ and $(16/3, -3200/7)$
$g_{2b}(x) = f_b(x+15)$	$x^3 + 19x^2 + 55x - 75$	$(-11, 288)$ and $(-5/3, -3200/27)$

Table 2: Cubics constructed with Eisenstein integers $(8, 3, 7)$ and $(10, 16, 14)$, each with two more from their family of translations.

The link between Eisenstein triples and Pythagorean triples. A Pythagorean triple (a, b, c) consists of integers satisfying $a^2 + b^2 = c^2$. They are most commonly understood through Pythagoras' theorem concerning triangles with an angle of 90 degrees, which can in turn be interpreted as a special case of the cosine rule

$$c^2 = a^2 - 2ab\cos(90°) + b^2 = a^2 + b^2.$$

Similarly, the Eisenstein equation (†) can also be interpreted as a special case of the cosine rule for triangles with an angle of 60°,

$$c^2 = a^2 - 2ab\cos(60°) + b^2 = a^2 - ab + b^2.$$

Consider these triangles in the complex plane with one vertex at the origin, the edge of length a on the real axis, and the edge of length b forming the angle with edge a. See Figure 1.

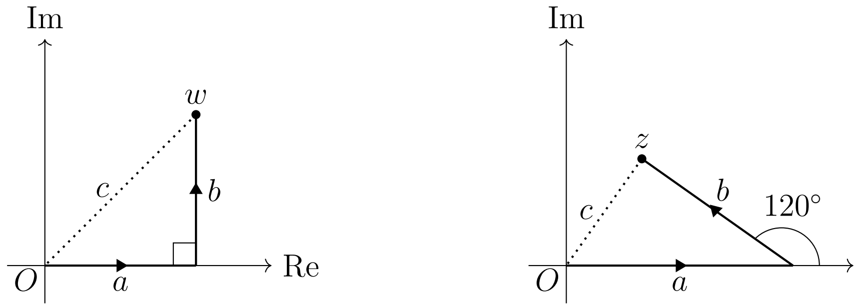

Figure 1: Triangles in the complex plane corresponding to (a) the Pythagorean triple (a, b, c) with an angle of 90°, and (b) the Eisenstein triple (a, b, c) with an angle of 60°.

These triangles will help us form expressions for the length c through the points w and z. It should be clear that $w = a + ib$ in Figure 1(a), but it will be helpful to be have a particular viewpoint. Think of w as the endpoint of two paths where

(a) first we travel from the origin to a on the real axis,
(b) then rotate by 90° counter-clockwise,
(c) then travel b units in this direction.

Rotation by θ in the complex plane is equivalent to multiplication by $\exp(i\theta)$. So, additively combining these steps, $w = a + \exp(i90°) \cdot b$. Euler's formula says

$$\exp(i90°) = \cos(90°) + i\sin(90°) = i,$$

so $w = a + ib$.

Now repeat these steps for the point z in Figure 1(b). The process is analogous: we find $z = a + \exp(i60°) \cdot b$, and

$$\exp(i60°) = i\sin(60°) + \cos(60°) := \omega$$

where ω is a cube root of unity. So $z = a + b\omega$. Two important properties ω has are (a) $\overline{w} = \omega^2$, and (b) $\omega + \omega^2 = -1$. The former is well reasoned geometrically, and for the latter consider

$$\omega^3 = 1 \implies (\omega - 1)(\omega^2 + \omega + 1) = 0 \implies \omega^2 + \omega = -1$$

as $\omega \neq 1$[2].

We are trying to use information about $z = a + b\omega$ to tell us about the length c. The bridge here is the norm

$$\begin{aligned} c^2 = |z|^2 = z\overline{z} &= (a + b\omega)\overline{(a + b\omega)} \\ &= (a + b\omega)(a + b\omega^2) = a^2 - ab + b^2. \end{aligned}$$

[2]We are working with a *primitive* cube root of unity $\omega \neq 1$; in general, $\omega = 1$ is a root of unity.

Look familiar? All that remains is to ensure that c is an integer, or equivalently, that c^2 is a square number. The final step here is to notice that norm is multiplicative, i.e. $|zw| = |z||w|$. In particular, *the norm of a squared complex number is itself square.* So instead of taking the norm of $z = a + b\omega$, use

$$\begin{aligned} z^2 = (a+b\omega)^2 &= a^2 + 2ab\omega + b^2\omega^2 \\ &= a^2 + 2ab\omega - b^2(1+\omega) = a^2 - b^2 + (2ab - b^2)\omega \end{aligned}$$

which, for ease of notation, we will define as $z^2 = \alpha = \beta\omega$. In sum,

$$|z^2|^2 = |\alpha + \beta\omega|^2 = \alpha^2 - \alpha\beta + \beta^2$$

is, by construction, square. Therefore $\alpha = a^2 - b^2$ and $\beta = 2ab - b^2$ forms the legs of an Eisenstein triple for any integers a and b, ensuring the cubic

$$f(x) = x(x-\alpha)(x-\beta)$$

has rational turning points.

References

A. Cuoco. "Meta-Problems in Mathematics." In: *The College Mathematics Journal* (2000), pp. 373–378. DOI https://doi.org/10.1080/07468342.2000.11974176