Fred Dickinson fredsterdickinson@gmail.com

Completing the Square and the Quadratic Formula.

Your task: try to understand where the quadratic formula comes from. You don't need to be able to reproduce this result.

This worksheet assumes that you have strong prior knowledge of 'completing the square.' If you haven't already, please read the worksheet titled 'An Introduction to Completing the Square.'

Consider any quadratic equation in its general form

$$ax^2 + bx + c = 0.$$

Whilst it is quite algebra-intensive, there is nothing stopping us from completing the square on this general form! That's precisely what we're going to do here.

First, we factor out a so that we have a quadratic with a leading coefficient of 1. We can leave the constant term c alone, so

$$a\left(x^2 + \frac{b}{a}\right) + c = 0.$$

Now we can complete the square for $x^2 + \frac{b}{a}x$ as usual. Note that $\frac{b}{a} \div 2$ is $\frac{b}{2a}$. So

$$x^2 + \frac{b}{a} = \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$$

and therefore

$$a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] + c = 0.$$

All we need to do now is re-arrange for x. This is slightly tricky. First, verify these steps

$$\begin{aligned} a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right] &= -c \\ \left(x + \frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} \\ \left(x + \frac{b}{2a}\right)^2 &= \left(\frac{b}{2a}\right)^2 - \frac{c}{a} \\ &= \frac{b^2}{4a^2} - \frac{c}{a}. \end{aligned}$$

We want to combine the fractions on the right hand side. To do so, multiply $\frac{c}{a}$ by a factor of $4a$ on the top and bottom. Then

$$\begin{aligned} \left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\ x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{\sqrt{4a^2}} \end{aligned}$$

and note that $\sqrt{4a^2} = 2a$ (you can check that $(2a)^2 = 2^2a^2 = 4a^2$). All that's left is to subtract the $\frac{b}{2a}$ from the left hand side, so

$$\begin{aligned} x + \frac{b}{2a} &= \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ x &= -\frac{b}{2a} + \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

This is the quadratic formula!

The quadratic formula is a powerful tool, and easy to memorise. It works for any quadratic and all you need are the coefficients a, b, and c.

That being said, it's important to understand where these things come from. You don't need to be able to completely understand this derivation, nor reproduce it; but now every time you chant "x equals minus b plus or minus the square root of ...", take a second to remember that it's really just a glorified version of completing the square.