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The Fundamentals of Probability

Read these notes and attempt the exercises as you go. Focus more on the ideas at hand than the way things are written; the notation has tried to be introduced gently, but it takes time to get used to!

Exercise solutions are found at the end of the document.

Probability is something we all talk about in real life, seen through statements such as "what are the chances?!" or 'wow, that's really *unlucky*'. Mathematics, being what it is, looks to formalise these ideas with a concrete set of rules. To begin, we have the three building blocks of probability.

(a) The set of *all* things that could possibly happen (**sample space**).
(b) Individual things that could happen (**events**).
(c) A function that tells you how likely an individual thing is to happen (**the probability**, denoted $P(\text{event})$.)

The first two basic rules associated to these building blocks (the 'axioms') are as follows.

(a) The probability that *something* from the set of all possible things happens is 1. Equivalently, the probabilities of all possible events must sum to 1.
(b) The probability of an individual event is between 0 and 1. We write this as $P(\text{event}) \in [0, 1]$. Probabilities *cannot be negative.*

The third axiom will be introduced later.

It is always helpful to have an example, so let's look at the rolls of a six-sided die. Assume that this die is perfectly weighted so that each side is equally likely to roll. Then all possible events that could happen are

$$\text{dice roll} \in \{1, 2, 3, 4, 5, 6\}.$$

We will introduce the so-called 'random variable' X to represent the roll of the dice. So $X \in \{1, 2, 3, 4, 5, 6\}$. This set is the **sample space**, denoted with the (capital) Greek letter 'Omega', $\Omega = \{1, 2, 3, 4, 5, 6\}$.

There are a handful of possible events we can look at. For example, an event could be a particular roll of the die, say 6. Because the die is evenly weighted, we know the probability of this particular event is

$$P(\{6\}) = \frac{1}{6}.$$

The choice to use notation $P(\{6\})$ is intentional, there to indicate that this event is a *subset* of the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. All events are subsets of Ω. Consider the event where we roll an *even* number. This is written $P(\{2, 4, 6\})$, and

$$\begin{aligned} P(\{2,4,6\}) &\overset{??}{=} P(\{2\}) + P(\{4\}) + P(\{6\}) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}. \end{aligned}$$

This event $\{2, 4, 6\}$ is also a subset of Ω. Hopefully you agree that it has probability one half. However, the ?? in the equation is there for a reason. We have separated the event $\{2, 4, 6\}$ into its parts $\{2\}$, $\{4\}$, and $\{6\}$. Why are we allowed to do this?

The answer is because these events are **mutually exclusive**. This means that each individual event has nothing to do with the other - there is no 'overlap'. Some authors prefer to say that the events are 'disjoint'. In general, if A and B are mutually exclusive, then

$$P(A \text{ or } B) = P(A) + P(B).$$

This is the third axiom.

Exercise 1.1. *Let $X \in \{1, 2, 3, 4, 5, 6\}$ be the possible rolls of a weighted die. The probability of even numbers $2, 4,$ and 6 rolling is $\frac{1}{5}$. The probability of odd numbers $1, 3,$ and 5 rolling is $\frac{2}{15}$.*

(a) Verify that the total probability of events adds to 1.

Let A be the event where a number less than or equal to 3 is rolled. Let B be the event where a 6 is rolled.

(b) Are these events mutually exclusive?
(c) What is the probability that A or B happen?
(d) What is the probability that neither A or B happen?
(e) How could you check your answer to (d)?

Dependent and Independent Events

If I have a banana for breakfast, is your bus more likely to be late? Probably not, unless I'm your bus driver; and you definitely don't want that. So, let's say my breakfast choice has absolutely no impact on your journey. Then we can say that the events

$$\{\text{fred has a banana for breakfast}\} \text{ and } \{\text{your bus arrives late}\}$$

are **independent.** To be precise, events are independent if one happening has no impact on the likelihood of the other.

Conversely, suppose we're playing a game of cards. If I play ten red cards in a row, will this affect the chance that I play a red card on my next turn? Yes, because there's less red cards left for me to play! So these events are *dependent.*

If two events are independent, then the probability they both happen **is the product of the individual probabilities** both happen. With the same example, let A be the event that I have a banana for breakfast, and B be the event where your bus is late. Then

$$P(A \text{ and } B) = P(A) \times P(B) = P(A)P(B).$$

It is more common to use the intersection $\cap$ to indicate 'and', i.e. $P(A \cap B) = P(A)P(B)$ if the events A and B are independent.

Exercise 1.2. *Suppose the probability I wear a white t-shirt on any given day is 0.35. Also suppose that the probability I wear black trousers is 0.8.*

If these two events have no impact on the likelihood of the other, what is the probability that I wear a white t-shirt and black trousers?

If events are dependent, i.e. one happening has an impact on the probability of the other happening, then we enter the realm of 'conditional probability'. We will not cover this in detail here, but you may recognise the formula

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

or have heard the term "Bayes' rule".