MATH10010 - The Role of Visualisation in University Mathematics

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Visualisation is an integral part of the learning of mathematics, playing a far more important role in today's mathematics than is generally known (Dreyfus, 1991). Courtesy of Presmeg (1986), we define visual methods to be methods that use visual imagery, with or without a diagram, as an essential part of the solution. Visual methods may or may not include analytic or algebraic methods, and visualisation refers to the employment of a visual method.

This essay looks to explore the role visualisation plays in learning mathematics at the university level. First, we explore visualisation within the transition from secondary to tertiary education, and then we evaluate its role for any mathematics student at university.

The Transition to University Mathematics

The secondary-tertiary transition in mathematics, typically referring to the entry into university level mathematics, is a widely discussed topic in mathematics education. It is not always clear when this 'transition' takes place (Gueudet, 2008), but the majority of literature acknowledges it as a major problem. Niss (2003) identifies challenges within the culture and identity of mathematics across different educational levels, while Moore (1994) explores the experiences new undergraduate students find in writing proofs. He suggests that the abrupt shift into developing formal and rigorous arguments is a source of difficulty for many, echoed by Tall (1991) and Artigue (2009).

There are several factors involved in the secondary-tertiary transition, but of particular interest is the quality of learning strategies students posses (Rach & Heinze, 2016). The environment in which one learns mathematics at university is dramatically different to that of secondary education, with a shift from semi-permanent classroom-style teaching to intermittent lectures, a style of teaching *'commonly criticised for being a poor way to provide notes'* (Pritchard, 2010). The reduction in contact hours and an ever increasing need to keep up with course content creates a demand for independent study most students would not have seen before university. It is, therefore, unsurprising that students find the transition an 'insurmountable struggle' (Di Martino & Gregorio, 2019), and we suggest that finding quality learning strategies is a necessary step all students must take.

To this end, we look to explore the role visualisation plays in the secondary-tertiary transition. First, consider the notion of *preformal proofs* as coined by Blum and Kirsch

(1991). These are *'a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises.'* Preformal thinking is suggested not to superficially make mathematics easier for students, but rather to encourage a natural line of reasoning, particularly for those who are less experienced. The authors emphasise that the lack of formality does not demand a lack of validity and rigour, and indeed, preformal thinking often assists students in progressing towards the production of formalised arguments.

We propose that the role of visualisation shares many characteristics with the role of preformal thinking, and by consequence, should be considered as a valuable learning strategy in its own right. Uesaka et al. (2007) suggest that the use of diagrams is one of the most efficient strategies for problem solving in mathematics, while Arcavi (2003) encourages visualisation as an excellent means to foster the insight students develop. Fischbein (1999) reflects these views by highlighting visualisation, and the immediate realisation it brings, as an essential part of intuitive understanding. For a new, undergraduate student in mathematics, the ability to visualise the concepts at hand will certainly assist their progress in building necessary foundations in several key areas of the subject. Indeed, Pinto and Tall (2002) conducted a study that saw how an undergraduate student uses 'natural' and pictorial understanding of concepts in real analysis to build a variety of formal definitions. These authors propose that the use of a well constructed visual image was instrumental in the student's ability to refine and reconstruct their understanding, eventually being able to formalise concepts with great accuracy. We suggest that students struggling to find quality learning strategies should consider employing visualisation, perhaps in conjunction with preformal thinking, as a means to develop rigour and formality whilst reducing the intensity of independent learning.

Another major challenged faced by students in the secondary-tertiary transition is the immediate increase of symbolism, generalisations and abstraction; a phenomenon coined as 'abstraction shock' (Di Martino & Gregorio, 2019). Previous discussion has outlined the utility of visualisation for students struggling with the transition into university mathematics; to what extent can it support students facing abstraction shock?

Visualisation and Abstraction

A high level of symbolism and abstraction is seen across many areas of mathematics. For new undergraduates, first encounters with differentiation from first principles or, courtesy of Artigue (2009), a formal notion of equality

$$x = y \iff \forall \varepsilon > 0, \ |x - y| < \varepsilon$$

are both concepts that demand a change in how one thinks about mathematical objects, yet this is almost always done so symbolically. On the other hand, if 'the first isomorphism theorem' or 'the Mandelbrot set' was referenced to a more experienced mathematician, a diagram or image would likely come to mind. Certainly, visualisation can be incredibly valuable for the understanding and communication of abstract topics, and the aim of the succeeding discussion is to identify why many areas of mathematics choose to use visual imagery as aids for learning and communication.

1. *Visualisation is a natural and immediate process.* Visualisation is natural in the sense that it aligns with how we see the world, through sight and spatial reasoning. Its immediacy is best highlighted through modern media platforms that allow us to quickly access large amounts of information at once. Fischbein (1999) highlights the importance of immediacy in mathematics, suggesting visual representations provide students with personal and creative involvement into the concepts at hand. These representations are incredibly helpful for the communication of mathematics, well illustrated by Arcavi (2003) through a series of problems explained with major pictorial support.

2. *Visualisation often contains little information that would be superfluous to a student's understanding.* The use of diagrams, pictures, and graphs can often remove a large level of symbolism that may otherwise obstruct a student from accessing the 'bigger picture' of the concepts at hand (Nelsen, 1993). Consequently, we suggest that the visualisation of a mathematical object is often a 'low-noise' representation of such object in the sense of Skemp (1987). These representations are invaluable for communication and teaching because they provide the opportunity to accurately assess a student's understanding of the concept being taught.

3. *Visual imagery is easily adaptable.* In this sense, we refer not to the literal scenario, but instead the ability to change a mental image quickly. Understanding a concept symbolically, perhaps through a formal definition or proof, may involve complex and intricate parts. If this understanding was incorrect, or needed to be applied in a similar context, the student may struggle to identify how to modify their existing understanding. On the other hand, visualisaton provides an easily adaptable framework that can be continuously modified upon finding flaws, as demonstrated by Pinto and Tall (2002). Additionally, well constructed visual images can be readily transformed to fit a wide range of different scenarios, providing students with the means to further expand their existing knowledge in problem-solving situations.

4. *Visualisation is motivating and encourages deeper understanding.* A widespread dislike of mathematics can be found across all stages of compulsory education (Aiken Jr, 1970). There are several reasons for this, but we focus on one: mathematics can be abstract and difficult, and consequently demotivating for many (Hwang & Son, 2021). A study performed by Alcock and Simpson (2004) suggests that students with a tendency to visualise are motivated to study mathematics, but also proposes that these students think of mathematical concepts as objects that can be compared. In this sense, Alcock suggests that visualisation allows students to organise their knowledge and gain relational understanding in the sense of Skemp (1987), a deeper and far more valuable understanding for students to obtain.

Critical Perspectives on Visualisation

Visualisation is an important part of learning mathematics, but it is certainly not infallible. Indeed, Presmeg (1986) undertook an investigation that suggests the majority of mathematically gifted students are *not* 'visualisers', individuals who prefer to use visual methods when attempting mathematical problems. The remainder of this essay serves to explore this statistic by critically evaluating the effectiveness of visualisation in the learning of mathematics.

The first and most easily understood challenge visualisation faces is simple; visual reasoning is difficult (Dreyfus, 1991). This is best illustrated by considering extreme examples such as infinitesimally small mathematical objects, or objects in high dimensional space. Advancements in technology continue to provide greater potential for mathematical visualisation, and the open source Python library *Manim* has provided great opportunity for many mathematics or mathematics-adjacent students to gain valuable insight into a range of topics (Marković & Kaštelan, 2024). However, for the majority of learning that happens each day, technology is not yet established to provide consistent support in visualising all areas of mathematics.

Dreyfus (1991) identifies another key challenge visualisation faces: socio-cultural barriers. He suggests that although a large number of mathematicians use visual imagery in their arguments, they choose to not publicise their images and diagrams because they would likely not be judged acceptable. Indeed, formal proof has historically been one of the most prominent characteristics of mathematics (Hanna, 1991) and only recently are attitudes among mathematicians starting to slowly change (Weber & Czocher, 2019). It's plausible that there will be an increased use of visual proofs in the lower levels of mathematics because the gap errors are less significant, and the visual imagery encourages problem solving in younger mathematicians. However, at the university level and above, we mirror the opinion of Dreyfus and propose that whilst visualisation may be an efficient learning aid for a proof, it should rarely be more than that.

A further obstacle visualisation faces is cognitive difficulty (Arcavi, 2003). Previously, visualisation has been described to greatly support a student's understanding, but at what point does visual imagery become more difficult than its symbolic counterpart? Arcavi suggests concepts that demand extreme detail in their illustrations should perhaps not be approached from a visual lens. This is echoed when he identifies the need for visual imagery to attain *'flexible and competent translation'* into its analytic representation. Illustrations of highly complex objects may provide little insight into the formal, analytic representation of the same object, or the analytic representation of highly complex objects may be too difficult to illustrate visually.

The issue of cognitive difficulty branches once more, this time into what is consistently considered a great issue of visual reasoning: you cannot always trust what you see. Kondratieva (2022) explores a plethora of visual fallacies, and several other authors emphasise this challenge ((Arcavi, 2003), (Davis, 1993), (Dreyfus, 1991)).

Closing

Visualisation is an integral part of learning mathematics for all students, and we suggest this is particularly highlighted at university. There are undoubtedly scenarios in which visual imagery is inappropriate, such as the aforementioned high dimensional spaces or objects with infinitesimal attributes. We recognise the challenges in using and interpreting visual imagery in proofs. Nonetheless, we suggest there is great merit in the use of visualisation for all mathematics students. For those in the secondary-tertiary transition, visualisation can act as a valuable learning strategy and assist students develop strong foundations whilst adjusting to university mathematics. For those more experienced students, visualisation can help in the effective communication or mathematics, or as an opportunity to expand their sample space.

We hope that, particularly with the advancements in technology, societal opinions on visual aids in mathematics continue to change for the better. To this end, we encourage mathematics students at university to employ visualisation where possible, remaining aware of its limitations and benefiting from the merit it brings.

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