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The Discriminant.

Your task: read the following summary on the discriminant. Then attempt both exercises. Solutions are given at the end of the document, but please give a good effort and check your work before looking at them!

Quadratic equations can be simplified so that they always take the form $ax^2+bx+c=0$. We call a, b, and c the coefficients of the quadratic equation.

Typically, the primary focus is finding the solutions to these equations. For example, $x^2+4=0$ has *no real solutions,* whereas $x^2-4=0$ has *two real solutions* ($x=\pm 2$). The discriminant is a way in which we can completely characterise just how many solutions any quadratic has.

We will use Δ (the Greek letter 'Delta') to represent the discriminant. It is defined

$$\Delta = b^2 - 4ac.$$

You should recognise this as part of the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-b \pm \sqrt{\Delta}}{2a}.$$

This correspondence is a good way to remember how the discriminant works. Remember, the square root of a negative number does not exist[1]. That is to say, $\sqrt{-4}$ is not defined as a real number. So then if

$$\Delta < 0 \qquad (b^2 - 4ac < 0)$$

there are *no real roots* to the quadratic equation. Furthermore, if $\Delta = 0$ then the quadratic formula tells us that

$$x = \frac{-b \pm \sqrt{0}}{2a} = -\frac{b}{2a}$$

i.e. there is only *one* (repeated[2]) root. Finally, if $\Delta > 0$, then there are two real roots to the equation as given by the quadratic formula.

In summary,

$$\begin{cases} \Delta > 0: & \text{two real roots,} \\ \Delta = 0: & \text{repeated real roots,} \\ \Delta < 0: & \text{no real roots.} \end{cases}$$

Applications of the discriminant. Common exam questions involve applying these ideas of the discriminant to intersections between lines and curves. For example, we might want to know if the line and curve represented by

$$\ell \ : \ y = x - 4, \qquad C \ : \ y = x^2 + 6x - 4$$

[1]They exist, but they are complex numbers; numbers with an imaginary part i.

[2]Repeated roots make more sense after factorising. Consider $x^2 - 6x + 9 = 0$. Then $(x-3)^2 = 0$, or equivalently, $(x-3)(x-3) = 0$. There are two brackets here - two (the same, but repeated) ways for the function to be zero. Technically all quadratics have two roots: they are either two real roots, *repeated* real roots, or two complex roots.

intersect, and if so, how many times they intersect. We can answer this question by equating the line and curve (intersections are just points where $\ell = C$) and finding the discriminant of the resultant quadratic. In this example, we have

$$\begin{aligned} x^2 + 6x - 4 &= x - 4 \\ x^2 + 5x &= 0 \qquad \Longrightarrow \qquad \Delta = 25 - 4 = 21 > 0 \end{aligned}$$

and hence we can conclude that there are two solutions to this quadratic, or equivalently, that the line and curve intersect twice.

A more involved question would involve another parameter, say k. Suppose we now have a curve defined $C : x^2 + 3x - 4$ and a line $\ell : y = x + 2k$. For what value(s) of k do the line and curve intersect exactly once? Again, by equating $\ell = C$ we have

$$\begin{aligned} x^2 + 3x - 4 &= x + 2k \\ x^2 + 2x + (-4 - 2k) &= 0 \qquad \Longrightarrow \qquad \Delta = 4 - 4(-4 - 2k) = 20 - 8k. \end{aligned}$$

We know that a quadratic has exactly one root if $\Delta = 0$, so we need $20 - 8k = 0$. Hence $k = 20/8 = 2.5$.

Exercise 1.1. *Consider the two quadratic curves in Figure 1. For each, determine if the determinant is postive, negative, or zero.*

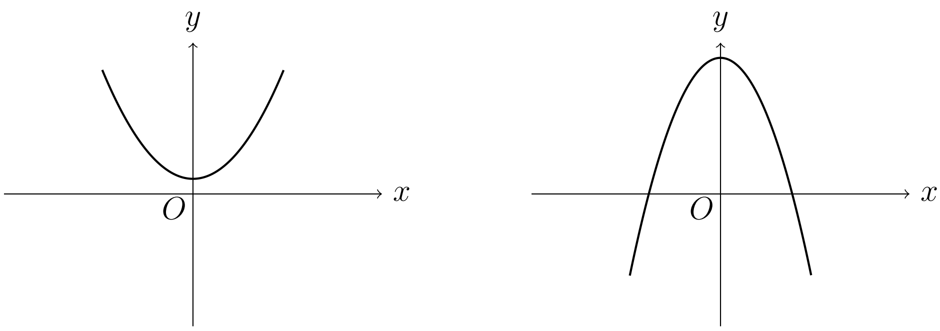

Figure 1: Two quadratic curves plotted in the real plane.

Exercise 1.2. *(May 2018, q14c) Consider a curve C defined by $y = 6 - (x - 3)^2$, and a line ℓ defined by $x + y = k$. This line intersects C twice. State the range of values of k, writing your answer in set notation.* ***(5)***

Solutions. **(1)** Figure 1(a) does not intersect the x-axis, and thus it has no real roots. Hence $\Delta < 0$. On the other hand, Figure 1(b) intersects the x-axis in two distinct places, and thus has two real roots. Hence $\Delta > 0$.

(2) First write ℓ as $y = k - x$. Then, by equating $\ell = C$,

$$\begin{aligned} 6 - (x-3)^2 &= k - x \\ 6 - (x^2 - 6x + 9) &= k - x \\ 0 &= x^2 - 7x + (k+3) \qquad \Longrightarrow \qquad \Delta = 49 - 4(k+3) = 37 - 4k. \end{aligned}$$

We need $\Delta > 0$ and thus $k < \frac{37}{4}$. Written in set notation, $\{k \in \mathbb{R} : k < \frac{37}{4}\}$.