

## HIDDEN QUADRATICS

*Your task: read the following examples to gain exposure and understanding of 'hidden quadratics'. There are no exercises here as you'll see them coming; instead, try to keep these ideas in mind when stuck solving an equation.*

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If you're working through a problem and end up with an equation of the type

$$x^2 + 6 = -7x,$$

you'll recognise this as a quadratic and easily solve for  $x = -6$  and  $x = -1$ . What I'd like to convince you of is that these quadratics are hiding in more places than you realise. Consider the following examples.

- (a) **Hiding in a polynomial.** Consider  $x^4 - 6x^2 - 7 = 0$  with the substitution  $y = x^2$ , yielding

$$y^2 - 6y - 7 = 0 \implies y = 7 \quad \text{or} \quad y = -1,$$

and hence  $x = \pm\sqrt{7}$  only (as  $x^2 \neq -1$ ).

- (b) **Hiding in trigonometry.** Let's solve  $\cos(2x) + 4\sin^2(x) = 4 - \sin(x)$  for  $x \in [0, 2\pi]$ . To see how this is a hidden quadratic, recall

$$\begin{aligned} \cos(2x) &= 2\cos^2(x) - 1 = 2(1 - \sin^2(x)) - 1 \\ &= 1 - 2\sin^2(x), \end{aligned}$$

and therefore the equation becomes

$$2\sin^2(x) + \sin(x) - 3 = 0.$$

With the substitution  $y = \sin(x)$ , we have  $2y^2 + y - 3 = 0$  which gives two solutions  $y = -3/2$  and  $y = 1$ .

Be careful here:  $\sin(x) \in [-1, 1]$  so the solution  $y = -3/2$  is not relevant. Therefore  $\sin(x) = 1$ , giving  $x = \pi/2$  as the only solution in  $[0, 2\pi]$ .

- (c) **Hiding in exponents or logarithms.** Finally, consider  $4^x + 2^{x+1} = -3$ . With the usual exponent laws, noting  $4 = 2^2$ , we have

$$(2^x)^2 + 2 \cdot 2^x - 3 = 0$$

which gives rise to the quadratic  $y^2 + 2y - 3 = 0$  with  $y = 2^x$ . Hence  $y = -3$  or  $y = 1$ , and as  $2^x \geq 0$ , we must have  $2^x = 1$ . Therefore  $x = 0$  is the only solution.