# The Basel Problem

The Basel problem concerns the infinite sum of squared reciprocals, often expressed in terms of the zeta function

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

It is named after the city of Basel in Switzerland, home to Euler and the Bernoulli family. Euler solved the problem in 1734 after many great minds, including Leibniz, Jacob Bernoulli, and Johann Bernoulli had failed to do so. His original proof uses a factorisation of $\sin(x)/x$, only proven by Weierstrass over a century later! (See 'With the Hadamard Factorisation').

No proofs here are original; many are textbook. For more, see Robin Chapman's (University of Exeter) list or Wikipedia's page on the topic.

*Unfinished.*

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**1. With Parseval's Identity.** If $f \in L^2[a,b]$, i.e. $f$ is square-integrable, then

$$||f||^2_{L^2[a,b]} = \frac{1}{b-a}\int_a^b |f(x)|^2 \mathrm{d}x = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2,$$

where the Fourier coefficients $\hat{f}(n)$ are found by calculating

$$\hat{f}(n) = \frac{1}{b-a}\int_a^b f(x)e^{-inx}\mathrm{d}x.$$

This is one formulation of *Parseval's identity* from the theory of Fourier analysis. Let $f(x) = x$ on $[0,1]$ be our square-integrable function of choice. Then

$$\hat{f}(0) = \int_0^1 x\mathrm{d}x = \frac{1}{2}, \quad \hat{f}(n) = \int_0^1 xe^{-2\pi inx}\mathrm{d}x = \frac{i}{2\pi n}, \quad ||f||^2_{L^2[0,1]} = \int_0^1 x^2\mathrm{d}x = \frac{1}{3},$$

and as the sum is symmetric, Parseval's identity asserts

$$\frac{1}{3} = \frac{1}{4} + 2\sum_{n=1}^{\infty}\frac{1}{4\pi^2n^2} \implies \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

**2. With Hadamard Factorisation.** The Hadamard factorisation theorem is a powerful tool from complex analysis. It can be viewed as an extension of the Weierstrass factorisation theorem, which itself can be viewed as an extension of the fundamental theorem of algebra.

Hadamard factorisation applies to functions of *finite order*, meaning that the function grows at most like the exponential. More formally, $f$ is of finite order if there exists $\rho > 0$ such that

$$|f(z)| \leq Ce^{A|z|^\rho}, \quad A, C > 0, \tag{1}$$

for all $z \in \mathbb{C}$. The *order of growth* of $f$ is the infimum of all $\rho$ satisfying (1), denoted $\rho_f$.

**Theorem 1.1** (Hadamard Factorisation)**.** *Let $f$ be an entire function of finite order. Let $\ell \geq 0$ be the order of the zero at $z = 0$, and let $z_1, z_2, z_3, \dots$ be the nonzero zeroes of $f$ listed with multiplicities. Then*

$$f(z) = e^{Q(z)} z^\ell \prod_{n=1}^{\infty} E_p(z/z_n)$$

*where $Q(z)$ is a polynomial of degree at most $\rho_f$. The elementary factors $E_p$ correspond to the genus $p$ of the zeros $\{z_n\}$, and are defined*

$$E_p(w) = (1-w)\exp\left(\sum_{k=1}^{p} w^k/k\right).$$

A few more notes: entire functions are holomorphic on all of $\mathbb{C}$, and the genus $p$ of the zeroes $\{z_k\}$ is the smallest integer $p$ such that

$$\sum_{n=1}^{\infty} \frac{1}{z_k^{p+1}} < \infty.$$

**An example, covered in lectures by Professor Jim Wright,** finds the Hadamard factorisation of $\sinh(z)$. It is defined as a difference of exponentials

$$\sinh(z) = \frac{1}{2}(e^z - e^{-z}),$$

so it is certainly entire and of finite order, with growth rate $\rho_{\sinh} = 1$. By solving $e^{2z} = 1$, we find $z = n\pi i$ for $n \in \mathbb{Z}$ to be the zeroes of $\sinh(z)$. Therefore the genus $p_{\sinh} = 1$. It will be helpful to look at the Taylor expansion

$$\sinh(z) = \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{z^k}{k!} - \sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1-(-1)^k}{k!}z^k.$$

Only odd powers contribute to the sum, so

$$\sinh(z) = \sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots,$$

which makes it clear that $z = 0$ is a simple zero; that is, $\ell = 1$. So far, Hadamard's factorisation yields

$$\sinh(z) = \exp(az+b)z\prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} E_1\left(z/n\pi i\right), \tag{2}$$

and we can simplify the elementary factors by noticing

$$\begin{aligned} E_1(z/n\pi i)E_1(-z/n\pi i) &= (1 - z/n\pi i)\exp(z/n\pi i)(1 + z/n\pi i)\exp(-z/n\pi i) \\ &= (1 + z^2/\pi^2 n^2) \end{aligned}$$

since $i^2 = -1$. Therefore the product simplifies to

$$\prod_{\substack{n\in\mathbb{Z}\\ n\neq 0}} E_1\left(z/n\pi i\right) = \prod_{n=1}^{\infty} E_1(z/n\pi i)E_1(-z/n\pi i) = \prod_{n=1}^{\infty}(1 + z^2/\pi^2 n^2).$$

It remains to find the constants $a$ and $b$ in (2). Let

$$\sinh(z)/z = 1 + z^2/3! + z^4/5! + \cdots =: g(z) \tag{3}$$

and notice $g(0) = 1$. Furthermore, $E_1(0) = 1$ and therefore $\exp(b) = 1 \implies b = 0$. To find $a$, consider the parity of each term in the factorisation,

$$\underbrace{\sinh(z)}_{\text{odd}} = \exp(az) \underbrace{z}_{\text{odd}} \prod_{n=1}^{\infty} \underbrace{1 + z^2/\pi^2 n^2}_{\text{even}}.$$

So $\exp(az)$ *must* be even for all $z \in \mathbb{C}$. By the definition of an even function, $e^{az} = e^{-az}$ or $e^{2az} = 1$ for all $z \in \mathbb{C}$! Necessarily $a = 0$.

With that, the Hadamard factorisation of $\sinh(z)$ is complete,

$$\sinh(z) = z \prod_{n=1}^{\infty} (1 + z^2/\pi^2 n^2). \tag{4}$$

**An application to the Basel problem.** We have already defined $\sinh(z)$ through its Taylor expansion; recall (3),

$$\sinh(z)/z = 1 + z^2/6 + z^4/120 + \dots \ .$$

Consider the Hadamard factorisation (4). The coefficients of the infinite product can each be expressed as their own series. For example,

$$\begin{aligned}\sinh(z)/z &= (1 + z^2/\pi^2)(1 + z^2/\pi^2 2^2)(1 + z^2/\pi^2 3^2) \cdots \\ &= 1 + \left( \sum_{n=1}^{\infty} 1/\pi^2 n^2 \right) z^2 + \dots\end{aligned}$$

and therefore, matching the Taylor expansion, we have

$$\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6.$$

This is very similar to Euler's origin proof that found the factorisation of $\sin(x)/x$, proven by Weierstrass over a hundred years later!

**3. With residue theory.** Cauchy's residue theorem states that, for singularities $\{z_k\}_k$ inside a closed curve $\Gamma$,

$$\frac{1}{2\pi i} \int_\Gamma f(z) \mathrm{d}z = \sum_{z \in \{z_k\}} \mathrm{Res}(f; z).$$

The particular function

$$f(z) = \pi z^2 \cot(\pi z)$$

is of interest because it has singularities at every integer, with the nonzero integers being *simple* poles. For these simples poles,

$$\mathrm{Res}(f; n) = \lim_{z \to n} (z - n) f(z) = \lim_{z \to n} \frac{(z - n) \cos(\pi z)}{z^2 \sin(\pi z)} = \frac{1}{z^2},$$

and the residue at zero can be calculated to be $-\pi^2/3$ using the Laurent series expansion. Thus, if we can show the integral vanishes as $z$ becomes arbitrarily large, we have

$$-\frac{\pi^2}{3} + 2\sum_{n=1}^{\infty} \frac{1}{n^2} = 0 \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

We will do so by showing $|\cot \pi z|$ is bounded along the square contour $\Gamma_N$ with vertices $(N + 1/2)(\pm 1 \pm i)$, as then $f(z) \to 0$ with the factor of $1/z^2$ dominating. Consider $\pi z = x + iy$ and

$$|\cot(\pi z)|^2 = \frac{|\cos(x+iy)|^2}{|\sin(x+iy)|^2} = \frac{|\cos(x)\cosh(y) - i\sin(x)\sinh(y)|^2}{|\sin(x)\cosh(y) + \cos^2(x)\sinh^2(y)|^2}$$

via the complex addition formulae, which then simplifies to be

$$\frac{\cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y)}{\sin^2(x)\cosh^2(y) + \cos^2(x)\sinh^2(y)} = \frac{\cos^2(x) + \sinh^2(y)}{\sin^2(x) + \sinh^2(y)}$$

with $\cos^2 x + \sin^2 x = 1$. On the vertical edges of the contour, $x = \pi(\pm N \pm 1/2)$, so $\cos(x) = 0$ and $\sin^2(x) \leq 1$. Thus here

$$|\cot \pi z|^2 = \frac{\sinh^2(y)}{1 + \sinh^2(y)} < 1.$$

Moreover, on the horizontal edge we have

$$|\cot(\pi z)|^2 \leq \frac{1 + \sinh^2(y)}{\sinh^2(y)} = \coth^2(y)$$

because $\sin^2(x) \leq 1$ and $\cos^2(x) \leq 1$; but $\coth^2$ is decreasing with $y$ and is at most $\coth^2(1/2) =: K$. So $|\cot(\pi z)|^2$ is bounded on $\Gamma_N$, and therefore so is $|\cot(\pi z)|$. Hence

$$\lim_{N\to\infty} \frac{1}{2\pi i}\int_{\Gamma_N} f(z)\mathrm{d}z = \lim_{N\to\infty} \frac{1}{2\pi i}\int_{\Gamma_N} \frac{\pi\cot(\pi z)}{z^2}\mathrm{d}z = 0$$

and the solution to the Basel problem is $\pi^2/6$.

**4. With two integrals and a change in coordinate system.** See proofs from the book.

**5. With Gregory's formula.** See proof 14 from Robin Chapman and your notes.