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Hidden Quadratics

Your task: read the following examples to gain exposure and understanding of 'hidden quadratics'. There are no exercises here as you'll see them coming; instead, try to keep these ideas in mind when stuck solving an equation.

If you're working through a problem and end up with an equation of the type

$$x^2 + 6 = -7x,$$

you'll recognise this as a quadratic and easily solve for $x = -6$ and $x = -1$. What I'd like to convince you of is that these quadratics are hiding in more places than you realise. Consider the following examples.

(a) **Hiding in a polynomial.** Consider $x^4 - 6x^2 - 7 = 0$ with the substitution $y = x^2$, yielding

$$y^2 - 6y - 7 = 0 \implies y = 7 \quad \text{or} \quad y = -1,$$

and hence $x = \pm\sqrt{7}$ only (as $x^2 \neq -1$).

(b) **Hiding in trigonometry.** Let's solve $\cos(2x)+4\sin^2(x) = 4-\sin(x)$ for $x \in [0, 2\pi]$. To see how this is a hidden quadratic, recall

$$\begin{aligned}\cos(2x) &= 2\cos^2(x) - 1 = 2(1 - \sin^2(x)) - 1\\ &= 1 - 2\sin^2(x),\end{aligned}$$

and therefore the equation becomes

$$2\sin^2(x) + \sin(x) - 3 = 0.$$

With the substitution $y = \sin(x)$, we have $2y^2 + y - 3 = 0$ which gives two solutions $y = -3/2$ and $y = 1$.

Be careful here: $\sin(x) \in [-1, 1]$ so the solution $y = -3/2$ is not relevant. Therefore $\sin(x) = 1$, giving $x = \pi/2$ as the only solution in $[0, 2\pi]$.

(c) **Hiding in exponents or logarithms.** Finally, consider $4^x + 2^{x+1} = -3$. With the usual exponent laws, noting $4 = 2^2$, we have

$$(2^x)^2 + 2 \cdot 2^x - 3 = 0$$

which gives rise to the quadratic $y^2 + 2y - 3 = 0$ with $y = 2^x$. Hence $y = -3$ or $y = 1$, and as $2^x \geq 0$, we must have $2^x = 1$. Therefore $x = 0$ is the only solution.