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An Introduction to Completing the Square.

Your task: read this introduction to 'completing the square', and then complete all three exercises.

Quadratic equations take the general form $ax^2 + bx + c = 0$, where a, b, and c are real numbers. Solutions to the quadratic equation are values x that satisfy this equation.

The first method to solve quadratic equations students learn in school is the method of factorising. You manipulate your quadratic to be the product of two *factors*, multiplying together to be zero. These individual factors must then themselves be equal to zero. A quick example:

$$\begin{aligned} x^2 + 7x + 10 &= 0 \\ (x+5)(x+2) &= 0 \qquad \implies x = -5, x = -2. \end{aligned}$$

Completing the square is an alternative way to solve quadratic equations. The idea is that you encapsulate all of your x terms under a square term in the form $(x-k)^2$, and then re-arrange for x. This notion is best demonstrated through an example

$$\begin{aligned} x^2 - 10x + 24 &= 0 \\ (x-5)^2 - 25 + 24 &= 0 \\ (x-5)^2 &= 1 \\ x - 5 &= \pm\sqrt{1} \qquad (= \pm 1) \\ x &= 5 \pm 1, \end{aligned}$$

so in this instance, $x = 4$ or $x = 6$.

This method works for any quadratic. It's very formulaic, so with practice, it doesn't take long at all. Here's how it works.

(I. The case when $a = 1$.) Consider just the first two terms of a quadratic equation $x^2 + bx$, or in the previous example, $x^2 - 10x$. We *want* this to be represented by a square, so we write $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. Why? Because

$$\begin{aligned} \left(x + \frac{b}{2}\right)^2 &= x^2 + \frac{b}{2}x + \frac{b}{2}x + \frac{b^2}{4} \\ &= x^2 + bx + \left(\frac{b}{2}\right)^2. \end{aligned}$$

So the expression $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)$ is equivalent to $x^2 + bx$. All we need to do now is add the c term from our quadratic so that

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c. \qquad (\spadesuit)$$

This may seem quite complicated, but in practice it's quite simple - especially when b is even. Here's another example. Consider the quadratic $x^2 - 6x + 7$[1]. In this case b is

[1] Try factorising this quadratic! You might struggle.

negative: $b = -6$. Using the expression in equation (♠),

$$\begin{aligned} x^2 - 6x + 7 &= (x-3)^2 - (-3)^2 + 7 \\ &= (x-3)^2 - 9 + 7 \\ &= (x-3)^2 - 2. \end{aligned}$$

So, if we wanted to solve $x^2 - 6x + 7 = 0$ we could write

$$\begin{aligned} x^2 - 6x + 7 &= 0 \\ (x-3)^2 - 2 &= 0 \\ (x-3) &= \pm\sqrt{2}, \end{aligned}$$

and therefore $x = 3 \pm \sqrt{2}$.

(II. The case when $a \neq 1$.) The previous case gave an in depth explanation for when $a = 1$: that is, the *leading coefficient* of your quadratic equation is 1. What happens if a is not 1; for example, solving $3x^2 - 21x + 17 = 0$?[2]

The trick here is to introduce a new first step. Take your leading coefficient a and factorise it out from your expression. In this example,

$$3x^2 - 24x + 17 = 3\left(x^2 - 8x + \frac{17}{3}\right).$$

Now you have a quadratic where $a = 1$: in this case, $x^2 - 8x + \frac{17}{3}$. We can complete the square in the same way as before. I will leave you, the reader, to check

$$x^2 - 8x + \frac{17}{3} = (x-4)^2 - \frac{31}{3}$$

and that we can thus write

$$\begin{aligned} 3x^2 - 24x + 17 &= 3\left[(x-4)^2 - \frac{31}{3}\right] \\ &= 3(x-4)^2 - 31. \end{aligned}$$

Exercise 1.1. *Verify that $x^2 - 8x + \frac{17}{3} = (x-4)^2 - \frac{31}{3}$. Use the result of the previous example to solve the quadratic equation*

$$4x^2 - 24x + 16 = x^2 - 1,$$

giving your answers to two decimal places.

Exercise 1.2. *Rewrite the quadratic expression $5x^2 - 60x + 21$ by completing the square. Then find the solutions to $5x^2 - 60x + 21 = 0$, giving your answers to two decimal places. Suggest how you would check your answers.*

Exercise 1.3. *I am solving a quadratic equation by completing the square. So far, all my steps have been correct and my final workings tell me*

$$(x-7)^2 = -4.$$

How many solutions are there to my quadratic equation?

[2] Again, try factorising this quadratic; you will also struggle!