

# Solutions to the M337/B 2013 Exam Paper

Fred Youhanaie (Ed.)

18 June, 2013

## Copyright

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

See the by-nc-sa<sup>1</sup> page for details.

## Acknowledgements

This is intended to be a community developed project, in particular the M337 class of 2012/2013. To this end, I have tagged answers given by others using their initials listed below. The untagged solutions, including any potential errors, are mine - FY.

DC	Dominic Corbett
FY	Fred Youhanaie
JK	J K
LK	Liga Kauke
VC	Vikki Cookson

---

<sup>1</sup><http://creativecommons.org/licenses/by-nc-sa/3.0/>

## Solutions to Part I

### Solution 1

(a)

$$\begin{aligned}\exp\left(3 + \frac{1}{4}\pi i\right) &= e^3 \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right) \\ &= \frac{e^3}{\sqrt{2}} + i \frac{e^3}{\sqrt{2}}\end{aligned}$$

(b) Let

$$w^3 = -8 = 8(\cos \pi + i \sin \pi)$$

*Handbook A1, 3.3*

then,

$$\begin{aligned}w &= 8^{\frac{1}{3}}(\cos(\pi/3) + i \sin(\pi/3)) \\ &= 2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) \\ &= 1 + \sqrt{3}i\end{aligned}$$

*Handbook A2, 5.3*

(c) Using the Principal  $\alpha$ th power:

$$\begin{aligned}i^{1-2i} &= \exp((1-2i)\text{Log}(i)) \\ &= \exp((1-2i)(\log_e |i| + i\text{Arg}(i))) \\ &= \exp(i\pi/2 - 2i^2\pi/2) \\ &= \exp(\pi + i\pi/2) \\ &= e^\pi e^{i\pi/2} \\ &= e^\pi (\cos(\pi/2) + i \sin(\pi/2)) \\ &= ie^\pi\end{aligned}$$

*Handbook A2, 4.4*

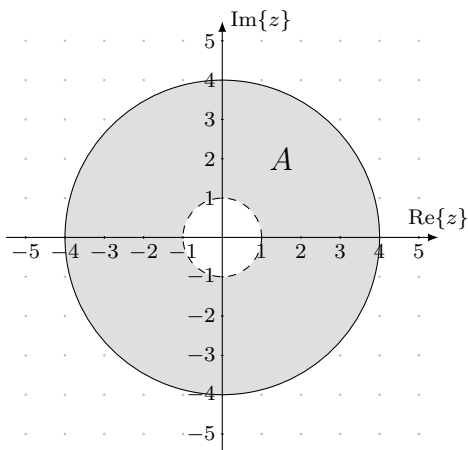
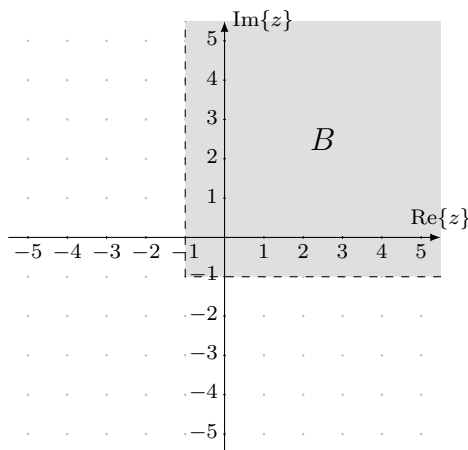
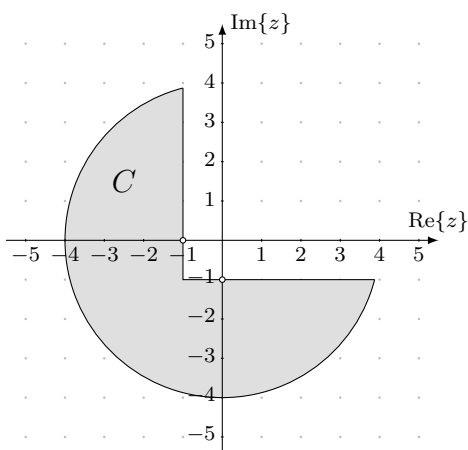
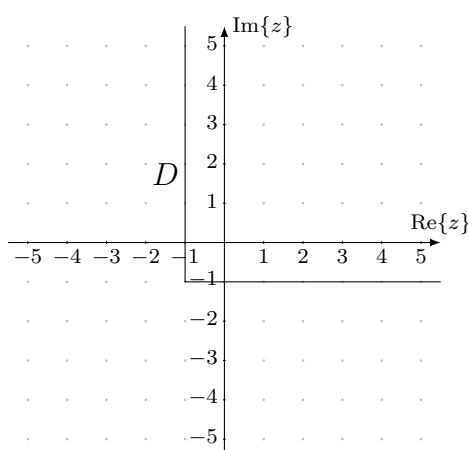
(d) Using the trigonometric functions

$$\begin{aligned}\cos(i \log_e 2) &= \frac{1}{2} (\exp(i^2 \log_e 2) + \exp(-i^2 \log_e 2)) \\ &= \frac{1}{2} (\exp(-\log_e 2) + \exp(\log_e 2)) \\ &= \frac{1}{2} (1/2 + 2) \\ &= \frac{5}{4}\end{aligned}$$

**Solution 2**

(a) [DC]

Below are the sketches of the four sets:

Sketch of the set  $A = \{z : 1 < |z| \leq 4\}$ .Sketch of the set  $B = \{z : \operatorname{Re} z > -1, \operatorname{Im} z > -1\}$ .Sketch of the set  $C = A - B$ .Sketch of the set  $D = \partial B$ .

(b) (i) [FY,LK]

 $A$  is not a region, not open $B$  is a region $C$  is not a region, not open $D$  is not a region, not open(ii)  $A$  is not compact, not closed $B$  is not compact, not closed $C$  is not compact, not closed

$D$  is not compact, not bounded

**Solution 3**

(a) By definition

*Handbook A2, 2.3*

$$\Gamma : \gamma(t) = 2(\cos(t) + i \sin(t)) = 2e^{it}, \quad (t \in [0, 2\pi])$$

(b) We use the polar form from above,

$$\overline{\gamma(t)} = 2e^{-it}$$

$$\gamma'(t) = 2ie^{it}$$

So,

*Handbook B1, 2.1*

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_0^{2\pi} (2e^{-it} 2ie^{it}) dt \\ &= \int_0^{2\pi} 4i dt \\ &= [4it]_0^{2\pi} \\ &= 8\pi i \end{aligned}$$

(c) [FY,LK] Let

$$f(z) = \frac{2 \sin z}{\bar{z}^2 + 1}$$

then,  $f$  is continuous on  $\Gamma$  by combination rules, where  $\Gamma$  has length  $L = 4\pi$

So, using the triangle inequality

*Handbook A1, 5.2(a)*

$$\begin{aligned} |2 \sin z| &= \left| 2 \frac{1}{2i} (e^{iz} - e^{-iz}) \right| \\ &= |-i (e^{iz} - e^{-iz})| \\ &\leq |e^{iz}| + |e^{-iz}| \\ &= e^{\operatorname{Re} z} + e^{-\operatorname{Re} z} \\ &< 2e^2 \end{aligned}$$

And, using the backward triangle inequality

*Handbook A1, 5.2(b)*

$$\begin{aligned}
 |\bar{z}^2 + 1| &\geq ||z|^2 - |1|| \\
 &= |4 - 1| \\
 &= 3
 \end{aligned}$$

So,

$$\begin{aligned}
 |f(z)| &= \left| \frac{2 \sin z}{\bar{z}^2 + 1} \right| \\
 &= \frac{|2 \sin z|}{|\bar{z}^2 + 1|} \\
 &\leq \frac{2e^2}{3} \\
 &= M
 \end{aligned}$$

*Handbook B1, 4.3*

then by the Estimation Theorem.

$$\left| \int_{\Gamma} \frac{2 \sin z}{\bar{z}^2 + 1} dz \right| \leq ML = \frac{2e^2}{3} 4\pi = \frac{8\pi e^2}{3}$$

**Solution 4**

(a) Let  $R = \{z : |z| < 2\}$ , and  $f(z) = \frac{\text{Log}(2-z)}{z^2+4}$ , then

1.  $R$  is a simply-connected region
2.  $f$  is analytic on  $R$
3.  $C$  is a closed contour in  $R$

Hence, by Cauchy's Theorem

*Handbook B2, 1.4*

$$\int_C \frac{\text{Log}(2-z)}{z^2+4} dz = 0$$

(b) [FY,LK]

Let  $R = \{z : |z| < 2\}$ , and  $f(z) = \frac{\text{Log}(2-z)}{z-2}$ , then

1.  $R$  is a simply-connected region
2.  $f$  is analytic on  $R$
3.  $C$  is a simple-closed contour in  $R$
4.  $z = 0$  is inside  $C$

Hence, by Cauchy's Integral Formula

*Handbook B2, 2.1*

$$\begin{aligned} f(0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz \\ &= \frac{1}{2\pi i} \int_C \frac{\text{Log}(2-z)/(z-2)}{z} dz \end{aligned}$$

where,

$$f(0) = \text{Log}(2)/(-2) = -\log_e 2/2$$

$$\int_C \frac{\text{Log}(2-z)}{z(z-2)} dz = -\frac{\log_e 2}{2} \times 2\pi i = -i\pi \log_e 2$$

(c) [FY,JK]

Let  $R = \{z : |z| < 2\}$ , and  $f(z) = \text{Log}(2-z)$ , then

1.  $R$  is a simply-connected region

2.  $f$  is analytic on  $R$
3.  $C$  is a simple-closed contour in  $R$
4.  $z = 0$  is inside  $C$
5.  $f$  is differentiable at  $z = 0$

Handbook B2, 3.1

Hence, by Cauchy's 2nd Derivative Formula

$$\begin{aligned} f^{(2)}(0) &= \frac{2!}{2\pi i} \int_C \frac{f(z)}{z^3} dz \\ &= \frac{1}{\pi i} \int_C \frac{\text{Log}(2-z)}{z^3} dz \end{aligned}$$

where,

$$\begin{aligned} f'(z) &= -\frac{1}{2-z} \\ f^{(2)}(z) &= -\frac{1}{(2-z)^2} \end{aligned}$$

So,

$$\int_C \frac{\text{Log}(2-z)}{z^3} dz = f^{(2)}(0)\pi i = -\frac{\pi i}{4}$$



**Solution 5**

- (a)  $f$  has three simple poles at 0,  $1/5$  and 5. We shall use the cover-up rule to obtain the residues.

*Handbook C1, 1.3*

$$\begin{aligned}\operatorname{Res}(f, 0) &= \frac{z^2 + 1}{(5z - 1)(z - 5)} \\ &= \frac{1}{(-1)(-5)} \\ &= \frac{1}{5}\end{aligned}$$

$$\begin{aligned}\operatorname{Res}(f, 1/5) &= \frac{z^2 + 1}{5z(z - 5)} \\ &= \frac{(1/5)^2 + 1}{5(1/5)(1/5 - 5)} \\ &= \frac{26/25}{-24/5} \\ &= -\frac{13}{60}\end{aligned}$$

$$\begin{aligned}\operatorname{Res}(f, 5) &= \frac{z^2 + 1}{z(5z - 1)} \\ &= \frac{25 + 1}{5(25 - 1)} \\ &= \frac{13}{60}\end{aligned}$$

- (b) We shall use the strategy for evaluating  $\int_0^{2\pi} \Phi(\cos t, \sin t) dt$ . After the replacements, we have, for  $C = \{z : |z| = 1\}$  *Handbook C1, 2.2*

$$\begin{aligned}\int_0^{2\pi} \frac{\cos t}{13 - 5 \cos t} dt &= \int_C \frac{\frac{1}{2}(z + 1/z)}{13 - \frac{5}{2}(z + 1/z)} \times \frac{1}{iz} dz \\ &= \int_C \frac{z^2 + 1}{26z - 5z^2 - 5} \times \frac{1}{iz} dz \\ &= i \int_C \frac{z^2 + 1}{z(5z^2 - 26z + 5)} dz \\ &= i \int_C \frac{z^2 + 1}{z(5z - 1)(z - 5)} dz \\ &= i \int_C f(z) dz\end{aligned}$$

Now,  $f(z)$  is analytic on  $\mathbb{C}$ , a simply-connected region, except for the three singularities. The unit circle  $C$  is a simple-closed contour in  $\mathbb{C}$ , which does not pass through  $f$ 's singularities, then by Cauchy's Residue Theorem

*Handbook C1, 2.1*

$$\begin{aligned}\int_C f(z) dz &= 2\pi i (\text{Res}(f, 0) + \text{Res}(f, 1/5)) \\ &= 2\pi i \left( \frac{1}{5} - \frac{13}{60} \right) \\ &= -\frac{\pi i}{30}\end{aligned}$$

Hence,

$$\int_0^{2\pi} \frac{\cos t}{13 - 5 \cos t} dt = i \int_C f(z) dz = i \left( -\frac{\pi i}{30} \right) = \frac{\pi}{30}$$

**Solution 6**

(a) We apply Rouché's Theorem for both cases.

*Handbook C2, 2.4*

(i) Let  $g_1(z) = iz^5$ , then

$$|f(z) - g_1(z)| = |5z^2 - 3i| \leq 5|z|^2 + |3i| = 23 < 32 = |g_1(z)|$$

Since,  $f$  and  $g_1$  are analytic on  $\mathbb{C}$  and  $C_1$  is a simple-closed contour in  $\mathbb{C}$ ,  $f$  has the same number of zeros as  $g_1$  inside  $C_1$ , namely 5, and none on  $C_1$ .

(ii) Let  $g_2(z) = 5z^2$ , then

$$|f(z) - g_2(z)| = |iz^5 - 3i| \leq |z|^5 + |-3i| = 4 < 5 = |g_2(z)|$$

Since,  $f$  and  $g_2$  are analytic on  $\mathbb{C}$  and  $C_2$  is a simple-closed contour in  $\mathbb{C}$ ,  $f$  has the same number of zeros as  $g_2$  inside  $C_2$ , namely 2, and none on  $C_2$ .

(b) [FY]

From part (a) we know that  $f$  has  $5 - 2 = 3$  zeros in the annulus  $\{z : 1 \leq |z| < 2\}$ . Now, since for  $|z| = 1$

$$|f(z)| = |iz^5 + 5z^2 - 3i| \geq |iz^5| - 5|z|^2 - |3i| = 9 > 0$$

then  $f$  has no zeros on  $C_2$ , so it has exactly 3 zeros in the open annulus  $\{z : 1 < |z| < 2\}$ , hence it follows that  $f(z) = 0$  has 3 solutions in the annulus.

(b) [JK]

$$|f(z)| > \left| |iz^5| - |3z^5 - 3i| \right| = |1 - 4| = 3$$

## Solution 7

Handbook D2, 1.14

- (a)  $q$  is continuous on  $\mathbb{C}$ , and its conjugate  $\bar{q}(z) = z + 1 + i$  is entire, hence  $q$  is a model fluid flow.

- (b) The complex potential function,  $\Omega(z)$ , for  $q$  is a primitive of  $\bar{q}$ , so

$$\Omega(z) = \frac{z^2}{2} + (1 + i)z$$

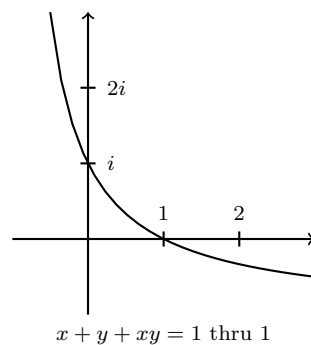
Now, for  $z = x + iy$

$$\begin{aligned}\Omega(x + iy) &= \frac{(x + iy)^2}{2} + (1 + i)(x + iy) \\ &= \frac{x^2 - y^2 + 2xyi}{2} + x + iy + ix - y \\ &= x^2/2 - y^2/2 + x - y + i(x + y + xy) \\ &= \Phi(x, y) + i\Psi(x, y)\end{aligned}$$

Handbook D2, 2.1

So,  $q$  has streamline  $\Psi(x, y) = x + y + xy = C$ , for constant  $C$ .

For the streamline through the point 1,  $\Psi(1, 0) = 1$ , so, the streamline through point 1 has the equation  $x + y + xy = 1$ , a hyperbola.



Since  $q(1) = 2 - i$ , the direction of the flow is from top-left to bottom-right.

Handbook D2, 1.10

Handbook B1, 2.1

- (c) Using the results from part (a) and part (b):

$$\begin{aligned}C_{\Gamma} &= \operatorname{Re} \int_{\Gamma} \bar{q}(z) \, dz \\&= \operatorname{Re} \int_{\Gamma} (z + 1 + i) \, dz \\&= \operatorname{Re} \int_0^4 (\gamma(t) + 1 + i) \gamma'(t) \, dt \\&= \operatorname{Re} \int_0^4 (t + 1 + i) \, dt \\&= \operatorname{Re} [t^2/2 + (1 + i)t]_0^4 \\&= \operatorname{Re}(16/2 + 4 + 4i) \\&= 12\end{aligned}$$

**Solution 8**

(a) The iteration sequence

$$z_{n+1} = 15z_n^2 + 3z_n + \frac{1}{16}$$

Handbook D3, 2.1

is conjugate to the iteration sequence

$$w_{n+1} = w_n + d$$

where

$$d = \frac{15}{16} + \frac{3}{2} - \frac{9}{4} = \frac{15 + 24 - 36}{16} = \frac{3}{16}$$

so,  $w_{n+1} = w_n + \frac{3}{16}$ . The conjugating function is

$$h(z) = 15z + \frac{1}{2} \times 3 = 15z + \frac{3}{2}$$

So,  $w_0 = h(z_0) = h(0) = 0 + \frac{3}{2} = \frac{3}{2}$

(b) [FY,LK]

$P_{\frac{3}{16}}$  has fixed points at  $z$ , where  $z^2 + \frac{3}{16} = z$ , these are the solutions to the equation

$$z^2 - z + \frac{3}{16} = 0$$

So

$$z = \frac{1 \pm \sqrt{1 - 12/16}}{2} = \frac{1 \pm \sqrt{1/4}}{2} = \frac{1}{2} \pm \frac{1}{4}$$

Hence, the fixed points of  $P_{\frac{3}{16}}$  are  $\frac{3}{4}$  and  $\frac{1}{4}$ .

Now,  $P'_{\frac{3}{16}}(z) = 2z$ , so

$$\left| P'_{\frac{3}{16}} \left( \frac{3}{4} \right) \right| = \frac{6}{4} = \frac{3}{2} > 1$$

and

$$\left| P'_{\frac{3}{16}} \left( \frac{1}{4} \right) \right| = \frac{2}{4} = \frac{1}{2} < 1$$

Handbook D3, 1.5

Hence,  $\frac{1}{4}$  is an attracting fixed point and  $\frac{3}{4}$  is a repelling one.

(b) [VC]

Alternatively, to solve the quadratic equation, we can multiply both sides by 16, so

$$\begin{aligned} z^2 - z + \frac{3}{16} &= 0 \\ \Leftrightarrow 16z^2 - 16z + 3 &= 0 \\ \Leftrightarrow (4z - 1)(4z - 3) &= 0 \end{aligned}$$

Hence, the roots are  $\frac{1}{4}$  and  $\frac{3}{4}$ .<sup>2</sup>

(c) Let  $c = -\frac{3}{2} + i$ , then it appears from the diagram that  $c$  is outside the Mandelbrot set.

*Handbook D3, 4.3*

Using the specification for  $M$

*Handbook D3, 4.5*

$$|P_c(0)| = |-3/2 + i| = \sqrt{9/4 + 1} = \sqrt{13/4} < 2$$

We go for the next iteration:

$$\begin{aligned} |P_c^{(2)}(0)| &= |(-3/2 + i)^2 - 3/2 + i| \\ &= |9/4 - 1 - 3i - 3/2 + i| \\ &= |-1/4 - 2i| \\ &= \sqrt{1/16 + 4} \\ &= \sqrt{65/4} \\ &\simeq 4.0 > 2 \end{aligned}$$

Hence,  $c$  lies outside the Mandelbrot set,  $c \notin M$ .

---

<sup>2</sup>and no pesky formula in sight!

## Solutions to Part II

### Solution 9

- (a) (i) Let  $z = x + iy$ , then

$$\begin{aligned} f(x + iy) &= (x + iy)(3 + \overline{x + iy}) + \operatorname{Re}(x + iy) \\ &= 3(x + iy) + x^2 + y^2 + x \\ &= x^2 + y^2 + 4x + i3y \\ &= u(x, y) + iv(x, y) \end{aligned}$$

where,  $u(x, y) = x^2 + y^2 + 4x$  and  $v(x, y) = 3y$ .

- (ii) The function  $f$  is defined on  $\mathbb{C}$ . For  $u$  and  $v$ , we have

$$\frac{\partial u}{\partial x} = 2x + 4 \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial y} = 3$$

The Cauchy-Riemann equations for the above partial derivatives hold when  $x = -\frac{1}{2}$  and  $y = 0$ .

Now,  $\alpha = (-\frac{1}{2}, 0)$ , as  $f$  is defined on  $\mathbb{C}$  and the partial derivatives for  $u$  and  $v$ :

1. exist on  $\mathbb{C}$
2. are continuous at  $\alpha$
3. satisfy the Cauchy-Riemann equations at  $\alpha$

then, by the Cauchy-Riemann Converse Theorem,  $f$  is differentiable at  $\alpha$ .

Since  $f$  is only differentiable at  $\alpha$ , then there is no region where  $f$  is analytic and contains  $\alpha$ , hence  $f$  is not analytic at  $(-\frac{1}{2}, 0)$ .

- (iii) From the Cauchy-Riemann Converse Theorem:

$$\begin{aligned} f' \left( -\frac{1}{2} \right) &= \frac{\partial u}{\partial x} \left( -\frac{1}{2}, 0 \right) + i \frac{\partial v}{\partial x} \left( -\frac{1}{2}, 0 \right) \\ &= 2 \left( -\frac{1}{2} \right) + 4 + i0 \\ &= 3 \end{aligned}$$

- (b) (i) Since  $g$  is analytic on  $\mathbb{C} - \{0\}$ , with  $g'(z) = 1 - \frac{i}{z^2}$ , and, since  $g'(1) = 1 - i \neq 0$ , then  $g$  is conformal at 1.

Handbook A4, 2.1

Handbook A4, 2.3

Handbook A4, 1.3

Handbook A4, 2.3

Handbook A4, 4.6



- (ii) With  $g(1) = 1 + i$ ,  $|g'(1)| = |1 - i| = \sqrt{2}$  and  $\text{Arg}(g'(1)) = -\frac{\pi}{4}$ , the effect of  $g$  on a small disc centred at 1 is to move it to  $1 + i$ , scale it by  $\sqrt{2}$  and rotate it by  $\frac{\pi}{4}$  clockwise.

Handbook A4, 1.11

- (iii) Since,  $\gamma_1(0) = e^{i0} = 1$  and  $\gamma_2(1) = (1 - 1)i + 1 = 1$ , then  $\gamma_1$  and  $\gamma_2$  meet at  $t = 0$  and  $t = 1$  respectively.

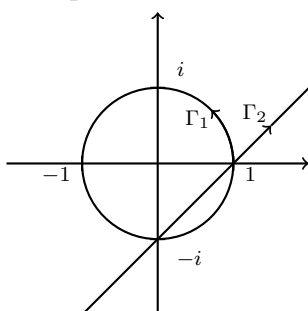
Let  $\theta$  be the angle from  $\Gamma_1$  to  $\Gamma_2$  at 1, then<sup>3</sup>

Handbook A4, 1.12

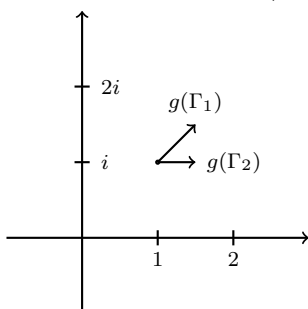
$$\theta = \text{Arg} \left( \frac{\gamma_2'(1)}{\gamma_1'(0)} \right) = \text{Arg} \left( \frac{ie^{i0}}{1+i} \right) = \text{Arg} \left( \frac{1}{2} + \frac{i}{2} \right) = \frac{\pi}{4}$$

Hence, at the point of intersection, the angle from  $\Gamma_1$  to  $\Gamma_2$  is  $\frac{\pi}{4}$ .

- (iv) The paths are shown below:



- (v) The directions of  $g(\Gamma_1)$  and  $g(\Gamma_2)$  are shown below:



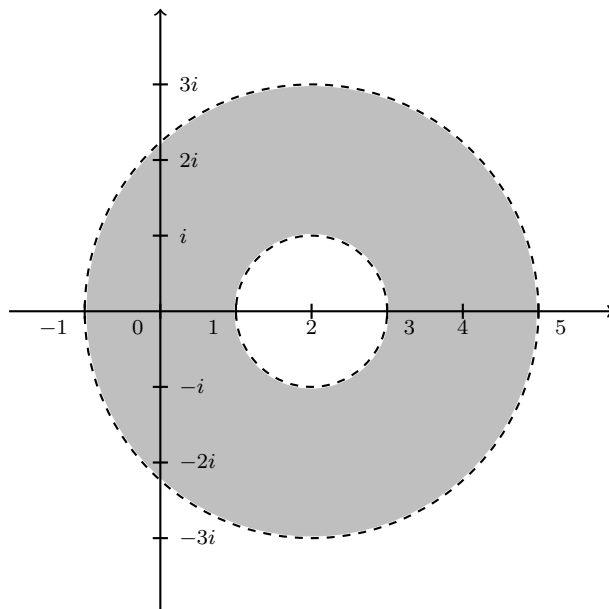
- (vi) TODO

---

<sup>3</sup>From FY's copy of the handbook!

**Solution 10**

- (a) (i) The function  $f(z)$  has two simple poles at  $z = 1$  and  $z = 5$ .  
(ii) A sketch of the annulus is shown below:



TODO

- (b) (i) TODO

- (ii) TODO

**Solution 11**

(a) [DC]

Note that

$$f(z) = \frac{\pi \cos(\pi z)}{(4z + 3i)(4z - 3i) \sin(\pi z)}$$

so by the cover-up rule (*HB28—1.3*)

$$\begin{aligned} \operatorname{Res}(f, -i\frac{3}{4}) &= \frac{\pi \cos(-\pi i\frac{3}{4})}{4(4(-i\frac{3}{4}) - 3i) \sin(-\pi i\frac{3}{4})} \\ &= \frac{\pi \cosh(-\frac{3}{4}\pi)}{-24i^2 \sinh(-\frac{3}{4}\pi)} \\ &= \underline{\underline{-\frac{\pi}{24} \coth(\frac{3}{4}\pi)}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Res}(f, i\frac{3}{4}) &= \frac{\pi \cos(\pi i\frac{3}{4})}{4(4(i\frac{3}{4}) + 3i) \sin(\pi i\frac{3}{4})} \\ &= \frac{\pi \cosh(\frac{3}{4}\pi)}{24i^2 \sinh(\frac{3}{4}\pi)} \\ &= \underline{\underline{-\frac{\pi}{24} \coth(\frac{3}{4}\pi)}} \end{aligned}$$

and by the  $g/h$  rule (*HB28—1.2*)

$$\begin{aligned} \operatorname{Res}(f, 0) &= \frac{\pi \cos(\pi(0))}{(4(0) + 3i)(4(0) - 3i) \pi \cos(\pi(0))} \\ &= \frac{1}{(3i)(-3i)} \\ &= \underline{\underline{\frac{1}{9}}} \end{aligned}$$

(b) [DC]

$\phi(n) = \frac{1}{16n^2+9}$  is an even function which is analytic on  $\mathbb{C}$  except for poles at the points  $\frac{3}{4}i$  and  $-\frac{3}{4}i$ . Now, if  $S_N$  is the square contour with vertices at  $(N + \frac{1}{2})(\pm 1 \pm i)$ , then its length is  $L = 4(2N + 1)$  and

$$|\cot \pi z| \leq 2 \quad \text{for } z \in S_N \quad (\text{HB30—4.2})$$

and since  $|z| \geq N + \frac{1}{2}$  for  $z \in S_N$ , we have

$$\begin{aligned} |16z^2 + 9| &\geq ||16z^2| - 9| \quad (HB11-5.2) \\ &= |16|z|^2 - 9| \\ &\geq |16(N + \frac{1}{2})^2 - 9| \quad \text{for } z \in S_N \\ \therefore |f(z)| &= \left| \frac{\pi \cot(\pi z)}{16z^2 + 9} \right| \leq \frac{2\pi}{16(N + \frac{1}{2})^2 - 9} = M \quad \text{for } z \in S_N \end{aligned}$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{16(N + \frac{1}{2})^2 - 9} \cdot 4(2N + 1),$$

which tends to 0 as  $N \rightarrow \infty$ , that is

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0$$

It follows by *HB30-4.1* that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} &= -\frac{1}{2} (\text{Res}(f, 0) + \text{Res}(f, -i\frac{3}{4}) + \text{Res}(f, i\frac{3}{4})) \\ &= -\frac{1}{2} \left( \frac{1}{9} - \frac{\pi}{24} \coth(\frac{3}{4}\pi) - \frac{\pi}{24} \coth(\frac{3}{4}\pi) \right) \\ &= \underline{\underline{\frac{\pi}{24} \coth(\frac{3}{4}\pi) - \frac{1}{18}}} \end{aligned}$$

(c) [DC]

Since

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 9} &= \sum_{n=-\infty}^{-1} \frac{1}{16n^2 + 9} + \frac{1}{16(0)^2 + 9} + \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \\ &= \sum_{n=\infty}^1 \frac{1}{16(-n)^2 + 9} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \\ &= \frac{1}{9} + 2 \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \quad (\text{sum of positive reals indp't of order}) \end{aligned}$$

we can simply substitute in our result from part (b) to give

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 9} &= \frac{1}{9} + \frac{2\pi}{24} \coth(\frac{3}{4}\pi) - \frac{2}{18} \\ &= \underline{\underline{\frac{\pi}{12} \coth(\frac{3}{4}\pi)}} \quad \text{QED} \end{aligned}$$

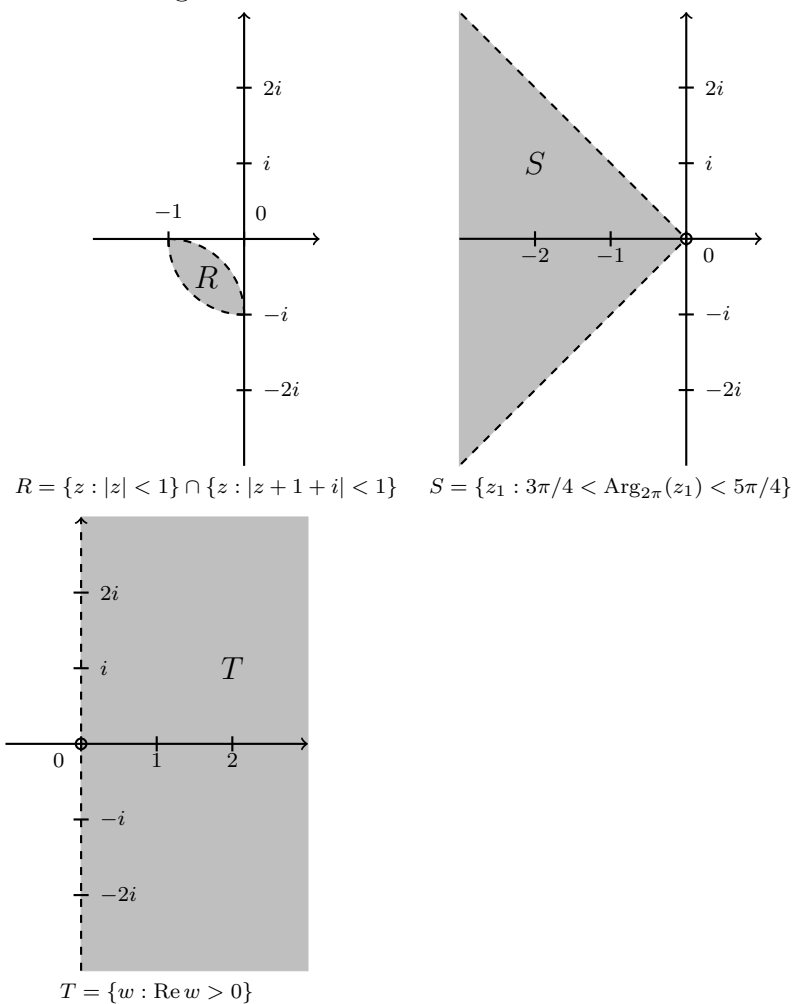
**Solution 12**

- (a) We have the mapping from  $\alpha = -1$ ,  $\beta = \infty$  and  $\gamma = -i$  to the standard triple of points,  $\alpha' = 0$ ,  $\beta' = 1$  and  $\gamma' = \infty$  respectively, hence

*Handbook D1, 2.11*

$$\begin{aligned}\hat{f}(z) &= \frac{(z - \alpha)(\beta - \gamma)}{(z - \gamma)(\beta - \alpha)} \\ &= \frac{(z - (-1))(\infty - (-i))}{(z - (-i))(\infty - (-1))} \\ &= \frac{z + 1}{z + i}\end{aligned}$$

- (b) (i) The three regions are shown below:



- (ii) The two boundaries (arcs) on  $R$  map to the two boundaries (rays) in  $S$ .

$R$  is bounded, and  $-i$  on  $R$  is mapped to  $\infty$  on  $S$ , which is unbounded.

The angle of intersection of the arcs on  $R$  at  $-1$  is  $\frac{\pi}{2}$ , this angle is preserved on the intersection of the two boundary rays of  $S$  at  $0$ .

The point  $\infty$  is outside the bounded region  $R$ , this point is mapped to  $1$ , which is also outside  $S$ .

The point  $\frac{1}{2}(-1-i)$  is in  $R$ , this point maps to  $-1$ , which is in  $S$ . Hence,  $f$  is a conformal mapping from  $R$  to  $S$ .

- (iii) We can map the region  $S$  to the region  $T$  with the square function,  $f_2(z_1) = z_1^2$ , so  $z_1 \in S$  is mapped to  $w \in T$ .

Since the Möbius Transformation,  $f_1$ , is one-one and conformal on  $R$  and the square function,  $f_2$ , is one-one and conformal on  $S$ , we can use their composite to map  $R$  to  $T$ ,  $f = f_2 \circ f_1$ , hence

$$f(z) = f_2(f_1(z)) = \left( \frac{z+1}{z+i} \right)^2$$

is a one-one conformal mapping from  $R$  to  $T$ .

- (iv) For the inverse function we have  $f^{-1} = f_1^{-1} \circ f_2^{-1}$ , where

$$z_1 = f_2^{-1}(w) = -\sqrt{w}$$

and

$$z = f_1^{-1}(z_1) = \frac{iz_1 - 1}{-z_1 + 1}$$

Hence,

$$z = f^{-1}(w) = \frac{i(-\sqrt{w}) - 1}{-(-\sqrt{w}) + 1} = \frac{-i\sqrt{w} - 1}{\sqrt{w} + 1}$$