Solutions to the M337/B 2013 Exam Paper

Fred Youhanaie (Ed.)

15 July, 2013

Copyright

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.
See the by-nc-sa[1] page for details.

Acknowledgements

This is intended to be a community developed project, in particular the M337 class of 2012/2013. To this end, I have tagged answers given by others using their initials listed below. The untagged solutions, including any potential errors, are mine - FY.

DC	Dominic Corbett
FY	Fred Youhanaie
JK	J K
LK	Liga Kauke
VC	Vikki Cookson

[1] http://creativecommons.org/licenses/by-nc-sa/3.0/

Solutions to Part I

Solution 1

(a)

$$\begin{aligned}\exp(3+\frac{1}{4}\pi i) &= e^3\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\\ &= \frac{e^3}{\sqrt{2}}+i\frac{e^3}{\sqrt{2}}\end{aligned}$$

(b) Let

$$w^3=-8=8(\cos\pi+i\sin\pi)$$

Handbook A1, 3.3

then,

$$\begin{aligned}w &= 8^{\frac{1}{3}}(\cos(\pi/3)+i\sin(\pi/3))\\ &= 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\ &= 1+\sqrt{3}\,i\end{aligned}$$

Handbook A2, 5.3

(c) Using the Principal αth power:

$$\begin{aligned}i^{1-2i} &= \exp((1-2i)\mathrm{Log}(i))\\ &= \exp((1-2i)(\log_e|i|+i\mathrm{Arg}(i)))\\ &= \exp(i\pi/2-2i^2\pi/2)\\ &= \exp(\pi+i\pi/2)\\ &= e^{\pi}e^{i\pi/2}\\ &= e^{\pi}(\cos(\pi/2)+i\sin(\pi/2))\\ &= ie^{\pi}\end{aligned}$$

Handbook A2, 4.4

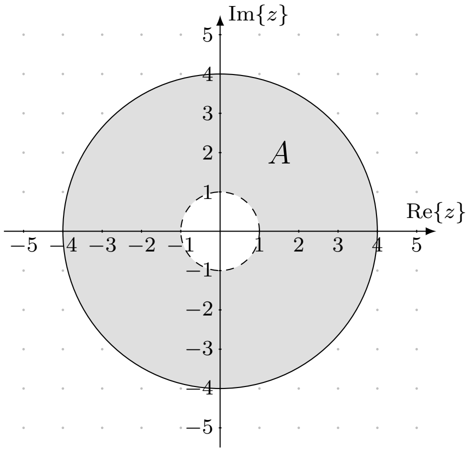
(d) Using the trigonometric functions

$$\begin{aligned}\cos(i\log_e 2) &= \frac{1}{2}\left(\exp(i^2\log_e 2)+\exp(-i^2\log_e 2)\right)\\ &= \frac{1}{2}\left(\exp(-\log_e 2)+\exp(\log_e 2)\right)\\ &= \frac{1}{2}(1/2+2)\\ &= \frac{5}{4}\end{aligned}$$

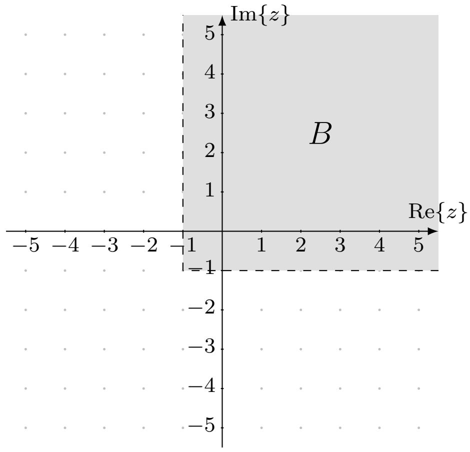
Solution 2

(a) [DC]

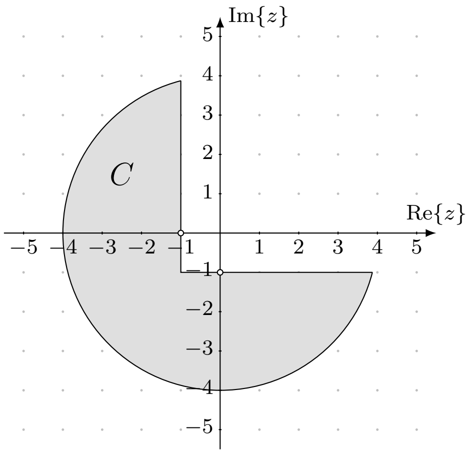
Below are the sketches of the four sets:



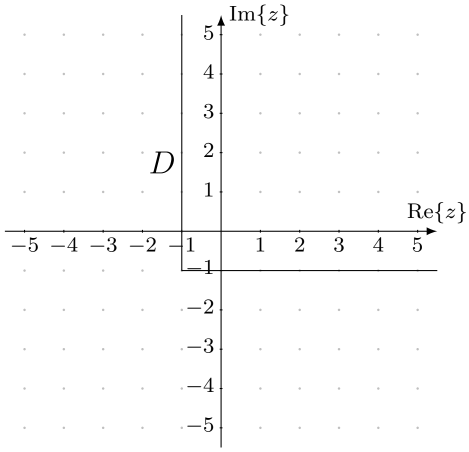
Sketch of the set $A = \{z : 1 < |z| \leq 4\}$.



Sketch of the set $B = \{z : \operatorname{Re} z > -1, \operatorname{Im} z > -1\}$.



Sketch of the set $C = A - B$.



Sketch of the set $D = \partial B$.

(b) (i) [FY,LK]

A is not a region, not open
B is a region
C is not a region, not open
D is not a region, not open

(ii) A is not compact, not closed
B is not compact, not closed
C is not compact, not closed

D is not compact, not bounded

Solution 3

(a) By definition *Handbook A2, 2.3*

$$\Gamma : \gamma(t) = 2(\cos(t) + i\sin(t)) = 2e^{it}, \quad (t \in [0, 2\pi])$$

(b) We use the polar form from above,

$$\begin{aligned} \overline{\gamma(t)} &= 2e^{-it} \\ \gamma'(t) &= 2ie^{it} \end{aligned}$$

So, *Handbook B1, 2.1*

$$\begin{aligned} \int_\Gamma \overline{z}\,dz &= \int_0^{2\pi} \left(2e^{-it}2ie^{it}\right)\,dt \\ &= \int_0^{2\pi} 4i\,dt \\ &= [4it]_0^{2\pi} \\ &= 8\pi i \end{aligned}$$

(c) [FY,LK] Let

$$f(z) = \frac{2\sin z}{\overline{z}^2 + 1}$$

then, f is continuous on Γ by combination rules, where Γ has length $L = 4\pi$

So, using the triangle inequality *Handbook A1, 5.2(a)*

$$\begin{aligned} |2\sin z| &= \left|2\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right| \\ &= \left|-i\left(e^{iz} - e^{-iz}\right)\right| \\ &\leq \left|e^{iz}\right| + \left|e^{-iz}\right| \\ &= e^{\operatorname{Re} z} + e^{-\operatorname{Re} z} \\ &< 2e^2 \end{aligned}$$

And, using the backward triangle inequality *Handbook A1, 5.2(b)*

$$\begin{aligned}\left|\overline{z}^2+1\right| &\geq \left||z|^2-|1|\right| \\ &= |4-1| \\ &= 3\end{aligned}$$

So,

$$\begin{aligned}|f(z)| &= \left|\frac{2\sin z}{\overline{z}^2+1}\right| \\ &= \frac{|2\sin z|}{|\overline{z}^2+1|} \\ &\leq \frac{2e^2}{3} \\ &= M\end{aligned}$$

Handbook B1, 4.3

then by the Estimation Theorem.

$$\left|\int_\Gamma \frac{2\sin z}{\overline{z}^2+1}\,dz\right| \leq ML = \frac{2e^2}{3}4\pi = \frac{8\pi e^2}{3}$$

Solution 4

(a) Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\text{Log}(2-z)}{z^2+4}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a closed contour in R

Hence, by Cauchy's Theorem *Handbook B2, 1.4*

$$\int_C \frac{\text{Log}(2-z)}{z^2+4}\,dz = 0$$

(b) [FY,LK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\text{Log}(2-z)}{z-2}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C

Hence, by Cauchy's Integral Formula *Handbook B2, 2.1*

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{\text{Log}(2-z)/(z-2)}{z}\,dz \end{aligned}$$

where,

$$f(0) = \text{Log}(2)/(-2) = -\log_e 2/2$$

$$\int_C \frac{\text{Log}(2-z)}{z(z-2)}\,dz = -\frac{\log_e 2}{2} \times 2\pi i = -i\pi \log_e 2$$

(c) [FY,JK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \text{Log}(2-z)$, then

1. R is a simply-connected region

2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C
5. f is differentiable at $z = 0$

Handbook B2, 3.1

Hence, by Cauchy's 2nd Derivative Formula

$$\begin{aligned} f^{(2)}(0) &= \frac{2!}{2\pi i}\int_C \frac{f(z)}{z^3}\,dz \\ &= \frac{1}{\pi i}\int_C \frac{\mathrm{Log}(2-z)}{z^3}\,dz \end{aligned}$$

where,

$$\begin{aligned} f'(z) &= -\frac{1}{2-z} \\ f^{(2)}(z) &= -\frac{1}{(2-z)^2} \end{aligned}$$

So,

$$\int_C \frac{\mathrm{Log}(2-z)}{z^3}\,dz = f^{(2)}(0)\pi i = -\frac{\pi i}{4}$$

Solution 5

(a) f has three simple poles at 0, 1/5 and 5. We shall use the cover-up rule to obtain the residues. *Handbook C1, 1.3*

$$\begin{aligned}
\operatorname{Res}(f,0) &= \frac{z^2+1}{(5z-1)(z-5)} \\
&= \frac{1}{(-1)(-5)} \\
&= \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f,1/5) &= \frac{z^2+1}{5z(z-5)} \\
&= \frac{(1/5)^2+1}{5(1/5)(1/5-5)} \\
&= \frac{26/25}{-24/5} \\
&= -\frac{13}{60}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f,5) &= \frac{z^2+1}{z(5z-1)} \\
&= \frac{25+1}{5(25-1)} \\
&= \frac{13}{60}
\end{aligned}$$

(b) We shall use the strategy for evaluating $\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt$. After the replacements, we have, for $C = \{z : |z| = 1\}$ *Handbook C1, 2.2*

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos t}{13 - 5\cos t}\, dt &= \int_C \frac{\frac{1}{2}(z+1/z)}{13 - \frac{5}{2}(z+1/z)} \times \frac{1}{iz}\, dz \\
&= \int_C \frac{z^2+1}{26z - 5z^2 - 5} \times \frac{1}{iz}\, dz \\
&= i \int_C \frac{z^2+1}{z(5z^2 - 26z + 5)}\, dz \\
&= i \int_C \frac{z^2+1}{z(5z-1)(z-5)}\, dz \\
&= i \int_C f(z)\, dz
\end{aligned}$$

Now, $f(z)$ is analytic on $\mathbb{C}$, a simply-connected region, except for the three singularities. The unit circle C is a simple-closed contour in $\mathbb{C}$, which does not pass through f's singularities, then by Cauchy's Residue Theorem

Handbook C1, 2.1

$$\begin{aligned}\int_C f(z)\,dz &= 2\pi i\left(\mathrm{Res}(f,0)+\mathrm{Res}(f,1/5)\right)\\ &= 2\pi i\left(\frac{1}{5}-\frac{13}{60}\right)\\ &= -\frac{\pi i}{30}\end{aligned}$$

Hence,

$$\int_0^{2\pi}\frac{\cos t}{13-5\cos t}\,dt = i\int_C f(z)\,dz = i\left(-\frac{\pi i}{30}\right)=\frac{\pi}{30}$$

Solution 6

(a) We apply Rouché's Theorem for both cases. *Handbook C2, 2.4*

(i) Let $g_1(z) = iz^5$, then

$$|f(z) - g_1(z)| = |5z^2 - 3i| \leq 5|z|^2 + |3i| = 23 < 32 = |g_1(z)|$$

Since, f and g_1 are analytic on $\mathbb{C}$ and C_1 is a simple-closed contour in $\mathbb{C}$, f has the same number of zeros as g_1 inside C_1, namely 5, and none on C_1.

(ii) Let $g_2(z) = 5z^2$, then

$$|f(z) - g_2(z)| = |iz^5 - 3i| \leq |z|^5 + |-3i| = 4 < 5 = |g_2(z)|$$

Since, f and g_2 are analytic on $\mathbb{C}$ and C_2 is a simple-closed contour in $\mathbb{C}$, f has the same number of zeros as g_2 inside C_2, namely 2, and none on C_2.

(b) [FY]

From part (a) we know that f has $5 - 2 = 3$ zeros in the annulus $\{z : 1 \leq |z| < 2\}$. Now, since for $|z| = 1$

$$|f(z)| = |iz^5 + 5z^2 - 3i| \geq |iz^5| - 5|z|^2 - |3i| = 9 > 0$$

then f has no zeros on C_2, so it has exactly 3 zeros in the open annulus $\{z : 1 < |z| < 2\}$, hence it follows that $f(z) = 0$ has 3 solutions in the annulus.

(b) [JK]

$$|f(z)| > \left| \left|iz^5\right| - \left|3z^5 - 3i\right| \right| = |1 - 4| = 3$$

Solution 7

Handbook D2, 1.14

(a) q is continuous on $\mathbb{C}$, and its conjugate $\overline{q}(z) = z + 1 + i$ is entire, hence q is a model fluid flow.

(b) The complex potential function, $\Omega(z)$, for q is a primitive of $\overline{q}$, so

$$\Omega(z) = \frac{z^2}{2} + (1+i)z$$

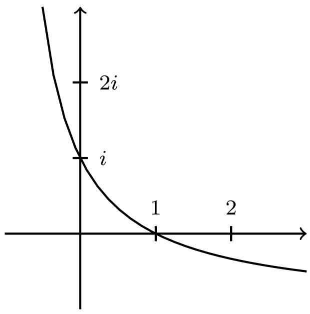
Now, for $z = x + iy$

$$\begin{aligned} \Omega(x+iy) &= \frac{(x+iy)^2}{2} + (1+i)(x+iy) \\ &= \frac{x^2 - y^2 + 2xyi}{2} + x + iy + ix - y \\ &= x^2/2 - y^2/2 + x - y + i(x + y + xy) \\ &= \Phi(x,y) + i\Psi(x,y) \end{aligned}$$

Handbook D2, 2.1

So, q has streamline $\Psi(x,y) = x + y + xy = C$, for constant C.

For the streamline through the point 1, $\Psi(1,0) = 1$, so, the streamline through point 1 has the equation $x + y + xy = 1$, a hyperbola.



$x + y + xy = 1$ thru 1

Since $q(1) = 2 - i$, the direction of the flow is from top-left to bottom-right.

Handbook D2, 1.10

Handbook B1, 2.1

(c) Using the results from part (a) and part (b):

$$
\begin{aligned}
C_\Gamma &= \operatorname{Re}\int_\Gamma \overline{q}(z)\,dz \\
&= \operatorname{Re}\int_\Gamma (z+1+i)\,dz \\
&= \operatorname{Re}\int_0^4 (\gamma(t)+1+i)\gamma'(t)\,dt \\
&= \operatorname{Re}\int_0^4 (t+1+i)\,dt \\
&= \operatorname{Re}\left[t^2/2+(1+i)t\right]_0^4 \\
&= \operatorname{Re}(16/2+4+4i) \\
&= 12
\end{aligned}
$$

Solution 8

(a) The iteration sequence

$$z_{n+1} = 15z_n^2 + 3z_n + \frac{1}{16}$$

Handbook D3, 2.1

is conjugate to the iteration sequence

$$w_{n+1} = w_n + d$$

where

$$d = \frac{15}{16} + \frac{3}{2} - \frac{9}{4} = \frac{15 + 24 - 36}{16} = \frac{3}{16}$$

so, $w_{n+1} = w_n + \frac{3}{16}$. The conjugating function is

$$h(z) = 15z + \frac{1}{2} \times 3 = 15z + \frac{3}{2}$$

So, $w_0 = h(z_0) = h(0) = 0 + \frac{3}{2} = \frac{3}{2}$

(b) [FY,LK]

$P_{\frac{3}{16}}$ has fixed points at z, where $z^2 + \frac{3}{16} = z$, these are the solutions to the equation

$$z^2 - z + \frac{3}{16} = 0$$

So

$$z = \frac{1 \pm \sqrt{1 - 12/16}}{2} = \frac{1 \pm \sqrt{1/4}}{2} = \frac{1}{2} \pm \frac{1}{4}$$

Hence, the fixed points of $P_{\frac{3}{16}}$ are $\frac{3}{4}$ and $\frac{1}{4}$.

Now, $P'_{\frac{3}{16}}(z) = 2z$, so

$$\left| P'_{\frac{3}{16}}\left(\frac{3}{4}\right) \right| = \frac{6}{4} = \frac{3}{2} > 1$$

and

$$\left| P'_{\frac{3}{16}}\left(\frac{1}{4}\right) \right| = \frac{2}{4} = \frac{1}{2} < 1$$

Handbook D3, 1.5

Hence, $\frac{1}{4}$ is an attracting fixed point and $\frac{3}{4}$ is a repelling one.

(b) [VC]

Alternatively, to solve the quadratic equation, we can multiply both sides by 16, so

$$\begin{aligned} & z^2 - z + \frac{3}{16} = 0 \\ \Leftrightarrow\quad & 16z^2 - 16z + 3 = 0 \\ \Leftrightarrow\quad & (4z-1)(4z-3) = 0 \end{aligned}$$

Hence, the roots are $\frac{1}{4}$ and $\frac{3}{4}$.[2]

(c) Let $c = -\frac{3}{2} + i$, then it appears from the diagram that c is outside the Mandelbrot set. *Handbook D3, 4.3*

Using the specification for M *Handbook D3, 4.5*

$$|P_c(0)| = |-3/2 + i| = \sqrt{9/4 + 1} = \sqrt{13/4} < 2$$

We go for the next iteration:

$$\begin{aligned} |P_c^{(2)}(0)| &= |(-3/2 + i)^2 - 3/2 + i| \\ &= |9/4 - 1 - 3i - 3/2 + i| \\ &= |-1/4 - 2i| \\ &= \sqrt{1/16 + 4} \\ &= \sqrt{65/4} \\ &\simeq 4.0 > 2 \end{aligned}$$

Hence, c lies outside the Mandelbrot set, $c \notin M$.

[2] and no pesky formula in sight!

Solutions to Part II

Solution 9

(a) (i) Let $z = x + iy$, then

$$\begin{aligned} f(x+iy) &= (x+iy)(3+\overline{x+iy}) + \mathrm{Re}(x+iy) \\ &= 3(x+iy) + x^2 + y^2 + x \\ &= x^2 + y^2 + 4x + i3y \\ &= u(x,y) + iv(x,y) \end{aligned}$$

where, $u(x,y) = x^2 + y^2 + 4x$ and $v(x,y) = 3y$.

(ii) The function f is defined on $\mathbb{C}$. For u and v, we have

$$\tfrac{\partial u}{\partial x} = 2x + 4 \quad \tfrac{\partial v}{\partial x} = 0$$

$$\tfrac{\partial u}{\partial y} = 2y \qquad \tfrac{\partial v}{\partial y} = 3$$

Handbook A4, 2.1

The Cauchy-Riemann equations for the above partial derivatives hold when $x = -\frac{1}{2}$ and $y = 0$.

Now, $\alpha = \left(-\frac{1}{2}, 0\right)$, as f is defined on $\mathbb{C}$ and the partial derivatives for u and v:

1. exist on $\mathbb{C}$
2. are continuous at α
3. satisfy the Cauchy-Riemann equations at α

Handbook A4, 2.3

then, by the Cauchy-Riemann Converse Theorem, f is differentiable at α.

Handbook A4, 1.3

Since f is only differentiable at α, then there is no region where f is analytic and contains α, hence f is not analytic at $\left(-\frac{1}{2}, 0\right)$.

Handbook A4, 2.3

(iii) From the Cauchy-Riemann Converse Theorem:

$$\begin{aligned} f'\left(-\frac{1}{2}\right) &= \frac{\partial u}{\partial x}\left(-\frac{1}{2}, 0\right) + i\frac{\partial v}{\partial x}\left(-\frac{1}{2}, 0\right) \\ &= 2\left(-\frac{1}{2}\right) + 4 + i0 \\ &= 3 \end{aligned}$$

Handbook A4, 4.6

(b) (i) Since g is analytic on $\mathbb{C} - \{0\}$, with $g'(z) = 1 - \frac{i}{z^2}$, and, since $g'(1) = 1 - i \neq 0$, then g is conformal at 1.

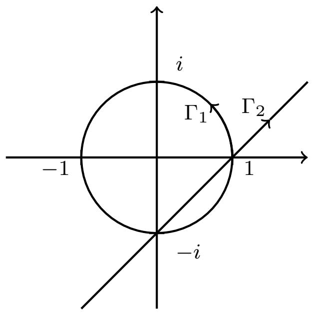
(ii) With $g(1) = 1 + i$, $|g'(1)| = |1 - i| = \sqrt{2}$ and $\operatorname{Arg}(g'(1)) = -\frac{\pi}{4}$, the effect of g on a small disc centred at 1 is to move it to $1 + i$, scale it by $\sqrt{2}$ and rotate it by $\frac{\pi}{4}$ clockwise. *Handbook A4, 1.11*

(iii) Since, $\gamma_1(0) = e^{i0} = 1$ and $\gamma_2(1) = (1 - 1)i + 1 = 1$, then γ_1 and γ_2 meet at $t = 0$ and $t = 1$ respectively.
Let θ be the angle from Γ_1 to Γ_2 at 1, then[3] *Handbook A4, 1.12*

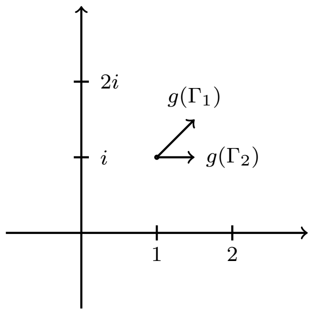
$$\theta = \operatorname{Arg}\left(\frac{\gamma_2'(1)}{\gamma_1'(0)}\right) = \operatorname{Arg}\left(\frac{ie^{i0}}{1+i}\right) = \operatorname{Arg}\left(\frac{1}{2} + \frac{i}{2}\right) = \frac{\pi}{4}$$

Hence, at the point of intersection, the angle from Γ_1 to Γ_2 is $\frac{\pi}{4}$.

(iv) The paths are shown below:



(v) The directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ are shown below:



(vi) [DC,FY]
The image of the unit circle, $\Gamma(t) = e^{it}$ for $t \in [0, 2\pi]$, under g is as follows: *Handbook A2, 2.5*

$$\begin{aligned}
g(\Gamma(t)) &= g(e^{it}) \\
&= e^{it} + ie^{-it} \\
&= \cos(t) + i\sin(t) + i(\cos(-t) + i\sin(-t)) \\
&= \cos(t) + i\sin(t) + i\cos(t) - i^2\sin(t) \\
&= (\cos(t) + \sin(t))(1 + i)
\end{aligned}$$

[3] From FY's copy of the handbook!

Solution 10

(a) [DC]

(i) f has exactly two singularities: a simple pole at $z = 1$ and another simple pole at $z = 5$.

(ii) Let $z - 2 = h$, so that $z = 2 + h$. Then, for $z \neq 1, 5$,

$$\begin{aligned}
f(z) &= \frac{1}{(1+h)(-3+h)} \\
&= -\frac{1}{4(1+h)} + \frac{1}{4(-3+h)} \\
&= -\frac{1}{4h}\frac{1}{(1+1/h)} - \frac{1}{12}\frac{1}{(1-h/3)} \\
&= -\frac{1}{4h}\left(1 + \left(-\frac{1}{h}\right) + \left(-\frac{1}{h}\right)^2 + \cdots\right) \\
&\quad - \frac{1}{12}\left(1 + \frac{h}{3} + \left(\frac{h}{3}\right)^2 + \cdots\right) \\
&\qquad\qquad\qquad\qquad \text{for } |-1| < |h| < |3| \\
&= -\frac{1}{4h} + \frac{1}{4h^2} - \frac{1}{12} - \frac{h}{36} - \frac{h^2}{108} + \cdots \\
&= \cdots + \tfrac{1}{4}(z-2)^{-2} - \tfrac{1}{4}(z-2)^{-1} - \tfrac{1}{12} \\
&\quad - \tfrac{1}{36}(z-2) - \tfrac{1}{108}(z-2)^2 + \cdots \\
&\qquad\qquad\qquad\qquad \text{for } 1 < |z-2| < 3
\end{aligned}$$

(b) [DC]

(i) Let

$$g = g_1 \circ (g_2 \cdot g_3)$$

Now, $g_3(z) = \sin z$ is represented by the basic Taylor series

$$z - \frac{z^3}{3!} + \cdots$$

on $\mathbb{C}$ (*HB25—3.5*). $g_2(z) = z$ is its own Taylor series and represents g_2 on $\mathbb{C}$. It follows by the Product Rule (*HB26—4.2*) that $g_2 \cdot g_3$ is represented by the Taylor series

$$z^2 - \frac{z^4}{3!} + \cdots$$

also on $\mathbb{C}$. Since $g_1(w) = \exp(w)$ is represented by the basic Taylor series

$$1 + w + \frac{w^2}{2!} + \cdots$$

on $\mathbb{C}$ (*HB25—3.5*), it follows from the above and the Composition Rule (*HB25—4.3*) that $g = g_1 \circ (g_2 \cdot g_3)$ is represented by the Taylor series

$$\begin{aligned} g(z) &= 1 + \left(z^2 - \frac{z^4}{3!} + \cdots\right) + \frac{1}{2!}\left(z^2 + \cdots\right)^2 \\ &= \underline{\underline{1 + z^2 + \frac{z^4}{3} + \cdots}} \qquad \text{for } |z| < r, \end{aligned}$$

where $r > 0$.

In addition, by the Chain Rule applied to standard derivatives (*HB18—1.6* & *HB19—3.1,4*) g is entire, so it follows by *HB25—3.3* that the Taylor series for g about any point converges to $f(z)$ for all $z \in \mathbb{C}$, and in particular that $1 + z^2 + \frac{z^4}{3} + \cdots$, the Taylor series about 0, $\underline{\underline{\text{represents } g \text{ on } \mathbb{C}}}$.

(ii) By composing the above Taylor series with $z \mapsto z^{-1}$ (which is its own Laurent series) and taking the product with $z \mapsto z^3$ (which is its own Taylor series), we have

$$\begin{aligned} z^3 g(1/z) &= z^3 \left(1 + z^{-2} + \frac{z^{-4}}{3} + \cdots\right) \\ &= z^3 + z + \frac{1}{3z} + \cdots \end{aligned}$$

which is analytic on the punctured disc $\mathbb{C} - \{0\}$ with centre 0. Since $C \in \mathbb{C}$ has centre 0, it follows by *HB28—4.2* that

$$\operatorname{Res}\left(z^3 g(1/z), 0\right) = 1/3$$

and that

$$\begin{aligned}\int_C z^3 g(1/z)\,\mathrm{d}z &= 2\pi i \times \operatorname{Res}\left(z^3 g(1/z), 0\right)\\ &= \underline{\underline{2\pi i/3}}\end{aligned}$$

Solution 11

(a) [DC]

Note that

$$f(z) = \frac{\pi\cos(\pi z)}{(4z+3i)\,(4z-3i)\sin(\pi z)}$$

so by the cover-up rule (*HB28—1.3*)

$$\begin{aligned}\operatorname{Res}(f,-i\tfrac{3}{4}) &= \frac{\pi\cos(-\pi i\frac{3}{4})}{4\left(4(-i\frac{3}{4})-3i\right)\sin(-\pi i\frac{3}{4})}\\ &= \frac{\pi\cosh(-\frac{3}{4}\pi)}{-24i^2\sinh(-\frac{3}{4}\pi)}\\ &= \underline{\underline{-\tfrac{\pi}{24}\coth(\tfrac{3}{4}\pi)}}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}(f,i\tfrac{3}{4}) &= \frac{\pi\cos(\pi i\frac{3}{4})}{4\left(4(i\frac{3}{4})+3i\right)\sin(\pi i\frac{3}{4})}\\ &= \frac{\pi\cosh(\frac{3}{4}\pi)}{24i^2\sinh(\frac{3}{4}\pi)}\\ &= \underline{\underline{-\tfrac{\pi}{24}\coth(\tfrac{3}{4}\pi)}}\end{aligned}$$

and by the g/h rule (*HB28—1.2*)

$$\begin{aligned}\operatorname{Res}(f,0) &= \frac{\pi\cos(\pi(0))}{(4(0)+3i)\,(4(0)-3i)\,\pi\cos(\pi(0))}\\ &= \frac{1}{(3i)(-3i)}\\ &= \underline{\underline{\tfrac{1}{9}}}\end{aligned}$$

(b) [DC]

$\phi(n) = \frac{1}{16n^2+9}$ is an even function which is analytic on $\mathbb{C}$ except for poles at the points $\frac{3}{4}i$ and $-\frac{3}{4}i$. Now, if S_N is the square contour with vertices at $(N+\frac{1}{2})(\pm 1 \pm i)$, then its length is $L = 4(2N+1)$ and

$$|\cot \pi z| \leq 2 \quad \text{for } z \in S_N \ (\textit{HB30—4.2})$$

and since $|z| \geq N + \frac{1}{2}$ for $z \in S_N$, we have

$$\begin{aligned}\left|16z^2+9\right| &\geq \left|\left|16z^2\right| - |9|\right| \quad (\textit{HB11—5.2})\\ &= \left|16\,|z|^2 - 9\right|\\ &\geq \left|16(N+\tfrac{1}{2})^2 - 9\right| \quad \text{for } z \in S_N\end{aligned}$$

$$\therefore |f(z)| = \left|\frac{\pi\cot(\pi z)}{16z^2+9}\right| \leq \frac{2\pi}{16(N+\frac{1}{2})^2-9} = M \quad \text{for } z \in S_N$$

Hence, by the Estimation Theorem,

$$\left|\int_{S_N} f(z)\,\mathrm{d}z\right| \leq \frac{2\pi}{16(N+\frac{1}{2})^2-9} \cdot 4(2N+1),$$

which tends to 0 as $N \to \infty$, that is

$$\lim_{N\to\infty} \int_{S_N} f(z)\,\mathrm{d}z = 0$$

It follows by *HB30—4.1* that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{16n^2+9} &= -\tfrac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}(f,-i\tfrac{3}{4}) + \operatorname{Res}(f,i\tfrac{3}{4})\right)\\ &= -\tfrac{1}{2}\left(\tfrac{1}{9} - \tfrac{\pi}{24}\coth(\tfrac{3}{4}\pi) - \tfrac{\pi}{24}\coth(\tfrac{3}{4}\pi)\right)\\ &= \underline{\underline{\tfrac{\pi}{24}\coth(\tfrac{3}{4}\pi) - \tfrac{1}{18}}}\end{aligned}$$

(c) [DC]

Since

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{16n^2+9} &= \sum_{n=-\infty}^{-1} \frac{1}{16n^2+9} + \frac{1}{16(0)^2+9} + \sum_{n=1}^{\infty} \frac{1}{16n^2+9}\\ &= \sum_{n=\infty}^{1} \frac{1}{16(-n)^2+9} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{1}{16n^2+9}\\ &= \frac{1}{9} + 2\sum_{n=1}^{\infty} \frac{1}{16n^2+9} \quad \text{(sum of positive reals indp't of order)}\end{aligned}$$

we can simply substitute in our result from part (b) to give

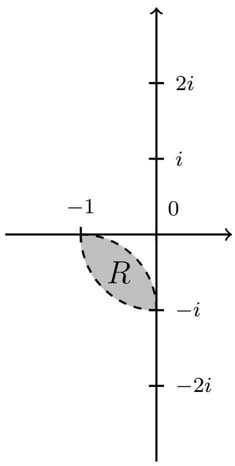
$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{16n^2+9} &= \frac{1}{9} + \tfrac{2\pi}{24}\coth(\tfrac{3}{4}\pi) - \tfrac{2}{18}\\ &= \underline{\underline{\tfrac{\pi}{12}\coth(\tfrac{3}{4}\pi)}} \quad \text{QED}\end{aligned}$$

Solution 12

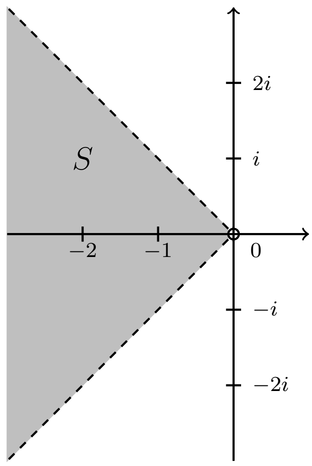
(a) We have the mapping from $\alpha = -1$, $\beta = \infty$ and $\gamma = -i$ to the standard triple of points, $\alpha' = 0$, $\beta' = 1$ and $\gamma' = \infty$ respectively, hence *Handbook D1, 2.11*

$$\begin{aligned}\hat{f}(z) &= \frac{(z-\alpha)(\beta-\gamma)}{(z-\gamma)(\beta-\alpha)} \\ &= \frac{(z-(-1))(\infty-(-i))}{(z-(-i))(\infty-(-1))} \\ &= \frac{z+1}{z+i}\end{aligned}$$

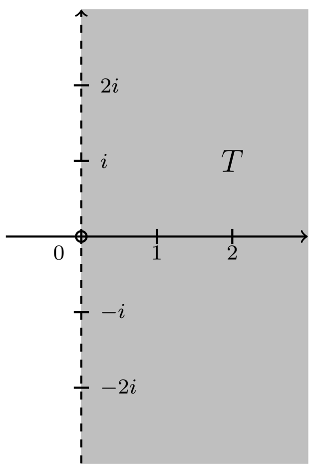
(b) (i) The three regions are shown below:



$R = \{z : |z| < 1\} \cap \{z : |z+1+i| < 1\}$



$S = \{z_1 : 3\pi/4 < \mathrm{Arg}_{2\pi}(z_1) < 5\pi/4\}$



$T = \{w : \mathrm{Re}\, w > 0\}$

(ii) The two boundaries (arcs) on R map to the two boundaries (rays) in S.

R is bounded, and $-i$ on R is mapped to ∞ on S, which is unbounded.

The angle of intersection of the arcs on R at -1 is $\frac{\pi}{2}$, this angle is preserved on the intersection of the two boundary rays of S at 0.

The point ∞ is outside the bounded region R, this point is mapped to 1, which is also outside S

The point $\frac{1}{2}(-1-i)$ is in R, this point maps to -1, which is in S.

Hence, f is a conformal mapping from R to S.

(iii) We can map the region S to the region T with the square function, $f_2(z_1) = z_1^2$, so $z_1 \in S$ is mapped to $w \in T$.

Since the Möbius Transformation, f_1, is one-one and conformal on R and the square function, f_2, is one-one and conformal on S, we can use their composite to map R to T, $f = f_2 \circ f_1$, hence

$$f(z) = f_2(f_1(z)) = \left(\frac{z+1}{z+i}\right)^2$$

is a one-one conformal mapping from R to T.

(iv) For the inverse function we have $f^{-1} = f_1^{-1} \circ f_2^{-1}$, where

$$z_1 = f_2^{-1}(w) = -\sqrt{w}$$

Handbook D1, 2.6

and

$$z = f_1^{-1}(z_1) = \frac{iz_1 - 1}{-z_1 + 1}$$

Hence,

$$z = f^{-1}(w) = \frac{i(-\sqrt{w}) - 1}{-(-\sqrt{w}) + 1} = \frac{-i\sqrt{w} - 1}{\sqrt{w} + 1}$$