Solutions to the M337/B 2013 Exam Paper

Fred Youhanaie (Ed.)

14 June, 2013

Copyright

Acknowledgements

This is intended to be a community developed project, in particular the M337 class of 2012/2013. To this end, I have tagged answers given by others using their initials listed below. The untagged solutions, including any potential errors, are mine - FY.

FY	Fred Youhanaie
JK	J K
LK	Liga Kauke
VC	Vikki Cookson

[1]http://creativecommons.org/licenses/by-nc-sa/3.0/

Solutions to Part I

Solution 1

(a)

$$\begin{aligned}\exp(3+\frac{1}{4}\pi i) &= e^3\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\\ &= \frac{e^3}{\sqrt{2}}+i\frac{e^3}{\sqrt{2}}\end{aligned}$$

(b) Let

$$w^3=-8=8(\cos\pi+i\sin\pi)$$

Handbook A1, 3.3

then,

$$\begin{aligned}w &= 8^{\frac{1}{3}}(\cos(\pi/3)+i\sin(\pi/3))\\ &= 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\\ &= 1+\sqrt{3}\,i\end{aligned}$$

Handbook A2, 5.3

(c) Using the Principal αth power:

$$\begin{aligned}i^{1-2i} &= \exp((1-2i)\mathrm{Log}(i))\\ &= \exp((1-2i)(\log_e|i|+i\mathrm{Arg}(i)))\\ &= \exp(i\pi/2-2i^2\pi/2)\\ &= \exp(\pi+i\pi/2)\\ &= e^{\pi}e^{i\pi/2}\\ &= e^{\pi}(\cos(\pi/2)+i\sin(\pi/2))\\ &= ie^{\pi}\end{aligned}$$

Handbook A2, 4.4

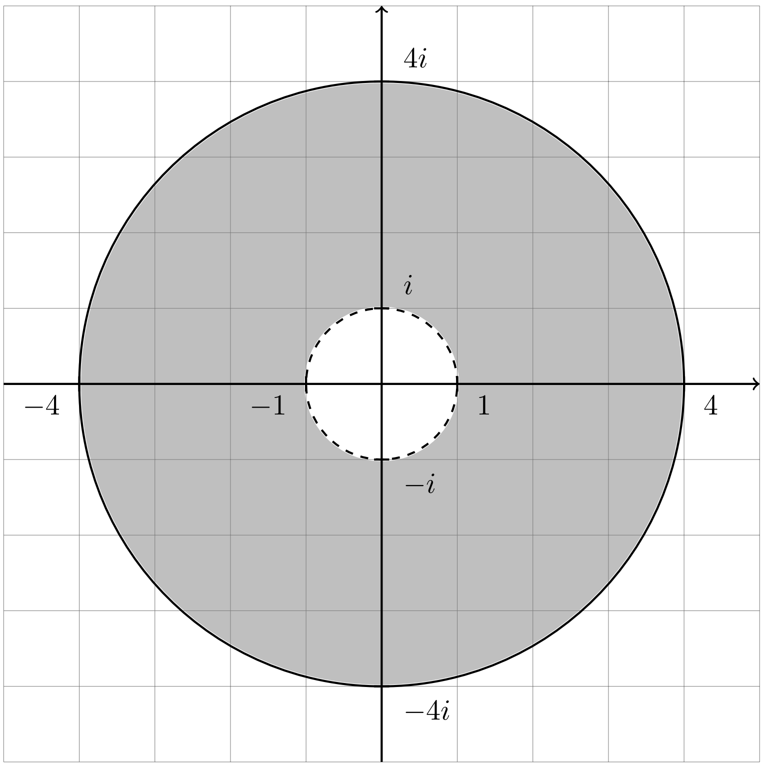
(d) Using the trigonometric functions

$$\begin{aligned}\cos(i\log_e 2) &= \frac{1}{2}\left(\exp(i^2\log_e 2)+\exp(-i^2\log_e 2)\right)\\ &= \frac{1}{2}\left(\exp(-\log_e 2)+\exp(\log_e 2)\right)\\ &= \frac{1}{2}\left(1/2+2\right)\\ &= \frac{5}{4}\end{aligned}$$

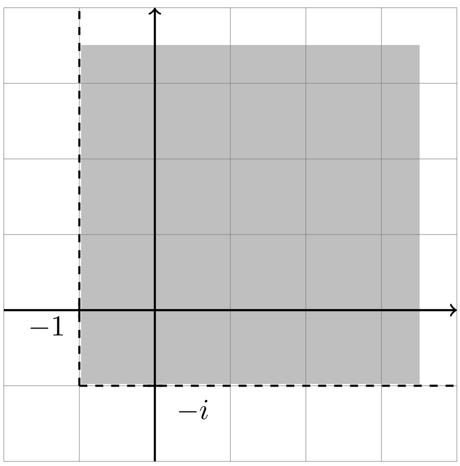
Solution 2

(a) [FY,LK]

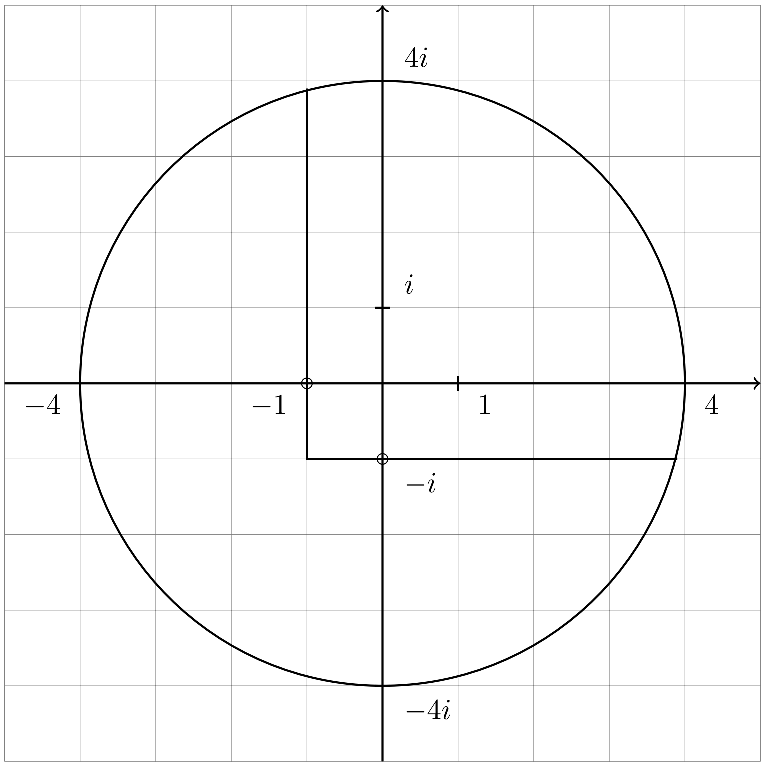
Below are the sketches of the four sets:



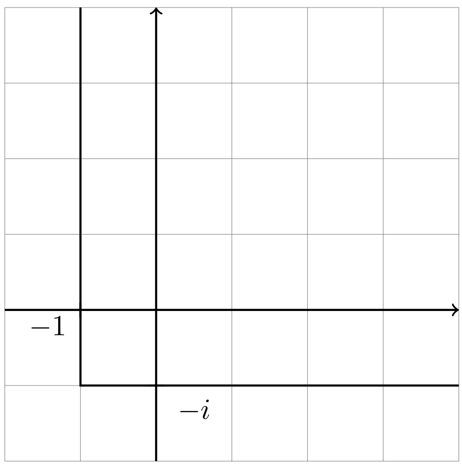
The set $A = \{z : 1 < |z| \leq 4\}$



The set $B = \{z : \mathrm{Re} z > -1, \mathrm{Im} z > -1\}$



The set $C = A - B$



The set $D = \partial B$

(b) (i) [FY,LK]

A is not a region, not open

B is a region

C is not a region, not open

D is not a region, not open

(ii) A is not compact, not closed
B is not compact, not closed
C is not compact, not closed
D is not compact, not bounded

Solution 3

Handbook A2, 2.3

(a) By definition

$$\Gamma : \gamma(t) = 2(\cos(t) + i\sin(t)) = 2e^{it}, \quad (t \in [0, 2\pi])$$

(b) We use the polar form from above,

$$\begin{aligned}\overline{\gamma(t)} &= 2e^{-it} \\ \gamma'(t) &= 2ie^{it}\end{aligned}$$

Handbook B1, 2.1

So,

$$\begin{aligned}\int_\Gamma \overline{z}\,dz &= \int_0^{2\pi} \left(2e^{-it}2ie^{it}\right)\,dt \\ &= \int_0^{2\pi} 4i\,dt \\ &= [4it]_0^{2\pi} \\ &= 8\pi i\end{aligned}$$

(c) [FY,LK] Let

$$f(z) = \frac{2\sin z}{\overline{z}^2 + 1}$$

then, f is continuous on Γ by combination rules, where Γ has length $L = 4\pi$

Handbook A1, 5.2(a)

So, using the triangle inequality

$$\begin{aligned}|2\sin z| &= \left|2\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)\right| \\ &= \left|-i\left(e^{iz} - e^{-iz}\right)\right| \\ &\leq \left|e^{iz}\right| + \left|e^{-iz}\right| \\ &= e^{\operatorname{Re} z} + e^{-\operatorname{Re} z} \\ &< 2e^2\end{aligned}$$

Handbook A1, 5.2(b)

And, using the backward triangle inequality

$$\begin{aligned}\left|\overline{z}^2+1\right| &\geq \left||z|^2-|1|\right| \\ &= |4-1| \\ &= 3\end{aligned}$$

So,

$$\begin{aligned}|f(z)| &= \left|\frac{2\sin z}{\overline{z}^2+1}\right| \\ &= \frac{|2\sin z|}{|\overline{z}^2+1|} \\ &\leq \frac{2e^2}{3} \\ &= M\end{aligned}$$

then by the Estimation Theorem. *Handbook B1, 4.3*

$$\left|\int_\Gamma \frac{2\sin z}{\overline{z}^2+1}\,dz\right| \leq ML = \frac{2e^2}{3}4\pi = \frac{8\pi e^2}{3}$$

Solution 4

(a) Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\mathrm{Log}(2-z)}{z^2+4}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a closed contour in R

Handbook B2, 1.4

Hence, by Cauchy's Theorem

$$\int_C \frac{\mathrm{Log}(2-z)}{z^2+4}\,dz = 0$$

(b) [FY,LK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\mathrm{Log}(2-z)}{z-2}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C

Handbook B2, 2.1

Hence, by Cauchy's Integral Formula

$$\begin{aligned} f(0) &= \frac{1}{2\pi i}\int_C \frac{f(z)}{z}\,dz \\ &= \frac{1}{2\pi i}\int_C \frac{\mathrm{Log}(2-z)/(z-2)}{z}\,dz \end{aligned}$$

where,

$$f(0) = \mathrm{Log}(2)/(-2) = -\log_e 2/2$$

$$\int_C \frac{\mathrm{Log}(2-z)}{z(z-2)}\,dz = -\frac{\log_e 2}{2} \times 2\pi i = -i\pi\log_e 2$$

(c) [FY,JK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \mathrm{Log}(2-z)$, then

1. R is a simply-connected region

2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C
5. f is differentiable at $z = 0$

Hence, by Cauchy's 2nd Derivative Formula *Handbook B2, 3.1*

$$\begin{aligned} f^{(2)}(0) &= \frac{2!}{2\pi i}\int_C \frac{f(z)}{z^3}\,dz \\ &= \frac{1}{\pi i}\int_C \frac{\mathrm{Log}(2-z)}{z^3}\,dz \end{aligned}$$

where,

$$\begin{aligned} f'(z) &= -\frac{1}{2-z} \\ f^{(2)}(z) &= -\frac{1}{(2-z)^2} \end{aligned}$$

So,

$$\int_C \frac{\mathrm{Log}(2-z)}{z^3}\,dz = f^{(2)}(0)\pi i = -\frac{\pi i}{4}$$

Solution 5

(a) f has three simple poles at 0, 1/5 and 5. We shall use the cover-up rule to obtain the residues.

Handbook C1, 1.3

$$\begin{aligned}
\operatorname{Res}(f,0) &= \frac{z^2+1}{(5z-1)(z-5)} \\
&= \frac{1}{(-1)(-5)} \\
&= \frac{1}{5}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f,1/5) &= \frac{z^2+1}{5z(z-5)} \\
&= \frac{(1/5)^2+1}{5(1/5)(1/5-5)} \\
&= \frac{26/25}{-24/5} \\
&= -\frac{13}{60}
\end{aligned}$$

$$\begin{aligned}
\operatorname{Res}(f,5) &= \frac{z^2+1}{z(5z-1)} \\
&= \frac{25+1}{5(25-1)} \\
&= \frac{13}{60}
\end{aligned}$$

Handbook C1, 2.2

(b) We shall use the strategy for evaluating $\int_0^{2\pi} \Phi(\cos t, \sin t)\, dt$.

After replacements, we have, for $C = \{z : |z| = 1\}$

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos t}{13 - 5\cos t}\, dt &= \int_C \frac{\frac{1}{2}(z+1/z)}{13 - \frac{5}{2}(z+1/z)} \times \frac{1}{iz}\, dz \\
&= \int_C \frac{z^2+1}{26z - 5z^2 - 5} \times \frac{1}{iz}\, dz \\
&= i\int_C \frac{z^2+1}{z(5z^2-26z+5)}\, dz \\
&= i\int_C \frac{z^2+1}{z(5z-1)(z-5)}\, dz
\end{aligned}$$

$$= \quad i \int_C f(z)\, dz$$

Now, $f(z)$ is analytic on $\mathbb{C}$, a simply-connected region, except for the three singularities. The unit circle C is a simple-closed contour in $\mathbb{C}$, which does not pass through f's singularities, then by Cauchy's Residue Theorem

Handbook C1, 2.1

$$\begin{aligned}
\int_C f(z)\, dz &= 2\pi i \left(\mathrm{Res}(f, 0) + \mathrm{Res}(f, 1/5)\right) \\
&= 2\pi i \left(\frac{1}{5} - \frac{13}{60}\right) \\
&= -\frac{\pi i}{30}
\end{aligned}$$

Hence,

$$\int_0^{2\pi} \frac{\cos t}{13 - 5\cos t}\, dt = i \int_C f(z)\, dz = i\left(-\frac{\pi i}{30}\right) = \frac{\pi}{30}$$

Solution 6

Handbook C2, 2.4

(a) We apply Rouché's Theorem for both cases.

(i) Let $g_1(z) = iz^5$, then

$$|f(z) - g_1(z)| = |5z^2 - 3i| \leq 5|z|^2 + |3i| = 23 < 32 = |g_1(z)|$$

Since, f and g_1 are analytic on $\mathbb{C}$ and C_1 is a simple-closed contour in $\mathbb{C}$, f has the same number of zeros as g_1 inside C_1, namely 5, and none on C_1.

(ii) Let $g_2(z) = 5z^2$, then

$$|f(z) - g_2(z)| = |iz^5 - 3i| \leq |z|^5 + |-3i| = 4 < 5 = |g_2(z)|$$

Since, f and g_2 are analytic on $\mathbb{C}$ and C_2 is a simple-closed contour in $\mathbb{C}$, f has the same number of zeros as g_2 inside C_2, namely 2, and none on C_2.

(b) [FY]

From part (a) we know that f has $5 - 2 = 3$ zeros in the annulus $\{z : 1 \leq |z| < 2\}$. Now, since for $|z| = 1$

$$|f(z)| = |iz^5 + 5z^2 - 3i| \geq |iz^5| - 5|z|^2 - |3i| = 9 > 0$$

then f has no zeros on C_2, so it has exactly 3 zeros in the open annulus $\{z : 1 < |z| < 2\}$, hence it follows that $f(z) = 0$ has 3 solutions in the annulus.

(b) [JK]

$$|f(z)| > \left|\, \left|iz^5\right| - \left|3z^5 - 3i\right| \,\right| = |1 - 4| = 3$$

Solution 7

(a) q is continuous on $\mathbb{C}$, and its conjugate $\overline{q}(z) = z + 1 + i$ is entire, hence q is a model fluid flow. *Handbook D2, 1.14*

(b) The complex potential function, $\Omega(z)$ for q is a primitive of $\overline{q}$, so

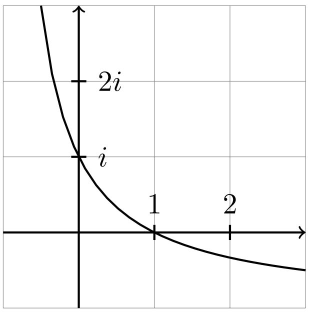
$$\Omega(z) = \frac{z^2}{2} + (1+i)z$$

Now, for $z = x + iy$

$$\begin{aligned}
\Omega(x+iy) &= \frac{(x+iy)^2}{2} + (1+i)(x+iy) \\
&= \frac{x^2 - y^2 + 2xyi}{2} + x + iy + ix - y \\
&= x^2/2 - y^2/2 + x - y + i(x + y + xy) \\
&= \Phi(x,y) + i\Psi(x,y)
\end{aligned}$$

So, q has streamline $\Psi(x,y) = x + y + xy = C$, for constant C. *Handbook D2, 2.1*

For the streamline through the point 1, $\Psi(1,0) = 1$, so, the streamline through point 1 has the equation $x + y + xy = 1$, a hyperbola.



$x + y + xy = 1$ thru 1

Since $q(1) = 2 - i$, the direction of the flow is from top-left to bottom-right.

(c) Using the results from part (a) and part (b): *Handbook D2, 1.10* *Handbook B1, 2.1*

$$\begin{aligned}
C_\Gamma &= \mathrm{Re} \int_\Gamma \overline{q}(z)\, dz \\
&= \mathrm{Re} \int_\Gamma (z + 1 + i)\, dz
\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Re}\int_0^4 (\gamma(t)+1+i)\gamma'(t)\,dt\\
&= \mathrm{Re}\int_0^4 (t+1+i)\,dt\\
&= \mathrm{Re}\left[t^2/2+(1+i)t\right]_0^4\\
&= \mathrm{Re}(16/2+4+4i)\\
&= 12
\end{aligned}
$$

Solution 8

(a) The iteration sequence

$$z_{n+1} = 15z_n^2 + 3z_n + \frac{1}{16}$$

is conjugate to the iteration sequence *Handbook D3, 2.1*

$$w_{n+1} = w_n + d$$

where

$$d = \frac{15}{16} + \frac{3}{2} - \frac{9}{4} = \frac{15 + 24 - 36}{16} = \frac{3}{16}$$

so, $w_{n+1} = w_n + \frac{3}{16}$. The conjugating function is

$$h(z) = 15z + \frac{1}{2} \times 3 = 15z + \frac{3}{2}$$

So, $w_0 = h(z_0) = h(0) = 0 + \frac{3}{2} = \frac{3}{2}$

(b) [FY,LK]

$P_{\frac{3}{16}}$ has fixed points at z, where $z^2 + \frac{3}{16} = z$, these are the solutions to the equation

$$z^2 - z + \frac{3}{16} = 0$$

So

$$z = \frac{1 \pm \sqrt{1 - 12/16}}{2} = \frac{1 \pm \sqrt{1/4}}{2} = \frac{1}{2} \pm \frac{1}{4}$$

Hence, the fixed points of $P_{\frac{3}{16}}$ are $\frac{3}{4}$ and $\frac{1}{4}$.

Now, $P'_{\frac{3}{16}}(z) = 2z$, so

$$\left| P'_{\frac{3}{16}}\left(\frac{3}{4}\right) \right| = \frac{6}{4} = \frac{3}{2} > 1$$

and

$$\left| P'_{\frac{3}{16}}\left(\frac{1}{4}\right) \right| = \frac{2}{4} = \frac{1}{2} < 1$$

Hence, $\frac{1}{4}$ is an attracting fixed point and $\frac{3}{4}$ is a repelling one. *Handbook D3, 1.5*

(b) [VC]

Alternatively, to solve the quadratic equation, we can multiply both sides by 16, so

$$\begin{aligned} & z^2 - z + \frac{3}{16} = 0 \\ \Leftrightarrow\ & 16z^2 - 16z + 3 = 0 \\ \Leftrightarrow\ & (4z-1)(4z-3) = 0 \end{aligned}$$

Hence, the roots are $\frac{1}{4}$ and $\frac{3}{4}$.[2]

(c) Let $c = -\frac{3}{2} + i$, then it appears from the diagram that c is outside the Mandelbrot set.

Handbook D3, 4.3

Handbook D3, 4.5

Using the specification for M

$$|P_c(0)| = |-3/2 + i| = \sqrt{9/4 + 1} = \sqrt{13/4} < 2$$

We go for the next iteration:

$$\begin{aligned} |P_c^{(2)}(0)| &= |(-3/2+i)^2 - 3/2 + i| \\ &= |9/4 - 1 - 3i - 3/2 + i| \\ &= |-1/4 - 2i| \\ &= \sqrt{1/16 + 4} \\ &= \sqrt{65/4} \\ &\simeq 4.0 > 2 \end{aligned}$$

Hence, c lies outside the Mandelbrot set, $c \notin M$.

[2]and no pesky formula in sight!

Solutions to Part II

Solution 9

(a) (i) $\boxed{\text{TODO}}$

(ii) $\boxed{\text{TODO}}$

(iii) $\boxed{\text{TODO}}$

(b) (i) $\boxed{\text{TODO}}$

(ii) $\boxed{\text{TODO}}$

(iii) $\boxed{\text{TODO}}$

(iv) $\boxed{\text{TODO}}$

(v) $\boxed{\text{TODO}}$

(vi) $\boxed{\text{TODO}}$

Solution 10

(a) (i) $\boxed{\text{TODO}}$

(ii) $\boxed{\text{TODO}}$

(b) (i) $\boxed{\text{TODO}}$

(ii) $\boxed{\text{TODO}}$

Solution 11

(a) $\boxed{\text{TODO}}$

(b) $\boxed{\text{TODO}}$

(c) $\boxed{\text{TODO}}$

Solution 12

(a) $\boxed{\text{TODO}}$

(b) (i) $\boxed{\text{TODO}}$

(ii) $\boxed{\text{TODO}}$

(iii) $\boxed{\text{TODO}}$

(iv) $\boxed{\text{TODO}}$