

Solutions to the M337/B 2013 Exam Paper

Fred Youhanaie (Ed.)

18 June, 2013

Copyright

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

See the by-nc-sa¹ page for details.

Acknowledgements

This is intended to be a community developed project, in particular the M337 class of 2012/2013. To this end, I have tagged answers given by others using their initials listed below. The untagged solutions, including any potential errors, are mine - FY.

DC	Dominic Corbett
FY	Fred Youhanaie
JK	J K
LK	Liga Kauke
VC	Vikki Cookson

¹<http://creativecommons.org/licenses/by-nc-sa/3.0/>

Solutions to Part I

Solution 1

(a)

$$\begin{aligned}\exp\left(3 + \frac{1}{4}\pi i\right) &= e^3 \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)\right) \\ &= \frac{e^3}{\sqrt{2}} + i \frac{e^3}{\sqrt{2}}\end{aligned}$$

(b) Let

$$w^3 = -8 = 8(\cos \pi + i \sin \pi)$$

Handbook A1, 3.3

then,

$$\begin{aligned}w &= 8^{\frac{1}{3}}(\cos(\pi/3) + i \sin(\pi/3)) \\ &= 2 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right) \\ &= 1 + \sqrt{3}i\end{aligned}$$

Handbook A2, 5.3

(c) Using the Principal α th power:

$$\begin{aligned}i^{1-2i} &= \exp((1-2i)\text{Log}(i)) \\ &= \exp((1-2i)(\log_e |i| + i\text{Arg}(i))) \\ &= \exp(i\pi/2 - 2i^2\pi/2) \\ &= \exp(\pi + i\pi/2) \\ &= e^\pi e^{i\pi/2} \\ &= e^\pi (\cos(\pi/2) + i \sin(\pi/2)) \\ &= ie^\pi\end{aligned}$$

Handbook A2, 4.4

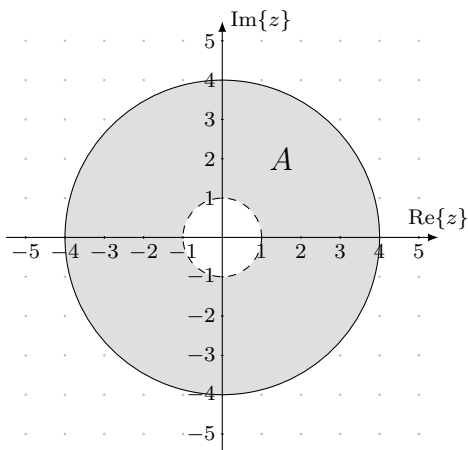
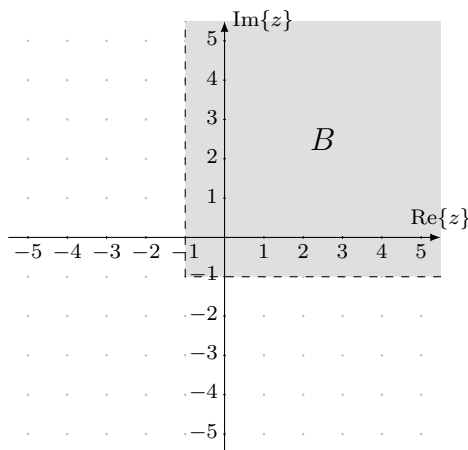
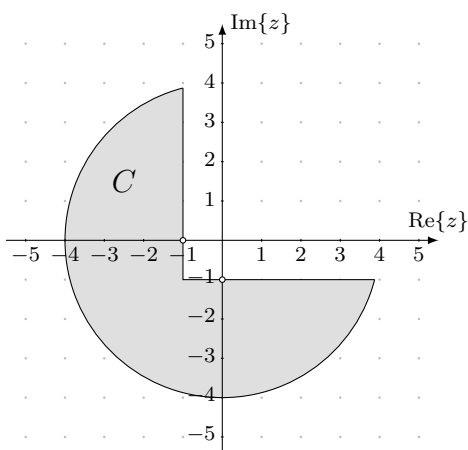
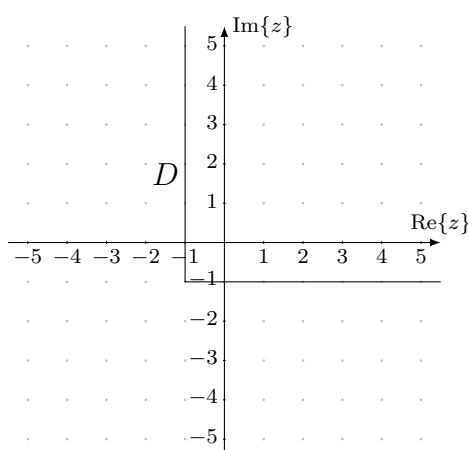
(d) Using the trigonometric functions

$$\begin{aligned}\cos(i \log_e 2) &= \frac{1}{2} (\exp(i^2 \log_e 2) + \exp(-i^2 \log_e 2)) \\ &= \frac{1}{2} (\exp(-\log_e 2) + \exp(\log_e 2)) \\ &= \frac{1}{2} (1/2 + 2) \\ &= \frac{5}{4}\end{aligned}$$

Solution 2

(a) [DC]

Below are the sketches of the four sets:

Sketch of the set $A = \{z : 1 < |z| \leq 4\}$.Sketch of the set $B = \{z : \operatorname{Re} z > -1, \operatorname{Im} z > -1\}$.Sketch of the set $C = A - B$.Sketch of the set $D = \partial B$.

(b) (i) [FY,LK]

 A is not a region, not open B is a region C is not a region, not open D is not a region, not open(ii) A is not compact, not closed B is not compact, not closed C is not compact, not closed

D is not compact, not bounded

Solution 3

(a) By definition

Handbook A2, 2.3

$$\Gamma : \gamma(t) = 2(\cos(t) + i \sin(t)) = 2e^{it}, \quad (t \in [0, 2\pi])$$

(b) We use the polar form from above,

$$\overline{\gamma(t)} = 2e^{-it}$$

$$\gamma'(t) = 2ie^{it}$$

So,

Handbook B1, 2.1

$$\begin{aligned} \int_{\Gamma} \bar{z} dz &= \int_0^{2\pi} (2e^{-it} 2ie^{it}) dt \\ &= \int_0^{2\pi} 4i dt \\ &= [4it]_0^{2\pi} \\ &= 8\pi i \end{aligned}$$

(c) [FY,LK] Let

$$f(z) = \frac{2 \sin z}{\bar{z}^2 + 1}$$

then, f is continuous on Γ by combination rules, where Γ has length $L = 4\pi$

So, using the triangle inequality

Handbook A1, 5.2(a)

$$\begin{aligned} |2 \sin z| &= \left| 2 \frac{1}{2i} (e^{iz} - e^{-iz}) \right| \\ &= |-i (e^{iz} - e^{-iz})| \\ &\leq |e^{iz}| + |e^{-iz}| \\ &= e^{\operatorname{Re} z} + e^{-\operatorname{Re} z} \\ &< 2e^2 \end{aligned}$$

And, using the backward triangle inequality

Handbook A1, 5.2(b)

$$\begin{aligned}
 |\bar{z}^2 + 1| &\geq ||z|^2 - |1|| \\
 &= |4 - 1| \\
 &= 3
 \end{aligned}$$

So,

$$\begin{aligned}
 |f(z)| &= \left| \frac{2 \sin z}{\bar{z}^2 + 1} \right| \\
 &= \frac{|2 \sin z|}{|\bar{z}^2 + 1|} \\
 &\leq \frac{2e^2}{3} \\
 &= M
 \end{aligned}$$

Handbook B1, 4.3

then by the Estimation Theorem.

$$\left| \int_{\Gamma} \frac{2 \sin z}{\bar{z}^2 + 1} dz \right| \leq ML = \frac{2e^2}{3} 4\pi = \frac{8\pi e^2}{3}$$

Solution 4

(a) Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\text{Log}(2-z)}{z^2+4}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a closed contour in R

Hence, by Cauchy's Theorem

Handbook B2, 1.4

$$\int_C \frac{\text{Log}(2-z)}{z^2+4} dz = 0$$

(b) [FY,LK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \frac{\text{Log}(2-z)}{z-2}$, then

1. R is a simply-connected region
2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C

Hence, by Cauchy's Integral Formula

Handbook B2, 2.1

$$\begin{aligned} f(0) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz \\ &= \frac{1}{2\pi i} \int_C \frac{\text{Log}(2-z)/(z-2)}{z} dz \end{aligned}$$

where,

$$f(0) = \text{Log}(2)/(-2) = -\log_e 2/2$$

$$\int_C \frac{\text{Log}(2-z)}{z(z-2)} dz = -\frac{\log_e 2}{2} \times 2\pi i = -i\pi \log_e 2$$

(c) [FY,JK]

Let $R = \{z : |z| < 2\}$, and $f(z) = \text{Log}(2-z)$, then

1. R is a simply-connected region

2. f is analytic on R
3. C is a simple-closed contour in R
4. $z = 0$ is inside C
5. f is differentiable at $z = 0$

Handbook B2, 3.1

Hence, by Cauchy's 2nd Derivative Formula

$$\begin{aligned} f^{(2)}(0) &= \frac{2!}{2\pi i} \int_C \frac{f(z)}{z^3} dz \\ &= \frac{1}{\pi i} \int_C \frac{\text{Log}(2-z)}{z^3} dz \end{aligned}$$

where,

$$\begin{aligned} f'(z) &= -\frac{1}{2-z} \\ f^{(2)}(z) &= -\frac{1}{(2-z)^2} \end{aligned}$$

So,

$$\int_C \frac{\text{Log}(2-z)}{z^3} dz = f^{(2)}(0)\pi i = -\frac{\pi i}{4}$$

Solution 5

- (a) f has three simple poles at 0, $1/5$ and 5. We shall use the cover-up rule to obtain the residues.

Handbook C1, 1.3

$$\begin{aligned}\operatorname{Res}(f, 0) &= \frac{z^2 + 1}{(5z - 1)(z - 5)} \\ &= \frac{1}{(-1)(-5)} \\ &= \frac{1}{5}\end{aligned}$$

$$\begin{aligned}\operatorname{Res}(f, 1/5) &= \frac{z^2 + 1}{5z(z - 5)} \\ &= \frac{(1/5)^2 + 1}{5(1/5)(1/5 - 5)} \\ &= \frac{26/25}{-24/5} \\ &= -\frac{13}{60}\end{aligned}$$

$$\begin{aligned}\operatorname{Res}(f, 5) &= \frac{z^2 + 1}{z(5z - 1)} \\ &= \frac{25 + 1}{5(25 - 1)} \\ &= \frac{13}{60}\end{aligned}$$

- (b) We shall use the strategy for evaluating $\int_0^{2\pi} \Phi(\cos t, \sin t) dt$. After the replacements, we have, for $C = \{z : |z| = 1\}$ *Handbook C1, 2.2*

$$\begin{aligned}\int_0^{2\pi} \frac{\cos t}{13 - 5 \cos t} dt &= \int_C \frac{\frac{1}{2}(z + 1/z)}{13 - \frac{5}{2}(z + 1/z)} \times \frac{1}{iz} dz \\ &= \int_C \frac{z^2 + 1}{26z - 5z^2 - 5} \times \frac{1}{iz} dz \\ &= i \int_C \frac{z^2 + 1}{z(5z^2 - 26z + 5)} dz \\ &= i \int_C \frac{z^2 + 1}{z(5z - 1)(z - 5)} dz \\ &= i \int_C f(z) dz\end{aligned}$$

Now, $f(z)$ is analytic on \mathbb{C} , a simply-connected region, except for the three singularities. The unit circle C is a simple-closed contour in \mathbb{C} , which does not pass through f 's singularities, then by Cauchy's Residue Theorem

Handbook C1, 2.1

$$\begin{aligned}\int_C f(z) dz &= 2\pi i (\text{Res}(f, 0) + \text{Res}(f, 1/5)) \\ &= 2\pi i \left(\frac{1}{5} - \frac{13}{60} \right) \\ &= -\frac{\pi i}{30}\end{aligned}$$

Hence,

$$\int_0^{2\pi} \frac{\cos t}{13 - 5 \cos t} dt = i \int_C f(z) dz = i \left(-\frac{\pi i}{30} \right) = \frac{\pi}{30}$$

Solution 6

(a) We apply Rouché's Theorem for both cases.

Handbook C2, 2.4

(i) Let $g_1(z) = iz^5$, then

$$|f(z) - g_1(z)| = |5z^2 - 3i| \leq 5|z|^2 + |3i| = 23 < 32 = |g_1(z)|$$

Since, f and g_1 are analytic on \mathbb{C} and C_1 is a simple-closed contour in \mathbb{C} , f has the same number of zeros as g_1 inside C_1 , namely 5, and none on C_1 .

(ii) Let $g_2(z) = 5z^2$, then

$$|f(z) - g_2(z)| = |iz^5 - 3i| \leq |z|^5 + |-3i| = 4 < 5 = |g_2(z)|$$

Since, f and g_2 are analytic on \mathbb{C} and C_2 is a simple-closed contour in \mathbb{C} , f has the same number of zeros as g_2 inside C_2 , namely 2, and none on C_2 .

(b) [FY]

From part (a) we know that f has $5 - 2 = 3$ zeros in the annulus $\{z : 1 \leq |z| < 2\}$. Now, since for $|z| = 1$

$$|f(z)| = |iz^5 + 5z^2 - 3i| \geq |iz^5| - 5|z|^2 - |3i| = 9 > 0$$

then f has no zeros on C_2 , so it has exactly 3 zeros in the open annulus $\{z : 1 < |z| < 2\}$, hence it follows that $f(z) = 0$ has 3 solutions in the annulus.

(b) [JK]

$$|f(z)| > \left| |iz^5| - |3z^5 - 3i| \right| = |1 - 4| = 3$$

Solution 7

- (a) q is continuous on \mathbb{C} , and its conjugate $\bar{q}(z) = z + 1 + i$ is entire, hence q is a model fluid flow.

Handbook D2, 1.14

- (b) The complex potential function, $\Omega(z)$, for q is a primitive of \bar{q} , so

$$\Omega(z) = \frac{z^2}{2} + (1 + i)z$$

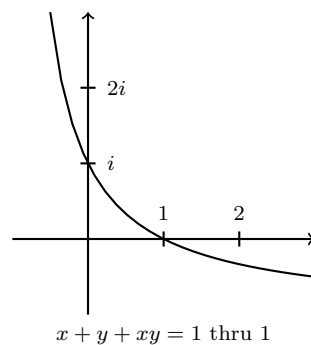
Now, for $z = x + iy$

$$\begin{aligned}\Omega(x + iy) &= \frac{(x + iy)^2}{2} + (1 + i)(x + iy) \\ &= \frac{x^2 - y^2 + 2xyi}{2} + x + iy + ix - y \\ &= x^2/2 - y^2/2 + x - y + i(x + y + xy) \\ &= \Phi(x, y) + i\Psi(x, y)\end{aligned}$$

Handbook D2, 2.1

So, q has streamline $\Psi(x, y) = x + y + xy = C$, for constant C .

For the streamline through the point 1, $\Psi(1, 0) = 1$, so, the streamline through point 1 has the equation $x + y + xy = 1$, a hyperbola.



Since $q(1) = 2 - i$, the direction of the flow is from top-left to bottom-right.

Handbook D2, 1.10

- (c) Using the results from part (a) and part (b):

Handbook B1, 2.1

$$\begin{aligned}C_{\Gamma} &= \operatorname{Re} \int_{\Gamma} \bar{q}(z) \, dz \\&= \operatorname{Re} \int_{\Gamma} (z + 1 + i) \, dz \\&= \operatorname{Re} \int_0^4 (\gamma(t) + 1 + i) \gamma'(t) \, dt \\&= \operatorname{Re} \int_0^4 (t + 1 + i) \, dt \\&= \operatorname{Re} [t^2/2 + (1 + i)t]_0^4 \\&= \operatorname{Re}(16/2 + 4 + 4i) \\&= 12\end{aligned}$$

Solution 8

(a) The iteration sequence

$$z_{n+1} = 15z_n^2 + 3z_n + \frac{1}{16}$$

Handbook D3, 2.1

is conjugate to the iteration sequence

$$w_{n+1} = w_n + d$$

where

$$d = \frac{15}{16} + \frac{3}{2} - \frac{9}{4} = \frac{15 + 24 - 36}{16} = \frac{3}{16}$$

so, $w_{n+1} = w_n + \frac{3}{16}$. The conjugating function is

$$h(z) = 15z + \frac{1}{2} \times 3 = 15z + \frac{3}{2}$$

So, $w_0 = h(z_0) = h(0) = 0 + \frac{3}{2} = \frac{3}{2}$

(b) [FY,LK]

$P_{\frac{3}{16}}$ has fixed points at z , where $z^2 + \frac{3}{16} = z$, these are the solutions to the equation

$$z^2 - z + \frac{3}{16} = 0$$

So

$$z = \frac{1 \pm \sqrt{1 - 12/16}}{2} = \frac{1 \pm \sqrt{1/4}}{2} = \frac{1}{2} \pm \frac{1}{4}$$

Hence, the fixed points of $P_{\frac{3}{16}}$ are $\frac{3}{4}$ and $\frac{1}{4}$.

Now, $P'_{\frac{3}{16}}(z) = 2z$, so

$$\left| P'_{\frac{3}{16}} \left(\frac{3}{4} \right) \right| = \frac{6}{4} = \frac{3}{2} > 1$$

and

$$\left| P'_{\frac{3}{16}} \left(\frac{1}{4} \right) \right| = \frac{2}{4} = \frac{1}{2} < 1$$

Handbook D3, 1.5

Hence, $\frac{1}{4}$ is an attracting fixed point and $\frac{3}{4}$ is a repelling one.

(b) [VC]

Alternatively, to solve the quadratic equation, we can multiply both sides by 16, so

$$\begin{aligned} z^2 - z + \frac{3}{16} &= 0 \\ \Leftrightarrow 16z^2 - 16z + 3 &= 0 \\ \Leftrightarrow (4z - 1)(4z - 3) &= 0 \end{aligned}$$

Hence, the roots are $\frac{1}{4}$ and $\frac{3}{4}$.²

(c) Let $c = -\frac{3}{2} + i$, then it appears from the diagram that c is outside the Mandelbrot set.

Handbook D3, 4.3

Using the specification for M

Handbook D3, 4.5

$$|P_c(0)| = |-3/2 + i| = \sqrt{9/4 + 1} = \sqrt{13/4} < 2$$

We go for the next iteration:

$$\begin{aligned} |P_c^{(2)}(0)| &= |(-3/2 + i)^2 - 3/2 + i| \\ &= |9/4 - 1 - 3i - 3/2 + i| \\ &= |-1/4 - 2i| \\ &= \sqrt{1/16 + 4} \\ &= \sqrt{65/4} \\ &\simeq 4.0 > 2 \end{aligned}$$

Hence, c lies outside the Mandelbrot set, $c \notin M$.

²and no pesky formula in sight!

Solutions to Part II

Solution 9

- (a) (i) Let $z = x + iy$, then

$$\begin{aligned} f(x + iy) &= (x + iy)(3 + \overline{x + iy}) + \operatorname{Re}(x + iy) \\ &= 3(x + iy) + x^2 + y^2 + x \\ &= x^2 + y^2 + 4x + i3y \\ &= u(x, y) + iv(x, y) \end{aligned}$$

where, $u(x, y) = x^2 + y^2 + 4x$ and $v(x, y) = 3y$.

- (ii) The function f is defined on \mathbb{C} . For u and v , we have

$$\frac{\partial u}{\partial x} = 2x + 4 \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \quad \frac{\partial v}{\partial y} = 3$$

The Cauchy-Riemann equations for the above partial derivatives hold when $x = -\frac{1}{2}$ and $y = 0$.

Now, $\alpha = (-\frac{1}{2}, 0)$, as f is defined on \mathbb{C} and the partial derivatives for u and v :

1. exist on \mathbb{C}
2. are continuous at α
3. satisfy the Cauchy-Riemann equations at α

then, by the Cauchy-Riemann Converse Theorem, f is differentiable at α .

Since f is only differentiable at α , then there is no region where f is analytic and contains α , hence f is not analytic at $(-\frac{1}{2}, 0)$.

- (iii) From the Cauchy-Riemann Converse Theorem:

$$\begin{aligned} f' \left(-\frac{1}{2} \right) &= \frac{\partial u}{\partial x} \left(-\frac{1}{2}, 0 \right) + i \frac{\partial v}{\partial x} \left(-\frac{1}{2}, 0 \right) \\ &= 2 \left(-\frac{1}{2} \right) + 4 + i0 \\ &= 3 \end{aligned}$$

- (b) (i) Since g is analytic on $\mathbb{C} - \{0\}$, with $g'(z) = 1 - \frac{i}{z^2}$, and, since $g'(1) = 1 - i \neq 0$, then g is conformal at 1.

Handbook A4, 2.1

Handbook A4, 2.3

Handbook A4, 1.3

Handbook A4, 2.3

Handbook A4, 4.6

- (ii) With $g(1) = 1 + i$, $|g'(1)| = |1 - i| = \sqrt{2}$ and $\text{Arg}(g'(1)) = -\frac{\pi}{4}$, the effect of g on a small disc centred at 1 is to move it to $1 + i$, scale it by $\sqrt{2}$ and rotate it by $\frac{\pi}{4}$ clockwise.

Handbook A4, 1.11

- (iii) Since, $\gamma_1(0) = e^{i0} = 1$ and $\gamma_2(1) = (1 - 1)i + 1 = 1$, then γ_1 and γ_2 meet at $t = 0$ and $t = 1$ respectively.

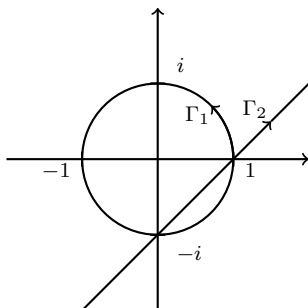
Let θ be the angle from Γ_1 to Γ_2 at 1, then³

Handbook A4, 1.12

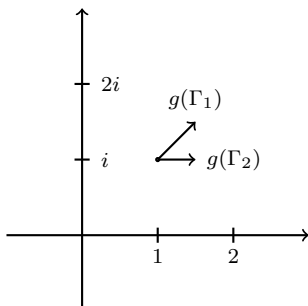
$$\theta = \text{Arg} \left(\frac{\gamma'_2(1)}{\gamma'_1(0)} \right) = \text{Arg} \left(\frac{ie^{i0}}{1+i} \right) = \text{Arg} \left(\frac{1}{2} + \frac{i}{2} \right) = \frac{\pi}{4}$$

Hence, at the point of intersection, the angle from Γ_1 to Γ_2 is $\frac{\pi}{4}$.

- (iv) The paths are shown below:



- (v) The directions of $g(\Gamma_1)$ and $g(\Gamma_2)$ are shown below:



- (vi) [DC,FY]

The image of the unit circle, $\Gamma(t) = e^{it}$ for $t \in [0, 2\pi]$, under g is as follows:

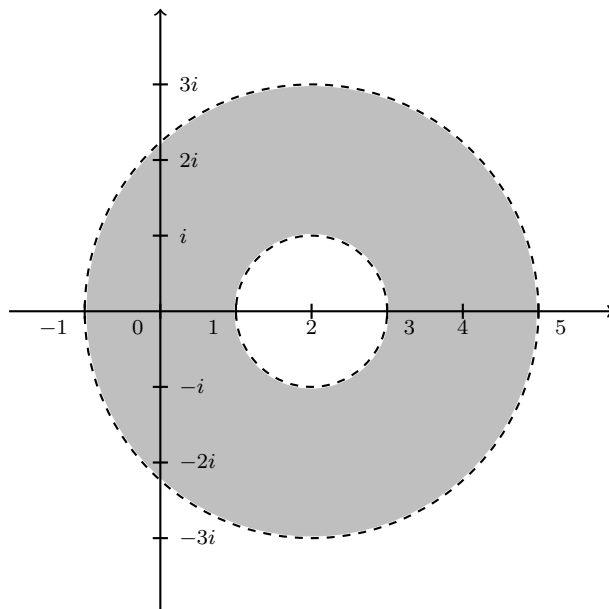
Handbook A2, 2.5

$$\begin{aligned} g(\Gamma(t)) &= g(e^{it}) \\ &= e^{it} + ie^{-it} \\ &= \cos(t) + i \sin(t) + i(\cos(-t) + i \sin(-t)) \\ &= \cos(t) + i \sin(t) + i \cos(t) - i^2 \sin(t) \\ &= (\cos(t) + \sin(t))(1 + i) \end{aligned}$$

³From FY's copy of the handbook!

Solution 10

- (a) (i) The function $f(z)$ has two simple poles at $z = 1$ and $z = 5$.
(ii) A sketch of the annulus is shown below:



TODO

- (b) (i) TODO

- (ii) TODO

Solution 11

(a) [DC]

Note that

$$f(z) = \frac{\pi \cos(\pi z)}{(4z + 3i)(4z - 3i) \sin(\pi z)}$$

so by the cover-up rule (*HB28—1.3*)

$$\begin{aligned} \operatorname{Res}(f, -i\frac{3}{4}) &= \frac{\pi \cos(-\pi i\frac{3}{4})}{4(4(-i\frac{3}{4}) - 3i) \sin(-\pi i\frac{3}{4})} \\ &= \frac{\pi \cosh(-\frac{3}{4}\pi)}{-24i^2 \sinh(-\frac{3}{4}\pi)} \\ &= \underline{\underline{-\frac{\pi}{24} \coth(\frac{3}{4}\pi)}} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Res}(f, i\frac{3}{4}) &= \frac{\pi \cos(\pi i\frac{3}{4})}{4(4(i\frac{3}{4}) + 3i) \sin(\pi i\frac{3}{4})} \\ &= \frac{\pi \cosh(\frac{3}{4}\pi)}{24i^2 \sinh(\frac{3}{4}\pi)} \\ &= \underline{\underline{-\frac{\pi}{24} \coth(\frac{3}{4}\pi)}} \end{aligned}$$

and by the g/h rule (*HB28—1.2*)

$$\begin{aligned} \operatorname{Res}(f, 0) &= \frac{\pi \cos(\pi(0))}{(4(0) + 3i)(4(0) - 3i) \pi \cos(\pi(0))} \\ &= \frac{1}{(3i)(-3i)} \\ &= \underline{\underline{\frac{1}{9}}} \end{aligned}$$

(b) [DC]

$\phi(n) = \frac{1}{16n^2+9}$ is an even function which is analytic on \mathbb{C} except for poles at the points $\frac{3}{4}i$ and $-\frac{3}{4}i$. Now, if S_N is the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$, then its length is $L = 4(2N + 1)$ and

$$|\cot \pi z| \leq 2 \quad \text{for } z \in S_N \quad (\text{HB30—4.2})$$

and since $|z| \geq N + \frac{1}{2}$ for $z \in S_N$, we have

$$\begin{aligned} |16z^2 + 9| &\geq ||16z^2| - 9| \quad (HB11-5.2) \\ &= |16|z|^2 - 9| \\ &\geq |16(N + \frac{1}{2})^2 - 9| \quad \text{for } z \in S_N \\ \therefore |f(z)| &= \left| \frac{\pi \cot(\pi z)}{16z^2 + 9} \right| \leq \frac{2\pi}{16(N + \frac{1}{2})^2 - 9} = M \quad \text{for } z \in S_N \end{aligned}$$

Hence, by the Estimation Theorem,

$$\left| \int_{S_N} f(z) dz \right| \leq \frac{2\pi}{16(N + \frac{1}{2})^2 - 9} \cdot 4(2N + 1),$$

which tends to 0 as $N \rightarrow \infty$, that is

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0$$

It follows by *HB30-4.1* that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} &= -\frac{1}{2} (\text{Res}(f, 0) + \text{Res}(f, -i\frac{3}{4}) + \text{Res}(f, i\frac{3}{4})) \\ &= -\frac{1}{2} \left(\frac{1}{9} - \frac{\pi}{24} \coth(\frac{3}{4}\pi) - \frac{\pi}{24} \coth(\frac{3}{4}\pi) \right) \\ &= \underline{\underline{\frac{\pi}{24} \coth(\frac{3}{4}\pi) - \frac{1}{18}}} \end{aligned}$$

(c) [DC]

Since

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 9} &= \sum_{n=-\infty}^{-1} \frac{1}{16n^2 + 9} + \frac{1}{16(0)^2 + 9} + \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \\ &= \sum_{n=\infty}^1 \frac{1}{16(-n)^2 + 9} + \frac{1}{9} + \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \\ &= \frac{1}{9} + 2 \sum_{n=1}^{\infty} \frac{1}{16n^2 + 9} \quad (\text{sum of positive reals indp't of order}) \end{aligned}$$

we can simply substitute in our result from part (b) to give

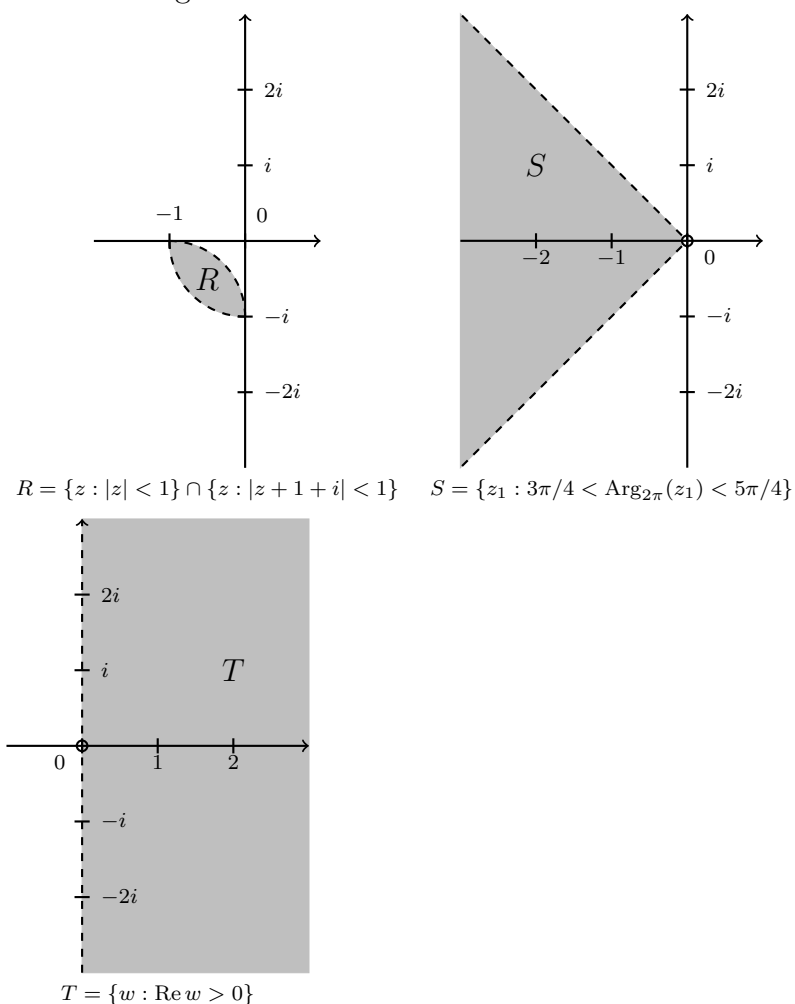
$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{16n^2 + 9} &= \frac{1}{9} + \frac{2\pi}{24} \coth(\frac{3}{4}\pi) - \frac{2}{18} \\ &= \underline{\underline{\frac{\pi}{12} \coth(\frac{3}{4}\pi)}} \quad \text{QED} \end{aligned}$$

Solution 12

- (a) We have the mapping from $\alpha = -1$, $\beta = \infty$ and $\gamma = -i$ to the standard triple of points, $\alpha' = 0$, $\beta' = 1$ and $\gamma' = \infty$ respectively, hence

$$\begin{aligned}\hat{f}(z) &= \frac{(z - \alpha)(\beta - \gamma)}{(z - \gamma)(\beta - \alpha)} \\ &= \frac{(z - (-1))(\infty - (-i))}{(z - (-i))(\infty - (-1))} \\ &= \frac{z + 1}{z + i}\end{aligned}$$

- (b) (i) The three regions are shown below:



- (ii) The two boundaries (arcs) on R map to the two boundaries (rays) in S .

R is bounded, and $-i$ on R is mapped to ∞ on S , which is unbounded.

The angle of intersection of the arcs on R at -1 is $\frac{\pi}{2}$, this angle is preserved on the intersection of the two boundary rays of S at 0 . The point ∞ is outside the bounded region R , this point is mapped to 1 , which is also outside S .

The point $\frac{1}{2}(-1-i)$ is in R , this point maps to -1 , which is in S . Hence, f is a conformal mapping from R to S .

- (iii) We can map the region S to the region T with the square function, $f_2(z_1) = z_1^2$, so $z_1 \in S$ is mapped to $w \in T$.

Since the Möbius Transformation, f_1 , is one-one and conformal on R and the square function, f_2 , is one-one and conformal on S , we can use their composite to map R to T , $f = f_2 \circ f_1$, hence

$$f(z) = f_2(f_1(z)) = \left(\frac{z+1}{z+i} \right)^2$$

is a one-one conformal mapping from R to T .

- (iv) For the inverse function we have $f^{-1} = f_1^{-1} \circ f_2^{-1}$, where

$$z_1 = f_2^{-1}(w) = -\sqrt{w}$$

and

$$z = f_1^{-1}(z_1) = \frac{iz_1 - 1}{-z_1 + 1}$$

Hence,

$$z = f^{-1}(w) = \frac{i(-\sqrt{w}) - 1}{-(-\sqrt{w}) + 1} = \frac{-i\sqrt{w} - 1}{\sqrt{w} + 1}$$