Class Problems 6

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Exercise 1

$$(P \land \neg Q) \lor (P \land Q)$$

P	Q	$\neg Q$	$(P \land \neg Q) \lor (P \land Q)$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	F

This table shows the truth values for the expression $(P \land \neg Q) \lor (P \land Q)$. As we can see, the expression is equivalent to P.

$$A = (A - B) \cup (A \cap B)$$

$$x \in A \iff x \in (A - B) \cup (A \cap B)$$

$$\iff x \in (A - B) \lor x \in (A \cap B)$$

$$\iff (x \in A \land x \notin B) \lor (x \in A \land x \in B)$$

$$\iff (x \in A) \land (x \in B \lor x \notin B)$$

$$\iff (x \in A)$$

Exercise 2

$$(x = y) ::= \forall z.(z \in x \iff z \in y)$$

a.

$$(x = \emptyset) ::= \forall z.(z \notin x)$$

b.

$$(x = y, z) ::= \forall t.(t \in x \iff t = y \lor t = z)$$

c.

$$(x \subseteq y) ::= \forall z.(z \in x \implies z \in y)$$

d.

$$(x = z \cup y) ::= \forall t.(t \in x \iff t \in y \lor t \in z)$$

e.

$$(x = z - y) ::= \forall t.(t \in x \iff t \in z \land t \notin y)$$

f.

$$(x = pow(y)) ::= \forall z.(z \in x \iff z \subseteq y)$$

g.

$$x = \bigcup_{z \in y} z ::= \forall t.\exists z(t \in z \land z \in \bigcup_{z \in y} z)$$

Exercise 3

a.

$$(a, b) = \{a, b\}$$

We can not define the order of elements in a set, so there is no way to define if the sequence represented by the set $\{a, b\}$ is (a, b) or (b, a).

b.

$$(a, b) = \{a, \{b\}\}$$

Let's assume a is the set $\{1\}$, and b is 2. With this definition, the pair would be represented by $\{\{1\}, \{2\}\}$. However, this definition is ambiguos, it can represent either the pair $(\{1\}, 2)$ or the pair $(\{2\}, 1)$.

c.

$$(a, b) = \{a, \{a, b\}\}$$

using the axiom of regularity, it is possible to say:

$$\forall x(x \neq \emptyset \implies (\exists y \in x)(y \cap x = \emptyset))$$

If $a = c$ and $b = d$, then $\{a, \{a, b\}\} = \{c, \{c, d\}\}$. As a belongs to the left hand side, it must belong to the right hand side as well, so either $a = c$ or $a = \{c, d\}$. if $a = \{c, d\}$, then $\{a, b\} = c$ or $\{a, b\} = \{c, d\}$.

if $\{a, b\} = c$ then $c \in \{c, d\} = a$ and $a \in c$, and this contradicts the axiom of regularity.

if $\{a, b\} = \{c, d\}$, then a is an element of a, again contradicting regularity. Hence $a = c$ must hold.

So either $\{a, b\} = \{c, d\}$ or $\{a, b\} = c$. if $\{a, b\} = c$ and $a = c$, then c is element of c, contradicting regularity. So, $a = c$ and $\{a, b\} = \{c, d\}$. So, $\{b\} = \{a, b\} - \{a\} = \{c, d\} - \{c\} = \{d\}$, so $b = d$.

Exercise 4

The first player can choose three options: a set with one element, two elements, three elements. Let's analyze by case.

a. Set with one element.

   The second player then can choose a set with three elements, and the game will happen the same way the three elements game. if he chooses a one element, the game will be won by player one. If he chooses two, the game will be won player one.

b. Set with two elements.

   The second player must choose the set with the other two elements, so he wins the game.

c. Set with three elements.

   The second player must choose the set with the other one element, so he wins the game.